# GRAVITATIONAL WAVES IN THERMAL STATES

A thesis submitted for the degree of

DOCTOR OF PHILOSOPHY IN PHYSICS

BY

BASEM GHAYOUR

SCHOOL OF PHYSICS
UNIVERSITY OF HYDERABAD
HYDERABAD 500 046
INDIA

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# GRAVITATIONAL WAVES IN THERMAL STATES

# Declaration

I hereby declare that, this thesis entitled **GRAVITATIONAL WAVES IN THERMAL STATES** is based on the work done by me at the School of Physics, University of Hyderabad under the supervision of Prof. P. K.Suresh. No part of this thesis has been previously submitted for a degree or diploma or any other qualification at this university or any other.

(Basem Ghayour)

Hyderabad,
Date:

# Certificate

This is to certify that, this thesis entitled **GRAVITATIONAL WAVES IN THERMAL STATES** is based on the work done by Mr. Ghayour under my supervision at the School of Physics, University of Hyderabad, in partial fulfillment of the requirements for the award of the degree of **Doctor of Philosophy in Physics**. No part of this thesis has been previously submitted for a degree or diploma or any other qualification at this university or any other.

(Prof. P.K. Suresh)
Thesis Supervisor

(Dean, School of Physics)

# Preface

The thesis is dedicated to the study of the amplitude and spectral energy density of relic gravitational waves (GWs) in thermal vacuum state. It mainly deals with the role of thermal vacuum state on the amplitude and spectral energy density of the relic GWs in the expanding flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe.

One of the remarkable predictions of the general theory of relativity is the existence of GWs. They are generated by the dynamics of various massive astrophysical objects and the variable gravitational field of early universe known as relic GWs. The relic GWs are very importance in cosmology because they provide information about early universe. These waves are not directly detected yet but in recent times the quest for GWs becoming more exciting than ever before due to progress in the observational techniques.

In its early evolutionary stage the universe underwent a rapid expansion known as inflation and is supposed as an important era of creation of the relic GWs. The standard model of inflation predicts non-thermal GWs, however the existence of GWs with thermal feature is not ruled out completely. The thermal GWs probably generated during the inflationary stage due to the stimulated emission process also. The pre-inflationary period of the universe is another potential era that possibly generated the thermal GWs. Also theories with higher dimensions, the evaporation of mini black holes in the early universe, etc; predict the existence of thermal GWs. The computation of the amplitude of the GWs and hence its spectral energy density studied so far without the considerations of thermal GWs. But the relic thermal GWs can also contribute to the amplitude and hence spectral energy density in the expanding universe. This is due to the fact that the various evolution stages of the universe certainly have been affecting the amplitude of the GWs spectrum. Thus in the present study we compute the amplitude of GWs and hence its spectral energy density in the thermal vacuum states for FLRW universe. The thermal GWs due to the extra dimensional effect can also be contributed to the amplitude and spectral energy density. Thus the corresponding modified amplitude is possible to compare with the sensitivity of the various GWs detectors.

It is believed that the inflationary stage played a crucial role on the further evolution stages of the universe. The inflation brought temperature of the universe below the required for trigging the thermo nuclear reactions. But the created particles, at end of the inflationary period, collided each other and thus achieved the required temperature known as the reheating stage of the universe. To include the reheating effect a new stage called z-stage is introduced to allow a general reheating epoch. Therefore it is interesting to study the effect of z-stage on the amplitude and spectral energy density of GWs.

Though the GWs are not directly observed the anisotropy of CMB with WMAP provides an upper bound on the angular power spectrum of relic GWs, known as the $B$ mode angular spectrum of CMB. And the results of various other missions that measured the CMB anisotropy show that there exist a discrepancy between the theoretical and estimated $B$ mode angular power spectrum from observations. This situation demand for an enhancement of $B$ mode angular power spectrum of CMB. Therefore the present work is also addressing this issue by considering the relic GWs in thermal squeezed vacuum state and explore the possibility of enhancing further the $B$ mode angular power spectrum of CMB in comparison to its zero temperature counter part.

