OSCILLATION THEORY OF NEUTRAL DELAY DYNAMIC EQUATIONS ON TIME SCALES

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RAMI REDDY PASUPULA

SCHOOL OF MATHEMATICS AND STATISTICS
UNIVERSITY OF HYDERABAD
HYDERABAD - 500 046

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CERTIFICATE

School of Mathematics and Statistics
University of Hyderabad
Hyderabad- 500 046

Date:

This is to certify that I, P. RAMI REDDY have carried out the research embodied in the present thesis entitled **OSCILLATION THEORY OF NEUTRAL DELAY DYNAMIC EQUATIONS ON TIME SCALES** for the full period prescribed under Ph.D ordinance of the University.

I declare that, to the best of my knowledge, no part of this thesis was earlier submitted for the award of research degree of any university.

P RAMI REDDY
Candidate (Reg. No. 08MMPP03)

Dr. S. Panigrahi
Supervisor

Dean of the School

Dedicated to

my beloved

***family members, teachers, relatives, and friends**,*

who mean everything to me.

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- P. Rami Reddy

List of Symbols

$\mathbb{T}$	time scale
$\mathbb{N}$	the set of all positive integers
$\mathbb{N}_0$ or $\mathbb{W}$	the set of all non-negative integers
$\mathbb{Z}$	the set of all integers
$\mathbb{Q}$	the set of all rational numbers
$\mathbb{R}$	the set of all real numbers
σ	forward jump operator
ρ	backward jump operator
μ	graininess function
f^{Δ}	delta derivative of f
$C_{rd}(\mathbb{T})$	the set of all rd-continuous functions defined on $\mathbb{T}$
$C_{rd}^{n}(\mathbb{T})$	the set of all functions whose n^{th} order delta-derivative is in $C_{rd}(\mathbb{T})$
$C(\mathbb{R}, \mathbb{R})$	the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$
$BC_{rd}(\mathbb{T}, \mathbb{R})$	the set of all bounded rd-continuous functions defined on $\mathbb{T}$
$NDDEs$	Neutral Delay Dynamic Equations
$\mid \quad \mid$	modulus
$\| \quad \|$	norm
sgn	signature function
$\not\equiv$	not equivalent
$\sum$	summation

Abstract

This thesis deals with the oscillation theory of a class of nonlinear Neutral Delay Dynamic Equations (NDDEs) on time scales. We establish some sufficient conditions under which all solutions of a NDDEs either oscillates or tends to zero for large t. In this thesis we have extended and generalizes the results obtained in the fourth order to the NDDEs of higher order. Our results in this thesis are not only new for differential and difference equations, but are also new for the generalized difference and q-difference equations and many other dynamic equations on time scales. From the structural point of view, this thesis is organized in six chapters, and we shall present them subsequently, pointing out the important results.

Chapter 1 is concerned with basic concepts and terminology of time scale calculus and an applications of NDDEs in various field. Some interesting results are also mentioned without giving the proof which are existing in the literature and necessary in the subsequent chapters.

Chapter 2 deals with oscillatory and asymptotic behavior of fourth order nonlinear neutral delay dynamic equations of the following forms

$$(r(t)(y(t)+p(t)y(\alpha(t)))^{\Delta^2})^{\Delta^2}+q(t)G(y(\beta(t)))=0 \tag{E$_1$}$$

and

$$(r(t)(y(t)+p(t)y(\alpha(t)))^{\Delta^2})^{\Delta^2}+q(t)G(y(\beta(t)))=f(t), \tag{E$_2$}$$

for $t\in[t_0,\infty)_{\mathbb{T}}$, $t_0(\geq 0)\in\mathbb{T}$, where $\mathbb{T}$ is a time scale such that $\sup\mathbb{T}=\infty$ under the assumptions

$$\int_{t_0}^{\infty}\frac{t}{r(t)}\Delta t=\infty \quad or \quad \int_{t_0}^{\infty}\frac{\sigma(t)}{r(t)}\Delta t<\infty,$$

for various ranges of $p(t)$. Sufficient conditions are also obtained for the existence of bounded positive solutions of the equation (E_2) by using **Schauder's fixed point theorem**. Examples are illustrated to show the feasibility and effectiveness of our results on different time scales.

Chapter 3 is an extension of Chapter 2. This chapter deals with the study of oscillatory and asymptotic behavior of fourth order nonlinear NDDEs with positive and negative coefficients equations of the following forms

$$(r(t)(y(t)+p(t)y(\alpha(t)))^{\Delta^2})^{\Delta^2}+q(t)G(y(\beta(t)))-h(t)H(y(\gamma(t)))=0 \qquad (\mathrm{E}_3)$$

and

$$(r(t)(y(t)+p(t)y(\alpha(t)))^{\Delta^2})^{\Delta^2}+q(t)G(y(\beta(t)))-h(t)H(y(\gamma(t)))=f(t), \qquad (\mathrm{E}_4)$$

for $t\in[t_0,\infty)_{\mathbb{T}}$, $t_0(\geq 0)\in\mathbb{T}$, where $\mathbb{T}$ is a time scale such that $\sup\mathbb{T}=\infty$ under the assumptions

$$\int_{t_0}^{\infty}\frac{t}{r(t)}\Delta t=\infty \quad or \quad \int_{t_0}^{\infty}\frac{\sigma(t)}{r(t)}\Delta t<\infty,$$

for various ranges of $p(t)$. Sufficient conditions are obtained for the existence of bounded positive solutions of the equation (E_4) by using **Krasnosel'skii's fixed point theorem**. Examples are presented to show the feasibility and effectiveness of our results on different time scales.

Chapter 4 is concerned with the study of oscillatory behavior of fourth order mixed (both delay & advanced) neutral dynamic equations of the following forms

$$\left(\frac{1}{a(t)}\left((y(t)+p(t)y(\alpha(t)))^{\Delta^2}\right)^m\right)^{\Delta^2}=q(t)f(y(\beta(t)))+r(t)g(y(\gamma(t))), \qquad (\mathrm{E}_5)$$

for $t\in[t_0,\infty)_{\mathbb{T}}$, $t_0(\geq 0)\in\mathbb{T}$, where $\mathbb{T}$ is a time scale such that $\sup\mathbb{T}=\infty$ under the assumptions

$$\int_{t_0}^{\infty}(a(t))^{\frac{1}{m}}\Delta t=\infty \quad or \quad \int_{t_0}^{\infty}(a(t))^{\frac{1}{m}}\Delta t<\infty,$$

where m is a ratio of odd positive integers, for various ranges of $p(t)$. Examples illustrating the results are included for different time scales.

Chapter 5 and Chapter 6 are extensions of Chapter 2 and Chapter 3. Chapter 5 is devoted to study the oscillatory and asymptotic behavior of higher order neutral delay dynamic equations of the following forms

$$(r(t)(y(t)+p(t)y(\alpha(t)))^{\Delta^n})^{\Delta^m}+q(t)G(y(\beta(t)))-h(t)H(y(\gamma(t)))=0 \qquad (\mathrm{E}_6)$$

and

$$(r(t)(y(t)+p(t)y(\alpha(t)))^{\Delta^n})^{\Delta^m}+q(t)G(y(\beta(t)))-h(t)H(y(\gamma(t)))=f(t), \qquad (\mathrm{E}_7)$$

for $t \in [t_0, \infty)_{\mathbb{T}}$, $t_0(\geq 0) \in \mathbb{T}$, where $\mathbb{T}$ is a time scale with $\sup \mathbb{T} = \infty$ and $m,\ n \in \mathbb{N}$ under the assumption

$$\int_{t_0}^{\infty} \frac{1}{r(t)} \Delta t = \infty,$$

for various ranges of $p(t)$. Sufficient conditions are obtained for the existence of bounded positive solutions of the equation (E_7) by using **Krasnosel'skii's fixed point theorem**. Examples are included to illustrate the validiation of the results.

This Chapter 6 deals with higher order NDDEs of the following forms

$$(r(t)(y(t)+p(t)y(\alpha(t)))^{\Delta^n})^{\Delta^2}+q(t)G(y(\beta(t)))-h(t)H(y(\gamma(t)))=0 \qquad (\mathrm{E}_8)$$

and

$$(r(t)(y(t)+p(t)y(\alpha(t)))^{\Delta^n})^{\Delta^2}+q(t)G(y(\beta(t)))-h(t)H(y(\gamma(t)))=f(t), \qquad (\mathrm{E}_9)$$

for $t \in [t_0, \infty)_{\mathbb{T}}$, $t_0(\geq 0) \in \mathbb{T}$, where $\mathbb{T}$ is a time scale with $\sup \mathbb{T} = \infty$ and $n \in \mathbb{N}$ under the assumption

$$\int_{t_0}^{\infty} \frac{(\sigma(t))^{n-1}}{r(t)} \Delta t < \infty,$$

for various ranges of $p(t)$. Sufficient conditions are obtained for the existence of bounded positive solutions of the equation (E_9) by using **Krasnosel'skii's fixed point theorem**. Examples are included to illustrate the validiation of the results.

Publications related to this thesis

1. S. Panigrahi, and P. Rami Reddy, On oscillatory and asymptotic behavior of fourth order non-linear neutral delay dynamic equations, Comput. Math. Appl. 62 (2011), 4258 - 4271. (Elsevier, USA)

2. S. Panigrahi, and P. Rami Reddy, Oscillatory and asymptotic behavior of fourth order non-linear neutral delay dynamic equations, Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis 20 (2013), 143 - 163. (Watam Press, Canada)

3. S. Panigrahi, John R. Graef, and P. Rami Reddy, Oscillation results for fourth-order non-linear neutral dynamic equations, Commun. Math. Anal. 15 (2013), 11 - 28. (Mathematical Research Publisher, USA)

4. John R. Graef, S. Panigrahi, and P. Rami Reddy, On oscillatory and asymptotic behavior of fourth order non-linear neutral delay dynamic equations with positive and negative coefficients, Math. Slovaca 64 (2) (2014), 347 - 366. (Springer, USA)

5. A. K. Tripathy, S. Panigrahi, and P. Rami Reddy, Oscillation of a class of fourth order functional dynamic equations, Functional Differential Equations 21 (1-2) (2014), 91-108. (Ariel University Center of Samariya, ISRAEL)

6. A. K. Tripathy, S. Panigrahi, and P. Rami Reddy, Oscillation of a class of fourth order mixed delay dynamic equations, Functional Differential Equations 22 (1-2) (2015), 69-91. (Ariel University Center of Samariya, ISRAEL).

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Contents

Chapter 1

Introduction

The classical mathematical models of natural occurrences were either entirely continuous or discrete. These models worked well for continuous behavior such as population growth and biological phenomena, and for a discrete behaviour such as applications of Newton's method and discretization of partial differential equations. However, these models are deficient when the behavior is some times continuous and some times discrete. The existence of both continuous and discrete behavior created the need for a different type of model. This is the concept behind the dynamic equations on time scales.

Calculus has been historically fragmented into multiple distinct theories such as differential calculus, difference calculus, quantum calculus, and many others. These theories are all about the concept of what it means to "change", but in various contexts. Time scale theory bridges the gap between the continuous and discrete analysis and expands on both theories. Differential equations are defined on an interval of the set of real numbers, while difference equations are defined on discrete sets. However, some physical systems are modelled by what is called dynamic equations because they are either differential equations, difference equations or a combination of both. This means that dynamic equations are defined on connected, discrete or combination of both type of sets. Hence time scale calculus provide a generalization of differential and difference analysis.

According E. T. Bell (1883-1960), one of the main tasks in mathematics is to hormonizing the continuous and the discrete notions including them in one comprehensive field, eliminating in this way common obscurities.

The unification and extension of continuous calculus, discrete calculus, q-calculus, and indeed arbitrary real-number calculus to time scale calculus was first accomplished by Stefan Hilger in 1998 [37], supervised by Bernd Aulbach and subsequent landmark papers ([38], [39]), as a way to unify the seemingly disparate fields of discrete dynamical systems (i.e., difference equations) and continuous dynamical systems (i.e., differential equations). Since Stefan Hilger introduced the time scale calculus, several authors have expounded on various aspects of this new theory ([3], [4]), and the monograps by Bohner and Peterson ([16], [17]), and the references cited therein. This theory is very important and useful in the mathematical modelling of several important dynamic process. As a result that the theory of a dynamic systems on time scales is developed in ([6], [7]).

As is well-known, the theory of time scales unifies continuous and discrete analysis. Calculus on time scales can be applied in a variety of important fields as in biology ([23], [40]), in Economy and in general in the field of the inequalities [16], in control theory [13], variational calculus [9], multiobjective optimization [51] and so on. The theory of dynamic equations unifies the theories of differential equations and difference equations, and it also extends these classical cases to cases “in between”, e.g., to the so-called q-difference equations, of course many other interesting time scales exists, and they give rise to plenty of applications ([3], [8], [16], [17], [20], [46], [68], [67]). For example, neutral differential and difference equations arises from many areas of applied mathematics, such as population dynamics [28], stability theory [84], circuit theory [14], bifurcation analysis [12], dynamical behavior of delayed network systems [87], and so on. Some more neutral delay differential equations appear in modeling of the network containing lossless transmission lines (as in high-speed computers in which the lossless transmission lines are used to interconnect switching circuits) ([19], [66]), in the study of vibrating mass attached to an elastic bar, as Euler type of equations in some variational problems, in the study of automatic control, and in nueromechanical systems in which inertia plays a major role ([35], [22], [58], [15]). In ([1], [2]), several authors from diverse fields have been illustrated the importance of the serious qualitative as well as quantitative study of the difference equations, in applications of the quantum calculus, in physics ([69], [67], [41], [68]), economics ([8]), and a time scale of the form of a union of disjoint closed real intervals constitutes a good background for the study of population (of plants, insects, etc.) models. Such models appear, for example, when a plant population exhibits exponential growth during the months of spring and summer, and at the begining of autumn all plants die while the seeds, remain in the ground. Similar examples concerning insect population, where all the adults die before the babies are born can be found in ([16], [68]). Interest in functional differential and difference equations is growing due to the development in science and technology and the varied applications in many areas. For examples, equations involving delay and those involving advance and a combination of both arise in nerve conduction (Life Sciences), organizational behaviour (Social Sciences), signal processing pantograph equations (Mechanical Engineering), to mention a few ([15, 22, 35, 58]). Study of such equations has been an active area of research for many researchers and recently an importance is given to the unification of continuous and discrete aspects of analysis on time scales.

There are applications of dynamic equations on time scales to quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatories [68]. Delay dynamic equations are very interesting on their own since they appear in the mathematical modeling of real world problems and in many application fields of many science branches. In fact, neutral delay dynamic equations appear in modeling of the networks containing lossless transmission lines (as in high-speed computers where the lossless transmission lines are used to interconnect switching circuits), in the study of vibrating masses attached to an elastic bar, as Euler type equations in some variational problems, in the theory of automatic control and in neuro-mechanical systems in which inertia plays a major rule. The population of a certain species, especially the ones that hibernate, can be also formulated of the form

$$(x(t) - A(t)x(\alpha(t)))^{\Delta} + B(t)x(\beta(t)) - C(t)x(\gamma(t)) = 0 \quad \text{for} \quad t \in [t_0, \infty)_{\mathbb{T}},$$

for instance, roughly talking, the coefficient inside the neutral part means that the current growth rate of the population depends on the earlier growth rate, one of the coefficients out side the neutral part is the birth rate, the other one is the death rate of the population, on the other hand, the delay associated with the coefficient related to the birth rate can be considered to be the pregnancy period of the adult individuals in the population.

The following equation of the form

$$\Delta^2(x_n + ax_{n-\tau_1} \pm bx_{n+\tau_2})^\alpha = q_n x^\beta_{n-\sigma_1} + p_n x^\beta_{n+\sigma_2},$$

where $n \in \mathbb{N}(n_0) = \{n_0, n_0+1, \cdots\}$, n_0 is a non negative integer, a, b are real non negative constants, τ_1, τ_2, σ_1, and σ_2 are positive integers, $\{q_n\}$ and $\{p_n\}$ are positive real sequences and α, β are ratio of odd positive integers with $\beta \geq 1$, arises in a number of important applications such as problems in population dynamics when maturation and gestation are included, in cobweb models, in economics where demand depends on the price at an earlier time and in electric networks containing lossless transmission lines etc.

Fourth order NDDEs which has application in the study of mathematical models of deflection beams. These beams, which appear in many structures, deflect under their own weight or under the influence of some external forces. In [34], explained the bending of an elastic beams with simply-supported ends under an external force $e(x)$ is described by the boundary-value problem

$$\frac{d^4u}{dx^4} = e(x), \quad 0 < x < 1$$

under the assumption $u(0) = u(1) = u''(0) = u''(1) = 0$. Moreover, in [83] proved that linear bounary value problem

$$\frac{d^4u}{dx^4} + g(x)u = e(x), \quad 0 < x < 1$$

under the assumption $u(0) = u(1) = u''(0) = u''(1) = 0$, $g(x)$, $e(x)$ are given real-valued continuous functions on $[0, 1]$ has exactly one solution provided $\inf_x f(x) = -\eta > \pi^4$.

The following equation of the form

$$(y(t) + P(t)y(\alpha(t)))^{(n)} + \delta Q(t)f(y(h(t))) = 0$$

arise in a variety of applications, one of the best known of which is the current in a lossless transmission line connected to a nonlinear circuit ([18], [19]). Such equations also arises as Euler equations for the minimization of functionals involving a time delay [24].

1.1 Basic calculus on time scales

This section gives an account of the terminology and definitions that will be used throughout the thesis. For more details on time scale calculus readers are advised to refer ([16], [17]) and the papers cited there in.

Definition 1.1.1. A **time scale** is an arbitrary nonempty closed subset of real numbers. The time scale is denoted by $\mathbb{T}$. Examples of time scales are

- $\mathbb{T} = \mathbb{R}$, i.e., the set of real numbers.
- $\mathbb{T} = \mathbb{N}$, i.e., the set of natural numbers.
- $\mathbb{T} = \mathbb{Z}$, i.e., the set of integers.
- $\mathbb{T} = h\mathbb{Z} = \{hz : \quad z \in \mathbb{Z}\}$, where h is a fixed positive real number.
- $\mathbb{T} = [0, 1] \cup [2, 3]$.
- $\mathbb{T} = q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\}$, where $q > 1$ is fixed and $\mathbb{N}_0$ is a non negative integer.
- $\mathbb{T} = \mathbb{N}^2 = \{t^2 : t \in \mathbb{N}\}$.
- $\mathbb{T} = P_{a,b} = \cup_{k=0}^{\infty}[k(a + b), k(a + b) + b]$, where a , $b \in \mathbb{R}$.
- $\mathbb{T} = \{\sqrt{n} : n \in \mathbb{N}_0\}$.
- $\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$, where $H_0 = 0$ and $H_n = \sum_{k=1}^{n} 1/k$, for $n \in \mathbb{N}$. But the rational numbers $\mathbb{Q}$, the complex numbers $\mathbb{C}$, and the open interval $(0, 1)$ are not time scales.

In this thesis, $\mathbb{T}$ is a time scale which is unbounded above, and $t_0 \in \mathbb{T}$ with $t_0 \geq 0$, we define the time scale interval in $\mathbb{T}$

$$[a, b]_{\mathbb{T}} := \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Other types of intervals are defined similarly. We assume that $\mathbb{T}$ has the topology which inherits from the standard topology on $\mathbb{R}$.

Definition 1.1.2. On a *time scale* $\mathbb{T}$, the **forward jump operator**, the **backward jump operator**, and the **graininess function** are defined as follows respectively:

$$\sigma(t) := \inf(t, \infty)_{\mathbb{T}}, \quad \rho(t) := \sup(-\infty, t)_{\mathbb{T}} \quad \text{and} \quad \mu(t) := \sigma(t) - t \quad \text{for} \quad t \in \mathbb{T},$$

where $\inf \phi = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \phi = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t), here ϕ denotes the empty set.

Definition 1.1.3. A point $t \in \mathbb{T}$ is said to be **left-dense** if $\inf \mathbb{T} < t$ and $\rho(t) = t$, **right-dense** if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, **left-scattered** if $\rho(t) < t$, **right-scattered** if $\sigma(t) > t$, and **isolated** if $\rho(t) < t < \sigma(t)$.

1.2 Derivative on time scale

Now we consider a function $f : \mathbb{T} \to \mathbb{R}$ and define the so-called **delta** (or **Hilger**) derivative of f at a point $t \in \mathbb{T}^k$. $\mathbb{T}^k$ is a new term derived from $\mathbb{T}$, if $\mathbb{T}$ has left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^k = \mathbb{T}$.

Definition 1.2.1. If $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by

$$f^\sigma(t) = f(\sigma(t)) \quad \text{for all} \quad t \in \mathbb{T}.$$

Definition 1.2.2. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that, given $\epsilon > 0$ there is a neighbourhood U of t $(i.e., U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0)$ such that

$$\mid [f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s] \mid \leq \epsilon \mid \sigma(t) - s \mid \quad \text{for all } s \in U.$$

We call $f^\Delta(t)$ the **delta (or Hilger)** derivative of f on $\mathbb{T}^k$.

Theorem 1.2.3. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then the following statements are hold.*

(*i*) *If f is differentiable at t, then f is continuous at t.*

(ii) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) If t is right dense, then f is differentiable at t if and only if the limit

$$\lim_{s\to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case $f^{\Delta}(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s}$.

(iv) If f is differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t).$$

Theorem 1.2.4. *Assume $f,\ g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. Then the following results are hold.*

(i) The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(f + g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

(ii) For any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$

(iii) The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).$$

(iv) If $f(t)f(\sigma(t)) \neq 0$, then $1/f$ is differentiable at t with

$$(1/f)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.$$

(v) If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is differentiable at t with

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

1.3 Integration on time scale

Definition 1.3.1. A function $f : \mathbb{T} \to \mathbb{R}$ is called **regulated** provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Definition 1.3.2. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be **right-dense continuous (rd-continuous)** provided it is continuous at each right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of all such rd- continuous functions is denoted by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are (delta) differentiable and whose (delta) derivative is rd-continuous is denoted by

$$C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R}).$$

Theorem 1.3.3. *Assume $f : \mathbb{T} \to \mathbb{R}$. Then the following statements are hold.*

(*i*) *If f is continuous, then f is rd-continuous.*

(*ii*) *If f is rd-continuous, then f is regulated.*

(*iii*) *The jump operator σ is rd-continuous.*

(*iv*) *If f is regulated or rd-continuous, then so is f^σ.*

(*v*) *Assume f is continuous. If $g : \mathbb{T} \to \mathbb{R}$ is regulated or rd-continuous, then $f \circ g$ has the property too.*

Definition 1.3.4. A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called **pre-differentiable** with (region of differentiation) D, provided $D \subset \mathbb{T}^k$, $\mathbb{T}^k - D$ is countable and contains no right-scattered elements of $\mathbb{T}$, and f is differentiable at each $t \in D$.

Theorem 1.3.5. ***(Existence of Pre-antiderivatives)***. *Let f be regulated. Then there exists a function F which is pre-differentiable with region of differentiation D such that*

$$F^\Delta(t) = f(t) \quad \textit{holds for all}\, t \in D$$

Definition 1.3.6. Assume $f : \mathbb{T} \to \mathbb{R}$ is regulated function. Any function F is a pre-antiderivative of f. We define the **indefinite integral** of a regulated function f by

$$\int f(t)\Delta t = F(t) + C,$$

where C is an arbitrary constant and F is a pre-antiderivative of f.

Definition 1.3.7. (i) The **Cauchy integral** of f is defined as

$$\int_r^s f(t)\Delta t = F(s) - F(r) \quad \text{for all} \quad r,\ s \in \mathbb{T},$$

where F is an anti-derivative of f.

(ii) Let $F : \mathbb{T} \to \mathbb{R}$ be an rd-continuous function, for $t_0 \in \mathbb{T}$ and $t \in \mathbb{T}^k$,

$$F(t) = \int_{t_0}^t f(\tau)\Delta\tau$$

is an antiderivative of f.

(iii) If $f \in C_{rd}$ and $t \in \mathbb{T}^k$,

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

Theorem 1.3.8. *If* $a,\ b,\ c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, *and* $f,\ g \in C_{rd}$, *then*

(I_1) $\int_a^b (f(t) + g(t))\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t,$

(I_2) $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t,$

(I_3) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t,$

(I_4) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t,$

(I_5) $\int_a^b f(\sigma(t))g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(t)\Delta t,$

(I_6) $\int_a^b f(t)g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b g(\sigma(t))f^\Delta(t)\Delta t,$

(I_7) $\int_a^a f(t)\Delta t = 0,$

(I_8) *if* $| f(t) | \leq g(t)$ *on* $[a, b)$, *then* $| \int_a^b f(t)\Delta t | \leq \int_a^b g(t)\Delta t$,

(I_9) *if* $f(t) \geqslant 0$ *for all* $a \leq t \leq b$, *then* $\int_a^b f(t)\Delta t \geqslant 0$,

(I_{10}) *the infinite integral of* f *are defined as* $\int_a^\infty f(s)\Delta s = \lim_{t\to\infty} \int_a^t f(s)\Delta s$.

Theorem 1.3.9. *Let* $a,\ b \in \mathbb{R}$ *and* $f \in C_{rd}$.

(i) *If* $\mathbb{T} = \mathbb{R}$, *then*

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

where the integral on the right is the usual Riemann integral from calculus.

(ii) *If* $[a, b]$ *consists only isolated points, then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t\in[a,b)} \mu(t)f(t) & \text{if} \quad a < b \\ 0 & \text{if} \quad a = b \\ -\sum_{t\in[b,a)} \mu(t)f(t) & \text{if} \quad a > b. \end{cases}$$

We now give some examples of what we have discussed so far. First when $\mathbb{T} = \mathbb{R}$, we have

$$\sigma(t) = t,\ \mu(t) \equiv 0,\ f^\Delta(t) = f'(t),\ \text{and} \int_a^b f(t)\Delta t = \int_a^b f(t)dt.$$

When $\mathbb{T} = \mathbb{Z}$, we have

$$\sigma(t) = t + 1,\ \mu(t) \equiv 1,\ f^\Delta(t) = \Delta f(t) = f(t + 1) - f(t),\ \text{and} \int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t).$$

When $\mathbb{T} = h\mathbb{Z} = \{t : t = hk, k \in \mathbb{Z}\}$ with $h(> 0)$ is a fixed real number, we have

$$\sigma(t) = t + h,\ \mu(t) \equiv h,\ f^\Delta(t) = \Delta_h f(t) = \frac{f(t + h) - f(t)}{h}, \quad \text{and}$$
$$\int_a^b f(t)\Delta t = h \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh).$$

When $\mathbb{T} = q^{\mathbb{N}_0} = \{t : t = q^k, k \in \mathbb{N}_0\}$, with $q(> 1)$ is a fixed real number, we have

$$\sigma(t) = qt,\ \mu(t) = (q-1)t,\ f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t}, \quad \text{and}$$

$$\int_a^b f(t)\Delta t = (q-1) \sum_{k=\log_q^a}^{\log_q^b - 1} f(q^k)q^k.$$

When $\mathbb{T} = \mathbb{N}_0^q = \{t : t = k^q, k \in \mathbb{N}\} \cup \{0\}$, with $q(> 0)$ is a fixed real number, we have

$$\sigma(t) = (t^{1/q} + 1)^q,\ \mu(t) = (t^{1/q} + 1)^q - t,\ f^{\Delta}(t) = \frac{f((t^{1/q} + 1)^q) - f(t)}{(t^{1/q} + 1)^q - t}, \quad \text{and}$$

$$\int_a^b f(t)\Delta t = \sum_{k=a^{1/q}}^{b^{1/q}-1} f(k^q)((k+1)^q - k^q).$$

When $\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$, where H_n is the set of hormonic numbers defined by $H_0 = 0$, $H_n = \sum_{k=1}^{n} 1/k$, $n \in \mathbb{N}_0$, we have

$$\sigma(H_n) = H_{n+1},\ \mu(H_n) = \frac{1}{n+1}, \quad \text{and} \quad f^{\Delta}(H_n) = (n+1)(f(H_{n+1}) - f(H_n)).$$

Definition 1.3.10. Let f be a real-valued function defined on an interval $[a, b]_{\mathbb{T}}$. We say that f is **increasing**, **decreasing**, **non-increasing**, and **non-decreasing** on $[a, b]_{\mathbb{T}}$ if for every t_1, $t_2 \in [a, b]_{\mathbb{T}}$ such that $t_2 > t_1$ imply $f(t_2) > f(t_1)$, $f(t_2) < f(t_1)$, $f(t_2) \leqslant f(t_1)$, and $f(t_2) \geqslant f(t_1)$ respectively. Let f be a differentiable function on $[a, b]_{\mathbb{T}}$. Then f is increasing, decreasing, non-increasing and non-decreasing on $[a, b]_{\mathbb{T}}$ if $f^{\Delta}(t) > 0$, $f^{\Delta}(t) < 0$, $f^{\Delta}(t) \leqslant 0$, and $f^{\Delta}(t) \geqslant 0$ for all $t \in [a, b]_{\mathbb{T}}$ respectively.

Theorem 1.3.11. ***(Chain Rule)*** *Assume $g : \mathbb{R} \to \mathbb{R}$ is continuous, $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}^k$, and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists c in the real interval $[t, \sigma(t)]$ with*

$$(f \circ g)^{\Delta}(t) = f'(g(c))g^{\Delta}(t).$$

1.4 Polynomials

In the sequel, we introduce the alternative definition of the generalized polynomials on time scales ([16, Section 1.6]) $h_k : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ as follows:

$$h_k(t,s) := \begin{cases} 1, & k = 0 \\ \int_s^t h_{k-1}(\theta, s)\Delta\theta, & k \in \mathbb{N} \end{cases} \tag{1.1}$$

for $s,\ t \in \mathbb{T}$. Note that, for all $s,\ t \in \mathbb{T}$ and $k \in \mathbb{N}_0 := \{n \in \mathbb{Z} : n \geqslant 0\}$, the function h_k satisfies

$$\frac{\partial}{\Delta t} h_k(t,s) = \begin{cases} 0, & k = 0 \\ h_{k-1}(t,s), & k \in \mathbb{N}. \end{cases} \tag{1.2}$$

In particular, for $\mathbb{T} = \mathbb{Z}$, we have $h_k(t,s) = (t-s)^{(k)}/k!$ for all $s,\ t \in \mathbb{Z}$ and $k \in \mathbb{N}_0$, where $^{(\cdot)}$ is the usual factorial function, and for $\mathbb{T} = \mathbb{R}$, we have $h_k(t,s) = (t-s)^k/k!$ for all $s,\ t \in \mathbb{R}$ and $k \in \mathbb{N}_0$. Note that

$$(t-s)^{(k)} := \frac{\Gamma(t-s+1)}{\Gamma(t-s-k+1)},$$

where for all $t,\ k \in \mathbb{C}$ such that the right hand side of the above equation well defined, where Γ is the gamma function.

Property 1.4.1. ([43, Property 2.1]) *Using induction and the definition given by* (1.1), *it is easy to see that* $h_k(t,s) \geqslant 0$ *holds for all* $k \in \mathbb{N}_0$ *and* $s, t \in \mathbb{T}$ *with* $t \geqslant s$ *and* $(-1)^k h_k(t,s) \geqslant 0$ *holds for all* $k \in \mathbb{N}$ *and* $s, t \in \mathbb{T}$ *with* $t \leq s$. *In view of the fact* (1.2), *for all* $k \in \mathbb{N}$, $h_k(t,s)$ *is increasing in* t *provided that* $t \geqslant s$, *and* $(-1)^k h_k(t,s)$ *is decreasing in* t *provided that* $t \leq s$. *Moreover, for all* $s,\ t \in \mathbb{T}$ *and all* $k,\ l \in \mathbb{N}_0$ *with* $l \leq k$, $h_k(t,s) \leq (t-s)^{k-l} h_l(t,s)$ *holds when* $t \geqslant s$, *while* $(-1)^k h_k(t,s) \leq (-1)^k (t-s)^{k-l} h_l(t,s)$ *when* $t \leq s$.

Lemma 1.4.2. **(Change of order of integration** ([44, Lemma 1])

Assume that $s,\ t \in \mathbb{T}$ and $f \in C_{rd}(\mathbb{T} \times \mathbb{T},\ \mathbb{R})$. Then

$$\int_s^t \int_\theta^t f(\theta, \xi)\Delta\xi\Delta\theta = \int_s^t \int_s^{\sigma(\xi)} f(\theta,\xi)\Delta\theta\Delta\xi. \tag{1.3}$$

Corollary 1.4.3. ([44, Corollary 1]) *Assume that* $n \in \mathbb{N}$*,* $s,\ t \in \mathbb{T}$ *and* $f \in C_{rd}(\mathbb{T}, \mathbb{R})$*. Then*

$$\int_s^t \int_{\theta_{n+1}}^t \cdots \int_{\theta_2}^t f(\theta_1)\Delta\theta_1\Delta\theta_2 \ldots \Delta\theta_{n+1} = (-1)^n \int_s^t h_n(s, \sigma(\theta)) f(\theta)\Delta\theta. \tag{1.4}$$

As an immediate consequence of Theorem 1.3.8 and Corollary 1.4.3, we have

Corollary 1.4.4. *Assume that* $n \in \mathbb{N}$*,* $s,\ t \in \mathbb{T}$ *and* $f \in C_{rd}(\mathbb{T}, \mathbb{R})$*. Then*

$$\int_s^t \int_s^{\theta_{n+1}} \cdots \int_s^{\theta_2} f(\theta_1)\Delta\theta_1\Delta\theta_2 \cdots \Delta\theta_{n+1} = \int_s^t h_n(t, \sigma(\theta)) f(\theta)\Delta\theta. \tag{1.5}$$

The following is the dynamic generalization of the well-known Taylor's formula from the continuous and the discrete calculus.

Lemma 1.4.5. (**Taylor's formula** (see [16, Theorem 1.113]) If $n \in \mathbb{N}$, $s \in \mathbb{T}$ and $f \in C_{rd}^n(\mathbb{T}, \mathbb{R})$, then

$$f(t) = \sum_{k=0}^{n-1} h_k(t, s) f^{\Delta^k}(s) + \int_s^t h_{n-1}(t, \sigma(\theta)) f^{\Delta^n}(\theta)\Delta\theta \quad \text{for } t \in \mathbb{T}.$$

Applying Lemma 1.4.5, for the function $f = h_{k+l}(.\ , s)$, where $s \in \mathbb{T}$, $k \in \mathbb{N}$ and $l \in \mathbb{N}_0$, we get the following result.

Lemma 1.4.6. ([47, Lemma 2]) If $k \in \mathbb{N}$, $l \in \mathbb{N}_0$ and $s \in \mathbb{T}$, then

$$h_{k+l}(t, s) = \int_s^t h_{k-1}(t, \sigma(\theta)) h_l(\theta, s)\Delta\theta \quad \text{for } t \in \mathbb{T}. \tag{1.6}$$

As an immediate consequence of Lemma 1.4.6, we can give the following alternative definition of the generalized polynomials:

$$h_k(t, s) := \begin{cases} 1, & k = 0 \\ \int_s^t h_{k-1}(t, \sigma(\theta))\Delta\theta, & k \in \mathbb{N} \end{cases} \tag{1.7}$$

for $s, t \in \mathbb{T}$. From (1.7), we obtain

$$\frac{\partial}{\Delta s} h_k(t, s) = \begin{cases} 0, & k = 0 \\ -h_{k-1}(t, \sigma(s)), & k \in \mathbb{N}. \end{cases} \tag{1.8}$$

Lemma 1.4.7. ([43, Lemma 2.2]) Let $n \in \mathbb{N}_0$, $\sup \mathbb{T} = \infty$, $f \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ and let $s \in \mathbb{T}$, then each of the following is true:

(I_1) $(-1)^n \int_s^\infty h_n(s, \sigma(\theta)) f(\theta) \Delta\theta < \infty$ implies $(-1)^n \int_t^\infty h_n(t, \sigma(\theta)) f(\theta) \Delta\theta < \infty$ for all $t \in \mathbb{T}$,

(I_2) $(-1)^n \int_s^\infty h_n(s, \sigma(\theta)) f(\theta) \Delta\theta = \infty$ implies $(-1)^n \int_t^\infty h_n(t, \sigma(\theta)) f(\theta) \Delta\theta = \infty$ for all $t \in \mathbb{T}$.

Lemma 1.4.8. ([43, Lemma 2.3]) Let $n \in \mathbb{N}_0$, $\sup \mathbb{T} = \infty$, $f \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$, and let $s \in \mathbb{T}$. Then

$$(-1)^n \int_s^\infty h_n(s, \sigma(\theta)) f(\theta) \Delta\theta < \infty$$

implies that each of the following is true:

(I_1) $(-1)^j \int_t^\infty h_j(t, \sigma(\theta)) f(\theta) \Delta\theta$ is decreasing for all $t \in \mathbb{T}$ and all $j \in [0, n]_{\mathbb{Z}}$,

(I_2) $\lim_{t\to\infty} \int_t^\infty h_j(t, \sigma(\theta)) f(\theta) \Delta\theta = 0$ for all $j \in [0, n]_{\mathbb{Z}}$,

(I_3) $(-1)^j \int_s^\infty h_j(s, \sigma(\theta)) f(\theta) \Delta\theta < \infty$ for all $j \in [0, n)_{\mathbb{Z}}$.

1.5 Important Lemmas and Theorems

In the following we would like to state some important lemmas and theorems which will be used to obtain our main results, in subsequent chapters.

Lemma 1.5.1. ([62, Lemma 1]) Let g be a continuous monotone function with $\lim_{t\to\infty} g(t) = \infty$. Set

$$z(t) = x(t) + c(t)x(g(t)).$$

If $x(t)$ is eventually positive, $\liminf_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} z(t) = L \in \mathbb{R}$ exists, then $L = 0$ provided that for some real numbers c_1, c_2, c_3, and c_4 the functions $c(t)$ is in one of the following ranges:

(a) $-\infty < c_1 \le c(t) \le 0$, (b) $0 \le c(t) \le c_2 < 1$, (c) $1 < c_3 \le c(t) \le c_4 < \infty$.

Theorem 1.5.2. *(**Schauder's fixed point theorem** [33, P. 29]) Let M be a closed, convex and non-empty subset of a Banach space X. Let $T : M \to M$ be a continuous function such that TM is a relatively compact subset of X. Then T has at least one fixed point in M. That is, there exists an $x \in M$ such that $Tx = x$.*

Theorem 1.5.3. *(**Krasnosel'skii's fixed point theorem** [45, Lemma 3]) Let S be a bounded, convex and closed subset of the Banach space X. Suppose there exists two operators $A,\ B : S \to X$ such that*

(i) $Ax + By \in S$ for all $x, y \in S$,

(ii) A is a contraction mapping,

(iii) B is completely continuous.

Then $A + B$ has a fixed point in S, that is, $Ax + Bx = x$ for some $x \in S$.

Lemma 1.5.4. (**Kiguradze's Lemma** [29, Lemma 2.1]) (This lemma is a particular case of Lemma 5.1.2, which is in Chapter 5) Let $u(t) \in C^n_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ and $u^{\Delta^n}(t) \leq (\geq) \not\equiv 0$ on $[t_1, \infty)_{\mathbb{T}}$ for any $t_1 \geq t_0$. Then there exists a $t_x \geq t_1$ and an integer $l \in [0, n]_{\mathbb{Z}}$ such that $n + l$ is odd (even) and

$$l > 0 \text{ implies } u^{\Delta^i}(t) > 0 \text{ for } t \geq t_x \ (i = 1,\ 2,\ 3,\ \cdots,\ l - 1),$$

$$l \leq n - 1 \text{ implies } (-1)^{l+i} u^{\Delta^i}(t) > 0 \text{ for } t \geq t_x \ (i = l,\ l + 1,\ l + 2,\ \cdots,\ n - 1).$$

1.6 Important remarks

The following remarks will be needed throughout the thesis.

Remark 1.6.1. All functional inequalities considered in this thesis are assumed to hold eventually, that is they are satisfied for all t large enough.

Remark 1.6.2. Throughout this thesis by $t \geq s$ for $t,\ s \in \mathbb{T}$ we shall mean $t \in [s, \infty) \cap \mathbb{T} := [s, \infty)_{\mathbb{T}}$.

Remark 1.6.3. The equations (E_1), (E_2), (E_3), (E_4), (E_5), (E_6), (E_7), (E_8), and (E_9) are used in this thesis that are mentioned in the abstract of our work.

Remark 1.6.4. Sufficient conditions which considered in each chapter are mutually exclusive from other chapters.

1.7 Brief overview of the thesis

Most publications on time scales are either focused on the generalization of results well known for either differential or difference equations (or both) to the relevant models on time scales or develop the theory of equations on time scales independently, thus contributing to the theory of both discrete and continuous equations. From the literature survey it is evident that there are fewer results concerning the oscillatory and asymptotic behaviour of solution of fourth order non-linear neutral delay dynamic equation on time scales. However, many researchers have studied first/second/third order differential, difference and dynamic equations independently and some of them generalized the results to higher order ([10], [11], [30], [42], [52], [59], [60], [64], [73]) and the references cited there in. From the structural point of view, this thesis organized in six chapters.

In Chapter 2, we establish the oscillatory and asymptotic behaviour of the solutions of a fourth order homogeneous NDDEs (E_1) and its associated forced equation (E_2). This chapter consists of five sections. To study the behaviour of solutions of (E_1) and (E_2) with the assumption

$$\int_{t_0}^{\infty} \frac{t}{r(t)} \Delta t = \infty,$$

we have developed two lemma's namely Lemma 2.1.1 and Lemma 2.1.2 in Section 2.1. In Section 2.1, we have studied the oscillation criteria of (E_1) for various ranges of $p(t)$ with $0 \le p(t) \le p < \infty$, $-1 < p \le p(t) \le 0$ and $-\infty < p_1 \le p(t) \le p_2 < -1$. In Section 2.2, we have studied the oscillatory and asymptotic behaviour of solution of (E_2) for various ranges of $p(t)$ with $0 \le p(t) \le p < \infty$, $-1 < p \le p(t) \le 0$, $-\infty < p \le p(t) \le 0$

and $0 \leq p(t) \leq p_1 < 1$ or $1 < p_2 \leq p(t) \leq p_3 < \infty$ and $-\infty < p_1 \leq p(t) \leq p_2 < -1$. Moreover, we have established the existence of bounded positive solution of (E_2) by using **Schauder's fixed point theorem** for the range of $p(t)$ with $0 \leq p(t) \leq p < 1$. In Section 2.3 and Section 2.4, we have studied the oscillatory behaviour of solutions of the equations (E_1) and (E_2) under the assumption

$$\int_{t_0}^{\infty} \frac{\sigma(t)}{r(t)} \Delta t < \infty$$

with the help of Lemma 2.3.1, Lemma 2.3.2 and Lemma 2.3.4. In Section 2.3, we have studied the oscillation criteria of (E_1) for various ranges of $p(t)$ with $0 \leq p(t) \leq p < \infty$, $0 \leq p(t) \leq p < 1$, $-1 < p \leq p(t) \leq 0$ and $-\infty < p_1 \leq p(t) \leq p_2 < -1$. In Section 2.4, we have studied the oscillation criteria of (E_2) for various ranges of $p(t)$ with $0 \leq p(t) \leq p < \infty$, $-1 < p \leq p(t) \leq 0$ and $-\infty < p \leq p(t) \leq 0$. Moreover, we have established the existence of bounded positive solution of (E_2) by using **Schauder's fixed point theorem** for the range of $p(t)$ with $0 \leq p(t) \leq p < 1$ and $-1 < p \leq p(t) \leq 0$. Section 2.5 deals with the conclusion.

Chapter 3, deals with the oscillatory and asymptotic behaviour of the solutions of a fourth order homogeneous NDDEs with positive and negative coefficients (E_3) and its associated forced equation (E_4) under the assumption

$$\int_{t_0}^{\infty} \frac{t}{r(t)} \Delta t = \infty$$

with the help of Lemma 2.1.1 and Lemma 2.1.2. In Section 3.1, we have studied the oscillation criteria of (E_3) for various ranges of $p(t)$ with $0 \leq p(t) \leq p_1 < 1$ or $1 < p_2 \leq p(t) \leq p_3 < \infty$, $-1 < p_4 \leq p(t) \leq 0$ and $-\infty < p_5 \leq p(t) \leq p_6 < -1$. In Section 3.2, we have shown that under certain sufficient conditions the solution of (E_4) is either oscillation or tends to zero for large t for various ranges of $p(t)$ with $0 \leq p(t) \leq p < \infty$ and $-1 < p(t) \leq 0$. Moreover, we have established the existence of bounded positive solution of (E_4) by using **Krasnosel'skii's fixed point theorem** for $p(t)$ with $0 < p(t) < 1$. In Section 3.3 and Section 3.4, we have studied the behaviour of solutions of equations (E_3) and (E_4) respectively with the assumption

$$\int_{t_0}^{\infty} \frac{\sigma(t)}{r(t)} \Delta t < \infty$$

with the help of Lemma 2.3.1, Lemma 2.3.2 and Lemma 2.3.4. In Section 3.3, we studied the oscillation criteria of (E_3) for various ranges of $p(t)$ with $0 \leq p(t) \leq p < \infty$, $0 \leq p(t) \leq p < 1$, $-1 < p_4 \leq p(t) \leq 0$, $-\infty < p_5 \leq p(t) \leq p_6 < -1$. In Section 3.4, we have studied the oscillation criteria of (E_2) for various ranges of $p(t)$ with $0 \leq p(t) \leq p < \infty$, $-1 < p(t) \leq 0$ and $-1 < p_4 \leq p(t) \leq 0$. Moreover, we have established the existence of bounded positive solution of (E_4) by using **Krasnosel'skii's fixed point theorem** for $1 < p_1 \leq p(t) \leq p_2 < \frac{1}{2}p_1^2 < \infty$. Section 3.5 deals with the conclusion.

In Chapter 4, we have studied the oscillation criteria of a fourth order mixed delay dynamic equation (E_5). In Section 4.1 and Section 4.2, we have studied the equation (E_5) under the assumptions

$$\int_{t_0}^{\infty} (a(t))^{1/m} \Delta t = \infty \; and \; \int_{t_0}^{\infty} (a(t))^{1/m} \Delta t < \infty$$

respectively, for various ranges of $p(t)$ with $0 \leq p(t) \leq a < \infty$, $-\infty < -b \leq p(t) \leq 0$ and with some additional conditions with various techniques. Section 4.3 deals with the conclusion.

In Chapter 5, we have studied the oscillatory and asymptotic behaviour of solutions of a higher order homogeneous NDDEs with positive and negative coefficients (E_6) and its associated forced equation (E_7) under the assumption

$$\int_{t_0}^{\infty} \frac{1}{r(t)} \Delta t = \infty.$$

This chapter consists of three sections. To study the behaviour of solutions of equations (E_6) and (E_7), first we have developed four lemma's namely Lemma 5.1.1, Lemma 5.1.2, Lemma 5.1.3 and Lemma 5.1.4 in Section 5.1. In Section 5.1, we have studied the oscillation criteria and asymptotic behaviour of (E_6) for various ranges of $p(t)$ with

$0 \le p(t) \le p < \infty$, $-1 < p_4 \le p(t) \le 0$ and $-\infty < p_5 \le p(t) \le p_6 < -1$. Section 5.2, deals with the oscillation criteria and asymptotic behaviour of solutions of (E_7) for various ranges of $p(t)$ with $0 \le p(t) \le p < \infty$, and $-\infty < p \le p(t) \le 0$. Moreover, we have established the existence of bounded positive solutions of (E_7) by using **Krasnosel'skii's fixed point theorem** for $0 \le p(t) < 1$. Section 5.3 deals with the conclusion.

Chapter 6, deals with the oscillatory and asymptotic behaviour of solutions of a higher order homogeneous NDDEs with positive and negative coefficients (E_8) and its associated forced equation (E_9) have studied under the assumption

$$\int_{t_0}^{\infty} \frac{(\sigma(t))^{(n-1)}}{r(t)} \Delta t < \infty.$$

This chapter consists of three sections. In Section 6.1, we have developed new lemma's those are Lemma 6.1.3, Lemma 6.1.4, Lemma 6.1.5 and Lemma 6.1.6. Also we have studied the oscillation criteria of (E_8) for various ranges of $p(t)$ with $0 \le p(t) \le p < \infty$, $0 \le p(t) \le p < 1$, $-1 < p_1 \le p(t) \le 0$ and $-\infty < p_2 \le p(t) \le p_3 < -1$. In Section 6.2, we have studied the oscillation criteria of (E_9) for various ranges of $p(t)$ with $0 \le p(t) \le p < \infty$, $-1 < p(t) \le 0$ and $-1 < p_4 \le p(t) \le 0$. Moreover, we have established the existence of bounded positive solution of (E_9) by using **Krasnosel'skii's fixed point theorem** for $1 < p_5 \le p(t) \le p_6 < \frac{1}{2}p_5^2 < \infty$. Section 6.3, deals with the conclusion.

Chapter 2

Fourth Order Neutral Delay Dynamic Equations

In ([48],[49]), Kusano and Naito have studied oscillatory behaviour of solutions of a class of fourth order nonlinear differential equations of the form

$$(r(t)y''(t))'' + yF(y^2(t), t) = 0,$$

where r and F are continuous and positive on $[0, \infty)$ and $[0, \infty) \times [0, \infty)$ respectively under the assumptions

$$\int_{t_0}^{\infty} \frac{t}{r(t)} dt = \infty \quad or \quad \int_{t_0}^{\infty} \frac{t}{r(t)} dt < \infty.$$

Jurang and Bin [27] have been investigated the behaviour of solutions of the fourth order non-linear difference equation of the form

$$\Delta^2(r_n \Delta^2 y_n) + f(n, y_n) = 0, \quad n \in N(n_0), \tag{2.1}$$

under the assumption

$$\sum_{n=n_0}^{\infty} \frac{n}{r_n} = \infty,$$

where $n \in \mathbb{N}_0 = \{n_0,\ n_0 + 1,\ n_0 + 2,\ \dots\}$, n_0 is a nonnegative integer and Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$, and the real sequences $\{r_n\}$ and the function f satisfying the following conditions:

(a) $r_n > 0$ for all $n \in \mathbb{N}_0$,

(b) $f : \mathbb{N}(n_0) \times \mathbb{R} \to \mathbb{R}$ is a continuous function $uf(n,u) > 0$ for all $u \neq 0$ and all $n \in \mathbb{N}_0$, and $f(n,.) \not\equiv 0$ eventually.

In order to complete the study of equation (2.1), again Thandapani and Arockiasamy [72] has been studied the equation (2.1), in the light of

$$\sum_{n=n_0}^{\infty} \frac{n}{r_n} < \infty.$$

In order to extend/generalize the results obtained in [27], again Thandapani and Arockiasamy [71] has considered the fourth order nonlinear difference equation of the form

$$\Delta^2(r_n \Delta^2(y_n + p_n y_{n-k})) + f(n, y_{\sigma(n)}) = 0, \quad n \in N(n_0), \tag{2.2}$$

where $f : N(n_0) \times \mathbb{R} \to \mathbb{R}$ is a continuous function with $uf(n,u) > 0$ for all $u \neq 0$, $\{r_n\}$ and $\{p_n\}$ are positive real sequences, $\{\sigma_n\}$ is an increasing sequence of integers and k is a non negative integer. They have obtained necessary and sufficient conditions for (2.2), when $0 \leqslant p_n < p < 1$ for all $n \in N(n_0)$. Clearly, if we consider $f(n, y_{\sigma(n)}) = q(n)G(y(n-k))$, then the work in [71], is a particular case of the work [75] as the ranges of $p(n)$ is concerned. Here an attempt is made to unify the results of ([56], [57], [75], [76]). Moreover, the results obtained in this chapter generalizes and extended the results of ([56], [57], [75], [76]), which are already existing in the literature.

The object of this chapter is to study the oscillatory and asymptotic behavior of solutions of a class of fourth order neutral delay dynamic equations of the following form

$$(r(t)(y(t) + p(t)y(\alpha(t)))^{\Delta^2})^{\Delta^2} + q(t)G(y(\beta(t))) = 0 \tag{E$_1$}$$

and

$$(r(t)(y(t) + p(t)y(\alpha(t)))^{\Delta^2})^{\Delta^2} + q(t)G(y(\beta(t))) = f(t), \tag{E$_2$}$$

for $t \in [t_0, \infty]_{\mathbb{T}}$, $t_0 \in \mathbb{T}$, under the assumptions

$$\int_{t_0}^{\infty} \frac{t}{r(t)} \Delta t = \infty \ \ or \ \ \int_{t_0}^{\infty} \frac{\sigma(t)}{r(t)} \Delta t < \infty,$$

where $\mathbb{T}$ is a time scale with $\sup \mathbb{T} = \infty$, $r \in C_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$, $p, f \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $q \in C_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$, $G \in C(\mathbb{R},\mathbb{R})$ is non decreasing function such that $uG(u) > 0 \ for \ u \neq 0$, $\alpha, \ \beta \in C([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ are strictly increasing delay functions such that $\lim_{t\to\infty}\alpha(t) = \infty = \lim_{t\to\infty}\beta(t)$, $\alpha(t) \leqslant t, \ \beta(t) \leqslant t$. α has the inverse $\alpha^{-1} \in C(\mathbb{T},\mathbb{T})$ when it is required.

Let $t_{-1} = \inf_{t\in[t_0,\infty)_{\mathbb{T}}}\{\alpha(t),\beta(t)\}$. By a solution of $(\mathrm{E}_1)/(\mathrm{E}_2)$, we mean a function $y \in C_{rd}([t_{-1},\infty)_{\mathbb{T}},\mathbb{R})$ such that $y(t)+p(t)y(\alpha(t)) \in C^2_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $r(t)(y(t)+p(t)y(\alpha(t)))^{\Delta^2} \in C^2_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ and satisfies $(\mathrm{E}_1)/(\mathrm{E}_2)$ identically on $[t_0,\infty)_{\mathbb{T}}$. A solution of $(\mathrm{E}_1)/(\mathrm{E}_2)$ is called oscillatory if it is neither eventually positive nor eventually negative, othewise it is nonoscillatory. In this study we do not focus our attention to eventually vanishing solutions of $(\mathrm{E}_1)/(\mathrm{E}_2)$. The equation $(\mathrm{E}_1)/(\mathrm{E}_2)$ is called oscillatory if all its solutions are oscillatory.

For example, if $\mathbb{T} = \mathbb{R}$, then $(\mathrm{E}_1)/(\mathrm{E}_2)$ becomes the fourth order nonlinear neutral delay differential equation of the form

$$(r(t)(y(t)+p(t)y(\alpha(t)))'')'' + q(t)G(y(\beta(t))) = 0$$

and

$$(r(t)(y(t)+p(t)y(\alpha(t)))'')'' + q(t)G(y(\beta(t))) = f(t).$$

If $\mathbb{T} = \mathbb{Z}$, then $(\mathrm{E}_1)/(\mathrm{E}_2)$ becomes the fourth order nonlinear neutral delay difference equation of the form

$$\Delta^2(r(t)\Delta^2(y(t)+p(t)y(\alpha(t)))) + q(t)G(y(\beta(t))) = 0$$

and

$$\Delta^2(r(t)\Delta^2(y(t)+p(t)y(\alpha(t)))) + q(t)G(y(\beta(t))) = f(t).$$

If $\mathbb{T} = h\mathbb{Z}, \ h > 0$, then $(\mathrm{E}_1)/(\mathrm{E}_2)$ becomes the fourth order nonlinear neutral delay h-difference equation of the form

$$\Delta_h^2(r(t)\Delta_h^2(y(t)+p(t)y(\alpha(t)))) + q(t)G(y(\beta(t))) = 0$$

and

$$\Delta_h^2(r(t)\Delta_h^2(y(t)+p(t)y(\alpha(t))))+q(t)G(y(\beta(t)))=f(t).$$

If $\mathbb{T}=q^{\mathbb{N}_0}=\{t:t=q^k,k\in\mathbb{N}_0\}$ with $q>1$, then $(\mathrm{E}_1)/(\mathrm{E}_2)$ becomes a fourth order q-neutral delay difference equation of the form

$$\Delta_q^2(r(t)\Delta_q^2(y(t)+p(t)y(\alpha(t))))+q(t)G(y(\beta(t)))=0$$

and

$$\Delta_q^2(r(t)\Delta_q^2(y(t)+p(t)y(\alpha(t))))+q(t)G(y(\beta(t)))=f(t).$$

If $\mathbb{T}=\mathbb{N}_0^2=\{t^2:t\in\mathbb{N}_0\}$, then equations $(\mathrm{E}_1)/(\mathrm{E}_2)$ becomes fourth order neutral delay difference equation of the form

$$\Delta_{\mathbb{N}}^2(r(t)\Delta_{\mathbb{N}}^2(y(t)+p(t)y(\alpha(t))))+q(t)G(y(\beta(t)))=0$$

and

$$\Delta_{\mathbb{N}}^2(r(t)\Delta_{\mathbb{N}}^2(y(t)+p(t)y(\alpha(t))))+q(t)G(y(\beta(t)))=f(t).$$

The following assumption (Λ) is considered throughout this chapter, where

(Λ) $(\alpha\circ\beta)(t)=(\beta\circ\alpha)(t)$ for all $t\in[t_0,\infty)_{\mathbb{T}}$ and $Q(t)=\min\{q(t),q(\alpha(t))\}$ for $t\in[t^*,\infty)_{\mathbb{T}}$, $t^*>t_0$.

2.1 Sufficient conditions for oscillation of (E_1) with $\int_{t_0}^{\infty}\frac{t}{r(t)}\Delta t=\infty$.

In this section, sufficient conditions are obtained for the oscillation of solutions of (E_1) under the assumption

$$(H_1)\qquad \int_{t_0}^{\infty}\frac{t}{r(t)}\Delta t=\infty.$$

We need the following lemmas for our work in the sequel.

Lemma 2.1.1. Let (H_1) hold. Let u be a twice rd-continuously differentiable function on $[t_0,\infty)_{\mathbb{T}}$ such that $r(t)u^{\Delta^2}(t) \in C^2_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ and $(r(t)u^{\Delta^2}(t))^{\Delta^2} \leq (\not\equiv)0$ holds for large $t \in [t_0,\infty)_{\mathbb{T}}$, $t_0 \in \mathbb{T}$. If $u(t) > 0$ ultimately, then one of the cases (a) or (b) holds for large t and if $u(t) < 0$ ultimately, then one of the cases (b), (c), (d) or (e) holds for large t, where

$$
\begin{aligned}
&(a) \quad u^{\Delta}(t) > 0, \quad u^{\Delta^2}(t) > 0 \text{ and } \quad (r(t)u^{\Delta^2}(t))^{\Delta} > 0,\\
&(b) \quad u^{\Delta}(t) > 0, \quad u^{\Delta^2}(t) < 0 \text{ and } \quad (r(t)u^{\Delta^2}(t))^{\Delta} > 0,\\
&(c) \quad u^{\Delta}(t) < 0, \quad u^{\Delta^2}(t) < 0 \text{ and } \quad (r(t)u^{\Delta^2}(t))^{\Delta} > 0,\\
&(d) \quad u^{\Delta}(t) < 0, \quad u^{\Delta^2}(t) < 0 \text{ and } \quad (r(t)u^{\Delta^2}(t))^{\Delta} < 0,\\
&(e) \quad u^{\Delta}(t) < 0, \quad u^{\Delta^2}(t) > 0 \text{ and } \quad (r(t)u^{\Delta^2}(t))^{\Delta} > 0.
\end{aligned}
$$

Proof. Since $(r(t)u^{\Delta^2}(t))^{\Delta^2} \leq 0$ for large t, then $u(t)$, $u^{\Delta}(t)$, $u^{\Delta^2}(t)$, $r(t)u^{\Delta^2}(t)$ and $(r(t)u^{\Delta^2}(t))^{\Delta}$ are monotonic and eventually of one sign on $[t_1,\infty)_{\mathbb{T}}$ $(t_1 > t_0)$, and hence there are eight possibilities. Let $u(t) > 0$ for $t \geq t_1 > t_0$. It is enough to show that cases (c), (d), (e) and the following cases, viz.,

$$
\begin{aligned}
&(f) \quad u^{\Delta}(t) < 0, \quad u^{\Delta^2}(t) > 0 \text{ and } \quad (r(t)u^{\Delta^2}(t))^{\Delta} < 0,\\
&(g) \quad u^{\Delta}(t) > 0, \quad u^{\Delta^2}(t) > 0 \text{ and } \quad (r(t)u^{\Delta^2}(t))^{\Delta} < 0,\\
&(h) \quad u^{\Delta}(t) > 0, \quad u^{\Delta^2}(t) < 0 \text{ and } \quad (r(t)u^{\Delta^2}(t))^{\Delta} < 0,
\end{aligned}
$$

do not hold. Indeed, in each of the cases (c) and (d), since $u^{\Delta^2}(t) < 0$ for large t, then $u^{\Delta}(t) < u^{\Delta}(t_2)$ for $t > t_2 \geq t_1$. By integrating from t_2 to t, we get $u(t) < 0$ for large t, which is a contradiction. In case (e), $r(t)u^{\Delta^2}(t)$ is increasing so $r(t)u^{\Delta^2}(t) > r(t_2)u^{\Delta^2}(t_2)$ for $t > t_2 \geq t_1$. That is, $u^{\Delta^2}(t) > r(t_2)u^{\Delta^2}(t_2)\frac{1}{r(t)}$. Multiplying the inequality with t on both sides and then integrating it by using integration by parts [see Theorem 1.3.8] from t_2 to t, we obtain

$$tu^{\Delta}(t) - t_2u^{\Delta}(t_2) - \int_{t_2}^{t} u^{\Delta\sigma}(\theta)\Delta\theta \geq r(t_2)u^{\Delta^2}(t_2)\int_{t_2}^{t} \frac{\theta}{r(\theta)}\Delta\theta.$$

Since $u^{\Delta^2}(t) > 0$, then $u^{\Delta}(t)$ is increasing. Therefore, the last inequality reduces to

$$tu^{\Delta}(t) - t_2u^{\Delta}(t_2) - \int_{t_2}^{t} u^{\Delta}(\theta)\Delta\theta \geq r(t_2)u^{\Delta^2}(t_2)\int_{t_2}^{t} \frac{\theta}{r(\theta)}\Delta\theta.$$

That is,

$$tu^{\Delta}(t) - t_2u^{\Delta}(t_2) + u(t_2) \geq u(t) + r(t_2)u^{\Delta^2}(t_2)\int_{t_2}^{t}\frac{\theta}{r(\theta)}\Delta\theta,$$

implies that

$$tu^{\Delta}(t) - t_2u^{\Delta}(t_2) + u(t_2) \geq r(t_2)u^{\Delta^2}(t_2)\int_{t_2}^{t}\frac{\theta}{r(\theta)}\Delta\theta.$$

Therefore, $u^{\Delta}(t) > 0$ due to (H_1), which is a contradiction. Since $(r(t)u^{\Delta^2}(t))^{\Delta}$ is monotonically decreasing and $(r(t)u^{\Delta^2}(t))^{\Delta} < 0$, then in each of the cases (f) and (g), we obtain $u^{\Delta^2}(t) < 0$ for large t, which is a contradiction. In case (h), since $(r(t)u^{\Delta^2}(t))^{\Delta}$ is decreasing, then $r(t)u^{\Delta^2}(t) \leq r(t_2)u^{\Delta^2}(t_2) + (r(t_2)u^{\Delta^2}(t_2))^{\Delta}(t - t_2)$ for $t > t_2 \geq t_1$. Hence $r(t)u^{\Delta^2}(t) < -vt$ for $t > t_3 \geq t_2$, where $v > 0$. This implies that $u^{\Delta^2}(t) < -v\frac{t}{r(t)}$ for $t > t_3 \geq t_2$. By integrating from t_3 to t, we obtain $u^{\Delta}(t) < 0$ for large t due to (H_1), which is a contradiction.

Next, suppose that $u(t) < 0$ for $t \geq t_1 > t_0$. It is enough to show that cases: (a), (f), (g) and (h) do not hold. The case (a), does not occur because in this $u(t) > 0$ ultimately. Proceeding as above we obtain a contradiction in the cases (f), (g) and (h). Thus the lemma is proved. □

Lemma 2.1.2. Let the conditions of Lemma 2.1.1 hold. If $u(t) > 0$ ultimately, then

$$u(t) > R_T(t)(r(t)u^{\Delta^2}(t))^{\Delta} \; for \; t \geq T > t_0,$$

where

$$R_T(t) = \int_{T}^{t}\frac{(t - \sigma(s))(s - T)}{r(s)}\Delta s.$$

Proof. Let $u(t) > 0$ for $t \geqslant t_1 > t_0$. From Lemma 2.1.1, it follows that one of the cases (a) or (b) holds. Suppose that case (a) holds. Since $(ru^{\Delta^2})^{\Delta}(t)$ is decreasing for $t \geq t_1$, then we have

$$r(t)u^{\Delta^2}(t) > \int_{T}^{t}(r(s)u^{\Delta^2}(s))^{\Delta}\Delta s > (t - T)(r(t)u^{\Delta^2}(t))^{\Delta}$$

for $t > T \geq t_1$. Hence,

$$u^{\Delta^2}(t) > (r(t)u^{\Delta^2}(t))^{\Delta}\;\frac{t - T}{r(t)}.$$

Integrating between T to t twice successively, we obtain

$$\begin{aligned} u(t) &> (r(t)u^{\Delta^2}(t))^{\Delta} \int_T^t \left(\int_T^{\theta} \frac{s-T}{r(s)} \Delta s \right) \Delta\theta \\ &= (r(t)u^{\Delta^2}(t))^{\Delta} \int_T^t \frac{(t-\sigma(s))\ (s-T)}{r(s)} \Delta s \\ &= (r(t)u^{\Delta^2}(t))^{\Delta} R_T(t), \end{aligned}$$

due to Corollary 1.4.4.

Next suppose that case (b) holds. For $t \geq T > t_1$, by integrating $R_T(t)(r(t)u^{\Delta^2}(t))^{\Delta^2} \leq 0$, by using integration by parts, we obtain

$$\begin{aligned} 0 &\geqslant \int_T^t R_T(s)(r(s)u^{\Delta^2}(s))^{\Delta^2} \Delta s \\ &= -\int_T^t R_T^{\Delta}(s)(ru^{\Delta^2})^{\Delta\sigma}(s)\Delta s + R_T(t)(r(t)u^{\Delta^2}(t))^{\Delta}, \end{aligned}$$

that is,

$$\begin{aligned} R_T(t)(r(t)u^{\Delta^2}(t))^{\Delta} &\leq \int_T^t R_T^{\Delta}(s)(ru^{\Delta^2})^{\Delta}(\sigma(s))\Delta s \\ &\leq \int_T^t R_T^{\Delta}(s)(ru^{\Delta^2})^{\Delta}(s)\Delta s \\ &\leq -\int_T^t R_T^{\Delta^2}(s)(ru^{\Delta^2})(\sigma(s))\Delta s \\ &\leq -\int_T^t \frac{s-T}{r(s)}\ (r(s)u^{\Delta^2}(s))\Delta s \\ &= -\ [(s-T)u^{\Delta}(s)]_T^t + \int_T^t u^{\Delta\sigma}(s)\Delta s \\ &< \int_T^t u^{\Delta\sigma}(s) \\ &< \int_T^t u^{\Delta}(s)\Delta s \\ &< u(t). \end{aligned}$$

Hence $u(t) > (r(t)u^{\Delta^2}(t))^{\Delta} R_T(t)$. Thus the lemma is proved. □

Remark 2.1.3. Notice that $R_T(t)$ is an increasing function.

Theorem 2.1.4. *Let $0 \leq p(t) \leqslant p < \infty$, $\beta(t) \leq \alpha(t)$, (Λ) and (H_1) hold. Suppose that*

(H_2) there exists $\lambda > 0$ such that $G(u) + G(v) \geqslant \lambda G(u+v)$ for $u > 0$ and $v > 0$,

(H_3) $G(u)G(v) = G(uv)$ for $u,\ v \in \mathbb{R}$,

(H_4) $\int_0^{\pm c} \frac{du}{G(u)} < \infty$ for all $c > 0$,

(H_5) $\int_{T'}^{\infty} G(R_T(\beta(t)))Q(t)\Delta t = \infty$, where $T' > T \geqslant t_0$ such that $\beta(t) \geq T$ for all $t \in [T', \infty)_{\mathbb{T}}$.

Then every solution of (E_1) *oscillates.*

Proof. If possible, let $y(t)$ be a nonoscillatory solution of (E_1). First, let $y(t)$ be an eventually positive solution. Then there exists $t_1 \in [t^*, \infty)_{\mathbb{T}}$ such that $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, $y(\alpha(\alpha(t)))$, $y(\beta(\alpha(t)))$ are all positive for all $t \in [t_1, \infty)_{\mathbb{T}}$. Define the following function

$$z(t) = y(t) + p(t)y(\alpha(t)), \tag{2.3}$$

we obtain

$$0 < z(t) \leqslant y(t) + py(\alpha(t)), \tag{2.4}$$

and

$$(r(t)z^{\Delta^2}(t))^{\Delta^2} = -q(t)G(y(\beta(t))) \leq (\not\equiv)0 \tag{2.5}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. Clearly $z(t) > 0$, $z(\alpha(t)) > 0$ for $t \geqslant t_1$. Then one of the cases (a) or (b) of Lemma 2.1.1 holds. The use of (H_2), (H_3) and (Λ) yields

$$\begin{aligned} 0 &= (r(t)z^{\Delta^2}(t))^{\Delta^2} + q(t)G(y(\beta(t))) + G(p)(r(\alpha(t))z^{\Delta^2}(\alpha(t)))^{\Delta^2} \\ &\quad + G(p)q(\alpha(t))G(y(\beta(\alpha(t)))) \\ &\geqslant (r(t)z^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))z^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(y(\beta(t)) + py(\alpha(\beta(t)))) \\ &\geqslant (r(t)z^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))z^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(z(\beta(t))) \end{aligned} \tag{2.6}$$

for $t \geqslant t_1$. Take $T' \in [t_0, \infty)_{\mathbb{T}}$ such that $\beta(t) \geqslant T > t_1$ for all $t \in [T', \infty)_{\mathbb{T}}$. Using ($H_3$) and Lemma 2.1.2, the above inequality will become

$$\begin{aligned} 0 \geqslant\ & (r(t)z^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))z^{\Delta^2}(\alpha(t)))^{\Delta^2} + \\ & \lambda Q(t)G(R_T(\beta(t)))G((r(\beta(t))z^{\Delta^2}(\beta(t)))^{\Delta}) \end{aligned}$$

for $t \geqslant T'$. Hence,

$$\begin{aligned}
&\lambda Q(t)G(R_T(\beta(t))) \\
&\qquad \leqslant -[G((r(\beta(t))z^{\Delta^2}(\beta(t)))^{\Delta})]^{-1}\Big((r(t)z^{\Delta^2}(t))^{\Delta^2}+G(p)(r(\alpha(t))z^{\Delta^2}(\alpha(t)))^{\Delta^2}\Big) \\
&\qquad \leqslant -[G((r(t)z^{\Delta^2}(t))^{\Delta})]^{-1}\big(r(t)z^{\Delta^2}(t)\big)^{\Delta^2} \\
&\qquad\quad - G(p)[G((r(\alpha(t))z^{\Delta^2}(\alpha(t)))^{\Delta})]^{-1}\big(r(\alpha(t))z^{\Delta^2}(\alpha(t))\big)^{\Delta^2}.
\end{aligned}$$

Since $\lim_{t\to\infty}(r(t)z^{\Delta^2}(t))^{\Delta}$ exists, then the use of (H_4) yields

$$\int_{T'}^{\infty} Q(t)G(R_T(\beta(t)))\Delta t < \infty,$$

which contradicts (H_5). Next, let $y(t)$ be an eventually negative solution. We set $x(t) = -y(t)$, to obtain $x(t)$ be an eventually positive solution of the following equation

$$(r(t)(x(t)+p(t)x(\alpha(t))\ )^{\Delta^2})^{\Delta^2} + q(t)G(x(\beta(t))) = 0.$$

Proceeding as above we arrive a contradiction. Thus the proof of the theorem is completed. □

Example 2.1.5. Consider

$$[t(y(t)+2y(t-c_1))^{\Delta^2}]^{\Delta^2} + (t+c_1+\sigma(t+c_1))y^{1/3}(t-c_2) = 0, \tag{2.7}$$

for $t \in [t_0,\infty)_{\mathbb{T}}$, $t_0 > 0$ and $t_0 \in \mathbb{T}$ ($\mathbb{T}$ is any $\mathbb{Z}$, $h\mathbb{Z}$, $\mathbb{R}$). Here $r(t) = t$, $p(t) = 2$, $\alpha(t) = t - c_1$, $q(t) = t + c_1 + \sigma(t + c_1)$, $G(u) = u^{1/3}$, $\beta(t) = t - c_2$ and $c_1, c_2 \in [t_0,\infty)_{\mathbb{T}}$ such that $c_1 \leq c_2$. $Q(t) = min\{q(t), q(t-c_1)\} = t+\sigma(t)$ and $\beta(t) = t-c_2 \geqslant T > t_0+c_2$ for all $t \in [T + c_2,\infty)_{\mathbb{T}}$.

$$\begin{aligned}
\int_{T+c_2}^{\infty} G(R_T(\beta(t)))Q(t)\Delta t &= \int_{T+c_2}^{\infty} G(R_T(t-c_2))(t+\sigma(t))\Delta t \\
&= \int_{T+c_2}^{T'} G(R_T(t-c_2))(t+\sigma(t))\Delta t \\
&\quad + \int_{T'}^{\infty} G(R_T(t-c_2))(t+\sigma(t))\Delta t \\
&> G(R_T(T'-c_2))\int_{T'}^{\infty}(t+\sigma(t))\Delta t \\
&= G(R_T(T'-c_2))[t^2]_{T'}^{\infty} = \infty.
\end{aligned}$$

It is easy to see that equation (2.7) satisfies all the conditions of Theorem 2.1.4. Hence every solution of equation (2.7) oscillates.

Remark 2.1.6. Note that (H_3) implies $G(-u) = -G(u)$, $u \in \mathbb{R}$.

Theorem 2.1.7. *Let $0 \leqslant p(t) \leqslant p < \infty$. Suppose that ($H_1$), ($H_2$) and ($\Lambda$) hold. If*

(H_3') $G(u)G(v) \geqslant G(uv)$ for $u > 0,\ v > 0$,

(H_6) $G(-u) = -G(u)$, $u \in \mathbb{R}$,

(H_7) $\int_{t^*}^{\infty} Q(t)\Delta t = \infty$,

then every solution of (E_1) *oscillates.*

Proof. Let $y(t)$ be a nonoscillatory solution of (E_1). First, let $y(t)$ be an eventually positive solution. There exists $t_1 \in [t^*, \infty)_{\mathbb{T}}$ such that $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, $y(\alpha(\alpha(t)))$, $y(\beta(\alpha(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. The proof is similar for the eventually negative solution. Setting $z(t)$ as in (2.3), we obtain (2.4) and (2.5) for $t \in [t_1, \infty)_{\mathbb{T}}$. From Lemma 2.1.1, it follows that one of the cases (a) or (b) holds. Proceeding as in the proof of Theorem 2.1.4, using (H_2), (H_3') and (Λ), we obtain (2.6), that is

$$0 \geqslant (r(t)z^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))z^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda G(z(\beta(t)))Q(t)$$

for $t \geqslant t_1$. Since $z(t)$ is increasing, then for some $k > 0$ such that $z(t) > k$ for $t \geqslant t_2 > t_1$. Take $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $\beta(t) \geqslant t_2$ for all $t \in [t_3, \infty)_{\mathbb{T}}$. Hence the above inequality will become

$$0 \geqslant (r(t)z^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))z^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda G(k)Q(t)$$

for $t \geqslant t_3$. Hence $\int_{t_3}^{\infty} Q(t)\Delta t < \infty$, which is a contradiction to (H_7). Thus the theorem is proved. ☐

Example 2.1.8. Let $\mathbb{T} = \{q^n : n \in \mathbb{N}_0\}$ with $q > 1$ is fixed, and consider the following q-difference equation

$$\Delta_q^2\Big(t\Delta_q^2\big(y(t) + q^{16}y(t/q^2)\big)\Big) + q^{8/9}\Big(\frac{2(q^8+1)(q^7+1)^2(q^6+1)}{(q-1)^4}t^{37/9}\Big)y^{1/9}(t/q) = 0 \quad (2.8)$$

for $t \in [q^2, \infty)_{\mathbb{T}}$, where $q(t) = q^{8/9}\Big(\frac{2(q^8+1)(q^7+1)^2(q^6+1)}{(q-1)^4}t^{37/9}\Big)$, $p(t) = q^{16}$, $r(t) = t$, $G(u) = u^{1/9}$. It is easy to see that equation (2.8) satisfies all the conditions of Theorem 2.1.7. Hence every solution of equation (2.8) oscillates. In particular, $y(t) = t^8(-1)^{\log_q^t}$ is an oscillatory solution of equation (2.8).

Examples 2.1.5 and 2.1.8 works for both the Theorems 2.1.4 and 2.1.7, where as the following example works for Theorem 2.1.4.

Example 2.1.9. Let $\mathbb{T} = h\mathbb{Z}$, h is a ratio of odd positive integers. Consider the following h-difference equation

$$\Delta_h^2\Big(e^{-4t-t/5}\Delta_h^2\big(y(t) + e^{10h}y(t-2h)\big)\Big) + 2\Big(\frac{e^{5h}+1}{h}\Big)^2\Big(\frac{e^{4h/5}+1}{h}\Big)^2 e^{\frac{-t}{5}+7h}y^{1/5}(t-7h) = 0, \quad (2.9)$$

for $t \in [7h, \infty)_{\mathbb{T}}$. Here $r(t) = e^{-4t-t/5}$, $p(t) = e^{10h}$, $q(t) = 2\Big(\frac{e^{5h}+1}{h}\Big)^2\Big(\frac{e^{4h/5}+1}{h}\Big)^2 e^{\frac{-t}{5}+7h}$, and $G(u) = u^{1/5}$. It is easy to see that equation (2.9) satisfies all the conditions of Theorem 2.1.4. Hence every solution of equation (2.9) is oscillatory. In particular, $y(t) = (-1)^t e^{5t}$ is an oscillatory solution of equation (2.9).

Remark 2.1.10. (H_3') and (H_6) need not imply (H_3). Indeed, if

$$G(u) = (a + b \mid u \mid^{\lambda}) \mid u \mid^{\mu}\, sgn\ u, \quad where\ \ \lambda \geqslant 0,\ \mu > 0,\ a \geqslant 1,\ b \geqslant 1,$$

then (H_3') and (H_6) are satisfied, but (H_3) fails to hold.

Remark 2.1.11. The prototype of G satisfying (H_2), (H_3') and (H_6) is

$$G(u) = (a+ \mid u \mid^{\lambda}) \mid u \mid^{\mu}\, sgn\ u, \quad where\ \ \lambda > 0,\ \mu > 0,\ \lambda + \mu \geqslant 1,\ a \geqslant 1.$$

For verifying it we may use the well known inequality (see [36], p.292)

$$u^p + v^p \geq \begin{cases} (u+v)^p, & 0 \leq p \leq 1 \\ 2^{1-p}(u+v)^p, & p \geq 1. \end{cases}$$

Remark 2.1.12. In Theorem 2.1.7, G could be **superlinear**, **sublinear** or **linear**. However, (H_7) implies (H_5), because $R_T^{\Delta}(t) > 0$ for $t \geqslant T$.

Theorem 2.1.13. *Let $0 \leqslant p(t) \leqslant p < \infty$, $\beta(t) \leqslant \alpha(t)$, (Λ) and (H_1)–(H_3) hold. Suppose that*

(H_8) $G(x_1)/x_1^\gamma \geqslant G(x_2)/x_2^\gamma$ for $x_1 > x_2 > 0$ and $\gamma \geq 1$,

(H_9) $\int_{T'}^\infty R_T^\gamma(\beta(t))Q(t)\Delta t = \infty$, where $T' \geq T \geqslant t_0$ such that $\beta(t) \geq T$ for all $t \in [T', \infty)_{\mathbb{T}}$.

Then every solution of (E_1) *oscillates.*

Proof. Proceeding as in the proof of Theorem 2.1.4, we obtain (2.6), that is

$$(r(t)z^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))z^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(z(\beta(t))) \leqslant 0$$

for $t \geqslant t_1$. Since $z(t)$ is increasing, then $z(t) > k > 0$ for $t \geqslant t_2$. Take $T' \in [T, \infty)_{\mathbb{T}}$ such that $\beta(t) \geq T > t_2$ for all $t \in [T', \infty)_{\mathbb{T}}$. Using (H_8) and Lemma 2.1.2, we obtain

$$\begin{aligned} G(z(\beta(t))) &= \left(\frac{G(z(\beta(t)))}{z^\gamma(\beta(t))}\right) z^\gamma(\beta(t)) \\ &\geqslant \left(\frac{G(k)}{k^\gamma}\right) z^\gamma(\beta(t)) \\ &> \left(\frac{G(k)}{k^\gamma}\right) R_T^\gamma(\beta(t))((r(\beta(t))z^{\Delta^2}(\beta(t)))^\Delta)^\gamma \end{aligned}$$

for $t \geqslant T' > T$. Hence (2.6) yields

$$\begin{aligned} &\lambda\Big(\frac{G(k)}{k^\gamma}\Big) R_T^\gamma(\beta(t))Q(t) \\ &\qquad < -((r(\beta(t))z^{\Delta^2}(\beta(t)))^\Delta)^{-\gamma}[(r(t)z^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))z^{\Delta^2}(\alpha(t)))^{\Delta^2}] \\ &\qquad < -((r(t)z^{\Delta^2}(t))^\Delta)^{-\gamma}(r(t)z^{\Delta^2}(t))^{\Delta^2} \\ &\qquad\qquad - G(p)((r(\alpha(t))z^{\Delta^2}(\alpha(t)))^\Delta)^{-\gamma}(r(\alpha(t))z^{\Delta^2}(\alpha(t)))^{\Delta^2}. \end{aligned}$$

Since $\lim_{t\to\infty}(r(t)z^{\Delta^2}(t))^\Delta$ exists, then proceeding as in the proof of Theorem 2.1.4, we obtain

$$\int_{T'}^\infty R_T^\gamma(\beta(t))Q(t)\Delta t < \infty,$$

a contradiction to (H_9). Thus the theorem is proved. □

Example 2.1.14. Consider

$$\left[\frac{t}{t+\sigma(t)}(y(t)+\frac{1}{t+2}y(t-c_1))^{\Delta^2}\right]^{\Delta^2}+(t+c_1+\sigma(t+c_1))y^3(t-c_2)=0 \quad (2.10)$$

for $t \in [t_0,\infty)_{\mathbb{T}}$, $t_0>0$ and $t_0 \in \mathbb{T}$ ($\mathbb{T}$ is any $\mathbb{Z}$, $h\mathbb{Z}$, $\mathbb{R}$). Here $r(t)=\frac{t}{t+\sigma(t)}$, $p(t)=\frac{1}{t+2}$, $\alpha(t)=t-c_1$, $q(t)=t+c_1+\sigma(t+c_1)$, $G(u)=u^3$ and $c_1,c_2 \in [t_0,\infty)_{\mathbb{T}}$ such that $c_1 \leqslant c_2$. $Q(t)=min\{q(t),q(t-c_1)\}=t+\sigma(t)$ and $\beta(t)=t-c_2 \geqslant T > t_0+c_2$ for all $t \in [T+c_2,\infty)_{\mathbb{T}}$. Take $\gamma=2$, we have

$$\begin{aligned}
\int_{T+c_2}^{\infty} R_T^{\gamma}(\beta(t))Q(t)\Delta t &= \int_{T+c_2}^{\infty} R_T^2(t-c_2)(t+\sigma(t))\Delta t\\
&= \int_{T+c_2}^{T'} R_T^2(t-c_2)(t+\sigma(t))\Delta t+\int_{T'}^{\infty} R_T^2(t-c_2)(t+\sigma(t))\Delta t\\
&> R_T^2(T'-c_2)\int_{T'}^{\infty}(t+\sigma(t))\Delta t\\
&= R_T^2(T'-c_2)[t^2]_{T'}^{\infty}=\infty.
\end{aligned}$$

It is easy to see that equation (2.10) satisfies all the conditions of Theorem 2.1.13. Hence every solution of equation (2.10) oscillates.

Theorem 2.1.15. *Let $-1<p\leqslant p(t)\leqslant 0$. If (H_1), (H_3), (H_4) and*

$$(H_{10}) \quad \int_{t_0}^{\infty} q(t)\Delta t=\infty$$

hold, then every solution of (E_1) *oscillates or tends to zero as $t\to\infty$.*

Proof. Let $y(t)$ be a nonoscillatory solution of (E_1). In view of (H_3), it is enough to consider $y(t)$ be an eventually positive solution. There exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $y(t)$, $y(\alpha(t))$, $y(\beta(t))$ are all positive for all $t \in [t_1,\infty)_{\mathbb{T}}$. Setting $z(t)$ as in (2.3), we obtain (2.5) for $t \in [t_1,\infty)_{\mathbb{T}}$. Thus, $z(t)>0$ or $z(t)<0$ for $t \geqslant t_2 > t_1$. Let $z(t)>0$ for $t \geqslant t_2$. From Lemma 2.1.1, it follows that one of the *cases* (a) or (b) are holds. Hence $z(t) > R_T(t)(r(t)z^{\Delta^2}(t))^{\Delta}$ for $t \geqslant T > t_2$ by Lemma 2.1.2. Since $z(t) \leqslant y(t)$ and $(r(t)z^{\Delta^2}(t))^{\Delta}$ is monotonic decreasing, take $t_3 \in [T,\infty)_{\mathbb{T}}$ such that $\beta(t)\geqslant T$ for all $t \in [t_3,\infty)_{\mathbb{T}}$. So (2.5) yields, for $t \geqslant t_3 > T$,

$$q(t)G(R_T(\beta(t)))G((r(t)z^{\Delta^2}(t))^{\Delta}) \leq -(r(t)z^{\Delta^2}(t))^{\Delta^2}.$$

Hence,

$$\int_{t_3}^{\infty} q(t)G(R_T(\beta(t)))\Delta t < \infty.$$

Since $R_T^{\Delta}(t) > 0$ and $R_T(t) > 0$, then $\int_{t_3}^{\infty} q(t)\Delta t < \infty$, which is a contradiction to (H_{10}). Hence $z(t) < 0$ for $t \in [t_2, \infty)_{\mathbb{T}}$. This implies that $y(t) < -p(t)y(\alpha(t)) < y(\alpha(t))$. Hence $y(t)$ is bounded. Consequently, $z(t)$ is bounded. One of the cases (b)-(e) holds by Lemma 2.1.1. Let the case (b) hold. If $\lim_{t\to\infty} z(t) = l$, then $-\infty < l \leqslant 0$. Suppose that $-\infty < l < 0$. Hence $z(t) < m < 0$ for $t \geqslant t_3 > t_2$. Further, $z(t) > py(\alpha(t))$ for $t \geqslant t_2$ and hence $0 < p^{-1}m < y(\alpha(t))$ for $t \geqslant t_3$. Since α has an inverse, then there exists $t_4 \in [t_3, \infty)_{\mathbb{T}}$ such that $\beta(t) \geqslant \alpha(t_3)$ for all $t \in [t_4, \infty)_{\mathbb{T}}$. Hence $y(\beta(t)) > p^{-1}m > 0$, for $t \geqslant t_4 > t_3$. Consequently, (2.5) yields

$$q(t)G(p^{-1}m) \leqslant -(r(t)z^{\Delta^2}(t))^{\Delta^2}.$$

Integrating the above inequality, we obtain $\int_{t_4}^{\infty} q(t)\Delta t < \infty$, a contradiction. Hence $l = 0$. Consequently,

$$\begin{aligned} 0 &= \lim_{t\to\infty} \sup z(t) = \lim_{t\to\infty} \sup(y(t) + p(t)y(\alpha(t))) \geqslant \lim_{t\to\infty} \sup y(t) + \lim_{t\to\infty} \inf(py(\alpha(t))) \\ &= \lim_{t\to\infty} \sup y(t) + p \lim_{t\to\infty} \sup y(\alpha(t)) = (1+p) \lim_{t\to\infty} \sup y(t). \end{aligned}$$

Since $(1 + p) > 0$, then $\lim_{t\to\infty} \sup y(t) = 0$.

In each of the cases (c) and (d) $\lim_{t\to\infty} z(t) = -\infty$, which contradicts the boundedness of $z(t)$.

Suppose the *case* (e) holds. Since $z(t)$ is bounded, then $\lim_{t\to\infty} z(t)$ exists. Further, $t > t_3 \geqslant t_2$ implies that $z^{\Delta^2}(t) > (r(t_3)z^{\Delta^2}(t_3)/r(t)$. Multiplying the inequality through by t and then integrating it as in case (e) of Lemma 2.1.1, we obtain

$$tz^{\Delta}(t) - t_3 z^{\Delta}(t_3) - z(t) + z(t_3) \geq r(t_3)z^{\Delta^2}(t_3) \int_{t_3}^{t} \frac{\theta}{r(\theta)}\Delta\theta.$$

This shows that $z(t)$ is unbounded due to (H_1), because the right hand side is positive. This contradicts the fact that $z(t)$ is bounded. Thus the proof is complete. □

Theorem 2.1.16. *Let $-\infty < p_1 \leqslant p(t) \leqslant p_2 < -1$. If ($H_1$) and ($H_{10}$) hold, then every bounded solution of* (E_1) *oscillates or tends to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a bounded nonoscillatory solution of (E_1). First, let $y(t)$ be an eventually positive solution. There exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $y(t)$, $y(\alpha(t))$, $y(\beta(t))$ are all positive for all $t \in [t_1, \infty)_{\mathbb{T}}$. Setting $z(t)$ as in (2.3), we obtain (2.5) for $t \geqslant t_1$ and hence $z(t) > 0$ or < 0 for $t \geqslant t_2$.

Let $z(t) > 0$ for $t \geqslant t_2$. So one of the cases (a) or (b) of Lemma 2.1.1 holds and $y(t) > -p(t)y(\alpha(t)) > y(\alpha(t))$. Hence $\liminf_{t\to\infty} y(t) > 0$. From (2.5), it follows that $\int_{t_2}^{\infty} q(t)\Delta t < \infty$, which is a contradiction to (H_{10}). Hence $z(t) < 0$ for $t \geqslant t_2$. Since $y(t)$ is bounded, then $z(t)$ is bounded and hence none of the cases (c), (d), (e) of Lemma 2.1.1 occurs. Suppose that the case (b) of Lemma 2.1.1 holds. If $-\infty < \lim_{t\to\infty} z(t) < 0$, then proceeding as in the proof of Theorem 2.1.15, we arrive a contradiction. Hence $\lim_{t\to\infty} z(t) = 0$. Consequently,

$$\begin{aligned}0 &= \liminf_{t\to\infty} z(t) \leqslant \liminf_{t\to\infty}(y(t) + p_2 y(\alpha(t))) \leqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_2 y(\alpha(t)))\\ &= (1 + p_2)\limsup_{t\to\infty} y(t).\end{aligned}$$

Since $(1+p_2) < 0$, then $\lim_{t\to\infty} y(t) = 0$. The remaining *cases* (c), (d) and (e) hold as in Theorem 2.1.15. Next, let $y(t)$ be an eventually negative solution. We set $x(t) = -y(t)$, to obtain $x(t)$ be an eventually positive solution of the following equation

$$(r(t)(x(t) + p(t)x(\alpha(t))\ )^{\Delta^2})^{\Delta^2} + q(t)\tilde{G}(x(\beta(t))) = 0,$$

where $\tilde{G}(u) = -G(-u)$. Proceeding as above we obtain $\lim_{t\to\infty} x(t) = 0$ and hence $\lim_{t\to\infty} y(t) = 0$. Thus the theorem is proved. □

Example 2.1.17. Let $\mathbb{T}$=h$\mathbb{Z}$, h is a fixed positive real number and consider the following difference equation

$$\Delta_h^2[2^{-t}\Delta_h^2(y(t) - (2 + 2^{-t})y(t - \tau))] + q(t)y^3(t) = 0, \tag{2.11}$$

for $t \in [\tau, \infty)_{\mathbb{T}}$, where $q(t) = 2^{3\tau}\Big((\frac{2^{-2h}-1}{h})^2(\frac{2^{-3h}-1}{h})^2 + (2 - 2^{-\tau})(\frac{2^{-h}-1}{h})^2(\frac{2^{-2h}-1}{h})^2 2^t\Big)$, fixed $\tau > 0$ and $\tau \in h\mathbb{Z}$ such that $2 - 2^{-\tau} > 0$ with

$$\int_{t_0}^{\infty} \frac{t}{r(t)}\Delta t = \sum_{t=t_0}^{\infty} t2^t\mu(t) = h\sum_{t=t_0}^{\infty} t2^t = \infty$$

and

$$\int_{t_0}^{\infty} q(t)\Delta t = \sum_{t=t_0}^{\infty} q(t)\mu(t) = h\sum_{t=t_0}^{\infty} q(t) > q(t_0)\sum_{t=t_0}^{\infty} 1 = \infty.$$

It is easy to see that equation (2.11) satisfies all the conditions of Theorem 2.1.16. Hence every solution of equation (2.11) oscillates or tends to zero as $t \to \infty$. In particular, $y(t) = 2^{-(t+\tau)}$ is a solution of equation (2.11).

2.2 Sufficient conditions for oscillation of (E_2) with $\int_{t_0}^{\infty} \frac{t}{r(t)}\Delta t = \infty$.

In the following, we obtain sufficient conditions for the oscillation of solutions of forced equation (E_2). Let

(H_{11}) there exists $F \in C_{rd}^2([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that $F(t)$ changes sign, $rF^{\Delta^2} \in C_{rd}^2([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ and $(rF^{\Delta^2})^{\Delta^2} = f$,

(H_{12}) there exists $F \in C_{rd}^2([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that $F(t)$ changes sign with $-\infty < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \infty$, $rF^{\Delta^2} \in C_{rd}^2([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ and $(rF^{\Delta^2})^{\Delta^2} = f$,

(H_{13}) there exists $F \in C_{rd}^2([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that $F(t)$ does not change sign, $\lim_{t\to\infty} F(t) = 0$, $rF^{\Delta^2} \in C_{rd}^2([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ and $(rF^{\Delta^2})^{\Delta^2} = f$,

(H_{14}) there exists $F \in C_{rd}^2([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that $\lim_{t\to\infty} F(t) = 0$, $rF^{\Delta^2} \in C_{rd}^2([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ and $(rF^{\Delta^2})^{\Delta^2} = f$.

Remark 2.2.1. If $\lim_{t\to\infty} F(t) = b \neq 0$ in (H_{13}), then we may proceed as follows: We set $\tilde{F}(t) = F(t) - b$ to obtain, $\tilde{F} \in C_{rd}^2([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $\tilde{F}^{\Delta^2}(t) = F^{\Delta^2}(t)$ and $\lim_{t\to\infty} \tilde{F}(t) = 0$. If $\tilde{F}(t)$ changes sign, then it comes under (H_{11}). If $\tilde{F}(t)$ does not change sign, then it comes under (H_{13}).

Theorem 2.2.2. *Let $0 \leqslant p(t) \leqslant p < \infty$. Suppose that (H_1), (H_2), (H_3'), (H_6), (H_{11}) and (Λ) hold. If*

(H_{15}) $\int_{t^*}^{\infty} Q(t)G(F^{+}(\beta(t)))\Delta t = \infty$ *and* $\int_{t^*}^{\infty} Q(t)G(F^{-}(\beta(t)))\Delta t = \infty$,

where $F^{+}(t) = \max\{F(t), 0\}$ *and* $F^{-}(t) = \max\{-F(t), 0\}$, *then all solutions of* (E_2) *oscillates.*

Proof. Let $y(t)$ be a nonoscillatory solution of (E_2). First, let $y(t)$ be an eventually positive solution. There exists $t_1 \in [t^*, \infty)_{\mathbb{T}}$ such that $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, $y(\beta(\alpha(t)))$ are all positive for all $t \in [t_1, \infty)_{\mathbb{T}}$. Setting $z(t)$ as in (2.3), we obtain (2.5) for $t \geqslant t_1$. Let

$$w(t) = z(t) - F(t). \tag{2.12}$$

Hence,

$$(r(t)w^{\Delta^2}(t))^{\Delta^2} = -q(t)G(y(\beta(t))) \leq (\not\equiv)0 \tag{2.13}$$

for $t \geqslant t_1$. From Lemma 2.1.1, we have $w(t) > 0$ or < 0 for $t \geqslant t_2$. First, let $w(t) > 0$ for $t \geq t_2$. Hence one of the cases (a) or (b) of Lemma 2.1.1 holds. Take $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $w(\alpha(t)) > 0$ for $t \geqslant t_3$. For $t \geqslant t_3$, using (H_2), (H_3') and (Λ) yields

$$\begin{aligned}
0 &= (r(t)w^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2} \\
&\quad + q(t)G(y(\beta(t))) + G(p)q(\alpha(t))G(y(\beta(\alpha(t)))) \\
&\geqslant (r(t)w^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2} \\
&\quad + \lambda Q(t)G(y(\beta(t)) + py(\alpha(\beta(t)))) \\
&\geqslant (r(t)w^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(z(\beta(t))).
\end{aligned} \tag{2.14}$$

Since $w(t) > 0$ for $t \in [t_2, \infty)_{\mathbb{T}}$. So $z(t) > F(t)$ imply $z(t) > F^{+}(t)$. Take $t_4 \in [t_3, \infty)_{\mathbb{T}}$ such that $\beta(t) \geqslant t_3$ for all $t \in [t_4, \infty)_{\mathbb{T}}$. The above inequality, (2.14) will become

$$0 \geqslant (r(t)w^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(F^{+}(\beta(t)))$$

for $t \geqslant t_4$. Hence,

$$\int_{t_4}^{\infty} Q(t)G(F^{+}(\beta(t)))\Delta t < \infty,$$

which is a contradiction to (H_{15}). Hence $w(t) < 0$ for $t \geq t_2$. Then $z(t) < F(t)$ for $t \geq t_2$, which is a contradiction to the positiveness of $z(t)$.

Next, let $y(t)$ be an eventually negative solution. We set $x(t) = -y(t)$, to obtain $x(t)$ be an eventually positive solution of the following equation

$$(r(t)(x(t) + p(t)x(\alpha(t)))^{\Delta^2})^{\Delta^2} + q(t)G(x(\beta(t))) = \tilde{f}(t),$$

where $\tilde{f}(t) = -f(t)$. If $\tilde{F}(t) = -F(t)$, then $\tilde{F}(t)$ changes sign, $\tilde{F}^+(t) = F^-(t)$ and $(r(t)\tilde{F}^{\Delta^2}(t))^{\Delta^2} = \tilde{f}(t)$. Proceeding as above we obtain a contradiction. Thus the theorem is proved. □

Example 2.2.3. Let $\mathbb{T} = \mathbb{R}$ and consider the following differential equation

$$[e^{-t}(y(t) + 2y(t-\pi))'']'' + (1 + 2e^{-t})y(t - \frac{3}{2}\pi) = -cos\ t, \tag{2.15}$$

for $t \in [3\pi/2, \infty)$. Here $Q(t) = min\{1 + 2e^{-(t-\pi)}, 1 + 2e^{-t}\} = 1 + 2e^{-t}$. Further $F(t) = \frac{1}{2}e^t sin\ t$ implies that $(r(t)F''(t))'' = -cos\ t$. Since $F(t - \frac{3}{2}\pi) = \frac{1}{2}e^{t-\frac{3}{2}\pi}cos\ t$, then

$$F^+(t - \tfrac{3}{2}\pi) = \begin{cases} 0 & \text{for } (4n-3)\frac{\pi}{2} \leqslant t \leqslant (4n-1)\frac{\pi}{2} \\ \frac{1}{2}e^{t-\frac{3}{2}\pi}cost & \text{for } (4n-1)\frac{\pi}{2} \leqslant t \leqslant (4n+1)\frac{\pi}{2} \end{cases}$$

and

$$F^-(t - \tfrac{3}{2}\pi) = \begin{cases} -\frac{1}{2}e^{t-\frac{3}{2}\pi}cos\ t & \text{for } (4n-3)\frac{\pi}{2} \leqslant t \leqslant (4n-1)\frac{\pi}{2} \\ 0 & \text{for } (4n-1)\frac{\pi}{2} \leqslant t \leqslant (4n+1)\frac{\pi}{2} \end{cases}$$

for $n = 1, 2, \cdots$. Thus,

$$\begin{aligned} \int_{\frac{3\pi}{2}}^{\infty} Q(t)G(F^+(\beta(t)))dt &= \int_{\frac{3\pi}{2}}^{\infty} (1 + 2e^{-t})F^+\Big(t - \frac{3}{2}\pi\Big)dt \\ &= \frac{1}{2}\sum_{n=1}^{\infty}\int_{(4n-1)\frac{\pi}{2}}^{(4n+1)\frac{\pi}{2}} (1 + 2e^{-t})e^{t-\frac{3\pi}{2}} cos\ t\ dt \\ &= \frac{1}{2}e^{\frac{-3\pi}{2}}\sum_{n=1}^{\infty}\Big[\frac{e^t}{2}(cos\ t + sin\ t) + 2sin\ t\Big]_{(4n-1)\frac{\pi}{2}}^{(4n+1)\frac{\pi}{2}} \\ &= \frac{1}{2}e^{\frac{-3\pi}{2}}\sum_{n=1}^{\infty}\Big[e^{2n\pi}\frac{(e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}})}{2} + 4\Big] \\ &> \frac{1}{2}e^{\frac{-3\pi}{2}}\sum_{n=1}^{\infty} 4 \\ &= 2e^{\frac{-3\pi}{2}}\sum_{n=1}^{\infty} 1 = \infty. \end{aligned}$$

Similarly, $\int_{\frac{3\pi}{2}}^{\infty} Q(t)G(F^{-}(\beta(t)))dt = \infty$. It is easy to see that equation (2.15) satisfies all the conditions of Theorem 2.2.2. Hence every solution of equation (2.15) oscillates. In particular, $y(t) = -sin\ t$ is an oscillatory solution.

Theorem 2.2.4. *Let $-1 < p \leqslant p(t) \leqslant 0$. Suppose that ($H_1$) and ($H_{12}$) hold. If*

$$(H_{16}) \quad \int_{t_1}^{\infty} q(t)G(F^{+}(\beta(t)))\Delta t = \infty \text{ and } \int_{t_1}^{\infty} q(t)G(F^{-}(\alpha^{-1}(\beta(t))))\Delta t = \infty$$

and

$$(H_{17}) \quad \int_{t_1}^{\infty} q(t)G(F^{-}(\beta(t)))\Delta t = \infty \text{ and } \int_{t_1}^{\infty} q(t)G(F^{+}(\alpha^{-1}(\beta(t))))\Delta t = \infty,$$

then every solution of (E_2) *oscillates.*

Proof. Proceeding as in the proof of Theorem 2.2.2, we obtain $w(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$ when $y(t) > 0$ for $t \geqslant t_1$. Let $w(t) > 0$ for $t \geqslant t_2$. Hence one of the cases (a) or (b) of Lemma 2.1.1 holds. Further, $w(t) > 0$ implies that $y(t) \geqslant y(t)+p(t)y(\alpha(t)) > F(t)$ and hence $y(t) \geqslant F^{+}(t)$. Take $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $\beta(t) \geqslant t_2$ for all $t \in [t_3, \infty)_{\mathbb{T}}$. From (2.13), we obtain $\int_{t_3}^{\infty} q(t)G(F^{+}(\beta(t)))\Delta t < \infty$, a contradiction to ($H_{16}$). Hence $w(t) < 0$ for $t \geqslant t_2$. Then one of the cases (b)-(e) of Lemma 2.1.1 holds. Let the case (b) holds. Since $w(t) < 0$, then $y(t) > F^{-}(\alpha^{-1}(t))$ for all $t \in [t_2, \infty)_{\mathbb{T}}$. Take $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $\beta(t) \geqslant t_2$ for all $t \in [t_3, \infty)_{\mathbb{T}}$. Therefore, $y(\beta(t)) > F^{-}(\alpha^{-1}(\beta(t)))$ for all $t \in [t_3, \infty)_{\mathbb{T}}$. From (2.13) we get

$$\int_{t_3}^{\infty} q(t)G(F^{-}(\alpha^{-1}(\beta(t))))\Delta t < \infty,$$

a contradiction. If $y(t)$ is unbounded, then there exists an increasing sequence $\{\tau_n\} \subset [t_1, \infty)_{\mathbb{T}}$ such that $\tau_n \to \infty$ as $n \to \infty$ and $y(\tau_n) = max\{y(t) : t_1 \leqslant t \leqslant \tau_n\}$. We may choose n large enough such that $\alpha(\tau_n) > t_1$. Hence, $w(\tau_n) \geqslant y(\tau_n)+py(\alpha(\tau_n))-F(\tau_n) \geqslant (1+p)y(\tau_n) - F(\tau_n)$. Since $F(t)$ is bounded and $(1+p) > 0$, then $w(\tau_n) > 0$ for large n, which is a contradiction. Hence $y(t)$ is bounded. Consequently, $w(t)$ is bounded. This implies that the cases (c) and (d) of Lemma 2.1.1 fails to hold. The boundedness of $w(t)$ and (H_1), imply that the case (e) of Lemma 2.1.1 does not hold as in Theorem 2.1.15.

Next, let $y(t)$ be an eventually negative solution. We set $x(t) = -y(t)$ to obtain $x(t)$ be an eventually positive solution of the following equation

$$(r(t)(x(t) + p(t)x(\alpha(t)))^{\Delta^2})^{\Delta^2} + q(t)\tilde{G}(x(\beta(t))) = \tilde{f}(t),$$

where $\tilde{G}(u) = -G(-u)$ and $\tilde{f}(t) = -f(t)$. If $\tilde{F}(t) = -F(t)$, then $\tilde{F}(t)$ changes sign with $-\infty < \liminf_{t\to\infty} \tilde{F}(t) < 0 < \limsup_{t\to\infty} \tilde{F}(t) < \infty$, $\tilde{F}^+(t) = F^-(t)$, $\tilde{F}^-(t) = F^+(t)$ and $(r(t)\tilde{F}^{\Delta^2}(t)) = \tilde{f}(t)$. Proceeding as above we obtain a contradiction. □

Theorem 2.2.5. *Let $-\infty < p \leqslant p(t) \leqslant 0$. Suppose that ($H_1$), ($H_3$), ($H_{12}$), ($H_{16}$) and ($H_{17}$) hold. Then every solution of* (E_2) *oscillates or tends to $\pm\infty$ as $t \to \infty$.*

Proof. Proceeding as in the proof of Theorem 2.2.4, we obtain a contradiction if $w(t) > 0$ for $t \geqslant t_2$. Hence $w(t) < 0$ for $t \geqslant t_2$. So one of the cases $(b) - (e)$ of Lemma 2.1.1 holds. Suppose that the case (b) holds. Since $w(t) < 0$, then $py(\alpha(t)) < F(t)$, that is, $y(t) \geqslant (-p^{-1})F^-(\alpha^{-1}(t))$ for $t \geqslant t_2$. Take $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $\beta(t) \geqslant t_2$. Hence $y(\beta(t)) \geqslant (-p^{-1})F^-(\alpha^{-1}(\beta(t)))$ for all $t \in [t_3, \infty)_{\mathbb{T}}$. By using this inequality and (H_3) in (2.13), then by integrating we obtain $\int_{t_3}^{\infty} q(t)G(F^-(\alpha^{-1}(\beta(t))))\Delta t < \infty$, a contradiction to ($H_{16}$). In each of the cases (c) and (d), $\lim_{t\to\infty} w(t) = -\infty$. If in the case (e), $-\infty < \lim_{t\to\infty} w(t) < 0$, then we obtain a contradiction due to (H_1). Thus, $\lim_{t\to\infty} w(t) = -\infty$ in each of the cases $(c) - (e)$. Consequently, $py(\alpha(t)) < w(t) + F(t)$ implies that $\lim_{t\to\infty} \sup(py(\alpha(t))) \leqslant \lim_{t\to\infty} w(t) + \lim_{t\to\infty} \sup F(t)$, that is, $p \liminf_{t\to\infty} y(t) = -\infty$ due to (H_{12}). Hence $\lim_{t\to\infty} y(t) = \infty$. The proof for the case $y(t)$ be an eventually negative solution is similar. Thus the theorem is proved. □

Corollary 2.2.6. *If the conditions of Theorem 2.2.5 are satisfied, then every bounded solution of* (E_2) *oscillates.*

Theorem 2.2.7. *Let $0 \leqslant p(t) \leqslant p_1 < 1$ or $1 < p_2 \leqslant p(t) \leqslant p_3 < \infty$ and let (H_1), (H_2), (H_3'), (H_6), (H_{13}) and (Λ) hold. If $\int_{t^*}^{\infty} Q(t)G(\mid F(\beta(t)) \mid)\Delta t = \infty$, then every bounded solution of* (E_2) *oscillates or tends to zero as $t \to \infty$.*

Proof. Proceeding as in the proof of Theorem 2.2.2, we obtain $w(t) > 0$ or < 0 for $t \geqslant t_2$. First we prove for the range of $p(t)$ is $0 \leqslant p(t) \leqslant p_1 < 1$. The proof for the remaining range of $p(t)$ is similar. Let $w(t) > 0$ for $t \geqslant t_2$. Hence $z(t) > F(t)$. Take $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $w(\alpha(t)) > 0$ for all $t \in [t_3, \infty)_{\mathbb{T}}$. Suppose that $F(t) > 0$ for $t \geqslant t_4$. Take $t_5 \in [t_4, \infty)_{\mathbb{T}}$ such that $\beta(t) \geqslant t_4$ for all $t \in [t_5, \infty)_{\mathbb{T}}$. From (2.14), it follows by Lemma 2.1.1, that $\int_{t_5}^{\infty} Q(t)G(F(\beta(t)))\Delta t < \infty$, which is a contradiction. Hence $F(t) < 0$ for $t \geqslant t_4$. From (2.13), we obtain $\int_{t_5}^{\infty} q(t)G(y(\beta(t)))\Delta t < \infty$ due to Lemma 2.1.1. Hence $\liminf_{t\to\infty} y(t) = 0$, because $\int_{t^*}^{\infty} Q(t)G(|\ F(\beta(t))\ |)\Delta t = \infty$ implies that $\int_{t_0}^{\infty} q(t)\Delta t = \infty$, because of ($H_{13}$). Since $w(t)$ is bounded and monotonic, then $\lim_{t\to\infty} w(t)$ exists and hence $\lim_{t\to\infty} z(t)$ exists. Thus $\lim_{t\to\infty} z(t) = 0$ due to Lemma 1.5.1. As $z(t) \geqslant y(t)$, then $\lim_{t\to\infty} y(t) = 0$. Suppose that $w(t) < 0$ for $t \geqslant t_1$. Hence $y(t) \leqslant z(t) < F(t)$. Consequently, $\lim_{t\to\infty} y(t) = 0$. Thus the theorem is proved. □

Theorem 2.2.8. *Let $-1 < p \leqslant p(t) \leqslant 0$. If ($H_1$), ($H_6$), ($H_{10}$) and ($H_{13}$) hold, then every solution of* (E_2) *oscillates or tends to zero as $t \to \infty$.*

Proof. Proceeding as in the proof of Theorem 2.2.2, we have $w(t) > 0$ or < 0 for $t \geqslant t_2$. Let $w(t) > 0$ for $t \geqslant t_2$. From (2.13) and Lemma 2.1.1, we obtain that

$$\int_{t_2}^{\infty} q(t)G(y(\beta(t)))\Delta t < \infty. \tag{2.16}$$

Hence $\liminf_{t\to\infty} y(t) = 0$. On the other hand, $\lim_{t\to\infty} w(t) = \infty$ in the case (a) of Lemma 2.1.1. Hence $\lim_{t\to\infty} z(t) = \infty$. However, $y(t) \geqslant z(t)$ implies that $\lim_{t\to\infty} y(t) = \infty$, which is a contradiction. In the case (b) of Lemma 2.1.1, $\lim_{t\to\infty} w(t) = l$, where $0 < l \leqslant \infty$. If $l = \infty$, we obtain a contradiction as above. Hence $0 < l < \infty$. Consequently, $\lim_{t\to\infty} z(t) = l$ due to Lemma 1.5.1, $l = 0$, which is a contradiction. Hence $w(t) < 0$ for $t \geq t_2$. We claim that $y(t)$ is bounded. Indeed, if $y(t)$ is unbounded, then there exists an increasing sequence $\{\tau_n\}_{n=1}^{\infty}$ such that $\tau_n \in [t_1, \infty)_{\mathbb{T}}$, $\tau_n \to \infty$, $y(\tau_n) \to \infty$ as $n \to \infty$ and $y(\tau_n) = \max\{y(t) : t_1 \leqslant t \leqslant \tau_n\}$. Hence,

$$w(\tau_n) \geqslant y(\tau_n) + py(\alpha(\tau_n)) - F(\tau_n) \geqslant (1+p)y(\tau_n) - F(\tau_n).$$

Consequently, $w(\tau_n) > 0$ for large n, which is a contradiction. Thus $w(t)$ is bounded. In each of the cases (c) and (d) of Lemma 2.1.1, $\lim_{t\to\infty} w(t) = -\infty$, which is a contradiction. In the case (e) of Lemma 2.1.1, by proceeding as case (e), when $w(t) < 0$ in the Theorem 2.1.15, we obtain $w(t)$ is unbounded, which is a contradiction. In the case (b) of Lemma 2.1.1, (2.16) holds and hence $\liminf_{t\to\infty} y(t) = 0$ and $\lim_{t\to\infty} w(t)$ exists. Consequently, $\lim_{t\to\infty} z(t)$ exists. From Lemma 1.5.1, $\lim_{t\to\infty} z(t) = 0$. Hence

$$0 = \lim_{t\to\infty} z(t) = \limsup_{t\to\infty}[y(t) + p(t)y(\alpha(t))] \geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty} py(\alpha(t))$$
$$= (1+p)\limsup_{t\to\infty} y(t).$$

As $(1+p) > 0$, then $\lim_{t\to\infty} y(t) = 0$. The proof of the theorem is complete. □

Theorem 2.2.9. *Let $-\infty < p_1 \le p(t) \le p_2 < -1$. If (H_1), (H_6), (H_{10}) and (H_{13}) hold, then every bounded solution of* (E_2) *oscillates or tends to zero as $t \to \infty$.*

The proof is similar to that of Theorem 2.2.8 and hence is omitted.

Corollary 2.2.10. *Suppose that the conditions of Theorem 2.2.9 are satisfied. Then every nonoscillatory solutions of* (E_2) *which does not tend to zero as $t \to \infty$ is unbounded.*

In the following sufficient conditions are obtained for the **existence of bounded positive solutions** of (E_2) by using well known **Schauder's fixed point theorem**.

Theorem 2.2.11. *Let $0 \leqslant p(t) \leqslant p < 1$. Suppose that (H_{12}) holds with*

$$\frac{-(1-p)}{8} < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \frac{1-p}{2}$$

and G is Lipschitzian on intervals of the form $[a,b]$, $0 < a < b < \infty$. If

$$(H_{18}) \quad \int_{t_0}^{\infty} \frac{\sigma(t)}{r(t)}\left(\int_t^{\infty} \sigma(s)q(s)\Delta s\right)\Delta t < \infty,$$

then (E_2), *admits a positive bounded solution.*

Proof. It is possible, to choose $T_1 \in [t_0, \infty)_{\mathbb{T}}$ sufficiently large such that

$$L\int_{T_1}^{\infty} \frac{\sigma(t)}{r(t)}\left(\int_t^{\infty} \sigma(s)q(s)\Delta s\right)\Delta t < \frac{1-p}{4},$$

where $L = max\{L_1, G(1)\}$ and L_1 is the Lipschitz constant of G on $[\frac{1-p}{8}, 1]$. Let $X = BC_{rd}([T_1,\infty)_{\mathbb{T}}, \mathbb{R})$. Then X is **Banach space** with respect to **norm** defined by $\|y\| = \sup_{t\in[T_1,\infty)_{\mathbb{T}}}\{| y(t) |\}$. Let

$$S = \{y \in X : \frac{1-p}{8} \leqslant y(t) \leqslant 1,\ t \geqslant T_1\}.$$

Hence S is a **complete metric space.** Take $T_2 \in [T_1,\infty)_{\mathbb{T}}$ such that $\alpha(t), \beta(t) \geqslant T_1$ for all $t \in [T_2,\infty)_{\mathbb{T}}$. For $y \in S$, we define

$$Ty(t) = \begin{cases} Ty(T_2) & \text{for } t \in [T_1, T_2)_{\mathbb{T}} \\ -p(t)y(\alpha(t)) + \frac{1+p}{2} + F(t) & \\ -\int_t^\infty \frac{\sigma(s)-t}{r(s)}\Big(\int_s^\infty (\sigma(u)-s)q(u)G(y(\beta(u)))\Delta u\Big)\Delta s & \text{for } t \in [T_2,\infty)_{\mathbb{T}}. \end{cases}$$

Hence $Ty(t) < \frac{1+p}{2} + \frac{1-p}{2} = 1$ for $t \geqslant T_1$ and $Ty(t) > -p + \frac{1+p}{2} - \frac{1-p}{8} - \frac{1-p}{4} = \frac{1-p}{8}$ for $t \geqslant T_1$. Consequently, $Ty \in S$, that is, $T : S \to S$. Next we show that T is continuous. Let $y_k \in S$ such that $\lim_{k\to\infty}\|y_k(t) - y(t)\| = 0$ for all $t \geqslant T_1$. Because S is closed, $y(t) \in S$. Indeed,

$$\begin{aligned} &\|(Ty_k) - (Ty)\| \\ &\quad\leqslant \|p(t)(y(\alpha(t)) - y_k(\alpha(t)))\| \\ &\qquad + \Big\|\int_t^\infty \frac{\sigma(s)-t}{r(s)}\Big(\int_s^\infty (\sigma(u)-s)q(u)[G(y(\beta(u))) - G(y_k(\beta(u)))]\Delta u\Big)\Delta s\Big\| \\ &\quad\leqslant p\|y_k - y\| + L_1\|y_k - y\| \int_t^\infty \frac{\sigma(s)}{r(s)}\Big(\int_s^\infty \sigma(u)q(u)\Delta u\Big)\Delta s \\ &\quad\leqslant p\|y_k - y\| + \frac{1-p}{4}\|y_k - y\| \\ &\quad\leqslant \big(p + \frac{1-p}{4}\big)\|y_k - y\|. \end{aligned}$$

That is, $\|Ty_k - Ty\| \leqslant \big(p + \frac{1-p}{4}\big)\|y_k - y\| \to 0$ as $t \to \infty$.

Hence T is continuous. In order to apply Schauder's fixed point theorem [see Theorem 1.5.2] we need to show that Ty is **pre-compact**. Let $y \in S$. For $t_2 \geqslant t_1 \geqslant T_1$,

$$\begin{aligned} Ty(t_1) - Ty(t_2) \;=\;& p(t_2)y(\alpha(t_2)) - p(t_1)y(\alpha(t_1)) + F(t_1) - F(t_2) \\ & - \int_{t_1}^\infty \frac{\sigma(s)-t_1}{r(s)}\Big(\int_s^\infty (\sigma(u)-s)q(u)G(y(\beta(u)))\Delta u\Big)\Delta s \\ & + \int_{t_2}^\infty \frac{\sigma(s)-t_2}{r(s)}\Big(\int_s^\infty (\sigma(u)-s)q(u)G(y(\beta(u)))\Delta u\Big)\Delta s, \end{aligned}$$

that is,

$$\begin{aligned}
Ty(t_1) - Ty(t_2) &= p(t_2)y(\alpha(t_2)) - p(t_1)y(\alpha(t_1)) + F(t_1) - F(t_2) \\
&\quad - \int_{t_1}^{\infty}\int_{\theta}^{\infty}\frac{1}{r(s)}\Big(\int_s^{\infty}(\sigma(u)-s)q(u)G(y(\beta(u)))\Delta u\Big)\Delta s\Delta\theta \\
&\quad + \int_{t_2}^{\infty}\int_{\theta}^{\infty}\frac{1}{r(s)}\Big(\int_s^{\infty}(\sigma(u)-s)q(u)G(y(\beta(u)))\Delta u\Big)\Delta s\Delta\theta \\
&= p(t_2)y(\alpha(t_2)) - p(t_1)y(\alpha(t_1)) + F(t_1) - F(t_2) \\
&\quad - \int_{t_1}^{t_2}\int_{\theta}^{\infty}\frac{1}{r(s)}\Big(\int_s^{\infty}(\sigma(u)-s)q(u)G(y(\beta(u)))\Delta u\Big)\Delta s\Delta\theta.
\end{aligned}$$

and

$$\begin{aligned}
\mid Ty(t_2) - Ty(t_1) \mid &\leqslant \mid p(t_2)y(\alpha(t_2)) - p(t_1)y(\alpha(t_1)) \mid + \mid F(t_1) - F(t_2) \mid \\
&\quad + \mid \int_{t_1}^{t_2}\int_{\theta}^{\infty}\frac{1}{r(s)}\Big(\int_s^{\infty}(\sigma(u)-s)q(u)G(y(\beta(u)))\Delta u\Big)\Delta s\Delta\theta \mid \\
&\leqslant \mid p(t_2)y(\alpha(t_2)) - p(t_1)y(\alpha(t_1)) \mid + \mid F(t_1) - F(t_2) \mid \\
&\quad + G(1)\int_{t_1}^{\infty}\frac{1}{r(s)}\Big(\int_s^{\infty}\sigma(u)q(u)\Delta u\Big)\Delta s \ \mid t_1 - t_2 \mid .
\end{aligned}$$

Therefore, $\mid Ty(t_2) - Ty(t_1) \mid \to 0$ as $t_2 \to t_1$. Thus Ty is pre-compact. By Schauder's fixed point theorem, T has a fixed point, that is, Ty=y. Consequently, $y(t)$ is the solution of (E_2) with $\frac{1-p}{8} \leqslant y(t) \leqslant 1$. This complete the proof of the theorem. □

Theorem 2.2.12. *Let $0 \leqslant p(t) \leqslant p < 1$. Suppose that (H_{14}) and (H_{18}) hold. If G is Lipschitzian on intervals of the form $[a, b]$, $0 < a < b < \infty$, then* (E_2) *admits a bounded positive solution.*

Proof. We may choos $T_1 > t_0$ sufficiently large such that $\mid F(t) \mid < \frac{1-p}{10}$ and

$$L\int_{T_1}^{\infty}\frac{\sigma(t)}{r(t)}\Big(\int_t^{\infty}\sigma(s)q(s)\Delta s\Big)\Delta t < \frac{1-p}{20},$$

where $L = max\{L_1, G(1)\}$ and L_1 is the Lipshitz constant of G on $[\frac{1-p}{20}, 1]$. Let $X = BC_{rd}([T_1, \infty)_{\mathbb{T}}, \mathbb{R})$. This X is a Banach space with respect to the norm defined by $\|y\| = \sup_{t\in[T_1,\infty)_{\mathbb{T}}}\{\mid y(t) \mid\}$ and $S = \{y \in X : \frac{1-p}{20} \leqslant y(t) \leqslant 1, t \geqslant T_1\}$. For $y \in S$, we define

$$Ty(t) = \begin{cases} Ty(T_2) & \text{for } t \in [T_1, T_2)_{\mathbb{T}} \\ -p(t)y(\alpha(t)) + \frac{1+4p}{5} + F(t) & \\ -\int_t^{\infty}\frac{\sigma(s)-t}{r(s)}\Big(\int_s^{\infty}(\sigma(u)-s)q(u)G(y(\beta(u)))\Delta u\Big)\Delta s & \text{for } t \in [T_2, \infty)_{\mathbb{T}}. \end{cases}$$

Proceeding as in the proof of Theorem 2.2.11, we may show that T is continuous and Ty is pre-compact. By Schauder's fixed point theorem T has a fixed point. Consequently, $y(t)$ is the solution of (E_2) with $\frac{1-p}{20} \leqslant y(t) \leqslant 1$. This complete the proof of the theorem. □

2.3 Sufficient conditions for oscillation of (E_1) with $\int_{t_0}^{\infty} \frac{\sigma(t)}{r(t)} \Delta t < \infty$.

In this section, sufficient conditions are obtained for the oscillatory and asymptotic behaviour of solutions of (E_1) under the assmption

$$(H_{19}) \quad \int_{t_0}^{\infty} \frac{\sigma(t)}{r(t)} \Delta t < \infty.$$

We need the following lemma's for our work in the sequel.

Lemma 2.3.1. Let (H_{19}) hold. Let u be a real-valued twice rd-continuously differentiable function on $[t_0, \infty)_{\mathbb{T}}$ such that $r(t)u^{\Delta^2}(t) \in C^2_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ and $(r(t)u^{\Delta^2}(t))^{\Delta^2} \leq (\not\equiv) 0$ for large $t \in [t_0, \infty)_{\mathbb{T}}$, $t_0 \in \mathbb{T}$. If $u(t) > 0$ ultimately, then one of the cases (a), (b), (c) or (d) holds for large t. If $u(t) < 0$ ultimately, then one of the cases (b), (c), (d), (e) or (f) holds for large t, where

$$\begin{aligned}
&(a) \quad u^{\Delta}(t) > 0, \quad u^{\Delta^2}(t) > 0 \text{ and } \quad (r(t)u^{\Delta^2}(t))^{\Delta} > 0,\\
&(b) \quad u^{\Delta}(t) > 0, \quad u^{\Delta^2}(t) < 0 \text{ and } \quad (r(t)u^{\Delta^2}(t))^{\Delta} > 0,\\
&(c) \quad u^{\Delta}(t) > 0, \quad u^{\Delta^2}(t) < 0 \text{ and } \quad (r(t)u^{\Delta^2}(t))^{\Delta} < 0,\\
&(d) \quad u^{\Delta}(t) < 0, \quad u^{\Delta^2}(t) > 0 \text{ and } \quad (r(t)u^{\Delta^2}(t))^{\Delta} > 0,\\
&(e) \quad u^{\Delta}(t) < 0, \quad u^{\Delta^2}(t) < 0 \text{ and } \quad (r(t)u^{\Delta^2}(t))^{\Delta} > 0,\\
&(f) \quad u^{\Delta}(t) < 0, \quad u^{\Delta^2}(t) < 0 \text{ and } \quad (r(t)u^{\Delta^2}(t))^{\Delta} < 0.
\end{aligned}$$

Proof. Since $(r(t)u^{\Delta^2}(t))^{\Delta^2} \leq 0$ for large t, then $u(t)$, $u^{\Delta}(t)$, $u^{\Delta^2}(t)$, $r(t)u^{\Delta^2}(t)$ and $(r(t)u^{\Delta^2}(t))^{\Delta}$ are monotonic and hence there are eight possibilities. Let $u(t) > 0$ for

$t \geq t_1 > t_0$. It is enough to show that (e), (f) and the following cases, viz.,

$$(g) \quad u^{\Delta}(t) > 0, \quad u^{\Delta^2}(t) > 0 \text{ and } \quad (r(t)u^{\Delta^2}(t))^{\Delta} < 0,$$

$$(h) \quad u^{\Delta}(t) < 0, \quad u^{\Delta^2}(t) > 0 \text{ and } \quad (r(t)u^{\Delta^2}(t))^{\Delta} < 0,$$

do not hold. Indeed, in each of the cases (e) and (f), since $u^{\Delta^2}(t) < 0$ for large t, then $u^{\Delta}(t) < u^{\Delta}(t_2)$ for $t > t_2 \geq t_1$. By integrating from t_2 to t, we get $u(t) < 0$ for large t, which is a contradiction. In the cases (g) and (h), since $(r(t)u^{\Delta^2}(t))^{\Delta}$ is monotonically decreasing, then $(r(t)u^{\Delta^2}(t))^{\Delta} \leq (r(t_2)u^{\Delta^2}(t_2))^{\Delta}$ for $t > t_2 \geq t_1$. Integrating, we obtain $u^{\Delta^2}(t) < 0$ for large t, which is a contradiction.

Next suppose that $u(t) < 0$ for $t \geq t_1 > t_0$. It is enough to show that cases (a), (g), (h) are not hold. In the case (a), $u^{\Delta^2}(t) > 0$ for $t \geq t_1 > t_0$, so $u^{\Delta}(t)$ is increasing. For $t > t_2 \geq t_1$, we have $u^{\Delta}(t) > u^{\Delta}(t_2)$. Integrating we obtain $u(t) > 0$ for large t, which is a contradiction. Proceeding as above the cases (g) and (h) do not hold. Thus, the lemma is proved. □

Lemma 2.3.2. Let (H_{19}) hold. Assume that $u(t)$ is a positive real valued rd-continuously delta differentiable function such that $r(t)u^{\Delta^2}(t)$ is twice continuously delta differentiable and $(r(t)u^{\Delta^2}(t))^{\Delta^2} \leqslant 0$ for large t. Then

(i) Suppose that the case (c) of Lemma 2.3.1 holds. Then there is constant $k \in (0, 1)$ such that the following inequalities holds for large t.

(I_1) $u^{\Delta}(t) \geqslant -(r(t)u^{\Delta^2}(t))^{\Delta} R_1(t)$,

(I_2) $u^{\Delta}(t) \geqslant -(r(t)u^{\Delta^2}(t)) \int_t^{\infty} \frac{1}{r(s)} \Delta s$,

(I_3) $u(t) \geqslant ktu^{\Delta}(t)$,

(I_4) $u(t) \geqslant -k(r(t)u^{\Delta^2}(t))^{\Delta} tR_1(t)$, where $R_1(t) = \int_t^{\infty} \frac{s-t}{r(s)} \Delta s$.

(ii) $u(t) \geqslant (r(t)u^{\Delta^2}(t))R_2(t)$ for large t in case (d) of Lemma 2.3.1, where $R_2(t) = \int_t^{\infty} \frac{\sigma(s)-t}{r(s)} \Delta s$.

Proof. (i) We may note that $R_1(t) < \infty$ and $R_2(t) < \infty$ due to (H_{19}). Let $u(t) > 0$ for $t \geqslant t_1 > t_0$. Since $(r(t)u^{\Delta^2}(t))^{\Delta^2} \leqslant 0$ for $t \geqslant t_1 > t_0$, then $(r(t)u^{\Delta^2}(t))^{\Delta}$ is monotonic decreasing. For $s > t \geqslant t_1$, $(r(s)u^{\Delta^2}(s))^{\Delta} \leqslant (r(t)u^{\Delta^2}(t))^{\Delta}$ and hence

$r(s)u^{\Delta^2}(s) \leqslant r(t)u^{\Delta^2}(t) + (r(t)u^{\Delta^2}(t))^{\Delta}(s-t)$. Thus,

$$0 < u^{\Delta}(s) \leqslant u^{\Delta}(t) + (r(t)u^{\Delta^2}(t))^{\Delta} \int_t^s \frac{\theta - t}{r(\theta)} \Delta\theta.$$

Taking limit as $s \to \infty$, the inequality (I_1) is obtained.

For $s > t \geqslant t_1$, $r(s)u^{\Delta^2}(s) \leqslant r(t)u^{\Delta^2}(t)$ and hence

$$0 < u^{\Delta}(s) \leqslant u^{\Delta}(t) + (r(t)u^{\Delta^2}(t)) \int_t^s \frac{1}{r(\theta)} \Delta\theta.$$

Taking limit as $s \to \infty$, the inequality (I_2) is obtained. Since $u^{\Delta^2}(t) < 0$ for $t > t_2 \geq t_1$, then $u^{\Delta}(t)$ is monotonic decreasing and we have

$$u(t) \geqslant u(t) - u(t_2) = \int_{t_2}^t u^{\Delta}(s)\Delta s > u^{\Delta}(t)(t - t_2) = u^{\Delta}(t)t\left(1 - \frac{t_2}{t}\right) > ktu^{\Delta}(t),$$

where $0 < k < 1$. Hence $u(t) \geqslant ktu^{\Delta}(t)$, for $t > t_2 \geqslant t_1$. Thus the inequality (I_3) is obtained. From (I_1) and (I_3), we have

$$u(t) \geqslant ktu^{\Delta}(t) \geqslant kt(-r(t)u^{\Delta^2}(t))^{\Delta} \int_t^{\infty} \frac{\theta - t}{r(\theta)} \Delta\theta = kt(-r(t)u^{\Delta^2}(t))^{\Delta} R_1(t).$$

Thus the inequality (I_4) is obtained.

(ii) For $s > \theta > x \geqslant t$, $r(\theta)u^{\Delta^2}(\theta) > r(x)u^{\Delta^2}(x)$. Integrating from x to s, we obtain

$$-u^{\Delta}(x) > r(x)u^{\Delta^2}(x) \int_x^s \frac{1}{r(\theta)} \Delta\theta.$$

On further integrating from t to s yields

$$\begin{aligned} u(t) &\geqslant \int_t^s (r(x)u^{\Delta^2}(x)) \left(\int_x^s \frac{1}{r(\theta)} \Delta\theta \right) \Delta x \\ &\geqslant r(t)u^{\Delta^2}(t) \int_t^s \left(\int_x^s \frac{1}{r(\theta)} \Delta\theta \right) \Delta x \\ &> r(t)u^{\Delta^2}(t) \int_t^s \frac{\sigma(x) - t}{r(x)} \Delta x. \end{aligned}$$

Taking limit as $s \to \infty$, we get

$$u(t) > (r(t)u^{\Delta^2}(t)) \int_t^{\infty} \left(\frac{\sigma(x) - t}{r(x)} \right) \Delta x = (r(t)u^{\Delta^2}(t)) R_2(t).$$

Thus, the lemma is proved. □

Remark 2.3.3. Since $R_1(t) < \int_t^\infty \frac{s}{r(s)}\Delta s$ and $R_2(t) < \int_t^\infty \frac{\sigma(s)}{r(s)}\Delta s$, then $R_1(t),\ R_2(t) \to 0$ as $t \to \infty$ and $\int_{t_0}^\infty \frac{1}{r(t)}\Delta t < \infty$ in view of (H_{19}). Clearly $R_1(t) \leqslant R_2(t)$ and $R_1(t)$, $R_2(t)$ are monotonic decreasing.

Lemma 2.3.4. Let (H_{19}) holds. Assume that the conditions of Lemma 2.3.1 hold. If $u(t) > 0$ for large t, then there exists constants $k_1 > 0$ and $k_2 > 0$ such that

$$k_1 R_2(t) \leqslant u(t) \leqslant k_2 t \; for \; large \;\; t.$$

Proof. Let $u(t) > 0$ for $t \geqslant T_1 > 1$. From Lemma 2.3.1 one of the four cases (a), (b), (c), or (d) holds. Since $(r(t)u^{\Delta^2}(t))^{\Delta^2} \leqslant 0$ for large t, then $(r(t)u^{\Delta^2}(t))^{\Delta}$ is decreasing. Then for $t > T \geq T_1$, we have

$$r(t)u^{\Delta^2}(t) \leq r(T)u^{\Delta^2}(T) + (r(T)u^{\Delta^2}(T))^{\Delta}(t-T),$$

that is,

$$u^{\Delta^2}(t) \leq r(T)u^{\Delta^2}(T)\frac{1}{r(t)} + (r(T)u^{\Delta^2}(T))^{\Delta}\frac{(t-T)}{r(t)}.$$

Integrating the last inequality twice for $t > T \geqslant T_1$, we obtain

$$u(t) - u(T) \leqslant u^{\Delta}(T)(t-T) + (r(T)u^{\Delta^2}(T)) \int_T^t \Big(\int_T^s \frac{1}{r(\theta)}\Delta\theta \Big) \Delta s + (r(T)u^{\Delta^2}(T))^{\Delta} \int_T^t \Big(\int_T^s \frac{\theta - T}{r(\theta)}\Delta\theta \Big) \Delta s,$$

that is,

$$u(t) \leqslant u(T) + u^{\Delta}(T)(t-T) + (r(T)u^{\Delta^2}(T)) \int_T^t \frac{t-\sigma(s)}{r(s)}\Delta s + (r(T)u^{\Delta^2}(T))^{\Delta} \int_T^t \frac{(t-\sigma(s))(s-T)}{r(s)}\Delta s. \tag{2.17}$$

In the case (a) of lemma 2.3.1, equation (2.17) become

$$\begin{aligned} u(t) &\leqslant u(T) + u^{\Delta}(T)(t-T) + (r(T)u^{\Delta^2}(T))t \int_T^t \frac{1}{r(s)}\Delta s \\ &\quad + (r(T)u^{\Delta^2}(T))^{\Delta} t \int_T^t \frac{s}{r(s)}\Delta s \\ &\leqslant \Big(u(T) + u^{\Delta}(T) + (r(T)u^{\Delta^2}(T)) \int_T^t \frac{1}{r(s)}\Delta s + (r(T)u^{\Delta^2}(T))^{\Delta} \int_T^t \frac{s}{r(s)}\Delta s \Big) t \\ &\leqslant \Big(u(T) + u^{\Delta}(T) + (r(T)u^{\Delta^2}(T)) \int_T^\infty \frac{1}{r(s)}\Delta s + (r(T)u^{\Delta^2}(T))^{\Delta} \int_T^\infty \frac{s}{r(s)}\Delta s \Big) t \\ &= k_{21} t, \end{aligned}$$

where $k_{21} = u(T) + u^{\Delta}(T) + (r(T)u^{\Delta^2}(T)) \int_T^{\infty} \frac{1}{r(s)} \Delta s + (r(T)u^{\Delta^2}(T))^{\Delta} \int_T^{\infty} \frac{s}{r(s)} \Delta s$. Similarly in the cases (b), (c) and (d) of Lemma 2.3.1, we have $u(t) \leqslant \Big(u(T) + u^{\Delta}(T) + (r(T)u^{\Delta^2}(T))^{\Delta} \int_T^{\infty} \frac{s}{r(s)} \Delta s\Big)t = k_{22}t$, $u(t) \leqslant (u(T) + u^{\Delta}(T))t = k_{23}t$ and $u(t) \leqslant \Big(u(T) + (r(T)u^{\Delta^2}(T)) \int_T^{\infty} \frac{1}{r(s)} \Delta s + (r(T)u^{\Delta^2}(T))^{\Delta} \int_T^{\infty} \frac{s}{r(s)} \Delta s\Big)t = k_{24}t$ respectively. Therefore $u(t) \leqslant k_2 t$ for large t, where $k_2 = \max\{k_{21}, k_{22}, k_{23}, k_{24}\}$. On the otherhand suppose that one of the cases (a), (b) or (c) of Lemma 2.3.1 hold. For $t > T_2 \geqslant T_1$, we have $u(t) > u(T_2)$. Since $u(T) > 0$ and $R_2(t) \to 0$ as $t \to \infty$, then there exists $k_{11} > 0$ and T_3 such that $T_3 \in [T_2, \infty)_{\mathbb{T}}$ and $u(t) > u(T_2) \geqslant k_{11}R_2(t)$ for all $t \in [T_3, \infty)_{\mathbb{T}}$. Hence $u(t) > k_{11}R_2(t)$. In the case of (d) of Lemma 2.3.1, there is $k_{12} > 0$ such that

$$u(t) \geqslant r(t)u^{\Delta^2}(t)R_2(t) \geqslant r(T_3)u^{\Delta^2}(T_3)R_2(t) = k_{12}R_2(t).$$

Therefore $k_1R_2(t) \leqslant u(t)$ for $t \in [T_3, \infty)_{\mathbb{T}}$, where $k_1 = \min\{k_{11}, k_{12}\}$. Hence,

$$k_1R_2(t) \leqslant u(t) \leqslant k_2t$$

for large t. Thus the lemma is proved. □

Theorem 2.3.5. *Let $0 \leqslant p(t) \leqslant p < \infty$. Suppose that ($H_2$), ($H_3$), ($H_{19}$) and (Λ) hold. If*

$$(H_{20}) \quad \int_{t^*}^{\infty} h(t)G(R_2(\beta(t)))Q(t)\Delta t = \infty,$$

where $h(t) = min\{R_1^{\gamma}(\sigma(t)), R_1^{\gamma}(\sigma(\alpha(t)))\}$ and $\gamma > 1$, then all solutions of (E_1) *oscillates.*

Remark 2.3.6. Note that (H_{20}) implies (H'_{20}), where (H'_{20}) $\int_{t^*}^{\infty} G(R_2(\beta(t)))Q(t)\Delta t = \infty$.

Proof. (**proof of Theorem 2.3.5**) Proceeding as in the proof of Theorem 2.1.4, by defining $z(t)$ as in (2.3), we obtain (2.5). Clearly $z(t) > 0$, $z(\alpha(t)) > 0$ for all $t \geq t_1$. From Lemma 2.3.1, one of the cases (a), (b), (c) or (d) are holds. Suppose the cases either (a), (b) or (d) holds. Then proceeding as in the proof of Theorem 2.1.4, using (H_2), (H_3) and (Λ), we obtain (2.6), that is,

$$0 \geq (r(t)z^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))z^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(z(\beta(t)))$$

for $t \geq t_1$. From Lemma 2.3.4, for $t \geq t_2 > t_1$, the above inequality becomes

$$0 \geq (r(t)z^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))z^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(k_1 R_2(\beta(t))),$$

and

$$\lambda G(k_1)G(R_2(\beta(t)))Q(t) \leqslant -(r(t)z^{\Delta^2}(t))^{\Delta^2} - G(p)(r(\alpha(t))z^{\Delta^2}(\alpha(t)))^{\Delta^2}.$$

Hence,

$$\int_{t_2}^{\infty} G(R_2(\beta(t)))Q(t)\Delta t < \infty,$$

which is a contradiction to (H'_{20}). Suppose the case (c) holds. The use of Lemma 2.3.2 and Lemma 2.3.4 yields,

$$k(-r(t)z^{\Delta^2}(t))^{\Delta} tR_1(t) \leqslant z(t) \leqslant k_2 t, \tag{2.18}$$

for $t \geqslant t_2 > t_1$. Define $f \in C(\mathbb{R}, \mathbb{R})$ such that $f(x) = x^{1-\gamma}$ is continuous and $g(t) = (-r(t)z^{\Delta^2}(t))^{\Delta}$. Using the "chain Rule" [see Theorem 1.3.11], in time scale for $t \geqslant t_2$ there exists c in the real interval $[t, \sigma(t)]$ and $g(c) = l$. Since g is increasing, then for $t < c < \sigma(t)$ we have $g(t) \leq g(c) \leq g(\sigma(t))$ imply $g(t) \leq l \leq g(\sigma(t))$. With this and (2.5), we obtain

$$\begin{aligned} -[((-r(t)z^{\Delta^2}(t))^{\Delta})^{1-\gamma}]^{\Delta} &= -(1-\gamma)l^{-\gamma}(-(r(t)z^{\Delta^2}(t))^{\Delta^2}) \\ &= (\gamma-1)l^{-\gamma}q(t)G(y(\beta(t))) \\ &\geqslant (\gamma-1)g^{-\gamma}(\sigma(t))q(t)G(y(\beta(t))). \end{aligned} \tag{2.19}$$

From (2.18) and (2.19), we obtain

$$-[((-r(t)z^{\Delta^2}(t))^{\Delta})^{1-\gamma}]^{\Delta} \geqslant (\gamma-1)L^{\gamma}R_1^{\gamma}(\sigma(t))q(t)G(y(\beta(t))), \tag{2.20}$$

where $L = \frac{k}{k_2}$. Take $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $\alpha(t) \geqslant t_2$ for all $t \in [t_3, \infty)_{\mathbb{T}}$. The use of ($H_2$), ($H_3$), ($\Lambda$) and Lemma 2.3.4 yields

$$\begin{aligned} &-[((-r(t)z^{\Delta^2}(t))^{\Delta})^{1-\gamma}]^{\Delta} - G(p)[((-r(\alpha(t))z^{\Delta^2}(\alpha(t)))^{\Delta})^{1-\gamma}]^{\Delta} \\ &\quad\geqslant (\gamma-1)L^{\gamma}\big(R_1^{\gamma}(\sigma(t))q(t)G(y(\beta(t))) + G(p)R_1^{\gamma}(\sigma(\alpha(t)))q(\alpha(t))G(y(\beta(\alpha(t))))\big) \\ &\quad\geqslant (\gamma-1)L^{\gamma}h(t)Q(t)G(z(\beta(t))) \\ &\quad\geq (\gamma-1)L^{\gamma}G(k_1)h(t)Q(t)G(R_2(\beta(t))), \end{aligned}$$

for $t \geqslant t_3$. Consequently,

$$\int_{t_3}^{\infty} h(t)G(R_2(\beta(t)))Q(t)\Delta t < \infty,$$

which is a contradiction to (H_{20}). The proof of $y(t)$ is an eventually negative is similar. Thus proof of the theorem is complete. □

Example 2.3.7. Suppose $\mathbb{T} = h\mathbb{Z}$, where h is a ratio of odd positive integers. Consider the following h-difference equation

$$\Delta_h^2\Big(e^{t/2}\Delta_h^2\big(y(t) + e^{2h}y(t-2h)\big)\Big) + 2e^h\Big(\frac{e^h+1}{h}\Big)^2\Big(\frac{e^{3h/2}+1}{h}\Big)^2 e^{7t/6}y^{1/3}(t-3h) = 0, \quad (2.21)$$

for $t \in [3h,\infty)_{\mathbb{T}}$. Here $r(t) = e^{t/2}$, $p(t) = e^{2h}$, $q(t) = 2e^h\Big(\frac{e^h+1}{h}\Big)^2\Big(\frac{e^{3h/2}+1}{h}\Big)^2 e^{7t/6}$ and $G(u) = u^{1/3}$. It is easy to see that equation (2.21), satisfies all the conditions of Theorem 2.3.5. Hence every solution of equation (2.21), oscillates. In particular, $y(t) = (-1)^t e^t$ is an oscillatory solution of (2.21).

Theorem 2.3.8. *Let $0 \leqslant p(t) \leqslant p < 1$. Suppose that ($H_3$), ($H_{19}$) and*

$$(H_{21}) \quad \int_{t_1}^{\infty} R_1^{\gamma}(\sigma(t))G(R_2(\beta(t)))q(t)\Delta t = \infty, \ \gamma > 1$$

hold. Then every solution of (E_1) *oscillates or tends to zero as $t \to \infty$.*

Remark 2.3.9. Note that (H_{21}) implies (H'_{21}), where (H'_{21}) $\int_{t_1}^{\infty} G(R_2(\beta(t)))q(t)\Delta t = \infty$. Also ($H'_{21}$) implies $\int_{t_1}^{\infty} q(t)\Delta t = \infty$.

Proof. (**Proof of the Theorem 2.3.8**) Let $y(t)$ be a nonoscillatory solution of (E_1) on $[t_0,\infty)_{\mathbb{T}}$, say $y(t)$ is an eventually positive solution. There exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, $y(\alpha(\alpha(t)))$ are all positive for all $t \in [t_1,\infty)_{\mathbb{T}}$. The case, when $y(t)$ be an eventually negative solution can be similarly dealt with. We set $z(t)$ as in (2.3) and $z(t) > 0$, $z(\alpha(t)) > 0$ for $t \geqslant t_1 > t_0$. Hence, from Lemma 2.3.1 one of the cases (a), (b), (c) or (d) are holds. Consider the cases (a) and (b) of Lemma 2.3.1. In either case $z(t)$ is increasing. Hence for $t \geq t_1 > t_0$,

$$(1-p)z(t) < z(t) - p(t)z(\alpha(t)) = y(t) - p(t)p(\alpha(t))y(\alpha(\alpha(t))) \leqslant y(t). \quad (2.22)$$

Thus, $y(\beta(t)) \geqslant (1-p)k_1R_2(\beta(t))$ for $t \geq t_2 > t_1$ by Lemma 2.3.4. Consequently, from (2.5) we obtain

$$\int_{t_2}^{\infty} G(R_2(\beta(t)))q(t)\Delta t < \infty,$$

which is a contradiction to (H'_{21}). For the case (c) of Lemma 2.3.1, we proceed as in the proof of Theorem 2.3.5, to obtain (2.20). From (2.22) and Lemma 2.3.4, we have $y(t) > (1-p)k_1R_2(t)$ for $t \geqslant t_2$. Consequently,

$$-[((-r(t)z^{\Delta^2}(t))^{\Delta})^{1-\gamma}]^{\Delta} \geqslant (\gamma-1)L^{\gamma}G((1-p)k_1)R_1^{\gamma}(\sigma(t))G(R_2(\beta(t)))q(t)$$

for $t \geqslant t_3 > t_2$. Integrating the above inequality, we get

$$\int_{t_3}^{\infty} R_1^{\gamma}(\sigma(t))G(R_2(\beta(t)))q(t)\Delta t < \infty,$$

a contradiction to (H_{21}). In the case (d) of Lemma 2.3.1, $\lim_{t\to\infty} z(t)$ exists. If $\liminf_{t\to\infty} y(t) > 0$, then from (2.5) it follows that

$$\int_{t_1}^{\infty} q(t)\Delta t < \infty,$$

which is a contradiction to the Remark 2.3.9. Hence $\liminf_{t\to\infty} y(t) = 0$. Consequently, $\lim_{t\to\infty} z(t) = 0$ by Lemma 1.5.1. Since $z(t) \geqslant y(t)$, then $\lim_{t\to\infty} y(t) = 0$. Thus the theorem is proved. □

Theorem 2.3.10. *Let $-1 < p \leqslant p(t) \leqslant 0$. If (H_3), (H_{19}) and (H_{21}) hold, then every solution of* (E_1) *oscillates or tends to zero as $t \to \infty$.*

Proof. If possible, let $y(t)$ be a nonoscillatory solution of (E_1) on $[t_0, \infty)_{\mathbb{T}}$. First, let $y(t)$ be an eventually positive solution. Setting $z(t)$ as in (2.3), we obtain (2.5) for $t \geqslant t_1$. Hence from Lemma 2.3.1, $z(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$. Let $z(t) > 0$ for $t \geqslant t_2$. Suppose that one of the cases (a), (b), (d) of Lemma 2.3.1 holds. From Lemma 2.3.4, we have $y(t) \geq z(t) \geq k_1R_2(t)$ for $t \geq t_3 > t_2$ and take $t_4 \in [t_3, \infty)_{\mathbb{T}}$ such that $\beta(t) \geq t_3$ for all $t \in [t_4, \infty)_{\mathbb{T}}$. Hence (2.5) yields, $\int_{t_4}^{\infty} G(R_2(\beta(t)))q(t)\Delta t < \infty$, $t_4 > t_3$, a contradiction to (H'_{21}). Suppose the case (c) of Lemma 2.3.1 holds. Proceeding as in the proof of Theorem 2.3.5, we obtain (2.20). Further, $y(t) \geqslant z(t) \geqslant k_1 R_2(t)$ for $t \geqslant t_3$ by Lemma 2.3.4. Hence, for $t \geqslant t_4 > t_3$,

$$-[((-r(t)z^{\Delta^2}(t))^{\Delta})^{1-\gamma}]^{\Delta} \geqslant (\gamma - 1)L^{\gamma}G(k_1 R_2(\beta(t)))R_1^{\gamma}(\sigma(t))q(t).$$

Integrating the above inequality yields,

$$\int_{t_4}^{\infty} R_1^{\gamma}(\sigma(t))G(R_2(\beta(t)))q(t)\Delta t < \infty,$$

a contradiction to (H_{21}). If $z(t) < 0$ for $t \geqslant t_2 > t_1$, then $y(t) < y(\alpha(t))$ and hence $y(t)$ is bounded. Thus $z(t)$ is bounded. Consequently, none of the cases (e) and (f) of Lemma 2.3.1 hold. In the cases (b) and (c), $-\infty < \lim_{t\to\infty} z(t) \leqslant 0$. Then,

$$\begin{aligned} 0 \geqslant \lim_{t\to\infty} z(t) &= \lim_{t\to\infty} \sup(y(t) + p(t)y(\alpha(t))) \\ &\geqslant \lim_{t\to\infty} \sup y(t) + \lim_{t\to\infty} \inf(py(\alpha(t))) \\ &= \lim_{t\to\infty} \sup y(t) + p \lim_{t\to\infty} \sup y(\alpha(t)) = (1+p) \lim_{t\to\infty} \sup y(t). \end{aligned}$$

Since $(1+p) > 0$, then $\lim_{t\to\infty} y(t) = 0$. In the case (d), $z(t) < \delta < 0$ for $t \geqslant t_2 > t_1$. Hence $z(t) > py(\alpha(t))$ implies that $y(t) > \delta/p$ for $t \geqslant t_3 > t_2$. Consequently, from (2.5) we obtain

$$G(\delta/p)\int_{t_3}^{\infty} q(t)\Delta t < \infty,$$

for $t_3 > t_2$, a contradiction to Remark 2.3.9. □

Example 2.3.11. Suppose $\mathbb{T} = \mathbb{Z}$, and consider the following difference equation

$$\Delta^2\Big(e^n \Delta^2\big(y(n) - 4e^{-2}y(n-1)\big)\Big) + 5(e^{-2}+1)^2(e^{-1}+1)^2 e^{9n-30} y^5(n-3) = 0, \quad (2.23)$$

for $n \in [3,\infty)_{\mathbb{Z}}$. Here $r(n) = e^n$, $p(n) = -4e^{-2}$, $q(n) = 5(e^{-2}+1)^2(e^{-1}+1)^2 e^{9n-30}$, $G(u) = u^5$, $\alpha(n) = n-1$ and $\beta(n) = n-3$. It is easy to see that equation (2.23), satisfies all the conditions of Theorem 2.3.10. Hence, every solution of (2.23) oscillates or tends to zero. In particular, $y(n) = (-1)^n e^{-2n}$ is an oscillatory solution of (2.23).

Theorem 2.3.12. *Suppose that* $-\infty < p_1 \leqslant p(t) \leqslant p_2 < -1$. *If* (H_3), (H_{19}) *and* (H_{21}) *hold, then every bounded solution of* (E_1) *oscillates or tends to zero as* $t \to \infty$ *or* $\liminf_{t\to\infty} \mid y(t) \mid > 0$.

Proof. If possible, let $y(t)$ be a bounded nonoscillatory solution of (E_1) on $[t_0, \infty)_{\mathbb{T}}$. Then $y(t)$ be an eventually positive or negative solution. First, let $y(t)$ be an eventually positive solution. There exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $y(t)$, $y(\alpha(t))$, $y(\beta(t))$ are all positive for all $t \in [t_1, \infty)_{\mathbb{T}}$. From (2.5), it follows that $z(t) > 0$ or < 0 for $t \in [t_2, \infty)_{\mathbb{T}}$, where $z(t)$ is given by (2.3). If $z(t) > 0$ for $t \geqslant t_2 > t_1$, then one of the cases $(a) - (d)$ of Lemma 2.3.1 holds and we arrive a contradiction in each case proceeding as in the proof of Theorem 2.3.10. Suppose that $z(t) < 0$ for $t \geqslant t_2$. In the case (b) or (c) of Lemma 2.3.1, $-\infty < \lim_{t\to\infty} z(t) \leqslant 0$. If $\lim_{t\to\infty} z(t) = 0$, then from the boundedness of $y(t)$ it follows that

$$\begin{aligned} 0 = \lim_{t\to\infty} z(t) &= \liminf_{t\to\infty} [y(t) + p(t)y(\alpha(t))] \\ &\leqslant \liminf_{t\to\infty} [y(t) + p_2 y(\alpha(t))] \\ &\leqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty} (p_2 y(\alpha(t))) \\ &= \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} y(\alpha(t)) = (1 + p_2) \limsup_{t\to\infty} y(t). \end{aligned}$$

Since $(1 + p_2) < 0$, then $\lim_{t\to\infty} y(t) = 0$. Let $-\infty < \lim_{t\to\infty} z(t) < 0$. Then there exists $\delta < 0$ such that $\delta > z(t) > p_1 y(\alpha(t))$. Hence, in the case (b) of Lemma 2.3.1, it follows from (2.5) that

$$G(\delta/p_1) \int_{t_2}^{\infty} q(t)\Delta t < \infty,$$

which is a contradiction to Remark 2.3.9. However, such a contradiction cannot be obtained in the case (c) of Lemma 2.3.1. Since $\delta > z(t) > p_1 y(\alpha(t))$, then $\liminf_{t\to\infty} y(t) > 0$. Further, in the case (d), one may proceed as in the proof of Theorem 2.3.10, to get a contradiction. However, either in the case (e) or in the case (f), $\lim_{t\to\infty} z(t) = -\infty$. Since $z(t) > p(t)y(\alpha(t))$, then $\lim_{t\to\infty} y(t) = \infty$, a contradiction to the boundedness of $y(t)$. The case when $y(t)$ is an eventually negative solution may similarly be dealt with. Thus the theorem is proved. □

Example 2.3.13. Suppose $\mathbb{T} = q^{\mathbb{N}_0}$, where $q > 1$ is a fixed real number and consider the following q-difference equation

$$\Delta_q^2\Big(t^4\Delta_q^2\big(y(t) - q^{27}(1+1/t)y(t/q^3)\big)\Big) + q\Big(\frac{2(q^9+1)(q^8+1)(q^{11}+1)(q^{10}+1)}{(q-1)^4}t^8 + \frac{(q^8+1)(q^7+1)(q^{10}+1)(q^9+1)}{(q-1)^4}t^7\Big)y^{1/9}(t/q) = 0, \quad (2.24)$$

for $t \in [q^3,\infty)_{\mathbb{T}}$. Here $r(t) = t^4$, $p(t) = -q^{27}(1+1/t)$, $\alpha(t) = \frac{t}{q^3}$, $\beta(t) = t/q$, $G(u) = u^{1/9}$ and $q(t) = q\Big(\frac{2(q^9+1)(q^8+1)(q^{11}+1)(q^{10}+1)}{(q-1)^4}t^8 + \frac{(q^8+1)(q^7+1)(q^{10}+1)(q^9+1)}{(q-1)^4}t^7\Big)$. It is easy to see that equation (2.24) satisfies all the conditions of Theorem 2.3.12. Hence, every solution of (2.24) oscillates. In particular, $y(t) = (-1)^{\log_q^t}t^9$ is an oscillatory solution of (2.24).