The thesis is organized as follows, Chapter 1 gives an introduction and background of the present study.

In Chapter 2 we present a brief account of the GWs and the GWs in the expanding FLRW universe.

In Chapter 3, we consider the GWs in thermal vacuum state. And hence, we compute the amplitude and spectral energy density for the accelerated as well as decelerated FLRW universe. We compare the upper bound of spectral energy density of the GWs in the thermal vacuum state with its bound put by the nucleosynthesis calculation. In second part of the Chapter 3, we study the contribution of high frequency thermal GWs, due to extra dimensional effects, to the amplitude and spectral energy of GWs. The corresponding modified amplitude of the spectrum is compared with the sensitivity of Adv.LIGO, ET and LISA missions.

Chapter 4 deals with the reheating effects on the amplitude and spectral energy density for the non-thermal GWs in the accelerated and decelerated FLRW universe. The chapter mainly discusses an estimate of the index of power law of expansion for the inflation and reheating stages with the non-thermal GWs.

Chapter 5 contains an investigation on the enhancement of the $B$ mode of angular power spectrum of CMB in the thermal squeezed vacuum state.

Summary and conclusions are given in Chapter 6.

# Acknowledgements

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Finally, I thank all who were directly or indirectly involved in preparing this thesis and have made every moment of my stay at the university a cherished memory.

# Notations

Greek indices $\mu, \nu, ...$ run from 0 to 3.

Latin indices $i, j$, ... run from 1 to 3.

We use Einstein's summation convention.

Partial derivative $\partial_\mu = \frac{\partial}{\partial x^\mu}$.

Covariant derivative $\nabla_\mu$.

We follow the natural unit $c = \hbar = k_B = 1$, unless it is mentioned.

To my wife

**FATEMEH**

# Contents

# Chapter 1

# Introduction

The gravitational waves (GWs) are the ripples of spacetime and its existence is one of the finest predictions of Einstein's general theory of relativity. The systems with time varying mass quadrupole moments are the potential sources to generate GWs such as binary star systems, black hole mergers, etc [1]. In recent times the quest for GWs is becoming more exciting than ever before due to progress in the observational techniques [2]-[6]. One of the current research interests in GWs is the detection of the waves that were generated in the early universe called relic GWs [7].

The relic GWs are of paramount interest in cosmology, the science that deals with origin and evolution of the large scale structures in the universe, because it provides valuable information about the physical conditions of early universe. The inflationary period [8], during which the universe underwent quasi exponential expansion, is supposed as the main source of the relic GWs. According to the standard model of inflation the relic GWs created during the inflationary epoch are mainly non-thermal in nature because the standard inflationary models are energetically not in favor of generating thermal GWs. However the existence of GWs with thermal features is not ruled out completely. Even during the inflationary stage the thermal GWs probably generated due to the stimulated emission process and are studied widely [9]. The pre-inflationary period of the universe is another potential era that possible generated the thermal GWs [10]. Also theories with higher dimensions [11], more than $(3+1)$ dimensions, predicts the existence of thermal GWs with a specific peak temperature to be observed today [12], [13]. The idea of extra dimensions have received much attention in high energy physics, specially in the context of hierarchy problem, gravity and cosmology. There are some other alternative scenarios and candidate like the evaporation of mini black holes in the early universe, also predict the existence of thermal GWs. In the present study, we mainly consider the thermal GWs that were probably generated in the pre-inflationary era, during the inflationary era due to stimulated emission mechanism and from the extra dimensional effects in the early universe.

The inflationary scenario predicts a stochastic cosmic background of GWs with a nearly scale invariant spectrum [14]. The spectrum of these relic GWs depends not only on the details of expansion during the inflationary era but also on the subsequent stages such as reheating, radiation, matter and acceleration. The generated relic GWs went through several stages of evolution of the universe and at various epoch of the universe, dominated with various form of energy. Therefore the different stages of evolution of the universe are to be taken into account for the study of amplitude and spectral energy density of relic GWs.