2.4 Sufficient conditions for oscillation of (E_2) with $\int_{t_0}^{\infty}\frac{\sigma(t)}{r(t)}\Delta t < \infty$.

Theorem 2.4.1. *Let $0 \leqslant p(t) \leqslant p < \infty$. Suppose that ($H_2$), ($H_3'$), ($H_6$), ($H_{11}$), ($H_{19}$) and ($\Lambda$) hold. If*

$$(H_{22}) \quad \int_{t^*}^{\infty} h(t)Q(t)G(F^+(\beta(t)))\Delta t = \infty = \int_{t^*}^{\infty} h(t)Q(t)G(F^-(\beta(t)))\Delta t,$$

where $h(t) = min\{R_1^\gamma(\sigma(t)), R_1^\gamma(\sigma(\alpha(t)))\}$, $\gamma > 1$, then all solutions of (E_2) *oscillates.*

Remark 2.4.2. Note that (H_{22}) implies (H_{22}'), where

$$(H_{22}') \quad \int_{t^*}^{\infty} Q(t)G(F^+(\beta(t)))\Delta t = \infty = \int_{t^*}^{\infty} Q(t)G(F^-(\beta(t)))\Delta t.$$

Proof. (**proof of Theorem 2.4.1**) Proceeding as in the proof of Theorem 2.2.2, by defining $z(t)$ and $w(t)$ as in (2.3) and (2.12), we obtain (2.13), that is,

$$(r(t)w^{\Delta^2}(t))^{\Delta^2} = -q(t)G(y(\beta(t))) \leqslant (\not\equiv)0$$

for $t \in [t_1,\infty)_{\mathbb{T}}$. Hence $w(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$. However, $w(t) < 0$ implies that $0 < z(t) < F(t)$, a contradiction due to (H_{11}). Therefore, $w(t) > 0$ for $t \geqslant t_2$. Further,

$z(t) \geqslant F^+(t)$, $t \geqslant t_2$. Take $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $\alpha(t)$, $\beta(t) \geqslant t_2$ for all $t \in [t_3, \infty)_{\mathbb{T}}$. The use of ($H_2$), ($H_3'$) and ($\Lambda$) yields,

$$\begin{aligned} 0 &\geqslant (r(t)w^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(z(\beta(t))) \\ &\geqslant (r(t)w^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(F^+(\beta(t))). \end{aligned}$$

If one of the cases (a), (b), (d) of Lemma 2.3.1 holds, then integrating the above inequality we get

$$\int_{t_3}^{\infty} Q(t)G(F^+(\beta(t)))\Delta t < \infty,$$

a contradiction to (H_{22}'). If the case (c) of Lemma 2.3.1 holds. By proceeding as in case (c) of Theorem 2.3.5, using Lemma 2.3.2 and Lemma 2.3.4, from (2.13), it follows that

$$-[((-r(t)w^{\Delta^2}(t))^{\Delta})^{1-\gamma}]^{\Delta} \geqslant (\gamma - 1)L^{\gamma}R_1^{\gamma}(\sigma(t))q(t)G(y(\beta(t))), \tag{2.25}$$

for $t \geq t_3 > t_2$, where $L = k/k_2$. Taking $t_4 \in [t_3, \infty)_{\mathbb{T}}$ such that $\alpha(t) \geqslant t_3$ and $\beta(t) \geqslant t_3$ for all $t \in [t_4, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned} -[((-r(t)w^{\Delta^2}(t))^{\Delta})^{1-\gamma}]^{\Delta} \quad - \quad & G(p)[((-r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta})^{1-\gamma}]^{\Delta} \\ & \geqslant \lambda L^{\gamma}(\gamma - 1)h(t)Q(t)G(z(\beta(t))) \\ & \geqslant \lambda L^{\gamma}(\gamma - 1)h(t)Q(t)G(F^+(\beta(t))) \end{aligned}$$

for $t \geqslant t_4 > t_3$. Integrating the above inequality, we obtain

$$\int_{t_4}^{\infty} h(t)Q(t)G(F^+(\beta(t)))\Delta t < \infty,$$

a contradiction to (H_{22}). The proof of $y(t)$ is an eventually negative is similar. Hence the proof of the theorem is complete. □

Example 2.4.3. Suppose $\mathbb{T} = \mathbb{R}$,

$$[e^t(y(t) + (1 + e^{-t})y(t - 3\pi))'']'' + (2 + e^{3t})y(t - \frac{7\pi}{2}) = e^{3t}\cos t, \tag{2.26}$$

for $t \in [\frac{7\pi}{2}, \infty)_{\mathbb{R}}$. Here $1 < p(t) = 1+e^{-t} < 2$, $Q(t) = min\{2+e^{3t}, 2+e^{3t-9\pi}\} = 2+e^{3t-9\pi}$ and $R_1(t) = e^{-t}$. Taking $\gamma = 2$, we get $h(t) = e^{-2t}$. Further, $F(t) = \frac{e^{2t}\sin t}{50}$ and

$F(t-7\pi/2) = e^{2(t-7\pi/2)}\sin(t-7\pi/2) = e^{2(t-7\pi/2)}\cos t$. From this,

$$F^+(t-\tfrac{7}{2}\pi) = \begin{cases} 0 & \text{for } (4n-3)\frac{\pi}{2} \leqslant t \leqslant (4n-1)\frac{\pi}{2} \\ \\ e^{2t-7\pi}cost & \text{for } (4n-1)\frac{\pi}{2} \leqslant t \leqslant (4n+1)\frac{\pi}{2} \end{cases}$$

and

$$F^-(t-\tfrac{7}{2}\pi) = \begin{cases} -e^{2t-7\pi}cost & \text{for } (4n-3)\frac{\pi}{2} \leqslant t \leqslant (4n-1)\frac{\pi}{2} \\ \\ 0 & \text{for } (4n-1)\frac{\pi}{2} \leqslant t \leqslant (4n+1)\frac{\pi}{2}, \end{cases}$$

for $n = 0, 1, 2, \cdots$. Thus,

$$\begin{aligned} \int_{\frac{7\pi}{2}}^{\infty} h(t)Q(t)G(F^+(\beta(t)))dt &= \int_{\frac{7\pi}{2}}^{\infty} e^{-2t}(2+e^{3t-9\pi})F^+(t-7\pi/2)dt \\ &= \sum_{n=2}^{\infty}\int_{(4n-1)\frac{\pi}{2}}^{(4n+1)\frac{\pi}{2}} e^{-2t}(2+e^{3t-9\pi})e^{2t-7\pi}cos\ t\ dt \\ &= e^{-7\pi}\sum_{n=2}^{\infty}\int_{(4n-1)\frac{\pi}{2}}^{(4n+1)\frac{\pi}{2}} (2+e^{3t-9\pi})cos\ t\ dt \\ &> 2e^{-7\pi}\sum_{n=2}^{\infty}\int_{(4n-1)\frac{\pi}{2}}^{(4n+1)\frac{\pi}{2}} cos\ t = \infty. \end{aligned}$$

Similarly, $\int_{\frac{7\pi}{2}}^{\infty} h(t)Q(t)G(F^-(\beta(t)))dt = \infty$. It is easy to see that equation (2.26) satisfies all the conditions of Theorem 2.4.1. Hence every solution of equation (2.26) oscillates. In particular $y(t) = sin\ t$ is an oscillatory solution of (2.26).

Theorem 2.4.4. *Let $0 \leqslant p(t) \leqslant p < \infty$. Suppose that ($H_2$), ($H_3'$), ($H_6$), ($H_{13}$), ($H_{19}$), ($H_{22}$) and* ($\Lambda$) *hold. Then every solution of* (E_2) *oscillates or tends to zero as $t \to \infty$.*

Proof. Proceeding as in the proof of Theorem 2.2.2, we obtain (2.13). Hence $w(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$. If $w(t) > 0$ for $t \geqslant t_2$, then we obtain a contradiction as in the proof of Theorem 2.4.1. If $w(t) < 0$ for $t \geqslant t_2$, then $y(t) \leqslant z(t) < F(t)$ and hence $\limsup_{t\to\infty} y(t) \leqslant 0$ by (H_{13}). Consequently, $\lim_{t\to\infty} y(t) = 0$. Thus the proof of the theorem is complete. □

Theorem 2.4.5. *Let $0 \leqslant p(t) \leqslant p < \infty$. Let ($H_2$), ($H_3'$), ($H_6$), ($H_{13}$), ($H_{19}$) and* ($\Lambda$) *hold. If*

$$(H_{23}) \quad \int_{t^*}^{\infty} h(t)Q(t)G(\mid F(\beta(t)) \mid)\Delta t = \infty,$$

then every bounded solution of (E_2) *oscillates or tends to zero as* $t \to \infty$.

Remark 2.4.6. Note that (H_{23}) implies (H'_{23}) where,

$$(H'_{23}) \quad \int_{t^*}^{\infty} Q(t)G(\mid F(\beta(t)) \mid)\Delta t = \infty.$$

Proof. (**proof of Theorem 2.4.5**) Proceeding as in the proof of Theorem 2.2.2, we obtain (2.13). Hence $w(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$. Let $w(t) > 0$. Thus $z(t) > F(t)$, for $t \geqslant t_2$. Let $F(t) \geqslant 0$ and $\alpha(t) \geqslant t_2$, $\beta(t) \geqslant t_2$ for $t \geqslant t_3 > t_2$. The use of (H_2), (H'_3) and (Λ) yields

$$0 \geq \; (r(t)w^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(F(\beta(t)))$$

for $t \geqslant t_3 > t_2$. If one of the cases (a), (b), (d) of Lemma 2.3.1 holds, then

$$\int_{t_3}^{\infty} Q(t)G(F(\beta(t)))\Delta t < \infty,$$

which is a contradiction to (H_{23}) due to Remark 2.4.6. If the case (c) of Lemma 2.3.1 holds, then we may proceed as in the proof of Theorem 2.4.1, we obtain

$$\int_{t_4}^{\infty} h(t)Q(t)G(F(\beta(t)))\Delta t < \infty,$$

a contradiction to (H_{23}). Let $F(t) \leqslant 0$ for $t \geqslant t_3$. Since $w(t)$ is bounded, then the case (a) of Lemma 2.3.1 does not hold. Further $\lim_{t\to\infty} w(t)$ exists in each of the cases (b) and (d). From (2.13), it follows that

$$\int_{t_3}^{\infty} q(t)G(y(\beta(t)))\Delta t < \infty$$

in each of the cases (b) and (d). Hence, $\liminf_{t\to\infty} y(t) = 0$, because (H_{23}) implies that $\int_{t_3}^{\infty} q(t)\Delta t = \infty$. In the case (c) of Lemma 2.3.1, proceeding as in the proof of Theorem 2.4.1, to obtain (2.25), which yields

$$-[((-r(t)w^{\Delta^2}(t))^{\Delta})^{1-\gamma}]^{\Delta} \geqslant (\gamma-1)L^{\gamma}h(t)q(t)G(y(\beta(t)))$$

that is,

$$\int_{t_3}^{\infty} h(t)q(t)G(y(\beta(t)))\Delta t < \infty.$$

Hence $\liminf_{t\to\infty} y(t) = 0$; otherwise, $\int_{t_3}^{\infty} h(t)q(t)\Delta t < \infty$, a contradiction to ($H_{23}$). From Lemma 1.5.1, it follows that $\lim_{t\to\infty} z(t) = 0$ and hence $\lim_{t\to\infty} y(t) = 0$, since $y(t) \le z(t)$. Let $w(t) < 0$ for $t \geqslant t_2$. Let $F(t) \geqslant 0$. Since $w(t) < 0$, then we have $y(t) < z(t) < F(t)$. Consequently, $\liminf_{t\to\infty} y(t) = 0$. In each of the cases (b), (c) and (d) of Lemma 2.3.1, $\lim_{t\to\infty} w(t)$ exists and hence $\lim_{t\to\infty} z(t)$ exists. Since $y(t)$ is bounded, then $w(t)$ is bounded. From Lemma 1.5.1, it follows that $\lim_{t\to\infty} z(t) = 0$. Since $z(t) > y(t)$, then $\lim_{t\to\infty} y(t) = 0$. The cases (e) and (f) of Lemma 2.3.1 do not hold, because $w(t)$ is bounded. Let $F(t) \leqslant 0$ for $t \geqslant t_3$. In this case, $w(t) < 0$ implies that $0 < z(t) < F(t)$, which is a contradiction. The case $y(t)$ be an eventually negative solution is similarly dealt with. Thus the theorem is proved. □

Theorem 2.4.7. *Let $-1 < p \leqslant p(t) \leqslant 0$. Suppose that ($H_6$), ($H_{12}$) and ($H_{19}$) hold. If*

$$(H_{24}) \quad \int_{t_1}^{\infty} R_1^{\gamma}(\sigma(t))q(t)G(F^{+}(\beta(t)))\Delta t = \infty = \int_{t_1}^{\infty} q(t)G(F^{-}(\alpha^{-1}(\beta(t))))\Delta t,$$

and

$$(H_{25}) \quad \int_{t_1}^{\infty} R_1^{\gamma}(\sigma(t))q(t)G(-F^{-}(\beta(t)))\Delta t = -\infty = \int_{t_1}^{\infty} q(t)G(-F^{+}(\alpha^{-1}(\beta(t))))\Delta t,$$

then a solution $y(t)$ of (E_2) *oscillates or* $\liminf_{t\to\infty}(y(t) - y(\alpha(t))) < 0$.

Proof. Proceeding as in the proof of Theorem 2.2.2, we obtain (2.13). Hence $w(t) > 0$ or < 0 for $t \geqslant t_2$. If $w(t) > 0$, then $y(t) > F(t)$ and hence $y(t) \geqslant F^{+}(t)$, $t \geqslant t_2$. Take $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $\beta(t) \geqslant t_2$ for all $t \in [t_3, \infty)_{\mathbb{T}}$. In each of the cases (a), (b) and (d) of Lemma 2.3.1, we obtain from (2.13), that

$$\int_{t_3}^{\infty} q(t)G(F^{+}(\beta(t)))\Delta t < \infty,$$

a contradiction. In the case (c) of Lemma 2.3.1, we obtain from (2.25), that

$$\int_{t_4}^{\infty} R_1^{\gamma}(\sigma(t))q(t)G(F^{+}(\beta(t)))\Delta t < \infty,$$

a contradiction. Hence $w(t) < 0$ for $t \geqslant t_2$. We claim that $y(t)$ is bounded. If not, then there exists an increasing sequence $\{\tau_n\}_{n=1}^{\infty} \subseteq [t_2, \infty)_{\mathbb{T}}$ such that $\tau_n \to \infty$ and $y(\tau_n) \to \infty$ as $n \to \infty$ and $y(\tau_n) = \max\{y(t) : t_2 \leqslant t \leqslant \tau_n\}$. Hence

$$\begin{aligned} w(\tau_n) &\geqslant y(\tau_n) + py(\alpha(\tau_n)) - F(\tau_n)) \\ &\geqslant (1+p)y(\tau_n) - F(\tau_n) \end{aligned}$$

implies that $w(\tau_n) > 0$ for large n because $1 + p > 0$ and $F(t)$ is bounded. This is a contradiction. Hence $w(t)$ is bounded. Thus, none of the cases (e) and (f) of Lemma 2.3.1 holds. Since $w(t) < 0$, then $y(t) > F^{-}(\alpha^{-1}(t))$. Take $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $\beta(t) \geq t_2$ for all $t \in [t_3, \infty)_{\mathbb{T}}$. Hence, in the cases (b) and (d) of Lemma 2.3.1, we obtain from (2.13) that

$$\int_{t_3}^{\infty} q(t)G(F^{-}(\alpha^{-1}(\beta(t))))\Delta t < \infty,$$

a contradiction. Suppose that the case (c) of Lemma 2.3.1 holds. None of the above considerations is possible in this case. However, $w(t) < 0$ implies that $(y(t)-y(\alpha(t))) < F(t)$. Hence $\liminf_{t\to\infty}(y(t) - y(\alpha(t))) \leqslant \liminf_{t\to\infty} F(t) < 0$. Next, let $y(t)$ be an eventually negative solution, then one may proceed as above. Thus the proof of the theorem is complete. □

Theorem 2.4.8. *Suppose that all the conditions of Theorem 2.4.7 are satisfied except* (H_{12})*, which is replaced by* (H_{13})*. Then every solution of* (E_2) *oscillates or tends to zero as* $t \to \infty$*.*

Proof. If $w(t) > 0$, then a contradiction is obtained in each of the cases (a)-(d) of Lemma 2.3.1. Hence $w(t) < 0$ for $t \geqslant t_2 > t_1$ imply that $z(t) < F(t)$. Since $z(t) \geqslant y(t) + py(\alpha(t))$, $(1 + p) > 0$ and $\limsup_{t\to\infty} z(t) \leqslant 0$, $\lim_{t\to\infty} y(t) = 0$. Thus the proof of the theorem is complete. □

Theorem 2.4.9. *Let* $-\infty < p \leqslant p(t) \leqslant 0$*. If* (H_3), (H_{12}), (H_{19}), (H_{24}) *and* (H_{25}) *hold, then a solution* $y(t)$ *of* (E_2) *oscillates or* $\mid y(t) \mid\to \infty$ *as* $t \to \infty$ *or* $\liminf_{t\to\infty}(y(t) - py(\alpha(t))) < 0$*.*

Proof. This proof is similar to that of Theorem 2.4.7 and hence is omitted. □

Theorem 2.4.10. *Let* $-1 < p \leqslant p(t) \leqslant 0$*. Suppose that* (H_3), (H_{13}) *and* (H_{19}) *hold. If*

$$(H_{26}) \quad \int_{t_1}^{\infty} R_1^{\gamma}(\sigma(t))q(t)G(\mid F(\beta(t)) \mid)\Delta t = \infty, \quad \gamma > 1,$$

then every solution of (E_2) *oscillates or tends to zero or tends to* $\pm\infty$ *as* $t \to \infty$*.*

Proof. Proceeding as in the proof of Theorem 2.2.2, to obtain (2.13). Hence $w(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$. Let $w(t) > 0$ for $t \geqslant t_2$. From (H_{26}), it follows that

$$\int_{t_1}^{\infty} q(t)G(\mid F(\beta(t)) \mid)\Delta t = \infty, \int_{t_1}^{\infty} R_1^{\gamma}(\sigma(t))q(t)\Delta t = \infty \text{ and } \int_{t_1}^{\infty} q(t)\Delta t = \infty$$

because, $F(t) \to 0$ and $R_1(\sigma(t)) \to 0$ as $t \to \infty$. Let $F(t) \geqslant 0$ for $t \geqslant t_3 > t_2$. Consequently, $y(t) \geqslant z(t) \geqslant F(t)$. In each of the cases (a), (b) and (d) of Lemma 2.3.1, it follows from (2.13), that

$$\int_{t_3}^{\infty} q(t)G(F(\beta(t)))\Delta t < \infty,$$

which is a contradiction. In the case (c) of Lemma 2.3.1, by proceeding as in the proof of Theorem 2.4.1, we obtain (2.25), which yields

$$\int_{t_3}^{\infty} R_1^{\gamma}(\sigma(t))q(t)G(F(\beta(t)))\Delta t < \infty,$$

a contradiction to (H_{26}). Let $F(t) \leqslant 0$ for $t \geqslant t_3 > t_2$. If $w(t) > 0$ for $t \geqslant t_2$, then $\lim_{t\to\infty} w(t) = \infty$ in the case (a) of Lemma 2.3.1. Hence $\lim_{t\to\infty} z(t) = \infty$. Since $y(t) > z(t)$, then $\lim_{t\to\infty} y(t) = \infty$. In each of the cases (b), (d) and (c) of Lemma 2.3.1, $0 \leqslant \delta \leqslant \infty$, where $\delta = \lim_{t\to\infty} w(t)$. If $\delta = \infty$, then $\lim_{t\to\infty} y(t) = \infty$. If $0 < \delta < \infty$, then $\lim_{t\to\infty} z(t) = \delta$. From (2.13), we get

$$\int_{t_3}^{\infty} q(t)G(y(\beta(t)))\Delta t < \infty,$$

in the cases (b) and (d). Further, in the case (c), (2.25) yields

$$\int_{t_3}^{\infty} R_1^{\gamma}(\sigma(t))q(t)G(y(\beta(t)))\Delta t < \infty.$$

Hence $\liminf_{t\to\infty} y(t) = 0$. From Lemma 1.5.1, it follows that $\delta = 0$, a contradiction. Then $\lim_{t\to\infty} z(t) = 0$. Since $z(t) \geqslant y(t) + py(\alpha(t))$ and $(1 + p) > 0$, then $y(t)$ is bounded and hence $\limsup_{t\to\infty} y(t) = 0$. Hence $\lim_{t\to\infty} y(t) = 0$. Hence $w(t) < 0$ for $t \geqslant t_2$. The following analysis holds for $F(t) \geqslant 0$ or $\leqslant 0$. As in the proof of Theorem 2.4.7, we may show that $y(t)$ is bounded and hence $w(t)$ is bounded. This implies that the cases (e) and (f) of Lemma 2.3.1 do not hold. In each of the cases (b), (c) and (d)

of Lemma 2.3.1, we proceed as follows. Since $w(t) < 0$, then $z(t) < F(t)$ and hence $\limsup_{t\to\infty} z(t) \leqslant 0$. Thus,

$$0 \geqslant \limsup_{t\to\infty} \Big(y(t) + py(\alpha(t)) \Big) \geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty} (py(\alpha(t))) = (1+p) \limsup_{t\to\infty} y(t).$$

Since $(1+p) > 0$, then $\lim_{t\to\infty} y(t) = 0$. The proof for the case when $y(t)$ is an eventually negative solution is similar. Thus the theorem is proved. □

Corollary 2.4.11. *Suppose that the conditions of Theorem 2.4.10 hold. Then every bounded solution of* (E_2) *oscillates or tends to zero as* $t \to \infty$.

In the following we have established the **existence of bounded positive solutions** of (E_2).

Theorem 2.4.12. *Let* $0 \leqslant p(t) \leqslant p < 1$. *Suppose that* G *is Lipschitzian on intervals of the form* $[a, b]$, $0 < a < b < \infty$, $F(t)$ *changes sign and*

$$-(1-p)/8 \leqslant F(t) \leqslant (1-p)/2,$$

where F *is same as in* (H_{11}). *If* (H_{19}) *and*

$$(H_{27}) \qquad \int_{t_0}^{\infty} \sigma(t)q(t)\Delta t < \infty$$

hold, then (E_2) *admits a positive bounded solution.*

Proof. Let $T_1 \in [t_0, \infty)_{\mathbb{T}}$ sufficiently large such that

$$L \int_{T_1}^{\infty} \sigma(t)q(t)\Delta t < \frac{1}{2}(1-p) \quad \textit{and} \quad \int_{T_1}^{\infty} \frac{\sigma(t)}{r(t)} \Delta t < \frac{1}{2},$$

where $L = \max\{L_1, G(1)\}$ and L_1 is the Lipschitz constant of G on $[\frac{1-p}{8}, 1]$. Let $X = BC_{rd}([T_1, \infty)_{\mathbb{T}}, \mathbb{R})$, then X is a Banach space with respect to norm defined by $\|y\| = \sup_{t\in[T_1,\infty)_{\mathbb{T}}}\{\mid y(t) \mid\}$. Let

$$S = \{y \in X : \frac{1-p}{8} \leqslant y(t) \leqslant 1,\ t \geqslant T_1\}.$$

Hence S is a complete metric space with the metric induced by the norm. Take $T_2 \in [T_1, \infty)_{\mathbb{T}}$ such that $\alpha(t),\ \beta(t) \geqslant T_1$ for all $t \in [T_2, \infty)_{\mathbb{T}}$. For $y \in S$, we define

$$Ty(t) = \begin{cases} Ty(T_2) & \text{for } t \in [T_1, T_2)_{\mathbb{T}} \\ -p(t)y(\alpha(t)) + \frac{1+p}{2} + F(t) & \\ -\int_t^\infty \frac{\sigma(s)-t}{r(s)}\Big(\int_s^\infty (\sigma(u)-s)q(u)G(y(\beta(u)))\Delta u\Big)\Delta s & \text{for } t \in [T_2, \infty)_{\mathbb{T}}. \end{cases}$$

Hence, $Ty(t) \leqslant \frac{1+p}{2} + \frac{1-p}{2} = 1$ for $t \geqslant T_1$ and $Ty(t) \geqslant -p + \frac{1+p}{2} - \frac{1-p}{8} - \frac{1-p}{4} = \frac{1-p}{8}$ for $t \geqslant T_1$. Because, for $t \geqslant T_1$,

$$\begin{aligned} &\int_t^\infty \frac{\sigma(s)-t}{r(s)}\Big(\int_s^\infty (\sigma(u)-s)q(u)G(y(\beta(u)))\Delta u\Big)\Delta s \\ &\qquad \le G(1)\int_t^\infty \frac{\sigma(s)}{r(s)}\Big(\int_s^\infty \sigma(u)q(u)\Delta u\Big)\Delta s \\ &\qquad \leqslant G(1)\Big(\int_{T_1}^\infty \frac{\sigma(t)}{r(t)}\Delta t\Big)\Big(\int_{T_1}^\infty \sigma(t)q(t)\Delta t\Big) \\ &\qquad \leqslant \frac{1}{4}(1-p). \end{aligned}$$

Consequently, $Ty \in S$, that is, $T : S \to S$. Next we show that T is continuous. Let $y_k(t) \in S$ such that $\lim_{k\to\infty}\|y_k - y\| = 0$ for all $t \geqslant T_1$. Because S is closed, $y(t) \in S$. Indeed,

$$\begin{aligned} &\|(Ty_k) - (Ty)\| \\ &\quad \leqslant \|p(t)(y(\alpha(t)) - y_k(\alpha(t)))\| \\ &\qquad + \Big\|\int_t^\infty \frac{\sigma(s)-t}{r(s)}\Big(\int_s^\infty (\sigma(u)-s)q(u)\big(G(y(\beta(u))) - G(y_k(\beta(u)))\big)\Delta u\Big)\Delta s\Big\| \\ &\quad \leqslant p\|y_k - y\| + L\|y_k - y\|\int_t^\infty \frac{\sigma(s)}{r(s)}\Big(\int_s^\infty \sigma(u)q(u)\Delta u\Big)\Delta s \\ &\quad \leqslant p\|y_k - y\| + \frac{1-p}{4}\|y_k - y\| \leqslant \big(p + \frac{1-p}{4}\big)\|y_k - y\|. \end{aligned}$$

Thus, $\|Ty_k - Ty\| \leqslant \big(p + \frac{1-p}{4}\big)\|y_k - y\| \to 0$ as $k \to \infty$. Hence T is continuous. In order to apply **Schauder's fixed point theorem** we need to show that Ty is pre-compact. Let $y \in S$. For $t_2 \geqslant t_1 \geqslant T_1$,

$$\begin{aligned} Ty(t_1) - Ty(t_2) \;=\; & p(t_2)y(\alpha(t_2)) - p(t_1)y(\alpha(t_1)) + F(t_1) - F(t_2) \\ & -\int_{t_1}^\infty \frac{\sigma(s)-t_1}{r(s)}\Big(\int_s^\infty (\sigma(u)-s)q(u)G(y(\beta(u)))\Delta u\Big)\Delta s \\ & +\int_{t_2}^\infty \frac{\sigma(s)-t_2}{r(s)}\Big(\int_s^\infty (\sigma(u)-s)q(u)G(y(\beta(u)))\Delta u\Big)\Delta s, \end{aligned}$$

that is,

$$\begin{aligned} Ty(t_1) - Ty(t_2) &= p(t_2)y(\alpha(t_2)) - p(t_1)y(\alpha(t_1)) + F(t_1) - F(t_2) \\ &\quad - \int_{t_1}^{\infty}\int_{\theta}^{\infty}\frac{1}{r(s)}\Big(\int_s^{\infty}(\sigma(u)-s)q(u)G(y(\beta(u)))\Delta u\Big)\Delta s\Delta\theta \\ &\quad + \int_{t_2}^{\infty}\int_{\theta}^{\infty}\frac{1}{r(s)}\Big(\int_s^{\infty}(\sigma(u)-s)q(u)G(y(\beta(u)))\Delta u\Big)\Delta s\Delta\theta, \end{aligned}$$

implies that,

$$\begin{aligned} \Big|Ty(t_2) - Ty(t_1)\Big| &\leqslant \Big|p(t_2)y(\alpha(t_2)) - p(t_1)y(\alpha(t_1))\Big| + \mid F(t_1) - F(t_2) \mid \\ &\quad + \Big|\int_{t_1}^{t_2}\int_{\theta}^{\infty}\frac{1}{r(s)}\Big(\int_s^{\infty}(\sigma(u)-s)q(u)G(y(\beta(u)))\Delta u\Big)\Delta s\Delta\theta\Big| \\ &\leq \Big|p(t_2)y(\alpha(t_2)) - p(t_1)y(\alpha(t_1))\Big| + \mid F(t_1) - F(t_2) \mid \\ &\quad + G(1)\Big|\int_{t_1}^{\infty}\frac{1}{r(s)}\Big(\int_s^{\infty}\sigma(u)q(u)\Delta u\Big)\Delta s\Big| \mid t_1 - t_2 \mid, \end{aligned}$$

$\to 0$ as $t_2 \to t_1$. Thus Ty is pre-compact. By **Schauder's fixed point theorem**, T has a **fixed point**, that is, Ty=y. Consequently, $y(t)$ is the solution of (E_2) with $\frac{1-p}{8} \leqslant y(t) \leqslant 1$. The proof of the theorem is complete. □

Theorem 2.4.13. *Let $-1 < p \leqslant p(t) \leqslant 0$. If ($H_{19}$) and ($H_{27}$) hold, G satisfies Lipschitz on intervals of the form $[a,b]$, $0 < a < b < \infty$, $F(t)$ changes sign such that*

$$-\frac{1}{8}(1+p) \leqslant F(t) \leqslant \frac{1}{2}(1+p),$$

then (E_2) *admits a positive bounded solution.*

Proof. It is possible to choose $T_1 \in [t_0, \infty)_{\mathbb{T}}$ sufficiently large such that

$$L\int_{T_1}^{\infty}\sigma(t)q(t)\Delta t < \frac{1}{2}(1+p) \quad and \quad \int_{T_1}^{\infty}\frac{\sigma(t)}{r(t)}\Delta t < \frac{1}{2},$$

where $L = max\{L_1, G(1)\}$ and L_1 is the Lipschitz constant of G on $[\frac{1+p}{8}, 1]$. Let $X = BC_{rd}([T_1,\infty)_{\mathbb{T}}, \mathbb{R})$. Then X is a Banach space with respect to norm defined by $\|y\| = \sup_{t\in[T_1,\infty)_{\mathbb{T}}}\{\mid y(t) \mid\}$. Let

$$S = \{y \in X : \frac{1+p}{8} \leqslant y(t) \leqslant 1,\ t \geqslant T_1\}.$$

Hence, S is a complete metric space. Take $T_2 \in [T_1, \infty)_{\mathbb{T}}$ such that $\alpha(t), \beta(t) \geqslant T_1$ for all $t \in [T_2, \infty)_{\mathbb{T}}$. For $y \in S$, we define

$$Ty(t) = \begin{cases} Ty(T_2) & \text{for } t \in [T_1, T_2)_{\mathbb{T}} \\ -p(t)y(\alpha(t)) + \frac{1+p}{2} + F(t) & \\ -\int_t^\infty \frac{\sigma(s)-t}{r(s)}\big(\int_s^\infty (\sigma(u)-s)q(u)G(y(\beta(u)))\Delta u\big)\Delta s & \text{for } t \in [T_2, \infty)_{\mathbb{T}}. \end{cases}$$

Rest of the proof is similar to that of Theorem 2.4.12. □

Theorem 2.4.14. *Let* $0 \leqslant p(t) \leqslant p < 1$ *and* G *satisfies Lipschitz property on intervals of the form* $[a, b]$, $0 < a < b < \infty$. *If* (H_{13}), (H_{19}) *and* (H_{27}) *hold, then* (E_2) *admits a positive bounded solution.*

Proof. This proof is similar to that of Theorem 2.4.12. However, there are some changes in the setting. Let T_1 be sufficiently large so that

$$| F(t) | < \frac{1}{10}(1-p) \ for \ t \geqslant T_1, L\int_{T_1}^\infty \sigma(t)q(t)\Delta t < \frac{1}{10}(1-p) \quad and \quad \int_{T_1}^\infty \frac{\sigma(t)}{r(t)}\Delta t < \frac{1}{2},$$

where $L = max\{L_1, G(1)\}$ and L_1 is the Lipschitz constant of G on $[\frac{1-p}{20}, 1]$. Let $X = BC_{rd}([T_1, \infty)_{\mathbb{T}}, \mathbb{R})$. Then X is a Banach space with respect to norm defined by $\|y\| = \sup_{t\in[T_1,\infty)_{\mathbb{T}}}\{| y(t) |\}$. Let $S = \{y \in Bc_{rd}([T_1, \infty)_{\mathbb{T}}, \mathbb{R}) : \frac{1-p}{20} \leqslant y(t) \leqslant 1\}$. Hence S is a complete metric space. Take $T_2 \in [T_1, \infty)_{\mathbb{T}}$ such that $\alpha(t), \beta(t) \geqslant T_1$ for all $t \in [T_2, \infty)_{\mathbb{T}}$. For $y \in S$, we define

$$Ty(t) = \begin{cases} Ty(T_2) & \text{for } t \in [T_1, T_2)_{\mathbb{T}} \\ -p(t)y(\alpha(t)) + \frac{1+4p}{5} + F(t) & \\ -\int_t^\infty \frac{\sigma(s)-t}{r(s)}\big(\int_s^\infty (\sigma(u)-s)q(u)G(y(\beta(u)))\Delta u\big)\Delta s & \text{for } t \in [T_2, \infty)_{\mathbb{T}}. \end{cases}$$

Rest of the proof is similar to that of Theorem 2.4.12. □

2.5 Conclusion

This chapter devided into five sections. In Section 2.1, we have studied the oscillatory and asymptotic behavior of solutions of homogeneous equation (E_1) with $\int_{t_0}^\infty \frac{t}{r(t)}\Delta t = \infty$ for different ranges of $p(t)$. In the Theorems 2.1.4, 2.1.7 and 2.1.13, we have proved that every solution of (E_1) oscillates for $p(t)$ with $0 \leq p(t) \leq p < \infty$. In the Theorem 2.1.15, we have shown that every solution of (E_1) oscillates or tends to zero as $t \to \infty$

for $p(t)$ with $-1 < p \leq p(t) \leq 0$. In Theorem 2.1.16, we have proved that every bounded solution of (E_1) oscillates or tends to zero as $t \to \infty$ for $p(t)$ with $-\infty < p_1 \leq p(t) \leq p_2 < -1$.

In Section 2.2, we have studied the oscillatory and asymptotic behavior of solutions of (E_2) with $\int_{t_0}^{\infty} \frac{t}{r(t)} \Delta t = \infty$ for different ranges of $p(t)$. In the Theorems 2.2.2 and 2.2.4, we have proved that every solution of (E_2) is oscillatory for $p(t)$ with $0 \leq p(t) \leq p < \infty$ and $-1 < p \leq p(t) \leq 0$ respectively. In Theorem 2.2.5, we have established that every solution of (E_2) oscillates or tends to $\pm\infty$ as $t \to \infty$ for $p(t)$ with $-\infty < p \leq p(t) \leq 0$. In Theorem 2.2.8, we have shown that every solution of (E_2) oscillates or tends to zero as $t \to \infty$ for $p(t)$ with $-1 < p \leq p(t) \leq 0$, whereas in Theorem 2.2.7, we have proved that every bounded solution of (E_2) oscillates or tends to zero as $t \to \infty$ for $p(t)$ with $0 \leq p(t) \leq p_1 < 1$ or $1 < p_2 \leq p(t) \leq p_3 < \infty$. In Theorems 2.2.11 and 2.2.12, we have obtained the sufficient conditions for the existence of bounded positive solution for the range $p(t)$ with $0 \leq p(t) \leq p \leq 1$.

In Section 2.3, we have studied the oscillatory and asymptotic behavior of homogeneous equation (E_1) with $\int_{t_0}^{\infty} \frac{\sigma(t)}{r(t)} \Delta t < \infty$ for different ranges of $p(t)$. In Theorem 2.3.5 and Theorem 2.3.8, we have proved that every solution of (E_1) is oscillatory for $p(t)$ with $0 \leq p(t) \leq p < \infty$ and $0 \leq p(t) \leq p < 1$ respectively. In Theorem 2.3.10, we have shown that every solution of (E_1) oscillates or tends to zero as $t \to \infty$ for $p(t)$ with $-1 < p \leq p(t) \leq 0$, where in Theorem 2.3.12, we have proved that every bounded solution of (E_1) oscillates or tends to zero as $t \to \infty$ or $\liminf_{t\to\infty} \mid y(t) \mid > 0$ for $p(t)$ with $-\infty < p_1 \leq p(t) \leq p_2 < -1$.

In Section 2.4, we have studied the oscillatory and asymptotic behavior of solution of (E_2) with $\int_{t_0}^{\infty} \frac{\sigma(t)}{r(t)} \Delta t < \infty$ for different ranges of $p(t)$. In Theorem 2.4.1, we have proved that every solution of (E_2) oscillatory, whereas in Theorem 2.4.4, we have established that every solution of (E_2) oscillatory or tends to zero as $t \to \infty$ for the range of $p(t)$ with $0 \leq p(t) \leq p < \infty$. Note that in Theorem 2.4.1, we used (H_{11}) and in Theorem 2.4.4 we used (H_{13}) instead of (H_{11}). In Theorem 2.4.7, we have shown that every solution of (E_2) oscillates or $\liminf_{t\to\infty}(y(t) - y(\alpha(t))) < 0$ whereas in Theorem 2.4.10, we have established that every solution of (E_2) oscillates or tends to zero as $t \to \infty$

or tends to $\pm\infty$ as $t \to \infty$ for $p(t)$ with $-1 < p \leq p(t) \leq 0$. In Theorems 2.4.12 and 2.4.14, we have obtained the sufficient conditions for the existence of bounded positive solution for the range of $p(t)$ with $0 \leq p(t) \leq p < 1$, whereas in Theorem 2.4.13, we have obtained the sufficient conditions for the existence of bounded positive solution for the range of $p(t)$ with $-1 < p \leq p(t) \leq 0$.

It would be interesting to study the oscillatory/nonoscillatory/asymptotic behavior of (E_1) and (E_2) if $p(t)$ oscillates and one has to obtained the sufficient conditions under which all solutions of (E_1) and (E_2) are oscillatory for all the ranges of $p(t)$ other than $0 \leq p(t) \leq p < \infty$.

Chapter 3

Fourth Order Neutral Delay Dynamic Equations with Positive and Negative Coefficients

This chapter is an extension of Chapter 2. In this chapter, we would like to study the oscillatory and asymptotic properties of solutions of fourth order nonlinear neutral delay dynamic equations of the the following form

$$(r(t)(y(t)+p(t)y(\alpha(t)))^{\Delta^2})^{\Delta^2}+q(t)G(y(\beta(t)))-h(t)H(y(\gamma(t)))=0 \qquad (E_3)$$

and

$$(r(t)(y(t)+p(t)y(\alpha(t)))^{\Delta^2})^{\Delta^2}+q(t)G(y(\beta(t)))-h(t)H(y(\gamma(t)))=f(t) \qquad (E_4)$$

under the assumptions

$$\int_{t_0}^{\infty}\frac{t}{r(t)}\Delta t=\infty \quad or \quad \int_{t_0}^{\infty}\frac{\sigma(t)}{r(t)}\Delta t<\infty$$

for $t\in[t_0,\infty]_{\mathbb{T}}$, $t_0\geqslant 0$, where $\mathbb{T}$ is a time scale such that sup $\mathbb{T}=\infty$ and $t_0\in\mathbb{T}$. This formulation is quite general in that it includes as special cases the well known fourth order Emden-Fowler type ordinary differential equation

$$y^{(4)}+q(t)|y|^{\gamma}\text{sgn y}=0$$

and its discrete analog, i.e., the difference equation

$$\Delta^4 y_n + q_n |y_n|^\gamma \mathrm{sgn\ y_n} = 0.$$

Variations on these equations such as those with time delays, forcing terms, or equations involving neutral terms have been widely studied in the literature and these are included in the forms of (E_3) and (E_4) as well. What is also important to point out here is that using the framework of time scales, a single result may apply to differential equations and difference equations at the same time. At the same time, the results would apply to other time scales such as the quantum time scale $\mathbb{T} = q^{\mathbb{N}_0} = \{t : t = q^k,\ k \in \mathbb{N}_0\}$ with $q > 1$, $\mathbb{T} = \mathbb{N}_0^2 = \{t^2 : t \in \mathbb{N}_0\}$, $\mathbb{T} = \{\sqrt{n} : n \in \mathbb{N}_0\}$, etc. We define $q(t)$ in Chapter 2 is positive. If $q(t) < 0$, then it would be possible to obtain analogous results for the oscillation and asymptotic behavior of solutions of (E_1) and (E_2). The problem remains open as to what happens if $q(t)$ is allowed to change signs. Note that if $q(t) = q^+(t) - q^-(t)$, where $q^+(t) = \max\{0, q(t)\}$ and $q^-(t) = \max\{0, -q(t)\}$, then ($E_1$) and ($E_2$) can be viewed as

$$(r(t)(y(t) + p(t)y(\alpha(t)))^{\Delta^2})^{\Delta^2} + q^+(t)G(y(\beta(t))) - q^-(t)G(y(\beta(t))) = 0 \tag{3.1}$$

and

$$\begin{aligned}(r(t)(y(t) + p(t)y(\alpha(t)))^{\Delta^2})^{\Delta^2} + q^+(t)G(y(\beta(t)))\\ -q^-(t)G(y(\beta(t))) = f(t),\end{aligned} \tag{3.2}$$

respectively.

Clearly, (3.1) and (3.2) are particular cases of (E_3) and (E_4) respectively. To the best of our knowledge there are no papers to date on fourth order nonlinear dynamic equations with positive and negative coefficients. The results obtained in this chapter are new and generalize the earlier work of (see [53], [54], [56], [57], [75], [76]).

In equations (E_3) and (E_4), we assume that $r,\ q \in C_{rd}([t_0,\infty)_{\mathbb{T}}, (0,\infty))$, $p,\ f \in C_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$, $h \in C_{rd}([t_0,\infty)_{\mathbb{T}}, [0,\infty))$, $G,\ H \in C(\mathbb{R},\mathbb{R})$ with $uG(u) > 0$ and $uH(u) > 0$ for $u \neq 0$, G is nondecreasing, H is bounded, $\alpha, \beta, \gamma \in C_{rd}(\mathbb{T},\mathbb{T})$ are strictly increasing functions such that

$$\lim_{t\to\infty} \alpha(t) = \lim_{t\to\infty} \beta(t) = \lim_{t\to\infty} \gamma(t) = \infty, \quad \text{and} \quad \alpha(t),\ \beta(t),\ \gamma(t) \leqslant t.$$

We also denote the inverse of α by $\alpha^{-1} \in C_{rd}(\mathbb{T}, \mathbb{T})$.

Let $t_{-1} = \inf_{t\in[t_0,\infty)_{\mathbb{T}}}\{\alpha(t), \beta(t), \gamma(t)\}$. By a *solution* of $(E_3)/(E_4)$ we mean a function $y \in C_{rd}([t_{-1}, \infty)_{\mathbb{T}}, \mathbb{R})$ with $y(t) + p(t)y(\alpha(t)) \in C^2_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, $r(t)(y(t) + p(t)y(\alpha(t)))^{\Delta^2} \in C^2_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ and such that $(E_3)/(E_4)$ is satisfies identically on $[t_0, \infty)_{\mathbb{T}}$. A solution is called *oscillatory* if it is neither eventually positive nor eventually negative, and it is called *nonoscillatory* otherwise. In this chapter, we do not discuss eventually identically vanishing solutions. An equation will be called oscillatory if all its solutions are oscillatory.

The following assumption (Λ) is considered throughout this chapter, where

(Λ) $(\alpha \circ \beta)(t) = (\beta \circ \alpha)(t)$ for all $t \in [t_0, \infty)_{\mathbb{T}}$ and $Q(t) = \min\{q(t), q(\alpha(t))\}$ for $t \in [t^*, \infty)_{\mathbb{T}}$, $t^* > t_0$.

3.1 Sufficient conditions for oscillation of (E_3) with $\int_{t_0}^{\infty} \frac{t}{r(t)}\Delta t = \infty$.

In this section, sufficient conditions are obtained for the oscillation of solutions of (E_3) under the assumption

$$(H_0) \quad \int_{t_0}^{\infty} \frac{t}{r(t)}\Delta t = \infty.$$

The results in this chapter will make use of the following conditions on the functions in equations (E_3) and (E_4).

(H_1) $\int_{t_0}^{\infty} \frac{\sigma(s)}{r(s)} \int_s^{\infty} \sigma(t)h(t)\Delta t\Delta s < \infty$,

(H_2) there exists $\lambda > 0$ such that $G(u) + G(v) \geqslant \lambda G(u + v)$ for $u > 0$ and $v > 0$,

(H_3) $G(u)G(v) = G(uv)$ for $u, v \in \mathbb{R}$ and $H(-u) = -H(u)$ for $u \in \mathbb{R}$,

(H_4) $\int_0^{\pm c} \frac{du}{G(u)} < \infty$ for all $c > 0$,

(H_5) $\int_{t*}^{\infty} Q(t)\Delta t = \infty$.

Theorem 3.1.1. *Assume that conditions (H_0)–(H_5) and* (Λ) *hold,* $\beta(t) \leq \alpha(t)$*, and* p_1*,* p_2*, and* p_3 *are positive real numbers. If (i)* $0 \leqslant p(t) \leqslant p_1 < 1$ *or (ii)* $1 < p_2 \leqslant p(t) \leqslant p_3 < \infty$ *holds, then every solution of* (E_3) *is either oscillatory or converges to zero as* $t \to \infty$*.*

Proof. Let $y(t)$ be a nonoscillatory solution of (E_3), say $y(t)$ is an eventually positive solution. (The proof in case $y(t) < 0$ eventually, is similar and will be omitted.) There exists $t_1 \in [t^*, \infty)_{\mathbb{T}}$ such that $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, $y(\gamma(t))$ and $y(\beta(\alpha(t)))$ are all positive for all $t \in [t_1, \infty)_{\mathbb{T}}$. Define the functions

$$z(t) = y(t) + p(t)y(\alpha(t)), \tag{3.3}$$

$$k(t) = \int_t^\infty \frac{\sigma(s) - t}{r(s)} \int_s^\infty (\sigma(\theta) - s)h(\theta)H(y(\gamma(\theta)))\Delta\theta\Delta s. \tag{3.4}$$

Notice that condition (H_1), and the fact that H is a bounded function imply that $k(t)$ exists for all t. Now if

$$w(t) = z(t) - k(t) = y(t) + p(t)y(\alpha(t)) - k(t), \tag{3.5}$$

then a calculation shows

$$(r(t)w^{\Delta^2}(t))^{\Delta^2} = -q(t)G(y(\beta(t))) \leq (\not\equiv)0, \tag{3.6}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. Clearly, $w(t)$, $w^{\Delta}(t)$, $(r(t)w^{\Delta^2}(t))$ and $(r(t)w^{\Delta^2}(t))^{\Delta}$ are monotonic functions on $[t_1, \infty)_{\mathbb{T}}$. In view of Lemma 2.1.1, we have two cases to consider, namely $w(t) > 0$ or $w(t) < 0$ for $t \geqslant t_2$ for some $t_2 > t_1$. First, we prove this theorem for $p(t)$ with range $0 \leqslant p(t) \leqslant p_1 < 1$. The proof for the remaining range of $p(t)$ is similar.

Suppose that $w(t) > 0$ for $t \geqslant t_2$. Then there exists $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $w(\alpha(t))$, $w(\beta(t)) > 0$ for $t \in [t_3, \infty)_{\mathbb{T}}$. Using ($H_2$), ($H_3$) and ($\Lambda$) gives

$$\begin{aligned} 0 &= (r(t)w^{\Delta^2}(t))^{\Delta^2} + q(t)G(y(\beta(t))) + G(p_1)(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2} \\ &\quad + G(p_1)q(\alpha(t))G(y(\beta(\alpha(t)))) \\ &\geqslant (r(t)w^{\Delta^2}(t))^{\Delta^2} + G(p_1)(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(y(\beta(t)) \\ &\quad + p_1 y(\alpha(\beta(t)))) \\ &\geqslant (r(t)w^{\Delta^2}(t))^{\Delta^2} + G(p_1)(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(z(\beta(t))) \end{aligned} \tag{3.7}$$

for $t \geqslant t_3$. From (3.4), it follows that $k(t) > 0$ and $k^{\Delta}(t) < 0$, so $w(\beta(t)) > 0$ for $t \geqslant t_3$ implies that $w(\beta(t)) < z(\beta(t))$ for $t \geqslant t_3$. Therefore, (3.7) yields

$$(r(t)w^{\Delta^2}(t))^{\Delta^2} + G(p_1)(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(w(\beta(t))) \leqslant 0 \tag{3.8}$$

for $t \geqslant t_3$. Choose $T' \in [T, \infty)_{\mathbb{T}}$ so that $\beta(t) \geqslant T \geqslant t_3$ for all $t \in [T', \infty)_{\mathbb{T}}$. Applying (H_3) and Lemma 2.1.2, inequality (3.8) yields

$$\begin{aligned} 0 \geqslant (r(t)w^{\Delta^2}(t))^{\Delta^2} &+ G(p_1)(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2} \\ &+ \lambda Q(t)G(R_T(\beta(t)))G((r(\beta(t))w^{\Delta^2}(\beta(t)))^{\Delta}) \end{aligned}$$

for $t \geqslant T'$. Hence,

$$\begin{aligned} &\lambda Q(t)G(R_T(\beta(t))) \\ &\quad\leqslant -[G((r(\beta(t))w^{\Delta^2}(\beta(t)))^{\Delta})]^{-1}\{(r(t)w^{\Delta^2}(t))^{\Delta^2} \\ &\qquad + G(p_1)(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2}\} \\ &\quad\leqslant -[G((r(t)w^{\Delta^2}(t))^{\Delta})]^{-1}(r(t)w^{\Delta^2}(t))^{\Delta^2} \\ &\qquad - G(p_1)[G((r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta})]^{-1}(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2}. \end{aligned}$$

Since $\lim_{t\to\infty}(r(t)w^{\Delta^2}(t))^{\Delta}$ exists, applying (H_4) to the above inequality gives

$$\int_{T'}^{\infty} Q(t)G(R_T(\beta(t)))\Delta t < \infty,$$

which contradicts (H_5), since $R_T(t)$ is a monotone increasing function.

Next, we suppose that $w(t) < 0$ for $t \geqslant t_2 > t_1$. Then $z(t) - k(t) < 0$ implies $y(t) \leqslant z(t) = y(t) + p(t)y(\alpha(t)) < k(t)$. Thus, $y(t)$ is bounded. By Lemma 2.1.1, any one of the cases (b), (c), (d) or (e) holds. Consider the case (b). Since $\lim_{t\to\infty} k(t)$ exists, $\lim_{t\to\infty} w(t)$ exists, and so $\lim_{t\to\infty} z(t)$ exists. Furthermore, $\lim_{t\to\infty}(r(t)w^{\Delta^2}(t))^{\Delta}$ exists, and an integration of (3.6) implies

$$\int_{t_2}^{\infty} Q(t)G(y(\beta(t)))\Delta t \leq \int_{t_2}^{\infty} q(t)G(y(\beta(t)))\Delta t < \infty.$$

Hence, it is easy to verify that $\liminf_{t\to\infty} y(t) = 0$ due to (H_5). It then follows from Lemma 1.5.1 that $\lim_{t\to\infty} z(t) = 0$. Thus, $\lim_{t\to\infty} y(t) = 0$ since $z(t) \geqslant y(t)$.

To see that cases (c) and (d) are not possible, first note that $w(t) < 0$, $y(t)$ is bounded, and $\lim_{t\to\infty} k(t)$ exists and so $\lim_{t\to\infty} w(t)$ exists. On the other hand, integrating $w^{\Delta^2}(t) < 0$ twice from t_2 to t, yields $\lim_{t\to\infty} w(t) = -\infty$, which is a contradiction. In case (e), $r(t)w^{\Delta^2}(t)$ is nondecreasing on $[t_2,\infty)_{\mathbb{T}}$. Hence, for $t > t_3 \geqslant t_2$, $r(t)w^{\Delta^2}(t) \geqslant r(t_3)w^{\Delta^2}(t_3)$, and we have

$$tw^{\Delta^2}(t) \geqslant r(t_3)w^{\Delta^2}(t_3)\frac{t}{r(t)}.$$

Integrating the above inequality from t_3 to t, we obtain

$$tw^{\Delta}(t) - t_3w^{\Delta}(t_3) - \int_{t_3}^{t} w^{\Delta}(\sigma(s))\Delta s \geqslant r(t_3)w^{\Delta^2}(t_3)\int_{t_3}^{t}\frac{s}{r(s)}\Delta s,$$

implies

$$tw^{\Delta}(t) - t_3w^{\Delta}(t_3) - \int_{t_3}^{t} w^{\Delta}(s)\Delta s \geqslant r(t_3)w^{\Delta^2}(t_3)\int_{t_3}^{t}\frac{s}{r(s)}\Delta s.$$

By (H_0),

$$w(t) \leqslant tw^{\Delta}(t) - t_3w^{\Delta}(t_3) + w(t_3) - (r(t_3)w^{\Delta^2}(t_3))\int_{t_3}^{t}\frac{s}{r(s)}\Delta s \to -\infty$$

as $t\to\infty$, which is a contradiction. This completes the proof of the theorem. □

The following corollary is immediate.

Corollary 3.1.2. *Under the conditions of Theorem 3.1.1, every unbounded solution of* (E_3) *oscillates.*

Example 3.1.3. Let $\mathbb{T} = \mathbb{R}$ and consider the differential equation

$$(y(t) + 81y(t/3))'''' + y^{1/3}(t/6) - \left(e^{-t} + e^{-t/3} + e^{-t/18}\right)e^{t/36}(1+e^{-t/36})\frac{y(t/36)}{1+|y(t/36)|} = 0, \quad (3.9)$$

for $t\in[t_0,\infty)_{\mathbb{T}}$, $t_0 > 0$. We have $r(t) = 1$, $p(t) = 81$, $q(t) = 1$, $h(t) = \left(e^{-t} + e^{-t/3} + e^{-t/18}\right)$ $e^{t/36}(1+e^{-t/36})$, $G(u) = u^{1/3}$ and $H(u) = \frac{u}{1+|u|}$. Also, $\gamma(t) = t/36$ and $\beta(t) = t/6 < t/3 = \alpha(t)$. It is easy to see that conditions (H_0)–(H_5) hold. For simplicity in showing that (H_1) holds, we only consider one term from $h(t)$. We have

$$\int_{t_0}^{\infty}\frac{\sigma(s)}{r(s)}\int_{s}^{\infty}\sigma(t)h(t)dtds = \int_{t_0}^{\infty} s\int_{s}^{\infty} t\left(e^{-t} + e^{-t/3} + e^{-t/18}\right)e^{t/36}(1+e^{-t/36})dtds < \infty.$$

By Theorem 3.1.1, any solution of (3.9) is either oscillates or tends to zero as $t\to\infty$. Here, $y(t) = e^{-t}$ is a solution.

In our next theorem, we replace condition (H_4) by a different type of growth condition on the function G.

Theorem 3.1.4. *Assume that conditions* $(H_0) - (H_3)$, (H_5) *and* (Λ) *hold,* $\beta(t) \leq \alpha(t)$ *and*

(H_6) $G(x_1)/x_1^\gamma \geqslant G(x_2)/x_2^\gamma$ *for* $x_1 > x_2 > 0$ *and some* $\gamma \geq 1$.

If (i) $0 \leqslant p(t) \leqslant p_1 < 1$ *or (ii)* $1 < p_2 \leqslant p(t) \leqslant p_3 < \infty$ *holds, then every solution of* (E_3) *is either oscillatory or converges to zero as* $t \to \infty$.

Proof. Proceeding as in the proof of Theorem 3.1.1, in case $w(t) > 0$ we can again obtain inequality (3.8) for $t \geqslant t_3$. In view of (3.6), and Lemma 2.1.1, $w(t)$ is increasing, and since $\beta(t)$ is increasing, there exists $k > 0$ and $t_4 \in [t_3, \infty)_{\mathbb{T}}$ such that $w(\beta(t)) > k$ for all $t \geqslant t_4$. Using (H_6) and Lemma 2.1.2, we obtain

$$\begin{aligned} G(w(\beta(t))) &= \left(\frac{G(w(\beta(t)))}{w^\gamma(\beta(t))}\right) w^\gamma(\beta(t)) \\ &\geqslant \left(\frac{G(k)}{k^\gamma}\right) w^\gamma(\beta(t)) \\ &> \left(\frac{G(k)}{k^\gamma}\right) R_T^\gamma(\beta(t))((r(\beta(t))w^{\Delta^2}(\beta(t)))^\Delta)^\gamma \end{aligned}$$

for $t > t_4$. Thus, (3.8) yields,

$$\begin{aligned} \lambda\left(\frac{G(k)}{k^\gamma}\right) & Q(t) R_T^\gamma(\beta(t))((r(\beta(t))w^{\Delta^2}(\beta(t)))^\Delta)^\gamma \\ &\leqslant \lambda Q(t) G(w(\beta(t))) \\ &\leqslant -(r(t)w^{\Delta^2}(t))^{\Delta^2} - G(p_1)(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2} \end{aligned}$$

for $t \geq t_4$. Since $\lim_{t\to\infty}(r(t)w^{\Delta^2}(t))^\Delta$ exists and $R_T(t)$ is nondecreasing, proceeding as in the proof of Theorem 3.1.1, we obtain

$$\int_{t_4}^{\infty} R_T^\gamma(\beta(t)) Q(t) \Delta t < \infty,$$

which contradicts (H_5). The proof in case $w(t) < 0$ is the same as in Theorem 3.1.1. □

We again have a corollary.

Corollary 3.1.5. *Under the conditions of Theorem 3.1.4, every unbounded solution of* (E_3) *oscillates.*

Example 3.1.6. Let $\mathbb{T} = \mathbb{Z}$ and consider the difference equation

$$\Delta^2[n\Delta^2(y(n) + (1 + (-1)^n)y(n-4))] + 32(n + e^{-n} + 1)y^3(n-5) - 64e^{-n}\frac{y(n-1)}{1 + y^2(n-1)} = 0, \quad (3.10)$$

for $n \in [6, \infty)_{\mathbb{Z}}$. Here $r(n) = n$, $p(n) = 1+(-1)^n$, $q(n) = 32(n+e^{-n}+1)$, $h(n) = 64e^{-n}$, $G(u) = u^3$ and $H(u) = \frac{u}{1+u^2}$. We also have $\gamma(n) = n - 1$ and $\beta(n) = n - 5 < n - 4 = \alpha(n)$. Notice that condition (H_1) becomes

$$\begin{aligned}
&\sum_{n=n_0}^{\infty}\left(\frac{n+1}{n}\sum_{s=n}^{\infty}64(s+1)e^{-s}\right)\\
&\quad\le \sum_{n=n_0}^{\infty}\left(2\sum_{s=n}^{\infty}64(2s)e^{-s}\right)\\
&\quad\le 256\sum_{n=n_0}^{\infty}\left(\sum_{s=n}^{\infty}se^{-s}\right)\\
&\quad= 256\left(\sum_{s=n_0}^{\infty}se^{-s} + \sum_{s=n_0+1}^{\infty}se^{-s} + \dots\right)\\
&\quad= 256e^{-n_0}\left(n_0 + 2(n_0+1)e^{-1} + 3(n_0+2)e^{-2} + \dots\right)\\
&\quad= 256e^{-n_0}\left[n_0 e\sum_{n=1}^{\infty}ne^{-n} + \sum_{n=1}^{\infty}n(n+1)e^{-n}\right] < \infty.
\end{aligned}$$

It is now easy to see that equation (3.10) satisfies all the conditions of Theorem 3.1.4. Hence, any solution of equation (3.10) oscillates or converges to 0 as $t \to \infty$. In particular, $y(n) = (-1)^n$ is a solution of equation (3.10).

In our next theorem we are able to replace conditions (H_3) and (H_4) in Theorem 3.1.1 with conditions (H_7) and (H_8) below.

Theorem 3.1.7. *Assume that conditions (H_0)–(H_2), (H_5), (Λ) and*

(H_7) $G(u)G(v) \geqslant G(uv)$ *for* $u, v > 0$,

(H_8) $G(-u) = -G(u)$ *and* $H(-u) = -H(u)$ *for* $u \in \mathbb{R}$.

hold. If (i) $0 \leqslant p(t) \leqslant p_1 < 1$ *or (ii)* $1 < p_2 \leqslant p(t) \leqslant p_3 < \infty$ *holds, then every solution of* (E_3) *is either oscillatory or converges to zero as* $t \to \infty$*.*

Proof. Proceeding as in the proof of Theorem 3.1.1, in case $w(t) > 0$ we again have (3.8) for $t \geqslant t_3$. Since $w(\beta(t))$ is nondecreasing, there exists $k > 0$ and $t_4 > t_3$ such that $w(\beta(t)) > k$ for $t \geqslant t_4$, so $z(\beta(t)) \geq w(\beta(t)) \geq k$ for $t \geq t_4$. Consequently, (3.7) yields

$$\lambda G(k) \int_{t_4}^{\infty} Q(t)\Delta t < \infty$$

contradicting (H_5). The remaining part of the proof is similar to the proof of Theorem 3.1.1. □

We again have a corollary for the unbounded solutions.

Corollary 3.1.8. *Under the conditions of Theorem 3.1.7, every unbounded solution of* (E_3) *oscillates.*

Remark 3.1.9. Notice that in Theorem 3.1.1 and Corollary 3.1.2, G is sublinear, whereas in Theorem 3.1.4 and Corollary 3.1.5, G is superlinear. But in Theorem 3.1.7 and Corollary 3.1.8, G could be linear, sublinear, or superlinear.

Next, we consider the case where $p(t)$ is negative. Here p_4, p_5, and p_6 are negative real numbers.

Theorem 3.1.10. *Let* $-1 < p_4 \leqslant p(t) \leqslant 0$ *and conditions* (H_0), (H_1), (H_3), (H_4), *and*

$$(H_9) \quad \int_{t_0}^{\infty} q(t)\Delta t = \infty$$

hold. Then every solution of (E_3) *is either oscillatory or tends to zero as* $t \to \infty$*.*

Proof. Let $y(t)$ be a nonoscillatory solution of (E_3), say $y(t) > 0$ eventually. From (3.3)–(3.5), we obtain (3.6) for $t \geqslant t_1$. By Lemma 2.1.1, $w(t)$ is monotonic. If $w(t) > 0$ for $t \geqslant t_2 > t_1$, then either case (a) or case (b) of Lemma 2.1.1 holds. Consequently, $w(t) \geqslant R_T(t)(r(t)w^{\Delta^2}(t))^{\Delta}$ for $t \geqslant T > t_1$ by Lemma 2.1.2. Moreover, $w(t) \leqslant y(t)$

since $p(t) \leqslant 0$. Choose $t_3 \in [T, \infty)_{\mathbb{T}}$ such that $\beta(t) \geqslant T$ for all $t \in [t_3, \infty)_{\mathbb{T}}$. Then, $y(\beta(t)) \geqslant R_T(\beta(t))(r(\beta(t))w^{\Delta^2}(\beta(t)))^{\Delta}$ for $t \geqslant t_3$ and (3.6) becomes

$$\int_{t_3}^{\infty} q(t)G(R_T(\beta(t)))\Delta t < \infty,$$

which contradicts (H_9) since G, R_T, and β are increasing functions.

Hence, $w(t) < 0$ for $t \geqslant t_2$, and so one of the cases (b), (c), (d), or (e) of Lemma 2.1.1 holds. We claim that $y(t)$ is bounded. If this is not the case, then there is an increasing sequence $\{\tau_n\}_{n=1}^{\infty} \subset [t_2, \infty)_{\mathbb{T}}$ such that $\tau_n \to \infty$, $y(\tau_n) \to \infty$ as $n \to \infty$, and $y(\tau_n) = \max\{y(t) : t_2 \leqslant t \leqslant \tau_n\}$. We may choose τ_1 large enough so that $\alpha(\tau_1) \geqslant t_2$. Hence,

$$0 \geqslant w(\tau_n) \geqslant y(\tau_n) + p(\tau_n)y(\alpha(\tau_n)) - k(\tau_n) \geqslant (1 + p_4)y(\tau_n) - k(\tau_n).$$

Since $k(\tau_n)$ is bounded and $1 + p_4 > 0$, we have $w(\tau_n) > 0$ for large n, which is a contradiction. Thus, our claim holds.

The proof that cases (c), (d), and (e) cannot hold are similar to the corresponding cases in the proof of Theorem 3.1.1. If (b) holds, then as in the proof of Theorem 3.1.1, we obtain $\liminf_{t\to\infty} y(t) = 0$. Hence, $\lim_{t\to\infty} z(t) = 0$ by Lemma 1.5.1. Consequently,

$$\begin{aligned} 0 = \limsup_{t\to\infty} z(t) &\geqslant \limsup_{t\to\infty}(y(t) + p_4 y(\alpha(t))) \\ &\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_4 y(\alpha(t))) \\ &= \limsup_{t\to\infty} y(t) + p_4 \limsup_{t\to\infty} y(\alpha(t)) \\ &= (1 + p_4)\limsup_{t\to\infty} y(t). \end{aligned}$$

Since $(1 + p_4) > 0$, $\limsup_{t\to\infty} y(t) = 0$. Hence, $\lim_{t\to\infty} y(t) = 0$. This completes the proof of the theorem. □

Corollary 3.1.11. *Under the conditions of Theorem 3.1.10, every unbounded solution of* (E_3) *oscillates.*

Example 3.1.12. Let $\mathbb{T} = h\mathbb{Z}$, where h is a ratio of odd positive integers and consider the h-difference equation

$$\Delta_h^2[e^{-2t/3}\Delta_h^2(y(t)-e^{-2t+h}y(t-h))]+e^{5h/3}\Big(\frac{e^h+1}{h}\Big)^2\Big(\frac{e^{h/3}+1}{h}\Big)^2 y^{1/3}(t-5h)$$
$$-e^{2h/3}\Big(\frac{e^{-h}+1}{h}\Big)^2\Big(\frac{e^{-5h/3}+1}{h}\Big)^2 e^{-2t}(1+e^{(t-2h)/3})\frac{y^{1/3}(t-2h)}{1+\mid y^{1/3}(t-2h)\mid}=0, \quad (3.11)$$

for $t(\geq 5h)\in\mathbb{T}$. Here $r(t)=e^{-2t/3}$, $p(t)=-e^{-2t+h}$, $q(t)=e^{5h/3}\Big(\frac{e^h+1}{h}\Big)^2\Big(\frac{e^{h/3}+1}{h}\Big)^2$, $G(u)=u^{1/3}$, $h(t)=e^{2h/3}\Big(\frac{e^{-h}+1}{h}\Big)^2\Big(\frac{e^{-5h/3}+1}{h}\Big)^2 e^{-2t}(1+e^{(t-2h)/3})$ and $H(u)=\frac{u^{1/3}}{1+|u^{1/3}|}$. The equation (3.11) satisfies the hypotheses of Theorem 3.1.10 and Corollary 3.1.11. In particular, $y(t)=(-1)^t e^t$ is an unbounded oscillatory solution.

Theorem 3.1.13. *Assume that conditions* (H_0)*,* (H_1)*,* (H_3)*,* (H_4)*, and* (H_9) *hold. If* $-\infty<p_5\leqslant p(t)\leqslant p_6<-1$*, then every bounded solution of* (E_3) *either oscillates or tends to zero as* $t\to\infty$*.*

Proof. Let $y(t)$ be a bounded nonoscillatory solution of (E_3), say $y(t)$ is eventually positive. With (3.3), (3.4), and (3.5), we obtain (3.6) for $t\geqslant t_1$. Hence, from Lemma 2.1.1, $w(t)$ is monotonic. If $w(t)>0$ for $t\geqslant t_2>t_1$, then one of the cases (a) or (b) of Lemma 2.1.1 holds. Consequently, $w(t)>R_T(t)(r(t)w^{\Delta^2}(t))^{\Delta}$ for $t\geqslant T>t_2$ by Lemma 2.1.2. Moreover, $w(t)\leqslant y(t)$. Choose $t_3\in[T,\infty)_{\mathbb{T}}$ such that $\beta(t)\geqslant T$ for all $t\in[t_3,\infty)_{\mathbb{T}}$. Then, $y(\beta(t))>R_T(\beta(t))(r(\beta(t))w^{\Delta^2}(\beta(t)))^{\Delta}$ for $t\geqslant t_3$, and (3.6) becomes

$$\int_{t_3}^{\infty} q(t)G(R_T(\beta(t)))\Delta t<\infty,$$

contradicting (H_9), since G, R_T, and β are increasing. Hence, $w(t)<0$ for $t\geqslant t_2$, so one of the cases (b), (c), (d), or (e) of Lemma 2.1.1 holds.

In case (b), since $w(t)<0$, $w^{\Delta}(t)>0$, and $\lim_{t\to\infty}k(t)$ exists, we have $\lim_{t\to\infty}z(t)$ exists. Furthermore, $\lim_{t\to\infty}(r(t)w^{\Delta^2}(t))^{\Delta}$ exists. Integrating (3.6) from t_3 to ∞, we obtain

$$\int_{t_3}^{\infty} q(t)G(y(\beta(t)))\Delta t<\infty,$$

which implies that $\liminf_{t\to\infty} y(t) = 0 = \liminf_{t\to\infty} y(\beta(t))$ due to (H_9). Hence, $\lim_{t\to\infty} z(t) = 0$ by Lemma 1.5.1. Therefore,

$$\begin{aligned} 0 = \liminf_{t\to\infty} z(t) &= \liminf_{t\to\infty}(y(t) + p(t)y(\alpha(t))) \\ &\leqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p(t)y(\alpha(t))) \\ &= \limsup_{t\to\infty} y(t) + p_6 \limsup_{t\to\infty} y(\alpha(t)) \\ &= (1+p_6)\limsup_{t\to\infty} y(t). \end{aligned}$$

Since $(1+p_6) < 0$, we have $\limsup_{t\to\infty} y(t) \leq 0$, so $\lim_{t\to\infty} y(t) = 0$.

Cases (c) and (d) are not possible since $y(t)$ is bounded, and $\lim_{t\to\infty} k(t)$ exists and so $\lim_{t\to\infty} w(t)$ exists. On the other hand, integrating $w^{\Delta^2}(t) < 0$ twice from t_2 to t, yields $\lim_{t\to\infty} w(t) = -\infty$, which is a contradiction.

If case (e) holds, then by proceeding as case (e) in the proof of Theorem 3.1.1 when $w(t) < 0$, we obtain $\lim_{t\to\infty} w(t) = -\infty$. This implies $\lim_{t\to\infty} z(t) = -\infty$, contradicting the fact that $y(t)$ is bounded. This completes the proof of the theorem. □

3.2 Sufficient conditions for oscillation of (E_4) with $\int_{t_0}^{\infty} \frac{t}{r(t)} \Delta t = \infty$.

This section is devoted to study the oscillatory and asymptotic behavior of solutions of the forced equation (E_4) with suitable forcing functions. Our attention is restricted to forcing functions that eventually change signs. We will make use of the following hypotheses on $f(t)$.

(H_{10}) There exists $F \in C_{rd}^2([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that $-\infty < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \infty$, $rF^{\Delta^2} \in C_{rd}^2([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, and $(rF^{\Delta^2})^{\Delta^2} = f$.

(H_{11}) There exists $F \in C_{rd}^2([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that $\liminf_{t\to\infty} F(t) = -\infty$, $\limsup_{t\to\infty} F(t) = \infty$, $rF^{\Delta^2} \in C_{rd}^2([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, and $(rF^{\Delta^2})^{\Delta^2} = f$.

Theorem 3.2.1. *Let $0 \leqslant p(t) \leqslant p < \infty$. Assume that ($H_0$)–($H_2$), ($H_7$), ($H_8$), ($H_{11}$), and ($\Lambda$) hold. If*

(H$_{12}$) $\limsup_{t\to\infty} \int_{t^*}^{t} Q(s)G(F(\beta(s)))\Delta s = +\infty$, *and*
$\liminf_{t\to\infty} \int_{t^*}^{t} Q(s)G(F(\beta(s)))\Delta s = -\infty$,

then equation (E$_4$) *is oscillatory.*

Proof. Let $y(t)$ be a nonoscillatory solution of (E$_4$), say, $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, $y(\gamma(t))$, and $y(\beta(\alpha(t)))$ are all positive for $t \in [t_1,\infty)_{\mathbb{T}}$, for some $t_1 \in [t^*,\infty)_{\mathbb{T}}$, $t^* > t_0$. Defining $z(t)$, $k(t)$, and $w(t)$ as in (3.3), (3.4), and (3.5) respectively, equation (E$_4$) becomes

$$(r(t)w^{\Delta^2}(t))^{\Delta^2} + q(t)G(y(\beta(t))) = f(t). \tag{3.12}$$

Let

$$v(t) = w(t) - F(t) = z(t) - k(t) - F(t). \tag{3.13}$$

Then, for $t \geqslant t_1$, equation (3.12) becomes

$$(r(t)v^{\Delta^2}(t))^{\Delta^2} = -q(t)G(y(\beta(t))) \leq (\not\equiv)0. \tag{3.14}$$

Thus, $v(t)$ is monotonic on $[t_1,\infty)_{\mathbb{T}}$.

Suppose $v(t) > 0$ for $t \geqslant t_2$ so that cases (a) or (b) of Lemma 2.1.1 holds. Then $z(t) > k(t) + F(t) \geq F(t)$ for $t \geqslant t_2$. Choose $t_3 \in [t_2,\infty)_{\mathbb{T}}$ such that $\beta(t) \geqslant t_2$ for all $t \in [t_3,\infty)_{\mathbb{T}}$. Then, $z(\beta(t)) > F(\beta(t))$ for $t \geqslant t_3$. Applying (H$_2$), (H$_7$) and (Λ) yields

$$\begin{aligned}
0 &\geqslant (r(t)v^{\Delta^2}(t))^{\Delta^2} + q(t)G(y(\beta(t))) + G(p)[(r(\alpha(t))v^{\Delta^2}(\alpha(t)))^{\Delta^2} \\
&\quad + q(\alpha(t))G(y(\beta(\alpha(t))))] \\
&\geqslant (r(t)v^{\Delta^2}(t))^{\Delta^2} + G(p)[(r(\alpha(t))v^{\Delta^2}(\alpha(t)))^{\Delta^2}] + \lambda Q(t)G(y(\beta(t)) \\
&\quad + py(\alpha(\beta(t)))) \\
&\geqslant (r(t)v^{\Delta^2}(t))^{\Delta^2} + G(p)[(r(\alpha(t))v^{\Delta^2}(\alpha(t)))^{\Delta^2}] + \lambda Q(t)G(z(\beta(t))) \\
&\geqslant (r(t)v^{\Delta^2}(t))^{\Delta^2} + G(p)[(r(\alpha(t))v^{\Delta^2}(\alpha(t)))^{\Delta^2}] + \lambda Q(t)G(F(\beta(t)))
\end{aligned}$$

for $t \geqslant t_3$. Integrating the above inequality, we obtain

$$\limsup_{t\to\infty} \int_{t_3}^{t} Q(s)G(F(\beta(s)))\Delta s < \infty,$$

which contradicts (H_{12}).

Therefore, $v(t) < 0$ for $t \geqslant t_2$ and one of the cases (b), (c), (d), or (e) of Lemma 2.1.1 holds. In each of these cases $z(t) < k(t) + F(t)$. This implies $\liminf_{t\to\infty} z(t) < 0$, which is a contradiction. This completes the proof of the theorem. □

Example 3.2.2. Let $\mathbb{T} = \mathbb{Z}$ and consider the difference equation

$$\begin{aligned}\Delta^2[n\Delta^2(y(n) + (1 + (-1)^n)y(n-4))] \\ + \Big(32\big(n + 1 + e^{-n}\big) + 8\big((n+2)^2 + (n+1)^2\big)\Big)y^3(n-3) \\ - 64e^{-n}\frac{y(n-1)}{1 + y^2(n-1)} = -8\big((n+2)^2 + (n+1)^2\big)(-1)^n, \quad (3.15)\end{aligned}$$

for $n \in [6, \infty)_{\mathbb{Z}}$. Here $r(n) = n$, $p(n) = 1 + (-1)^n$, $\alpha(n) = n - 4$, $\beta(n) = n - 3$ $q(n) = 32\big(n + 1 + e^{-n}\big) + 8\big((n+2)^2 + (n+1)^2\big)$, $h(n) = 64e^{-n}$, $G(u) = u^3$ and $H(u) = \frac{u}{1+u^2}$, $\gamma(n) = n - 1$ and $f(n) = -8\big((n+2)^2 + (n+1)^2\big)(-1)^n$. We consider $F(n) = -(-1)^n n$ such that $\Delta^2(r(n)\Delta^2 F(n)) = f(n)$. It is now easy to see that equation (3.15) satisfies all the conditions of Theorem 3.2.1. Hence, any solution of equation (3.15) oscillates or converges to 0 as $n \to \infty$. In particular, $y(n) = (-1)^n$ is a solution of this equation.