Computation of the GWs spectrum in the decelerated expanding model is usually done with matter dominated universe [15]-[20] and the resulting spectrum is used for putting constraint on the detection of GWs originating from sources other than early universe epoch. The result of astronomical observations on SN Ia [21]-[22] shows that the universe is currently undergoing accelerated expansion indicating a non-zero cosmological constant. According to the ΛCDM concordance model, the observed acceleration of the present universe is supposed to be driven by an unknown form of energy, called the dark energy. Effect of the current acceleration on the nature of GWs the spectrum and spectral energy density are studied by several people [16], [23] and shown that the current acceleration phase of the universe does change the shape, amplitude and spectrum of the GWs [23].

In the present work, we study the amplitude of the GWs and hence its spectral energy density in the thermal vacuum states. The contribution of thermal GWs to its amplitude and spectral energy density can be studied for the accelerated as well as decelerated Friedmann-Lemaître-Robertson-Walker (FLRW) universe. The contribution of very high frequency thermal GWs from the extra dimensional effect can be included on the amplitude and spectral energy density of GWs and the corresponding modified amplitude is possible to compare with the sensitivity of the GWs detector such as Advanced Laser Interferometer GWs Observatory (Adv.LIGO)[2], Einstein Telescope (ET) [3] and Laser Interferometer Space Antenna (LISA) [4]. The effect of the very high frequency thermal GWs on its total spectral energy density also be examined that whether it exceed the upper bound put by the thermal nucleosynthesis calculation.

It is believed that the inflationary stage played a crucial role on the further evolution stages of the universe because particles were created at end of the inflationary period. And moreover collisions of the created particles were responsible for reheating the universe. The reheating was essential for the nucleosynthesis process since the inflation brought temperature of the universe below the required for taking place the thermo nuclear reactions. Towards the end of inflation, during the reheating, the equation of state of energy of the universe is considered as quite complicated and also model dependent. Hence a new stage called z-stage is introduced to allow a general reheating epoch. Therefore it is interesting to study the effect of z-stage on the amplitude and spectral energy density of GWs. In the present work, we study the GWs spectrum due to z-stage for the non-thermal GWs case only.

The GWs are not observed directly as on today but may be possible with the upcoming Planck or similar missions. The thermal GWs may be observed with the 21-cm hydrogen lines. The other but indirect way to observe the existence of relic GWs is through the cosmic microwave background radiation (CMB). This is possible because the GWs definitely had influenced the various anisotropies of the CMB. Though the GWs are not directly observed the anisotropy of CMB provides an upper bound, by using Wilkinson Microwave Anisotropy Probe (WMAP) data, on the angular power spectrum of relic GWs known as the $B$ mode of CMB. The other observations that measured the CMB anisotropy such as DASI, Boomerang, Maxipol, QUaD, CBI and Capmap are giving even higher upper bound on the $B$ mode angular power spectrum. These results show that there exist a discrepancy between the theoretical and estimated angular $B$ mode power spectrum from the various observed CMB anisotropy data. This situation is demanding for a plausible mechanism to enhance the power spectrum of GWs. Therefore the present work is also address as this issue by considering the relic GWs in thermal squeezed vacuum state and study the possibility of enhancing the $B$ mode angular power spectrum of CMB.

The discussed reasons so far in the introduction are the primary motivation to consider the GWs in thermal states. And moreover the earlier studies are not properly addressed the role of thermal vacuum states on the GWs in cosmology.

The present thesis is mainly to study the amplitude and spectral energy density of the thermal GWs in FLRW universe. Since both GWs and FLRW universe are developed through the celebrated Einstein field equations, which are in turn built on the foundations of general theory of relativity, we next discuss the Einstein field equations and feature of FLRW universe briefly.

## 1.1 Einstein's Field Equations

Einstein's general theory of relativity describes the gravitation in terms of the curvature of space-time with an appropriate source. The curvature and the source are related through the Einstein's field equations, given by

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}, \tag{1.1}$$

where $G$ is the Newton's gravitational constant and

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\, R, \tag{1.2}$$

is the Einstein tensor. And the Ricci tensor $R_{\mu\nu}$ is given by

$$R_{\mu\nu} = \partial_\nu \Gamma^\sigma_{\mu\sigma} - \partial_\sigma \Gamma^\sigma_{\mu\nu} + \Gamma^\rho_{\mu\sigma}\, \Gamma^\sigma_{\rho\nu} - \Gamma^\rho_{\mu\nu}\, \Gamma^\sigma_{\rho\sigma}, \tag{1.3}$$

where the Christoffel symbol of second kind $\Gamma^\sigma_{\mu\nu}$ is related to the fundamental metric tensor $g_{\mu\nu}$ as follows

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}\right). \tag{1.4}$$

In the Einstein tensor (1.2) $R$ is the Ricci scalar given by

$$R = g_{\mu\nu}\, R^{\mu\nu}. \tag{1.5}$$

In the field equation (1.1) the term $T_{\mu\nu}$ is called the energy-momentum tensor and it provides the source for gravity.

## 1.2 Standard Cosmology

Theoretical foundation of the modern cosmology is based on the Einstein's general theory of relativity. The standard model of cosmology is based on the general theory of relativity and the cosmological principle which states that, averaged on sufficiently large cosmological scales, the universe is homogeneous and isotropic [1, 24]. This requirement leads to the metric that governs space-time known as the FLRW given by

$$ds^2 = dt^2 - S(t)^2 \Big( \frac{dr^2}{1 - Kr^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \Big), \tag{1.6}$$

where $t$ is the cosmological time, $r, \theta$, and $\phi$ are the comoving coordinates and $S(t)$ the scale factor of the universe. And where $K$ is called the curvature parameter which take three different values, $K = 0, +1, -1$ respectively corresponding to flat, closed and open FLRW universe.

To solve the Einstein's field equations [1, 25], we assume a perfect fluid as source for gravity and its energy-momentum tensor is given by

$$T_{\mu\nu} = (p+\rho)\, u_\mu u_\nu - p\, g_{\mu\nu}, \tag{1.7}$$

where $u^\mu = (1, 0, 0, 0)$ is comoving velocity of the fluid and

$$T_{00} = \rho, \tag{1.8}$$

$$T_{ii} = p, \tag{1.9}$$

$$T^0_i = T^i_0 = T^i_j = 0, \quad (i \neq j), \tag{1.10}$$

where $\rho$ and $p$ respectively the energy density and pressure of the fluid.

With the perfect fluid and FLRW metric, the 00 component of the Einstein's field equation leads to

$$\left(\frac{\dot{S}}{S}\right)^2 + \frac{K}{S^2} = \frac{8\pi G}{3}\,\rho, \tag{1.11}$$

and the $ii$ components of the field equation is

$$2\frac{\ddot{S}}{S} + \left(\frac{\dot{S}}{S}\right)^2 + \frac{K}{S^2} = -8\pi G\, p. \tag{1.12}$$

The eqs.(1.11) and (1.12) are called the Friedmann's equations. Using eqs.(1.11) and (1.12), we get

$$\frac{\ddot{S}}{S} = -\frac{4\pi G}{3}\,(\rho + 3p), \tag{1.13}$$

and it provides the condition for acceleration, i.e; $\ddot{S}/S > 0$, of the universe.

The eq.(1.11) can be rewritten as follows

$$\Omega - 1 = \frac{K}{S^2 H^2}, \tag{1.14}$$

where

$$\Omega = \frac{\rho}{\rho_c}, \tag{1.15}$$

is the density contrast defined as fraction of total density $\rho$ to the critical density $\rho_c$ of the universe. The critical density is defined as

$$\rho_c = \frac{3H^2}{8\pi G}, \tag{1.16}$$

which is the density that makes the universe exactly flat, and $H = \dot{S}/S$ is the Hubble parameter. The current observational data is in favour of flat FLRW universe, and the present study is restricted to the flat FLRW universe only.