Theorem 3.2.3. *Let $-1 < p(t) \leqslant 0$. Suppose that (H_0), (H_1), (H_8) and (H_{11}) hold. If*

(H_{13}) $$\limsup_{t\to\infty} \int_{t_3}^{t} q(s)G(F(\beta(s)))\Delta s = +\infty \quad \textit{and}$$
$$\liminf_{t\to\infty} \int_{t_3}^{t} q(s)G(F(\beta(s)))\Delta s = -\infty,$$

then any solution of (E_4) *is either oscillatory or satisfies* $\limsup_{t\to\infty} \mid y(t) \mid= \infty$.

Proof. Proceeding as in the proof of Theorem 3.2.1, we obtain (3.14) for $t \in [t_1, \infty)_{\mathbb{T}}$. Thus, $v(t)$ is monotonic, so $v(t) > 0$ or $v(t) < 0$ for large t. If $v(t) > 0$ for $t \geqslant t_2 > t_1$, then either case (a) or case (b) of Lemma 2.1.1 holds for $t \geqslant t_2$. Since $v(t) > 0$ then, $z(t) > z(t) - k(t) > F(t)$ implies that $z(t) > F(t)$, so $y(t) > z(t) > F(t)$ for $t \geqslant t_2$. Choose $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $\beta(t) \geqslant t_2$ for all $t \in [t_3, \infty)_{\mathbb{T}}$. Hence, for $t \geqslant t_3$, $y(\beta(t)) > z(\beta(t)) > F(\beta(t))$. From (3.14), we have

$$q(t)G(F(\beta(t))) \leqslant q(t)G(y(\beta(t))) = -(r(t)v^{\Delta^2}(t))^{\Delta^2}$$

for $t \geqslant t_3$. An integration yields a contradiction to (H_{13}).

Now assume $v(t) < 0$ for $t \geqslant t_2$. Thus, $z(t) - k(t) < F(t)$, and condition (H_{11}) implies that $\liminf_{t\to\infty} z(t) = -\infty$, so $\limsup_{t\to\infty} y(t) = \infty$. This completes the proof of the theorem. □

Remark 3.2.4. We can drop the condition (H_{13}), from the hypothesis of Theorem 3.2.3 and obtain that bounded solutions oscillates. If $v(t) > 0$, then $z(t) > k(t) + F(t) > F(t)$, and condition (H_{11}) contradicts the boudedness of y. In case if $v(t) < 0$, then $z(t) < k(t) + F(t)$ and condition (H_{11}) contradicts the boudedness of $z(t)$, because $y(t)$ is bounded.

Theorem 3.2.5. *Assume that* (H_0)*–*(H_2)*,* (H_7)*,* (H_8)*,* (H_{10})*,* (H_{12})*, and* (Λ) *hold. If* $0 \leqslant p(t) \leqslant p < \infty$*, then every unbounded solution of* (E_4) *oscillates.*

Proof. Let $y(t)$ be an unbounded nonoscillatory solution of (E_4), say $y(t)$ is an eventually positive solution. Using (3.3), (3.4), (3.5), and (3.13), we obtain inequality (3.14). Thus, $v(t)$ is monotonic, so first assume $v(t) > 0$ for all $t \geqslant t_2 > t_1$. Proceeding as in the proof of Theorem 3.2.1, we again obtain a contradiction.

Next, let $v(t) < 0$ for $t \geqslant t_2 > t_1$. From Lemma 2.1.1, it follows that one of the cases (b), (c), (d), or (e) holds. In case (b), $\lim_{t\to\infty} v(t)$ exists and hence $z(t) = v(t) + k(t) + F(t)$ implies $y(t) \leqslant v(t) + k(t) + F(t)$. That is, $y(t)$ is bounded, which is a contradiction.

For each of the cases (c), (d) and (e), $v(t)$ is nonincreasing on $[t_2, \infty)_{\mathbb{T}}$, so let $\lim_{t\to\infty} v(t) = l$ for some $l \in [-\infty, 0)$. If $l = -\infty$, then $y(t) \leqslant v(t) + k(t) + F(t)$, which in view of (H_{10}), implies that $y(t)$ eventually becomes negative. If $-\infty < l < 0$, then in cases (c) and (d), $v^{\Delta}(t)$ is decreasing. Successive integrations of $v^{\Delta^2}(t)$ again show that $\lim_{t\to\infty} v(t) = -\infty$. If case (e) holds, $y(t) \leqslant v(t) + k(t) + F(t) \leqslant k(t) + F(t)$, which contradicts the unboundedness of $y(t)$. This completes the proof of the theorem. □

Example 3.2.6. Let $\mathbb{T} = q^{\mathbb{N}_0}$, with $q > 1$ is a fixed real number, and consider the q-difference equation

$$\Delta_q^2\left(\frac{(q-1)^4}{2(q^{10}+1)(q^9+1)(q^7+1)(q^6+1)t}\Delta_q^2\left(y(t)+q^{20}y(t/q^2)\right)\right)$$
$$+q^2(t^3+1/t^{32}+1/t^7)y^{1/5}(t/q)-\frac{q^{30}}{t^{40}}(1+t^{20}/q^{60})\frac{y(t/q^3)}{1+y^2(t/q^3)}$$
$$=-1/t^5(-1)^{\log_q^t}, \quad (3.16)$$

for $t \in [q^3,\infty)_{\mathbb{T}}$. Here $r(t)=\frac{(q-1)^4}{2(q^{10}+1)(q^9+1)(q^7+1)(q^6+1)t}$, $p(t)=q^{20}$, $G(u)=u^{1/5}$, $q(t)=q^2(t^3+1/t^{32}+1/t^7)$, $h(t)=\frac{q^{30}}{t^{40}}(1+t^{20}/q^{60})$, $H(u)=\frac{u}{1+u^2}$, and $f(t)=-1/t^5(-1)^{\log_q^t}$. We consider $F(t)=-\frac{(q^{10}+1)(q^9+1)(q^7+1)(q^6+1)}{(1/q+1)(1/q^3+1)(1/q^4+1)}(-1)^{\log_q^t}$ such that $\Delta_q^2(r(t)\Delta_q^2F(t))=-1/t^5(-1)^{\log_q^t}$. It is easy to see that equation (3.16) satisfies all the conditions of Theorem 3.2.5. Hence every solution of (3.16) is oscillatory. In particular, $y(t)=t^{10}(-1)^{\log_q^t}$ is a solution of equation (3.16).

In the following we would like to give proof for the **existence of bounded positive solution** of (E_4) for the range $0 \le p(t) < 1$.

Theorem 3.2.7. *Assume that* $0 \leqslant p(t) \leqslant p_1 < 1$, *and* ($H_1$) *and* ($H_{10}$) *hold with*

$$\frac{-1}{8}(1-p_1) < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \frac{1}{4}(1-p_1).$$

In addition, assume that G and H are Lipschitzian on $\mathbb{R}$ with Lipschitz constants G_1 and H_1, respectively. If

$$\int_{t_0}^{\infty}\frac{\sigma(t)}{r(t)}\int_t^{\infty}\sigma(s)q(s)\Delta s\Delta t<\infty,$$

then (E_4) *admits a positive bounded solution.*

Proof. Choose $t_1 > t_0$ large enough so that

$$\int_{t_1}^{\infty}\frac{\sigma(t)}{r(t)}\int_t^{\infty}\sigma(s)h(s)\Delta s\Delta t<\frac{1-p_1}{4H(1)},$$

and

$$\int_{t_1}^{\infty}\frac{\sigma(t)}{r(t)}\int_t^{\infty}\sigma(s)q(s)\Delta s\Delta t<\frac{1-p_1}{4G(1)}.$$

Let $X = BC_{rd}([t_1,\infty)_{\mathbb{T}},\mathbb{R})$ be the Banach Space of all bounded rd-continuous functions on $[t_1,\infty)_{\mathbb{T}}$ with the norm

$$\|x\| = \sup\{|\ x(t)\ | : t \in [t_1,\infty)_{\mathbb{T}}\},$$

and let

$$S = \{x \in X : \frac{1}{8}(1-p_1) \leqslant x(t) \leqslant 1,\ t \in [t_1,\infty)_{\mathbb{T}}\}.$$

Then, S is a closed, bounded, and convex subset of X. Take $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that $\alpha(t)$, $\beta(t)$, $\gamma(t) \geqslant t_1$ for all $t \in [t_2,\infty)_{\mathbb{T}}$. Define the mappings A, $B : S \to S$ by

$$Ax(t) = \begin{cases} Ax(t_2) & \text{for } t \in [t_1,t_2)_{\mathbb{T}} \\ -p(t)x(\alpha(t)) + \frac{1+p_1}{2} & \text{for } t \in [t_2,\infty)_{\mathbb{T}}. \end{cases}$$

and

$$Bx(t) = \begin{cases} Bx(t_2) & \text{for } t \in [t_1,t_2)_{\mathbb{T}} \\ -\int_t^\infty \frac{\sigma(s)-t}{r(s)}\Big(\int_s^\infty (\sigma(u)-s)q(u)G(x(\beta(u)))\Delta u\Big)\Delta s & \\ \qquad +F(t)+k(t) & \text{for } t \in [t_2,\infty)_{\mathbb{T}}. \end{cases}$$

For $x \in S$, we have

$$\begin{aligned} k(t) &= \int_t^\infty \frac{\sigma(s)-t}{r(s)} \int_s^\infty (\sigma(u)-s)h(u)H(x(\gamma(u)))\Delta u \Delta s \\ &\leqslant H(1)\int_t^\infty \frac{\sigma(s)}{r(s)} \int_s^\infty \sigma(u)h(u)\Delta u \Delta s \\ &< \frac{1}{4}(1-p_1). \end{aligned}$$

and

$$\int_t^\infty \frac{\sigma(s)-t}{r(s)} \int_s^\infty (\sigma(u)-s)q(u)G(x(\beta(u)))\Delta u \Delta s < \frac{1-p_1}{4}.$$

For all x, $y \in S$ and all $t \in [t_2,\infty)_{\mathbb{T}}$, we have

$$Ax(t)+By(t) \leqslant \frac{1+p_1}{2} + \frac{1}{4}(1-p_1) + \frac{1}{4}(1-p_1) = 1,$$

and

$$Ax(t)+By(t) \geqslant -p_1 + \frac{1+p_1}{2} - \frac{1}{8}(1-p_1) - \frac{1}{4}(1-p_1) = \frac{1-p_1}{8}.$$

Thus, $Ax + By \in S$.

To show that A is a contraction mapping on S, first notice that

$$\begin{aligned}\|Ax - Ay\| &= \left\| -p(t)x(\alpha(t)) + \frac{1+p_1}{2} + p(t)y(\beta(t)) - \frac{1+p_1}{2} \right\| \\ &= \|-p(t)(x(\alpha(t)) - y(\alpha(t)))\| \\ &\leqslant p_1\|x(\alpha(t)) - y(\alpha(t))\| \\ &= p_1\|x(t) - y(t)\|.\end{aligned}$$

Since $p_1 < 1$, A is a contraction mapping.

To show that B is completely continuous on S, we need to show that B is continuous and maps bounded sets into relatively compact sets. In order to show that B is continuous, let x, $x_k = x_k(t) \in S$ be such that $\|x_k - x\| = \sup_{t \geqslant t_1}\{| x_k(t) - x(t) |\} \to 0$. Since S is closed then $x(t) \in S$. For $t \geqslant t_1$, we have

$$\begin{aligned}\|(Bx_k) - (Bx)\| = \Big\| \, & F(t) + \int_t^\infty \frac{\sigma(s) - t}{r(s)} \int_s^\infty (\sigma(u) - s)h(u)H(x_k(\gamma(u)))\Delta u \Delta s \\ & - \int_t^\infty \frac{\sigma(s) - t}{r(s)} \int_s^\infty (\sigma(u) - s)q(u)G(x_k(\beta(u)))\Delta u \Delta s \\ & - F(t) - \int_t^\infty \frac{\sigma(s) - t}{r(s)} \int_s^\infty (\sigma(u) - s)h(u)H(x(\gamma(u)))\Delta u \Delta s \\ & + \int_t^\infty \frac{\sigma(s) - t}{r(s)} \int_s^\infty (\sigma(u) - s)q(u)G(x(\beta(u)))\Delta u \Delta s \Big\| \\ = \Big\| & \int_t^\infty \frac{\sigma(s) - t}{r(s)} \int_s^\infty (\sigma(u) - s)h(u)(H(x_k(\gamma(u))) - H(x(\gamma(u))))\Delta u \Delta s \\ & + \int_t^\infty \frac{\sigma(s) - t}{r(s)} \int_s^\infty (\sigma(u) - s)q(u)(G(x(\beta(u))) - G(x_k(\beta(u))))\Delta u \Delta s \Big\| \\ \leqslant \; & H_1\|x_k - x\| \int_t^\infty \frac{\sigma(s)}{r(s)} \int_s^\infty \sigma(u)h(u)\Delta u \Delta s \\ & + G_1\|x_k - x\| \int_t^\infty \frac{\sigma(s)}{r(s)} \int_s^\infty \sigma(u)q(u)\Delta u \Delta s \\ \leqslant \; & \frac{1}{2}(1 - p_1)\|x - x_k\|.\end{aligned}$$

Since for all $t \geqslant t_1$, $\{x_k\}$, converges uniformly to x as $k \to \infty$, we have $\lim_{k\to\infty}\|(Bx_k) - (Bx)\| = 0$. Thus, B is continuous.

To show that BS is relatively compact, it suffices to show that the family of functions $\{Bx : x \in S\}$ is uniformly bounded and equicontinuous on $[t_1, \infty)_{\mathbb{T}}$. The uniform boundedness is clear. To show that BS is equicontinuous, let $x \in S$ and t'', $t' \geqslant t_1$, say $t^{''} \geq t^{'}$. Then

$$
\begin{aligned}
&|(Bx)(t^{''}) - (Bx)(t^{'})| \\
&= \Big| F(t^{''}) + k(t^{''}) - \int_{t^{''}}^{\infty} \frac{\sigma(s) - t^{''}}{r(s)} \int_{s}^{\infty} (\sigma(u) - s) q(u) G(x(\beta(u))) \Delta u \Delta s \\
&\quad - F(t^{'}) - k(t^{'}) + \int_{t^{'}}^{\infty} \frac{\sigma(s) - t^{'}}{r(s)} \int_{s}^{\infty} (\sigma(u) - s) q(u) G(x(\beta(u))) \Delta u \Delta s \Big| \\
&= |F(t^{''}) - F(t^{'})| + |k(t^{''}) - k(t^{'})| + G(1) \int_{t^{'}}^{t^{''}} \int_{\theta}^{\infty} \frac{1}{r(s)} \int_{s}^{\infty} \sigma(u) q(u) \Delta u \Delta s \\
&\leq |F(t^{''}) - F(t^{'})| + |k(t^{''}) - k(t^{'})| + G(1)|t^{'} - t^{''}| \int_{t^{'}}^{\infty} \frac{1}{r(s)} \int_{s}^{\infty} \sigma(u) q(u) \Delta u \Delta s.
\end{aligned}
$$

So $|\ (Bx)(t^{''}) - (Bx)(t^{'})\ | \to 0$ as $t^{'} \to t^{''}$. Therefore, $\{Bx : x \in S\}$ is uniformly bounded and equicontinuous on $[t_1, \infty)_{\mathbb{T}}$. Hence, BS is relatively compact. By **Krasnosel'skii's fixed point theorem**, there exists $x \in S$ such that $Ax + Bx = x$. Thus, the theorem is proved. □

Remark 3.2.8. Results similar to Theorem 3.2.7 can be proved for other ranges of $p(t)$.

3.3 Sufficient conditions for oscillation of (E_3) with $\int_{t_0}^{\infty} \frac{\sigma(t)}{r(t)} \Delta t < \infty$.

In this section, sufficient conditions are obtained for the oscillatory and asymptotic behavior of (E_3) under the assumption

$$
(H_{14}) \quad \int_{t_0}^{\infty} \frac{\sigma(t)}{r(t)} \Delta t < \infty.
$$

The results in this section will make use of the following conditions on the functions in equation (E_3).

(H_{15}) for some $l > 1$, $\int_{t^*}^{\infty} d(t) G(R_2(\beta(t))) Q(t) \Delta t = \infty$,
where $d(t) = \min\{R_1^l(\sigma(t)), R_1^l(\sigma(\alpha(t)))\}$, $R_1(t) = \int_t^{\infty} \frac{s-t}{r(s)} \Delta s$, $R_2(t) = \int_t^{\infty} \frac{\sigma(s)-t}{r(s)} \Delta s$,

(H_{16}) for some $l > 1$, $\int_{t_1}^{\infty} R_1^l(\sigma(t)) G(R_2(\beta(t))) q(t) \Delta t = \infty$.

Remark 3.3.1. Notice that (H_{15}) implies (H'_{15}), where (H'_{15}) $\int_{t^*}^{\infty} G(R_2(\beta(t)))Q(t)\Delta t = \infty$; and ($H_{16}$) implies ($H'_{16}$), where ($H'_{16}$) $\int_{t_0}^{\infty} G(R_2(\beta(t)))q(t)\Delta t = \infty$, which in turn implies $\int_{t_0}^{\infty} q(t)\Delta t = \infty$.

Theorem 3.3.2. *Assume that conditions (H_1)–(H_3), (H_{14}), (H_{15}) and (Λ) hold, and p is positive real numbers. If $0 \leq p(t) \leq p < \infty$ holds, then any solution of (E_3) is either oscillatory or converges to zero as $t \to \infty$.*

Proof. Proceeding as in the proof of Theorem 3.1.1, by defining $z(t)$, $k(t)$, and $w(t)$ as in (3.3), (3.4), and (3.5) respectively, we obtain (3.6), that is

$$(r(t)w^{\Delta^2}(t))^{\Delta^2} = -q(t)G(y(\beta(t))) \leq (\not\equiv)0,$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. Thus, $w(t)$ is monotonic for $t \in [t_1, \infty)_{\mathbb{T}}$, we have to consider the two cases $w(t) > 0$ or $w(t) < 0$.

Suppose that $w(t) > 0$ for $t \geq t_2$, for some $t_2 > t_1$, then there exists $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $w(\alpha(t)) > 0$, $w(\beta(t)) > 0$ for $t \in [t_3, \infty)_{\mathbb{T}}$. By Lemma 2.3.1, one of the cases (a), (b), (c) or (d) holds. If (a), (b) or (d) holds, then applying (H_2), (H_3) and (Λ) to equation (E_3) gives

$$\begin{aligned} 0 &= (r(t)w^{\Delta^2}(t))^{\Delta^2} + q(t)G(y(\beta(t))) + G(p)(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2} \\ &\quad + G(p)q(\alpha(t))G(y(\beta(\alpha(t)))) \\ &\geqslant (r(t)w^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(y(\beta(t)) + py(\alpha(\beta(t)))) \\ &\geqslant (r(t)w^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(z(\beta(t))) \end{aligned} \tag{3.17}$$

for $t \geqslant t_2 > t_1$, where we have used the fact that $z(t) \leqslant y(t) + py(\alpha(t))$. From (3.4), it follows that $k(t) > 0$ and $k^{\Delta}(t) < 0$. Hence, $w(\beta(t)) > 0$ for $t \geq t_3$ implies that $w(\beta(t)) < z(\beta(t))$ for $t \geq t_3$. From (3.17), we have

$$(r(t)w^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(w(\beta(t))) \leqslant 0, \tag{3.18}$$

for $t \geqslant t_3 > t_2$. Applying Lemma 2.3.4 and (H_3) to (3.18) gives

$$(r(t)w^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))w^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda G(k_1)Q(t)G(R_2(\beta(t))) \leqslant 0,$$

for $t \geqslant t_4 > t_3$. Now $\lim_{t\to\infty}(r(t)w^{\Delta^2}(t))^\Delta$ exists, so integrating the above inequality implies

$$\lambda G(k_1)\int_{t_4}^{\infty} G(R_2(\beta(t)))Q(t)\Delta t < \infty,$$

which contradicts (H'_{15}).

Next, suppose case (c) holds. Using (I_4) of Lemma 2.3.2, and Lemma 2.3.4, we have

$$k(-r(t)w^{\Delta^2}(t))^\Delta tR_1(t) \leqslant w(t) \leqslant k_2 t \tag{3.19}$$

for $t \geqslant t_3 > t_2$. Choose $f(x) = x^{1-l}$ with $l > 1$, which is continuous on $(0,\infty)$, and take $g(t) = (-r(t)w^{\Delta^2}(t))^\Delta$. Applying the chain rule (Lemma 1.3.11), in time scale for $t \geq t_3$, there exists "c" in the real interval $[t,\sigma(t)]$ and $g(c) = L$. Since g is increasing, then for $t \leq c \leq \sigma(t)$ imply $g(t) \leq L \leq g(\sigma(t))$. With this and (3.6), we have

$$\begin{aligned} -[((-r(t)w^{\Delta^2}(t))^\Delta)^{1-l}]^\Delta &= (l-1)L^{-l}(-r(t)w^{\Delta^2}(t))^{\Delta^2} \\ &= (l-1)L^{-l}q(t)G(y(\beta(t))) \\ &\geqslant (l-1)g^{-l}(\sigma(t))q(t)G(y(\beta(t))). \end{aligned} \tag{3.20}$$

From (3.19), $kg(t)R_1(t) \leqslant k_2$ for $t \geqslant t_3$, so $kg(\sigma(t))R_1(\sigma(t)) \leqslant k_2$ for $t \geqslant t_3$. Thus, (3.20) becomes

$$-[((-r(t)w^{\Delta^2}(t))^\Delta)^{1-l}]^\Delta \geqslant (l-1)L_1^l R_1^l(\sigma(t))q(t)G(y(\beta(t))), \tag{3.21}$$

where $L_1 = k/k_2$. Choose $t_4 \in [t_3,\infty)_{\mathbb{T}}$ such that $\alpha(t) \geq t_3$ for all $t \in [t_4,\infty)_{\mathbb{T}}$. Using ($H_2$), ($H_3$), ($\Lambda$) and Lemma 2.3.4, we have

$$\begin{aligned} - \quad & [((-r(t)w^{\Delta^2}(t))^\Delta)^{1-l}]^\Delta - G(p)[((-r(\alpha(t))w^{\Delta^2}(\alpha(t)))^\Delta)^{1-l}]^\Delta \\ \geq \quad & (l-1)L_1^l R_1^l(\sigma(t))q(t)G(y(\beta(t))) + G(p)(l-1)L_1^l R_1^l(\sigma(\alpha(t)))q(\alpha(t))G(y(\beta(\alpha(t)))) \\ \geq \quad & \lambda(l-1)L_1^l d(t)Q(t)G(z(\beta(t))) \\ \geq \quad & \lambda(l-1)L_1^l d(t)Q(t)G(w(\beta(t))) \\ \geq \quad & \lambda(l-1)L_1^l d(t)Q(t)G(k_1R_2(\beta(t))) \end{aligned}$$

for $t \geqslant t_4$. Therefore,

$$\int_{t_4}^{\infty} d(t)G(R_2(\beta(t)))Q(t)\Delta t < \infty,$$

which contradicts (H_{15}).

Now we suppose that $w(t) < 0$ for $t \geq t_2$. Then $z(t) - k(t) < 0$ implies $y(t) \leq z(t) = y(t) + p(t)y(\alpha(t)) < k(t)$. Thus, $y(t)$ is bounded. By Lemma 2.3.1, it follows that one of the cases (b), (c), (d), (e), or (f) holds for $t \geq t_2$. In cases (e) and (f), $\lim_{t\to\infty} w(t) = -\infty$ which contradicts the boundedness of $y(t)$. In the cases (b) and (c), $w(t)$ is increasing and $w(t) < 0$, so $\lim_{t\to\infty} w(t)$ exists. Consequently,

$$\begin{aligned} 0 \geqslant \lim_{t\to\infty} w(t) &= \limsup_{t\to\infty}[z(t) - k(t)] \\ &\geq \limsup_{t\to\infty}[y(t) - k(t)] \\ &\geq \limsup_{t\to\infty} y(t) - \lim_{t\to\infty} k(t), \end{aligned}$$

implying that $\lim_{t\to\infty} y(t) = 0$ since $\lim_{t\to\infty} k(t) = 0$. Finally, let case (d) of Lemma 2.3.1 hold. Then $w(t) < 0$ is decreasing so $\lim_{t\to\infty} w(t) = L$ with $-\infty \leqslant L < 0$. Since $k(t) \to 0$, this implies $z(t)$ eventually becomes negative, which is a contradiction. This completes the proof of the theorem. □

The following corollary is immediate.

Corollary 3.3.3. *Under the conditions of Theorem 3.3.2, every unbounded solution of* (E_3) *oscillates.*

Example 3.3.4. Let $\mathbb{T} = \mathbb{Z}$ and consider the difference equation

$$\begin{aligned} \Delta^2[e^n\Delta^2(y(n) + e^{-5n}y(n-2))] + e^{1/3}(e+1)^2(e^2+1)^2e^{5n/3}y^{1/3}(n-1) \\ - e^{-2}(e^{-4}+1)^2(e^{-3}+1)^2(1+e^{2n})e^{-4n}\frac{y(n)}{1+y^2(n)} = 0, \quad (3.22) \end{aligned}$$

for $n(\geq 2) \in \mathbb{Z}$. Here $r(n) = e^n$, $p(n) = e^{-5n}$, $q(n) = e^{1/3}(e+1)^2(e^2+1)^2e^{5n/3}$, $h(n) = e^{-2}(e^{-4}+1)^2(e^{-3}+1)^2(1+e^{2n})e^{-4n}$, $G(u) = u^{1/3}$ and $H(u) = \frac{u}{1+u^2}$. Equation (3.22) satisfies all the conditions of Theorem 3.3.2 and Corollary 3.3.3. In particular, $y(n) = (-1)^n e^n$ is an unbounded oscillatory solution.

Our next theorem gives sufficient conditions for all unbounded solutions to oscillate.

Theorem 3.3.5. *Let $0 \leqslant p(t) \leqslant p < 1$. If ($H_1$), ($H_3$), ($H_{14}$) and ($H_{16}$) hold, then every unbounded solution of* (E_3) *oscillates.*

Proof. Let $y(t)$ be an unbounded nonoscillatory solution of (E_3), say $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, $y(\gamma(t))$ and $y(\alpha(\alpha(t)))$ are all positive for $t \in [t_1, \infty)_{\mathbb{T}}$, for some $t_1 > t_0$. We set $z(t)$, $k(t)$ and $w(t)$ as in (3.3)–(3.5), to obtain (3.6) for $t \geq t_1$. Consequently, $w(t)$, $w^{\Delta}(t)$, $(r(t)w^{\Delta^2}(t))$, and $(r(t)w^{\Delta^2}(t))^{\Delta}$ are of constant signs on $[t_2, \infty)_{\mathbb{T}}$, $t_2 \geqslant t_1$.

Assume that $w(t) > 0$ for $t \geq t_2$. By Lemma 2.3.1, one of the cases (a), (b), (c) or (d) holds. First suppose that (a) or (b) holds. Then $0 < w^{\Delta}(t) = z^{\Delta}(t) - k^{\Delta}(t)$. If $z^{\Delta}(t)$ oscillates, then $w^{\Delta}(t)$ cannot be monotonic. Thus, $z^{\Delta}(t)$ is monotonic. Then $z^{\Delta}(t) > 0$ or $z^{\Delta}(t) < 0$ for large t. If $z^{\Delta}(t) < 0$ then $z(t)$ is bounded, which is a contradiction to $z(t)$ is unbounded, because $y(t)$ is unbounded. So $z^{\Delta}(t) > 0$ for large t, say $t \geq t_3 > t_2$, which gives that $z(t)$ is increasing for $t \geq t_3$. Hence, in these two cases,

$$
\begin{aligned}
(1-p)z(t) \leq (1-p(t))z(t) &< z(t) - p(t)z(\alpha(t)) \\
&= y(t) - p(t)p(\alpha(t))y(\alpha(\alpha(t))) < y(t),
\end{aligned}
$$

that is,

$$
y(t) > (1-p)z(t) > (1-p)w(t) \tag{3.23}
$$

for $t \geqslant t_3 > t_2$. Thus, (3.6) implies

$$
G((1-p)w(\beta(t)))q(t) \leqslant -(r(t)w^{\Delta^2}(t))^{\Delta^2},
$$

and applying Lemma 2.3.4 and (H_3) gives

$$
G(k_1(1-p))G(R_2(\beta(t)))q(t) \leqslant -(r(t)w^{\Delta^2}(t))^{\Delta^2}. \tag{3.24}
$$

Integrating (3.24) from t_3 to ∞, we have

$$
\int_{t_3}^{\infty} G(R_2(\beta(t)))q(t)\Delta t < \infty,
$$

which contradicts (H'_{16}).

If case (d) holds, then $w^{\Delta}(t) < 0$. This implies that w and z are bounded which cannot happen if y is unbounded.

If case (c) of Lemma 2.3.1 holds, we proceed as in the proof of Theorem 3.3.2 to obtain (3.21). From (3.21), (3.23), we have

$$-[((-r(t)w^{\Delta^2}(t))^{\Delta})^{1-l}]^{\Delta} \geqslant (l-1)L_1^l G((1-p)w(\beta(t)))q(t)R_1^l(\sigma(t))$$

for $t \geqslant t_3$. Thus, the last inequality reduces to

$$-[((-r(t)w^{\Delta^2}(t))^{\Delta})^{1-l}]^{\Delta} \geqslant (l-1)L_1^l G((1-p)k_1)q(t)R_1^l(\sigma(t))G(R_2(\beta(t)))$$

for $t > t_4 \geq t_2$ due to Lemma 2.3.4. Integrating the last inequality from t_4 to ∞, we obtain

$$\int_{t_4}^{\infty} q(t)R_1^l(\sigma(t))G(R_2(\beta(t)))\Delta t < \infty,$$

contradicting (H_{16}).

Finally, we see that since $y(t)$ is unbounded, the case $w(t) < 0$ does not arise because $w(t) = z(t) - k(t) < 0$ implies $0 < z(t) < k(t)$, so again $z(t)$ is bounded. This completes the proof of the theorem. □

Our next two results are for the case where $p(t)$ is negative.

Theorem 3.3.6. *Let $-1 < p_4 \leqslant p(t) \leqslant 0$ and conditions (H_1), (H_3), (H_{14}) and (H_{16}) hold. Then any solution of* (E_3) *is either oscillatory or converges to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a nonoscillatory solution of (E_3), say $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, $y(\gamma(t))$ are all positive for all $t \in [t_1, \infty)_{\mathbb{T}}$, $t_1 \geqslant t_0$. Setting $z(t)$, $k(t)$, and $w(t)$ as in (3.3)–(3.5), we obtain (3.6) for $t \geqslant t_1$. Hence, $w(t)$ is monotonic for large $t \in [t_1, \infty)_{\mathbb{T}}$. Let $w(t) > 0$ for $t \geqslant t_2$, for some $t_2 \geqslant t_1$ and assume that one of the cases (a), (b), or (d) of Lemma 2.3.1 holds for $t \geqslant t_2$. By Lemma 2.3.4, we have $y(\beta(t)) \geqslant w(\beta(t)) \geqslant k_1 R_2(\beta(t))$ for $t \geqslant t_3 > t_2$, with which (3.6) yields

$$\int_{t_3}^{\infty} G(R_2(\beta(t)))q(t)\Delta t < \infty,$$

which is a contradiction to (H'_{16}).

Next, we consider the case (c). Proceeding as in the proof of Theorem 3.3.2, we obtain (3.21). Furthermore, $y(t) \geqslant w(t) \geqslant k_1 R_2(\beta(t))$ for $t \geqslant t_3$ by Lemma 2.3.4.

Consequently, for $t \geqslant t_4 > t_3$,

$$-[((-r(t)w^{\Delta^2}(t))^{\Delta})^{1-l}]^{\Delta} \geqslant (l-1)L_1^l G(k_1)q(t)R_1^l(\sigma(t))G(R_2(\beta(t))). \tag{3.25}$$

An integration of (3.25) gives

$$\int_{t_4}^{\infty} G(R_2(\beta(t)))q(t)R_1^l(\sigma(t))\Delta t < \infty,$$

contradicting (H_{16}).

Now suppose $w(t) < 0$ for $t \geqslant t_2$. We claim that $y(t)$ is bounded. If not, then there is an increasing sequence $\{\tau_n\}_{n=1}^{\infty} \subset [t_2, \infty)_{\mathbb{T}}$ such that $\tau_n \to \infty$, $y(\tau_n) \to \infty$ as $n \to \infty$, and $y(\tau_n) = \max\{y(t) : t_2 \leqslant t \leqslant \tau_n\}$. We choose τ_1 large enough so that $\alpha(\tau_1) \geqslant t_2$. Hence,

$$0 \geqslant w(\tau_n) \geqslant y(\tau_n) + p(\tau_n)y(\alpha(\tau_n)) - k(\tau_n) \geqslant (1+p_4)y(\tau_n) - k(\tau_n).$$

Since $k(\tau_n)$ is bounded and $1 + p_4 > 0$, we have $w(\tau_n) > 0$ for large n, which is a contradiction, so our claim is true. Hence, $z(t)$ and $w(t)$ are bounded. Clearly, cases (e) and (f) of Lemma 2.3.1 cannot arise.

In the cases (b) and (c), $-\infty < \lim_{t\to\infty} w(t) \leqslant 0$. Using the fact that $\lim_{t\to\infty} k(t) = 0$, we have $\lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t)$. Hence,

$$\begin{aligned} 0 \geq \lim_{t\to\infty} w(t) &= \lim_{t\to\infty} z(t) \\ &= \limsup_{t\to\infty}[y(t) + p(t)y(\alpha(t))] \\ &\geq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_4 y(\alpha(t))) \\ &= \limsup_{t\to\infty} y(t) + p_4 \limsup_{t\to\infty} y(\alpha(t)) \\ &= (1+p_4)\limsup_{t\to\infty} y(t), \end{aligned}$$

which implies that $\limsup_{t\to\infty} y(t) = 0$, that is, $\lim_{t\to\infty} y(t) = 0$.

If case (d) holds, then $\lim_{t\to\infty}(r(t)w^{\Delta^2}(t))^{\Delta}$ exists and so (3.6) gives

$$\int_{t_2}^{\infty} q(t)G(y(\beta(t)))\Delta t < \infty. \tag{3.26}$$

If $\liminf_{t\to\infty} y(t) > 0$, then it follows from (3.26) that

$$\int_{t_2}^{\infty} q(t)\Delta t < \infty,$$

which contradicts Remark 3.3.1. Hence, $\liminf_{t\to\infty} y(t) = 0$. Using Lemma 1.5.1, we conclude that $\lim_{t\to\infty} w(t) = 0 = \lim_{t\to\infty} z(t)$. Proceeding as above, we may show that $\limsup_{t\to\infty} y(t) = 0$ and hence $\lim_{t\to\infty} y(t) = 0$. This completes the proof of the theorem. □

Example 3.3.7. Let $\mathbb{T} = \mathbb{R}$, and consider the differential equation

$$\left(e^{\frac{t}{2}}\left(y(t) - \frac{1}{2}e^{-t}y(t/2)\right)''\right)'' + e^{\frac{9t}{2}}y^3(t) - \frac{11}{2}e^{-t}(1+e^{-t})\frac{y(t/4)}{1+y^2(t/4)} = 0, \quad (3.27)$$

for $t \in [t_0, \infty)_{\mathbb{T}}$, $t_0 > 0$. Here $r(t) = e^{t/2}$, $p(t) = -1/2e^{-t}$, $\alpha(t) = t/2$, $\beta(t) = t$, $\gamma(t) = t/4$, $q(t) = e^{9t/2}$, $G(u) = u^3$, $h(t) = \frac{11}{2}e^{-t}(1+e^{-t})$ and $H(u) = \frac{u}{1+u^2}$. It is easy to see that (3.27) satisfies all the conditions of Theorem 3.3.6. Indeed, $y(t) = e^{-2t}$ is a solution of (3.27) that converges to 0 as $t \to \infty$.

Theorem 3.3.8. *Assume there are constants p_5 and p_6 such that $-\infty < p_5 \leqslant p(t) \leqslant p_6 < -1$ and conditions (H_1), (H_3), (H_{14}), and (H_{16}) hold. Then any solution of* (E_3) *is either oscillatory, or satisfies $\liminf_{t\to\infty} \mid y(t) \mid = 0$, or satisfies $\mid y(t) \mid \to \infty$ as $t \to \infty$.*

Proof. Proceeding as in the proof of Theorem 3.3.6 in the cases (a), (b), (c) or (d) for $w(t) > 0$, we again obtain a contradictions to (H_{16}).

Next we consider case $w(t) < 0$ for $t \geqslant t_2$. Then one of the the cases (b), (c), (d), (e) or (f) holds. Suppose the cases (b) or (d) are holds. Since $\lim_{t\to\infty}(r(t)w^{\Delta^2}(t))^{\Delta}$ exists, (3.6) gives

$$\int_{t_2}^{\infty} q(t)G(y(\beta(t)))\Delta t < \infty.$$

If $\liminf_{t\to\infty} y(t) > 0$, then it follows that

$$\int_{t_2}^{\infty} q(t)\Delta t < \infty,$$

contradicting Remark 3.3.1. Hence, $\liminf_{t\to\infty} y(t) = 0$. If case (c) holds, then as in the proof of case (c) of Theorem 3.3.2, choose $f(x) = x^{1-l}$ and $g(t) = (-r(t)w^{\Delta^2}(t))^{\Delta}$. By Lemma 1.3.11, there exists 'c' in the real interval $[t, \sigma(t)]$ with $g(c) = L$ such that

$$\begin{aligned} -[((-r(t)w^{\Delta^2}(t))^{\Delta})^{1-l}]^{\Delta} &= (l-1)L^{-1}(-r(t)w^{\Delta^2}(t))^{\Delta^2} \\ &= (l-1)L^{-l}q(t)G(y(\beta(t))). \end{aligned}$$

Integrating, we obtain

$$\int_{t_2}^{\infty} q(t)G(y(\beta(t)))\Delta t < \infty.$$

Hence, $\liminf_{t\to\infty} y(t) = 0$.

Finally, in cases (e) and (f), we have $w^{\Delta^2}(t) < 0$ for $t \geqslant t_2$, and integrating twice from t_3 to t, we obtain $w(t) \to -\infty$ as $t \to \infty$. From (3.5), $z(t) \to -\infty$ as $t \to \infty$, so $\lim_{t\to\infty} y(\alpha(t)) = \lim_{t\to\infty} y(t) = \infty$. This completes the proof of the theorem. □

Example 3.3.9. Let $\mathbb{T} = h\mathbb{Z}$, where $h = p/q$ (p and q are odd positive integers) and consider the h-difference equation

$$\Delta_h^2[e^t\Delta_h^2(y(t) - e^h(1+e^{-5t})y(t-h))] + 2e^h\Big(\frac{e^h+1}{h}\Big)^2\Big(\frac{e^{2h}+1}{h}\Big)^2 e^{5t/3}y^{1/3}(t-3h)$$
$$- e^{2h}\Big(\frac{e^{-4h}+1}{h}\Big)^2\Big(\frac{e^{-3h}+1}{h}\Big)^2(1+e^{t-2h})e^{-4t}\frac{y(t-2h)}{1+\mid y(t-2h)\mid} = 0, \quad (3.28)$$

for $t(\geq 3h) \in \mathbb{T}$. Here $r(t) = e^t$, $p(t) = -e^h(1+e^{-5t})$, $q(t) = 2e^h\Big(\frac{e^h+1}{h}\Big)^2\Big(\frac{e^{2h}+1}{h}\Big)^2 e^{5t/3}$, $h(t) = e^{2h}\Big(\frac{e^{-4h}+1}{h}\Big)^2\Big(\frac{e^{-3h}+1}{h}\Big)^2(1+e^{t-2h})e^{-4t}$, $G(u) = u^{1/3}$, and $H(u) = \frac{u}{1+|u|}$. Equation (3.28) satisfies the hypotheses of Theorem 3.3.8. Here, we have $y(t) = (-1)^t e^t$ is an unbounded oscillatory solution.

3.4 Sufficient conditions for oscillation of (E_4) with $\int_{t_0}^{\infty}\frac{\sigma(t)}{r(t)}\Delta t < \infty$.

This section is concerned with the oscillatory and asymptotic behavior of solutions of equation (E_4) for suitable forcing functions $f(t)$. We restrict our forcing funtion to those that change signs.

Theorem 3.4.1. *Let* $0 \leqslant p(t) \leqslant p < \infty$, *and assume that conditions* (H_1)–(H_3), (H_{11}) *and* (H_{14}) *hold. If*

$$(H_{17})\ \limsup_{t\to\infty}\int_{t^*}^{t} d(s)Q(s)G(F(\beta(s)))\Delta s = +\infty$$
$$\textit{and}\ \liminf_{t\to\infty}\int_{t^*}^{t} d(s)Q(s)G(F(\beta(s)))\Delta s = -\infty,$$

then every solution of (E_4) *oscillates.*

Remark 3.4.2. Notice that condition (H_{17}) implies

$$(H'_{17})\ \limsup_{t\to\infty}\int_{t^*}^{t} Q(s)G(F(\beta(s)))\Delta s = +\infty \text{ and } \liminf_{t\to\infty}\int_{t^*}^{t} Q(s)G(F(\beta(s)))\Delta s = -\infty.$$

Proof. (**proof of Theorem 3.4.1**) Proceeding as in the proof of Theorem 3.2.1, by defining $z(t)$, $k(t)$, $w(t)$, and $v(t)$ as in (3.3)–(3.5), (3.13), we obtain (3.14). Thus, $v(t)$ is monotonic on $[t_2,\infty)_{\mathbb{T}}$, for some $t_2 > t_1$. If $v(t) > 0$ for $t \geqslant t_2$, then $z(t) > k(t) + F(t) > F(t)$. In view of (3.14), ($H_2$), ($H_3$) and ($\Lambda$), it is easy to see that

$$\begin{aligned}
0 &= (r(t)v^{\Delta^2}(t))^{\Delta^2} + q(t)G(y(\beta(t))) + G(p)(r(\alpha(t))v^{\Delta^2}(\alpha(t)))^{\Delta^2}\\
&\quad + G(p)q(\alpha(t))G(y(\beta(\alpha(t))))\\
&\ge (r(t)v^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))v^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(z(\beta(t)))\\
&\ge (r(t)v^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha(t))v^{\Delta^2}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(F(\beta(t))), \qquad (3.29)
\end{aligned}$$

for $t \geqslant t_3 \geqslant t_2$. Let (a), (b) or (d) of Lemma 2.3.1 hold. Integrating (3.29), we obtain

$$\limsup_{t\to\infty}\int_{t_3}^{t} Q(t)G(F(\beta(t)))\Delta t < \infty$$

contradicting (H'_{17}).

Let case (c) of Lemma 2.3.1 hold. Then proceeding as in the proof of case (c) in Theorem 3.3.2, we obtain an inequality similar to (3.21) from which it follows that

$$\begin{aligned}
-[((-r(t)v^{\Delta^2}(t))^{\Delta})^{1-l}]^{\Delta} \ &- \ G(p)[((-r(\alpha(t))v^{\Delta^2}(\alpha(t)))^{\Delta})^{1-l}]^{\Delta}\\
&\geqslant \ \lambda(l-1)L_1^l d(t)Q(t)G(z(\beta(t)))\\
&\geqslant \ \lambda(l-1)L_1^l d(t)Q(t)G(F(\beta(t)))
\end{aligned}$$

for $t \geqslant t_3$. An integration shows

$$\limsup_{t\to\infty}\int_{t_3}^{t} d(s)Q(s)G(F(\beta(s)))\Delta s < +\infty$$

contradicting (H_{17}).

Therefore, $v(t) < 0$ for $t \geqslant t_2$ and one of the cases (b)–(f) of Lemma 2.3.1 holds. In each of these cases $z(t) \le k(t) + F(t)$, which implies $\liminf_{t\to\infty} z(t) < 0$. This contradiction completes the proof of the theorem. ◻

Example 3.4.3. Let $\mathbb{T} = \mathbb{R}$ and consider the equation

$$\Big(e^t\big(y(t) + e^{-4t}y(t-\pi)\big)''\Big)'' + 8e^{t+2\pi}y(t-2\pi)$$
$$- 50e^{-3t+\pi/2}\left(1 + e^{2t-3\pi}\cos^2 t\right)\frac{y(t-3\pi/2)}{1+y^2(t-3\pi/2)} = 6e^{2t}\cos t, \quad t \geq 2\pi \quad (3.30)$$

for $t \in [2\pi, \infty)$. Here $r(t) = e^t$, $p(t) = e^{-4t}$, $q(t) = 8e^{t+2\pi}$, $G(u) = u$, $h(t) = 50e^{-3t+\pi/2}\left(1 + e^{2t-3\pi}\cos^2 t\right)$, $H(u) = \frac{u}{1+u^2}$, and $f(t) = 6e^{2t}\cos t$. We consider $F(t) = \frac{3e^t}{25}(3\sin t - 4\cos t) = \frac{3e^t}{5}\sin(t-\theta)$, *where* $\theta = tan^{-1}(4/3)$ such that $(r(t)F''(t))'' = f(t)$. Equation (3.30) satisfies the hypotheses of Theorem 3.4.1, and all solutions are oscillatory. Here $y(t) = e^t \sin t$ is such an oscillatory solution.

Remark 3.4.4. We can drop condition (H_{17}) from the hypotheses of Theorem 3.4.1 and obtain that bounded solutions oscillate. In case $v(t) < 0$, the proof is the same. If $v(t) > 0$, then $z(t) > k(t) + F(t) > F(t)$ and condition (H_{11}) contradicts the boundedness of $y(t)$.

Our next two results are for the case where $p(t) \leqslant 0$.

Theorem 3.4.5. *Let* $-1 < p(t) \leqslant 0$ *and conditions* (H_1), (H_8), (H_{11}), *and* (H_{14}) *hold. If*

$$(H_{18}) \ \limsup_{t\to\infty} \int_{t^*}^{t} R_1^l(\sigma(s))q(s)G(F(\beta(s)))\Delta s = +\infty$$
$$\text{and} \ \liminf_{t\to\infty} \int_{t^*}^{t} R_1^l(\sigma(s))q(s)G(F(\beta(s)))\Delta s = -\infty,$$

then any solution y *of equation* (E_4) *is either oscillatory or satisfies* $\limsup_{t\to\infty}|y(t)| = \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (E_4) say $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, and $y(\gamma(t))$ are all positive on $[t_1,\infty)_{\mathbb{T}}$, $t_1 \geqslant t_0$. Define $v(t)$ as in (3.13), so that we obtain (3.14). Consequently, $v(t)$ is monotonic on $[t_1,\infty)_{\mathbb{T}}$. Let $v(t) > 0$ for $t \geqslant t_2$. Then one of the cases (a)–(d) of Lemma 2.3.1 holds.
Now, $v(t) > 0$ implies

$$y(t) > z(t) > k(t) + F(t) > F(t) \quad (3.31)$$

for $t \geqslant t_2 > t_1$. If any one of the cases (a), (b), or (d) holds, then using (3.31) in (3.14), we obtain

$$\limsup_{t\to\infty} \int_{t_3}^{t} q(s)G(F(\beta(s)))\Delta s < \infty$$

contradicting (H_{18}).

Assume that case (c) holds. Proceeding as in the proof of case (c) when $v(t) > 0$ in Theorem 3.3.2, we obtain an inequality similar to (3.21) is

$$-[((-r(t)v^{\Delta^2}(t))^{\Delta})^{1-l}]^{\Delta} \geqslant (l-1)L_1^l R_1^l(\sigma(t))q(t)G(y(\beta(t))), \tag{3.32}$$

and using (3.31) and (3.32), this becomes

$$-[((-r(t)v^{\Delta^2}(t))^{\Delta})^{1-l}]^{\Delta} \geqslant (l-1)L_1^l R_1^l(\sigma(t))q(t)G(F(\beta(t))), \tag{3.33}$$

for $t \geqslant t_3 > t_2$. An integration yields a contradiction to (H_{18}).

We must have $v(t) < 0$ for $t \geqslant t_2$. Now, $z(t) - k(t) < F(t)$ which implies that $\liminf_{t\to\infty} z(t) = -\infty$ so $\limsup_{t\to\infty} y(t) = +\infty$, which completes the proof of the theorem. □

Example 3.4.6. Let $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$ and consider the equation

$$\Delta_q^2\left(\frac{3(q-1)^4t^4}{8(1/q+1)(q^2+1)(q+1)}\Delta_q^2\left(y(t) - 1/3y(t/q^3)\right)\right) + (t^4 + 1/t^4 + 1)y(t/q) - \frac{2}{t^4}\frac{y(t/q^5)}{1+y^2(t/q^5)} = -t^4(-1)^{\log_q^t}, \tag{3.34}$$

for $t \in [q^3, \infty)_{\mathbb{T}}$. Here we choose $F(t) = -\frac{8(1+1/q)(q^2+1)(q+1)}{3(q^4+1)(q^3+1)(q^6+1)(q^5+1)}t^4(-1)^{\log_q^t}$ such that $\Delta_q^2(r(t)\Delta_q^2 F(t)) = -t^4(-1)^{\log_q^t}$. Equation (3.34) satisfies all the conditions of Theorem 3.4.5. In particular, $y(t) = (-1)^{\log_q^t}$ is an oscillatory solution of (3.34).

Theorem 3.4.7. *Let $-1 < p_4 \leqslant p(t) \leqslant 0$ and conditions (H_1), (H_8), (H_{10}), (H_{14}), and (H_{18}) hold. Then every unbounded solution of* (E_4) *oscillates.*

Proof. Let $y(t)$ be a positive unbounded nonoscillatory solution of (E_4) on $[t_0, \infty)_{\mathbb{T}}$. Proceeding as in the proof of Theorem 3.4.5, we have the required contradiction if $v(t) > 0$ for $t \geqslant t_2$.

Next, we suppose that $v(t) < 0$ for $t \geqslant t_2$. Since $y(t)$ is unbounded, then there exists $\{\tau_n\}_{n=1}^{\infty} \subset [t_2, \infty)_{\mathbb{T}}$ such that $\tau_n \to \infty$, $y(\tau_n) \to \infty$ as $n \to \infty$, and

$$y(\tau_n) = \max\{y(t) : t_2 \leqslant t \leqslant \tau_n\}.$$

We may choose n large enough so that $\alpha(\tau_n) \geqslant t_2$. Hence,

$$z(\tau_n) \geqslant (1 + p_4)y(\tau_n).$$

By Lemma 2.3.1, one of the cases (b)–(f) are holds. Now $z(t) = v(t) + k(t) + F(t)$ implies that $z(t) < k(t) + F(t)$, and so

$$\begin{aligned} \infty = (1 + p_4) \limsup_{n\to\infty} y(\tau_n) &\leqslant \limsup_{n\to\infty}[k(\tau_n) + F(\tau_n)] \\ &\leq \lim_{t\to\infty} k(\tau_n) + \limsup_{n\to\infty} F(\tau_n) \\ &< \infty. \end{aligned}$$

This contradiction completes the proof of the theorem. □

The final theorem in this section gives sufficient conditions for the equation (E_4) to have a **bounded positive solution.**

Theorem 3.4.8. *Assume that* $1 < p_1 \leq p(t) \leq p_2 < \frac{1}{2}p_1^2 < \infty$ *and* (H_1) *hold. Suppose that* (H_{10}) *holds with* $\frac{-(p_1-1)}{16p_2} \leq F(t) \leq \frac{p_1-1}{8p_2}$. *In addition, assume that* G *and* H *are Lipschitz on* $\mathbb{R}$ *with Lipschitz constants* G_1 *and* H_1 *respectively. If*

$$\int_{t_0}^{\infty} \frac{\sigma(t)}{r(t)} \int_t^{\infty} \sigma(s)q(s)\Delta s\Delta t < \infty,$$

then (E_4) *admits a positive bounded solution.*

Proof. We can choose $t_1 > t_0$ large enough so that

$$\int_{t_1}^{\infty} \frac{\sigma(t)}{r(t)} \int_t^{\infty} \sigma(s)h(s)\Delta s\Delta t < \frac{p_1 - 1}{4p_1H(1)},$$

by (H_1) and

$$\int_{t_1}^{\infty} \frac{\sigma(t)}{r(t)} \int_t^{\infty} \sigma(s)q(s)\Delta s\Delta t < \frac{p_1 - 1}{16p_2G(1)},$$

by hypotheses. Let $X = BC_{rd}([t_1,\infty)_{\mathbb{T}},\mathbb{R})$ be the Banach space of all bounded rd-continuous functions on $[t_1,\infty)_{\mathbb{T}}$ with the norm

$$\|x\| = \sup\{\mid x(t)\mid : t \in [t_1,\infty)_{\mathbb{T}}\},$$

and let

$$S = \{x \in X : \frac{p_1-1}{8p_1p_2} \leqslant x(t) \leqslant 1,\ t \in [t_1,\infty)_{\mathbb{T}}\}.$$

Then, S is a closed, bounded, and convex subset of X. Take $t_2 \in [t_1,\infty)_{\mathbb{T}}$ so that $\alpha(t)$, $\beta(t)$, $\gamma(t) \geqslant t_1$ for all $t \in [t_2,\infty)_{\mathbb{T}}$. Define the mappings A, $B : S \to S$ by

$$Ax(t) = \begin{cases} Ax(t_2) & \text{for } t \in [t_1,t_2)_{\mathbb{T}} \\ -\frac{x(\alpha^{-1}(t))}{p(\alpha^{-1}(t))} + \frac{2p_1^2+p_1-1}{4p_1p(\alpha^{-1}(t))} & \text{for } t \in [t_2,\infty)_{\mathbb{T}}. \end{cases}$$

and

$$Bx(t) = \begin{cases} Bx(t_2) & \text{for } t \in [t_1,t_2)_{\mathbb{T}} \\ \frac{F(\alpha^{-1}(t))}{p(\alpha^{-1}(t))} + \frac{k(\alpha^{-1}(t))}{p(\alpha^{-1}(t))} - & \\ \quad \frac{1}{p(\alpha^{-1}(t))}\int_{\alpha^{-1}(t)}^{\infty} \frac{\sigma(s)-\alpha^{-1}(t)}{r(s)}\left(\int_s^{\infty}(\sigma(u)-s)q(u)G(x(\beta(u)))\Delta u\right)\Delta s & \text{for } t \in [t_2,\infty)_{\mathbb{T}}. \end{cases}$$

For $x \in S$, we have

$$\begin{aligned} k(t) &= \int_t^{\infty}\frac{\sigma(s)-t}{r(s)}\int_s^{\infty}(\sigma(u)-s)h(u)H(x(\gamma(u)))\Delta u\Delta s \\ &\leqslant H(1)\int_t^{\infty}\frac{\sigma(s)}{r(s)}\int_s^{\infty}\sigma(u)h(u)\Delta u\Delta s \\ &< \frac{1}{4p_1}(p_1-1). \end{aligned}$$

and

$$\frac{1}{p(\alpha^{-1}(t))}\int_{\alpha^{-1}(t)}^{\infty}\frac{\sigma(s)-\alpha^{-1}(t)}{r(s)}\int_s^{\infty}(\sigma(u)-s)q(u)G(x(\beta(u)))\Delta u\Delta s \le \frac{p_1-1}{16p_1p_2}.$$

For all x, $y \in S$ and all $t \in [t_2,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned} Ax(t)+By(t) &\leqslant -\frac{p_1-1}{8p_1p_2^2} + \frac{1}{4p_1^2}(2p_1^2+p_1-1) + \frac{1}{8p_1p_2}(p_1-1) + \frac{1}{4p_1^2}(p_1-1) \\ &\le \frac{1}{8p_1p_2^2} + \frac{1}{2} + \frac{p_1-1}{8p_2} + \frac{1}{8p_1p_2}(p_1-1) + \frac{1}{8p_2}(p_1-1) \\ &\le 1/8 + 1/2 + 1/8 + 1/8 + 1/8 \\ &< 1, \end{aligned}$$

and

$$\begin{aligned}
Ax(t)+By(t) &\geqslant -\frac{1}{p_1}+\frac{1}{4p_1p_2}(2p_1^2+p_1-1)-\frac{1}{16p_1p_2}(p_1-1)-\frac{1}{16p_1p_2}(p_1-1)\\
&\geq -\frac{1}{p_1}+\frac{p_1}{2p_2}+\frac{p_1-1}{4p_1p_2}-\frac{p_1-1}{8p_1p_2}\\
&= -\frac{1}{p_1}+\frac{p_1}{2p_2}+\frac{p_1-1}{8p_1p_2}\\
&= \frac{p_1^2-2p_2}{2p_1p_2}+\frac{p_1-1}{8p_1p_2}\\
&\geq \frac{p_1-1}{8p_1p_2}.
\end{aligned}$$

Thus, $Ax+By\in S$.

To show that A is a contraction mapping on S, first notice that

$$\begin{aligned}
\|Ax-Ay\| &= \left\|-\frac{x(\alpha^{-1}(t))}{p(\alpha^{-1}(t))}+\frac{2p_1^2+p_1-1}{4p_1p(\alpha^{-1}(t))}+\frac{y(\alpha^{-1}(t))}{p(\alpha^{-1}(t))}-\frac{2p_1^2+p_1-1}{4p_1p(\alpha^{-1}(t))}\right\|\\
&= \left\|-\frac{x(\alpha^{-1}(t))}{p(\alpha^{-1}(t))}+\frac{y(\alpha^{-1}(t))}{p(\alpha^{-1}(t))}\right\|\\
&\leqslant \frac{1}{p_1}\left\|x(t)-y(t)\right\|.
\end{aligned}$$

Since $p_1>1$, A is a contraction mapping.

To show that B is completely continuous on S, we need to show that B is continuous and maps bounded sets into relatively compact sets. In order to show that B is continuous, let x, $x_k=x_k(t)\in S$ be such that $\|x_k-x\|=\sup_{t\geqslant t_1}\{|x_k(t)-x(t)|\}\to 0$. Since S is closed then $x(t)\in S$. For $t\geqslant t_1$, we have

$$\begin{aligned}
&\|(Bx_k)-(Bx)\|\\
&=\Bigg\|\frac{1}{p(\alpha^{-1}(t))}\int_{\alpha^{-1}(t)}^{\infty}\frac{\sigma(s)-\alpha^{-1}(t)}{r(s)}\int_s^{\infty}(\sigma(u)-s)h(u)H(x_k(\gamma(u)))\Delta u\Delta s\\
&\quad-\frac{1}{p(\alpha^{-1}(t))}\int_{\alpha^{-1}(t)}^{\infty}\frac{\sigma(s)-\alpha^{-1}(t)}{r(s)}\int_s^{\infty}(\sigma(u)-s)q(u)G(x_k(\beta(u)))\Delta u\Delta s\\
&\quad-\frac{1}{p(\alpha^{-1}(t))}\int_{\alpha^{-1}(t)}^{\infty}\frac{\sigma(s)-\alpha^{-1}(t)}{r(s)}\int_s^{\infty}(\sigma(u)-s)h(u)H(x(\gamma(u)))\Delta u\Delta s\\
&\quad+\frac{1}{p(\alpha^{-1}(t))}\int_{\alpha^{-1}(t)}^{\infty}\frac{\sigma(s)-\alpha^{-1}(t)}{r(s)}\int_s^{\infty}(\sigma(u)-s)q(u)G(x(\beta(u)))\Delta u\Delta s\Bigg\|,
\end{aligned}$$

that is,

$$\begin{aligned}
&\|(Bx_k)-(Bx)\| \\
&\leq \frac{1}{p_1}\Bigg\| \int_{\alpha^{-1}(t)}^{\infty} \frac{\sigma(s)-\alpha^{-1}(t)}{r(s)}\Big(\int_s^{\infty}(\sigma(u)-s)h(u)\big(H(x_k(\gamma(u)))-H(x(\gamma(u)))\big)\Delta u\Big)\Delta s \\
&\quad + \frac{1}{p_1}\int_{\alpha^{-1}(t)}^{\infty} \frac{\sigma(s)-\alpha^{-1}(t)}{r(s)}\Big(\int_s^{\infty}(\sigma(u)-s)q(u)\big(G(x(\beta(u)))-G(x_k(\beta(u)))\big)\Delta u\Big)\Delta s\Bigg\| \\
&\leqslant \frac{1}{p_1}H_1\|x_k-x\|\int_{\alpha^{-1}(t)}^{\infty}\frac{\sigma(s)}{r(s)}\int_s^{\infty}\sigma(u)h(u)\Delta u\Delta s \\
&\quad + \frac{1}{p_1}G_1\|x_k-x\|\int_{\alpha^{-1}(t)}^{\infty}\frac{\sigma(s)}{r(s)}\int_s^{\infty}\sigma(u)q(u)\Delta u\Delta s \\
&\leqslant \frac{1}{4p_1^2}(p_1-1)\|x-x_k\| + \frac{1}{16p_1p_2}(p_1-1)\|x-x_k\|.
\end{aligned}$$

Since for all $t \geqslant t_1$, $\{x_k(t)\}$ converges uniformly to $x(t)$ as $k \to \infty$, $\lim_{k\to\infty}\|(Bx_k)-(Bx)\|= 0$. Thus, B is continuous.

To show that BS is relatively compact, it suffices to show that the family of functions $\{Bx : x \in S\}$ is uniformly bounded and equicontinuous on $[t_1,\infty)_{\mathbb{T}}$. The uniform boundedness is clear. To show that BS is equicontinuous, let $x \in S$ and $t'', t' \geqslant t_1$ such that $t'' \geq t'$. Then

$$\begin{aligned}
&|(Bx)(t'')-(Bx)(t')| \\
&\leqslant \left|\frac{F(\alpha^{-1}(t''))}{p(\alpha^{-1}(t''))}-\frac{F(\alpha^{-1}(t'))}{p(\alpha^{-1}(t'))}\right| + \left|\frac{k(\alpha^{-1}(t''))}{p(\alpha^{-1}(t''))}-\frac{k(\alpha^{-1}(t'))}{p(\alpha^{-1}(t'))}\right| \\
&\quad + \Bigg|\frac{1}{p(\alpha^{-1}(t'))}\int_{\alpha^{-1}(t')}^{\infty}\Big(\frac{\sigma(s)-\alpha^{-1}(t')}{r(s)}\int_s^{\infty}(\sigma(u)-s)q(u)G(x(\beta(u)))\Delta u\Big)\Delta s \\
&\quad - \frac{1}{p(\alpha^{-1}(t''))}\int_{\alpha^{-1}(t'')}^{\infty}\Big(\frac{\sigma(s)-\alpha^{-1}(t'')}{r(s)}\int_s^{\infty}(\sigma(u)-s)q(u)G(x(\beta(u)))\Delta u\Big)\Delta s\Bigg| \\
&= \left|\frac{F(\alpha^{-1}(t''))}{p(\alpha^{-1}(t''))}-\frac{F(\alpha^{-1}(t'))}{p(\alpha^{-1}(t'))}\right| + \left|\frac{k(\alpha^{-1}(t''))}{p(\alpha^{-1}(t''))}-\frac{k(\alpha^{-1}(t'))}{p(\alpha^{-1}(t'))}\right| \\
&\quad + \Bigg|\frac{1}{p(\alpha^{-1}(t'))}\int_{\alpha^{-1}(t')}^{\infty}\int_{\theta}^{\infty}\frac{1}{r(s)}\int_s^{\infty}(\sigma(u)-s)q(u)G(x(\beta(u)))\Delta u\Delta s\Delta\theta \\
&\quad - \frac{1}{p(\alpha^{-1}(t''))}\int_{\alpha^{-1}(t'')}^{\infty}\int_{\theta}^{\infty}\frac{1}{r(s)}\int_s^{\infty}(\sigma(u)-s)q(u)G(x(\beta(u)))\Delta u\Delta s\Delta\theta\Bigg|,
\end{aligned}$$

that is,

$$\begin{aligned}
&|(Bx)(t^{''}) - (Bx)(t^{'})| \\
&\leq \left|\frac{F(\alpha^{-1}(t''))}{p(\alpha^{-1}(t''))} - \frac{F(\alpha^{-1}(t'))}{p(\alpha^{-1}(t'))}\right| + \left|\frac{k(\alpha^{-1}(t''))}{p(\alpha^{-1}(t''))} - \frac{k(\alpha^{-1}(t'))}{p(\alpha^{-1}(t'))}\right| \\
&\quad + G(1)\left|\Big(\frac{1}{p(\alpha^{-1}(t'))} - \frac{1}{p(\alpha^{-1}(t''))}\Big)\int_{\alpha^{-1}(t'')}^{\infty}\frac{\sigma(s)}{r(s)}\int_s^{\infty}\sigma(u)q(u)\Delta u\Delta s\right| \\
&\quad + G(1)\left|\frac{1}{p(\alpha^{-1}(t'))}\int_{\alpha^{-1}(t')}^{\infty}\frac{1}{r(s)}\int_s^{\infty}\sigma(u)q(u)\Delta u\Delta s\right|\left|\alpha^{-1}(t^{'}) - \alpha^{-1}(t^{''})\right|.
\end{aligned}$$

So $|(Bx)(t^{''}) - (Bx)(t^{'})| \to 0$ as $t^{'} \to t^{''}$. Therefore, $\{Bx : x \in S\}$ is uniformly bounded and equicontinuous on $[t_1, \infty)_{\mathbb{T}}$. Hence, BS is relatively compact. By **Krasnosel'skii's fixed point theorem** there exists $x \in S$ such that $Ax + Bx = x$. Thus, the theorem is proved. □

Remark 3.4.9. Results similar to Theorem 3.4.8 can be proved for other ranges of values for $p(t)$.

3.5 Conclusion

In Section 3.1, we have studied the oscillatory and asymptotic behavior of (E_3) with the assumption $\int_{t_0}^{\infty}\frac{t}{r(t)}\Delta t = \infty$ for various ranges of $p(t)$. In Theorems 3.1.1, 3.1.4 and 3.1.7, we have established that every solution of (E_3) oscillates or converges to zero as $t \to \infty$ for $p(t)$ ranges with $0 \leq p(t) \leq p_1 < 1$ or $1 < p_2 \leq p(t) \leq p_3 < \infty$. In Theorem 3.1.10, we have proved that every solution of (E_3) oscillates or converges to zero as $t \to \infty$ for $p(t)$ with $-1 < p_4 \leq p(t) \leq 0$ and in Theorem 3.1.13, we have proved that every bounded solution of (E_3) oscillates or tends to zero as $t \to \infty$ for $p(t)$ range with $-\infty < p_5 \leq p(t) \leq p_6 < -1$.

In Section 3.2, we have studied the oscillatory and asymptotic behavior of (E_4) with the assumption $\int_{t_0}^{\infty}\frac{t}{r(t)}\Delta t = \infty$ for various ranges of $p(t)$ with the help of (H_{10}) and (H_{11}). In Theorem 3.2.1, we have proved that every solution of (E_4) is oscillatory and in Theorem 3.2.5, we have shown that every unbounded solution of (E_4) oscillates for $p(t)$ with $0 \leq p(t) \leq p < \infty$. In Theorem 3.2.3, we have proved that every solution y

of (E_4) oscillatory or $\limsup_{t\to\infty} \mid y(t) \mid= \infty$ for the range $p(t)$ with $-1 < p(t) \leq 0$. In Theorem 3.2.7, we have obtined the sufficient conditions for existence of bounded positive solutions of (E_4) with $0 \leq p(t) \leq p_1 < 1$.

In Section 3.3, we have studied the oscillatory and asymptotic behavior of (E_3) with the assumption $\int_{t_0}^{\infty} \frac{\sigma(t)}{r(t)} \Delta t < \infty$ for various ranges of $p(t)$. In Theorem 3.3.2, we have proved that every solution of (E_3) is either oscillatory or converges to zero as $t \to \infty$ for $p(t)$ with $0 \leq p(t) \leq p < \infty$. In Theorem 3.3.5, we have proved that every unbounded solution of (E_3) oscillates for $p(t)$ with $0 \leq p(t) \leq p < 1$. In Theorem 3.3.6, we have proved that every solution of (E_3) is either oscillates or converges to zero as $t \to \infty$ for $p(t)$ with $-1 < p_4 \leq p(t) \leq 0$. In Theorem 3.3.8, we have proved that every solution $y(t)$ of (E_3) is either oscillatory or $\liminf_{t\to\infty} \mid y(t) \mid= 0$ or $\lim_{t\to\infty} \mid y(t) \mid= \infty$.

In Section 3.4, we have established that the oscillatory and asymptotic behavior of (E_4) with the assumption $\int_{t_0}^{\infty} \frac{\sigma(t)}{r(t)} \Delta t < \infty$ for various ranges of $p(t)$ and the help of (H_{10}) and (H_{11}). In Theorem 3.4.1, we have shown that every solution of (E_4) is oscillatory for $p(t)$ with $0 \leq p(t) \leq p < \infty$. In Theorem 3.4.5, we have established that every solution $y(t)$ of (E_4) is either oscillatory or satisfies $\limsup_{t\to\infty} \mid y(t) \mid= \infty$ for $p(t)$ with $-1 < p(t) \leq 0$ and $F(t)$ as defined in (H_{11}), whereas in Theorem 3.4.7, we proved that every unbounded solution of (E_4) oscillates for $p(t)$ with $-1 < p_4 \leq p(t) \leq 0$ and $F(t)$ as defined in (H_{10}). In Theorem 3.4.8, we obtained the sufficient conditions for existence of bounded positive solutions of (E_4) with $1 < p_1 \leq p(t) \leq p_2 < \frac{1}{2}p_1^2 < \infty$.

It would be interesting to study the oscillatory/nonoscillatory/asymptotic behavior of (E_3) and (E_4) if $p(t)$ oscillates and every solution of (E_3) and (E_4) are oscillatory for all ranges of $p(t)$.

Chapter 4

Fourth Order Nonlinear Mixed Neutral Delay Dynamic Equations

In this chapter, an attempt is made to study the oscillatory behavior of solutions of nonlinear functional mixed delay dynamic equations of the form

$$\left(\frac{1}{a(t)}\left((y(t)+p(t)y(\alpha(t)))^{\Delta^2}\right)^m\right)^{\Delta^2} = q(t)f(y(\beta(t))) + r(t)g(y(\gamma(t))) \qquad (E_5)$$

under the assumptions

$$\int_{t_0}^{\infty}(a(t))^{\frac{1}{m}}\Delta t = \infty \quad or \quad \int_{t_0}^{\infty}(a(t))^{\frac{1}{m}}\Delta t < \infty,$$

where $t_0(>0) \in \mathbb{T}$, m is a quotient of odd positive integers, $a,\ q,\ r \in C_{rd}(\mathbb{T},\mathbb{R})$ are positive functions and $\alpha,\ \beta,\ \gamma \in C_{rd}(\mathbb{T},\mathbb{T})$ such that $\alpha(t) \le t,\ \beta(t) \le t,\ \gamma(t) \ge t$, and $\lim_{t\to\infty}\alpha(t) = \infty = \lim_{t\to\infty}\beta(t) = \lim_{t\to\infty}\gamma(t)$, $f,\ g \in C(\mathbb{R},\mathbb{R})$ are continuous functions with $uf(u) > 0,\ vg(v) > 0$ for all $u, v \neq 0$, and $p \in C_{rd}(\mathbb{T},\mathbb{R})$.
If $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, then (E_5) reduces to

$$\left(\frac{1}{a(t)}\left((y(t)+p(t)y(\alpha(t)))''\right)^m\right)'' = q(t)f(y(\beta(t))) + r(t)g(y(\gamma(t))) \qquad (4.1)$$

and

$$\Delta^2\left(\frac{1}{a(n)}\left(\Delta^2\big(y(n)+p(n)y(\alpha(n))\big)\right)^m\right) = q(n)f(y(\beta(n))) + r(n)g(y(\gamma(n))) \qquad (4.2)$$

respectively. In [77, 81], the author has considered the equations (4.1) and (4.2) and established the sufficient conditions for oscillation under the assumptions

$$\int_{t_0}^{\infty} (a(t))^{\frac{1}{m}}\, dt = \infty,$$

and

$$\sum_{n=n_0}^{\infty} (a(n))^{\frac{1}{m}} = \infty$$

respectively. It is interesting to study the unification of continuous and discrete aspects of (4.1) and (4.2) through the dynamic equations on time scales.

We may note that, (E_5) includes a class of differential or difference equations with delay as well as advanced argument of neutral type. In the last few years, there has been an increasing interest in obtaining sufficient conditions for the oscillatory and nonoscillatory behaviour of solutions of different classes of neutral dynamic equations. We refer the reader to some of the papers [5, 65, 78, 79] and the references cited therein.

By a solution of (E_5), we mean a nontrivial real valued function y on $[T_y,\ \infty)_{\mathbb{T}}$ such that $y(t) + p(t)y(\alpha(t)) \in C^2_{rd}(\mathbb{T}, \mathbb{R})$, $\left(\frac{1}{a(t)}\left((y(t) + p(t)y(\alpha(t)))^{\Delta^2}\right)^m\right) \in C^2_{rd}(\mathbb{T}, \mathbb{R})$ and satisfies (E_5), for $T_y \geq t_0 > 0$. In this chapter, we do not consider the solutions that eventually vanish identically. A solution y of (E_5) is said to be *oscillatory* if it is neither eventually positive nor eventually negative and it is *nonoscillatory* otherwise. We define the operators as follows:

$$L_0 z(t) = z(t),\ L_1 z(t) = L_0^{\Delta} z(t),\ L_2 z(t) = \frac{1}{a(t)}\left(L_1^{\Delta} z(t)\right)^m,$$
$$L_3 z(t) = L_2^{\Delta} z(t),\ L_4 z(t) = L_3^{\Delta} z(t),\ where\ z(t) = y(t) + p(t)y(\alpha(t)).$$

Then equation (E_5) becomes

$$L_4 z(t) = q(t) f(y(\beta(t))) + r(t) g(y(\gamma(t))). \tag{4.3}$$

We note that the study of oscillatory and asymptotic behaviour of solutions of dynamic equation (E_5) is same as the study of equation (4.3). Throughout this chapter, we assume the following hypotheses for our work use in the sequel.

$$A[s,v] = \int_v^s (s - \sigma(t))(t - v)^{\frac{1}{m}} a^{\frac{1}{m}}(t)\Delta t,\ s > \sigma(t) > t > v \geq t_0,$$

$$B[s,t_2] = \int_{t_2}^{s} (s-\sigma(t))(s-t)^{\frac{1}{m}} a^{\frac{1}{m}}(t)\Delta t,\ s > \sigma(t) > t > t_2 \geq t_0,$$

$$C[v,u] = \int_{u}^{v} (\sigma(t)-u)(v-t)^{\frac{1}{m}} a^{\frac{1}{m}}(t)\Delta t,\ v > \sigma(t) > t > u \geq t_0,$$

$$D[u,v] = \int_{v}^{u} (\sigma(t)-v)(t-v)^{\frac{1}{m}} a^{\frac{1}{m}}(t)\Delta t,\ u > \sigma(t) > t > v \geq t_0.$$

and

(A_0) $\int_{t_0}^{\infty} \big(a(t)\big)^{1/m}\Delta t = \infty,\ t_0 \in \mathbb{T},$

(A_1) $f(uv) \geq f(u)f(v),\ g(uv) \geq g(u)g(v), for\ u,\ v \in \mathbb{R}\ and\ u,\ v > 0,$

(A_2) $f(-u) = -f(u),\ g(-u) = -g(u), for\ u \in \mathbb{R},$

(A_3) $\exists\ \lambda > 0, \mu > 0\ such\ that\ f(u) + f(v) \geq \lambda f(u+v),\ g(u) + g(v) \geq \mu g(u+v),$
$for\ u, v \in \mathbb{R}\ and\ u, v > 0,$

(A_4) $Q(t) = min\{q(t),\ q(\alpha(t))\},\ R(t) = min\{r(t),\ r(\alpha(t))\},\ for\ t \in [t^*, \infty)_{\mathbb{T}},$
$t^* \geq t_0,$

(A_5) $\frac{f(u^{\frac{1}{m}})}{u} \geq M_1 > 0,\ \frac{g(u^{\frac{1}{m}})}{u} \geq M_2 > 0,\ for\ u \neq 0,$

(A_6) $\limsup_{s\to\infty} \int_{\beta(s)}^{\alpha(s)} Q(\theta)f[B(\beta(\theta),\ t_2)]\Delta\theta > \frac{1+f(a)}{\lambda M_1},\ for\ some\ t_2 \geq t_0,$

(A_7) $\limsup_{s\to\infty} \int_{s}^{\gamma(s)} R(\theta)g[A(\gamma(\theta),\ \gamma(s))]\Delta\theta > \frac{1+g(a)}{\mu M_2},$

(A_8) $\limsup_{s\to\infty} \int_{\alpha^{-1}(\beta(s))}^{s} Q(\theta)f[C(\beta(s),\ \beta(\theta))]\Delta\theta > \frac{1+f(a)}{\lambda M_1},$

(A_9) $\alpha,\ \beta\ and\ \gamma\ are\ bijective\ functions\ satisfying\ the\ properties:$
$\alpha(\beta(t)) = \beta(\alpha(t)),\ \beta(\gamma(t)) = \gamma(\beta(t)),\ \gamma(\alpha(t)) = \alpha(\gamma(t)),$
$\alpha^{-1}(t) \geq t,\ \beta^{-1}(t) \geq t,\ \gamma^{-1}(t) \leq t,\ for\ every\ right-scattered\ point$
$t \in [t_0,\ \infty)_{\mathbb{T}},\ t_0 \geq 0,$

(A_{10}) $\limsup_{s\to\infty} \int_{s}^{\gamma(s)} r(t)g[A(\gamma(t),\ \gamma(s))]\Delta t > \frac{1}{M_2},$

(A_{11}) $\limsup_{s\to\infty} \int_{\beta(s)}^{s} q(t)f[B(\beta(t),\ t_2)]\Delta t > \frac{1}{M_1},\ \ for\ every\ t_2 \geq t_0,$

(A_{12}) $\limsup_{s\to\infty} \int_{\beta(s)}^{s} q(t)f[C(\beta(s),\ \beta(t))]\Delta t > \frac{1}{M_1},$

(A_{13}) $\limsup_{s\to\infty} \int_{s}^{\beta^{-1}(\alpha(s))} q(t)f[C(\alpha^{-1}(\beta(t)),\ \alpha^{-1}(\beta(s)))]\Delta t > \frac{1}{M_1 f(b^{-1})},\ \ b > 0,$

(A_{14}) $\frac{f^{\frac{1}{m}}(u)}{u} \geq M_3 > 0,\ \ \frac{g^{\frac{1}{m}}(u)}{u} \geq M_4 > 0,\ for\ u \neq 0.$

4.1 Oscillation results of (E_5) with $\int_{t_0}^{\infty}(a(t))^{\frac{1}{m}}\Delta t = \infty$.

This section deals with the oscillation criteria for (E_5) under the assumption (A_0). We need the following lemma for our work in the sequel.

Lemma 4.1.1. Suppose that (A_0) holds. Let u be a rd-continuously differentiable function on $[t_0,\infty)_{\mathbb{T}}$ such that $(\frac{1}{a(t)}(u^{\Delta^2}(t))^m) \in C_{rd}^{\Delta^2}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ and $(\frac{1}{a(t)}(u^{\Delta^2}(t))^m)^{\Delta^2} \geq (\not\equiv)0$, for large $t \in [t_0,\infty)_{\mathbb{T}}$. If $u(t) > 0$ ultimately, then one of cases (a), (b) or (c) holds for large t, and if $u(t) < 0$ ultimately, then one of cases (c) or (d) holds for large t, where

$$(a)\quad L_1u(t) > 0,\quad L_2u(t) > 0 \quad \text{and} \quad L_3u(t) > 0,$$
$$(b)\quad L_1u(t) > 0,\quad L_2u(t) > 0 \quad \text{and} \quad L_3u(t) < 0,$$
$$(c)\quad L_1u(t) < 0,\quad L_2u(t) > 0 \quad \text{and} \quad L_3u(t) < 0,$$
$$(d)\quad L_1u(t) < 0,\quad L_2u(t) < 0 \quad \text{and} \quad L_3u(t) < 0.$$

Proof. Since $(\frac{1}{a(t)}(u^{\Delta^2}(t))^m)^{\Delta^2} \geq 0$ for large t, then by using the operators $L_0u(t) = u(t)$, $L_1u(t) = L_0^{\Delta}u(t)$, $L_2u(t) = \frac{1}{a(t)}\left(L_1^{\Delta}u(t)\right)^m$, $L_3u(t) = L_2^{\Delta}u(t)$ and $L_4u(t) = L_3^{\Delta}u(t)$ we write it as $L_4u(t) \geq 0$ for large t. Then $u(t)$, $L_1u(t)$, $L_2u(t)$ and $L_3u(t)$ are monotonic and hence there are eight possibilities. Let $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that each $u(t)$, $L_1u(t)$, $L_2u(t)$ and $L_3u(t)$ one of constant sign for $t \in [t_1,\infty)_{\mathbb{T}}$.

First, let $u(t) > 0$ for $t \geq t_1 > t_0$. It is enough to show that (d) and the following cases, viz.,

$$(e)\quad L_1u(t) < 0,\ L_2u(t) < 0 \ \textit{and}\ L_3u(t) > 0,$$
$$(f)\quad L_1u(t) > 0,\ L_2u(t) < 0 \ \textit{and}\ L_3u(t) > 0,$$
$$(g)\quad L_1u(t) < 0,\ L_2u(t) > 0 \ \textit{and}\ L_3u(t) > 0,$$
$$(h)\quad L_1u(t) > 0,\ L_2u(t) < 0 \ \textit{and}\ L_3u(t) < 0,$$

do not hold. In the case (d), $L_2u(t) < 0$ for $t \geq t_1 > t_0$, this implies that $L_1u(t)$ is decreasing. Then for $t > t_2 \geq t_1$, we have $L_1u(t) \leq L_1u(t_2)$. By integrating this inequality from t_2 to t, we obtain $u(t) < 0$ for large t, which is a contradiction. In the case (e) and case (f), $L_3u(t)$ is increasing and $L_3u(t) > 0$. Then for $t > t_2$, we have $L_3u(t) \geq L_3u(t_2)$. By integrating this inequality from t_2 to t, we obtain $L_2u(t) > 0$ for large t, which is a contradiction. In case (g), from $L_2u(t) > 0$ and $L_2u(t)$ is increasing, because $L_3u(t) > 0$. For $t > t_2$, we have $L_2u(t) \geq L_2u(t_2)$. This implies that $L_1^{\Delta}u(t) \geq L_2^{1/m}u(t_2)a^{1/m}(t)$. By integrating this inequality from t_2 to t, we obtain $L_1u(t) > 0$ for large t due to (A_0), which is a contradiction. In the case (h), $L_2u(t) < 0$ and $L_2u(t)$ is decreasing, because $L_3u(t) < 0$. For $t > t_2$, we have $L_2u(t) \leq L_2u(t_2)$. This implies that $L_1^{\Delta}u(t) \leq L_2^{1/m}u(t_2)a^{1/m}(t)$. By integrating this inequality from t_2 to t, we obtain $L_1u(t) < 0$ for large t due to (A_0), which is a contradiction.

Next, suppose that $u(t) < 0$ for $t \geq t_1 > t_0$. It is enough to show that cases (a), (b), (e), (f), (g) and (h) do not hold. In the cases (a) and (b), $L_2u(t) > 0$ for $t \geq t_1 > t_0$, this implies that $L_1u(t)$ is increasing. Then for $t > t_2 \geq t_1$, we have $L_1u(t) \geq L_1u(t_2)$. By integrating this inequality from t_2 to t, we obtain $u(t) > 0$ for large t, which is a contradiction. Proceeding as above we obtain a contradiction in each of the cases (e), (f), (g) and (h). Thus the lemma is proved. □

Theorem 4.1.2. *Let* $0 \leq p(t) \leq a < \infty$, *and* $\beta(t) \leq \alpha(t)$ *for* $t \in [t_0, \infty)_{\mathbb{T}}$. *If* $(A_0) - (A_9)$ *hold, then* (E_5) *is oscillatory.*

Proof. Suppose on the contrary that $y(t)$ is a non-oscillatory solution of (E_5) on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, there exists $t_1 \in [t^*, \infty)_{\mathbb{T}}$, sufficiently large such that $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, $y(\alpha(\beta(t)))$ are all positive for $t \in [t_1, \infty)_{\mathbb{T}}$. From (4.3), it follows that $L_4z(t) > 0$, for $t \geq t_1$. Hence, we can find a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $L_iz(t), i = 1, 2, 3$ are eventually of one sign on $[t_2, \infty)_{\mathbb{T}}$. In what follows, we consider the cases (a), (b), or (c) of Lemma 4.1.1, since $z(t) > 0$.

Let case (a) holds. For $u > v \geq t_2$,

$$L_2z(u) - L_2z(v) = \int_v^u L_3z(s)\Delta s \geq (u - v)L_3z(v),$$

that is,

$$L_2z(u) \geq (u - v)L_3z(v).$$

Hence,

$$z^{\Delta^2}(u) \geq a^{\frac{1}{m}}(u)(u-v)^{\frac{1}{m}} L_3^{\frac{1}{m}} z(v). \tag{4.4}$$

For $s > \sigma(t) > t > t_2$, it is easy to verify that

$$\int_{t_2}^{s} (s-\sigma(t)) z^{\Delta^2}(t)\Delta t = z(s) - z(t_2) - (s-t_2) z^{\Delta}(t_2).$$

Thus,

$$z(s) > \int_{t_2}^{s} (s-\sigma(t)) z^{\Delta^2}(t)\Delta t, \tag{4.5}$$

implies that

$$\begin{aligned} z(s) &> \int_{t_2}^{s} (s-\sigma(t)) a^{\frac{1}{m}}(t)(t-v)^{\frac{1}{m}} L_3^{\frac{1}{m}} z(v)\Delta t \\ &\geq L_3^{\frac{1}{m}} z(v) \int_{v}^{s} (s-\sigma(t)) a^{\frac{1}{m}}(t)(t-v)^{\frac{1}{m}}\Delta t \\ &= L_3^{\frac{1}{m}} z(v) A[s,v], \ for\ s > v \geq t_2, \end{aligned}$$

due to (4.4). Letting $s = \gamma(\theta)$ and $v = \gamma(s)$, we get

$$z(\gamma(\theta)) \geq L_3^{\frac{1}{m}} z(\gamma(s)) A[\gamma(\theta),\ \gamma(s)], \tag{4.6}$$

for $\gamma(\theta) > \gamma(s) \geq t_2$. From (4.3), it follows that

$$L_4 z(t) \geq r(t) g(y(\gamma(t))),$$

for $t \geq t_2$ and

$$\begin{aligned} L_4 z(t) + g(a) L_4 z(\alpha(t)) &\geq r(t) g(y(\gamma(t))) + g(a) r(\alpha(t)) g(y(\gamma(\alpha(t)))) \\ &\geq \mu R(t) g(y(\gamma(t)) + a y(\gamma(\alpha(t)))) \\ &= \mu R(t) g(y(\gamma(t)) + a y(\alpha(\gamma(t)))) \\ &\geq \mu R(t) g(z(\gamma(t))) \end{aligned}$$

due to (A_1), (A_3), (A_4), and (A_9), where we have used the fact that $z(\gamma(t)) \leq y(\gamma(t)) + ay(\alpha(\gamma(t)))$. Using (4.6), the last inequality becomes

$$\begin{aligned} L_4 z(\theta) + g(a) L_4 z(\alpha(\theta)) &\geq \mu R(\theta) g[A(\gamma(\theta),\ \gamma(s)) L_3^{\frac{1}{m}} z(\gamma(s))] \\ &\geq \mu R(\theta) g[A(\gamma(\theta),\ \gamma(s))] g[L_3^{\frac{1}{m}} z(\gamma(s))]. \end{aligned}$$

Integrating the last inequality from s to $\gamma(s)$, we obtain

$$\begin{aligned}\int_s^{\gamma(s)} R(\theta)g[A(\gamma(\theta),\ \gamma(s))]\Delta\theta &\leq \frac{1}{\mu g[L_3^{\frac{1}{m}} z(\gamma(s))]}\int_s^{\gamma(s)}[L_4z(\theta)+g(a)L_4z(\alpha(\theta))]\Delta\theta \\ &< \frac{L_3z(\gamma(s))+g(a)L_3z(\alpha(\gamma(s)))}{\mu g[L_3^{\frac{1}{m}} z(\gamma(s))]} \\ &\leq \frac{L_3z(\gamma(s))+g(a)L_3z(\gamma(s))}{\mu g[L_3^{\frac{1}{m}} z(\gamma(s))]},\end{aligned}$$

where we have used the fact that $\alpha(\gamma(s)) \leq \gamma(s)$. By applying ($A_5$), we obtain

$$\begin{aligned}\int_s^{\gamma(s)} R(\theta)g[A(\gamma(\theta),\ \gamma(s))]\Delta\theta &\leq \frac{(1+g(a))L_3z(\gamma(s))}{\mu g[L_3^{\frac{1}{m}} z(\gamma(s))]} \\ &\leq \frac{(1+g(a))}{\mu M_2}\end{aligned}$$

contradicts our hypothesis (A_7).

Let case (b) holds. For $s > u \geq t_2$, we have

$$-L_2z(u) < L_2z(s) - L_2z(u) = \int_u^s L_3z(t)\Delta t,$$

that is,

$$L_2z(u) > \int_u^s -L_3z(t)\Delta t \geq (s-u)(-L_3z(s)).$$

Consequently,

$$z^{\Delta^2}(u) \geq (s-u)^{\frac{1}{m}} a^{\frac{1}{m}}(u)(-L_3^{\frac{1}{m}} z(s)). \tag{4.7}$$

Therefore, (4.5) can be viewed as

$$\begin{aligned}z(s) &> (-L_3^{\frac{1}{m}} z(s))\int_{t_2}^s (s-\sigma(t))(s-t)^{\frac{1}{m}} a^{\frac{1}{m}}(t)\Delta t \\ &= (-L_3^{\frac{1}{m}} z(s))B[s,\ t_2],\ for\ s > \sigma(t) > t > t_2.\end{aligned}$$

Letting $s = \beta(\theta)$, we obtain

$$z(\beta(\theta)) > (-L_3^{\frac{1}{m}} z(\beta(\theta)))B[\beta(\theta),\ t_2],\ for\ \beta(\theta) > \sigma(t) > t > t_2. \tag{4.8}$$

From (4.3), it follows that

$$L_4z(t) \geq q(t)f(y(\beta(t)))$$

and hence proceeding as in case (a), we obtain

$$L_4z(t) + f(a)L_4z(\alpha(t)) \geq \lambda Q(t)f(z(\beta(t))), \; for \; t \geq t_2$$

due to (A_1), (A_3), (A_4) and (A_9). Using (4.8), the above inequality yields that

$$L_4z(\theta) + f(a)L_4z(\alpha(\theta)) \geq \lambda Q(\theta)f[B(\beta(\theta), t_2)]f[-L_3^{\frac{1}{m}}z(\beta(\theta))].$$

Integrating the last inequality from $\beta(s)$ to $\alpha(s)$, we obtain

$$\begin{aligned}\lambda f[-L_3^{\frac{1}{m}}z(\beta(\alpha(s)))]\int_{\beta(s)}^{\alpha(s)} Q(\theta)f[B(\beta(\theta), t_2)]\Delta\theta &\leq -L_3z(\beta(s)) - f(a)L_3z(\alpha(\beta(s)))\\ &\leq (1 + f(a))(-L_3z(\alpha(\beta(s)))),\end{aligned}$$

due to $-L_3z$ is decreasing. With (A_5) and (A_9), the last inequality reduces to

$$\begin{aligned}\int_{\beta(s)}^{\alpha(s)} Q(\theta)f[B(\beta(\theta), t_2)]\Delta\theta &\leq \Big(\frac{1 + f(a)}{\lambda}\Big)\frac{-L_3z(\alpha(\beta(s)))}{f[-L_3^{\frac{1}{m}}z(\alpha(\beta(s)))]}\\ &\leq \frac{1 + f(a)}{\lambda M_1},\end{aligned}$$

which is a contradiction to (A_6).

Let case (c) holds. For $v > \sigma(t) > t > u \geq t_2$, it is easy to verify that

$$\begin{aligned}z(u) &= z(v) - (v - u)z^{\Delta}(v) + \int_u^v (\sigma(t) - u)z^{\Delta^2}(t)\Delta t\\ &> \int_u^v (\sigma(t) - u)z^{\Delta^2}(t)\Delta t.\end{aligned}$$

On the otherhand, we ensure that (4.7) holds and hence the above inequality becomes

$$\begin{aligned}z(u) &> \int_u^v (\sigma(t) - u)(v - t)^{\frac{1}{m}}a^{\frac{1}{m}}(t)(-L_3^{\frac{1}{m}}z(v))\Delta t\\ &\geq (-L_3^{\frac{1}{m}}z(v))\int_u^v (\sigma(t) - u)(v - t)^{\frac{1}{m}}a^{\frac{1}{m}}(t)\Delta t\\ &= (-L_3^{\frac{1}{m}}z(v))C[v, \; u], \; for \; v > \sigma(t) > t > u \geq t_2.\end{aligned}$$

Letting v and u by $\beta(v)$ and $\beta(\theta)$ respectively in the last inequality, we get

$$z(\beta(\theta)) > (-L_3^{\frac{1}{m}}z(\beta(v)))C[\beta(v), \; \beta(\theta)], \; for \; \beta(v) \geq s > \sigma(t) > t > \beta(\theta) \geq t_2.$$

Proceeding as in case (b), and integrating the resulting inequality from $\alpha^{-1}(\beta(v))$ to v, we obtain a contradiction to (A_8).

If $y(t) < 0$ for sufficiently large t on $[t_0,\ \infty)_{\mathbb{T}}$, then $-y(t)$ is also a solution of (E_5) due to (A_2). Hence the details are omitted. This completes the proof of the theorem. □

Example 4.1.3. On $\mathbb{T} = \mathbb{R}$, and consider the differential equation

$$(y(t) + 2y(t-\pi))'''' = \frac{1}{4}y(t-7\pi) + \frac{3}{4}y(t+3\pi), \tag{4.9}$$

where $m = 1,\ a(t) = 1,\ p(t) = 2,\ \alpha(t) = t - \pi,\ \beta(t) = t - 7\pi,\ \gamma(t) = t + 3\pi,\ q(t) = \frac{1}{4},\ and\ r(t) = \frac{3}{4}$. Clearly, all the conditions of Theorem 4.1.2 are satisfied for (4.9) when $\mathbb{T} = \mathbb{R}$. Hence (4.9) is oscillatory. Indeed, $y(t) = sint$ is an oscillatory solution of (4.9).

Theorem 4.1.4. *Let $-\infty < -b \le p(t) \le 0$, and $\beta(t) \le \alpha(t)$ for $t \in [t_0, \infty)_{\mathbb{T}}$, $b > 0$. If $(A_0) - (A_2)$, (A_5), $(A_9) - (A_{13})$ and*

$$\int_{t_0}^{\infty} q(t)\Delta t = \infty$$

hold, then every solution of (E_5) *oscillates.*

Proof. Proceeding as in the proof of Theorem 4.1.2, we get $L_4 z(t) > 0$ on $[t_1,\ \infty)_{\mathbb{T}}$. Hence $z(t)$ is monotonic, then we consider the cases when $z(t) > 0$ and $z(t) < 0$. Suppose there exists a $t_2 \in [t_1,\ \infty)_{\mathbb{T}}$ such that $z(t) > 0$, for $t \ge t_2$. By Lemma 4.1.1, one of the cases (a), (b), or (c) are hold. When case (a) holds, we have (4.6), for $\gamma(\theta) > \gamma(s) \ge t_2$. Since $z(t) > 0$, then $z(t) \le y(t)$ for $t \in [t_2,\ \infty)_{\mathbb{T}}$. Consequently, (4.3) becomes

$$\begin{aligned} L_4 z(\theta) &\ge r(\theta) g(y(\gamma(\theta))) \ge r(\theta) g(z(\gamma(\theta))) \\ &\ge r(\theta) g[A(\gamma(\theta),\ \gamma(s))] g[L_3^{\frac{1}{m}} z(\gamma(s))] \end{aligned}$$

due to (4.6), for $\theta > s \ge t_2$, $\gamma(\theta) > \gamma(s) \ge \gamma(t_2) \ge t_2$. Integrating the last inequality from s to $\gamma(s)$, we obtain a contradiction to (A_{10}) due to (A_5). The cases (b) and (c) can similarly be dealt as case (a), we obtain a contradiction to (A_{11}) and (A_{12}) respectively.

Next, we suppose that $z(t) < 0$, for $t \in [t_2,\ \infty)_{\mathbb{T}}$. Clearly, $z(t) \geq -by(\alpha(t))$, for $t \geq t_2$. This implies that there exists a $t_3 \in [t_2,\ \infty)_{\mathbb{T}}$ such that $y(t) \geq (-\frac{1}{b})z(\alpha^{-1}(t))$, for $t \in [t_3,\ \infty)_{\mathbb{T}}$ due to (A_9). Since $z(t) < 0$, then by Lemma 4.1.1, we have one of the cases (c), or (d) are hold. For case (c), we integrate (4.7) from u to s to obtain

$$-z^{\Delta}(u) \geq z^{\Delta}(s) - z^{\Delta}(u) \geq -L_3^{\frac{1}{m}} z(s) \int_u^s (s-t)^{\frac{1}{m}} a^{\frac{1}{m}}(t)\Delta t,$$

for $s > u > t_3$, that is,

$$\begin{aligned} -z(s) + z(u) &\geq -L_3^{\frac{1}{m}} z(s) \int_u^s \int_\theta^s (s-t)^{\frac{1}{m}} a^{\frac{1}{m}}(t)\Delta t\Delta\theta \\ &= -L_3^{\frac{1}{m}} z(s) \int_u^s (\sigma(\theta)-u)(s-\theta)^{\frac{1}{m}} a^{\frac{1}{m}}(\theta)\Delta\theta \\ &= -L_3^{\frac{1}{m}} z(s) C[s,\ u]. \end{aligned}$$

Therefore,

$$-z(s) \geq -L_3^{\frac{1}{m}} z(s) C[s,\ u],\ s > u > t_3. \tag{4.10}$$

Letting s and u by $\alpha^{-1}(\beta(v))$ and $\alpha^{-1}(\beta(s))$ respectively, the inequality (4.10) can be viewed as

$$-z(\alpha^{-1}(\beta(v))) \geq -L_3^{\frac{1}{m}} z(\alpha^{-1}(\beta(v)))C[\alpha^{-1}(\beta(v)),\ \alpha^{-1}(\beta(s))],$$

for $\alpha^{-1}(\beta(v)) > \alpha^{-1}(\beta(s)) > t_3$. From (4.3), it follows that

$$\begin{aligned} L_4 z(v) &\geq q(v) f(y(\beta(v))) \\ &\geq q(v) f\left(-\frac{1}{b} z(\alpha^{-1}(\beta(v)))\right). \end{aligned} \tag{4.11}$$

The inequality (4.11) reduces to

$$\begin{aligned} L_4 z(v) &\geq f\left(\frac{1}{b}\right) q(v) f(-z(\alpha^{-1}(\beta(v)))) \\ &\geq f\left(\frac{1}{b}\right) q(v) f(-L_3^{\frac{1}{m}} z(\alpha^{-1}(\beta(v)))) f[C(\alpha^{-1}(\beta(v)),\ \alpha^{-1}(\beta(s)))], \end{aligned}$$

for $\alpha^{-1}(\beta(v)) > \alpha^{-1}(\beta(s)) > t_3$. Integrating the last inequality from s to $\beta^{-1}(\alpha(s))$, we get a contradiction to (A_{13}), due to the fact that $L_3 z$ is increasing.

In case (d), we use the fact that $z(t)$ is nonincreasing for $t \geq t_3$. Hence there exists a constant $C > 0$ and $t_4 \geq t_3$ such that $z(t) \leq -C$, for $t \geq t_4$. As a result, (4.11) becomes

$$L_4 z(t) \geq q(t) f\Big(-\frac{1}{b} z(\alpha^{-1}(\beta(t)))\Big) \geq f(C/b) q(t),$$

for $t \geq t_4$ and on integration from t_4 to ∞, we get a contradiction due to the fact that $\int_{t_0}^{\infty} q(t)\Delta t = \infty$. This completes the proof of the theorem. □

Theorem 4.1.5. *Let* $0 \leq p(t) \leq a < \infty$ *and* $\beta(t) \leq \alpha(t)$, *for* $t \in [t_0, \infty)_{\mathbb{T}}$. *Assume that* $(A_0) - (A_4)$, (A_9) *and* (A_{14}) *hold. If*

(A_{15}) $\limsup_{s\to\infty} \int_s^{\gamma(s)} \int_s^{t_5} [a(t_4) \int_s^{t_4} \int_s^{t_3} R(t)\Delta t \Delta t_3]^{\frac{1}{m}} \Delta t_4 \Delta t_5 > \left(\frac{1+g(a)}{\mu}\right)^{\frac{1}{m}} \frac{1}{M_4}$,

for $t_5 > t_4 > t_3 > s \geq t_0$,

(A_{16}) $\limsup_{s\to\infty} B[\beta(s),\ t_2] \left(\int_s^{\infty} Q(t)\Delta t\right)^{\frac{1}{m}} > \frac{1}{M_3} \left(\frac{1+f(a)}{\lambda}\right)^{\frac{1}{m}}$,

and

(A_{17}) $\limsup_{s\to\infty} \int_{\alpha^{-1}(\beta(s))}^{s} \int_{t_2}^{s} [a(\alpha(t_3)) \int_{t_3}^{s} (\sigma(t_4) - t_3) Q(t_4)\Delta t_4]^{\frac{1}{m}} \Delta t_3 \Delta t_2 >$
$\left(\frac{1+f(a)}{\lambda}\right)^{\frac{1}{m}} \frac{1}{M_3}$, *where* $s > \sigma(t_4) > t_4 > t_3 > t_2 \geq t_0$

hold, then every solution of (E_5) *oscillates.*

Proof. On the contrary we proceed as in Theorem 4.1.2, we obtain $L_4 z(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Since $z(t) > 0$ then by Lemma 4.1.1, one of the cases (a), (b), or (c) holds. Assume that case (a) holds. Using (4.3) and we can find a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that

$$\begin{aligned} L_4 z(t) + g(a) L_4 z(\alpha(t)) &\geq r(t) g(y(\gamma(t))) + g(a) r(\alpha(t)) g(y(\gamma(\alpha(t)))) \\ &\geq R(t)[g(y(\gamma(t))) + g(a y(\alpha(\gamma(t))))] \\ &\geq \mu R(t) g(z(\gamma(t))),\ \ t \in [t_2, \infty)_{\mathbb{T}} \end{aligned}$$

holds due to (A_1), (A_3), (A_4) and (A_9). Integrating the last inequality from s to t_3, we get

$$\begin{aligned} L_3 z(t_3) + g(a) L_3 z(\alpha(t_3)) &\geq \mu \int_s^{t_3} R(t) g(z(\gamma(t)))\Delta t \\ &\geq \mu g(z(\gamma(s))) \int_s^{t_3} R(t)\Delta t,\ \ t_3 > t_2. \end{aligned}$$

On further integration to the above inequality, we obtain

$$L_2 z(t_4) + g(a) L_2 z(\alpha(t_4)) \geq \mu g(z(\gamma(s))) \int_s^{t_4} \int_s^{t_3} R(t) \Delta t \Delta t_3,$$

for $t_4 > t_3 > s \geq t_2$, that is,

$$(1 + g(a)) L_2 z(t_4) \geq \mu g(z(\gamma(s))) \int_s^{t_4} \int_s^{t_3} R(t) \Delta t \Delta t_3.$$

Consequently, for $t_4 > t_3 > s \geq t_2$,

$$z^{\Delta^2}(t_4) \geq \left(\frac{\mu}{1 + g(a)} \right)^{\frac{1}{m}} g^{\frac{1}{m}}(z(\gamma(s))) \Big[a(t_4) \int_s^{t_4} \int_s^{t_3} R(t) \Delta t \Delta t_3 \Big]^{\frac{1}{m}}.$$

Integrating the last inequality from s to $t_5 (t_5 > t_4)$, we get

$$z^{\Delta}(t_5) \geq \left(\frac{\mu}{1 + g(a)} \right)^{\frac{1}{m}} g^{\frac{1}{m}}(z(\gamma(s))) \int_s^{t_5} \Big[a(t_4) \int_s^{t_4} \int_s^{t_3} R(t) \Delta t \Delta t_3 \Big]^{\frac{1}{m}} \Delta t_4,$$

for $t_5 > t_4 > t_3 > t_2$. Hence,

$$\begin{aligned} &\int_s^{\gamma(s)} z^{\Delta}(t_5) \Delta t_5 \\ &\geq \left(\frac{\mu}{1 + g(a)} \right)^{\frac{1}{m}} g^{\frac{1}{m}}(z(\gamma(s))) \int_s^{\gamma(s)} \int_s^{t_5} \Big[a(t_4) \int_s^{t_4} \int_s^{t_3} R(t) \Delta t \Delta t_3 \Big]^{1/m} \Delta t_4 \Delta t_5, \end{aligned}$$

that is,

$$z(\gamma(s)) \geq \left(\frac{\mu}{1 + g(a)} \right)^{\frac{1}{m}} g^{\frac{1}{m}}(z(\gamma(s))) \int_s^{\gamma(s)} \int_s^{t_5} \Big[a(t_4) \int_s^{t_4} \int_s^{t_3} R(t) \Delta t \Delta t_3 \Big]^{\frac{1}{m}} \Delta t_4 \Delta t_5$$

implies that

$$\frac{z(\gamma(s))}{g^{\frac{1}{m}}(z(\gamma(s)))} \geq \left(\frac{\mu}{1 + g(a)} \right)^{\frac{1}{m}} \int_s^{\gamma(s)} \int_s^{t_5} \Big[a(t_4) \int_s^{t_4} \int_s^{t_3} R(t) \Delta t \Delta t_3 \Big]^{\frac{1}{m}} \Delta t_4 \Delta t_5.$$

Applying (A_{14}) to the last inequality, we get a contradiction to (A_{15}).

Let case (b) hold. Using (4.3), it is easy to verify that

$$\begin{aligned} L_4 z(t) + f(a) L_4 z(\alpha(t)) &\geq q(t) f(y(\beta(t))) + f(a) q(\alpha(t)) f(y(\beta(\alpha(t)))) \\ &\geq \lambda Q(t) f(z(\beta(t))), \ for \ t \in [t_2, \infty)_{\mathbb{T}}. \end{aligned}$$

Integrating the above inequality from s to ∞, we get

$$-L_3 z(s) - f(a) L_3 z(\alpha(s)) \geq \lambda f(z(\beta(s))) \int_s^{\infty} Q(t) \Delta t, s > t_2$$

that is,

$$-(1+f(a))L_3z(\beta(s)) \geq \lambda f(z(\beta(s))) \int_s^{\infty} Q(t)\Delta t.$$

Consequently, (4.8) becomes

$$\begin{aligned} z(\beta(s)) &\geq -L_3^{\frac{1}{m}} z(\beta(s)) B[\beta(s),\ t_2] \\ &\geq B[\beta(s),\ t_2] \left(\frac{\lambda}{1+f(a)}\right)^{\frac{1}{m}} f^{\frac{1}{m}}(z(\beta(s))) \left(\int_s^{\infty} Q(t)\Delta t\right)^{\frac{1}{m}}, \end{aligned}$$

that is,

$$\frac{z(\beta(s))}{f^{\frac{1}{m}}(z(\beta(s)))} \geq \left(\frac{\lambda}{1+f(a)}\right)^{\frac{1}{m}} B[\beta(s),\ t_2] \left(\int_s^{\infty} Q(t)\Delta t\right)^{\frac{1}{m}}. \tag{4.12}$$

Applying (A_{14}) to the inequality (4.12), it yields a contradiction to (A_{16}).

Finally, we consider case (c). Proceeding as in case (b), we obtain

$$(1+f(a))(-L_3z(\alpha(t_4))) \geq \lambda f(z(\beta(s))) \int_{t_4}^{s} Q(t)\Delta t,\ s > t_4 > t_2.$$

Thus for $s > \sigma(t_4) > t_4 > t_3 > t_2$,

$$\begin{aligned} (1+f(a)) \int_{t_3}^{s} (-L_3z(\alpha(t_4)))\Delta t_4 &\geq \lambda f(z(\beta(s))) \int_{t_3}^{s} \int_{t_4}^{s} Q(t)\Delta t \Delta t_4 \\ &= \lambda f(z(\beta(s))) \int_{t_3}^{s} (\sigma(t_4) - t_3) Q(t_4)\Delta t_4, \end{aligned}$$

that is,

$$(1+f(a))L_2z(\alpha(t_3)) \geq \lambda f(z(\beta(s))) \int_{t_3}^{s} (\sigma(t_4) - t_3) Q(t_4)\Delta t_4,$$

that is,

$$L_1^{\Delta} z(\alpha(t_3)) \geq \left(\frac{\lambda}{1+f(a)}\right)^{\frac{1}{m}} f^{\frac{1}{m}}(z(\beta(s))) \Big[a(\alpha(t_3)) \int_{t_3}^{s} (\sigma(t_4) - t_3) Q(t_4)\Delta t_4\Big]^{\frac{1}{m}}.$$

Integrating the last inequality from t_2 to s, we get

$$-z^{\Delta}(\alpha(t_2)) \geq \left(\frac{\lambda}{1+f(a)}\right)^{\frac{1}{m}} f^{\frac{1}{m}}(z(\beta(s))) \int_{t_2}^{s} \Big[a(\alpha(t_3)) \int_{t_3}^{s} (\sigma(t_4) - t_3) Q(t_4)\Delta t_4\Big]^{\frac{1}{m}} \Delta t_3$$

and hence further integration from $\alpha^{-1}(\beta(s))$ to s, we get a contradiction to (A_{17}), due to (A_{14}). Thus the theorem is proved. □

Example 4.1.6. On $\mathbb{T} = h\mathbb{Z}$, (where $h(< 1)$ is a quotient of odd positive integers) and consider the h-difference equation

$$\Delta_h^2\Big(e^{-t}\Delta_h^2\big(y(t)+e^{2h}y(t-2h)\big)\Big) = \frac{1}{h^2}\left(\frac{e^h+1}{h}\right)^2\Big(e^{-t+8h}y(t-8h)+7e^{-t-6h}y(t+6h)\Big), \quad (4.13)$$

for $t(\geq 8h) \in \mathbb{T}$, where $m = 1$, $a(t) = e^t$, $p(t) = e^{2h}$, $\alpha(t) = t - 2h$, $\beta(t) = t - 8h$, $\gamma(t) = t + 6h$, $q(t) = \frac{1}{h^2}\left(\frac{e^h+1}{h}\right)^2 e^{-t+8h}$, $r(t) = \frac{7}{h^2}\left(\frac{e^h+1}{h}\right)^2 e^{-t-6h}$, $f(u) = u$ and $g(u) = u$. Clearly, all the conditions of Theorem 4.1.5 are satisfied. Hence, (4.13) is oscillatory. Indeed, $y(t) = (-1)^t e^t$ is an oscillatory solution of (4.13).

Theorem 4.1.7. *Let $-\infty < -b \leq p(t) \leq 0$, for $t \in [t_0,\infty)_{\mathbb{T}}$, $b > 0$. Assume that (A_0), (A_2), (A_9), and (A_{14}) hold. If*

(A_{18}) $\limsup_{s\to\infty}\int_s^{\gamma(s)}\int_s^{t_6}[a(t_5)\int_s^{t_5}\int_s^{t_4} r(t)\Delta t\Delta t_4]^{\frac{1}{m}}\Delta t_5\Delta t_6 > \frac{1}{M_4}$,

where $t_6 > t_5 > t_4 > s \geq t_0$,

(A_{19}) $\limsup_{t\to\infty}[\int_t^\infty q(s)\Delta s]^{1/m}B[\beta(t), t_3] > \frac{1}{M_3}$,

(A_{20}) $\limsup_{s\to\infty}\int_{\beta(s)}^s\int_{t_3}^s[a(t_4)\int_{t_4}^s(\sigma(t_5)-t_4)q(t_5)\Delta t_5]^{\frac{1}{m}}\Delta t_4\Delta t_3 > \frac{1}{M_3}$,

where $s > \sigma(t_5) > t_5 > t_4 > t_3 \geq t_0$, and

(A_{21}) $\int_{t_0}^\infty q(t)\Delta t = \infty$

hold, then every solution of (E_5) *oscillates.*

Proof. Proceeding as in the proof of Theorem 4.1.4, we consider two cases, that is, $z(t) > 0$ and $z(t) < 0$. Suppose there exists a $t_3 \in [t_2,\ \infty)_{\mathbb{T}}$ such that $z(t) > 0$, for $t \geq t_3$. By Lemma 4.1.1, one of the cases (a), (b), or (c) are hold. Clearly, $z(t) \leq y(t)$, for $t \in [t_3,\ \infty)_{\mathbb{T}}$. Consider case (a). From (4.3), it follows that

$$L_4 z(t) \geq r(t)g(y(\gamma(t))) \geq r(t)g(z(\gamma(t))),$$

for $t \geq t_3$. Integrating the above inequality from s to t_4 $(t_4 > t_3)$, we obtain

$$L_3 z(t_4) \geq g(z(\gamma(s)))\int_s^{t_4} r(t)\Delta t,$$

that is,

$$\int_s^{t_5} L_3 z(t_4)\Delta t_4 \geq g(z(\gamma(s)))\int_s^{t_5}\Big(\int_s^{t_4} r(t)\Delta t\Big)\Delta t_4,$$

where $t_5 > t_4$. Consequently, for $t_6 > t_5$

$$z^{\Delta^2}(t_5) \geq a^{\frac{1}{m}}(t_5) g^{\frac{1}{m}}(z(\gamma(s)))\Big[\int_s^{t_5}\int_s^{t_4} r(t)\Delta t\Delta t_4\Big]^{\frac{1}{m}}$$

and therefore,

$$\int_s^{t_6} z^{\Delta^2}(t_5)\Delta t_5 \geq g^{\frac{1}{m}}(z(\gamma(s)))\int_s^{t_6}\left[a(t_5)\int_s^{t_5}\int_s^{t_4} r(t)\Delta t\Delta t_4\right]^{\frac{1}{m}}\Delta t_5,$$

that is,

$$z^{\Delta}(t_6) \geq g^{\frac{1}{m}}(z(\gamma(s)))\int_s^{t_6}\left[a(t_5)\int_s^{t_5}\int_s^{t_4} r(t)\Delta t\Delta t_4\right]^{\frac{1}{m}}\Delta t_5.$$

Hence,

$$\int_s^{\gamma(s)} z^{\Delta}(t_6)\Delta t_6 \geq g^{\frac{1}{m}}(z(\gamma(s)))\int_s^{\gamma(s)}\int_s^{t_6}\left[a(t_5)\int_s^{t_5}\int_s^{t_4} r(t)\Delta t\Delta t_4\right]^{\frac{1}{m}}\Delta t_5\Delta t_6$$

implies that

$$z(\gamma(s)) \geq g^{\frac{1}{m}}(z(\gamma(s)))\int_s^{\gamma(s)}\int_s^{t_6}\left[a(t_5)\int_s^{t_5}\int_s^{t_4} r(t)\Delta t\Delta t_4\right]^{\frac{1}{m}}\Delta t_5\Delta t_6.$$

Applying (A_{14}), to the last inequality, we get a contradiction to (A_{18}).
Consider case (b). From (4.3), it follows that

$$L_4 z(t) \geq q(t) f(y(\beta(t))) \geq q(t) f(z(\beta(t))),$$

$t \in [t_3,\ \infty)_{\mathbb{T}}$. Integrating the above inequality from t to ∞,

$$-L_3 z(t) \geq f(z(\beta(t)))\int_t^{\infty} q(s)\Delta s,$$

this implies,

$$-L_3 z(\beta(t)) \geq -L_3 z(t) \geq f(z(\beta(t)))\int_t^{\infty} q(s)\Delta s, \tag{4.14}$$

since $L_3 z$ is increasing. Using (4.14) in (4.8), it follows that

$$\begin{aligned} z(\beta(t)) &\geq -L_3^{1/m} z(\beta(t)) B[\beta(t), t_2] \\ &\geq f^{1/m}(z(\beta(t))) B[\beta(t), t_2]\Big(\int_t^{\infty} q(s)\Delta s\Big)^{1/m}, \end{aligned}$$

which leads a contradiction to (A_{19}). Consider case (c), using (4.3) and for $t_3 < t_4 < t_5 < s$, we obtain

$$-L_3 z(t_5) \geq \int_{t_5}^{s} q(\theta) f(y(\beta(\theta)))\Delta\theta \geq \int_{t_5}^{s} q(\theta) f(z(\beta(\theta)))\Delta t_5$$
$$\geq f(z(\beta(s))) \int_{t_5}^{s} q(\theta)\Delta\theta,$$

that is,

$$L_2 z(t_4) \geq \quad f(z(\beta(s))) \textstyle\int_{t_4}^{s}\int_{t_5}^{s} q(\theta)\Delta\theta\Delta t_5,$$

that is,

$$-L_1 z(t_3) \geq \quad f^{1/m}(z(\beta(s))) \textstyle\int_{t_3}^{s}[a(t_4)\int_{t_4}^{s}(\sigma(t_5)-t_4)q(t_5)\Delta t_5]^{1/m}\Delta t_4.$$

Integrating the last inequality from $\beta(s)$ to s, which leads a contradiction to (A_{20}).

Next, we suppose that $z(t) < 0$, for $t \in [t_3, \infty)_{\mathbb{T}}$. Then by Lemma 4.1.1, one of the *cases* (c) or (d) are hold. Using the same type of reasoning as in Theorem 4.1.4, we can find $t_4 \in [t_3, \infty)_{\mathbb{T}}$ such that $y(\beta(t)) \geq (-\frac{1}{b})z(\alpha^{-1}(\beta(t)))$ for $t \in [t_4, \infty)_{\mathbb{T}}$ due to (A_9). Then (4.3) reduces to

$$L_4 z(t) > q(t) f(y(\beta(t))) \geq q(t) f(-\frac{1}{b}z(\alpha^{-1}(\beta(t)))) \geq q(t) f(-\frac{1}{b}z(\beta(t))),$$

for $t \geq t_4$. For the *cases* (c) and (d), by integrating the last inequality from t_4 to ∞, we obtain

$$f(-\frac{1}{b}z(\beta(t_4))) \int_{t_4}^{\infty} q(t)\Delta t < -L_3 z(t_4),$$

which yields a contradiction to (A_{21}). This completes the proof of the theorem. □

Example 4.1.8. On $\mathbb{T} = \mathbb{Z}$, consider the difference equation

$$\Delta^2\Big(\big(\Delta^2(y(n) - ey(n-1))\big)^3\Big)$$
$$= \big(e+1\big)^6\big(e^3+1\big)^2\big(e^{18}y^3(n-6) + 7e^{-12}y^3(n+4)\big), \quad (4.15)$$

for $n(\geq 6) \in \mathbb{T}$, where $m = 3$, $a(n) = 1$, $p(n) = -e$, $\alpha(n) = n-1$, $\beta(n) = n-6$, $\gamma(n) = n+4$, $q(n) = \big(e+1\big)^6\big(e^3+1\big)^2 e^{18}$, $r(n) = 7(e+1)^6(e^3+1)^2 e^{-12}$, $f(u) = u^3$ and $g(u) = u^3$. Clearly, all the conditions of Theorem 4.1.7 are satisfied. Hence, (4.15) is oscillatory. Indeed, $y(n) = (-1)^n e^n$ is an oscillatory solution of (4.15).

Theorem 4.1.9. *Let $-\infty < -b \leq p(t) \leq 0,\ b > 0,\ $ for $t \in [t_0, \infty)_{\mathbb{T}}$. Assume that (A_0)-(A_2), (A_9), and (A_{21}) hold. For $\psi : \mathbb{T} \to \mathbb{T}$ is bijection such that $\psi(t) \leq t$, ψ is increasing, if*

(i) $x^{\Delta}(t) - r(t)g[A(\gamma(t), \gamma(\psi(t)))]g(x^{\frac{1}{m}}(\gamma(\psi(t)))) = 0,$

(ii) $u^{\Delta}(t) + q(t)f[B(\beta(t), t_2)]f(u^{\frac{1}{m}}(\beta(t))) = 0,\ for\ every\ \ t_2 > t_0,$

(iii) $v^{\Delta}(t) + q(t)f[C(\psi^{-1}(\beta(t)), \beta(t))]f(v^{\frac{1}{m}}(\psi^{-1}(\beta(t)))) = 0,$

(iv) $w^{\Delta}(t) - f(\frac{1}{b})q(t)f[C(\alpha^{-1}(\beta(t)), \alpha^{-1}(\psi(\beta(t))))]f(-w^{\frac{1}{m}}(\alpha^{-1}(\beta(t)))) = 0$

are oscillatory, then every solution of (E_5) *oscillates.*

Proof. If possible, let $y(t)$ be a nonoscillatory solution of (E_5) on $[t_0, \infty)_{\mathbb{T}}$. Using the same type of reasoning as in the proof of Theorem 4.1.4, we have three cases (a), (b) and (c) for $z(t) > 0$ and the cases (c) and (d) for $z(t) < 0$ on $[t_2, \infty)_{\mathbb{T}}$. Let $z(t) > 0$ for $t \geq t_2$. Consider case (a). Following to the proof of Theorem 4.1.2, we obtain

$$z(\gamma(t)) \geq A[\gamma(t), \gamma(\psi(t))]L_3^{\frac{1}{m}} z(\gamma(\psi(t))),$$

for $\gamma(t) > \gamma(\psi(t)) \geq t_2$. Since $L_4 z(t) \geq r(t)g(y(\gamma(t)))$, then $z(t) \leq y(t)$ implies that

$$L_4 z(t) \geq r(t)g[A(\gamma(t), \gamma(\psi(t)))]g(L_3^{\frac{1}{m}} z(\gamma(\psi(t)))), t \geq\ t_2$$

due to (A_1). As a result, $L_3 z(t)$ is an eventually positive solution of

$$x^{\Delta}(t) - r(t)g[A(\gamma(t), \gamma(\psi(t)))]g(x^{\frac{1}{m}}(\gamma(\psi(t)))) \geq 0,\ t \geq t_2,$$

a contradiction. The proof for the cases (b) and (c) are similar as in case (a).

Next, we suppose that $z(t) < 0$ on $[t_2, \infty)_{\mathbb{T}}$. It happens that, cases (c) and (d) are the required cases. *Case* (d) follows from Theorem 4.1.4. For case (c), by proceeding as in the proof of Theorem 4.1.4, and using (4.10) in (4.11), we obtain

$$L_4 z(t) \geq f\left(\frac{1}{b}\right) q(t) f(-L_3^{\frac{1}{m}} z(\alpha^{-1}(\beta(t)))) f[C(\alpha^{-1}(\beta(t)), \alpha^{-1}(\psi(\beta(t))))],$$

for $t \in [t_4, \infty)_{\mathbb{T}}$, that is, $L_3 z(t)$ is a negative solution of

$$w^{\Delta}(t) - f\left(\frac{1}{b}\right) q(t) f[C(\alpha^{-1}(\beta(t)), \alpha^{-1}(\psi(\beta(t))))] f(-w^{\frac{1}{m}}(\alpha^{-1}(\beta(t)))) \geq 0$$

on $[t_4, \infty)_{\mathbb{T}}$, a contradiction. Hence the theorem is proved. □

Theorem 4.1.10. *Let $0 \leq p(t) \leq a < 1$, for $t \in [t_0, \infty)_{\mathbb{T}}$. Assume that $(A_0) - (A_2)$ hold. For $\psi : \mathbb{T} \to \mathbb{T}$ such that $\psi(t) \leq t$ and ψ is increasing, if*

$$x^{\Delta}(t) - g(1-a)r(t)g[A(\gamma(t), \gamma(\psi(t)))]g(x^{\frac{1}{m}}(\gamma(\psi(t)))) = 0$$

and

$$u^{\Delta}(t) + f(1-a)q(t)f[B(\beta(t), t_2)]f(u^{\frac{1}{m}}(\beta(t))) = 0, \ for\ every\ t_2 > t_0$$

are oscillatory, then every unbounded solution of (E_5) *oscillates.*

Proof. Let $y(t)$ be an unbounded nonoscillatory solution of (E_5) on $[t_0, \infty)_{\mathbb{T}}$. Proceeding as in the proof of Theorem 4.1.2, we consider the three cases (a), (b) and (c). For each of the cases (a) and (b), $z(t)$ is nondecreasing on $[t_2, \infty)_{\mathbb{T}}$. So there exists a $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that

$$\begin{aligned}(1-a)z(t) \leq (1-p(t))z(t) &\leq z(t) - p(t)z(\alpha(t)) \\ &= y(t) + p(t)y(\alpha(t)) - p(t)y(\alpha(t)) - p(t)p(\alpha(t))y(\alpha(\alpha(t))) \\ &= y(t) - p(t)p(\alpha(t))y(\alpha(\alpha(t))) < y(t),\end{aligned}$$

for $t \geq t_3$. The rest of the proof follows from the proof of Theorems 4.1.2 and 4.1.9. For case (c), $z(t)$ happens to be bounded and hence a contradiction to our supposition. This completes the proof of the theorem. □

4.2 Oscillation results of (E_5) with $\int_{t_0}^{\infty} (a(t))^{\frac{1}{m}} \Delta t < \infty$.

This section deals with the oscillation criteria for (E_5) under the assumption

$$(A_0') \quad \int_{t_0}^{\infty} (a(t))^{\frac{1}{m}} \Delta t < \infty.$$

We need the following lemma for our work in the sequel.

Lemma 4.2.1. Suppose that (A_0') holds. Let u be a rd-continuously differentiable function on $[t_0, \infty)_{\mathbb{T}}$ such that $(\frac{1}{a(t)}(u^{\Delta^2}(t))^m) \in C_{rd}^{\Delta^2}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ and $(\frac{1}{a(t)}(u^{\Delta^2}(t))^m)^{\Delta^2} \geq$

$(\not\equiv)0$, for large $t \in [t_0, \infty)_{\mathbb{T}}$. If $u(t) > 0$ ultimately, then one of the cases (a), (b), (c), (d), or (e) holds for large t, and if $u(t) < 0$ ultimately, then one of the cases (b), (d), (e), or (f) holds for large t, where

$$(a)\ \ L_1u(t) > 0,\ L_2u(t) > 0,\ \textit{and}\ L_3u(t) > 0,$$

$$(b)\ \ L_1u(t) < 0,\ L_2u(t) > 0,\ \textit{and}\ L_3u(t) > 0,$$

$$(c)\ \ L_1u(t) > 0,\ L_2u(t) > 0,\ \textit{and}\ L_3u(t) < 0,$$

$$(d)\ \ L_1u(t) < 0,\ L_2u(t) > 0,\ \textit{and}\ L_3u(t) < 0,$$

$$(e)\ \ L_1u(t) > 0,\ L_2u(t) < 0,\ \textit{and}\ L_3u(t) < 0,$$

$$(f)\ \ L_1u(t) < 0,\ L_2u(t) < 0,\ \textit{and}\ L_3u(t) < 0.$$

Proof. The proof of this lemma is same as the proof of Lemma 4.1.1. Hence the details are omitted. □

Theorem 4.2.2. *Let* $0 \leq p(t) \leq a < \infty$ *and* $\alpha(t) \geq \beta(t)$, *for* $t \in [t_0,\ \infty)_{\mathbb{T}}$. *Assume that* (A_0') *and* $(A_1) - (A_9)$ *hold. Furthermore, assume that*
(A_{22}) $\limsup_{s\to\infty} \int_{\gamma^{-1}(s)}^{s} R(\theta)g[D(\gamma(s),\ \gamma(\theta))]\Delta\theta > \frac{1+g(a)}{\mu M_2}$,
and
(A_{23}) $\limsup_{s\to\infty} \int_{s}^{\beta^{-1}(\alpha(s))} Q(\theta)f[D(\beta(\theta), \beta(s))]\Delta\theta > \frac{1+f(a)}{\lambda M_1}$,
hold. Then every solution of (E_5) *oscillates.*

Proof. Suppose on the contrary that $y(t)$ is a non-oscillatory solution of (E_5) on $[t_0,\ \infty)_{\mathbb{T}}$. Without loss of generality, there exists a $t_1 \in [t^*, \infty)_{\mathbb{T}}$, sufficiently large such that $y(t)$, $y(\alpha(t))$, $y(\beta(t))$ and $y(\alpha(\beta(t)))$ are all positive for all on $[t_1, \infty)_{\mathbb{T}}$. From (4.3), it follows that $L_4z(t) > 0$, for $t \geq t_1$. Hence, we can find a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $L_iz(t), i = 1, 2, 3$ are eventually of one sign on $[t_2,\ \infty)_{\mathbb{T}}$. In what follows, we consider the first five cases of Lemma 4.2.1.
Consider case (b) hold. For $u > v \geq t_2$,

$$L_2z(u) - L_2z(v) = \int_v^u L_3z(s)\Delta s \geq (u-v)L_3z(v),$$

that is, $L_2z(u) \geq (u-v)L_3z(v)$. Hence,

$$z^{\Delta^2}(u) \geq a^{\frac{1}{m}}(u)(u-v)^{\frac{1}{m}}L_3^{\frac{1}{m}}z(v). \tag{4.16}$$

For $u > \sigma(t) > v \geq t_2$,

$$\int_\theta^u z^{\Delta^2}(t)\Delta t = z^\Delta(u) - z^\Delta(\theta).$$

Integrating the above equation from v to u, we get

$$\int_v^u \int_\theta^u z^{\Delta^2}(t)\Delta t\Delta\theta = z(v) - z(u) + z^\Delta(u)(u-v),$$

that is,

$$\int_v^u (\sigma(t) - v)z^{\Delta^2}(t)\Delta t = z(v) - z(u) + (u-v)z^\Delta(u). \tag{4.17}$$

Therefore,

$$\int_v^u (\sigma(t) - v)z^{\Delta^2}(t)\Delta t \leq z(v)$$

implies that

$$\begin{aligned} z(v) &\geq \int_v^u (\sigma(t) - v)a^{\frac{1}{m}}(t)(t-v)^{\frac{1}{m}} L_3^{\frac{1}{m}} z(v)\Delta t \\ &= L_3^{\frac{1}{m}} z(v) D[u, v] \end{aligned}$$

due to (4.16). Letting $v = \gamma(\theta)$ and $u = \gamma(s)$, we get

$$z(\gamma(\theta)) \geq L_3^{\frac{1}{m}} z(\gamma(\theta)) D[\gamma(s),\ \gamma(\theta)], \tag{4.18}$$

for $\gamma(s) > \gamma(\theta) \geq t_2$. From (4.3), it follows that

$$L_4 z(t) \geq r(t)g(y(\gamma(t))),$$

for $t \geq t_2$. Hence,

$$\begin{aligned} L_4 z(t) + g(a)L_4 z(\alpha(t)) &\geq r(t)g(y(\gamma(t))) + g(a)r(\alpha(t))g(y(\gamma(\alpha(t)))) \\ &\geq \mu R(t)g(y(\gamma(t)) + ay(\gamma(\alpha(t)))) \\ &= \mu R(t)g(y(\gamma(t)) + ay(\alpha(\gamma(t)))) \\ &\geq \mu R(t)g(z(\gamma(t))) \end{aligned} \tag{4.19}$$

due to (A_1), (A_3), (A_4), *and* (A_9), where we have used the fact that $z(\gamma(t)) \leq y(\gamma(t)) + ay(\alpha(\gamma(t)))$. Using (4.18) in (4.19), we obtain

$$\begin{aligned} L_4 z(\theta) + g(a) L_4 z(\alpha(\theta)) &\geq \mu R(\theta) g[D(\gamma(s),\ \gamma(\theta)) L_3^{\frac{1}{m}} z(\gamma(\theta))] \\ &\geq \mu R(\theta) g[D(\gamma(s),\ \gamma(\theta))] g[L_3^{\frac{1}{m}} z(\gamma(\theta))]. \end{aligned} \tag{4.20}$$

Integrating the inequality (4.20) from $\gamma^{-1}(s)$ to s, we obtain

$$\begin{aligned} g[L_3^{\frac{1}{m}} z(s)] \int_{\gamma^{-1}(s)}^{s} R(\theta) g[D(\gamma(s),\ \gamma(\theta))] \Delta\theta &\leq \frac{1}{\mu} \int_{\gamma^{-1}(s)}^{s} [L_4 z(\theta) + g(a) L_4 z(\alpha(\theta))] \Delta\theta \\ &\leq \frac{L_3 z(s) + g(a) L_3 z(\alpha(s))}{\mu} \\ &\leq \frac{L_3 z(s) + g(a) L_3 z(s)}{\mu}. \end{aligned}$$

As a result,

$$\begin{aligned} \int_{\gamma^{-1}(s)}^{s} R(\theta) g[D(\gamma(s),\ \gamma(\theta))] \Delta\theta &\leq \frac{(1+g(a)) L_3 z(s)}{\mu g[L_3^{\frac{1}{m}} z(s)]} \\ &\leq \frac{(1+g(a))}{\mu M_2} \end{aligned}$$

contradicts our hypothesis (A_{22}).

Consider case (e) hold. For $s > u \geq t_2$, we have

$$L_2 z(s) \leq L_2 z(s) - L_2 z(u) = \int_u^s L_3 z(t) \Delta t \leq (s-u) L_3 z(s),$$

that is,

$$-z^{\Delta^2}(s) \geq (s-u)^{\frac{1}{m}} a^{\frac{1}{m}}(s) (-L_3^{\frac{1}{m}} z(s)). \tag{4.21}$$

On the other hand,

$$z(u) = z(v) + (u-v) z^{\Delta}(u) - \textstyle\int_v^u (\sigma(t) - v) z^{\Delta^2}(t) \Delta t, \tag{4.22}$$

that is,

$$z(u) \geq -\int_v^u (\sigma(t) - v) z^{\Delta^2}(t) \Delta t, \quad u > v \geq t_2$$

implies that

$$\begin{aligned} z(u) &\geq (-L_3^{\frac{1}{m}} z(u)) \int_v^u (\sigma(t)-v)(t-v)^{\frac{1}{m}} a^{\frac{1}{m}}(t)\Delta t \\ &= (-L_3^{\frac{1}{m}} z(u)) D[u,\ v],\ for\ u > \sigma(t) > v > t_2. \end{aligned}$$

Letting $u = \beta(\theta)$ and $v = \beta(s)$, we obtain

$$z(\beta(\theta)) \geq (-L_3^{\frac{1}{m}} z(\beta(\theta))) D[\beta(\theta),\ \beta(s)], \tag{4.23}$$

for $\beta(\theta) > \sigma(t) > t > \beta(s) > t_2$. From (4.3), it follows that

$$L_4 z(t) \geq q(t) f(y(\beta(t)))$$

and hence proceeding as in case (b), we obtain

$$L_4 z(t) + f(a) L_4 z(\alpha(t)) \geq \lambda Q(t) f(z(\beta(t))),\ for\ t \geq t_2$$

due to (A_1), (A_3), (A_4) and (A_9). Using (4.23), the above inequality yields that

$$L_4 z(\theta) + f(a) L_4 z(\alpha(\theta)) \geq \lambda Q(\theta) f[D(\beta(\theta), \beta(s))] f[-L_3^{\frac{1}{m}} z(\beta(\theta))].$$

Integrating the above inequality from s to $\beta^{-1}(\alpha(s))$, we get

$$\begin{aligned} \lambda \int_s^{\beta^{-1}(\alpha(s))} & Q(\theta) f[D(\beta(\theta), \beta(s))] f[-L_3^{\frac{1}{m}} z(\beta(\theta))] \Delta\theta \\ & \leq \int_s^{\beta^{-1}(\alpha(s))} [L_4 z(\theta) + f(a) L_4 z(\alpha(\theta))] \Delta\theta, \end{aligned}$$

that is,

$$\lambda f[-L_3^{\frac{1}{m}} z(\alpha(s))] \int_s^{\beta^{-1}(\alpha(s))} Q(\theta) f(D[\beta(\theta), \beta(s)]) \Delta\theta \leq -L_3 z(s) - f(a) L_3 z(\alpha(s)),$$

using the fact that $L_3 z(t) < 0$. Consequently,

$$\lambda f[-L_3^{\frac{1}{m}} z(\alpha(s))] \int_s^{\beta^{-1}(\alpha(s))} Q(\theta) f(D[\beta(\theta), \beta(s)]) \Delta\theta \leq -(1+f(a)) L_3 z(\alpha(s))$$

implies that

$$\int_s^{\beta^{-1}(\alpha(s))} Q(\theta) f(D[\beta(\theta), \beta(s)]) \Delta\theta \leq -\frac{(1+f(a)) L_3 z(\alpha(s))}{\lambda f[-L_3^{\frac{1}{m}} z(\alpha(s))]} \leq \frac{1+f(a)}{\lambda M_1},$$

a contradiction to (A_{23}) due to (A_5).

The remaining cases of Lemma 4.2.1 are similar to the above two cases and also follows from the Theorem 4.1.2, when (A_0) holds.

If $y(t) < 0$ for sufficiently large t on $[t_0, \infty)_{\mathbb{T}}$, then $-y(t)$ is also a solution of (E_5) due to (A_2). Hence the details are omitted. This completes the proof of the theorem. □

Theorem 4.2.3. *Let* $-\infty < -b \leq p(t) \leq 0$ *and* $\alpha(t) \geq \beta(t)$, *for* $t \in [t_0, \infty)_{\mathbb{T}}$, $b > 0$. *Assume that* $(A_0'), (A_1), (A_2), (A_5)$ *and* $(A_9) - (A_{13})$ *hold. Furthermore, assume that*

(A_{24}) $\limsup_{s\to\infty} \int_{\gamma^{-1}(s)}^{s} r(\theta)g(D[\gamma(s),\ \gamma(\theta)])\Delta\theta > \frac{1}{M_2}$,

(A_{25}) $\limsup_{s\to\infty} \int_{s}^{\beta^{-1}(s)} q(\theta)f(D[\beta(\theta), \beta(s)])\Delta\theta > \frac{1}{M_1}$,

(A_{26}) $\limsup_{s\to\infty} \int_{s}^{\alpha^{-1}(\gamma(s))} r(\theta)g(D[\alpha^{-1}(\gamma(\theta)),\ \alpha^{-1}(\gamma(s))])\Delta\theta > \frac{1}{M_2 g(\frac{1}{b})}$,

(A_{27}) $\limsup_{s\to\infty} \int_{\alpha^{-1}(\beta(s))}^{s} q(\theta)f(D[\alpha^{-1}(\beta(s)),\ \alpha^{-1}(\beta(\theta))])\Delta\theta > \frac{1}{M_1 f(\frac{1}{b})}$,

and

(A_{28}) $\limsup_{s\to\infty} \int_{s}^{\beta^{-1}(\alpha(s))} q(\theta)f(A[\alpha^{-1}(\beta(\theta)),\ \alpha^{-1}(\beta(s))])\Delta\theta > \frac{1}{f(b^{-1})M_1}$,

hold. Then every solution of (E_5) *oscillates.*

Proof. Let $y(t)$ be a non-oscillatory solution of (E_5) such that $y(t) > 0$, for $t \geq t_0$. Proceeding as in the proof of Theorem 4.2.2, we consider Lemma 4.2.1, for $t \geq t_1$. Further, it follows that $z(t)$ is a monotonic function on $[t_2, \infty)_{\mathbb{T}}, t_1 > t_0$. Assume that $z(t) > 0$, for $t \geq t_2$. In what follows, we shall consider the first five cases of Lemma 4.2.1.

Using the same type of reasoning as in case (b) of Theorem 4.2.2, we get the inequality (4.18). From (4.3), it follows that $L_4 z(t) \geq r(t)g(y(\gamma(t)))$ and hence

$$\begin{aligned} L_4 z(\theta) &\geq r(\theta)g(z(\gamma(\theta))) \\ &\geq r(\theta)g[D(\gamma(s),\ \gamma(\theta))]g[L_3^{\frac{1}{m}} z(\gamma(\theta))] \end{aligned}$$

due to (4.18). Integrating the last inequality from $\gamma^{-1}(s)$ to s, we obtain

$$g[L_3^{\frac{1}{m}} z(s)] \int_{\gamma^{-1}(s)}^{s} r(\theta)g[D(\gamma(s),\ \gamma(\theta))]\Delta\theta \leq L_3 z(s).$$

Hence,

$$\int_{\gamma^{-1}(s)}^{s} r(\theta)g[D(\gamma(s),\ \gamma(\theta))]\Delta\theta \ \leq \frac{L_3 z(s)}{g[L_3^{\frac{1}{m}} z(s)]} \leq \frac{1}{M_2},$$

a contradiction to (A_{24}). Consider case (e), from (4.3) and (4.23), it follows that

$$\begin{aligned} L_4 z(t) &\geq q(t)f(y(\beta(t))) \\ &\geq q(t)f(z(\beta(t))) \\ &\geq q(t)f(-L_3^{1/m} z(\beta(t))D[\beta(t),\ \beta(s)]) \\ &\geq q(t)f(-L_3^{1/m} z(\beta(t)))f(D[\beta(t),\ \beta(s)]). \end{aligned} \tag{4.24}$$

Integrating the inequality (4.24) from s to $\beta^{-1}(s)$, we obtain

$$\int_s^{\beta^{-1}(s)} q(t)f(-L_3^{1/m} z(\beta(t)))f(D[\beta(t),\beta(s)])\Delta t \leq L_3 z(\beta^{-1}(s)) - L_3 z(s),$$

that is,

$$f(-L_3^{1/m} z(s)) \int_s^{\beta^{-1}(s)} q(t)f(D[\beta(t),\beta(s)])\Delta t \leq -L_3 z(s),$$

a contradiction to (A_{25}) due to (A_5). The remaining cases of Lemma 4.2.1, are similar to the above two cases and also follow from the Theorem 4.1.4 when (A_0) holds.

Next, we suppose that $z(t) < 0$, for $t \in [t_2,\ \infty)_{\mathbb{T}}$. Clearly, $z(t) \geq -by(\alpha(t))$, for $t \geq t_2$ implies that there exists a $t_3 \in [t_2,\ \infty)_{\mathbb{T}}$ such that $y(t) \geq (-\frac{1}{b})z(\alpha^{-1}(t))$, for $t \in [t_3,\ \infty)_{\mathbb{T}}$ due to (A_9). In this case, we consider cases (b), (d), (e), and (f) of Lemma 4.2.1.

Consider case (b), for $u \geq \sigma(t) \geq v > t_3$. Proceeding as in Theorem 4.2.2, from (4.16) and (4.17), we obtain

$$\begin{aligned} -z(u) &\geq \int_v^u (\sigma(t) - v) z^{\Delta^2}(t)\Delta t \\ &= L_3^{\frac{1}{m}} z(v)D[u, v]. \end{aligned}$$

Letting u and v by $\alpha^{-1}(\gamma(\theta))$ and $\alpha^{-1}(\gamma(s))$ respectively, the last inequality can be viewed as

$$-z(\alpha^{-1}(\gamma(\theta))) \geq L_3^{\frac{1}{m}} z(\alpha^{-1}(\gamma(s)))D[\alpha^{-1}(\gamma(\theta)),\ \alpha^{-1}(\gamma(s))],$$

for $\alpha^{-1}(\gamma(\theta)) > \alpha^{-1}(\gamma(s)) > t_4 \geq t_3$. From (4.3), it follows that

$$\begin{aligned} L_4 z(\theta) &\geq r(\theta) g(y(\gamma(\theta))) \\ &\geq r(\theta) g\bigg(-\frac{1}{b} z(\alpha^{-1}(\gamma(\theta)))\bigg) \\ &\geq g\bigg(\frac{1}{b}\bigg) r(\theta) g(-z(\alpha^{-1}(\gamma(\theta)))) \\ &\geq g\bigg(\frac{1}{b}\bigg) r(\theta) g(L_3^{\frac{1}{m}} z(\alpha^{-1}(\gamma(s)))) g(D[\alpha^{-1}(\gamma(\theta)),\ \alpha^{-1}(\gamma(s))]), \end{aligned} \tag{4.25}$$

for $\alpha^{-1}(\gamma(\theta)) > \alpha^{-1}(\gamma(s)) > t_4$. Integrating the inequality (4.25) from s to $\alpha^{-1}(\gamma(s))$, we get

$$\begin{aligned} g\bigg(\frac{1}{b}\bigg) g(L_3^{\frac{1}{m}} z(\alpha^{-1}(\gamma(s)))) \int_s^{\alpha^{-1}(\gamma(s))} & r(\theta) g(D[\alpha^{-1}(\gamma(\theta)), \alpha^{-1}(\gamma(s))]) \Delta\theta \\ &\leq L_3 z(\alpha^{-1}(\gamma(s))). \end{aligned}$$

Using (A_5) in the above inequality, we obtain a contradiction to (A_{26}).

For case (e), from (4.22), it follows that

$$-z(v) \geq -\int_v^u (\sigma(t) - v) z^{\Delta^2}(t) \Delta t,$$

for $u \geq v > t_2$, and because of (4.21), we obtain

$$-z(v) \geq -L_3^{\frac{1}{m}} z(u) D[u,\ v].$$

Substitute $v = \alpha^{-1}(\beta(\theta))$ and $u = \alpha^{-1}(\beta(s))$, we obtain

$$-z(\alpha^{-1}(\beta(\theta))) \geq -L_3^{\frac{1}{m}} z(\alpha^{-1}(\beta(s))) D[\alpha^{-1}(\beta(s)),\ \alpha^{-1}(\beta(\theta))],$$

for $\alpha^{-1}(\beta(s)) \geq \alpha^{-1}(\beta(\theta)) \geq t_3$. Using the fact that $L_4 z(\theta) \geq q(\theta) f(y(\beta(\theta))))$, we obtain

$$\begin{aligned} L_4 z(\theta) &\geq f\bigg(\frac{1}{b}\bigg) q(\theta) f(-z(\alpha^{-1}(\beta(\theta)))) \\ &\geq f\bigg(\frac{1}{b}\bigg) q(\theta) f(-L_3^{\frac{1}{m}} z(\alpha^{-1}(\beta(s))) D[\alpha^{-1}(\beta(s)),\ \alpha^{-1}(\beta(\theta))]) \\ &\geq f\bigg(\frac{1}{b}\bigg) q(\theta) f(-L_3^{\frac{1}{m}} z(\alpha^{-1}(\beta(s)))) f(D[\alpha^{-1}(\beta(s)),\ \alpha^{-1}(\beta(\theta))]) \end{aligned} \tag{4.26}$$

due to (A_1). Integrating the inequality (4.26) from $\alpha^{-1}(\beta(s))$ to s, it follows that

$$f\left(\frac{1}{b}\right)f(-L_3^{\frac{1}{m}}z(\alpha^{-1}(\beta(s))))\int_{\alpha^{-1}(\beta(s))}^{s} q(\theta)f(D[\alpha^{-1}(\beta(s)),\,\alpha^{-1}(\beta(\theta))])\Delta\theta$$
$$\leq -L_3 z(\alpha^{-1}(\beta(s))),$$

that is,

$$\int_{\alpha^{-1}(\beta(s))}^{s} q(\theta)f(D[\alpha^{-1}(\beta(s)),\ \alpha^{-1}(\beta(\theta))])\Delta\theta \leq \frac{-L_3 z(\alpha^{-1}(\beta(s)))}{f(\frac{1}{b})f(-L_3^{\frac{1}{m}}z(\alpha^{-1}(\beta(s))))} \leq \frac{1}{M_1 f(\frac{1}{b})},$$

a contradiction to (A_{27}).

In case (f), $L_i z(t) < 0$, $i = 0, 1, 2, 3$, for $t \geq t_3 > t_2$. For $u > v > t_3$,

$$L_2 z(u) \leq L_2 z(u) - L_2 z(v) = \int_v^u L_3 z(t)\Delta t \leq L_3 z(u)(u - v).$$

That is,

$$-z^{\Delta^2}(u) \geq -L_3^{1/m} z(u)(u - v)^{1/m} a^{1/m}(u).$$

For $s > \sigma(t) > t > v \geq t_3$,

$$-z^{\Delta}(\theta) + z^{\Delta}(v) = -\int_v^{\theta} z^{\Delta^2}(t)\Delta t. \tag{4.27}$$

Integrating the inequality (4.27) from v to s, we obtain

$$-z(s) + z(v) + z^{\Delta}(v)(s - v) = -\int_v^s \int_v^{\theta} z^{\Delta^2}(t)\Delta t\Delta\theta,$$

that is,

$$-z(s) \geq -\int_v^s (s - \sigma(t))z^{\Delta^2}(t)\Delta t.$$

Hence,

$$\begin{aligned} -z(s) &\geq \int_v^s (s - \sigma(t))(-L_3^{1/m} z(t))(t - v)^{1/m} a^{1/m}(t)\Delta t \\ &\geq -L_3^{1/m} z(s) \int_v^s (s - \sigma(t))(t - v)^{1/m} a^{1/m}(t)\Delta t \\ &= -L_3^{1/m} z(s) A[s, v]. \end{aligned} \tag{4.28}$$

Letting s and v by $\alpha^{-1}(\beta(\theta))$ and $\alpha^{-1}(\beta(s))$ respectively, the inequality (4.28) can be viewed as

$$-z(\alpha^{-1}(\beta(\theta))) \geq -L_3^{\frac{1}{m}} z(\alpha^{-1}(\beta(\theta)))A[\alpha^{-1}(\beta(\theta)),\ \alpha^{-1}(\beta(s))],$$

for $\alpha^{-1}(\beta(\theta)) > \alpha^{-1}(\beta(s)) > t_3$. From (4.3), it follows that

$$\begin{aligned} L_4 z(\theta) &\geq q(\theta) f(y(\beta(\theta))) \\ &\geq q(\theta) f\Big(-\frac{1}{b} z(\alpha^{-1}(\beta(\theta)))\Big) \\ &\geq f\Big(\frac{1}{b}\Big) q(\theta) f(-z(\alpha^{-1}(\beta(\theta)))) \\ &\geq f\Big(\frac{1}{b}\Big) q(\theta) f(-L_3^{\frac{1}{m}} z(\alpha^{-1}(\beta(\theta)))) f(A[\alpha^{-1}(\beta(\theta)),\ \alpha^{-1}(\beta(s))]), \end{aligned} \tag{4.29}$$

for $\alpha^{-1}(\beta(\theta)) > \alpha^{-1}(\beta(s)) > t_3$. Integrating the inequality (4.29) from s to $\beta^{-1}(\alpha(s))$, we get a contradiction to (A_{28}). Case (d) follows from the Theorem 4.1.4 when (A_0) holds. This completes the proof of the theorem. □

Example 4.2.4. On $\mathbb{T} = \mathbb{Z}$, consider the difference equation

$$\begin{aligned} \Delta^2\Big(n^2\big(\Delta^2(y(n) - y(n-1))\big)\Big) &= 8\big((n+2)^2 + n^2\big)y(n-6) \\ &\quad +16(n+1)^2 y(n+6), \end{aligned} \tag{4.30}$$

for $n(\geq 7) \in \mathbb{T}$, where $m = 1$, $a(n) = 1/n^2$, $p(n) = -1$, $\alpha(n) = n-1$, $\beta(n) = n-6$, $\gamma(n) = n+6$, $q(n) = 8\big((n+2)^2 + n^2\big)$, $r(n) = 16(n+1)^2$, $f(u) = u$ and $g(u) = u$. Clearly, all the conditions of Theorem 4.2.3 are satisfied. Hence, (4.30) is oscillatory. Indeed, $y(n) = (-1)^n$ is an oscillatory solution of (4.30).

Theorem 4.2.5. *Let $0 \leq p(t) \leq a < \infty$ and $\gamma(t) \geq \beta^{-3}(t)$, for all $t \in [t_0,\ \infty)_{\mathbb{T}}$. Assume that (A_0'), $(A_1) - (A_4)$, (A_9) and (A_{14}) hold. If*

(A_{29}) $\limsup_{s\to\infty} \left[(\beta^{-3}(s) - \beta^{-2}(s)) \left(\int_s^{\beta^{-1}(s)} \int_s^{t_5} R(t)\Delta t \Delta t_5\right)^{\frac{1}{m}} \int_{\beta^{-1}(s)}^{\beta^{-2}(s)} a^{\frac{1}{m}}(\theta)\Delta\theta\right] > \left(\frac{1+g(a)}{\mu M_4^m}\right)^{\frac{1}{m}},$

(A_{30}) $\limsup_{t\to\infty} \int_t^{\beta^{-3}(t)} \int_u^{\beta^{-1}(u)} \left(a(\gamma(s)) \int_{t_2}^s \int_{t_2}^v Q(\theta)\Delta\theta\Delta v\right)^{\frac{1}{m}} \Delta s \Delta u > \left(\frac{1+f(a)}{\lambda M_3^m}\right)^{\frac{1}{m}},$

(A_{31}) $\int_{t^*}^\infty R(t) g(A(\gamma(t)))\Delta t = +\infty,\ \ A(t) = \int_t^\infty (a(s))^{\frac{1}{m}} \Delta s,$

and

(A_{32}) $\limsup_{s\to\infty} \int_{\alpha^{-1}(\beta(s))}^{\alpha^{-1}(s)} \int_{u}^{\alpha^{-1}(u)} \left[a(\alpha(t_3)) \int_{t_3}^{s} (\sigma(t)-t_3)Q(t)\Delta t\right]^{\frac{1}{m}} \Delta t_3 \Delta u > \left(\frac{1+f(a)}{\lambda M_3^m}\right)^{\frac{1}{m}}$

hold, then every solution of (E_5) *oscillates.*

Proof. If possible, let $y(t)$ be a non-oscillatory solution of (E_5) such that $y(t) > 0$, for $t \geq t^* > t_0$. From (4.3), it follows that $L_3 z(t)$ is nondecreasing on $[t_1, \infty)_{\mathbb{T}}$, $t_1 > t_0$. Thus, we have five cases as in Theorem 4.2.2. Assume that the case (a) holds. We may note that

$$z(t) = z(\beta(t)) + \int_{\beta(t)}^{t} z^{\Delta}(s)\Delta s > (t-\beta(t))z^{\Delta}(\beta(t)) \tag{4.31}$$

and

$$z^{\Delta}(t) = z^{\Delta}(\beta(t)) + \int_{\beta(t)}^{t} z^{\Delta^2}(\theta)\Delta\theta > \int_{\beta(t)}^{t} z^{\Delta^2}(\theta)\Delta\theta, \tag{4.32}$$

for $t \geq t_2 > t_1$. As a result, (4.32) reduces to

$$z^{\Delta}(t) > L_2^{\frac{1}{m}} z(\beta(t)) \int_{\beta(t)}^{t} a^{\frac{1}{m}}(\theta)\Delta\theta. \tag{4.33}$$

Taking (4.31) and (4.33) into account, we can find $t_3 > t_2$ such that

$$z(t) > (t-\beta(t))L_2^{\frac{1}{m}} z(\beta^2(t)) \int_{\beta^2(t)}^{\beta(t)} a^{\frac{1}{m}}(\theta)\Delta\theta,$$

for $t \geq t_3$, that is,

$$L_2^{\frac{1}{m}} z(t) < \left((\beta^{-2}(t)-\beta^{-1}(t)) \int_{t}^{\beta^{-1}(t)} a^{\frac{1}{m}}(\theta)\Delta\theta\right)^{-1} z(\beta^{-2}(t)), \tag{4.34}$$

for $t \geq t_4 > t_3$. Integrating the following inequality

$$L_4 z(t) + g(a)L_4 z(\alpha(t)) \geq \mu R(t) g(z(\gamma(t))) \tag{4.35}$$

from s to $t_5 (> t_4)$, we get

$$\begin{aligned} L_3 z(t_5) + g(a)L_3 z(\alpha(t_5)) &\geq \mu \int_{s}^{t_5} R(t) g(z(\gamma(t)))\Delta t \\ &\geq \mu g(z(\gamma(s))) \int_{s}^{t_5} R(t)\Delta t, \end{aligned} \tag{4.36}$$

$t_5 \geq s \geq t_4$ due to nondecreasing $z(t)$. Further integration of the inequality (4.36) from s to $\beta^{-1}(s)$, we obtain

$$\int_s^{\beta^{-1}(s)} [L_3 z(t_5) + g(a) L_3 z(\alpha(t_5))] \Delta t_5 \geq \mu \int_s^{\beta^{-1}(s)} g(z(\gamma(s))) \int_s^{t_5} R(t) \Delta t \Delta t_5,$$

that is,

$$L_2 z(\beta^{-1}(s)) + g(a) L_2 z(\alpha(\beta^{-1}(s))) \geq \mu g(z(\gamma(s))) \int_s^{\beta^{-1}(s)} \int_s^{t_5} R(t) \Delta t \Delta t_5$$

which implies that

$$(1 + g(a)) L_2 z(\beta^{-1}(s)) \geq \mu g(z(\gamma(s))) \int_s^{\beta^{-1}(s)} \int_s^{t_5} R(t) \Delta t \Delta t_5.$$

Hence,

$$\begin{aligned} & L_2^{\frac{1}{m}} z(\beta^{-1}(s)) \\ & \geq \left(\frac{\mu}{1 + g(a)} \right)^{\frac{1}{m}} g^{\frac{1}{m}}(z(\gamma(s))) \left(\int_s^{\beta^{-1}(s)} \int_s^{t_5} R(t) \Delta t \Delta t_5 \right)^{\frac{1}{m}}. \end{aligned} \tag{4.37}$$

Upon using (4.34) in (4.37), we get

$$\begin{aligned} & \left((\beta^{-3}(s) - \beta^{-2}(s)) \int_{\beta^{-1}(s)}^{\beta^{-2}(s)} a^{\frac{1}{m}}(\theta) \Delta \theta \right)^{-1} z(\beta^{-3}(s)) \\ & \geq \left(\frac{\mu}{(1 + g(a))} \right)^{\frac{1}{m}} g^{\frac{1}{m}}(z(\gamma(s))) \left(\int_s^{\beta^{-1}(s)} \int_s^{t_5} R(t) \Delta t \Delta t_5 \right)^{\frac{1}{m}}, \end{aligned}$$

that is,

$$\begin{aligned} & \frac{z(\beta^{-3}(s))}{g^{\frac{1}{m}}(z(\beta^{-3}(s)))} \\ & \quad \geq (\beta^{-3}(s) - \beta^{-2}(s)) \left(\frac{\mu}{(1 + g(a))} \int_s^{\beta^{-1}(s)} \int_s^{t_5} R(t) \Delta t \Delta t_5 \right)^{\frac{1}{m}} \int_{\beta^{-1}(s)}^{\beta^{-2}(s)} a^{\frac{1}{m}}(\theta) \Delta \theta. \end{aligned}$$

Applying (A_{14}) to the above inequality, it follows that

$$\begin{aligned} & \limsup_{s \to \infty} \left[(\beta^{-3}(s) - \beta^{-2}(s)) \left(\int_s^{\beta^{-1}(s)} \int_s^{t_5} R(t) \Delta t \Delta t_5 \right)^{\frac{1}{m}} \int_{\beta^{-1}(s)}^{\beta^{-2}(s)} a^{\frac{1}{m}}(\theta) \Delta \theta \right] \\ & \qquad \leq \left(\frac{1 + g(a)}{\mu M_4^m} \right)^{\frac{1}{m}}, \end{aligned}$$

a contradiction to our assumption (A_{29}).