The Friedmann's equations discussed so far were without the cosmological constant $\Lambda$ and with the $\Lambda$ the equations can be written as

$$\left(\frac{\dot{S}}{S}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{S^2} + \frac{1}{3}\Lambda, \tag{1.17}$$

$$\frac{\ddot{S}}{S} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{1}{3}\Lambda. \tag{1.18}$$

The result of astronomical observations on SN Ia [21, 22] shows that the universe is currently undergoing accelerated expansion indicating a non-zero cosmological constant. According to the $\Lambda$CDM concordance model, the observed acceleration of the present universe is supposed to be driven by an unknown form of energy known as the dark energy which dominates the present universe.

The contracted Bianchi identity $\nabla_\nu G^{\mu\nu} = 0$, gives us the conservation of the energy-momentum tensor $\nabla_\nu T^{\mu\nu} = 0$. Thus continuity equation is given by

$$\dot{\rho} + 3\frac{\dot{S}}{S}(\rho + p) = 0. \tag{1.19}$$

This tells us how fast the density of the universe dilutes as it expands.

### Radiation

Assume that the universe is filled with radiation and its energy density and pressure respectively are $\rho_r$, $p_r$ and $p_r = \rho_r/3$. Therefore, it follows from the eq (1.19) that

$$\rho_r \propto S^{-4}. \tag{1.20}$$

Thus the Friedmann's equation for the radiation filled flat FLRW universe is

$$\left(\frac{\dot{S}}{S}\right)^2 = \frac{8\pi G}{3}\frac{\rho_{r0}\, S_0^4}{S^4}, \tag{1.21}$$

where $\rho_{r0}$ and $S_0$ are the vales of the energy density and scale factor of the present universe. The evolution of the scale factor for the radiation filled universe can be obtained from eq.(1.21) as,

$$S \propto t^{1/2}. \tag{1.22}$$

Therefore the flat FLRW universe dominated with radiation continues to expand for all time but the Hubble parameter goes to zero as $t \to \infty$.

**Dust**

This case illustrates the expansion of the universe filled with non-interacting particles. Since the dust doest not exert pressure its energy density and pressure are $\rho_M$ and $p_M = 0$. Therefore follows from eq (1.19) that

$$\rho_M \propto S^{-3}, \tag{1.23}$$

and the corresponding Friedmann's equation is

$$\left(\frac{\dot{S}}{S}\right)^2 = \frac{8\pi G}{3}\frac{\rho_{M0}\, S_0^3}{S^3}, \tag{1.24}$$

where $\rho_{M0}$ is the value of the energy density of the present universe. And hence the evolution of the scale factor for the dust filled flat FLRW is

$$S \propto t^{2/3}. \tag{1.25}$$

**Vacuum**

This vacuum filled case of the universe is usually taken as the dark energy and is related to the cosmological constant. Therefore its energy density and pressure are respectively given by

$$\rho_\Lambda = \frac{\Lambda}{8\pi G}, \qquad p_\Lambda = -\rho_\Lambda. \tag{1.26}$$

Therefore the Friedmann's equation for the vacuum dominated universe is

$$\left(\frac{\dot{S}}{S}\right)^2 = \frac{8\pi G}{3}\,\rho_\Lambda = H_0^2, \tag{1.27}$$

thus

$$S \propto e^{H_0 t}, \tag{1.28}$$

where $H_0$ is the value of the Hubble parameter for the present universe. This case is also known as the de-Sitter universe, an empty space solution of Einstein's field equations with a cosmological constant [1].