Let case (b) hold. Using (A_1), (A_3), (A_4) and (A_9) in (4.3), it is easy to verify that

$$L_4 z(t) + f(a) L_4 z(\alpha(t)) \geq \lambda Q(t) f(z(\beta(t))), \tag{4.38}$$

for $t \geq t_2 > t_1$. Integrating (4.38), twice from t_2 to t and then using the fact that $z(t)$ is nonincreasing, we obtain

$$(1+f(a)) L_2 z(t) \geq L_2 z(t) + f(a) L_2 z(\alpha(t)) \geq \lambda f(z(\beta(t))) \int_{t_2}^{t} \int_{t_2}^{v} Q(\theta) \Delta\theta \Delta v,$$

that is,

$$(1+f(a)) L_2 z(\gamma(t)) \geq (1+f(a)) L_2 z(t) \geq \lambda f(z(\beta(t))) \int_{t_2}^{t} \int_{t_2}^{v} Q(\theta) \Delta\theta \Delta v,$$

because $L_2 z$ is increasing. The preceeding inequality reduces to,

$$z^{\Delta^2}(\gamma(t)) \geq \left(\frac{\lambda}{1+f(a)} \right)^{\frac{1}{m}} f^{\frac{1}{m}}(z(\beta(t))) \left(a(\gamma(t)) \int_{t_2}^{t} \int_{t_2}^{v} Q(\theta) \Delta\theta \Delta v \right)^{\frac{1}{m}}. \tag{4.39}$$

Integrating (4.39), from t to $\beta^{-1}(t)$, we get

$$-z^{\Delta}(\gamma(t)) \geq \left(\frac{\lambda}{1+f(a)} \right)^{\frac{1}{m}} f^{\frac{1}{m}}(z(t)) \int_{t}^{\beta^{-1}(t)} \left(a(\gamma(s)) \int_{t_2}^{s} \int_{t_2}^{v} Q(\theta) \Delta\theta \Delta v \right)^{\frac{1}{m}} \Delta s.$$

Finally, we integrate the above inequality from t to $\beta^{-3}(t)$, to obtain

$$z(\gamma(t)) \geq \left(\frac{\lambda f(z(\beta^{-3}(t)))}{(1+f(a))} \right)^{\frac{1}{m}} \int_{t}^{\beta^{-3}(t)} \int_{u}^{\beta^{-1}(u)} \left(a(\gamma(s)) \int_{t_2}^{s} \int_{t_2}^{v} Q(\theta) \Delta\theta \Delta v \right)^{\frac{1}{m}} \Delta s \Delta u.$$

Consequently,

$$\int_{t}^{\beta^{-3}(t)} \int_{u}^{\beta^{-1}(u)} \left(a(\gamma(s)) \int_{t_2}^{s} \int_{t_2}^{v} Q(\theta) \Delta\theta \Delta v \right)^{\frac{1}{m}} \Delta s \Delta u \leq \left(\frac{1+f(a)}{\lambda M_3^m} \right)^{\frac{1}{m}}$$

which contradicts (A_{30}).

In cases (c) and (e), $z(t)$ is nondecreasing on $[t_1, \infty)_{\mathbb{T}}$. Hence, there exists $C > 0$ and $t_2 > t_1$ such that $z(t) \geq C$, for $t \geq t_2$. Further, if we define $A(t) = \int_t^{\infty} (a(s))^{\frac{1}{m}} \Delta s$, then $A(0) < \infty$, $A(\infty) = 0$ and hence we can find $C_1 > 0$ and $t_3 > t_2$ such that $z(t) \geq C_1 A(t)$, for $t \geq t_3$. Therefore, (4.35) becomes

$$L_4 z(t) + g(a) L_4 z(\alpha(t)) \geq \mu g(C_1) R(t) g(A(\gamma(t))), \tag{4.40}$$

for $t \geq t_3$. Integrating (4.40) from t_3 to $+\infty$, we obtain a contradiction to (A_{31}). Finally, we consider case (d). Integrating (4.38) from t_2 to s and then, it follows that

$$(1+f(a))(-L_3 z(\alpha(t_2))) \geq \lambda f(z(\beta(s))) \int_{t_2}^{s} Q(t)\Delta t.$$

Thus, for $s > \sigma(t_2) > t_2 > t_3$,

$$\begin{aligned}(1+f(a)) \int_{t_3}^{s} (-L_3 z(\alpha(t_2)))\Delta t_2 &\geq \lambda f(z(\beta(s))) \int_{t_3}^{s}\int_{t_2}^{s} Q(t)\Delta t \Delta t_2 \\ &= \lambda f(z(\beta(s))) \int_{t_3}^{s} (\sigma(t_2)-t_3)Q(t_2)\Delta t_2,\end{aligned}$$

that is,

$$\begin{aligned}&(1+f(a))^{\frac{1}{m}} z^{\Delta^2}(\alpha(t_3)) \\ &\qquad \geq (\lambda f(z(\beta(s))))^{\frac{1}{m}} \left[a(\alpha(t_3)) \int_{t_3}^{s} (\sigma(t_2)-t_3)Q(t_2)\Delta t_2\right]^{\frac{1}{m}}. \qquad (4.41)\end{aligned}$$

Integrating (4.41), from u to $\alpha^{-1}(u)$, we get

$$-z^{\Delta}(\alpha(u)) \geq \left(\frac{\lambda f(z(\beta(s)))}{1+f(a)}\right)^{\frac{1}{m}} \int_{u}^{\alpha^{-1}(u)} \left[a(\alpha(t_3)) \int_{t_3}^{s} (\sigma(t_2)-t_3)Q(t_2)\Delta t_2\right]^{\frac{1}{m}} \Delta t_3.$$

Further integration of the above inequality from $\alpha^{-1}(\beta(s))$ to $\alpha^{-1}(s)$, we obtain

$$\begin{aligned}&z(\beta(s)) \\ &\geq \left(\frac{\lambda f(z(\beta(s)))}{1+f(a)}\right)^{\frac{1}{m}} \int_{\alpha^{-1}(\beta(s))}^{\alpha^{-1}(s)} \int_{u}^{\alpha^{-1}(u)} \left[a(\alpha(t_3)) \int_{t_3}^{s} (\sigma(t_2)-t_3)Q(t_2)\Delta t_2\right]^{\frac{1}{m}} \Delta t_3 \Delta u.\end{aligned}$$

Therefore,

$$\limsup_{s\to\infty} \int_{\alpha^{-1}(\beta(s))}^{\alpha^{-1}(s)} \int_{u}^{\alpha^{-1}(u)} \left[a(\alpha(t_3)) \int_{t_3}^{s} (\sigma(t_2)-t_3)Q(t_2)\Delta t_2\right]^{\frac{1}{m}} \Delta t_3 \Delta u \leq \left(\frac{1+f(a)}{\lambda M_3^m}\right)^{\frac{1}{m}}$$

due to (A_{14}), a contradiction to (A_{32}). Hence, the proof of the theorem is complete. □

Remark 4.2.6. The same conclusion of Theorem 4.2.5 holds true if we replace (A_1) by the following condition

$$f(u)f(v) \geq f(uv),\ g(u)g(v) \geq g(uv),\ for\ u,\ v \in \mathbb{R}\ and\ u,\ v > 0.$$

Example 4.2.7. On $\mathbb{T} = q^{\mathbb{N}_0}$, consider the following q-difference equation

$$\Delta_q^2\left(t^3\Delta_q^2\left(y(t)+\frac{1}{qt^2}y(t/q)\right)\right)=\frac{t^5}{q^6}y(t/q^6)\left(1+\frac{2(1/q+1)^2(1/q^2+1)}{(q-1)^4q^{36}}y^6(t/q^6)\right)$$
$$+q^{13}t^5y(q^{13}t)\left(1+\frac{(1/q^3+1)^2(1/q^4+1)(1/q^2+1)}{(q-1)^4}q^{104}y^8(q^{13}t)\right),\quad (4.42)$$

for $t \in [q^6,\infty)_{\mathbb{T}}$, where $m = 1$, $a(t) = 1/t^3$, $p(t) = \frac{1}{qt^2}$, $\alpha(t) = t/q$, $\beta(t) = t/q^6$, $\gamma(t) = q^{13}t$, $q(t) = \frac{t^5}{q^6}$, $r(t) = q^{13}t^5$, $f(u) = u\left(1+\frac{2(1/q+1)^2(1/q^2+1)}{(q-1)^4q^{36}}u^6\right)$ and $g(u) = u\left(1+\frac{(1/q^3+1)^2(1/q^4+1)(1/q^2+1)}{(q-1)^4}q^{104}u^8\right)$. Clearly, all the conditions of Remark 4.2.6 are satisfied. Hence, (4.42) is oscillatory. Indeed, $y(t) = \frac{(-1)^{log_q^t}}{t}$ is an oscillatory solution of (4.42).

Theorem 4.2.8. *Let $-\infty < -b \le p(t) \le 0$ and $\gamma(t) \ge \beta^{-3}(t)$, for $t \in [t_0,\infty)_{\mathbb{T}}$, $b > 0$. Assume that (A_0'), (A_1), (A_2), (A_9) and (A_{14}) hold. If*

(A_{33}) $\limsup_{s\to\infty}\left[(\beta^{-3}(s)-\beta^{-2}(s))\left(\int_s^{\beta^{-1}(s)}\int_s^{t_4} r(t)\Delta t\Delta t_4\right)^{\frac{1}{m}}\int_{\beta^{-1}(s)}^{\beta^{-2}(s)} a^{\frac{1}{m}}(\theta)\Delta\theta\right] > \frac{1}{M_4}$,

(A_{34}) $\limsup_{t\to\infty}\int_t^{\beta^{-3}(t)}\int_u^{\beta^{-1}(u)}\left(a(\gamma(s))\int_{t_2}^s\int_{t_2}^v q(\theta)\Delta\theta\Delta v\right)^{\frac{1}{m}}\Delta s\Delta u > \frac{1}{M_3}$,

(A_{35}) $\int_{t_0}^\infty r(t)g(A(\gamma(t)))\Delta t = +\infty$, $A(t) = \int_t^\infty (a(s))^{\frac{1}{m}}\Delta s$,

(A_{36}) $\limsup_{s\to\infty}\int_{\alpha^{-1}(\beta(s))}^{\alpha^{-1}(s)}\int_u^{\alpha^{-1}(u)}\left[a(\alpha(t_3))\int_{t_3}^s(\sigma(t)-t_3)q(t)\Delta t\right]^{\frac{1}{m}}\Delta t_3\Delta u > \frac{1}{M_3}$,

(A_{37}) $\limsup_{t\to\infty}\int_t^{\beta^{-1}(t)}\int_u^{\beta^{-1}(u)}\left[a(\beta^{-2}(s))\int_s^{\beta^{-1}(s)}\int_v^{\beta^{-1}(v)} r(\theta)\Delta\theta\Delta v\right]^{\frac{1}{m}}\Delta s\Delta u > \frac{1}{g^{1/m}(b)M_4}$,

and

(A_{38}) $\int_T^\infty q(t)f\left(\int_{\alpha^{-1}(\beta(t))}^\infty A(s)\Delta s\right)\Delta t = \infty$, $T \ge t_0 > 0$;

hold, then every solution of (E_5) *oscillates.*

Proof. Suppose on the contrary that $y(t)$ is a non-oscillatory solution of (E_5) such that $y(t) > 0$, for $t \ge t_0$. Then proceeding as in the proof of Theorem 4.2.3, we obtain

$$L_4z(t) \ge r(t)g(z(\gamma(t))),\quad (4.43)$$

for $t \ge t_2$. Using the same type of reasoning as in Theorem 4.2.5, by integrating (4.43) from s to t_4, it follows that

$$\int_s^{t_4} L_4z(t)dt \ge \int_s^{t_4} r(t)g(z(\gamma(t)))\Delta t \ge g(z(\gamma(s)))\int_s^{t_4} r(t)\Delta t$$

due to nondecreasing $z(t)$ with $z(t) > 0$ and $L_3 z(t) > 0$, for $t \geq t_1$. Consequently, the preceeding inequality yields

$$L_3 z(t_4) \geq g(z(\gamma(s))) \int_s^{t_4} r(t)\Delta t$$

which on further integration from s to $\beta^{-1}(s)$, we get

$$L_2 z(\beta^{-1}(s)) \quad \geq \quad g(z(\gamma(s))) \int_s^{\beta^{-1}(s)} \int_s^{t_4} r(t)\Delta t \Delta t_4.$$

The rest of this case follows from the case (a) of Theorem 4.2.5.

Consider case (b). From equation (4.3), it follows that

$$\int_{t_2}^{t} L_4 z(\theta)\Delta\theta \geq \int_{t_2}^{t} q(\theta) f(y(\beta(\theta)))\Delta\theta \geq f(z(\beta(t))) \int_{t_2}^{t} q(\theta)\Delta\theta,$$

where $z(t) \leq y(t)$, for $t \geq t_2$. Thus,

$$\int_{t_2}^{t} L_3 z(v)\Delta v \geq f(z(\beta(t))) \int_{t_2}^{t} \int_{t_2}^{v} q(\theta)\Delta\theta\Delta v,$$

that is,

$$L_2 z(t) \geq f(z(\beta(t))) \int_{t_2}^{t} \int_{t_2}^{v} q(\theta)\Delta\theta\Delta v.$$

This implies that

$$L_2 z(\gamma(t)) \geq L_2 z(t) \geq f(z(\beta(t))) \int_{t_2}^{t} \int_{t_2}^{v} q(\theta)\Delta\theta\Delta v,$$

because $L_2 z$ is increasing. This gives that

$$z^{\Delta^2}(\gamma(t)) \geq f^{\frac{1}{m}}(z(\beta(t))) \left(a(\gamma(t)) \int_{t_2}^{t} \int_{t_2}^{v} q(\theta)\Delta\theta\Delta v \right)^{\frac{1}{m}}. \tag{4.44}$$

Integrating (4.44) from t to $\beta^{-1}(t)$, we get

$$-z^{\Delta}(\gamma(t)) \geq f^{\frac{1}{m}}(z(t)) \int_t^{\beta^{-1}(t)} \left(a(\gamma(s)) \int_{t_2}^{s} \int_{t_2}^{v} q(\theta)\Delta\theta\Delta v \right)^{\frac{1}{m}} \Delta s. \tag{4.45}$$

Integrating the inequality (4.45), from t to $\beta^{-3}(t)$

$$\begin{aligned} &-z(\gamma(\beta^{-3}(t))) + z(\gamma(t)) \\ &\quad \geq f^{\frac{1}{m}}(z(\beta^{-3}(t))) \int_t^{\beta^{-3}(t)} \int_u^{\beta^{-1}(u)} \left(a(\gamma(s)) \int_{t_2}^{s} \int_{t_2}^{v} q(\theta)\Delta\theta\Delta v \right)^{\frac{1}{m}} \Delta s \Delta u, \end{aligned}$$

that is,

$$\begin{aligned} z(\gamma(t)) &\geq f^{\frac{1}{m}}(z(\beta^{-3}(t)))\int_t^{\beta^{-3}(t)}\int_u^{\beta^{-1}(u)}\Big(a(\gamma(s))\int_{t_2}^s\int_{t_2}^v q(\theta)\Delta\theta\Delta v\Big)^{\frac{1}{m}}\Delta s\Delta u, \\ &\geq f^{\frac{1}{m}}(z(\gamma(t)))\int_t^{\beta^{-3}(t)}\int_u^{\beta^{-1}(u)}\Big(a(\gamma(s))\int_{t_2}^s\int_{t_2}^v q(\theta)\Delta\theta\Delta v\Big)^{\frac{1}{m}}\Delta s\Delta u, \end{aligned}$$

a contradiction to (A_{34}) due to (A_{14}).

The proof for the cases (c) and (e) are same as in Theorem 4.2.5. Suppose that case (d) holds. From equation (4.3), it happens that

$$\int_{t_2}^s L_4 z(\theta)\Delta\theta \geq \int_{t_2}^s q(\theta)f(y(\beta(\theta)))\Delta\theta \geq \int_{t_2}^s f(z(\beta(\theta)))q(\theta)\Delta\theta,$$

that is,

$$-L_3 z(t_2) \geq f(z(\beta(s)))\int_{t_2}^s q(\theta)\Delta\theta.$$

Thus, for $s > t_2 \geq t_3$,

$$\begin{aligned} \int_{t_3}^s(-L_3 z(t_2))\Delta t_2 &\geq f(z(\beta(s)))\int_{t_3}^s\int_{t_2}^s q(t)\Delta t\Delta t_2 \\ &= f(z(\beta(s)))\int_{t_3}^s(\sigma(t_2)-t_3)q(t_2)\Delta t_2, \end{aligned}$$

implies that

$$L_2 z(\alpha(t_3)) \geq L_2 z(t_3) \geq f(z(\beta(s)))\int_{t_3}^s(\sigma(t_2)-t_3)q(t_2)\Delta t_2,$$

that is,

$$z^{\Delta^2}(\alpha(t_3)) \geq f^{\frac{1}{m}}(z(\beta(s)))\left[a(\alpha(t_3))\int_{t_3}^s(\sigma(t_2)-t_3)q(t_2)\Delta t_2\right]^{\frac{1}{m}}. \tag{4.46}$$

Integrating the inequality (4.46) from u to $\alpha^{-1}(u)$, we obtain

$$-z^{\Delta}(\alpha(u)) \geq f^{\frac{1}{m}}(z(\beta(s)))\int_u^{\alpha^{-1}(u)}\left[a(\alpha(t_3))\int_{t_3}^s(\sigma(t_2)-t_3)q(t_2)\Delta t_2\right]^{\frac{1}{m}}\Delta t_3. \tag{4.47}$$

Again, integrating the inequality (4.47) from $\alpha^{-1}(\beta(s))$ to $\alpha^{-1}(s)$, we obtain

$$\begin{aligned} &z(\beta(s)) \\ &\quad\geq f^{\frac{1}{m}}(z(\beta(s)))\int_{\alpha^{-1}(\beta(s))}^{\alpha^{-1}(s)}\int_u^{\alpha^{-1}(u)}\left[a(\alpha(t_3))\int_{t_3}^s(\sigma(t_2)-t_3)q(t_2)\Delta t_2\right]^{\frac{1}{m}}\Delta t_3\Delta u, \end{aligned}$$

a contradiction to (A_{36}) due to (A_{14}).

Next, we suppose that $z(t) < 0$ for $t \in [t_2, \infty)_{\mathbb{T}}$. Using the same type of reasoning as in Theorem 4.2.3, we can find $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $y(\gamma(t)) \geq (-\frac{1}{b})z(\alpha^{-1}(\gamma(t)))$, for $t \in [t_3, \infty)_{\mathbb{T}}$ due to (A_9). In this case, we consider cases (b), (d), (e) and (f) of Lemma 4.2.1. For case (b) of Lemma 4.2.1, it follows that

$$
\begin{aligned}
L_4 z(t) \geq r(t)g(y(\gamma(t))) &\geq r(t)g\left(-\frac{1}{b}z(\alpha^{-1}(\gamma(t)))\right) \\
&\geq r(t)g\left(-\frac{1}{b}z(\gamma(t))\right) \qquad (4.48)
\end{aligned}
$$

and hence for $t \geq t_3 > t_2$,

$$
\begin{aligned}
\int_v^{\beta^{-1}(v)} L_4 z(\theta)\Delta\theta &\geq \int_v^{\beta^{-1}(v)} r(\theta)g\left(-\frac{1}{b}z(\gamma(\theta))\right)\Delta\theta \\
&\geq g\left(-\frac{1}{b}z(\gamma(v))\right)\int_v^{\beta^{-1}(v)} r(\theta)\Delta\theta,
\end{aligned}
$$

that is,

$$
L_3 z(\beta^{-1}(v)) \geq g\left(-\frac{1}{b}z(\gamma(v))\right)\int_v^{\beta^{-1}(v)} r(\theta)\Delta\theta.
$$

Integrating the preceeding inequality from t to $\beta^{-1}(t)$, we obtain

$$
L_2 z(\beta^{-2}(t)) \geq g\left(-\frac{1}{b}z(\gamma(t))\right)\int_t^{\beta^{-1}(t)}\int_v^{\beta^{-1}(v)} r(\theta)\Delta\theta\Delta v
$$

implies that

$$
z^{\Delta^2}(\beta^{-2}(t)) \geq g^{\frac{1}{m}}\left(-\frac{1}{b}z(\gamma(t))\right)\left[a(\beta^{-2}(t))\int_t^{\beta^{-1}(t)}\int_v^{\beta^{-1}(v)} r(\theta)\Delta\theta\Delta v\right]^{\frac{1}{m}}. \qquad (4.49)
$$

Integrating the inequality (4.49) from t to $\beta^{-1}(t)$, we obtain

$$
-z^{\Delta}(\beta^{-2}(t)) \geq g^{\frac{1}{m}}\left(-\frac{1}{b}z(\gamma(t))\right)\int_t^{\beta^{-1}(t)}\left[a(\beta^{-2}(s))\int_s^{\beta^{-1}(s)}\int_v^{\beta^{-1}(v)} r(\theta)\Delta\theta\Delta v\right]^{\frac{1}{m}}\Delta s.
$$

Further integrating the above inequality from t to $\beta^{-1}(t)$, it follows that

$$
\begin{aligned}
&-z(\beta^{-3}(t)) \\
&\geq g^{\frac{1}{m}}\left(\frac{1}{b}\right)g^{1/m}(-z(\gamma(t)))\int_t^{\beta^{-1}(t)}\int_u^{\beta^{-1}(u)}\left[a(\beta^{-2}(s))\int_s^{\beta^{-1}(s)}\int_v^{\beta^{-1}(v)} r(\theta)\Delta\theta\Delta v\right]^{\frac{1}{m}}\Delta s\Delta u,
\end{aligned}
$$

that is,

$$\int_t^{\beta^{-1}(t)}\int_u^{\beta^{-1}(u)}\left[a(\beta^{-2}(s))\int_s^{\beta^{-1}(s)}\int_v^{\beta^{-1}(v)}r(\theta)\Delta\theta\Delta v\right]^{\frac{1}{m}}\Delta s\Delta u$$
$$\leq \frac{-z(\beta^{-3}(t))}{g^{\frac{1}{m}}\left(\frac{1}{b}\right)g^{1/m}(-z(\beta^{-3}(t)))} \leq \frac{1}{g^{1/m}(1/b)M_4},$$

which contradicts (A_{37}).

For cases (d) and (f), we use the inequality (4.48) for $t \geq t_2$. Since, $z(t)$ is nonincreasing on $[t_1,\infty)_{\mathbb{T}}$, then there exist $C > 0$ and $t_3 > t_2$ such that $z(t) \leq -C$, for $t \geq t_3$. With $A(t) = \int_t^{\infty}(a(s))^{\frac{1}{m}}\Delta s$ satisfying the properties: $A(0) < \infty$ and $A(\infty) = 0$, we can find $C_1 > 0$ and $t_4 > t_3$ such that $z(t) \leq -C_1A(t)$, for $t \geq t_4$. Hence, (4.48) becomes

$$L_4z(t) \geq r(t)g\left(\frac{C_1}{b}A(\gamma(t))\right),$$

for $t \geq t_4$, which on integration from t_4 to ∞, we obtain

$$\int_{t_4}^{\infty} r(t)g(A(\gamma(t)))\Delta t < \infty,$$

a contradiction to (A_{35}).

Finally, we consider case (e) of Lemma 4.2.1. From equation (4.3), it follows that

$$L_4z(t) \geq q(t)f(y(\beta(t))) \geq f\left(-\frac{1}{b}z(\alpha^{-1}(\beta(t)))\right)q(t), \tag{4.50}$$

for $t \geq t_2$. Since $L_2z(t)$ is nonincreasing, then there exists $K_1^m > 0$ and $t_3 > t_2$ such that $L_2z(t) \leq -K_1^m$, for $t \geq t_3$. Consequently, $z^{\Delta^2}(t) \leq -K_1a^{\frac{1}{m}}(t)$, for $t \geq t_3$ which on integration from t to ∞, we get $-z^{\Delta}(t) \leq -K_1A(t)$, for $t \geq t_3$. Therefore, $z(t) \leq -K_1\int_t^{\infty}A(s)\Delta s$, for $t \geq t_3$. Using this fact in (4.50), we have

$$L_4z(t) \geq q(t)f\left(\frac{K_1}{b}\int_{\alpha^{-1}(\beta(t))}^{\infty}A(s)\Delta s\right), \tag{4.51}$$

for $t \geq t_4 > t_3$. Integrating (4.51) from t_4 to ∞, we obtain a contradiction to (A_{38}). This completes the proof of the theorem. □

Example 4.2.9. On $\mathbb{T} = \mathbb{R}$, consider

$$\left(t^3\big(y(t) - 2y(t-\pi)\big)''\right)'' = 18t^2y(t - \frac{13\pi}{2}) + 3t(t^2-6)y(t+24\pi), \tag{4.52}$$

for $t \in [7\pi, \infty)$, where $m = 1$, $a(t) = \frac{1}{t^3}$, $p(t) = -2$, $\alpha(t) = t - \pi$, $\beta(t) = t - \frac{13\pi}{2}$, $\gamma(t) = t + 24\pi$, $q(t) = 18t^2$ and $r(t) = 3t(t^2 - 6)$. Clearly, all the conditions of Theorem 4.2.8 are satisfied for (4.52). Hence (4.52) is oscillatory. Indeed, $y(t) = sint$ is an oscillatory solution of (4.52).

4.3 Conclusion

We have studied the oscillatory criteria of (E_5) in two Sections namely 4.1 and 4.2. In Section 4.1, we studied the oscillatory criteria of (E_5) with $\int_{t_0}^{\infty}(a(t))^{1/m}\Delta t = \infty$ for different ranges of $p(t)$. In Theorems 4.1.2 and 4.1.5, we proved that every solution of (E_5) is oscillatory for the range of $p(t)$ with $0 \le p(t) \le a < \infty$. Note that the technique is used to prove the Theorem 4.1.2 is different from the Theorem 4.1.5. In Theorems 4.1.4 and 4.1.7, we proved that every solution of (E_5) is oscillatory for the range of $p(t)$ with $-\infty < -b \le p(t) \le 0$.

In Theorem 4.1.9, we have established that every solution of (E_5) is oscillatory for the range of $p(t)$ with $-\infty < -b \le p(t) \le 0$, whereas in Theorem 4.1.10, we have shown that every unbounded solution of (E_5) is oscillatory for the range of $p(t)$ with $0 \le p(t) \le a < 1$.

In Section 4.2, we have studied the oscillatory criteria of (E_5) with $\int_{t_0}^{\infty}(a(t))^{1/m}\Delta t < \infty$ for different ranges of $p(t)$. By employing the same technique as in Theorems 4.2.2 and 4.2.3, for the ranges of $p(t)$ with $0 \le p(t) \le a < \infty$ and $-\infty < -b \le p(t) \le 0$ respectively, we have shown that every solution of (E_5) oscillates. In Theorems 4.2.5 and 4.2.8, for the ranges of $p(t)$ with $0 \le p(t) \le a < \infty$ and $-\infty < -b \le p(t) \le 0$ respectively, under certain sufficient conditions, we have established that every solution of (E_5) oscillates.

It would be interesting to study the oscillatory/nonoscillatory/asymptotic behavior of (E_5) if $p(t)$ oscillates.

Chapter 5

Higher Order Neutral Delay Dynamic Equations with Positive and Negative Coefficients-I

In this chapter, we concerned the oscillatory and asymptotic properties of solutions of the following higher-order nonlinear NDDEs of the following form

$$(r(t)(y(t)+p(t)y(\alpha(t)))^{\Delta^n})^{\Delta^m}+q(t)G(y(\beta(t)))-h(t)H(y(\gamma(t)))=0 \qquad (\mathrm{E}_6)$$

and

$$(r(t)(y(t)+p(t)y(\alpha(t)))^{\Delta^n})^{\Delta^m}+q(t)G(y(\beta(t)))-h(t)H(y(\gamma(t)))=f(t), \qquad (\mathrm{E}_7)$$

for $t\in[t_0,\infty)_{\mathbb{T}}$, $t_0(\geqslant 0)\in\mathbb{T}$, under the assumption

$$\int_{t_0}^{\infty}\frac{1}{r(t)}=\infty,$$

where $\mathbb{T}$ is a time scale with $\sup\mathbb{T}=\infty$, and $m,\ n\in\mathbb{N}$, r, $q\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$, p, $f\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $h\in C_{rd}([t_0,\infty)_{\mathbb{T}},[0,\infty))$, G, $H\in C(\mathbb{R},\mathbb{R})$ satisfy $uG(u)>0$ and $uH(u)>0$ for $u\neq 0$, G is nondecreasing, H is bounded, and α, β, $\gamma\in C_{rd}(\mathbb{T},\mathbb{T})$ are strictly increasing functions such that

$$\lim_{t\to\infty}\alpha(t)=\lim_{t\to\infty}\beta(t)=\lim_{t\to\infty}\gamma(t)=\infty,\quad \alpha(t),\ \beta(t),\ \gamma(t)\leqslant t.$$

Let $t_{-1} = \inf_{t\in[t_0,\infty)_{\mathbb{T}}}\{\alpha(t), \beta(t), \gamma(t)\}$. By a *solution* of $(E_6)/(E_7)$, we mean a function $y \in C_{rd}([t_{-1}, \infty)_{\mathbb{T}}, \mathbb{R})$ such that $y(t) + p(t)y(\alpha(t)) \in C^n_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, $r(t)(y(t) + p(t)y(\alpha(t)))^{\Delta^n} \in C^m_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, and such that $(E_6)/(E_7)$ is satisfied identically on $[t_0, \infty)_{\mathbb{T}}$. A solution of $(E_6)/(E_7)$ is called *oscillatory* if it is neither eventually positive nor eventually negative, and it is *nonoscillatory* otherwise. In this paper, we do not consider solutions that eventually vanish identically. An equation will be called oscillatory if all its solutions are oscillatory. For recent devlopment of the works in this direction readers are advaised to (see [[29], [30], [32], [63], [73], [80], [85], [86]]) and the references cited therein.

The following assumption (Λ) is considered throughout this chapter, where

(Λ) $(\alpha \circ \beta)(t) = (\beta \circ \alpha)(t)$ for all $t \in [t_0, \infty)_{\mathbb{T}}$ and $Q(t) = \min\{q(t), q(\alpha(t))\}$ for $t \in [t^*, \infty)_{\mathbb{T}}$, $t^* > t_0$.

5.1 Sufficient conditions for oscillation of (E_6) with $\int_{t_0}^{\infty} \frac{1}{r(t)}\Delta t = \infty$.

In this section, sufficient conditions are obtained for the oscillation of solutions of (E_6) under the assumption

$$(H_0) \quad \int_{t_0}^{\infty} \frac{1}{r(t)}\Delta t = \infty.$$

We need the following assumptions for our work in the sequel.

(H_1) $\int_{t_0}^{\infty} \frac{(\sigma(s))^{n-1}}{r(s)} \int_s^{\infty} (\sigma(t))^{m-1} h(t)\Delta t\Delta s < \infty$,

(H_2) there exists $\lambda > 0$ such that $G(u) + G(v) \geqslant \lambda G(u + v)$ for $u > 0$ and $v > 0$,

(H_3) $G(u)G(v) = G(uv)$ for $u, v \in \mathbb{R}$ and $H(-u) = -H(u)$ for $u \in \mathbb{R}$,

(H_4) $\int_0^{\pm c} \frac{du}{G(u)} < \infty$ for all $c > 0$,

(H_5) $\int_{t^*}^{\infty} Q(t)\Delta t = \infty$.

Lemma 5.1.1. Let (H_0) hold and u be an $n+m$ times continuously delta-differentiable function on $[t_0,\infty)_{\mathbb{T}}$ such that $(r(t)u^{\Delta^n}(t))^{\Delta^m}$ exists on $[t_0,\infty)_{\mathbb{T}}$.

(I_1) If $u^{\Delta^n}(t) < (>)\ 0$ and $u^{\Delta^{n-1}}(t) < (>)\ 0$ $(n \geqslant 2)$ for large $t \in [t_0,\infty)_{\mathbb{T}}$, then $u^{\Delta^i}(t) < (>)\ 0$ and $\lim_{t\to\infty} u^{\Delta^i}(t) = -\infty\ (\infty)$ for all $i \in [0, n-2]_{\mathbb{Z}}$.

(I_2) If $u^{\Delta^n}(t) < (>)\ 0$ and $(r(t)u^{\Delta^n}(t))^{\Delta} < (>)\ 0$ $(n \geqslant 1)$ for large $t \in [t_0,\infty)_{\mathbb{T}}$, then $u^{\Delta^i}(t) < (>)\ 0$ and $\lim_{t\to\infty} u^{\Delta^i}(t) = -\infty\ (\infty)$ for all $i \in [0, n-1]_{\mathbb{Z}}$.

(I_3) If $(r(t)u^{\Delta^n}(t))^{\Delta^{m-1}} < (>)\ 0$ and $(r(t)u^{\Delta^n}(t))^{\Delta^m} < (>)\ 0$ $(n \geqslant 1,\ m \geqslant 2)$ for large $t \in [t_0,\infty)_{\mathbb{T}}$, then $u^{\Delta^i}(t) < (>)\ 0$, $\lim_{t\to\infty} u^{\Delta^i}(t) = -\infty\ (\infty)$ for all $i \in [0, n-1]_{\mathbb{Z}}$, and $(r(t)u^{\Delta^n}(t))^{\Delta^i} < (>)\ 0$, $\lim_{t\to\infty}(r(t)u^{\Delta^n}(t))^{\Delta^i} = -\infty\ (\infty)$ for all $i \in [0, m-2]_{\mathbb{Z}}$.

Proof. First, we prove the cases which are negative (The proof is similar for positive case). Let $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $u^{\Delta^n}(t) < 0$ and $u^{\Delta^{n-1}}(t) < 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$. Since $u^{\Delta^n}(t) < 0$, then $u^{\Delta^{n-1}}(t)$ is decreasing in $t \in [t_1,\infty)_{\mathbb{T}}$. For $t > T \geqslant t_1$,

$$u^{\Delta^{n-2}}(t) - u^{\Delta^{n-2}}(T) \leq u^{\Delta^{n-1}}(T)(t-T).$$

Hence $u^{\Delta^{n-2}}(t) < 0$ for large $t \in [T,\infty)_{\mathbb{T}}$ and $\lim_{t\to\infty} u^{\Delta^{n-2}}(t) = -\infty$. Since $u^{\Delta^{n-1}}(t) < 0$ and $u^{\Delta^{n-2}}(t) < 0$, then by repeating the procedure as above we obtain, $u^{\Delta^i}(t) < 0$ and $\lim_{t\to\infty} u^{\Delta^i}(t) = -\infty$ for all $i \in [0, n-3]_{\mathbb{Z}}$ if $n \geqslant 3$. Thus $u^{\Delta^i}(t) < 0$ and $\lim_{t\to\infty} u^{\Delta^i}(t) = -\infty$ for all $i \in [0, n-2]_{\mathbb{Z}}$. Hence (I_1) is proved.

For (I_2), take $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $u^{\Delta^n}(t) < 0$ and $(r(t)u^{\Delta^n}(t))^{\Delta} < 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$. Since $(r(t)u^{\Delta^n}(t))^{\Delta} < 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$, then $r(t)u^{\Delta^n}(t)$ is decreasing in $[t_1,\infty)_{\mathbb{T}}$. For $t > T \geqslant t_1$,

$$r(t)u^{\Delta^n}(t) \leq r(T)u^{\Delta^n}(T),$$

that is,

$$u^{\Delta^n}(t) \leq r(T)u^{\Delta^n}(T)\frac{1}{r(t)}.$$

By integrating the above inequality over $[T,t]_{\mathbb{T}} \subseteq [t_1,\infty)_{\mathbb{T}}$,

$$u^{\Delta^{n-1}}(t) - u^{\Delta^{n-1}}(T) \leq (r(T)u^{\Delta^n}(T)) \int_T^t \frac{1}{r(\theta)}\Delta\theta.$$

Therefore, $u^{\Delta^{n-1}}(t) < 0$ for large $t \in [T,\infty)_{\mathbb{T}}$ and $\lim_{t\to\infty} u^{\Delta^{n-1}}(t) = -\infty$ due to (H_0). Since $u^{\Delta^n}(t) < 0$ and $u^{\Delta^{n-1}}(t) < 0$, then by proceeding as in (I_1), we have $u^{\Delta^i}(t) < 0$ and $\lim_{t\to\infty} u^{\Delta^i}(t) = -\infty$ for all $i \in [0, n-2]_{\mathbb{Z}}$ if $n \geqslant 2$. Thus, $u^{\Delta^i}(t) < 0$ and $\lim_{t\to\infty} u^{\Delta^i}(t) = -\infty$ for all $i \in [0, n-1]_{\mathbb{Z}}$. Hence ($I_2$) is proved.

For (I_3), take $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $(r(t)u^{\Delta^n}(t))^{\Delta^m} < 0$ and $(r(t)u^{\Delta^n}(t))^{\Delta^{m-1}} < 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$. Since $(r(t)u^{\Delta^n}(t))^{\Delta^m} < 0$, then $(r(t)u^{\Delta^n}(t))^{\Delta^{m-1}}$ is decreasing in $t \in [t_1,\infty)_{\mathbb{T}}$. For $t > T \geqslant t_1$,

$$(r(t)u^{\Delta^n}(t))^{\Delta^{m-2}} - (r(T)u^{\Delta^n}(T))^{\Delta^{m-2}} \leq (r(T)u^{\Delta^n}(T))^{\Delta^{m-1}}(t-T).$$

Therefore, $(r(t)u^{\Delta^n}(t))^{\Delta^{m-2}} < 0$ for large $t \in [T,\infty)_{\mathbb{T}}$ and $\lim_{t\to\infty}(r(t)u^{\Delta^n}(t))^{\Delta^{m-2}} = -\infty$. Since $(r(t)u^{\Delta^n}(t))^{\Delta^{m-1}} < 0$ and $(r(t)u^{\Delta^n}(t))^{\Delta^{m-2}} < 0$, then by proceeding as in (I_1), we have $(r(t)u^{\Delta^n}(t))^{\Delta^i} < 0$ and $\lim_{t\to\infty}(r(t)u^{\Delta^n}(t))^{\Delta^i} = -\infty$ for all $i \in [0, m-3]_{\mathbb{Z}}$ if $m \geqslant 3$. From this we have, $u^{\Delta^n}(t) < 0$ and $(r(t)u^{\Delta^n}(t))^{\Delta} < 0$. Again, by proceeding as in (I_2), we have $u^{\Delta^i}(t) < 0$ and $\lim_{t\to\infty} u^{\Delta^i}(t) = -\infty$ for all $i \in [0, n-1]_{\mathbb{Z}}$. Thus, $u^{\Delta^i}(t) < 0$ and $\lim_{t\to\infty} u^{\Delta^i}(t) = -\infty$ for all $i \in [0, n-1]_{\mathbb{Z}}$, and $(r(t)u^{\Delta^n}(t))^{\Delta^i} < 0$, $\lim_{t\to\infty}(r(t)u^{\Delta^n}(t))^{\Delta^i} = -\infty$ for all $i \in [0, m-2]_{\mathbb{Z}}$. Hence ($I_3$) is proved. This completes the proof of the lemma. □

Lemma 5.1.2. Let (H_0) hold, $u(t) \in C^n_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $r(t)u^{\Delta^n}(t) \in C^m_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ and $(r(t)u^{\Delta^n}(t))^{\Delta^m} \leq (\geq) \not\equiv 0$ for large $t \in [t_0,\infty)_{\mathbb{T}}$. If $u(t) > 0$ eventually, then there exists an integer l, $0 \leq l \leq n+m$ such that $n+m+l$ is odd (even) and one of the following cases (a) or (b) holds for large $t \in [t_0,\infty)_{\mathbb{T}}$, where

(a) if $n < l \leq n+m$, then $u^{\Delta^i}(t) > 0$ $(i = 1, 2, \cdots, n)$, $(r(t)u^{\Delta^n}(t))^{\Delta^i} > 0$ $(i = 1, 2, \cdots, l-n-1)$ and $(-1)^{n+i+l}(r(t)u^{\Delta^n}(t))^{\Delta^i} > 0$ $(i = l-n, l-n+1, \cdots, m)$.

(b) if $0 \leq l \leq n$, then $u^{\Delta^i}(t) > 0$ $(i = 1, 2, \cdots, l-1)$, $(-1)^{i+l}u^{\Delta^i}(t) > 0$ $(i = l, l+1, \cdots, n)$ and $(-1)^{n+i+l}(r(t)u^{\Delta^n}(t))^{\Delta^i} > 0$ $(i = 1, 2, \cdots, m)$.

Proof. We have to consider the following cases.

$$Case\ \ I: \ (r(t)u^{\Delta^n}(t))^{\Delta^m} \leq 0 \ for\ large\ t \in [t_0,\infty)_{\mathbb{T}},$$

$$Case\ \ II: \ (r(t)u^{\Delta^n}(t))^{\Delta^m} \geq 0 \ for\ large\ t \in [t_0,\infty)_{\mathbb{T}}.$$

Case I. Suppose that $(r(t)u^{\Delta^n}(t))^{\Delta^m} \leq 0$ for large $t \in [t_0, \infty)_{\mathbb{T}}$. Then, we can find a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $u^{\Delta^i}(t)$ $(i = 0, 1, 2, \cdots, n-1)$ and $(r(t)u^{\Delta^n}(t))^{\Delta^i}$ $(i = 0, 1, 2, \cdots, m-1)$ are monotonic and eventually of one sign on $[t_1, \infty)_{\mathbb{T}}$. Therefore, $u(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$ as $u(t) > 0$ for large $t \in [t_0, \infty)_{\mathbb{T}}$. First, we shall prove that $(r(t)u^{\Delta^n}(t))^{\Delta^{m-1}} > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. If it is not possible, then there exists some $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $(r(t_2)u^{\Delta^n}(t_2))^{\Delta^{m-1}} < 0$. Since $(r(t)u^{\Delta^n}(t))^{\Delta^{m-1}}$ is decreasing and $(r(t_2)u^{\Delta^n}(t_2))^{\Delta^{m-1}} < 0$, then $(r(t)u^{\Delta^n}(t))^{\Delta^{m-1}} < 0$ for all $t \in [t_2, \infty)_{\mathbb{T}}$. But, from $(r(t)u^{\Delta^n}(t))^{\Delta^m} \leq 0$, $(r(t)u^{\Delta^n}(t))^{\Delta^{m-1}} < 0$ then by using Lemma 5.1.1, we can find $\lim_{t\to\infty} u(t) = -\infty$, which is a contradiction to $u(t) > 0$. Thus, $(r(t)u^{\Delta^n}(t))^{\Delta^{m-1}} > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$ and there exists a smallest integer l, $0 \leq l \leq n+m-1$ with $n+m+l$ is odd: first if $n < l \leq n+m-1$, then

$$(-1)^{n+i+l}(r(t)u^{\Delta^n}(t))^{\Delta^i} > 0 \quad \text{for all } i \in [l-n, m]_{\mathbb{Z}}. \tag{5.1}$$

Let $l > n+2$ and suppose that

$$(r(t)u^{\Delta^n}(t))^{\Delta^{l-n-1}} < 0 \quad \text{for } t \in [t_2, \infty)_{\mathbb{T}},\ t_2 \geqslant t_1, \tag{5.2}$$

then from Lemma 5.1.1, it follows that

$$(r(t)u^{\Delta^n}(t))^{\Delta^{l-n-2}} > 0. \tag{5.3}$$

Otherwise, if we consider $(r(t)u^{\Delta^n}(t))^{\Delta^{l-n-2}} < 0$, then by Lemma 5.1.1 follows that $u(t) < 0$ for large $t \in [t_2, \infty)_{\mathbb{T}}$ due to (5.2), which is a contradiction to $u(t) > 0$. Inequalities (5.1)-(5.3) can be unified as

$$(-1)^{n+i+l-2}(r(t)u^{\Delta^n}(t))^{\Delta^i} > 0 \quad \text{for all } i \in [l-n-2, m]_{\mathbb{Z}},$$

which is a contradiction to the definition of l. Hence (5.2) fails then $(r(t)u^{\Delta^n}(t))^{\Delta^{l-n-1}} > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Since $(r(t)u^{\Delta^n}(t))^{\Delta^{l-n}} > 0$ and $(r(t)u^{\Delta^n}(t))^{\Delta^{l-n-1}} > 0$, then by using Lemma 5.1.1, we have $u^{\Delta^i}(t) > 0$ $(i = 1, 2, \cdots, n)$ and $(r(t)u^{\Delta^n}(t))^{\Delta^i} > 0$ $(i = 1, 2, \cdots, l-n-2)$. Again, let $l = n+2$,

$$(-1)^{n+i+n+2}(r(t)u^{\Delta^n}(t))^{\Delta^i} > 0 \quad \text{for all } i \in [2, m]_{\mathbb{Z}}. \tag{5.4}$$

Now suppose

$$(r(t)u^{\Delta^n}(t))^{\Delta} < 0 \quad \text{for} \quad t \in [t_2, \infty)_{\mathbb{T}},\ t_2 > t_1, \tag{5.5}$$

then once from Lemma 5.1.1, we get

$$u^{\Delta^n}(t) > 0, \tag{5.6}$$

Inequalities (5.4)-(5.6) can be unified to

$$(-1)^{n+l-2}u^{\Delta^n}(t) > 0 \quad \text{and } (-1)^{n+i+l-2}(r(t)u^{\Delta^n}(t))^{\Delta^i} > 0\ (i = 1,\ 2,\ \cdots,\ m),$$

which is a contradiction to the definition of l. Hence, (5.5) fails and $(r(t)u^{\Delta^n}(t))^{\Delta} > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Since $(r(t)u^{\Delta^n}(t))^{\Delta^2} > 0$ and $(r(t)u^{\Delta^n}(t))^{\Delta} > 0$, then by using Lemma 5.1.1, we have $u^{\Delta^i}(t) > 0$ $(i = 1,\ 2,\ \cdots,\ n)$. The proof for $l = n + 1$ is same as $l = n + 2$. Thus (a) is proved.

Next, if $0 \le l \le n$, then

$$\begin{aligned} (-1)^{i+l}u^{\Delta^i}(t) &> 0 \quad \text{for all } i \in [l, n]_{\mathbb{Z}} \quad \text{and} \\ (-1)^{n+i+l}(r(t)u^{\Delta^n}(t))^{\Delta^i} &> 0 \quad \text{for all} \quad i \in [1, m]_{\mathbb{Z}}. \end{aligned} \tag{5.7}$$

Let $l > 2$ and suppose that

$$u^{\Delta^{l-1}}(t) < 0 \quad \text{for} \quad t \in [t_2, \infty)_{\mathbb{T}},\ t_2 > t_1, \tag{5.8}$$

then once from Lemma 5.1.1, we obtain

$$u^{\Delta^{l-2}}(t) > 0. \tag{5.9}$$

Inequalities (5.7)-(5.9) can be unified as

$$\begin{gathered} (-1)^{i+l-2}u^{\Delta^i}(t) > 0 \quad \text{for all } i \in [l-2, n]_{\mathbb{Z}} \quad \text{and} \\ (-1)^{n+i+l-2}(r(t)u^{\Delta^n}(t))^{\Delta^i} > 0 \quad \text{for all } i \in [1, m]_{\mathbb{Z}}, \end{gathered}$$

which is a contradiction to the definition of l. Hence, (5.8) fails and $u^{\Delta^{l-1}}(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Since $u^{\Delta^l}(t) > 0$ and $u^{\Delta^{l-1}}(t) > 0$, then by using Lemma 5.1.1, we have $u^{\Delta^i}(t) > 0$ $(i = 1,\ 2,\ \cdots,\ l-2)$. Thus (b) is proved. Hence *Case I* is proved.

Case II. Given that $(r(t)u^{\Delta^n}(t))^{\Delta^m} \geqslant 0$ for large $t \in [t_0, \infty)_{\mathbb{T}}$. Then, we can find a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $u^{\Delta^i}(t)$ $(i = 0,\ 1,\ 2,\ \cdots,\ n-1)$ and $(r(t)u^{\Delta^n}(t))^{\Delta^i}$ $(i = 0,\ 1,\ 2,\ \cdots,\ m-1)$ are monotonic and eventually of one sign on $[t_1, \infty)_{\mathbb{T}}$. Therefore, $u(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$, because $u(t) > 0$ for large $t \in [t_0, \infty)_{\mathbb{T}}$. Since $(r(t)u^{\Delta^n}(t))^{\Delta^m} \geqslant 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$, then $(r(t)u^{\Delta^n}(t))^{\Delta^{m-1}}$ is increasing on $[t_1, \infty)_{\mathbb{T}}$. Then, one of the following sub cases are holds.

$$Sub\ case\ (i)\ (r(t)u^{\Delta^n}(t))^{\Delta^{m-1}} > 0\ for\ t \in [t_1, \infty)_{\mathbb{T}},$$
$$Sub\ case\ (ii)\ (r(t)u^{\Delta^n}(t))^{\Delta^{m-1}} < 0\ for\ t \in [t_1, \infty)_{\mathbb{T}}.$$

If Sub case (i) holds, then from Lemma 5.1.1, $u^{\Delta^i}(t) > 0$ $(i = 1,\ 2,\ \cdots,\ n)$ and $(r(t)u^{\Delta^n}(t))^{\Delta^i} > 0$ $(i = 1,\ 2,\ \cdots,\ m-2)$. This proves the theorem for $l = n + m$ $(m + n + l = m + n + m + n$ is even). If Sub case (ii) holds, then from Case I, there exist $l \in [0, m + n - 1]_{\mathbb{Z}}$ with $m + n - 1 + l$ is odd (i.e., $m + n + l$ is even) such that one of the cases (a) or (b) are holds. Hence proof of the $Case\ II$ is completed. This completes the proof of the lemma. □

Lemma 5.1.3. Let (H_0) hold, $u(t) \in C^n_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, $r(t)u^{\Delta^n}(t) \in C^m_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ and $(r(t)u^{\Delta^n}(t))^{\Delta^m} \leq (\geq) \not\equiv 0$ for large $t \in [t_0, \infty)_{\mathbb{T}}$. If $u(t) < 0$ eventually, then there exists an integer l, $0 \leq l \leq n + m$ such that $n + m + l$ is even (odd) and one of the cases (c) or (d) holds for large $t \in [t_0, \infty)_{\mathbb{T}}$, where

- (c) if $n < l \leq n + m$, then $u^{\Delta^i}(t) < 0$ $(i = 1,\ 2,\ \cdots,\ n)$, $(r(t)u^{\Delta^n}(t))^{\Delta^i} < 0$ $(i = 1,\ 2,\ \cdots,\ l - n - 1)$ and $(-1)^{1+n+i+l}(r(t)u^{\Delta^n}(t))^{\Delta^i} > 0$ $(i = l - n,\ l - n + 1,\ \cdots,\ m)$.

- (d) if $0 \leq l \leq n$, then $u^{\Delta^i}(t) < 0$ $(i = 1,\ 2,\ \cdots,\ l - 1)$, $(-1)^{1+i+l}u^{\Delta^i}(t) > 0$ $(i = l,\ l + 1,\ \cdots,\ n)$ and $(-1)^{1+n+i+l}(r(t)u^{\Delta^n}(t))^{\Delta^i} > 0$ $(i = 1,\ 2,\ \cdots,\ m)$.

Proof. Let $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $(r(t)u^{\Delta^n}(t))^{\Delta^m} \leq (\geqslant) \not\equiv 0$ and $u(t) < 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. Let $u(t) = -v(t) < 0$, where $v(t) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$, then $(r(t)v^{\Delta^n}(t))^{\Delta^m} \geqslant (\leq) 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. Since $v(t) > 0$ and $(r(t)v^{\Delta^n}(t))^{\Delta^m} \geqslant (\leq) 0$, then from Lemma 5.1.2, there exists $l \in [0, n+m]_{\mathbb{Z}}$ with $n+m+l$ is even (odd) such that one of the cases (a) or (b) holds. Suppose (a) holds, then replace $v(t)$ by $-u(t)$ in (a), we obtain $u^{\Delta^i}(t) < 0$ $(i = 1,\ 2,\ \cdots,\ n)$, $(r(t)u^{\Delta^n}(t))^{\Delta^i} < 0$ $(i = 1,\ 2,\ \cdots,\ l-n-1)$ and $-(-1)^{n+i+l}(r(t)u^{\Delta^n}(t))^{\Delta^i} > 0$ (or $(-1)^{1+n+i+l}(r(t)u^{\Delta^n}(t))^{\Delta^i} > 0$) $(i = l-n,\ l-n+1,\ \cdots,\ m)$. Thus, (c) is proved. Similarly, we can prove (d), by replacing $v(t)$ with $-u(t)$ in (b), when $0 \le l \le n$. □

Lemma 5.1.4. Assume that the conditions of Lemma 5.1.2 hold and $(r(t)u^{\Delta^n}(t))^{\Delta^m} \le (\not\equiv) 0$ for large $t \in [t_0, \infty)_{\mathbb{T}}$. If $u(t) > 0$ eventually, then there exists $l \in [0, m+n]_{\mathbb{Z}}$ such that $m+n+l$ is odd, and

(i) if $1 \le l \le n+m$, then

$$u(t) > R_T(t)(r(t)u^{\Delta^n}(t))^{\Delta^{m-1}} \ for\ t\ \ge\ T\ >\ t_0$$

where,

$$R_T(t) = \begin{cases} \int_T^t \frac{h_{n-1}(t,\sigma(\theta))}{r(\theta)} \int_T^\theta h_{l-n-1}(\theta,\sigma(s))h_{m+n-l-1}(s,t)\Delta s\Delta\theta & \text{for } n < l \le n+m \\ \int_T^t \frac{h_{n-1}(t,\sigma(\theta))h_{m-1}(\theta,t)}{r(\theta)}\Delta\theta & \text{for } l = n \\ -\int_T^t h_{l-1}(t,\sigma(s)) \int_s^t \frac{h_{n-l-1}(s,\sigma(\theta))h_{m-1}(\theta,t)}{r(\theta)}\Delta\theta\Delta s & \text{for } 1 \le l < n. \end{cases}$$

(ii) if $l = 0$, then $u(s) > R_s(t)(r(t)u^{\Delta^n}(t))^{\Delta^{m-1}}$ for $t \ge s \geqslant t_0$, where

$$R_s(t) = -\int_s^t \frac{h_{n-1}(s,\sigma(\theta))h_{m-1}(\theta,t)}{r(\theta)}\Delta\theta.$$

Proof. If $(r(t)u^{\Delta^n}(t))^{\Delta^m} \le 0$ for large $t \in [t_0, \infty)_{\mathbb{T}}$, then we can find a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $u^{\Delta^i}(t)$ $(i = 0,\ 1,\ 2,\ \cdots,\ n-1)$ and $(r(t)u^{\Delta^n}(t))^{\Delta^i}$ $(i = 0,\ 1,\ 2,\ \cdots,\ m-1)$ are monotonic and eventually of one sign on $[t_1, \infty)_{\mathbb{T}}$. Hence, $u(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. From Lemma 5.1.2, there exists an integer l, $0 \le l \le n+m$ with $m+n+l$ is odd and one of the cases (a) or (b) are holds. Since $(r(t)u^{\Delta^n}(t))^{\Delta^m} \le 0$, then $(r(t)u^{\Delta^n}(t))^{\Delta^{m-1}}$ is decreasing. For $t > \theta > s > T \geqslant t_1$,

$$(r(\theta)u^{\Delta^n}(\theta))^{\Delta^{m-1}} \geqslant (r(t)u^{\Delta^n}(t))^{\Delta^{m-1}}. \tag{5.10}$$

If $n < l \le n+m$, then *case* (a) of Lemma 5.1.2 holds. Integrating the inequality (5.10) on $[s,t]_{\mathbb{T}} \subseteq [T,t]_{\mathbb{T}} \subseteq [t_1, \infty)_{\mathbb{T}}$ for $m+n-l-1$ times, we obtain

$$(-1)^{m+n-l-1}(r(s)u^{\Delta^n}(s))^{\Delta^{l-n}} \ \ge (r(t)u^{\Delta^n}(t))^{\Delta^{m-1}} \int_s^t \int_{s_{m+n-l-1}}^t \cdots \int_{s_2}^t \Delta s_1 \Delta s_2 \cdots \Delta s_{m+n-l-1},$$

that is,

$$\begin{aligned}(r(s)u^{\Delta^n}(s))^{\Delta^{l-n}} &\geqslant (r(t)u^{\Delta^n}(t))^{\Delta^{m-1}}\int_s^t\int_{s_{m+n-l-1}}^t\cdots\int_{s_2}^t \Delta s_1\Delta s_2\cdots\Delta s_{m+n-l-1}\\ &= (-1)^{m+n-l-2}(r(t)u^{\Delta^n}(t))^{\Delta^{m-1}}\int_s^t h_{m+n-l-2}(s,\sigma(\theta))\Delta\theta\\ &= (-1)^{m+n-l-2+1}(r(t)u^{\Delta^n}(t))^{\Delta^{m-1}}\int_t^s h_{m+n-l-2}(s,\sigma(\theta))\Delta\theta\\ &= (r(t)u^{\Delta^n}(t))^{\Delta^{m-1}}h_{m+n-l-1}(s,t), \qquad (5.11)\end{aligned}$$

due to (1.4), (1.7) and $m+n-l-1$ is even. Using Taylor's formula (see [Lemma 1.4.5]) for $\theta\in[s,t]_{\mathbb{T}}\subseteq[T,t]_{\mathbb{T}}$, and fact that $(r(t)u^{\Delta^n}(t))^{\Delta^i}>0$ $(i=1,\ 2,\ \cdots,\ l-n)$, we obtain

$$\begin{aligned}r(\theta)u^{\Delta^n}(\theta) &= \sum_{k=0}^{l-n-1}h_k(\theta,T)(r(T)u^{\Delta^n}(T))^{\Delta^k}+\int_T^\theta h_{l-n-1}(\theta,\sigma(s))(r(s)u^{\Delta^n}(s))^{\Delta^{l-n}}\Delta s\\ &> \int_T^\theta h_{l-n-1}(\theta,\sigma(s))(r(s)u^{\Delta^n}(s))^{\Delta^{l-n}}\Delta s,\end{aligned}$$

that is,

$$u^{\Delta^n}(\theta)>\frac{1}{r(\theta)}\int_T^\theta h_{l-n-1}(\theta,\sigma(s))(r(s)u^{\Delta^n}(s))^{\Delta^{l-n}}\Delta s. \qquad (5.12)$$

Again, by using Taylor's formula for $t\in[T,\infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned}u(t) &= \sum_{k=0}^{n-1}h_k(t,T)u^{\Delta^k}(T)+\int_T^t h_{n-1}(t,\sigma(\theta))u^{\Delta^n}(\theta)\Delta\theta\\ &> \int_T^t h_{n-1}(t,\sigma(\theta))u^{\Delta^n}(\theta)\Delta\theta. \qquad (5.13)\end{aligned}$$

By unifying the inequalities (5.11)–(5.13), we obtain

$$\begin{aligned}u(t) &> \int_T^t h_{n-1}(t,\sigma(\theta))\Big(\frac{1}{r(\theta)}\int_T^\theta h_{l-n-1}(\theta,\sigma(s))(r(s)u^{\Delta^n}(s))^{\Delta^{l-n}}\Delta s\Big)\Delta\theta\\ &> (r(t)u^{\Delta^n}(t))^{\Delta^{m-1}}\int_T^t\left(\frac{h_{n-1}(t,\sigma(\theta))}{r(\theta)}\int_T^\theta h_{l-n-1}(\theta,\sigma(s))h_{m+n-l-1}(s,t)\Delta s\right)\Delta\theta\\ &= (r(t)u^{\Delta^n}(t))^{\Delta^{m-1}}R_T(t).\end{aligned}$$

Next, if $l\le n$, then integrating the inequality (5.10) on $[\theta,t]\subseteq[s,t]\subseteq[T,\infty)_{\mathbb{T}}$ for $(m-1)$ times, we obtain

$$\begin{aligned}(-1)^{m-1}r(\theta)u^{\Delta^n}(\theta) &\geqslant (r(t)u^{\Delta^n}(t))^{\Delta^{m-1}}\int_\theta^t\int_{\theta_{m-1}}^t\cdots\int_{\theta_2}^t\Delta\theta_1\Delta\theta_2\cdots\Delta\theta_{m-1}\\ &= (-1)^{m-2}(r(t)u^{\Delta^n}(t))^{\Delta^{m-1}}\int_\theta^t h_{m-2}(\theta,\sigma(x))\Delta x,\end{aligned}$$

that is,

$$(-1)^{m-1}u^{\Delta^n}(\theta) \geqslant (-1)^{m-1}(r(t)u^{\Delta^n}(t))^{\Delta^{m-1}}\frac{h_{m-1}(\theta,t)}{r(\theta)} \tag{5.14}$$

due to (1.4) and (1.7). If $l = n$, then from (5.14), we obtain

$$u^{\Delta^n}(\theta) \geq (r(t)u^{\Delta^n}(t))^{\Delta^{m-1}}\frac{h_{m-1}(\theta,t)}{r(\theta)}. \tag{5.15}$$

By using Taylor's formula for $t \in [T,\infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned} u(t) &= \sum_{k=0}^{n-1} h_k(t,T)u^{\Delta^k}(T) + \int_T^t h_{n-1}(t,\sigma(\theta))u^{\Delta^n}(\theta)\Delta\theta \\ &> \int_T^t h_{n-1}(t,\sigma(\theta))u^{\Delta^n}(\theta)\Delta\theta. \end{aligned} \tag{5.16}$$

By unifying the inequalities (5.15) and (5.16), we obtain

$$u(t) > (r(t)u^{\Delta^n}(t))^{\Delta^{m-1}}\int_T^t \frac{h_{n-1}(t,\sigma(\theta))h_{m-1}(\theta,t)}{r(\theta)}\Delta\theta.$$

Integrating the inequality (5.14) on $[s,t] \subseteq [T,\infty)_{\mathbb{T}}$ for $(n-l)$ times, we obtain

$$\begin{aligned} &(-1)^{m-1+n-l}u^{\Delta^l}(s) \\ &\quad \geqslant (-1)^{m-1}(r(t)u^{\Delta^n}(t))^{\Delta^{m-1}}\int_s^t\int_{s_{n-l-1}}^t \cdots \int_{s_1}^t \frac{h_{m-1}(\theta,t)}{r(\theta)}\Delta\theta\Delta s_1\cdots\Delta s_{n-l-1} \\ &\quad = (-1)^{m-1+n-l-1}(r(t)u^{\Delta^n}(t))^{\Delta^{m-1}}\int_s^t \frac{h_{n-l-1}(s,\sigma(\theta))h_{m-1}(\theta,t)}{r(\theta)}\Delta\theta, \end{aligned}$$

that is,

$$u^{\Delta^l}(s) \;>\; -(r(t)u^{\Delta^n}(t))^{\Delta^{m-1}}\int_s^t \frac{h_{n-l-1}(s,\sigma(\theta))h_{m-1}(\theta,t)}{r(\theta)}\Delta\theta. \tag{5.17}$$

First, if $1 \leq l$, then from Taylor's formula for $t \in [T,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned} u(t) &= \sum_{k=0}^{l-1} h_k(t,T)u^{\Delta^k}(T) + \int_T^t h_{l-1}(t,\sigma(s))u^{\Delta^l}(s)\Delta s \\ &> \int_T^t h_{l-1}(t,\sigma(s))u^{\Delta^l}(s)\Delta s. \end{aligned} \tag{5.18}$$

By unifying the inequalities (5.17) and (5.18), we obtain

$$\begin{aligned} u(t) &> \int_T^t h_{l-1}(t,\sigma(s))\left(-(r(t)u^{\Delta^n}(t))^{\Delta^{m-1}}\int_s^t \frac{h_{n-l-1}(s,\sigma(\theta))h_{m-1}(\theta,t)}{r(\theta)}\Delta\theta\right)\Delta s \\ &> -(r(t)u^{\Delta^n}(t))^{\Delta^{m-1}}\int_T^t h_{l-1}(t,\sigma(s))\int_s^t \frac{h_{n-l-1}(s,\sigma(\theta))h_{m-1}(\theta,t)}{r(\theta)}\Delta\theta\Delta s \\ &= (r(t)u^{\Delta^n}(t))^{\Delta^{m-1}}R_T(t). \end{aligned}$$

Thus, (i) is proved. Next, if $l = 0$, then (5.17) yields

$$\begin{aligned} u(s) &> -(r(t)u^{\Delta^n}(t))^{\Delta^{m-1}} \int_s^t \frac{h_{n-1}(s,\sigma(\theta))h_{m-1}(\theta,t)}{r(\theta)}\Delta\theta \\ &= R_s(t)(r(t)u^{\Delta^n}(t))^{\Delta^{m-1}}. \end{aligned}$$

Thus, (ii) is proved. This completes the proof of the lemma. □

Remark 5.1.5. Notice that $R_T(t)$ is an increasing function in t.

Theorem 5.1.6. *Assume that conditions* (H_0)*–*(H_5) *and* (Λ) *hold,* $\beta(t) \leq \alpha(t)$*, and* p *is a positive real number. If* $0 \leq p(t) \leq p < \infty$ *hold, then every solution of* (E_6) *is either oscillatory or converges to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (E_6), say $y(t)$ is an eventually positive solution. (The proof in case $y(t) < 0$ eventually is similar and will be omitted.) There exists $t_1 \in [t^*, \infty)_{\mathbb{T}}$ such that $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, $y(\gamma(t))$ and $y(\beta(\alpha(t)))$ are all positive for all $t \in [t_1, \infty)_{\mathbb{T}}$. Define the functions

$$z(t) = y(t) + p(t)y(\alpha(t)), \tag{5.19}$$

$$k(t) = (-1)^{m+n-2} \int_t^\infty \frac{h_{n-1}(t,\sigma(s))}{r(s)} \int_s^\infty h_{m-1}(s,\sigma(\theta))h(\theta)H(y(\gamma(\theta)))\Delta\theta\Delta s. \tag{5.20}$$

Notice that condition (H_1) and the fact that H is a bounded function imply that $k(t)$ exists for all t. Now if we let

$$w(t) = z(t) - k(t) = y(t) + p(t)y(\alpha(t)) - k(t), \tag{5.21}$$

then a calculation shows

$$(r(t)w^{\Delta^n}(t))^{\Delta^m} = -q(t)G(y(\beta(t))) \leqslant (\not\equiv)0, \tag{5.22}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. Thus, $w(t)$ is monotonic on $[t_1, \infty)_{\mathbb{T}}$. Since $w(t)$ is monotonic then we have two cases to consider, namely $w(t) > 0$ or $w(t) < 0$ for $t \geqslant t_2$, for some $t_2 \geqslant t_1$. Suppose that $w(t) > 0$ for $t \geqslant t_2$. Then by Lemma 5.1.2, there exists an integer $l \in [0, n+m]_{\mathbb{Z}}$ such that $n + m + l$ is odd and one of the cases (a) or (b) holds.

Since $w(t) > 0$, then there exists $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $w(\alpha(t)) > 0, w(\beta(t)) > 0$ for $t \in [t_3, \infty)_{\mathbb{T}}$. Using ($H_2$), ($H_3$) and ($\Lambda$) gives

$$\begin{aligned}
0 &= (r(t)w^{\Delta^n}(t))^{\Delta^m} + q(t)G(y(\beta(t))) + G(p)(r(\alpha(t))w^{\Delta^n}(\alpha(t)))^{\Delta^m} \\
&\quad + G(p)q(\alpha(t))G(y(\beta(\alpha(t)))) \\
&\geqslant (r(t)w^{\Delta^n}(t))^{\Delta^m} + G(p)(r(\alpha(t))w^{\Delta^n}(\alpha(t)))^{\Delta^m} + \lambda Q(t)G(y(\beta(t)) \\
&\quad + py(\alpha(\beta(t)))) \\
&\geqslant (r(t)w^{\Delta^n}(t))^{\Delta^m} + G(p)(r(\alpha(t))w^{\Delta^n}(\alpha(t)))^{\Delta^m} + \lambda Q(t)G(z(\beta(t))) \qquad (5.23)
\end{aligned}$$

for $t \geqslant t_3$. From (5.20), it follows that $k(t) > 0$ and $k^{\Delta}(t) < 0$ and since $w(\beta(t)) > 0$ for $t \geqslant t_3$ implies $w(\beta(t)) < z(\beta(t))$ for $t \geqslant t_3$. Therefore, (5.23) yields

$$(r(t)w^{\Delta^n}(t))^{\Delta^m} + G(p)(r(\alpha(t))w^{\Delta^n}(\alpha(t)))^{\Delta^m} + \lambda Q(t)G(w(\beta(t))) \leqslant 0 \qquad (5.24)$$

for $t \geqslant t_3$. Choose $T' \in [T, \infty)_{\mathbb{T}}$ so that $\beta(t) > T \geqslant t_3$ for all $t \in [T', \infty)_{\mathbb{T}}$. First, we consider the *cases* (a) and (b) with $l \geq 1$. Then from Lemma 5.1.4 and (H_3), inequality (5.24) yields

$$\begin{aligned}
0 \geqslant\ & (r(t)w^{\Delta^n}(t))^{\Delta^m} + G(p)(r(\alpha(t))w^{\Delta^n}(\alpha(t)))^{\Delta^m} \\
&+ \lambda Q(t)G(R_T(\beta(t)))G((r(\beta(t))w^{\Delta^n}(\beta(t)))^{\Delta^{m-1}})
\end{aligned}$$

for $t \geqslant T'$. Hence,

$$\begin{aligned}
&\lambda Q(t)G(R_T(\beta(t))) \\
&\quad \leqslant -[G((r(\beta(t))w^{\Delta^n}(\beta(t)))^{\Delta^{m-1}})]^{-1}\{(r(t)w^{\Delta^n}(t))^{\Delta^m} \\
&\qquad + G(p)(r(\alpha(t))w^{\Delta^n}(\alpha(t)))^{\Delta^m}\} \\
&\quad \leqslant -[G((r(t)w^{\Delta^n}(t))^{\Delta^{m-1}})]^{-1}(r(t)w^{\Delta^n}(t))^{\Delta^m} \\
&\qquad - G(p)[G((r(\alpha(t))w^{\Delta^n}(\alpha(t)))^{\Delta^{m-1}})]^{-1}(r(\alpha(t))w^{\Delta^n}(\alpha(t)))^{\Delta^m}. \qquad (5.25)
\end{aligned}$$

Since $\lim_{t\to\infty}(r(t)w^{\Delta^n}(t))^{\Delta^{m-1}}$ exists, then applying (H_4) to the inequality (5.25), we obtain

$$\int_{T'}^{\infty} Q(t)G(R_T(\beta(t)))\Delta t < \infty,$$

which contradicts (H_5), since $R_T(t)$ is a monotonic increasing function.

If we consider the *case* (b) with $l = 0$ (it is evident from Lemma 5.1.2 that *case* (b) only possible), then from Lemma 5.1.2, $w^{\Delta}(t) < 0$. This implies that $w(t)$ is decreasing and bounded. By integrating, the inequality (5.24), over $[T', t]_{\mathbb{T}}$ yields

$$G(w(\beta(t))) \int_{T'}^{t} Q(t)\Delta t < \infty.$$

This implies that $w(\beta(t)) \to 0$ as $t \to \infty$, because G is continuous and due to (H_5). It follows that $\lim_{t\to\infty} w(t) = 0$, since $w(t) \leq w(\beta(t))$. Due to the fact (5.21), $\lim_{t\to\infty} z(t) = 0$. It follows that $\lim_{t\to\infty} y(t) = 0$, since $y(t) \leq z(t)$.

Next, we suppose that $w(t) < 0$ for $t \geqslant t_2 > t_1$. Since $w(t) < 0$, then $z(t) - k(t) < 0$ implies $y(t) \leqslant z(t) = y(t) + p(t)y(\alpha(t)) < k(t)$. Thus, $y(t)$ is bounded. It follows that $w(t)$ is bounded as well as $z(t)$ due to (5.21) and (5.19) respectively. Since $w(t) < 0$, then by Lemma 5.1.3, there exists an integer $l \in [0, n+m]_{\mathbb{Z}}$ with $n+m+l$ is even such that one of the cases (c) or (d) are holds. The *cases* (c) and (d) with $l > 1$ does not hold, because in these cases $w(t)$ will be unbounded by Lemma 5.1.1, which is a contradiction to the boundedness of $w(t)$. Therefore, the *case* (d) with $l \leq 1$ are hold. In these cases, $\lim_{t\to\infty} w(t) = k(\leq 0)$ say. Due to the fact (5.21), $\lim_{t\to\infty} z(t) = k(\leq 0)$. If $k < 0$, then $z(t)$ will be negative for large t, which is a contradiction to the positiveness of $z(t)$. Therefore, $k = 0$, it follows that $\lim_{t\to\infty} y(t) = 0$, since $y(t) \leq z(t)$. Thus, the theorem is proved. □

The following corollary is immediate.

Corollary 5.1.7. *Under the conditions of Theorem 5.1.6, every unbounded solution of* (E_6) *oscillates.*

Example 5.1.8. Let $\mathbb{T} = \mathbb{R}$ and consider the seond order differential equation

$$\Big(e^{-t/10}\left(y(t) + e^{-t/2}y(t/2)\right)'\Big)' + y^{1/3}(t/2) \\ - (11/5\ e^{-11t/10} + e^{-t/6})e^{t/20}(1 + e^{-t/10})\frac{y(t/20)}{1 + y^2(t/20)} = 0, \quad (5.26)$$

$t \in [t_0, \infty)_{\mathbb{T}}$. It is easy to see that equation (5.26) satisfies all the conditions of Theorem 5.1.6. Hence, any soluton of equation (5.26) oscillates or converge to zero as $t \to \infty$. In particular, $y(t) = e^{-t}$ is solution of (5.26) that converges to zero as $t \to \infty$.

Remark 5.1.9. In Theorem 5.1.6, if equation (E_6) is of even order ($n+m$ is even), then we can replace the condition (H_5) by

$$\int_{T'}^{\infty} G(R_T(\beta(t)))Q(t)\Delta t = \infty, \ where\ \beta(t) \geq T\ for\ all\ t \geq T'.$$

Remark 5.1.10. Note that (H_5) implies $\int_{T'}^{\infty} G(R_T(\beta(t)))Q(t)\Delta t = \infty$.

Theorem 5.1.11. *Assume that conditions (H_0)–(H_3), (H_5), (Λ) hold, $\beta(t) \leq \alpha(t)$, and*

(H_6) $G(x_1)/x_1^{\gamma} \geqslant G(x_2)/x_2^{\gamma}$ for $x_1 > x_2 > 0$ and some $\gamma \geq 1$.

If $0 \leq p(t) \leq p < \infty$ hold, then every solution of (E_6) *is either oscillatory or converges to zero as $t \to \infty$.*

Proof. Proceeding as in the proof of Theorem 5.1.6, when $w(t) > 0$, we can obtain inequality (5.24) for $t \geqslant t_3$. First, we consider the *cases* (a) and (b) with $l \geq 1$ are hold. In these cases $w(t)$ is increasing, because $w^{\Delta}(t) > 0$. From this we can find $k(>0) \in \mathbb{R}$ such that $w(t) \geqslant k$. We can find $t_4 \in [t_3, \infty)_{\mathbb{T}}$ such that $w(\beta(t)) \geqslant k$ and $\beta(t) \geq T$ for all $t \in [t_4, \infty)_{\mathbb{T}}$. Using ($H_6$) and Lemma 5.1.4, we obtain

$$\begin{aligned} G(w(\beta(t))) &= \left(\frac{G(w(\beta(t)))}{w^{\gamma}(\beta(t))}\right) w^{\gamma}(\beta(t)) \\ &\geqslant \left(\frac{G(k)}{k^{\gamma}}\right) w^{\gamma}(\beta(t)) \\ &> \left(\frac{G(k)}{k^{\gamma}}\right) R_T^{\gamma}(\beta(t))\Big((r(\beta(t))w^{\Delta^n}(\beta(t)))^{\Delta^{m-1}}\Big)^{\gamma} \end{aligned}$$

for $t \geq t_4$. Thus, (5.24) yields

$$\begin{aligned} &\lambda\left(\frac{G(k)}{k^{\gamma}}\right) Q(t) R_T^{\gamma}(\beta(t))\big((r(\beta(t))w^{\Delta^n}(\beta(t)))^{\Delta^{m-1}}\big)^{\gamma} \\ &\qquad < \lambda Q(t) G(w(\beta(t))) \\ &\qquad \leqslant -(r(t)w^{\Delta^n}(t))^{\Delta^m} - G(p)(r(\alpha(t))w^{\Delta^n}(\alpha(t)))^{\Delta^m} \end{aligned}$$

for $t \geq t_4$. Since $\lim_{t\to\infty}(r(t)w^{\Delta^n}(t))^{\Delta^{m-1}}$ exists and $R_T(t)$ is nondecreasing, then proceeding as in the proof of Theorem 5.1.6, we obtain

$$\int_{t_4}^{\infty} R_T^{\gamma}(\beta(t))Q(t)\Delta t < \infty,$$

which contradicts (H_5). The proof for the *case* (b) with $l = 0$ when $w(t) > 0$, and for the *cases* (c) and (d) when $w(t) < 0$ is follows from the proof of Theorem 5.1.6. This completes the proof of the theorem. □

We again have a corollary.

Corollary 5.1.12. *Under the conditions of Theorem 5.1.11, every unbounded solution of* (E_6) *oscillates.*

Remark 5.1.13. In Theorem 5.1.11, if the equation (E_6) is of even order ($n + m$ is even), then we can replace the condition (H_5) by

$$\int_{T'}^{\infty} R_T^{\gamma}(\beta(t))Q(t)\Delta t = \infty,\ \gamma \geq 1,\ where\ \beta(t) \geq T\ for\ all\ t \geq T'.$$

Remark 5.1.14. Note that (H_5) implies $\int_{T'}^{\infty} R_T^{\gamma}(\beta(t))Q(t)\Delta t = \infty$.

The following example is related to the Remark 5.1.13.

Example 5.1.15. Let $\mathbb{T} = \mathbb{R}$, and consider the differential equation

$$\Big(e^{-3t}\big(y(t) + e^{-2\pi}y(t-2\pi)\big)''\Big)'' + e^{-t-9\pi}y^3(t-3\pi) - 33e^{-\frac{11t+\pi}{3}}(1 + e^{-\frac{2(t-\pi)}{3}})\frac{y^{1/3}(t-\pi)}{1 + y^{2/3}(t-\pi)} = 0, \quad (5.27)$$

for $t(\geq 3\pi) \in \mathbb{R}$. Here $r(t) = e^{-3t}$, $p(t) = e^{-2\pi}$, $q(t) = e^{-t-9\pi}$, $h(t) = 33e^{-\frac{11t+\pi}{3}}(1 + e^{-\frac{2(t-\pi)}{3}})$, $G(u) = u^3$, and $H(u) = \frac{u^{1/3}}{1+u^{2/3}}$. It is now easy to see that equation (5.27) satisfies all the conditions of Remark 5.1.13. Hence, any solution of equation (5.27) oscillates or converges to 0 as $t \to \infty$. In particular, $y(t) = e^{-t}$ is a solution of equation (5.27).

In our next theorem we are able to replace conditions (H_3) and (H_4) in Theorem 5.1.6, with conditions (H_7) and (H_8) below.

Theorem 5.1.16. *Assume that conditions (H_0)–(H_2), (H_5), (Λ) hold, and*

(H_7) $G(u)G(v) \geqslant G(uv)$ *for* $u,\ v > 0$,

(H_8) $G(-u) = -G(u)$ *and* $H(-u) = -H(u)$ *for* $u \in \mathbb{R}$.

If $0 \leqslant p(t) \leqslant p < \infty$ *hold, then every solution of* (E_6) *is either oscillatory or converges to zero as* $t \to \infty$.

Proof. Proceeding as in the proof of Theorem 5.1.6 when $w(t) > 0$, we can again obtain inequality (5.24) for $t \geqslant t_3$. By Lemma 5.1.2, there exists an integer $l \in [0, n+m]_Z$ with $n+m+l$ is odd such that one of the cases (a) or (b) are holds. For the *case* (b) with $l = 0$, $w(t)$ is decreasing and bounded, because $w^\Delta(t) < 0$. From this, we have $\lim_{t\to\infty} w(t) = k(\geq 0) \in \mathbb{R}$. If $k > 0$, then we can find $t_4 \in [t_3, \infty)_\mathbb{T}$ such that $w(\beta(t)) \geqslant k_1$, where $k_1 = k/2$. Consequently, (5.24) yields

$$\lambda G(k_1) \int_{t_4}^{\infty} Q(t)\Delta t < \infty,$$

contradicting (H_5). If $k = 0$, then $\lim_{t\to\infty} w(t) = 0$. Since $k(t)$ is bounded and monotonic then from (5.21), we have $\lim_{t\to\infty} z(t) = 0$. Hence, $\lim_{t\to\infty} y(t) = 0$, since $y(t) \leqslant z(t)$.
Next consider the *cases* (a) and (b) with $l \geqslant 1$. Hence $w(t)$ is increasing since $w^\Delta(t) > 0$. Then we can find $t_4 \in [t_3, \infty)_\mathbb{T}$ and $k(> 0) \in \mathbb{R}$ such that $w(\beta(t)) \geqslant k$ for all $t \in [t_4, \infty)_\mathbb{T}$. Consequently, (5.24) yields

$$\lambda G(k) \int_{t_4}^{\infty} Q(t)\Delta t < \infty,$$

contradicting (H_5). The remaining part of the proof (i.e., when $w(t) < 0$) follows from the Theorem 5.1.6. □

We again have a corollary for the unbounded solutions.

Corollary 5.1.17. *Under the conditions of Theorem 5.1.16, every unbounded solution of* (E_6) *oscillates.*

Example 5.1.18. Let $\mathbb{T} = \mathbb{Z}$, and consider the difference equation

$$\Delta^2\Big(e^{-n/3}\Delta^2\big(y(n) + e^{-4n+2}y(n-2)\big)\Big) + e(e+1)^2(e^{2/3}+1)^2e^{n/3}y^{1/3}(n-3)$$
$$- e^2(e^{-3}+1)^2(e^{-10/3}+1)^2e^{-13n/3}(1+e^{n-2})\frac{y(n-2)}{1+\mid y(n-2)\mid} = 0, \quad (5.28)$$

for $n(\geq 3) \in \mathbb{T}$. Here $r(n) = e^{-n/3}$, $p(n) = e^{-4n+2}$, $q(n) = e(e+1)^2(e^{2/3}+1)^2e^{n/3}$, $h(n) = e^2(e^{-3}+1)^2(e^{-10/3}+1)^2e^{-13n/3}(1+e^{n-2})$, $G(u) = u^{1/3}$, and $H(u) = \frac{u}{1+|u|}$. It is easy to see that equation (5.28) satisfies all the conditions of Theorem 5.1.16. Hence, any soluton of equation (5.28) oscillates or converge to 0 as $t \to \infty$. In particular, $y(n) = (-1)^n e^n$ is a oscillatory solution of (5.28).

Remark 5.1.19. Notice that in Theorem 5.1.6 and Corollary 5.1.7, G is sublinear, whereas in Theorem 5.1.11 and Corollary 5.1.12, G is superlinear. But in Theorem 5.1.16 and Corollary 5.1.17, G could be linear, sublinear, or superlinear.

Next, we consider the case where $p(t)$ is negative. Here p_4, p_5, and p_6 are negative real numbers.

Theorem 5.1.20. *Let* $-1 < p_4 \leqslant p(t) \leqslant 0$ *hold,* $\beta(t) < t$ *and conditions* (H_0), (H_1), (H_3), (H_4),

(H_9) $\int_{t_3}^{\infty} q(t)G(R_T(\beta(t)))\Delta t = \infty,$ *and*

(H_{10}) $\int_{t_3}^{\infty} q(t)G(R_{\beta(t)}(t))\Delta t = \infty,$

hold. Then every solution of (E_6) *is either oscillatory or tends to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (E_6), say $y(t) > 0$ eventually. From (5.19)-(5.21), we obtain (5.22), for $t \geqslant t_1$. Since $w(t)$ is monotonic, then we have two cases to consider, namely $w(t) > 0$ or $w(t) < 0$ for $t \geqslant t_2$, for some $t_2 \geqslant t_1$. Suppose that $w(t) > 0$ for $t \geqslant t_2 > t_1$, then one of the *cases* (a) or (b) are holds by Lemma 5.1.2.

First, if we consider the *cases* (a) and (b) with $l \geq 1$, then $w(t) > R_T(t)(r(t)w^{\Delta^n}(t))^{\Delta^{m-1}}$ for $t > T \geqslant t_2$ by Lemma 5.1.4. Moreover, $w(t) \leq z(t) \leq y(t)$, since $p(t) \leq 0$. Choose $t_3 \in [T, \infty)_{\mathbb{T}}$ such that $\beta(t) \geq T$ for all $t \in [t_3, \infty)_{\mathbb{T}}$. Then, $y(\beta(t)) > R_T(\beta(t))(r(\beta(t))w^{\Delta^n}(\beta(t)))^{\Delta^{m-1}}$ for $t \geq t_3$, and (5.22) becomes

$$\int_{t_3}^{\infty} q(t)G(R_T(\beta(t)))\Delta t < \infty,$$

which contradicts (H_9) due to (H_4).
If $l = 0$, then from Lemma 5.1.4, we obtain

$$w(\beta(t)) > R_{\beta(t)}(t)(r(t)w^{\Delta^n}(t))^{\Delta^{m-1}}$$

for $t > \beta(t) \geqslant t_2$, $t \in [t_3, \infty)_{\mathbb{T}}$. Hence (5.22) yields

$$q(t)G(R_{\beta(t)}(t))G((r(t)w^{\Delta^n}(t))^{\Delta^{m-1}}) \leq -(r(t)w^{\Delta^n}(t))^{\Delta^m}.$$

Integrating the above inequality from t_3 to ∞, we obtain

$$\int_{t_3}^{\infty} q(t)G(R_{\beta(t)}(t))\Delta t < \infty,$$

which contradicts (H_{10}) due to (H_4). Hence $w(t) < 0$ for $t \geqslant t_2$. From Lemma 5.1.3 there exists $l \in [0, m+n]_{\mathbb{Z}}$ such that one of the cases (c) or (d) are holds. We claim that $y(t)$ is bounded. If this is not possible, then there is an increasing sequence $\{\tau_n\}_{n=1}^{\infty} \subset [t_2, \infty)_{\mathbb{T}}$ such that $\tau_n \to \infty$, $y(\tau_n) \to \infty$ as $n \to \infty$, where $y(\tau_n) = \max\{y(t) : t_2 \leqslant t \leqslant \tau_n\}$. We may choose τ_1 large enough so that $\alpha(\tau_1) \geqslant t_2$. Hence,

$$0 \geqslant w(\tau_n) \geqslant y(\tau_n) + p(\tau_n)y(\alpha(\tau_n)) - k(\tau_n) \geqslant (1 + p_4)y(\tau_n) - k(\tau_n).$$

Since $k(\tau_n)$ is bounded and $1 + p_4 > 0$, we have $w(\tau_n) > 0$ for large n, which is a contradiction. Thus, our claim holds. Since $y(t)$, $k(t)$, and $p(t)$ all are bounded, it follows that both $z(t)$ and $w(t)$ are bounded due to (5.19) and (5.21) respectively. For the *cases* (c) and (d) with $l > 1$, $w(t)$ will be unbounded by Lemma 5.1.1, which is a contradiction to $w(t)$ is bounded. Therefore, the *cases* when $l > 1$ is not possible. Again, the *case* (d) with $l \leq 1$, $\lim_{t\to\infty} w(t)$ exists (finite), which is non-positive. Consequently,

$$\begin{aligned}
0 \geqslant \limsup_{t\to\infty} w(t) &= \limsup_{t\to\infty} z(t) \\
&\geqslant \limsup_{t\to\infty} (y(t) + p_4 y(\alpha(t))) \\
&\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty} (p_4 y(\alpha(t))) \\
&= \limsup_{t\to\infty} y(t) + p_4 \limsup_{t\to\infty} y(\alpha(t)) \\
&= (1 + p_4) \limsup_{t\to\infty} y(t).
\end{aligned}$$

Since $(1 + p_4) > 0$, then $\limsup_{t\to\infty} y(t) = 0$. Hence $\lim_{t\to\infty} y(t) = 0$. This completes the proof of the theorem. □

Corollary 5.1.21. *Under the conditions of Theorem 5.1.20, every unbounded solution of* (E_6) *oscillates.*

Remark 5.1.22. In Theorem 5.1.20, if the equation (E_6) is of even order (m+n even), then the conditions $\beta(t) < t$ and (H_{10}) is not required.

Example 5.1.23. Let $\mathbb{T} = h\mathbb{Z}$, (where h is a ratio of odd positive integers) and consider the h-difference equation

$$\Delta_h\Big(e^{-t/3}\Delta_h^2\big(y(t) - (1-e^{-2t})y(t-h)\big)\Big) + (1+e^{-h})\Big(\frac{e^h+1}{h}\Big)^2\Big(\frac{e^{2h/3}+1}{h}\Big)e^{4h/3}e^{t/3}y^{1/3}(t-4h) - e^{h/5}\Big(\frac{e^{-h}+1}{h}\Big)^2\Big(\frac{e^{-4h/3}+1}{h}\Big)(1+e^{\frac{t-6h}{5}})e^{-23t/15}\frac{y^{1/5}(t-6h)}{(1+\mid y^{1/5}(t-6h)\mid)} = 0, \quad (5.29)$$

for $t \in [6h,\infty)_{\mathbb{T}}$. Here $r(t) = e^{-t/3}$, $q(t) = (1+e^{-h})\Big(\frac{e^h+1}{h}\Big)^2\Big(\frac{e^{2h/3}+1}{h}\Big)e^{4h/3}e^{t/3}$, $p(t) = -(1-e^{-2t})$, $h(t) = e^{h/5}\Big(\frac{e^{-h}+1}{h}\Big)^2\Big(\frac{e^{-4h/3}+1}{h}\Big)(1+e^{\frac{t-6h}{5}})e^{-23t/15}$, $G(u) = u^{1/3}$ and $H(u) = \frac{u^{1/5}}{(1+|u^{1/5}|)}$. It is easy to see that equation (5.29) satisfies all the conditions of Theorem 5.1.20. Hence, any soluton of equation (5.29) oscillates or converge to 0 as $t \to \infty$. In particular, $y(t) = (-1)^t e^t$ is an oscillatory solution of (5.29).

Next theorem deals with the behaviour of bounded solution of (E_6).

Theorem 5.1.24. *Assume that conditions* (H_0), (H_1), (H_8), *and*

$$(H_{11}) \quad \int_{t_0}^{\infty} q(t)\Delta t = \infty,$$

hold. If $-\infty < p_5 \leqslant p(t) \leqslant p_6 < -1$, *then every bounded solution of* (E_6) *either oscillates or tends to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a bounded nonoscillatory solution of (E_6), say $y(t)$ is an eventually positive. With (5.19), (5.20), and (5.21) as defined earlier in the Theorem 5.1.6, we obtain (5.22) for $t \geqslant t_1$. Since $y(t)$, $p(t)$, and $k(t)$ all are bounded, then from (5.19) and (5.21), follows that $z(t)$ and $w(t)$ are bounded. Then (5.22) follows that $w(t)$ is monotonic. First, suppose that $w(t) > 0$ for $t \geqslant t_2 > t_1$, then from Lemma 5.1.2, there exists $l \in [0, m+n]_{\mathbb{Z}}$ with $m+n+l$ is odd such that one of the cases (a) or (b) are holds. For the *cases* (a) and (b) with $l > 1$, $w(t)$ will be unbounded by Lemma 5.1.1, which is a contradiction to $w(t)$ is bounded. Again, if we consider the *case* (b) with $l \leq 1$, $\lim_{t\to\infty} w(t)$ exists (finite), which is non-negative, because $w(t)$ is monotonic, bounded and $w(t) > 0$. Consequently,

$$\begin{aligned} 0 \leq \liminf_{t\to\infty} w(t) &= \liminf_{t\to\infty} z(t) \\ &\leq \liminf_{t\to\infty}(y(t) + p_6 y(\alpha(t))) \\ &\leq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_6 y(\alpha(t))) \\ &= \limsup_{t\to\infty} y(t) + p_6 \limsup_{t\to\infty} y(\alpha(t)) \\ &= (1 + p_6) \limsup_{t\to\infty} y(t). \end{aligned}$$

Since $(1 + p_6) < 0$, then we have $\limsup_{t\to\infty} y(t) \leq 0$, so $\lim_{t\to\infty} y(t) = 0$.

Next, suppose that $w(t) < 0$ for $t \geqslant t_2$. By Lemma 5.1.3, there exists $l \in [0, m+n]_{\mathbb{Z}}$ such that one of the cases (c) or (d) are holds. In the *cases* (c) and (d) with $l > 1$, $w(t)$ will be unbounded by Lemma 5.1.1, which is a contradiction to $w(t)$ is bounded. Therefore, $l > 1$ is not possible. So, $l \leq 1$ are possible. In these cases $\lim_{t\to\infty} w(t)$ exists (finite), it follows that $\lim_{t\to\infty} z(t)$ exists (finite) due to (5.21). By integrating (5.22), from t_2 to ∞, we obtain

$$\int_{t_2}^{\infty} q(t)G(y(\beta(t)))\Delta t < \infty,$$

it follows that $\liminf_{t\to\infty} y(t) = 0$ due to (H_{11}). From Lemma 1.5.1, it follows that $\lim_{t\to\infty} z(t) = 0$. Consequently,

$$\begin{aligned} 0 = \liminf_{t\to\infty} z(t) &= \liminf_{t\to\infty}(y(t) + p(t)y(\alpha(t))) \\ &\leq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p(t)y(\alpha(t))) \\ &= \limsup_{t\to\infty} y(t) + p_6 \limsup_{t\to\infty} y(\alpha(t)) \\ &= (1 + p_6) \limsup_{t\to\infty} y(t). \end{aligned}$$

Since $1 + p_6 < 0$, then $\limsup_{t\to\infty} y(t) = 0$. Hence, $\lim_{t\to\infty} y(t) = 0$. This completes the proof of the theorem. □

Remark 5.1.25. Note that (H_{11}) implies (H_9).

5.2 Sufficient conditions for oscillation of (E_7) with $\int_{t_0}^{\infty} \frac{1}{r(t)} \Delta t = \infty$.

This section is devoted to study the oscillatory and asymptotic behavior of solutions of the forced equation (E_7) with suitable forcing functions. Our attention is restricted to forcing functions that eventually change signs. We will make use of the following hypotheses on $f(t)$.

(H_{12}) There exists $F \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that $-\infty < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \infty$, $rF^{\Delta^n} \in C_{rd}^m([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, and $(rF^{\Delta^n})^{\Delta^m} = f$.

(H_{13}) There exists $F \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that $\liminf_{t\to\infty} F(t) = -\infty$, $\limsup_{t\to\infty} F(t) = \infty$, $rF^{\Delta^n} \in C_{rd}^m([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, and $(rF^{\Delta^n})^{\Delta^m} = f$.

Theorem 5.2.1. *Let $0 \leqslant p(t) \leqslant p < \infty$. Assume that (H_0)–(H_2), (H_7), (H_8), (H_{13}) and (Λ) hold. If*

$$(H_{14}) \qquad \limsup_{t\to\infty} \int_{t^*}^{t} Q(s)G(F(\beta(s)))\Delta s = +\infty \text{ and } \liminf_{t\to\infty} \int_{t^*}^{t} Q(s)G(F(\beta(s)))\Delta s = -\infty,$$

then equation (E_7) *is oscillatory.*

Proof. Let $y(t)$ be a nonoscillatory solution of (E_7), say, $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, $y(\gamma(t))$ and $y(\beta(\alpha(t)))$ are all positive for $t \in [t_1,\infty)_{\mathbb{T}}$ for some $t^* \in [t_0,\infty)_{\mathbb{T}}$, $t^* > t_0$. Defining $z(t)$, $k(t)$, and $w(t)$ as in (5.19), (5.20) and (5.21) respectively, equation (E_7) becomes

$$(r(t)w^{\Delta^n}(t))^{\Delta^m} + q(t)G(y(\beta(t))) = f(t). \tag{5.30}$$

Let

$$v(t) = w(t) - F(t) = z(t) - k(t) - F(t). \tag{5.31}$$

Then for $t \geqslant t_2$, equation (5.30) becomes

$$(r(t)v^{\Delta^n}(t))^{\Delta^m} = -q(t)G(y(\beta(t))) \leqslant (\not\equiv) 0. \tag{5.32}$$

Thus, $v(t)$ is monotonic on $[t_1, \infty)_{\mathbb{T}}$.

Suppose $v(t) > 0$ for $t \geqslant t_2$. From Lemma 5.1.2, one of the cases (a) or (b) holds. If $v(t) > 0$, then $z(t) > k(t) + F(t) \geq F(t)$ for $t \geqslant t_2$. Choose $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $\beta(t) \geqslant t_2$ for all $t \in [t_3, \infty)_{\mathbb{T}}$. Then $z(\beta(t)) > F(\beta(t))$ for $t \geqslant t_3$. Applying (H_2), (H_7) and (Λ) yields

$$\begin{aligned}
0 \ &\geqslant \ (r(t)v^{\Delta^n}(t))^{\Delta^m} + q(t)G(y(\beta(t))) + G(p)[(r(\alpha(t))v^{\Delta^n}(\alpha(t)))^{\Delta^m} \\
&\quad + q(\alpha(t))G(y(\beta(\alpha(t))))] \\
&\geqslant \ (r(t)v^{\Delta^n}(t))^{\Delta^m} + G(p)(r(\alpha(t))v^{\Delta^n}(\alpha(t)))^{\Delta^m} + \lambda Q(t)G(y(\beta(t)) \\
&\quad + py(\alpha(\beta(t)))) \\
&\geqslant \ (r(t)v^{\Delta^n}(t))^{\Delta^m} + G(p)(r(\alpha(t))v^{\Delta^n}(\alpha(t)))^{\Delta^m} + \lambda Q(t)G(z(\beta(t))) \\
&\geqslant \ (r(t)v^{\Delta^n}(t))^{\Delta^m} + G(p)(r(\alpha(t))v^{\Delta^n}(\alpha(t)))^{\Delta^m} + \lambda Q(t)G(F(\beta(t)))
\end{aligned}$$

for $t \geqslant t_3$. Integrating the above inequality, we obtain

$$\limsup_{t\to\infty} \int_{t_3}^{t} Q(s)G(F(\beta(s)))\Delta s < \infty,$$

which contradicts (H_{14}).

Therefore, $v(t) < 0$ for $t \geqslant t_1$. By Lemma 5.1.3, there exists $l \in [0, m+n]_{\mathbb{Z}}$ such that one of the cases (c) or (d) are holds. In each of these cases (c) or (d) we have $z(t) < k(t) + F(t)$. This implies $\liminf_{t\to\infty} z(t) < 0$, which is a contradiction. This completes the proof of the theorem. □

Example 5.2.2. Let $\mathbb{T} = q^{\mathbb{N}_0}$ ($q > 1$ is fixed). Consider the q-difference equation

$$\begin{aligned}
\Delta_q^2\Big(1/t\Delta_q\big(y(t) + 1/2y(t/q)\big)\Big) + \frac{1}{(q-1)^3}\Big(\frac{1}{q^2}+1\Big)\Big(\frac{1}{q^3}+1\Big)(1/t^6+1)y(t/q^3) - \\
\frac{1}{(q-1)^3}\Big(\frac{1}{q^2}+1\Big)\Big(\frac{1}{q^3}+1\Big)2/t^6\frac{y(t/q^5)}{1+y^4(t/q^5)} = \\
-\frac{1}{(q-1)^3}\Big(\frac{1}{q^2}+1\Big)\Big(\frac{1}{q^3}+1\Big)\Big(1/t^4+1\Big)(-1)^{\log_q^t}, \quad (5.33)
\end{aligned}$$

for $t \in [q^5, \infty)_{\mathbb{T}}$. In this example, we choose

$$F(t) = (1+1/q^2)(1+1/q^3)\Big(\frac{t^4}{(q^4+1)(q^2+1)(q+1)} + \frac{1}{2(1+1/q^2)(1+1/q^3)}\Big)(-1)^{\log_q^t}$$

such that

$$\Delta_q^2\Big(\frac{1}{t}\Delta_q F(t)\Big) = -\frac{1}{(q-1)^3}\Big(\frac{1}{q^2}+1\Big)\Big(\frac{1}{q^3}+1\Big)\Big(1/t^4+1\Big)(-1)^{\log_q^t}.$$

It is easy to see that equation (5.33) satisfies all the conditions of Theorem 5.2.1. Hence, any soluton of equation (5.33) oscillates. In particular, $y(t) = (-1)^{log_q^t}$ is an oscillatory solution of (5.33).

Theorem 5.2.3. *Let* $-\infty < p \leq p(t) \leqslant 0$. *Suppose that* (H_0), (H_1), (H_8) *and* (H_{13}) *hold. If*

$$(H_{15}) \quad \limsup_{t\to\infty} \int_{t_3}^{t} q(s)G(F(\beta(s)))\Delta s = +\infty \quad \textit{and}$$
$$\liminf_{t\to\infty} \int_{t_3}^{t} q(s)G(F(\beta(s)))\Delta s = -\infty,$$

then every solution of (E_7) *is either oscillates or* $\limsup_{t\to\infty} \mid y(t) \mid= \infty$.

Proof. Proceeding as in the proof of Theorem 5.2.1, we obtain (5.32) for $t \in [t_1, \infty)_{\mathbb{Z}}$. Thus, $v(t)$ is monotonic, so $v(t) > 0$ or $v(t) < 0$ for large $t \in [t_1, \infty)_{\mathbb{T}}$. If $v(t) > 0$ for $t \geqslant t_2 > t_1$, then one of the cases (a) or (b) of Lemma 5.1.2 holds for $t \geqslant t_2$. In these cases, $z(t) > z(t) - k(t) > F(t)$ implies that $z(t) > F(t)$, so $y(t) > z(t) > F(t)$ for $t \geqslant t_2$. Choose $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $\beta(t) \geqslant t_2$ for all $t \in [t_3, \infty)_{\mathbb{T}}$. Hence, for $t \geqslant t_3$, $y(\beta(t)) > z(\beta(t)) > F(\beta(t))$. From (5.32), we have

$$q(t)G(F(\beta(t))) \leqslant q(t)G(y(\beta(t))) = -(r(t)v^{\Delta^n}(t))^{\Delta^m} \tag{5.34}$$

for $t \geqslant t_3$. Integrating the inequality (5.34), we obtain

$$\limsup_{t\to\infty} \int_{t_3}^{t} q(s)G(F(\beta(s)))\Delta s < \infty,$$

which contradicts (H_{15}).

Now assume $v(t) < 0$ for $t \geqslant t_2$. Thus, $z(t) - k(t) < F(t)$ and condition (H_{13}), then implies that $\liminf_{t\to\infty} z(t) = -\infty$. This implies that $\liminf_{t\to\infty}(py(\alpha(t))) \leq -\infty$. Consequently, $p \limsup_{t\to\infty} y(\alpha(t)) \leq -\infty$. Thus, $\limsup_{t\to\infty} y(t) = \infty$. □

Remark 5.2.4. We can drop the condition (H_{15}) from the hypotheses of Theorem 5.2.3, and obtain that bounded solutions oscillates. If $v(t) > 0$ then $y(t) > z(t) > F(t)$, and condition (H_{13}) contradicts the boundedness of $y(t)$. In case $v(t) < 0$ for $t \geq t_2$, $z(t) - k(t) < F(t)$ and condition (H_{13}) implies that $\liminf_{t\to\infty} z(t) = -\infty$, contradicts the boundedness of $y(t)$.

Theorem 5.2.5. *Assume that* (H_0)–(H_2), (H_7), (H_8), (H_{12}), (H_{14}) *and* (Λ) *hold. If* $0 \leqslant p(t) \leqslant p < \infty$ *holds, then every unbounded solution of* (E_7) *oscillates.*

Proof. Let $y(t)$ be an unbounded nonoscillatory solution of (E_7), say $y(t)$ is an eventually positive solution. Using (5.19), (5.20), (5.21), and (5.31), we obtain inequality (5.32). Thus, $v(t)$ is monotonic, so first assume $v(t) > 0$ for all $t \geqslant t_2$. Proceeding as in the proof of Theorem 5.2.1, we again obtain a contradiction.

Next, let $v(t) < 0$ for $t \geqslant t_2 > t_1$. From Lemma 5.1.3, it follows that one of the cases (c) or (d) are holds. In the *case* (d) with $l = 0$, $\lim_{t\to\infty} v(t)$ exists and hence $z(t) = v(t) + k(t) + F(t)$ implies $y(t) \leqslant v(t) + k(t) + F(t)$. That is, $y(t)$ is bounded, which is a contradiction to $y(t)$ is unbounded. In the *cases* (c) and (d) with $l \geqslant 1$, $v(t)$ is nondecreasing on $[t_2, \infty)_{\mathbb{T}}$, so let $\lim_{t\to\infty} v(t) = k$ for some $k \in [-\infty, 0)$. If $k = -\infty$, then $y(t) \leqslant v(t)+k(t)+F(t)$, which in view of (H_{12}), we obtain $y(t)$ eventually becomes negative. If $-\infty < k < 0$, then $v(t) + k(t) + F(t) > y(t)$. Here all $v(t)$, $k(t)$ and $F(t)$ are bounded but $y(t)$ is not bounded, which is a contradiction. Therefore $l \geqslant 1$ is not possible. This completes the proof of the theorem. □

Our final theorem in this section gives sufficient conditions for equation (E_7) to have a **bounded positive solution.**

Theorem 5.2.6. *Assume that* $0 \leqslant p(t) \leqslant p_1 < 1$, *(*$H_1$*) and (*$H_{12}$*) hold with*

$$\frac{-1}{8}(1 - p_1) < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \frac{1}{4}(1 - p_1).$$

In addition, assume that G *and* H *are Lipschitzian on* $\mathbb{R}$ *with Lipschitz constants* G_1 *and* H_1 *respectively. If*

$$\int_{t_0}^{\infty} \frac{(\sigma(t))^{n-1}}{r(t)} \int_t^{\infty} (\sigma(s))^{m-1} q(s) \Delta s \Delta t < \infty,$$

then (E_7) *admits a positive bounded solution.*

Proof. Choose $t_1 > t_0$ large enough so that

$$\int_{t_1}^{\infty} \frac{(\sigma(t))^{n-1}}{r(t)} \int_t^{\infty} (\sigma(s))^{m-1} h(s) \Delta s \Delta t < \frac{1-p_1}{4H(1)},$$

with (H_1) and

$$\int_{t_1}^{\infty} \frac{(\sigma(t))^{n-1}}{r(t)} \int_t^{\infty} (\sigma(s))^{m-1} q(s) \Delta s \Delta t < \frac{1-p_1}{4G(1)}.$$

Let $X = BC_{rd}([t_1,\infty)_{\mathbb{T}}, \mathbb{R})$ be the Banach space of all bounded rd-continuous functions on $[t_1,\infty)_{\mathbb{T}}$ with the norm

$$\|x\| = \sup\{| x(t) | : t \in [t_1,\infty)_{\mathbb{T}}\},$$

and let

$$S = \{x \in X : \frac{1}{8}(1-p_1) \leqslant x(t) \leqslant 1,\ t \in [t_1,\infty)_{\mathbb{T}}\}.$$

Then, S is a closed, bounded, and convex subset of X. Take $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that $\alpha(t)$, $\beta(t)$, $\gamma(t) \geqslant t_1$ for all $t \in [t_2,\infty)_{\mathbb{T}}$. Define the mappings A, $B : S \to S$ by

$$Ax(t) = \begin{cases} Ax(t_2) & \text{for } t \in [t_1,t_2)_{\mathbb{T}} \\ -p(t)x(\alpha(t)) + \frac{1+p_1}{2} & \text{for } t \in [t_2,\infty)_{\mathbb{T}}. \end{cases}$$

and

$$Bx(t) = \begin{cases} Bx(t_2) & \text{for } t \in [t_1,t_2)_{\mathbb{T}} \\ -(-1)^{m+n-2} \int_t^{\infty} \frac{h_{n-1}(t,\sigma(s))}{r(s)} \int_s^{\infty} h_{m-1}(s,\sigma(u)) q(u) G(x(\beta(u))) \Delta u \Delta s & \\ \quad +F(t) + k(t) & \text{for } t \in [t_2,\infty)_{\mathbb{T}}. \end{cases}$$

For $x \in S$, we have

$$\begin{aligned} k(t) &= (-1)^{m+n-2} \int_t^{\infty} \frac{h_{n-1}(t,\sigma(s))}{r(s)} \int_s^{\infty} h_{m-1}(s,\sigma(u)) h(u) H(x(\gamma(u))) \Delta u \Delta s \\ &\leqslant H(1)(-1)^{m+n-2} \int_t^{\infty} \frac{(t-\sigma(s))^{n-1}}{r(s)} \int_s^{\infty} (s-\sigma(u))^{m-1} h(u) \Delta u \Delta s \\ &= H(1) \int_t^{\infty} \frac{(\sigma(s)-t)^{n-1}}{r(s)} \int_s^{\infty} (\sigma(u)-s)^{m-1} h(u) \Delta u \Delta s \\ &\leqslant H(1) \int_t^{\infty} \frac{(\sigma(s))^{n-1}}{r(s)} \int_s^{\infty} (\sigma(u))^{m-1} h(u) \Delta u \Delta s \\ &< \frac{1}{4}(1-p_1), \end{aligned}$$

and similarly we can prove that,

$$(-1)^{m+n-2}\int_t^\infty \frac{h_{n-1}(t,\sigma(s))}{r(s)}\int_s^\infty h_{m-1}(s,\sigma(u))q(u)G(x(\beta(u)))\Delta u\Delta s \le \frac{1-p_1}{4},$$

due to the Property 1.4.1. For all x, $y \in S$ and all $t \in [t_2,\infty)_{\mathbb{T}}$, we have

$$Ax(t)+By(t) \leqslant \frac{1+p_1}{2}+\frac{1}{4}(1-p_1)+\frac{1}{4}(1-p_1)=1$$

and

$$Ax(t)+By(t) \geqslant -p_1+\frac{1+p_1}{2}-\frac{1}{8}(1-p_1)-\frac{1}{4}(1-p_1)=\frac{1-p_1}{8}.$$

Thus, $Ax+By \in S$. To show that A is a contraction mapping on S, first notice that

$$\begin{aligned}
\|Ax-Ay\| &= \left\|-p(t)x(\alpha(t))+\frac{1+p_1}{2}+p(t)y(\alpha(t))-\frac{1+p_1}{2}\right\| \\
&= \|-p(t)(x(\alpha(t))-y(\alpha(t)))\| \\
&\leqslant p_1\|x(\alpha(t))-y(\alpha(t))\| \\
&= p_1\|x(t)-y(t)\|.
\end{aligned}$$

Since $p_1 < 1$, A is a contraction mapping. To show that B is completely continuous on S, we need to show that B is continuous and maps bounded sets into relatively compact sets. In order to show that B is continuous, let x, $x_k = x_k(t) \in S$ be such that $\|x_k - x\| = \sup_{t\geqslant t_1}\{\mid x_k(t)-x(t)\mid\} \to 0$. Since S is closed, $x(t) \in S$. By using Property 1.4.1 and for $t \geqslant t_1$, we have

$$\begin{aligned}
&\|(Bx_k)-(Bx)\| \\
&= \Bigg\|F(t)+(-1)^{m+n-2}\int_t^\infty \frac{h_{n-1}(t,\sigma(s))}{r(s)}\int_s^\infty h_{m-1}(s,\sigma(u))h(u)H(x_k(\gamma(u)))\Delta u\Delta s \\
&\quad -(-1)^{m+n-2}\int_t^\infty \frac{h_{n-1}(t,\sigma(s))}{r(s)}\int_s^\infty h_{m-1}(s,\sigma(u))q(u)G(x_k(\beta(u)))\Delta u\Delta s \\
&\quad -F(t)-(-1)^{m+n-2}\int_t^\infty \frac{h_{n-1}(t,\sigma(s))}{r(s)}\int_s^\infty h_{m-1}(s,\sigma(u))h(u)H(x(\gamma(u)))\Delta u\Delta s \\
&\quad +(-1)^{m+n-2}\int_t^\infty \frac{h_{n-1}(t,\sigma(s))}{r(s)}\int_s^\infty h_{m-1}(s,\sigma(u))q(u)G(x(\beta(u)))\Delta u\Delta s\Bigg\|,
\end{aligned}$$

that is,

$$\begin{aligned}
&\|(Bx_k) - (Bx)\| \\
&= \Bigg\|(-1)^{m+n-2}\int_t^\infty \frac{h_{n-1}(t,\sigma(s))}{r(s)}\int_s^\infty h_{m-1}(s,\sigma(u))h(u)\Big(H(x_k(\gamma(u))) \\
&\quad -H(x(\gamma(u)))\Big)\Delta u\Delta s \\
&\quad + (-1)^{m+n-2}\int_t^\infty \frac{h_{n-1}(t,\sigma(s))}{r(s)}\int_s^\infty h_{m-1}(s,\sigma(u))q(u)\Big(G(x(\beta(u))) \\
&\quad - G(x_k(\beta(u)))\Big)\Delta u\Delta s\Bigg\| \\
&\leqslant H_1\|x_k - x\|(-1)^{m+n-2}\int_t^\infty \frac{(t-\sigma(s))^{n-1}}{r(s)}\int_s^\infty (s-\sigma(u))^{m-1}h(u)\Delta u\Delta s \\
&\quad + G_1\|x_k - x\|(-1)^{m+n-2}\int_t^\infty \frac{(t-\sigma(s))^{n-1}}{r(s)}\int_s^\infty (s-\sigma(u))^{m-1}q(u)\Delta u\Delta s \\
&= H_1\|x_k - x\|\int_t^\infty \frac{(\sigma(s)-t)^{n-1}}{r(s)}\int_s^\infty (\sigma(u)-s)^{m-1}h(u)\Delta u\Delta s \\
&\quad + G_1\|x_k - x\|\int_t^\infty \frac{(\sigma(s)-t)^{n-1}}{r(s)}\int_s^\infty (\sigma(u)-s)^{m-1}q(u)\Delta u\Delta s \\
&\leqslant H_1\|x_k - x\|\int_t^\infty \frac{(\sigma(s))^{n-1}}{r(s)}\int_s^\infty (\sigma(u))^{m-1}h(u)\Delta u\Delta s \\
&\quad + G_1\|x_k - x\|\int_t^\infty \frac{(\sigma(s))^{n-1}}{r(s)}\int_s^\infty (\sigma(u))^{m-1}q(u)\Delta u\Delta s \\
&\leqslant \frac{1}{2}(1-p_1)\|x - x_k\|.
\end{aligned}$$

Since for all $t \geqslant t_1$, $\{x_k(t)\}$ converges uniformly to $x(t)$ as $k \to \infty$, we have $\lim_{k\to\infty}\|(Bx_k)-(Bx)\|= 0$. Thus, B is continuous. To show that BS is relatively compact, it suffices to show that the family of functions $\{Bx : x \in S\}$ is uniformly bounded and equicontinuous on $[t_1,\infty)_{\mathbb{T}}$. The uniform boundedness is clear. To show that BS is equicontinuous, let $x \in S$ and $t^{''},\ t^{'} \geqslant t_1$ such that $t' > t''$. Then by the Corollary 1.4.3, we have

$$\begin{aligned}
&|(Bx)(t^{''}) - (Bx)(t^{'})| \\
&= \Bigg|F(t^{''}) + k(t^{''}) - (-1)^{m+n-2}\int_{t''}^\infty \frac{h_{n-1}(t^{''},\sigma(s))}{r(s)}\int_s^\infty h_{m-1}(s,\sigma(u))q(u)G(x(\beta(u)))\Delta u\Delta s \\
&\quad - F(t^{'}) - k(t^{'}) + (-1)^{m+n-2}\int_{t'}^\infty \frac{h_{n-1}(t^{'},\sigma(s))}{r(s)}\int_s^\infty h_{m-1}(s,\sigma(u))q(u)G(x(\beta(u)))\Delta u\Delta s\Bigg|,
\end{aligned}$$

that is,

$$
\begin{aligned}
&|(Bx)(t'') - (Bx)(t')| \\
=&\Big|F(t'') + k(t'') \\
&- (-1)^{m+n-3}\int_{t''}^{\infty}\int_{\theta}^{\infty}\frac{h_{n-2}(\theta,\sigma(s))}{r(s)}\int_{s}^{\infty}h_{m-1}(s,\sigma(u))q(u)G(x(\beta(u)))\Delta u\Delta s \\
&- F(t') - k(t') \\
&+ (-1)^{m+n-3}\int_{t'}^{\infty}\int_{\theta}^{\infty}\frac{h_{n-2}(\theta,\sigma(s))}{r(s)}\int_{s}^{\infty}h_{m-1}(s,\sigma(u))q(u)G(x(\beta(u)))\Delta u\Delta s\Big| \\
=&\Big|F(t'') + k(t'') - F(t') - k(t') \\
&- (-1)^{m+n-3}\int_{t''}^{t'}\int_{\theta}^{\infty}\frac{h_{n-2}(\theta,\sigma(s))}{r(s)}\int_{s}^{\infty}h_{m-1}(s,\sigma(u))q(u)G(x(\beta(u)))\Delta u\Delta s\Big| \\
\leq&|F(t'') - F(t')| + |k(t'') - k(t')| \\
&+ G(1)|t' - t''|(-1)^{m+n-3}\int_{t''}^{\infty}\frac{h_{n-2}(t'',\sigma(s))}{r(s)}\int_{s}^{\infty}h_{m-1}(s,\sigma(u))q(u)\Delta u\Delta s \\
\leq&|F(t'') - F(t')| + |k(t'') - k(t')| \\
&+ G(1)|t' - t''|(-1)^{m+n-3}\int_{t''}^{\infty}\frac{(t'' - \sigma(s))^{n-2}}{r(s)}\int_{s}^{\infty}(s - \sigma(u))^{m-1}q(u)\Delta u\Delta s \\
\leq&|F(t'') - F(t')| \\
&+ |k(t'') - k(t')| + G(1)|t' - t''|\int_{t''}^{\infty}\frac{(\sigma(s) - t'')^{n-2}}{r(s)}\int_{s}^{\infty}(\sigma(u) - s)^{m-1}q(u)\Delta u\Delta s \\
\leq&|F(t'') - F(t')| + |k(t'') - k(t')| + G(1)|t' - t''|\int_{t''}^{\infty}\frac{(\sigma(s))^{n-2}}{r(s)}\int_{s}^{\infty}(\sigma(u))^{m-1}q(u)\Delta u\Delta s.
\end{aligned}
$$

So $|(Bx)(t'') - (Bx)(t')| \to 0$ as $t'' \to t'$ if $n \geqslant 2$ (for $n = 1$ it is similar). Therefore, $\{Bx : x \in S\}$ is uniformly bounded and equicontinuous on $[t_1, \infty)_{\mathbb{T}}$. Hence, BS is relatively compact. By **Krasnosel'skii's fixed point theorem**, there exists $x \in S$ such that $Ax + Bx = x$. Thus, the theorem is proved. □

Remark 5.2.7. Results similar to Theorem 5.2.6 can be proved for other ranges of $p(t)$.

Remark 5.2.8. Note that $\int_{t_0}^{\infty}\frac{1}{r(t)}\Delta t = \infty$ implies $\int_{t_0}^{\infty}\frac{t}{r(t)}\Delta t = \infty$, $t_0 \in \mathbb{R}$.
Indeed, if $t_0 \geq 1$, then

$$\int_{t_0}^{\infty}\frac{t}{r(t)}\Delta t \geq \int_{t_0}^{\infty}\frac{1}{r(t)}\Delta t = \infty.$$

If $t_0 < 1$, then

$$\infty = \int_{t_0}^{\infty} \frac{1}{r(t)}\Delta t = \int_{t_0}^{1} \frac{1}{r(t)}\Delta t + \int_{1}^{\infty} \frac{1}{r(t)}\Delta t.$$

That is,

$$\int_{1}^{\infty} \frac{1}{r(t)} = \infty.$$

Hence,

$$\int_{t_0}^{\infty} \frac{t}{r(t)}\Delta t \geq \int_{t_0}^{1} \frac{t}{r(t)}\Delta t + \int_{1}^{\infty} \frac{1}{r(t)}\Delta t = \infty.$$

Remark 5.2.9. If $m = n = 2$ in $(E_6)/(E_7)$ with $\int_{t_0}^{\infty} \frac{1}{r(t)}\Delta t = \infty$, then some of the results of this chapter reduces to the results of Chapter 3.

5.3 Conclusion

We have studied the oscillatory and asymptotic behaviour of solutions of (E_6) and (E_7) in Section 4.1 and Section 4.2 respectively, under the assumption

$$\int_{t_0}^{\infty} \frac{1}{r(t)}\Delta t = \infty.$$

In Section 5.1, we have studied the oscillatory and asymptotic behavior of (E_6) for different ranges of $p(t)$. In the Theorems 5.1.6, 5.1.11 and 5.1.16, we have shown that every solution of (E_6) oscillates or tends to zere as t tends to infinity for the range of $p(t)$ with $0 \leq p(t) \leq p < \infty$. In Theorem 5.1.20, we have proved that every solution of (E_6) is either oscillatory or tends to zere as t tends to infinity for the range of $p(t)$ with $-1 < p_4 \leq p(t) \leq 0$. In Theorem 5.1.24, we have established that every bounded solution of (E_6) is either oscillatory or tends to zere as t tends to infinity for the range of $p(t)$ with $-\infty < p_5 \leq p(t) \leq p_6 < -1$.

In Section 5.2, we have studied the oscillatory and asymptotic behavior of (E_7) for different ranges of $p(t)$ and $F(t)$ according to (H_{12}) and (H_{13}). In the Theorems 5.2.1 and 5.2.5, we have shown that every solution of (E_7) is oscillatory for the range of $p(t)$ with $0 \leq p(t) \leq p < \infty$ and $F(t)$ with (H_{13}) and (H_{12}) respectively. In Theorem 5.2.3, we have proved that every solution of (E_7) is either oscillatory or $\limsup_{t\to\infty} \mid y(t) \mid = \infty$ for the range of $p(t)$ with $-\infty < p \leq p(t) \leq 0$. In Theorem 5.2.6, we have established that (E_7) has a nonoscillatory solution for the range of $p(t)$ with $0 \leq p(t) \leq p_1 < 1$.

It would be interesting to study the oscillatory/nonoscillatory/asymptotic behavior of solutions of (E_6) and (E_7) if $p(t)$ oscillates and every solution of (E_6) and (E_7) are oscillatory for all ranges of $p(t)$.

Chapter 6

Higher Order Neutral Delay Dynamic Equations with Positive and Negative Coefficients - II

The oscillatory and asymptotic behaviour of solution of equations (E_1) and (E_2) has been studied in Section 2.3 and Section 2.4 respectively under the assumption

$$\int_{t_0}^{\infty} \frac{\sigma(t)}{r(t)} \Delta t < \infty.$$

Later on, we have extended the results obtained in the Sections 2.3 and 2.4 to the nonlinear NDDEs (E_3) and (E_4) with positive and negative coefficients in the Sections 3.3 and 3.4 respectively. In order to generalize the work of Section 3.3 and Section 3.4, we concerned with the oscillatory and asymptotic behaviour of the neutral delay dynamic equations of the following form

$$(r(t)(y(t)+p(t)y(\alpha(t)))^{\Delta^n})^{\Delta^2} + q(t)G(y(\beta(t))) - h(t)H(y(\gamma(t))) = 0 \qquad (E_8)$$

and

$$(r(t)(y(t)+p(t)y(\alpha(t)))^{\Delta^n})^{\Delta^2} + q(t)G(y(\beta(t))) - h(t)H(y(\gamma(t))) = f(t) \qquad (E_9)$$

for $t \in [t_0, \infty)_{\mathbb{T}}$, $t_0 (\geqslant 0) \in \mathbb{T}$ under the assumption

$$\int_{t_0}^{\infty} \frac{(\sigma(t))^{n-1}}{r(t)} \Delta t < \infty,$$

where $\mathbb{T}$ is a time scale with $\sup \mathbb{T} = \infty$, and $n \in \mathbb{N}$, $r,\ q \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$, $p,\ f \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, $h \in C_{rd}([t_0, \infty)_{\mathbb{T}}, [0, \infty))$, $G,\ H \in C(\mathbb{R}, \mathbb{R})$ satisfy $uG(u) > 0$ and $uH(u) > 0$ for $u \neq 0$, G is nondecreasing, H is bounded, and $\alpha,\ \beta,\ \gamma \in C_{rd}(\mathbb{T}, \mathbb{T})$ are strictly increasing functions such that

$$\lim_{t\to\infty} \alpha(t) = \lim_{t\to\infty} \beta(t) = \lim_{t\to\infty} \gamma(t) = \infty, \quad \alpha(t),\ \beta(t),\ \gamma(t) \leqslant t.$$

Let $t_{-1} = \inf_{t\in[t_0,\infty)_{\mathbb{T}}}\{\alpha(t), \beta(t), \gamma(t)\}$. By a *solution* of $(E_8)/(E_9)$, we mean a function $y \in C_{rd}([t_{-1}, \infty)_{\mathbb{T}}, \mathbb{R})$ such that $y(t) + p(t)y(\alpha(t)) \in C^n_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, $r(t)(y(t) + p(t)y(\alpha(t)))^{\Delta^n} \in C^2_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, and such that $(E_8)/(E_9)$ is satisfied identically on $[t_0, \infty)_{\mathbb{T}}$. A solution of $(E_8)/(E_9)$ is called *oscillatory* if it is neither eventually positive nor eventually negative, and it is *nonoscillatory* otherwise. In this paper, we do not consider solutions that eventually vanish identically. An equation will be called oscillatory if all its solutions are oscillatory.

We need the following assumption (Λ) throughout our discussion, where

(Λ) $(\alpha \circ \beta)(t) = (\beta \circ \alpha)(t)$ for all $t \in [t_0, \infty)_{\mathbb{T}}$ and $Q(t) = \min\{q(t), q(\alpha(t))\}$ for $t \in [t^*, \infty)_{\mathbb{T}}$, $t^* > t_0$.

6.1 Sufficient conditions for oscillation of (E_8) with $\int_{t_0}^{\infty} \frac{(\sigma(t))^{n-1}}{r(t)} \Delta t < \infty$.

This section deals with the oscillatory and asymptotic behavior of solutions of (E_8) under the assumption

$$(H_1) \quad \int_{t_0}^{\infty} \frac{(\sigma(t))^{n-1}}{r(t)} \Delta t < \infty.$$

We need the following assumptions for our work in the sequel.

(H_2) $\int_{t_0}^{\infty} \frac{(\sigma(s))^{n-1}}{r(s)} \int_s^{\infty} \sigma(t)h(t)\Delta t \Delta s < \infty$,

(H_3) there exists $\lambda > 0$ such that $G(u) + G(v) \geqslant \lambda G(u + v)$ for $u > 0$ and $v > 0$,

(H_4) $G(u)G(v) = G(uv)$ for $u, v \in \mathbb{R}$ and $H(-u) = -H(u)$ for $u \in \mathbb{R}$,

(H_5) for some $\gamma > 1$, $\int_{t^*}^{\infty} d(t)Q(t)G(R(\beta(t)))\Delta t = \infty$, where $d(t) = \min\{\frac{R^\gamma(\sigma(t))}{(\sigma(t))^{\gamma(n-1)}}, \frac{R^\gamma(\sigma(\alpha(t)))}{(\sigma(\alpha(t)))^{\gamma(n-1)}}\}$ and $R(t) = (-1)^{n-1}\int_t^\infty \frac{h_{n-1}(t,\sigma(\theta))}{r(\theta)}\Delta\theta$,

(H_6) $G(-u) = -G(u)$ and $H(-u) = -H(u)$ for $u \in \mathbb{R}$,

(H_7) for some $\gamma > 1$, $\int_{t_3}^{\infty} \frac{R^\gamma(\sigma(t))}{(\sigma(t))^{\gamma(n-1)}} q(t)G(R(\beta(t)))\Delta t = \infty$.

Remark 6.1.1. Notice that (H_5) implies

(H_5') $\int_{t^*}^{\infty} Q(t)G(R(\beta(t)))\Delta t = \infty$,

which in turn implies $\int_{t^*}^{\infty} Q(t)\Delta t = \infty$, and ($H_7$) implies

(H_7') $\int_{t_3}^{\infty} q(t)G(R(\beta(t)))\Delta t = \infty$,

which in turn implies $\int_{t_0}^{\infty} q(t)\Delta t = \infty$.

Remark 6.1.2. Note that (H_1) implies (H_1'), where

$$(H_1') \quad \int_{t_0}^{\infty} \frac{1}{r(t)}\Delta t < \infty.$$

Lemma 6.1.3. Let (H_1) holds and $u(t)$ be n-times continuously differentiable function on $t \in [t_0, \infty)_{\mathbb{T}}$ such that $r(t)u^{\Delta^n}(t) \in C^2_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ and $(r(t)u^{\Delta^n}(t))^{\Delta^2} \leq (\geq) \not\equiv 0$ for large $t \in [t_0, \infty)_{\mathbb{T}}$. If $u(t) > 0$ eventually, then one of the *cases* (a), (b) or (c) holds for large t, where

(a) there exists $l \in [0, n]_{\mathbb{Z}}$ such that $n + l$ is even (even) and $u^{\Delta^i}(t) > 0$ $(i = 1, 2, \cdots, l-1)$, $(-1)^{l+i}u^{\Delta^i}(t) > 0$ $(i = l, l+1, \cdots, n)$ when $(r(t)u^{\Delta^n}(t))^{\Delta} > (>)0$.

(b) there exists $l \in [0, n]_{\mathbb{Z}}$ such that $n+l$ is odd (even) and $u^{\Delta^i}(t) > 0$ $(i = 1, 2, \cdots, l-1)$, $(-1)^{l+i}u^{\Delta^i}(t) > 0$ $(i = l, l+1, \cdots, n)$ when $(r(t)u^{\Delta^n}(t))^{\Delta} > (<)0$.

(c) there exists $l \in [0, n]_{\mathbb{Z}}$ such that $n+l$ is odd (odd) and $u^{\Delta^i}(t) > 0$ $(i = 1, 2, \cdots, l-1)$, $(-1)^{l+i}u^{\Delta^i}(t) > 0$ $(i = l, l+1, \cdots, n)$ when $(r(t)u^{\Delta^n}(t))^{\Delta} < (<)0$.

Proof. We complete the proof of this lemma by dividing it into two parts. In the first part we study with $(r(t)u^{\Delta^n}(t))^{\Delta^2} \leq 0$ for large $t \in [t_0, \infty)_{\mathbb{T}}$, and in the second part we study with $(r(t)u^{\Delta^n}(t))^{\Delta^2} \geq 0$ for large $t \in [t_0, \infty)_{\mathbb{T}}$.

First let $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $(r(t)u^{\Delta^n}(t))^{\Delta^2} \leq 0$ and $u(t) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. Then $u^{\Delta^i}(t)$ $(i = 1, 2, \cdots, n-1)$, $(r(t)u^{\Delta^n}(t))$ and $(r(t)u^{\Delta^n}(t))^{\Delta}$ all are monotonic and each one has constant sign for all $t \in [t_2, \infty)_{\mathbb{T}}$, $t_2 \geqslant t_1$. Since $(r(t)u^{\Delta^n}(t))^{\Delta}$ is monotonic for $t \geq t_2$, then one of the following sub cases are holds.

$$\textit{Sub case (i)} \;\; (r(t)u^{\Delta^n}(t))^{\Delta} > 0,$$

$$\textit{Sub case (ii)} \;\; (r(t)u^{\Delta^n}(t))^{\Delta} < 0.$$

For $t > T \geqslant t_2$,

$$r(t)u^{\Delta^n}(t) - r(T)u^{\Delta^n}(T) \leq (r(T)u^{\Delta^n}(T))^{\Delta}(t - T). \tag{6.1}$$

If Sub case (i) holds, then (6.1) implies either $u^{\Delta^n}(t) > 0$ or $u^{\Delta^n}(t) < 0$ for $t \geq t_2$. Since $(r(t)u^{\Delta^n}(t))^{\Delta} > 0$, then $r(t)u^{\Delta^n}(t)$ is increasing for $t \geq t_2$. For $t > T \geqslant t_2$, we have

$$u^{\Delta^{n-1}}(t) - u^{\Delta^{n-1}}(T) > (r(T)u^{\Delta^n}(T)) \int_T^t \frac{1}{r(\theta)} \Delta\theta. \tag{6.2}$$

If $u^{\Delta^n}(t) > 0$, then (6.2) implies that $u^{\Delta^{n-1}}(t) > 0$ or $u^{\Delta^{n-1}}(t) < 0$ due to the fact (H_1'). If $u^{\Delta^{n-1}}(t) > 0$, then $u^{\Delta^i}(t) > 0$ $(i = 1, 2, \cdots, n-2)$ by Lemma 5.1.1. Again, if $u^{\Delta^{n-1}}(t) < 0$, then by Lemma 1.5.4, there exists $l \in [0, n-1]_{\mathbb{Z}}$ with $n - 1 + l$ $(i.e., n + l$ even) is odd such that $u^{\Delta^i}(t) > 0$ $(i = 1, 2, \cdots, l-1)$, $(-1)^{l+i}u^{\Delta^i}(t) > 0$ $(i = l, l+1, \cdots, n-1)$. From this we can conclude that, $l \in [0, n]_{\mathbb{Z}}$ such that $n + l$ is even and $u^{\Delta^i}(t) > 0$ $(i = 1, 2, \cdots, l-1)$, $(-1)^{l+i}u^{\Delta^i}(t) > 0$ $(i = l, l+1, \cdots, n)$ when $(r(t)u^{\Delta^n}(t))^{\Delta} > 0$. Thus (a) is proved.

If $u^{\Delta^n}(t) < 0$ holds, then (6.2) implies that $u^{\Delta^{n-1}}(t) > 0$ or $u^{\Delta^{n-1}}(t) < 0$ is possible due to the fact (H_1'). By Lemma 5.1.1, $u^{\Delta^{n-1}}(t) > 0$ is possible. Otherwise, if $u^{\Delta^{n-1}}(t) < 0$, then $u^{\Delta^i}(t) < 0$ $(i = 0, 1, \cdots, n-2)$ for $n \geqslant 2$, which is a contradiction to $u(t) > 0$. Since $u^{\Delta^{n-1}}(t) > 0$, then by Lemma 1.5.4, there exists $l \in [0, n-1]_{\mathbb{Z}}$ such that $n-1+l$ $(i.e., n+l$ odd) is even and $u^{\Delta^i}(t) > 0$ $(i = 1, 2, \cdots, l-1)$, $(-1)^{l+i}u^{\Delta^i}(t) > 0$ $(i = l, l+1, \cdots, n-1)$. From this we can conclude that, $l \in [0, n]_{\mathbb{Z}}$ with $n + l$ is odd such that $u^{\Delta^i}(t) > 0$ $(i = 1, 2, \cdots, l-1)$, $(-1)^{l+i}u^{\Delta^i}(t) > 0$ $(i = l, l+1, \cdots, n)$ when $(r(t)u^{\Delta^n}(t))^{\Delta} > 0$. Thus (b) is proved.

If Sub case (ii) holds, then (6.1) implies $u^{\Delta^n}(t) < 0$ eventually. Since $(r(t)u^{\Delta^n}(t))^\Delta < 0$ for $t \geq t_2$, then $r(t)u^{\Delta^n}(t)$ is decreasing in $t \in [t_2, \infty)_{\mathbb{T}}$. For $t > T \geqslant t_2$, we have

$$u^{\Delta^{n-1}}(t) - u^{\Delta^{n-1}}(T) < (r(T)u^{\Delta^n}(T)) \int_T^t \frac{1}{r(\theta)} \Delta\theta.$$

This implies that $u^{\Delta^{n-1}}(t) > 0$ for large $t \in [t_2, \infty)_{\mathbb{T}}$. Otherwise, if $u^{\Delta^{n-1}}(t) < 0$ for $t \geq t_2$, then $u^{\Delta^i}(t) < 0$ for $t \geq t_2$ $(i = 0, 1, \cdots, n-2)$ by Lemma 5.1.1, which is a contradiction to the fact that $u(t) > 0$. Since $u^{\Delta^{n-1}}(t) > 0$, then by Lemma 1.5.4, there exists $l \in [0, n-1]_{\mathbb{Z}}$ such that $n - 1 + l$ $(i.e., n + l$ odd) is even and $u^{\Delta^i}(t) > 0$ $(i = 1, 2, \cdots, l-1)$, $(-1)^{l+i}u^{\Delta^i}(t) > 0$ $(i = l, l+1, \cdots, n-1)$. From this we can conclude that, $l \in [0, n]_{\mathbb{Z}}$ with $n+l$ is odd such that $u^{\Delta^i}(t) > 0$ $(i = 1, 2, \cdots, l-1)$, $(-1)^{l+i}u^{\Delta^i}(t) > 0$ $(i = l, l+1, \cdots, n)$ when $(r(t)u^{\Delta^n}(t))^\Delta < 0$. Thus (c) is proved.

In the second part, let $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $(r(t)u^{\Delta^n}(t))^{\Delta^2} \geq 0$ and $u(t) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. Since $(r(t)u^{\Delta^n}(t))^\Delta$ is monotonic, then one of Sub case (i) or Sub case (ii) holds and

$$r(t)u^{\Delta^n}(t) - r(T)u^{\Delta^n}(T) \geq (r(T)u^{\Delta^n}(T))^\Delta (t - T). \tag{6.3}$$

If Sub case (i) holds, then (6.3) implies $u^{\Delta^n}(t) > 0$ for large t. Since $(r(t)u^{\Delta^n}(t))^\Delta > 0$ for $t \geq t_2$, then $(r(t)u^{\Delta^n}(t))$ is increasing in $t \in [t_2, \infty)_{\mathbb{T}}$. Then for $t > T \geqslant t_2$, we have

$$u^{\Delta^{n-1}}(t) - u^{\Delta^{n-1}}(T) \geq (r(T)u^{\Delta^n}(T)) \int_T^t \frac{1}{r(\theta)} \Delta\theta.$$

This implies that $u^{\Delta^{n-1}}(t) > 0$ or $u^{\Delta^{n-1}}(t) < 0$, due to the fact (H_1'). Proceeding as in first part of this lemma, there exists $l \in [0, n]_{\mathbb{Z}}$ such that $n + l$ is even $u^{\Delta^i}(t) > 0$ $(i = 1, 2, \cdots, l-1)$, $(-1)^{l+i}u^{\Delta^i}(t) > 0$ $(i = l, l+1, \cdots, n)$ when $(r(t)u^{\Delta^n}(t))^\Delta > 0$ for large t. Thus (a) is proved.

If Sub case (ii) holds, then (6.3) implies that one of $u^{\Delta^n}(t) > 0$ or $u^{\Delta^n}(t) < 0$ holds for large $t \in [T, \infty)_{\mathbb{T}}$. Since $(r(t)u^{\Delta^n}(t))^\Delta < 0$ for $t \geq t_2$, then $r(t)u^{\Delta^n}(t)$ is decreasing in $t \in [t_2, \infty)_{\mathbb{T}}$. Then for $t > T \geqslant t_2$, we have

$$u^{\Delta^{n-1}}(t) - u^{\Delta^{n-1}}(T) \leq (r(T)u^{\Delta^n}(T)) \int_T^t \frac{1}{r(\theta)} \Delta\theta. \tag{6.4}$$

This implies that $u^{\Delta^{n-1}}(t) > 0$ or $u^{\Delta^{n-1}}(t) < 0$ for large $t \in [t_2, \infty)_{\mathbb{T}}$ when $u^{\Delta^n}(t) > 0$ or $u^{\Delta^n}(t) < 0$, due to the fact (H_1'). If $u^{\Delta^n}(t) > 0$, then by proceeding as in first part of this lemma, there exists $l \in [0, n]_{\mathbb{Z}}$ such that $n+l$ is even $u^{\Delta^i}(t) > 0$ $(i = 1, 2, \cdots, l-1)$, $(-1)^{l+i}u^{\Delta^i}(t) > 0$ $(i = l, l+1, \cdots, n)$ when $(r(t)u^{\Delta^n}(t))^{\Delta} < 0$ for large t. Thus (b) is proved.

If $u^{\Delta^n}(t) < 0$, then by proceeding as in first part of this lemma, there exists $l \in [0, n]_{\mathbb{Z}}$ such that $n + l$ is odd $u^{\Delta^i}(t) > 0$ $(i = 1, 2, \cdots, l-1)$, $(-1)^{l+i}u^{\Delta^i}(t) > 0$ $(i = l, l+1, \cdots, n)$ when $(r(t)u^{\Delta^n}(t))^{\Delta} < 0$ for large $t \in [t_0, \infty)_{\mathbb{T}}$. Thus (c) is proved. This completes the proof of the theorem. □

Lemma 6.1.4. Let (H_1) holds and $u(t)$ be n-times continuously differentiable function on $t \in [t_0, \infty)_{\mathbb{T}}$ such that $r(t)u^{\Delta^n}(t) \in C^2_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ and $(r(t)u^{\Delta^n}(t))^{\Delta^2} \leq (\geq) \not\equiv 0$ for large $t \in [t_0, \infty)_{\mathbb{T}}$. If $u(t) < 0$ eventually, then one of the cases (d), (e) or (f) are holds, where

(d) there exists $l \in [0, n]_{\mathbb{Z}}$ such that $n + l$ is even (even) and $u^{\Delta^i}(t) < 0$ $(i = 1, 2, \cdots, l-1)$, $(-1)^{1+l+i}u^{\Delta^i}(t) > 0$ $(i = l, l+1, \cdots, n)$ when $(r(t)u^{\Delta^n}(t))^{\Delta} < (<)0$.

(e) there exists $l \in [0, n]_{\mathbb{Z}}$ such that $n + l$ is even (odd) and $u^{\Delta^i}(t) < 0$ $(i = 1, 2, \cdots, l-1)$, $(-1)^{1+l+i}u^{\Delta^i}(t) > 0$ $(i = l, l+1, \cdots, n)$ when $(r(t)u^{\Delta^n}(t))^{\Delta} > (<)0$.

(f) there exists $l \in [0, n]_{\mathbb{Z}}$ such that $n + l$ is odd (odd) and $u^{\Delta^i}(t) < 0$ $(i = 1, 2, \cdots, l-1)$, $(-1)^{1+l+i}u^{\Delta^i}(t) > 0$ $(i = l, l+1, \cdots, n)$ when $(r(t)u^{\Delta^n}(t))^{\Delta} > (>)0$.

Proof. Let $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $(r(t)u^{\Delta^n}(t))^{\Delta^2} \leq (\geqslant)0$ and $u(t) < 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. Let $u(t) = -v(t) < 0$, where $v(t) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. Then, $(r(t)v^{\Delta^n}(t))^{\Delta^2} \geq (\leq)\ 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. Since $v(t) > 0$ and $(r(t)v^{\Delta^n}(t))^{\Delta^2} \geq (\leq)\ 0$ then from Lemma 6.1.3, one of the *cases* (a), (b) or (c) holds. Let *case* (a) holds. Then, there exists $l \in [0, n]_{\mathbb{Z}}$ such that $n + l$ is even (even) and $v^{\Delta^i}(t) > 0$ $(i = 1, 2, \cdots, l-1)$, $(-1)^{l+i}v^{\Delta^i}(t) > 0$ $(i = l, l+1, \cdots, n)$ when $(r(t)v^{\Delta^n}(t))^{\Delta} > (>)0$. Replacing $v(t)$ by $-u(t)$ in (a), we obtain $l \in [0, n]_{\mathbb{Z}}$ such that $n + l$ is even (even) and $u^{\Delta^i}(t) < 0$ $(i =$

$1, 2, \cdots, l-1)$, $(-1)^{1+l+i}u^{\Delta^i}(t) > 0$ $(i = l, l+1, \cdots, n)$ when $(r(t)u^{\Delta^n}(t))^{\Delta} < (<)0$. Thus (d) is proved. The proof for the *cases* (e) and (f) is similar to that of *case* (d). Hence proof for these cases are omitted. □

Lemma 6.1.5. Let (H_1) holds. Suppose that $u(t)$ be n-times continuously differentiable function on $t \in [t_0, \infty)_{\mathbb{T}}$ such that $r(t)u^{\Delta^n}(t) \in C^2_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ and $(r(t)u^{\Delta^n}(t))^{\Delta^2} \leq (\not\equiv)0$ for large $t \in [t_0, \infty)_{\mathbb{T}}$. Assume that $u(t) > 0$ eventually.

(I_1) If the *case* (c) of Lemma 6.1.3 holds, then

$$u(t) \geqslant kt(-r(t)u^{\Delta^n}(t))^{\Delta}R(t),$$

where $R(t) = (-1)^{n-1}\int_t^{\infty} \frac{h_{n-1}(t,\sigma(\theta))}{r(\theta)}\Delta\theta$ and $k > 0$.

(I_2) If the *cases* (a), (c) and the *case* (b) with $l \geqslant 1$ of Lemma 6.1.3 hold, then

$$u(t) \geqslant k_1 R(t),$$

where $k_1 > 0$.

(I_3) If the *case* (b) with $l = 0$ of Lemma 6.1.3 holds, then

$$u(s) \geqslant (r(t)u^{\Delta^n}(t))^{\Delta}R_s(t)$$

for $t > s$ and $t, s \in \mathbb{T}$, where $R_s(t) = (-1)^{n-1}\int_s^t \frac{h_{n-1}(s,\sigma(\theta))(t-\theta)}{r(\theta)}\Delta\theta$ (note that this case (b) with $l = 0$ arises when n is odd).

Proof. Let $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $(r(t)u^{\Delta^n}(t))^{\Delta^2} \leq 0$ and $u(t) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. Then, $u^{\Delta^i}(t)$ $(i = 1, 2, \cdots, n-1)$, $(r(t)u^{\Delta^n}(t))$ and $(r(t)u^{\Delta^n}(t))^{\Delta}$ all are monotonic and each one has of constant sign for all $t \in [t_2, \infty)_{\mathbb{T}}$, $t_2 \geqslant t_1$. From Lemma 6.1.3, one of the cases (a), (b) or (c) holds.

Suppose the *case* (c) holds. Since $(-r(t)u^{\Delta^n}(t))^{\Delta}$ is non-decreasing in $[t_2, \infty)_{\mathbb{T}}$. For $\theta > t \geqslant t_2$,

$$\begin{aligned} -r(\theta)u^{\Delta^n}(\theta) \geq -r(\theta)u^{\Delta^n}(\theta) + r(t)u^{\Delta^n}(t) &= \int_t^{\theta}(-r(x)u^{\Delta^n}(x))^{\Delta}\Delta x \\ &\geq (-r(t)u^{\Delta^n}(t))^{\Delta}(\theta - t), \end{aligned}$$

implies that,

$$-r(\theta)u^{\Delta^n}(\theta) \geq (-r(t)u^{\Delta^n}(t))^{\Delta}(\theta - t),$$

that is,

$$\begin{aligned} -u^{\Delta^n}(\theta) &\geqslant (-r(t)u^{\Delta^n}(t))^{\Delta}\frac{\theta(1-\frac{t}{\theta})}{r(\theta)} \\ &\geq k_{31}(-r(t)u^{\Delta^n}(t))^{\Delta}\frac{\theta}{r(\theta)} \end{aligned}$$

where $k_{31} = 1 - t/\theta,\ k_{31} \in (0,1)$. First, we consider the *case* (c) with $l = 0$. Integrating the above inequality on $[t,\infty)_{\mathbb{T}} \subseteq [t_2,\infty)_{\mathbb{T}}$ for n times, we obtain

$$\begin{aligned} u(t) &\geqslant k_{31}(-r(t)u^{\Delta^n}(t))^{\Delta}(-1)^{n-1}\int_t^{\infty}\frac{\theta h_{n-1}(t,\sigma(\theta))}{r(\theta)}\Delta\theta \\ &\geqslant k_{31}t(-r(t)u^{\Delta^n}(t))^{\Delta}(-1)^{n-1}\int_t^{\infty}\frac{h_{n-1}(t,\sigma(\theta))}{r(\theta)}\Delta\theta \\ &\geqslant k_{31}t(-r(t)u^{\Delta^n}(t))^{\Delta}R(t). \end{aligned}$$

Next, we consider the *case* (c) with $l \geqslant 1$ holds. In this case, $u^{\Delta}(t) > 0$ for $t \geq t_2$, then $u(t)$ is increasing for $t \geq t_2$. We can find $k_{32} > 0$ such that $u(t) \geqslant k_{32}t(-r(t)u^{\Delta^n}(t))^{\Delta}R(t)$. Thus,

$$u(t) \geqslant kt(-r(t)u^{\Delta^n}(t))^{\Delta}R(t),$$

where $k = \min\{k_{31}, k_{32}\}$. Thus (I_1) is proved.

Suppose the case (a) holds. Since $r(t)u^{\Delta^n}(t)$ is increasing in $t \in [t_2,\infty)_{\mathbb{T}}$, then for $\theta > t \geqslant t_2$,

$$r(\theta)u^{\Delta^n}(\theta) \geqslant r(t)u^{\Delta^n}(t),$$

that is,

$$u^{\Delta^n}(\theta) \geqslant r(t)u^{\Delta^n}(t)\frac{1}{r(\theta)}. \tag{6.5}$$

First, we consider the *case* (a) with $l = 0$. Integrating, the inequality (6.5) on $[t,\infty)_{\mathbb{T}} \subseteq$

$[t_2,\infty)_{\mathbb{T}}$ for n times, we obtain

$$\begin{aligned} u(t) &\geqslant r(t)u^{\Delta^n}(t)(-1)^{n-1}\int_t^\infty \frac{h_{n-1}(t,\sigma(\theta))}{r(\theta)}\Delta\theta \\ &= r(t)u^{\Delta^n}(t)R(t) \\ &\geqslant r(t_2)u^{\Delta^n}(t_2)R(t) \\ &= k_{11}R(t), \end{aligned}$$

where $k_{11} = r(t_2)u^{\Delta^n}(t_2)$.

Next, consider the *case* (a) with $l \geqslant 1$. In this case, $u^{\Delta}(t) > 0$, then $u(t)$ is increasing for $t \in [t_2,\infty)_{\mathbb{T}}$. Since $R(t)$ is decreasing and $u(t)$ is increasing, then there exists $k_{12} > 0$ such that $u(t) \geqslant k_{12}R(t)$.

Similarly, in the *case* (b) with $l \geqslant 1$, $u(t)$ is increasing for $t \geq t_2$. So, we can find $k_{13} > 0$ such that $u(t) \geqslant k_{13}R(t)$. Similarly, in the *case* (c), we find $k_{14} = kt_2(-r(t_2)u^{\Delta^n}(t_2))^{\Delta} > 0$ such that $u(t) \geqslant kt_2(-r(t_2)u^{\Delta^n}(t_2))^{\Delta}R(t) = k_{14}R(t)$. Thus,

$$u(t) \geqslant k_1 R(t),$$

where $k_1 = \min\{k_{11}, k_{12}, k_{13}, k_{14}\}$. Hence (I_2) is proved.