The major predictions of the standard cosmology are (i) expansion of the universe (ii) existence of CMB and (iii) abundance of light elements. Though the standard cosmology is spectacularly successful with its predictions but faced a number of unresolved problems known as the horizon problem, the flatness problem, unwanted relic problem and structure formation problem etc. There are no satisfactory solutions within in the standard cosmology and hence to solve the aforementioned problems, a scenario called inflation is introduced [26].

## 1.3 Inflation

According to the simplest inflationary scenario the universe underwent an exponential expansion during its early period of evolution [26]. The mechanism of inflation can be explained in a simple model of inflation with a single homogeneous scalar field, called inflaton. The scalar field can be described with the following Lagrangian

$$L = \sqrt{-g}\left(\frac{1}{2}\, g^{\mu\nu}\, \nabla_\mu \phi\, \nabla_\nu \phi - V(\phi)\right). \tag{1.29}$$

The scalar field is governed by the Klein-Gordon equation given by

$$\left(\Box + \frac{dV}{d\phi}\right)\phi = 0, \tag{1.30}$$

where $\Box = g^{\mu\nu}\, \nabla_\mu \nabla_\nu$.

In the FLRW spacetime where the field is close to spatially homogeneous, the spatial derivatives can be ignored. With the determinant $g$ for the metric (1.6), $\sqrt{-g} = S^3$ and therefore the Klein-Gordon equation becomes

$$\ddot{\phi} + 3\, H\, \dot{\phi} + \frac{dV(\phi)}{d\phi} = 0. \tag{1.31}$$

The energy-momentum tensor for the scalar field is given by

$$T_{\mu\nu} = \nabla_\mu \phi\, \nabla_\nu \phi - \frac{1}{2}\, g_{\mu\nu}\, [g^{\alpha\beta}\, \nabla_\alpha \phi\, \nabla_\beta \phi - V(\phi)]. \tag{1.32}$$

Therefore energy density and pressure of the scalar field are

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \qquad p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{1.33}$$

Hence the Friedmann's equation becomes,

$$H^2 = \frac{8\pi}{3m_{pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \tag{1.34}$$

where $m_{pl} = 1/\sqrt{G}$ is the Planck mass.

The condition for acceleration is $\rho + 3p < 0$, therefore to occur inflation, the inflaton field satisfies the condition that $V(\phi) > \dot{\phi}^2$. Therefore the inflation and indeed expansion of the universe is driven by the potential of the inflaton field. This is easily achievable with any suitably flat potential with the field displaced away from its minimum.

The conditions that define the slow-roll model of inflation are given by

$$\frac{1}{2}\dot{\phi}^2 \ll V(\phi), \qquad \ddot{\phi} \ll 3H\dot{\phi}. \tag{1.35}$$

Hence the Friedmann and Klein-Gordon equations become

$$H^2 \simeq \frac{8\pi}{3m_{pl}^2}V(\phi), \qquad 3H\dot{\phi} \simeq -V'(\phi). \tag{1.36}$$

Here $V'(\phi)$ is the derivative of the potential with respect to the field $\phi$. The approximations outlined in eq.(1.35) is usually considered with the two slow-roll parameters $\epsilon$ and $\tau$ defined respectively as follows,

$$\epsilon \equiv \frac{m_{pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2, \qquad \tau \equiv \frac{m_{pl}^2}{8\pi}\frac{V''}{V}. \tag{1.37}$$

The necessary but not sufficient conditions for the slow-roll approximation to hold are $\epsilon \ll 1$ and $|\tau| \ll 1$. Note that $\epsilon$ is positive by definition.

The slow-roll parameters make it easy to study the potentials for which the dynamical equations cannot be solved exactly. They also make it easy to see how inflation might begin and end for a given potential. For example the potential $V = m^2\phi^2/2$, $\epsilon = (m_{pl}^2/16\pi)(2/\phi)^2$, which is less than one as long as $\phi^2 > m_{pl}^2/4\pi$. When the field gets too close to the minimum, the slow-roll conditions are violated and inflation ends. Inflation can end in a different way also, as in hybrid inflation, where some extra field or effect ends inflation while the slow-roll conditions hold.