Consider the *case* (b) with $l = 0$. By Lemma 6.1.3, we have $(-1)^i u^{\Delta^n}(t) > 0$ $(i = 1,2,\cdots,n)$ and $(r(t)u^{\Delta^n}(t))^{\Delta} > 0$. Since $(r(t)u^{\Delta^n}(t))^{\Delta}$ is decreasing in $t \in [t_2,\infty)_{\mathbb{T}}$, then for $t > \theta > s \geqslant t_2$, we have

$$-r(\theta)u^{\Delta^n}(\theta) \geqslant (r(t)u^{\Delta^n}(t))^{\Delta}(t-\theta),$$

that is,

$$-u^{\Delta^n}(\theta) \geqslant (r(t)u^{\Delta^n}(t))^{\Delta}\frac{(t-\theta)}{r(\theta)}.$$

By integrating the above inequality on $[s,t]_{\mathbb{T}} \subseteq [t_2,\infty)_{\mathbb{T}}$ for n times, we obtain

$$\begin{aligned} u(s) &\geqslant (r(t)u^{\Delta^n}(t))^{\Delta}\int_s^t\int_{\theta_{n-1}}^t \cdots \int_{\theta_1}^t \frac{(t-\theta)}{r(\theta)}\Delta\theta\Delta\theta_1\cdots\Delta\theta_{n-1} \\ &= (r(t)u^{\Delta^n}(t))^{\Delta}(-1)^{n-1}\int_s^t \frac{h_{n-1}(s,\sigma(\theta))(t-\theta)}{r(\theta)}\Delta\theta \\ &= (r(t)u^{\Delta^n}(t))^{\Delta}R_s(t). \end{aligned}$$

Hence (I_3) is proved. This complete the proof of the lemma. □

Lemma 6.1.6. Let (H_1) holds and $u(t)$ be n-times continuously differentiable function on $t \in [t_0, \infty)_{\mathbb{T}}$ such that $r(t)u^{\Delta^n}(t) \in C^2_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ and $(r(t)u^{\Delta^n}(t))^{\Delta^2} \leq (\not\equiv) 0$ for large $t \in [t_0, \infty)_{\mathbb{T}}$. If $u(t) > 0$ eventually, then there exists a constant $k_2 > 0$ such that

$$u(t) \leq k_2 t^n \quad \text{for large } t \in [t_0, \infty)_{\mathbb{T}}.$$

Proof. Let $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $(r(t)u^{\Delta^n}(t))^{\Delta^2} \leq 0$ and $u(t) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. Then we can find $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $u^{\Delta^i}(t)$ $(i = 1, 2, \cdots, n-1)$, $(r(t)u^{\Delta^n}(t))$ and $(r(t)u^{\Delta^n}(t))^{\Delta}$ all are monotonic and each one has of constant sign for all $t \in [t_2, \infty)_{\mathbb{T}}$. Since $u(t) > 0$, then by Lemma 6.1.3 one of the cases (a), (b) or (c) hold. Since $(r(t)u^{\Delta^n}(t))^{\Delta}$ is decreasing for $t \in [t_2, \infty)_{\mathbb{T}}$, then for $t > T \geqslant t_2$, we have

$$(r(t)u^{\Delta^n}(t))^{\Delta} \leq (r(T)u^{\Delta^n}(T))^{\Delta}.$$

Integrating the above inequality between T to t, we obtain

$$u^{\Delta^n}(t) \leq r(T)u^{\Delta^n}(T)\frac{1}{r(t)} + (r(T)u^{\Delta^n}(T))^{\Delta}\frac{t-T}{r(t)}.$$

Integrating the above inequality on $[T, t]_{\mathbb{T}} \subset [t_2, \infty)_{\mathbb{T}}$ for n-times, we obtain

$$\begin{aligned}
u(t) \leq & u(T) + u^{\Delta}(T)\int_T^t \Delta\theta + u^{\Delta^2}(T)\int_T^t\int_T^{\theta_1} \Delta\theta\Delta\theta_1 + \cdots \\
& + u^{\Delta^{n-1}}(T)\int_T^t\int_T^{\theta_{n-2}}\cdots\int_T^{\theta_1} \Delta\theta\Delta\theta_1\cdots\Delta\theta_{n-2} \\
& + (r(T)u^{\Delta^n}(T))\int_T^t\int_T^{\theta_{n-1}}\cdots\int_T^{\theta_1} \frac{1}{r(\theta)}\Delta\theta\Delta\theta_1\cdots\Delta\theta_{n-1} \\
& + (r(T)u^{\Delta^n}(T))^{\Delta}\int_T^t\int_T^{\theta_{n-1}}\cdots\int_T^{\theta_1} \frac{\theta-T}{r(\theta)}\Delta\theta\Delta\theta_1\cdots\Delta\theta_{n-1} \\
\leq & u(T) + u^{\Delta}(T)(t-T) + u^{\Delta^2}(T)h_2(t,T) + \cdots + u^{\Delta^{n-1}}(T)h_{n-1}(t,T) \\
& + (r(T)u^{\Delta^n}(T))\int_T^t \frac{h_{n-1}(t,\sigma(\theta))}{r(\theta)}\Delta\theta \\
& + (r(T)u^{\Delta^n}(T))^{\Delta}\int_T^t \frac{h_{n-1}(t,\sigma(\theta))}{r(\theta)}(\theta-T)\Delta\theta,
\end{aligned}$$

due to the fact (1.5) and (1.7). Let $k = \mid u(T) \mid + \mid u^{\Delta}(T) \mid + \cdots + \mid u^{\Delta^{n-1}}(T) \mid + \mid$

$r(T)u^{\Delta^n}(T) \mid + \mid (r(T)u^{\Delta^n}(T))^{\Delta} \mid$. Hence, by using Property 1.4.1, we have

$$\begin{aligned} u(t) &\leq k\Big(1 + (t-T) + h_2(t,T) + \cdots + h_{n-1}(t,T) + \int_T^t \frac{h_{n-1}(t,\sigma(\theta))}{r(\theta)}\Delta\theta \\ &\quad + \int_T^t \frac{h_{n-1}(t,\sigma(\theta))}{r(\theta)}(\theta - T)\Delta\theta\Big) \\ &\leq k\Big(1 + (t-T) + (t-T)^2 + \cdots + (t-T)^{n-1} + \int_T^t \frac{(t-\sigma(\theta))^{n-1}}{r(\theta)}\Delta\theta \\ &\quad + \int_T^t \frac{(t-\sigma(\theta))^{n-1}}{r(\theta)}(\theta - T)\Delta\theta\Big) \\ &\leq k\Big(1 + t + t^2 + \cdots + t^{n-1} + \int_T^t \frac{t^{n-1}}{r(\theta)}\Delta\theta + \int_T^t \frac{t^{n-1}\theta}{r(\theta)}\Delta\theta\Big) \\ &\leq k\Big(1 + t + t^2 + \cdots + t^{n-1} + t^{n-1}\int_T^t \frac{1}{r(\theta)}\Delta\theta + t^n\int_T^t \frac{1}{r(\theta)}\Delta\theta\Big) \\ &\leq k_2 t^n, \end{aligned}$$

where $k_2 = k\Big(n + 2\int_T^\infty \frac{1}{r(\theta)}\Delta\theta\Big)$. Thus, the lemma is proved. □

Theorem 6.1.7. *Assume that the conditions (H_1)–(H_5), (Λ) hold, and p is a positive real number. If $0 \leqslant p(t) \leqslant p < \infty$ holds, then every solution of* (E_8) *is either oscillatory or converges to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a nonoscillatory solution of (E_8) on $[t_0, \infty)_{\mathbb{T}}$, say $y(t)$ is an eventually positive solution. (The proof in case $y(t) < 0$ eventually is similar and will be omitted.) Then, there exists $t_1 \in [t^*, \infty)_{\mathbb{T}}$ such that $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, $y(\gamma(t))$, $y(\alpha(\alpha(t)))$ and $y(\alpha(\beta(t)))$ are all positive for $t \geqslant t_1$. Set

$$z(t) = y(t) + p(t)y(\alpha(t)), \tag{6.6}$$

and

$$k(t) = (-1)^{n-1}\int_t^\infty \frac{h_{n-1}(t,\sigma(s))}{r(s)}\int_s^\infty (\sigma(\theta) - s)h(\theta)H(y(\gamma(\theta)))\Delta\theta\Delta s. \tag{6.7}$$

Notice that condition (H_2) and the fact that H is a bounded function imply that $k(t)$ exists for all t. Now if we let

$$w(t) = z(t) - k(t) = y(t) + p(t)y(\alpha(t)) - k(t), \tag{6.8}$$

then

$$(r(t)w^{\Delta^n}(t))^{\Delta^2} = -q(t)G(y(\beta(t))) \leqslant (\not\equiv)0, \tag{6.9}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. Clearly, $w^{\Delta^i}(t)$ $(i = 0, 1, \cdots, n-1)$, $(r(t)w^{\Delta^n}(t))$, and $(r(t)w^{\Delta^n}(t))^{\Delta}$ are monotonic on $[t_1, \infty)_{\mathbb{T}}$. Since $w(t)$ is monotonic, then we have to consider the two cases $w(t) > 0$ or $w(t) < 0$.

Suppose that $w(t) > 0$ for $t \geq t_2$, for some $t_2 > t_1$, then by Lemma 6.1.3 one of the *cases* (a), (b) or (c) holds. First, we consider the *cases* (a) and (b) are hold. Then applying (H_3), (H_4) and (Λ) in equation (E_8), we obtain

$$\begin{aligned}
0 &= (r(t)w^{\Delta^n}(t))^{\Delta^2} + q(t)G(y(\beta(t))) + G(p)(r(\alpha(t))w^{\Delta^n}(\alpha(t)))^{\Delta^2} \\
&\quad + G(p)q(\alpha(t))G(y(\beta(\alpha(t)))) \\
&\geqslant (r(t)w^{\Delta^n}(t))^{\Delta^2} + G(p)(r(\alpha(t))w^{\Delta^n}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(y(\beta(t)) + py(\alpha(\beta(t)))) \\
&\geqslant (r(t)w^{\Delta^n}(t))^{\Delta^2} + G(p)(r(\alpha(t))w^{\Delta^n}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(z(\beta(t)))
\end{aligned} \tag{6.10}$$

for $t \geqslant t_2 > t_1$, where we have used the fact that $z(t) \leqslant y(t) + py(\alpha(t))$. From (6.7), it follows that $k(t) > 0$ and $k^{\Delta}(t) < 0$. Hence, $w(\beta(t)) > 0$ for $t \geq t_3$ implies that $w(\beta(t)) < z(\beta(t))$ for $t \geq t_3$. From (6.10), we have

$$(r(t)w^{\Delta^n}(t))^{\Delta^2} + G(p)(r(\alpha(t))w^{\Delta^n}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(w(\beta(t))) \leqslant 0, \tag{6.11}$$

for $t \geq t_3 > t_2$. For the *case* (a) and the *case* (b) with $l \geqslant 1$, from (6.11), we obtain

$$(r(t)w^{\Delta^n}(t))^{\Delta^2} + G(p)(r(\alpha(t))w^{\Delta^n}(\alpha(t)))^{\Delta^2} + \lambda G(k_1)Q(t)G(R(\beta(t))) \leqslant 0,$$

for $t \geqslant t_4 > t_3$ due to Lemma 6.1.5. Now $\lim_{t\to\infty}(r(t)w^{\Delta^n}(t))^{\Delta}$ exists, so integrating the above inequality implies

$$\lambda G(k_1) \int_{t_4}^{\infty} Q(t)G(R(\beta(t)))\Delta t < \infty,$$

which contradicts (H_5'). For the *case* (b) with $l = 0$, (6.11) gives

$$G(w(\beta(t))) \int_{t_4}^{t} Q(\theta)\Delta\theta < \infty,$$

imply that $\lim_{t\to\infty} w(\beta(t)) = 0$, G is continuous and due to Remark 6.1.1. Since $w(t)$ is decreasing for $t \geq t_4$, then $\lim_{t\to\infty} w(t) = 0$. From (6.8), $\lim_{t\to\infty} z(t) = 0$. Therefore, $\lim_{t\to\infty} y(t) = 0$, since $y(t) \leq z(t)$.

Now we consider the *case* (*c*). The use of Lemma 6.1.5 and Lemma 6.1.6 yields

$$k(-r(t)w^{\Delta^n}(t))^{\Delta} tR(t) \leqslant w(t) \leqslant k_2 t^n, \tag{6.12}$$

for $t \geqslant t_2 > t_1$. Define $f \in C(\mathbb{R}, \mathbb{R})$ such that $f(x) = x^{1-\gamma}$ $(\gamma > 1)$, is continuous on $(0, \infty)$, and take $g(t) = (-r(t)w^{\Delta^n}(t))^{\Delta}$. Applying the chain rule (see [Lemma 1.3.11]), using (6.9) and the fact that g is increasing, means there is a c in the real interval $[t, \sigma(t)]$ with $g(c) = L$, (i.e., $g(t) \leq L \leq g(\sigma(t))$) such that

$$\begin{aligned} -[((-r(t)w^{\Delta^n}(t))^{\Delta})^{1-\gamma}]^{\Delta} &= (\gamma - 1)L^{-\gamma}(-r(t)w^{\Delta^n}(t))^{\Delta^2} \\ &= (\gamma - 1)L^{-\gamma}q(t)G(y(\beta(t))) \\ &\geqslant (\gamma - 1)g^{-\gamma}(\sigma(t))q(t)G(y(\beta(t))). \end{aligned} \tag{6.13}$$

From (6.12), $kg(t)R(t) \leqslant k_2 t^{n-1}$ for $t \geqslant t_3$, so $kg(\sigma(t))R(\sigma(t)) \leqslant k_2(\sigma(t))^{n-1}$ for $t \geqslant t_3$. Thus, (6.13) becomes

$$-[((-r(t)w^{\Delta^n}(t))^{\Delta})^{1-\gamma}]^{\Delta} \geqslant (\gamma - 1)L_1^{\gamma}\frac{R^{\gamma}(\sigma(t))}{(\sigma(t))^{\gamma(n-1)}}q(t)G(y(\beta(t))), \tag{6.14}$$

where $L_1 = k/k_2$. Choose $t_4 \in [t_3, \infty)_{\mathbb{T}}$ such that $\alpha(t) \geq t_3$ for all $t \in [t_4, \infty)_{\mathbb{T}}$. Using (H_3), (H_4), (Λ) and Lemma 6.1.5, we have

$$\begin{aligned} -[((-r(t)&w^{\Delta^n}(t))^{\Delta})^{1-\gamma}]^{\Delta} - G(p)[((-r(\alpha(t))w^{\Delta^n}(\alpha(t)))^{\Delta})^{1-\gamma}]^{\Delta} \\ &\geq (\gamma - 1)L_1^{\gamma}\frac{R^{\gamma}(\sigma(t))}{(\sigma(t))^{\gamma(n-1)}}q(t)G(y(\beta(t))) \\ &\quad + G(p)(\gamma - 1)L_1^{\gamma}\frac{R^{\gamma}(\sigma(\alpha(t)))}{(\sigma(\alpha(t)))^{\gamma(n-1)}}q(\alpha(t))G(y(\beta(\alpha(t)))) \\ &\geq \lambda(\gamma - 1)L_1^{\gamma}d(t)Q(t)G(z(\beta(t))) \\ &\geq \lambda(\gamma - 1)L_1^{\gamma}d(t)Q(t)G(w(\beta(t))) \\ &\geq \lambda(\gamma - 1)L_1^{\gamma}G(k_1)d(t)Q(t)G(R(\beta(t))) \end{aligned}$$

for $t \geqslant t_4$. Therefore,

$$\int_{t_4}^{\infty} d(t)Q(t)G(R(\beta(t)))\Delta t < \infty,$$

which contradicts (H_5).

Next, we suppose that $w(t) < 0$ for $t \geq t_2$. Then $z(t) - k(t) < 0$ implies $y(t) \leq z(t) = y(t) + p(t)y(\alpha(t)) < k(t)$. Thus, $y(t)$ is bounded. It follows that $z(t)$ and $w(t)$ are bounded due to (6.6) and (6.8) respectively, because $p(t)$ is bounded. By Lemma 6.1.4, one of the *cases* (d), (e) or (f) hold for $t \geq t_2$. For the *cases* (d), (e) and (f) with $l \leq 1$, $\lim_{t\to\infty} w(t) \leq 0$. From (6.8), $\lim_{t\to\infty} z(t) \leq 0$. If $\lim_{t\to\infty} z(t) < 0$, then $z(t)$ will be negative, which is a contradiction to $z(t) > 0$. Therefore, $\lim_{t\to\infty} z(t) = 0$. It follows that $\lim_{t\to\infty} y(t) = 0$, since $y(t) \leq z(t)$. The *cases* (d), (e) and (f) with $l > 1$ does not hold, because in these cases $w(t)$ will be unbounded by Lemma 5.1.1, which is a contradiction to $w(t)$ is bounded. This completes the proof of the theorem. □

The following corollary is immediate.

Corollary 6.1.8. *Under the conditions of Theorem 6.1.7, every unbounded solution of* (E_8) *oscillates.*

Example 6.1.9. Let $\mathbb{T} = \mathbb{R}$ and consider the differential equation

$$\left(e^{t/2}\left(y(t) + e^{-t}y(t/2)\right)'''\right)'' + 37e^{9t/2}y^3(t) - e^{-t}(1 + e^{-t})\frac{y(t/4)}{1 + y^2(t/4)} = 0, \quad (6.15)$$

$t(> 0) \in \mathbb{T}$. Here $r(t) = e^{t/2}$, $p(t) = e^{-t}$, $q(t) = 37e^{9t/2}$, $h(t) = e^{-t}(1 + e^{-t})$, $G(u) = u^3$, $\alpha(t) = t/2$, $\beta(t) = t$, $\gamma(t) = t/4$ and $H(u) = \frac{u}{1+u^2}$. It is easy to see that equation (6.15) satisfies all the conditions of Theorem 6.1.7. Hence, any soluton of equation (6.15) oscillates or converge to 0 as $t \to \infty$. In particular, $y(t) = e^{-2t}$ is a solution of (6.15) that converges to 0 as $t \to \infty$.

Our next theorem gives sufficient conditions for all unbounded solutions to oscillate.

Theorem 6.1.10. *Let* $0 \leqslant p(t) \leqslant p < 1$. *If* ($H_1$), ($H_2$), ($H_4$) *and* ($H_7$) *hold, then every unbounded solution of* (E_8) *oscillates.*

Proof. Let $y(t)$ be an unbounded nonoscillatory solution of (E_8), say $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, $y(\gamma(t))$ and $y(\alpha(\alpha(t)))$ are all positive for $t \in [t_1, \infty)_{\mathbb{T}}$, for some $t_1 \geqslant t_0$. We set $z(t)$, $k(t)$ and $w(t)$ as in (6.6)–(6.8) to obtain (6.9) for $t \geq t_1$. Consequently, $w^{\Delta^i}(t)$

$(i = 0, 1, \cdots, n-1)$, $(r(t)w^{\Delta^n}(t))$, and $(r(t)w^{\Delta^n}(t))^{\Delta}$ are monotonic and of constant sign on $[t_2, \infty)_{\mathbb{T}}$, $t_2 \geqslant t_1$. Assume that $w(t) > 0$ for $t \geq t_2$. By Lemma 6.1.3, one of the *cases* (a), (b) or (c) holds. If we consider the *cases* (a), (b) and (c) with $l \geq 1$, then $0 < w^{\Delta}(t) = z^{\Delta}(t) - k^{\Delta}(t)$. If $z^{\Delta}(t)$ oscillates, then $z^{\Delta}(t) \leq 0$ at some arbitrarily large values of t, which is a contradiction to $w^{\Delta}(t) > 0$, since $k^{\Delta}(t) < 0$ for all t and $\lim_{t\to\infty} k^{\Delta}(t) = 0$. Thus, z is monotonic and we cannot have $z^{\Delta}(t) \leq 0$, so $z^{\Delta}(t) > 0$ for all large t, say $t \geq t_3 > t_2$. Hence, in these cases

$$\begin{aligned}(1-p)z(t) \leq (1-p(t))z(t) &< z(t) - p(t)z(\alpha(t)) \\ &= y(t) - p(t)p(\alpha(t))y(\alpha(\alpha(t))) < y(t),\end{aligned}$$

that is,

$$y(t) > (1-p)z(t) > (1-p)w(t) \tag{6.16}$$

for $t \geqslant t_3 > t_2$. Thus, (6.9) implies

$$G((1-p)w(\beta(t)))q(t) \leqslant -(r(t)w^{\Delta^n}(t))^{\Delta^2}. \tag{6.17}$$

Applying Lemma 6.1.5 to the last inequality for the *cases* (a) and (b) with $l \geqslant 1$, the inequality (6.17) yields

$$G(k_1(1-p))G(R(\beta(t)))q(t) \leqslant -(r(t)w^{\Delta^n}(t))^{\Delta^2}. \tag{6.18}$$

Integrating (6.18), from t_3 to ∞, we have

$$\int_{t_3}^{\infty} q(t)G(R(\beta(t)))\Delta t < \infty,$$

which contradicts (H_7'). For the *case* (c) with $l \geqslant 1$ of Lemma 6.1.3, we proceed as in the proof of Theorem 6.1.7, to obtain (6.14). From (6.14), (6.16) and Lemma 6.1.5, we have

$$-[((-r(t)w^{\Delta^n}(t))^{\Delta})^{1-\gamma}]^{\Delta} \geqslant (\gamma-1)L_1^{\gamma}G((1-p)k_1)\frac{R^{\gamma}(\sigma(t))}{(\sigma(t))^{\gamma(n-1)}}q(t)G(R(\beta(t)))$$

for $t \geqslant t_3$. Integrating the last inequality from t_3 to ∞, we obtain

$$\int_{t_3}^{\infty} \frac{R^{\gamma}(\sigma(t))}{(\sigma(t))^{\gamma(n-1)}}q(t)G(R(\beta(t)))\Delta t < \infty,$$

contradicting (H_7). Since $y(t)$ is unbounded, then $z(t)$ and $w(t)$ are unbounded due to (6.6) and (6.8) respectively. Thus, the *cases* (a), (b) and (c) with $l = 0$ does not hold, because in these cases $w(t)$ will be bounded. Finally, we see that since $y(t)$ is unbounded, the case $w(t) < 0$ does not arise, because $w(t) = z(t) - k(t) < 0$ implies $0 < z(t) < k(t)$, so again $z(t)$ is bounded. This completes the proof of the theorem. □

Example 6.1.11. Let $\mathbb{T} = \mathbb{Z}$, and consider the difference equation

$$\Delta^2(e^n\Delta(y(n) + e^{-6n+4}y(n-4))) + e^{2/3}(e+1)(e^2+1)^2e^{5n/3}y^{1/3}(n-2) - e^3(e^{-5}+1)(e^{-4}+1)^2e^{-5n}(1+e^{n-3})\frac{y(n-3)}{1+\mid y(n-3)\mid} = 0, \quad (6.19)$$

for $n(\geq k) \in \mathbb{Z}$, where $k \in \mathbb{Z}$ such that $k \geqslant 4$ and $e^{-6n+4} < 1$ for all $n \geqslant k$. Here $r(n) = e^n$, $p(n) = e^{-6n+4}$, $q(n) = e^{2/3}(e+1)(e^2+1)^2e^{5n/3}$, $h(n) = e^3(e^{-5}+1)(e^{-4}+1)^2e^{-5n}(1+e^{n-3})$, $\alpha(n) = n-4$, $\beta(n) = n-2$, $\gamma(n) = n-3$, $G(u) = u^{1/3}$, and $H(u) = \frac{u}{1+|u|}$. It is easy to see that equation (6.19) satisfies all the conditions of Theorem 6.1.10. Hence, any unbouded soluton of equation (6.19) oscillates. In particular, $y(n) = (-1)^ne^n$ is a oscillatory solution of (6.19).

Theorem 6.1.12. *Let* $-1 < p_1 \leqslant p(t) \leqslant 0$, p_1 *is a negative real number,* $\beta(t) < t$, *(H_1), (H_2), (H_4), and (H_7) hold. If*

(H_8) $\int_0^{\pm c} \frac{du}{G(u)} < \infty$ *and*

(H_9) $\int_{t_4}^{\infty} G(R_{\beta(t)}(t))q(t)\Delta t = \infty$,

then any solution of (E_8) *is either oscillatory or converges to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (E_8), say $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, $y(\gamma(t))$ are all positive for all $t \in [t_1, \infty)_{\mathbb{T}}$, $t_1 \geqslant t_0$. Setting $z(t)$, $k(t)$, and $w(t)$ as in (6.6), (6.7), and (6.8), we obtain (6.9) for $t \geqslant t_1$. Hence, $w(t)$ is monotonic for large $t \in [t_1, \infty)_{\mathbb{T}}$. Let $w(t) > 0$ for $t \geqslant t_2$, for some $t_2 \geqslant t_1$ and assume that one of the cases (a), (b) or (c) of Lemma 6.1.3 holds. In the *cases* (a), (c) and *case* (b) with $l \geqslant 1$, we have $y(\beta(t)) \geqslant w(\beta(t)) \geqslant k_1R(\beta(t))$ for $t \geqslant t_3 > t_2$ due to Lemmma 6.1.5. First, we consider the *case* (a) and the *case* (b) with $l \geq 1$. Then (6.9) yields,

$$\int_{t_3}^{\infty} q(t)G(R(\beta(t)))\Delta t < \infty,$$

contradicting (H_7'). In the *case* (b) with $l = 0$, we have

$$y(\beta(t)) \geqslant w(\beta(t)) \geqslant (r(t)w^{\Delta^n}(t))^{\Delta} R_{\beta(t)}(t)$$

for $t > \beta(t) \geqslant t_3$ for $t \geqslant t_4$, $t_4 \in [t_3, \infty)_{\mathbb{T}}$ by Lemma 6.1.5. From (H_4), (6.9) yields

$$G((r(t)w^{\Delta^n}(t))^{\Delta})q(t)G(R_{\beta(t)}(t)) \leq -(r(t)w^{\Delta^n}(t))^{\Delta^2}.$$

This implies that

$$\int_{t_4}^{\infty} q(t)G(R_{\beta(t)}(t))\Delta t < \infty,$$

which contradicts to (H_9) due to (H_8). For the case (c) of Lemma 6.1.3, we have $y(t) \geqslant w(t) \geqslant k_1 R(t)$ for $t \geqslant t_3$ by Lemma 6.1.5. Consequently, (6.14) follows

$$-[((-r(t)w^{\Delta^n}(t))^{\Delta})^{1-\gamma}]^{\Delta} \geqslant (\gamma - 1)L_1^{\gamma} G(k_1) \frac{R^{\gamma}(\sigma(t))}{(\sigma(t))^{\gamma(n-1)}} q(t)G(R(\beta(t))),$$

for $t \geq t_4 > t_3$. An integration shows,

$$\int_{t_4}^{\infty} \frac{R^{\gamma}(\sigma(t))}{(\sigma(t))^{\gamma(n-1)}} q(t)G(R(\beta(t)))\Delta t < \infty,$$

contradicting to (H_7).

Now suppose that $w(t) < 0$ for $t \geqslant t_2$. We claim that $y(t)$ is bounded. If not, then there is an increasing sequence $\{\tau_n\}_{n=1}^{\infty} \subset [t_2, \infty)_{\mathbb{T}}$ such that $\tau_n \to \infty$, $y(\tau_n) \to \infty$ as $n \to \infty$, and $y(\tau_n) = \max\{y(t) : t_2 \leqslant t \leqslant \tau_n\}$. We choose τ_1 large enough so that $\alpha(\tau_1) \geqslant t_2$. Hence,

$$0 \geqslant w(\tau_n) \geqslant y(\tau_n) + p(\tau_n)y(\alpha(\tau_n)) - k(\tau_n) \geqslant (1 + p_1)y(\tau_n) - k(\tau_n).$$

Since $k(\tau_n)$ is bounded and $1 + p_1 > 0$, we have $w(\tau_n) > 0$ for large n, which is a contradiction, so our claim is true. Hence, $z(t)$ and $w(t)$ are bounded. Since $w(t) < 0$, then by Lemma 6.1.4 one of the *cases* (d), (e) or (f) are holds. The *cases* (d), (e) and (f) with $l > 1$ does not hold, because in these cases $w(t)$ will be unbounded by Lemma 5.1.1. So the *cases* (d), (e) and (f) with $l \leq 1$ are hold. In these cases, $\lim_{t\to\infty} w(t)$

exists finite number. Hence,

$$\begin{aligned}
0 \geq \lim_{t\to\infty} w(t) &= \lim_{t\to\infty} z(t) \\
&= \limsup_{t\to\infty}[y(t) + p(t)y(\alpha(t))] \\
&\geq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_1 y(\alpha(t))) \\
&= \limsup_{t\to\infty} y(t) + p_1 \limsup_{t\to\infty} y(\alpha(t)) \\
&= (1 + p_1) \limsup_{t\to\infty} y(t),
\end{aligned}$$

which implies that $\limsup_{t\to\infty} y(t) = 0$, that is, $\lim_{t\to\infty} y(t) = 0$. This completes the proof of the theorem. □

Remark 6.1.13. Under the conditions of Theorem 6.1.12, every unbounded solution of (E_8) oscillates.

Remark 6.1.14. In Theorem 6.1.12, equation (E_8) is of even order (n is even), then conditions $\beta(t) < t$, (H_8) and (H_9) need not be required.

Theorem 6.1.15. *Let $-\infty < p_2 \leqslant p(t) \leqslant p_3 < -1$, p_2, p_3 are negative real numbers. Assume that conditions (H_1), (H_2), (H_4), and (H_7) hold. Then any solution of* (E_8) *is either oscillatory, or satisfies* $\lim_{t\to\infty} y(t) = 0$, *or satisfies* $\mid y(t) \mid\to \infty$ *as* $t \to \infty$.

Proof. Proceeding as in the proof of Theorem 6.1.12, in the *cases* (a), (c), and the *case* (b) with $l \geqslant 1$ when $w(t) > 0$, we again obtain a contradiction to (H_7). Now for the *case* (b) with $l = 0$, $\lim_{t\to\infty} w(t)$ exists.

$$\begin{aligned}
0 \leq \lim_{t\to\infty} w(t) &= \lim_{t\to\infty} z(t) \\
&= \liminf_{t\to\infty}[y(t) + p(t)y(\alpha(t))] \\
&\leq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_3 y(\alpha(t)) \\
&= \limsup_{t\to\infty} y(t) + p_3 \limsup_{t\to\infty} y(\alpha(t)) \\
&= (1 + p_3) \limsup_{t\to\infty} y(t),
\end{aligned}$$

which implies that $\limsup_{t\to\infty} y(t) = 0$, that is, $\lim_{t\to\infty} y(t) = 0$.

Next, we consider the case $w(t) < 0$ for $t \geqslant t_2$. Then by Lemma 6.1.4, one of the *cases* (d), (e) or (f) holds. In these cases, if $\lim_{t\to\infty} w(t)$ exists, then $-\infty \leq$

$\lim_{t\to\infty} w(t) \leq 0$. If $-\infty < \lim_{t\to\infty} w(t) \leq 0$, then there exists a non-positive real number m such that $\lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t) = m$ due to (6.8). It follows that

$$\limsup_{t\to\infty}(p_2 y(\alpha(t))) \leq \limsup_{t\to\infty} z(t) = m.$$

That is, $\liminf_{t\to\infty} y(t) \geqslant 0$. In the *cases* ($e$) and ($f$), (6.9) yields

$$\int_{t_2}^{\infty} q(t)G(y(\beta(t)))\Delta t < \infty.$$

If $\liminf_{t\to\infty} y(t) > 0$, then it follows that

$$\int_{t_2}^{\infty} q(t)\Delta t < \infty,$$

contradicting to Remark 6.1.1. Hence, $\liminf_{t\to\infty} y(t) = 0$.

Consider the case (d). Then choose $f(x) = x^{1-l}$ and $g(t) = (-r(t)w^{\Delta^n}(t))^{\Delta}$. By Lemma 1.3.11, there exists c in the real interval $[t, \sigma(t)]$ with $g(c) = L$ such that

$$\begin{aligned} -[((-r(t)w^{\Delta^n}(t))^{\Delta})^{1-\gamma}]^{\Delta} &= (\gamma-1)L^{-\gamma}(-r(t)w^{\Delta^n}(t))^{\Delta^2} \\ &= (\gamma-1)L^{-\gamma}q(t)G(y(\beta(t))). \end{aligned}$$

Integrating, we obtain

$$\int_{t_2}^{\infty} q(t)G(y(\beta(t)))\Delta t < \infty.$$

If $\liminf_{t\to\infty} y(t) > 0$, then we obtain a contradiction to Remark 6.1.1. Hence, $\liminf_{t\to\infty} y(t) = 0$. By Lemma 1.5.1, we have $\lim_{t\to\infty} z(t) = 0$. Thus,

$$\begin{aligned} 0 = \lim_{t\to\infty} z(t) &= \liminf_{t\to\infty}(y(t) + p(t)y(\alpha(t))) \\ &\leq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_3 y(\alpha(t))) \\ &= \limsup_{t\to\infty} y(t) + p_3 \limsup_{t\to\infty}(y(\alpha(t))) \\ &= (1+p_3)\limsup_{t\to\infty} y(t). \end{aligned}$$

It follows that $\lim_{t\to\infty} y(t) = 0$, because $1 + p_3 < 0$. Consider the *cases* (d), (e) and (f) with $\lim_{t\to\infty} w(t) = -\infty$. From (6.8), it follows that $\lim_{t\to\infty} z(t) = -\infty$.

$$\begin{aligned} -\infty &= \limsup_{t\to\infty} z(t) \\ &\geqslant \liminf_{t\to\infty} y(t) + \limsup_{t\to\infty}(p_2 y(\alpha(t))) \\ &\geqslant (1+p_2)\liminf_{t\to\infty} y(t) \end{aligned}$$

This follows that $\lim_{t\to\infty} y(t) = \infty$, because $1 + p_2 < 0$. This completes the proof of the theorem. $\square$

Remark 6.1.16. Suppose if we will not use the Lemma 1.5.1 in the Theorem 6.1.15, then any solution of (E_8) is either oscillatory, or satisfies $\liminf_{t\to\infty} y(t) = 0$, or satisfies $| y(t) | \to \infty$ as $t \to \infty$.

6.2 Sufficient conditions for oscillation of (E_9) with $\int_{t_0}^{\infty} \frac{(\sigma(t))^{n-1}}{r(t)} \Delta t < \infty$.

This section is concerned with the oscillatory and asymptotic behavior of solutions of equation (E_9) for suitable forcing functions $f(t)$. We restrict our forcing functions to those that change signs. We will use the following conditions:

(H_{10}) There exists $F \in C_{rd}^n([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that $rF^{\Delta^n} \in C_{rd}^2([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $(rF^{\Delta^n})^{\Delta^2} = f$, and

$$-\infty < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \infty,$$

(H_{11}) There exists $F \in C_{rd}^n([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that $rF^{\Delta^n} \in C_{rd}^2([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $(rF^{\Delta^n})^{\Delta^2} = f$,

$$\liminf_{t\to\infty} F(t) = -\infty, \quad \text{and} \quad \limsup_{t\to\infty} F(t) = +\infty.$$

Theorem 6.2.1. *Let $0 \leqslant p(t) \leqslant p < \infty$, and assume that conditions (H_1)–(H_4), (Λ) and (H_{11}) hold. If*

$$(H_{12}) \quad \limsup_{t\to\infty} \int_{t^*}^{t} d(s)Q(s)G(F(\beta(s)))\Delta s = +\infty$$
$$\text{and} \quad \liminf_{t\to\infty} \int_{t^*}^{t} d(s)Q(s)G(F(\beta(s)))\Delta s = -\infty,$$

then every solution of (E_9) *oscillates.*

Remark 6.2.2. Notice that condition (H_{12}) implies

(H'_{12}) $\limsup_{t\to\infty}\int_{t^*}^{t} Q(s)G(F(\beta(s)))\Delta s = +\infty$ and $\liminf_{t\to\infty}\int_{t^*}^{t} Q(s)G(F(\beta(s)))\Delta s = -\infty.$

Proof. (**proof of Theorem 6.2.1**) Suppose that $y(t)$ is a nonoscillatory solution of (E_9) on $[t_0,\infty)_{\mathbb{T}}$ so that $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, $y(\gamma(t))$, and $y(\alpha(\beta(t)))$ are all positive on $[t_1,\infty)_{\mathbb{T}}$, for some $t_1 > t^* \geqslant t_0$. With z, k, and w as in (6.6)–(6.8), let

$$v(t) = w(t) - F(t) = z(t) - k(t) - F(t) \tag{6.20}$$

for $t \geq t_1$. Then (E_9) becomes

$$(r(t)v^{\Delta^n}(t))^{\Delta^2} = -q(t)G(y(\beta(t))) \leqslant (\not\equiv)0, \tag{6.21}$$

for $t \geq t_1$. Thus, $v(t)$ is monotonic on $[t_2,\infty)_{\mathbb{T}}$, for some $t_2 > t_1$. If $v(t) > 0$ for $t \geqslant t_2$, then $z(t) > k(t) + F(t) > F(t)$. By Lemma 6.1.3, one of the *cases* (a), (b), or (c) holds. First, we consider the cases (a) and (b). In view of (6.21), (H_3), (H_4) and (Λ), it is easy to see that

$$\begin{aligned}
0 &= (r(t)v^{\Delta^n}(t))^{\Delta^2} + q(t)G(y(\beta(t))) + G(p)(r(\alpha(t))v^{\Delta^n}(\alpha(t)))^{\Delta^2} \\
&\quad + G(p)q(\alpha(t))G(y(\beta(\alpha(t)))) \\
&\geq (r(t)v^{\Delta^n}(t))^{\Delta^2} + G(p)(r(\alpha(t))v^{\Delta^n}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(z(\beta(t))) \\
&\geq (r(t)v^{\Delta^n}(t))^{\Delta^2} + G(p)(r(\alpha(t))v^{\Delta^n}(\alpha(t)))^{\Delta^2} + \lambda Q(t)G(F(\beta(t))),
\end{aligned} \tag{6.22}$$

for $t \geqslant t_3 \geqslant t_2$. Integrating (6.22), we obtain

$$\limsup_{t\to\infty}\int_{t_3}^{t} Q(s)G(F(\beta(s)))\Delta s < \infty,$$

contradicting (H'_{12}).

Next, we consider the *case* (c). Then proceeding as in the proof of Theorem 6.1.7, we obtain an inequality similar to (6.14), from which it follows that

$$\begin{aligned}
-[((-r(t)v^{\Delta^n}(t))^{\Delta})^{1-\gamma}]^{\Delta} &- G(p)[((-r(\alpha(t))v^{\Delta^n}(\alpha(t)))^{\Delta})^{1-\gamma}]^{\Delta} \\
&\geqslant \lambda(\gamma-1)L_1^{\gamma}d(t)Q(t)G(z(\beta(t))) \\
&\geqslant \lambda(\gamma-1)L_1^{\gamma}d(t)Q(t)G(F(\beta(t)))
\end{aligned}$$

for $t \geqslant t_3$. An integration shows

$$\limsup_{t\to\infty}\int_{t_3}^{t} d(s)Q(s)G(F(\beta(s)))\Delta s < +\infty,$$

contradicting (H_{12}).

Therefore, $v(t) < 0$ for $t \geqslant t_2$. By Lemma 6.1.4 one of the cases (d), (e) or (f) holds. In each of these cases $z(t) \leq k(t) + F(t)$ which implies $\liminf_{t\to\infty} z(t) < 0$. This is a contradiction. Thus, completes the proof of the theorem. □

Remark 6.2.3. We can drop the condition (H_{12}) from the hypotheses of Theorem 6.2.1 and obtain that bounded solutions oscillate. In case $v(t) < 0$, the proof is the same. If $v(t) > 0$, then $z(t) > k(t) + F(t) > F(t)$ and condition (H_{11}) contradicts the boundedness of $y(t)$.

Our next two results are for the case where $p(t) \leqslant 0$.

Theorem 6.2.4. *Let $-1 < p(t) \leqslant 0$ and conditions (H_1), (H_2), (H_6), and (H_{11}) hold. If*

$$(H_{13})\ \limsup_{t\to\infty} \int_{t_3}^{t} \frac{R^\gamma(\sigma(s))}{(\sigma(s))^{\gamma(n-1)}} q(s)G(F(\beta(s)))\Delta s = +\infty$$

$$\text{and } \liminf_{t\to\infty} \int_{t_3}^{t} \frac{R^\gamma(\sigma(s))}{(\sigma(s))^{\gamma(n-1)}} q(s)G(F(\beta(s)))\Delta s = -\infty,$$

then every solution of (E_9) *is either oscillatory or satisfies* $\limsup_{t\to\infty} |y(t)| = \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (E_9), say $y(t)$, $y(\alpha(t))$, $y(\beta(t))$, and $y(\gamma(t))$ are all positive on $[t_1, \infty)_{\mathbb{T}}$, $t_1 \geqslant t_0$. Define $v(t)$ as in (6.20), so that we obtain (6.21). Consequently, $v(t)$ is monotonic on $[t_2, \infty)_{\mathbb{T}}$. Let $v(t) > 0$ for $t \geqslant t_2$. Then one of the cases (a), (b) or (c) of Lemma 6.1.3 holds. Now, $v(t) > 0$ implies

$$y(t) > z(t) > k(t) + F(t) > F(t) \tag{6.23}$$

for $t \geqslant t_2 > t_1$. First, we consider the cases (a) and (b), then using (6.23) in (6.21), we obtain

$$\limsup_{t\to\infty} \int_{t_3}^{t} q(s)G(F(\beta(s)))\Delta s < \infty,$$

a contradicting to the fact (H_{13}).

Assume that the *case* (c) holds. Proceeding as in the proof of Theorem 6.1.7, we obtain similar to (6.14), we have

$$-[((-r(t)v^{\Delta^n}(t))^\Delta)^{1-\gamma}]^\Delta \geqslant (\gamma - 1)L_1^\gamma \frac{R^\gamma(\sigma(t))}{(\sigma(t))^{\gamma(n-1)}} q(t)G(y(\beta(t))).$$

Using (6.23), and the above inequality yields

$$-[((-r(t)v^{\Delta^n}(t))^{\Delta})^{1-\gamma}]^{\Delta} \geqslant (\gamma-1)L_1^{\gamma}\frac{R^{\gamma}(\sigma(t))}{(\sigma(t))^{\gamma(n-1)}}q(t)G(F(\beta(t))),$$

for $t \geqslant t_3 > t_2$. An integration yields a contradiction to (H_{13}).

We must have $v(t) < 0$ for $t \geqslant t_2$. Now, $z(t) - k(t) < F(t)$ which implies that $\liminf_{t\to\infty} z(t) = -\infty$, so $\limsup_{t\to\infty} y(t) = +\infty$, which completes the proof of the theorem. □

Example 6.2.5. Let $\mathbb{T} = q^{\mathbb{N}_0}$, where $q > 1$ is fixed. Consider the q-difference equation

$$\Delta_q^2\Big(\frac{(q-1)^3}{(q^2+1)(q+1)}t^3\Delta_q\big(y(t)-\frac{1}{2}y(t/q)\big)\Big) + (t^4+\frac{1}{t^5}+3)y^{1/3}(t/q^2) - \frac{2}{t^5}\frac{y(t/q^4)}{1+y^2(t/q^4)} = t^4(-1)^{\log_q^t}, \quad (6.24)$$

for $t \in [q^4,\infty)_{\mathbb{T}}$. Here $r(t) = \frac{(q-1)^3}{(q^2+1)(q+1)}t^3$, $p(t) = -1/2$, $q(t) = t^4 + \frac{1}{t^5} + 3$, $h(t) = 2/t^5$, $G(u) = u^{1/3}$, and $H(u) = \frac{u}{1+u^2}$. We consider $F(t) = \frac{-(q^2+1)(q+1)}{(q^4+1)(q^6+1)(q^5+1)}t^4(-1)^{\log_q^t}$ such that $\Delta_q^2(r(t)\Delta_q F(t)) = t^4(-1)^{\log_q^t}$. It is easy to see that equation (6.24) satisfies all the conditions of Theorem 6.2.4. Hence, any soluton of equation (6.24) oscillates. In particular, $y(t) = (-1)^{log_q^t}$ is an oscillatory solution of (6.24).

Theorem 6.2.6. *Let $-1 < p_4 \leqslant p(t) \leqslant 0$, where p_4 is a negative real number. Assume that (H_1), (H_2), (H_6), (H_{10}), and (H_{13}) are hold. Then every unbounded solution of* (E_9) *oscillates.*

Proof. Let $y(t)$ be a positive unbounded nonoscillatory solution of (E_9) on $[t_0,\infty)_{\mathbb{T}}$. Proceeding as in the proof of Theorem 6.2.4, we obtain the required contradiction when $v(t) > 0$ for $t \geqslant t_2$.

Next, we suppose that $v(t) < 0$ for $t \geqslant t_2$. Since $y(t)$ is unbounded, then there exists $\{\tau_n\}_{n=1}^{\infty} \subset [t_2,\infty)_{\mathbb{T}}$ such that $\tau_n \to \infty$, $y(\tau_n) \to \infty$ as $n \to \infty$, and

$$y(\tau_n) = \max\{y(t) : t_2 \leqslant t \leqslant \tau_n\}.$$

We may choose n large enough so that $\alpha(\tau_n) \geqslant t_2$. Hence,

$$z(\tau_n) \geqslant (1+p_4)y(\tau_n).$$

By Lemma 6.2.3, one of the cases (d), (e) or (f) holds. Now $z(t) = v(t) + k(t) + F(t)$ implies that $z(t) < k(t) + F(t)$, and so

$$\begin{aligned} \infty = (1 + p_4) \limsup_{n\to\infty} y(\tau_n) &\leqslant \limsup_{n\to\infty} [k(\tau_n) + F(\tau_n)] \\ &\leq \lim_{t\to\infty} k(\tau_n) + \limsup_{n\to\infty} F(\tau_n) \\ &< \infty. \end{aligned}$$

This contradiction completes the proof of the theorem. □

The last Theorem in this section is related to the **existence of positive solution** of (E_9).

Theorem 6.2.7. *Assume that* $1 < p_5 \leq p(t) \leq p_6 < \frac{1}{2}p_5^2 < \infty$ *(*p_5*,* p_6 *are positive real numbers) and* (H_2) *hold. Suppose that* (H_{10}) *holds with* $\frac{-(p_5-1)}{16p_6} \leq F(t) \leq \frac{p_5-1}{8p_6}$*. In addition, assume that* G *and* H *are Lipschitz on* $\mathbb{R}$ *with Lipschitz constants* G_1 *and* H_1 *respectively. If*

$$\int_{t_0}^{\infty} \frac{(\sigma(t))^{n-1}}{r(t)} \int_t^{\infty} \sigma(s)q(s)\Delta s\Delta t < \infty,$$

then (E_9) *admits a positive bounded solution.*

Proof. By (H_2), we can choose $t_1 > t_0$ large enough so that

$$\int_{t_1}^{\infty} \frac{(\sigma(t))^{n-1}}{r(t)} \int_t^{\infty} \sigma(s)h(s)\Delta s\Delta t < \frac{p_5 - 1}{4p_5 H(1)},$$

and

$$\int_{t_1}^{\infty} \frac{(\sigma(t))^{n-1}}{r(t)} \int_t^{\infty} \sigma(s)q(s)\Delta s\Delta t < \frac{p_5 - 1}{16p_6 G(1)}.$$

Let $X = BC_{rd}([t_1, \infty)_{\mathbb{T}}, \mathbb{R})$ be the Banach space of all bounded rd-continuous functions on $[t_1, \infty)_{\mathbb{T}}$ with the norm

$$\|x\| = \sup\{| x(t) |: t \in [t_1, \infty)_{\mathbb{T}}\},$$

and let

$$S = \{x \in X : \frac{p_5 - 1}{8p_5p_6} \leqslant x(t) \leqslant 1, \ t \in [t_1, \infty)_{\mathbb{T}}\}.$$

Then, S is a closed, bounded, and convex subset of X. Take $t_2 \in [t_1, \infty)_{\mathbb{T}}$ so that $\alpha(t)$, $\beta(t)$, $\gamma(t) \geqslant t_1$ for all $t \in [t_2, \infty)_{\mathbb{T}}$. Define the mappings A, $B : S \to S$ by

$$Ax(t) = \begin{cases} Ax(t_2) & \text{for } t \in [t_1, t_2)_{\mathbb{T}} \\ -\frac{x(\alpha^{-1}(t))}{p(\alpha^{-1}(t))} + \frac{2p_5^2+p_5-1}{4p_5 p(\alpha^{-1}(t))} & \text{for } t \in [t_2, \infty)_{\mathbb{T}}. \end{cases}$$

and

$$Bx(t) = \begin{cases} Bx(t_2) & \text{for } t \in [t_1, t_2)_{\mathbb{T}} \\ \frac{F(\alpha^{-1}(t))}{p(\alpha^{-1}(t))} + \frac{k(\alpha^{-1}(t))}{p(\alpha^{-1}(t))} & \\ -\frac{(-1)^{n-1}}{p(\alpha^{-1}(t))} & \\ \int_{\alpha^{-1}(t)}^{\infty} \frac{h_{n-1}(\alpha^{-1}(t),\sigma(s))}{r(s)} \big(\int_s^{\infty} (\sigma(u) - s) q(u) G(x(\beta(u)))\Delta u \big) \Delta s & \text{for } t \in [t_2, \infty)_{\mathbb{T}}. \end{cases}$$

For $x \in S$, we have

$$\begin{aligned} k(t) &= (-1)^{n-1} \int_t^{\infty} \frac{h_{n-1}(t, \sigma(s))}{r(s)} \int_s^{\infty} (\sigma(u) - s) h(u) H(x(\gamma(u)))\Delta u \Delta s \\ &\leqslant H(1)(-1)^{n-1} \int_t^{\infty} \frac{(t - \sigma(s))^{n-1}}{r(s)} \int_s^{\infty} (\sigma(u) - s) h(u) \Delta u \Delta s \\ &\leqslant H(1) \int_t^{\infty} \frac{(\sigma(s))^{n-1}}{r(s)} \int_s^{\infty} \sigma(u) h(u) \Delta u \Delta s \\ &< \frac{1}{4p_5}(p_5 - 1), \end{aligned}$$

and similarly we can show that,

$$\frac{(-1)^{n-1}}{p(\alpha^{-1}(t))} \int_{\alpha^{-1}(t)}^{\infty} \Big(\frac{h_{n-1}(\alpha^{-1}(t), \sigma(s))}{r(s)} \int_s^{\infty} (\sigma(u) - s) q(u) G(x(\beta(u)))\Delta u \Big) \Delta s \leq \frac{p_5 - 1}{16 p_5 p_6}.$$

For all x, $y \in S$ and all $t \in [t_2, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned} Ax(t) + By(t) &\leqslant -\frac{p_5 - 1}{8p_5p_6^2} + \frac{1}{4p_5^2}(2p_5^2 + p_5 - 1) + \frac{1}{8p_5p_6}(p_5 - 1) + \frac{1}{4p_5^2}(p_5 - 1) \\ &\leq \frac{1}{8p_5p_6^2} + \frac{1}{4p_5^2}(2p_5^2 + p_5) + \frac{1}{8p_6} + \frac{1}{8p_5} p_5 \\ &\leq \frac{1}{8p_5p_6^2} + \frac{1}{2} + \frac{p_5}{8p_5} + \frac{1}{8p_6} + \frac{1}{8p_5} p_5 \\ &\leq 1/8 + 1/2 + 1/8 + 1/8 + 1/8 \\ &\leq 1. \end{aligned}$$

and

$$\begin{aligned}
Ax(t)+By(t) &\geqslant -\frac{1}{p_5}+\frac{1}{4p_5p_6}(2p_5^2+p_5-1)-\frac{1}{16p_5p_6}(p_5-1)-\frac{1}{16p_5p_6}(p_5-1)\\
&\geq -\frac{1}{p_5}+\frac{1}{4p_5p_6}(2p_5^2)+\frac{1}{4p_5p_6}(p_5-1)-\frac{1}{16p_5p_6}(p_5-1)-\frac{1}{16p_5p_6}(p_5-1)\\
&\geq -\frac{1}{p_5}+\frac{1}{4p_5p_6}(4p_6)+\frac{p_5-1}{8p_5p_6}\\
&\geq \frac{p_5-1}{8p_5p_6}.
\end{aligned}$$

Thus, $Ax+By\in S$. To show that A is a contraction mapping on S, first notice that

$$\begin{aligned}
\|Ax-Ay\| &= \left\|-\frac{x(\alpha^{-1}(t))}{p(\alpha^{-1}(t))}+\frac{2p_5^2+p_5-1}{4p_5p(\alpha^{-1}(t))}+\frac{y(\alpha^{-1}(t))}{p(\alpha^{-1}(t))}-\frac{2p_5^2+p_5-1}{4p_5p(\alpha^{-1}(t))}\right\|\\
&= \left\|-\frac{x(\alpha^{-1}(t))}{p(\alpha^{-1}(t))}+\frac{y(\alpha^{-1}(t))}{p(\alpha^{-1}(t))}\right\|\\
&\leqslant \frac{1}{p_5}\left\|x(t)-y(t)\right\|.
\end{aligned}$$

Since $p_5>1$, A is a contraction mapping. To show that B is completely continuous on S, we need to show that B is continuous and maps bounded sets into relatively compact sets. In order to show that B is continuous, let x, $x_k=x_k(t)\in S$ be such that $\|x_k-x\|=\sup_{t\geqslant t_1}\{|x_k(t)-x(t)|\}\to 0$. Since S is closed, $x(t)\in S$. For $t\geqslant t_1$, we have

$$\begin{aligned}
&\|(Bx_k)-(Bx)\|\\
&= \Bigg\|\frac{(-1)^{n-1}}{p(\alpha^{-1}(t))}\int_{\alpha^{-1}(t)}^{\infty}\frac{h_{n-1}(\alpha^{-1}(t),\sigma(s))}{r(s)}\int_s^{\infty}(\sigma(u)-s)h(u)H(x_k(\gamma(u)))\Delta u\Delta s\\
&\quad -\frac{(-1)^{n-1}}{p(\alpha^{-1}(t))}\int_{\alpha^{-1}(t)}^{\infty}\frac{h_{n-1}(\alpha^{-1}(t),\sigma(s))}{r(s)}\int_s^{\infty}(\sigma(u)-s)q(u)G(x_k(\beta(u)))\Delta u\Delta s\\
&\quad -\frac{(-1)^{n-1}}{p(\alpha^{-1}(t))}\int_{\alpha^{-1}(t)}^{\infty}\frac{h_{n-1}(\alpha^{-1}(t),\sigma(s))}{r(s)}\int_s^{\infty}(\sigma(u)-s)h(u)H(x(\gamma(u)))\Delta u\Delta s\\
&\quad +\frac{(-1)^{n-1}}{p(\alpha^{-1}(t))}\int_{\alpha^{-1}(t)}^{\infty}\frac{h_{n-1}(\alpha^{-1}(t),\sigma(s))}{r(s)}\int_s^{\infty}(\sigma(u)-s)q(u)G(x(\beta(u)))\Delta u\Delta s\Bigg\|,
\end{aligned}$$

that is,

$$
\begin{aligned}
&\|(Bx_k)-(Bx)\| \\
&= \Bigg\| \frac{(-1)^{n-1}}{p(\alpha^{-1}(t))}\int_{\alpha^{-1}(t)}^{\infty}\frac{h_{n-1}(\alpha^{-1}(t),\sigma(s))}{r(s)}\int_s^\infty(\sigma(u)-s)h(u)\Big(H(x_k(\gamma(u))) \\
&\qquad - H(x(\gamma(u)))\Big)\Delta u\Delta s \\
&\quad + \frac{(-1)^{n-1}}{p(\alpha^{-1}(t))}\int_{\alpha^{-1}(t)}^{\infty}\frac{h_{n-1}(\alpha^{-1}(t),\sigma(s))}{r(s)}\int_s^\infty(\sigma(u)-s)q(u)\Big(G(x(\beta(u))) \\
&- G(x_k(\beta(u)))\Big)\Delta u\Delta s\Bigg\| \\
&\leqslant \frac{1}{p_5}H_1\|x_k-x\|\int_{\alpha^{-1}(t)}^{\infty}\frac{(\sigma(s)-\alpha^{-1}(t))^{n-1}}{r(s)}\int_s^\infty(\sigma(u)-s)h(u)\Delta u\Delta s \\
&\quad + \frac{1}{p_5}G_1\|x_k-x\|\int_{\alpha^{-1}(t)}^{\infty}\frac{(\sigma(s)-\alpha^{-1}(t))^{n-1}}{r(s)}\int_s^\infty(\sigma(u)-s)q(u)\Delta u\Delta s \\
&\leqslant \frac{1}{p_5}H_1\|x_k-x\|\int_{\alpha^{-1}(t)}^{\infty}\frac{(\sigma(s))^{n-1}}{r(s)}\int_s^\infty\sigma(u)h(u)\Delta u\Delta s \\
&\quad + \frac{1}{p_5}G_1\|x_k-x\|\int_{\alpha^{-1}(t)}^{\infty}\frac{(\sigma(s))^{n-1}}{r(s)}\int_s^\infty\sigma(u)q(u)\Delta u\Delta s \\
&\leqslant \frac{1}{4p_5^2}(p_5-1)\|x-x_k\|+\frac{1}{16p_5p_6}(p_5-1)\|x-x_k\|.
\end{aligned}
$$

Since for all $t \geqslant t_1$, $\{x_k(t)\}$ converges uniformly to $x(t)$ as $k\to\infty$, $\lim_{k\to\infty}\|(Bx_k)-(Bx)\|=0$. Thus, B is continuous. To show that BS is relatively compact, it suffices to show that the family of functions $\{Bx : x\in S\}$ is uniformly bounded and equicontinuous on $[t_1,\infty)_{\mathbb{T}}$. The uniform boundedness is clear. To show that BS is equicontinuous, let $x\in S$ and $t^{''}, t^{'} \geqslant t_1$ such that $t^{'} > t^{''}$. Then by Property 1.4.1 and Corollarry 1.4.3, we obtain

$$
\begin{aligned}
&|(Bx)(t^{''})-(Bx)(t^{'})| \\
&\leqslant \left|\frac{F(\alpha^{-1}(t^{''}))}{p(\alpha^{-1}(t^{''}))}-\frac{F(\alpha^{-1}(t^{'}))}{p(\alpha^{-1}(t^{'}))}\right| + \left|\frac{k(\alpha^{-1}(t^{''}))}{p(\alpha^{-1}(t^{''}))}-\frac{k(\alpha^{-1}(t^{'}))}{p(\alpha^{-1}(t^{'}))}\right| \\
&\quad + \Bigg|\frac{(-1)^{n-1}}{p(\alpha^{-1}(t'))}\int_{\alpha^{-1}(t^{'})}^{\infty}\frac{h_{n-1}(\alpha^{-1}(t^{'}),\sigma(s))}{r(s)}\int_s^\infty(\sigma(u)-s)q(u)G(x(\beta(u)))\Delta u\Delta s \\
&\quad - \frac{(-1)^{n-1}}{p(\alpha^{-1}(t''))}\int_{\alpha^{-1}(t^{''})}^{\infty}\frac{h_{n-1}(\alpha^{-1}(t^{''}),\sigma(s))}{r(s)}\int_s^\infty(\sigma(u)-s)q(u)G(x(\beta(u)))\Delta u\Delta s\Bigg|
\end{aligned}
$$

$$\leqslant \left|\frac{F(\alpha^{-1}(t''))}{p(\alpha^{-1}(t''))} - \frac{F(\alpha^{-1}(t'))}{p(\alpha^{-1}(t'))}\right| + \left|\frac{k(\alpha^{-1}(t''))}{p(\alpha^{-1}(t''))} - \frac{k(\alpha^{-1}(t'))}{p(\alpha^{-1}(t'))}\right|$$
$$+ \left|\frac{(-1)^{n-2}}{p(\alpha^{-1}(t'))}\int_{\alpha^{-1}(t')}^{\infty}\int_{\theta}^{\infty}\frac{h_{n-2}(\theta,\sigma(s))}{r(s)}\int_{s}^{\infty}(\sigma(u)-s)q(u)G(x(\beta(u)))\Delta u\Delta s\Delta\theta\right.$$
$$\left. - \frac{(-1)^{n-2}}{p(\alpha^{-1}(t''))}\int_{\alpha^{-1}(t'')}^{\infty}\int_{\theta}^{\infty}\frac{h_{n-2}(\theta,\sigma(s))}{r(s)}\int_{s}^{\infty}(\sigma(u)-s)q(u)G(x(\beta(u)))\Delta u\Delta s\Delta\theta\right|$$

$$\leqslant \left|\frac{F(\alpha^{-1}(t''))}{p(\alpha^{-1}(t''))} - \frac{F(\alpha^{-1}(t'))}{p(\alpha^{-1}(t'))}\right| + \left|\frac{k(\alpha^{-1}(t''))}{p(\alpha^{-1}(t''))} - \frac{k(\alpha^{-1}(t'))}{p(\alpha^{-1}(t'))}\right|$$
$$+ \left|\Big(\frac{1}{p(\alpha^{-1}(t'))} - \frac{1}{p(\alpha^{-1}(t''))}\Big)\right|$$
$$\left|(-1)^{n-2}\int_{\alpha^{-1}(t')}^{\infty}\int_{\theta}^{\infty}\frac{h_{n-2}(\theta,\sigma(s))}{r(s)}\int_{s}^{\infty}(\sigma(u)-s)q(u)G(x(\beta(u)))\Delta u\Delta s\Delta\theta\right|$$
$$+ \left|\frac{(-1)^{n-2}}{p(\alpha^{-1}(t''))}\int_{\alpha^{-1}(t'')}^{\alpha^{-1}(t')}\int_{\theta}^{\infty}\frac{h_{n-2}(\theta,\sigma(s))}{r(s)}\int_{s}^{\infty}(\sigma(u)-s)q(u)G(x(\beta(u)))\Delta u\Delta s\Delta\theta\right|$$

$$\leqslant \left|\frac{F(\alpha^{-1}(t''))}{p(\alpha^{-1}(t''))} - \frac{F(\alpha^{-1}(t'))}{p(\alpha^{-1}(t'))}\right| + \left|\frac{k(\alpha^{-1}(t''))}{p(\alpha^{-1}(t''))} - \frac{k(\alpha^{-1}(t'))}{p(\alpha^{-1}(t'))}\right|$$
$$+ \left|\Big(\frac{1}{p(\alpha^{-1}(t'))} - \frac{1}{p(\alpha^{-1}(t''))}\Big)\right|$$
$$\left|(-1)^{n-1}\int_{\alpha^{-1}(t')}^{\infty}\frac{h_{n-1}(\alpha^{-1}(t'),\sigma(s))}{r(s)}\int_{s}^{\infty}(\sigma(u)-s)q(u)G(x(\beta(u)))\Delta u\Delta s\right|$$
$$+ \frac{1}{p_5}\left|(-1)^{n-2}\int_{\alpha^{-1}(t'')}^{\infty}\frac{h_{n-2}(\alpha^{-1}(t''),\sigma(s))}{r(s)}\int_{s}^{\infty}(\sigma(u)-s)q(u)G(x(\beta(u)))\Delta u\Delta s\right|$$
$$\mid \alpha^{-1}(t') - \alpha^{-1}(t'') \mid$$

$$\leqslant \left|\frac{F(\alpha^{-1}(t''))}{p(\alpha^{-1}(t''))} - \frac{F(\alpha^{-1}(t'))}{p(\alpha^{-1}(t'))}\right| + \left|\frac{k(\alpha^{-1}(t''))}{p(\alpha^{-1}(t''))} - \frac{k(\alpha^{-1}(t'))}{p(\alpha^{-1}(t'))}\right|$$
$$+ G(1)\left|\Big(\frac{1}{p(\alpha^{-1}(t'))} - \frac{1}{p(\alpha^{-1}(t''))}\Big)\right| \quad \left|\int_{\alpha^{-1}(t')}^{\infty}\frac{(\sigma(s))^{n-1}}{r(s)}\int_{s}^{\infty}\sigma(u)q(u)\Delta u\Delta s\right|$$
$$+ \frac{G(1)}{p_5}\left|\int_{\alpha^{-1}(t'')}^{\infty}\frac{(\sigma(s))^{n-2}}{r(s)}\int_{s}^{\infty}\sigma(u)q(u)\Delta u\Delta s\right| \mid \alpha^{-1}(t') - \alpha^{-1}(t'') \mid,$$

we obtain $|(Bx)(t'') - (Bx)(t')| \to 0$ as $t'' \to t'$. Therefore, $\{Bx : x \in S\}$ is uniformly bounded and equicontinuous on $[t_1, \infty)_{\mathbb{T}}$. Hence, BS is relatively compact. By Krasnosel'skii's fixed point theorem [see Theorem 1.5.3], there exists $x \in S$ such that $Ax + Bx = x$. Thus, the theorem is proved. □

Remark 6.2.8. Results similar to Theorem 6.2.7 can be proved for other ranges of $p(t)$.

6.3 Conclusion

In this chapter we have studied the oscillatory and asymptotic behaviour of solutions of (E_8) and (E_9) under the assumption

$$\int_{t_0}^{\infty} \frac{(\sigma(t))^{n-1}}{r(t)} \Delta t < \infty,$$

in Section 6.1 and Section 6.2, respectively.

In Section 6.1, we have proved in Theorem 6.1.7, that every solution of (E_8) oscillates or converges to zere as t tends to infinity for the range of $p(t)$ with $0 \leq p(t) \leq p < \infty$. In Theorem 6.1.10, we have shown that every unbounded solution of (E_8) is oscillatory for the range of $p(t)$ with $0 \leq p(t) \leq p < 1$. In Theorem 6.1.12, we have proved that every solution of (E_8) is either oscillatory or converges to zero as $t \to \infty$ for the range of $p(t)$ with $-1 < p_1 \leq p(t) \leq 0$. In Theorem 6.1.15, we have shown that every solution of (E_8) is either oscillatory, or satisfies $\lim_{t\to\infty} y(t) = 0$, or satisfies $| y(t) | \to \infty$ as $t \to \infty$ for the range of $p(t)$ with $-\infty < p_2 \leq p(t) \leq p_3 < -1$.

In Section 6.2, we have proved in Theorem 6.2.1, that every solution of (E_9) is oscillatory for the range of $p(t)$ with $0 \leq p(t) \leq p < \infty$. In Theorem 6.2.4, we have proved that every solution of (E_9) is either oscillatory or satisfies $\limsup_{t\to\infty} | y(t) | = \infty$ for the range of $p(t)$ with $-1 < p(t) \leq 0$. In Theorem 6.2.6, we have shown that every unbounded solution of (E_9) oscillates for the range of $p(t)$ with $-1 < p_4 \leq p(t) \leq 0$. In Theorem 6.2.7, we have obtained sufficient conditions for (E_9) have a nonoscillatory solution for $p(t)$ with $1 < p_5 \leq p(t) \leq p_6 < \frac{1}{2}p_5^2 < \infty$.

It would be interesting to study the oscillatory/nonoscillatory/asymptotic behavior of (E_8) and (E_9) if $p(t)$ oscillates and every solution of (E_8) and (E_9) are oscillatory for all ranges of $p(t)$.

Bibliography

[1] R. P. Agarwal, "Difference Equations and Inequalities: Theory, Methods, and Applications", Dekker, New york (1992).

[2] R. P. Agarwal, P. J. Y. wong, "Advanced topics in Difference Equations", Kluwer Academic, Dordrecht (1997).

[3] R. P. Agarwal, M. Bohner, *Basic calculus on time scales and some of its applications*, Results Math. **35** (1-2) (1999), 3–22.

[4] R. P. Agarwal, M. Bohner, D. O'Regan, A. Peterson, *Dynamic equations on time scales : A survey,* J. Compu. Appl. Math. **141** (2002), 1–26.

[5] R. P. Agarwal, D. O'Regan, S. H. Saker, *Oscillation criteria for second order nonlinear neutral delay dynamic equations,* J. Math. Anal. Appl. **300** (2004), 203–217.

[6] C. D. Ahlbrandt, M. Bohner, J. Ridenhour, *Hamiltonian systems on time scales and some of its applications*, J. Math. Anal. Appl. **250** (2) (2000), 561–578.

[7] F. M. Atici, G. SH. Guseinov, *On Green's functions and positive solutions for boundary value problems on time scales*, J. Comput. Appl. Math. **141** (2002), 75–99.

[8] F. M. Atici, D. C. Biles, and Alex Lebedinskky, *An applications of time scales to economics*, Math. Comput. Modelling **43** (2006), 718–726.

[9] F. M. Atici, F. Uysal, *A production inventory model of HMMS (Proposed by Holt, Modigliani, Muth and Simon) on time scales,* Appl. Math. Lett. **21** (2008), 236–243.

[10] B. Baculíková, J. Džurina, *Oscillation of third-order nonlinear differential equations,* Appl. Math. Lett. **24** (2011), 466–470.

[11] B. Baculíková, E. M. Elabbasy, S. H. Saker, J. Džurina, *Oscillation criteria for third-order nonlinear differential equations,* Math. Slovaca **58** (2) (2008), 201–220.

[12] A. G. Balanov, N. B. Janson, P. V. E. Mc Clintock, R. W. Tuckr and C. H. T. Wang, *Bifurcation analysis of a neutral delay differential equation modelling the torsional motion of a driven drill-string*, Chaos, Solitons and Fractals, **15** (2003), 381–394.

[13] Z. Bartosiewicz, E. Pawluszewicz, *Realization of nonlinear control systems on time scales,* IEEE Trans. Autom. Control, **53** (2) (2008), 571–575.

[14] A. Bellen, N. Guglielmi, A. E. Ruehli, *Methods for linear systems of circuit delay differential equations of neutral type*, IEEE Trans. Circ. Systems-I, **46** (1999), 212–216.

[15] E. Boe, H. C. Chang, *Dynamics of delayed systems under feedback control*, Chem. Engng. Sci. **44** (1989), 1281–1294.

[16] M. Bohner, A. Peterson, " Dynamic Equations on Time scales : An Introduction with Applications", Birkhäuser Boston inc., 2001.

[17] M. Bohner, A. Peterson, "Advances in Dynamic Equations on Time scales", Birkhäuser Boston inc., 2003.

[18] R. K. Brayton, *Bifurcation of periodic solutions in a nonlinear difference-differential equation of neutral type*, Quart. Appl. Math. **24** (1966), 215–224.

[19] R. K. Brayton, *Nonlinear oscillations in a distributed network*, Quart. Appl. Math. **24** (1967), 289–301.

[20] R. K. Brayton and R. A. Willoughby, *On the numerical integration of a symmetric system of difference-differential equations of neutral type*, J. Math. Anal. Appl. **18** (1967), 182–189.

[21] Da-Xue Chen, and Guang-Hui Liu, *Oscillation theorems for second-order nonlinear neutral dynamic equations with variable delays and damping,* World Academy of Science, Engineering and Technology **5** (2011-02-26).

[22] R. D. Driver, *A mixed neutral systems*, Nonlinear Analysis, Theory Methods and Applications (1984), 155–158.

[23] E. R. Duke, "Solving higher order dynamic equations on time scales as first order systems MSC thesis", Marshall University, 2006.

[24] L. E. Elsgolc, "Qualitative Methods in Mathematical Analysis, Transl. Math. Monographs **12**", Amer. Mathematical Soc., Providence, RI, 1964.

[25] L. H. Erbe, Q. K. Kong, and B. G. Zhang, "Oscillation Theory for Functional Differential Equations", Marcel Dekkar, New York, 1995.

[26] H. I. Freedom, J. Wo, *Periodic solutions of single species models with periodic delay*, SIAM Journal of Mathematical Analysis **23** (1992), 689–701.

[27] Y. Jurang, L. Bin, *Oscillatory and asymptotic behaviour of fourth order nonlinear difference equations*, Acta Mathematica Sinica, New Series **13** (1)(1997), 105–115.

[28] K. Gopalsamy, "Stability and Oscillation in Delay Differential Equations of Population Dynamics", Kluwer Acad. Pub. Dordrecht Boston, London, 1992.

[29] S. R. Grace, J. R. Graef, S. Panigrahi, and Ercan Tunc, *On the oscillatory behavior of even order neutral delay dynamic equations on time scales,* Electron. J. Qual. Theory Differ. Equ. **96** (2012), 1–12.

[30] Said R. Grace, John R. Graef, and Mohamed A. El-Beltagy, *On the oscillation of third order neutral delay dynamic equations on time scales*, J. Comput. Math. Appl. **63** (4) (2012), 775–782.

[31] J. R. Graef, M. K. Grammatikopoulos, and P. W. Spikes, *Asymptotic behavior of nonoscillatory solutions of neutral delay differential equations of arbitrary order*, Nonlinear Anal. **21** (1993), 23–42.

[32] J. R. Graef, S. Panigrahi, and P. Rami Reddy, *On oscillatory and asymptotic behavior of fourth order nonlinear neutral delay dynamic equations with positive and negative coefficients*, Math. Slovaca **64** (2)(2014), 347–366.

[33] I. Gyori, G. Ladas, "Oscillation Theory of Delay Differential Equations with Applications", Clarendon Press, Oxford, 1991.

[34] C. P. Gupta, *Existence and uniqueness results for the bending of an elastic beam equation at resonance*, J. Math. Anal. Appl. **35** (1988), 208–225.

[35] J. K. Hale, "Theory of Functional Differential Equations", Springer-Verlag, New York, 1977.

[36] T. H. Hilderbrandt, "Introduction to the Theory of Integrations" Academic Press, New York, 1963.

[37] S. Hilger, "Ein Maβkettenkalkül mit Anwendung auf Zentrusmannigfaltigkeiten", Ph.D Thesis, Universität, Würzburg, 1998.

[38] S. Hilger, *Analysis on measure chains: a unified approach to continuous and discrete calculus*, Results Math. **18** (1-2) (1990), 18–56.

[39] S. Hilger, *Differential and difference calculus unified*, Nonlinear Anal. 30 (5) (1997), 2683–2694.

[40] M. A. Jones, B. Song, D. M. Thomas, *Controlling wound healing through debridement*, Math. Comput. Modelling **40** (2004), 1057–1064.

[41] V. Kac, P. Cheung, "Quantum calculus" Universitext, Springer - Verlag, New York (2002).

[42] B. Karpuz, L. N. Padhy, R. Rath, *Oscillation and asymptotic behaviour of a higher order neutral differential equation with positive and negative coefficients*, Elect. J. Differ. Equ. **113** 2008 (2008), 1–15.

[43] B. Karpuz, *Asymptotic behaviour of bounded solutions of a class of higher-order neutral dynamic equations*, Appl. Math. Compt. **215** (2009), 2174–2183.

[44] B. Karpuz, *Unbounded oscillation of higher-order nonlinear delay dynamic equations of neutral type with oscillating coefficients*, Electron. J. Qual. Theory Differ. Equ. **34** (2009), 1–14.

[45] B. Karpuz and Ö. Öcalan, *Necessary and sufficient conditions on asymptotic behaviour of solutions of forced neutral delay dynamic equations*, Nonlinear Anal. **71** (2009), 3063–3071.

[46] Y. Kuang, "Delay Differential Equations with Applications in Population Dynamics", Academic Press, New Work, 1993.

[47] B. Karpuz, *Sufficient conditions for the oscillation and asymptotic behaviour of higher-order dynamic equations of neutral type*, Appl. Math. Comput. **221** (2013), 453–462.

[48] T. Kusano and M. Naito, *Non-linear oscillation of fourth order differential equations*, Canad. J. Math. **4** (1976), 840–852.

[49] T. Kusano and M. Naito, *On fourth order non linear oscillations*, J. London Math. Soc. **14** (2) (1976), 91–105.

[50] Malgorzata Migda, *On the discrete version of generalized Kiguradze's Lemma* , Fasc. Math. **35** (2005).

[51] A. B. Malinowska and D. F. M. Torres, *Necessary and sufficient conditions for local pareto optimality on time scales*, J. Math. Sci. **161** (6) (2009), 803–810.

[52] Özkan Öcalan, *Oscillation of neutral differential equation with positive and negative coefficients*, J. Math. Anal. Appl., **331** (2007), 644–654.

[53] S. Panigrahi, P. Rami Reddy, *On oscillatory and asymptotic behaviour of fourth order non-linear neutral delay dynamic equations*, Comput. Math. Appl. **62** (2011), 4258–4271.

[54] S. Panigrahi and P. Rami Reddy, *Oscillatory and asymptotic behaviour of fourth order non-linear neutral delay dynamic equations*, Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis **20** (2013), 143–163.

[55] S. Panigrahi, J. R. Graef and P. Rami Reddy, *Oscillation results for fourth order nonlinear neutral dynamic equations*, Commu. Math. Anal. **15** (2013), 11–28.

[56] N. Parhi, and A. K. Tripathy, *On oscillatory fourth order nonlinear neutral differential equations I*, Math. Slovaca **54** (4) (2004), 389–410.

[57] N. Parhi, and A. K. Tripathy, *On scillatory fourth order nonlinear neutral differential equations II*, Math. Slovaca **55** (2) (2005), 183–202.

[58] E. P. Popov, "Automatic Regulation and Control" Nauka Mascow (in Russian), 1966.

[59] R. N. Rath, P. P. Mishra, L. N. Padhy, *On oscillation and asymptotic behavior of a neutral differential equation of first order with positive and negative coefficients*, Electronic journal of differential equations (1) (2007), 1–7.

[60] R. N. Rath, N. Misra, L. N. Padhy, *Oscillatory and asymptotic behaviour of a nonlinear second order neutral differential equation*, Math. Slovaca **57** (2)(2007), 157–170.

[61] P. Rami Reddy, and S. Panigrahi, *Oscillatory behaviour of higher-order neutral delay dynamic equations with positive and negative coefficients-I* (communicated)

[62] Y. Sahiner Yilmaz, and Ağacik Zafer, *Bounded oscillation of nonlinear neutral differential equations of arbitrary order,* Czechoslovak Math. J. **51** (1) (2001), 185–195.

[63] S. H. Saker, *Oscillation of second-order nonlinear neutral delay dynamic equations on time scales*, J. Comput. Appl. Math. **187** (2006), 123–141.

[64] S. H. Saker, J. O. Alzabut, A. Mukheimer, *On the oscillatory behavior for a certain class of third order nonlinear delay difference equations*, Elect. J. Qual. Theory of Diff. Equ. **67** (2010), 1–16.

[65] S. H. Saker, D. O'Regan, *New oscillation criteria for second-order neutral functional dynamic equations via the generalized Riccati substitution*, Commun Nonlinear Sci Numer Simulat **16** (2011), 423–434.

[66] M. Slemrod and E. F. Infante, *Asymptotic stability criteria for linear systems of difference-differential equations on neutral type and their discrete analogues*, J. Math. Anal. Appl. **38** (1992), 399–415.

[67] C. Soria-Hoyo, F. Pontiga, and A. Castellanos, *A PIC based procedure for the integration of multiple time scale problems in gas discharge physics*, J. Comput. Phys. **228** (2009), 1017–1029.

[68] V. Spedding, *Taming Nature's Numbers*, New scientist, **179** (2003), 28–31.

[69] P. N. Swamy, *Deformed Heisenberg algebra*, origin of q-calculus, phys. **328** (2003), 145–153.

[70] E. Thandapani, P. Sundaram, J. R. Graef, and P. W. Spikes, *Asymptotic behavior and oscillation of solutions of neutral delay difference equations of arbitrary order*, Math. Slovaca **47** (1997), 539–551.

[71] E. Thandapani, I. M. Arockiasamy, *Oscillatory and asymptotic behaviour of fourth order non-linear neutral delay difference equations*, Indian J. Pure Appl. Math. **32** (2001), 109–123.

[72] E. Thandapani, I. M. Arockiasamy, *Fourth-order nonlinear oscillations of difference equations*, Comp. Math. Appl. **42** (2001), 357–368.

[73] E. Thandapani, M. Vijaya, and Tongxing Li, *On the oscillation of third order half-linear neutral type difference equations*, Electron. J. Qual. Theory Differ. Equ. **76** (2011), 1–13.

[74] C. J. Tian and S. S. Cheng, *Oscillatory criteria for delay neutral difference equations with positive and negative coefficients*, Bull. Soc. Parana Math. **21** (2003), 1–12.

[75] A. K. Tripathy, *Oscillation of fourth order nonlinear neutral difference equations - I*, Math. Slovaca **58** (2) (2008), 221–240.

[76] A. K. Tripathy, *Oscillation of fourth order nonlinear neutral difference equations II*, Math. Slovaca **58** (5) (2008), 581–604.

[77] A. K. Tripathy, *Oscillation of a class of fourth-order nonlinear mixed neutral difference equations,* Inter. J. Difference Eqns. **2** (2009), 253–272.

[78] A. K. Tripathy, *Some oscillation results for second order nonlinear dynamic equations of neutral type*, Nonlinear Analysis : Theory, Methods and Applications **71** (2009), e1727–e1735.

[79] A. K. Tripathy, T. Gnana Bhaskar, *Oscillation results for second order neutral delay dynamic equations*, J. Functional Differential Equations **(**3-4) (2010), 329–344.

[80] A. K. Tripathy, *Oscillatory behavior of a class of nonlinear second order mixed difference equations,* Electron. J. Qual. Theory Differ. Equ. **48** (2010), 1–19.

[81] A. K. Tripathy, *Oscillation criteria for a class of nonlinear fourth order neutral differential equations*, Mathematica Slovaca **63** (2013), 243–262.

[82] A. K. Tripathy, S. Panigrahi, P. R. Reddy, *Oscillation of a class of fourth order functional dynamic equations,* Functional Differential Equations **21** (1-2) (2014), 91–108.

[83] R. A. Usami, *A uniqueness theorem for a boundary value problem*, Proc. Amer. Math. Soc. **77** (1979), 329–335.

[84] W. Xiong and J. Liang, *Novel stability criteria for neutral systems with multiple time delays*, Chaos, Solitons and Fractals **32** (2007), 1735–1741.

[85] Yibing Sun, Zhenlai Han, Shurong Sun, and Chao Zhang, *Oscillation criteria for even order nonlinear neutral differential equations,* Electron. J. Qual. Theory Differ. Equ. **30** (2012), 1–12.

[86] A. Zafer, and R. S. Dahiya, *Oscillation of bounded solutions of neutral differential equations*, Appl. Math. Lett. **6** (2) (1993), 43–46.

[87] J. Zhou, T. Chen and L. Xiang, *Robust synchronisation of delayed neutral networks based on adaptive control and parameters identification*, Chaos, Solitons and Fractals **27** (2006), 905–913.

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