There are many models of inflation and collectively known as inflationary paradigm. The chief advantage of the inflationary scenario is that it provides the seed mechanism for the formation of large scale structures in the universe.

# Chapter 2

# Gravitational Waves

One of the remarkable predictions of the general theory of relativity is the existence of GWs. There are two types of sources mainly that generate GWs, one the astrophysical candidates such as dynamics of neutron star binaries, black hole mergers and the second is from the cosmological perturbations in the early universe [27, 28, 29]. The cosmologically generated GWs are also known as the relic GWs and were generated by the strong variable gravitational field of early universe [30]. The GWs are described by the gravitational wave equation and is the consequence of the linearized form of the Einstein field equations under suitable limiting conditions. Thus the main aim of this chapter is to present the Einstein linearized field equation, its solution and basic properties of the GWs briefly. A short discussion on the relic GWs in the expanding universe is also included. The creation of the relic GWs, during the inflationary era, through the parametric amplification mechanism also explained briefly.

## 2.1 Linearized Einstein Field Equations

Einstein's general theory of relativity of gravity leads to Newtonian gravity in the suitable limit conditions, when the gravitational field is weak, static and the particles in the gravitational field move slowly compared to the velocity of light. But consider a situation where the gravitational field is weak but not static, and there are no restrictions on the motion of particles in the gravitational field. Then the corresponding weak gravitational field can be considered as a small 'perturbation', $h_{\mu\nu}$, on the flat Minkowski metric $\eta_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}(h^2) + ..., \quad |h_{\mu\nu}| \ll 1. \tag{2.1}$$

Here, we consider only first-order terms in $h_{\mu\nu}$. In the absence of gravity, space-time is flat and is characterised by the Minkowski metric and following in the discussions, we consider it with the signature $(+, -, -, -)$. The coordinate systems involved in eq.(2.1) are called the Lorentz coordinate systems. Indices of any tensor, under the weak field approximation, can be raised or lowered using $\eta^{\mu\nu}$ or $\eta_{\mu\nu}$ respectively. Under a background Lorentz transformation, the perturbation transforms as a second-rank tensor:

$$h_{\alpha\beta} = \Lambda_\alpha{}^\mu \Lambda_\beta{}^\nu \; h_{\mu\nu}, \tag{2.2}$$

where $\Lambda_\alpha{}^\mu$ and $\Lambda_\beta{}^\nu$ are the Lorentz transformation matrices. The equations that govern $h_{\mu\nu}$ are obtained by taking the Einstein's field equations up to first order. The affine connection to the first order is given by [1]

$$\Gamma^{\lambda(1)}_{\mu\nu} = \frac{1}{2}\,\eta^{\lambda\rho}\,[\partial_\mu h_{\rho\nu} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}]. \tag{2.3}$$

Therefore, the Riemann curvature tensor reduces to first order as

$$R^{(1)}{}_{\mu\nu\rho\sigma} = \eta_{\mu\lambda}\,\partial_\rho \Gamma^{\lambda(1)}_{\nu\sigma} - \eta_{\mu\lambda}\,\partial_\sigma \Gamma^{\lambda(1)}_{\nu\rho}, \tag{2.4}$$

and the Ricci tensor is obtained to the first order as

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\,\left[\partial_\lambda \partial_\nu h^\lambda{}_\mu + \partial_\lambda \partial_\mu h^\lambda{}_\nu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu}\right], \tag{2.5}$$

where, $\Box \equiv \eta^{\lambda\rho}\,\partial_\lambda \partial_\rho$ is the D'Alembertian in flat space-time. Contracting eq.(2.5) with $\eta^{\mu\nu}$, gives the corresponding Ricci scalar as

$$R^{(1)} = \partial_\lambda \partial_\mu h^{\lambda\mu} - \Box h. \tag{2.6}$$

[1]Here onward superscript $^{(1)}$ means first order approximation.

Therefore the Einstein tensor $G_{\mu\nu}$ in the limit of weak gravitational field is

$$\begin{aligned} G^{(1)}{}_{\mu\nu} &= R^{(1)}{}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R^{(1)} \\ &= \frac{1}{2}\left[\partial_\lambda \partial_\nu h^\lambda_\mu + \partial_\lambda \partial_\mu h^\lambda{}_\nu - \partial_\mu \partial_\nu h - \eta_{\mu\nu}\left(\partial_\lambda \partial_\sigma h^{\lambda\sigma} - \Box h\right) - \Box h_{\mu\nu}\right]. \end{aligned} \tag{2.7}$$

Therefore the linearized Einstein's field equations are

$$G^{(1)}{}_{\mu\nu} = 8\pi G\, T_{\mu\nu}. \tag{2.8}$$

Note that while deriving the above linearized form of the Einstein field equation the source term assumed as unperturbed. The linearized field equations (2.8) have no unique solutions as any solution to these equations will not remain invariant under a 'gauge' transformation. As a result, equations (2.8) can have infinitely many solutions. In other words, the decomposition eq.(2.1) of $g_{\mu\nu}$ in the weak gravitational field approximation does not completely specify the coordinate system. When a system that is invariant under a gauge transformation, then the gauge can be fixed and work in that selected coordinate system. One such coordinate system is the harmonic coordinate system and the gauge condition is given by

$$g^{\mu\nu}\Gamma^\lambda{}_{\mu\nu} = 0. \tag{2.9}$$

In the weak field limit, this condition reduces to

$$\partial_\lambda h^\lambda{}_\mu = \frac{1}{2}\partial_\mu h. \tag{2.10}$$

This condition is called the Lorentz gauge. In this selected gauge, the linearized Einstein's equations simplify and reduces to

$$\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h = -\,16\pi G\, T_{\mu\nu}. \tag{2.11}$$

The 'trace-reversed' perturbation, $\bar{h}_{\mu\nu}$, is defined as,

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\, h. \tag{2.12}$$

Thus the harmonic gauge condition further reduces to

$$\partial_\mu \bar{h}^\mu{}_\lambda = 0. \tag{2.13}$$

Therefore the linearized Einstein's equations become

$$\Box \bar{h}_{\mu\nu} = -\,16\pi G\, T_{\mu\nu}. \tag{2.14}$$

**Plane wave solution**

The propagation of GWs in vacuum is regarded as a superposition of plane waves. The GW has two independent polarization states and their explicit form is displayed in a particular coordinate system, the transverse-traceless (TT) gauge. The linearized Einstein's equation in vacuum, can be written as

$$\Box \bar{h}_{\mu\nu} = 0. \tag{2.15}$$

Since the trace $\bar{h} = -h$ satisfies the same wave equation,

$$\Box h_{\mu\nu} = 0. \tag{2.16}$$

Consider a plane wave solution in the form of

$$h_{\mu\nu}(x) = \epsilon_{\mu\nu}\, e^{in_\alpha\, x^\alpha}, \tag{2.17}$$

where $\epsilon_{\mu\nu}$ is the symmetric polarization tensor, i.e,

$$\epsilon_{\mu\nu} = \epsilon_{\nu\mu}, \tag{2.18}$$

and $n^\alpha$ is the $4-$ wavevector $n^\alpha = (\omega, \mathbf{n})$. Substituting eq.(2.17) in eq.(2.16), obtain

$$n^2\, \epsilon_{\mu\nu}\, e^{inx} = 0, \tag{2.19}$$

thus the wavevector is a null-vector, and

$$n^2 = n_\alpha n^\alpha = -\,\omega^2 + \mathbf{n}^2 = 0. \tag{2.20}$$

GWs propagate at the same speed $\omega/|\mathbf{n}| = c = 1$ as electromagnetic waves. Furthermore, since the wave eq.(2.16) valid only in the coordinates satisfying the Lorentz gauge condition eq.(2.10) and the polarization tensor is transverse:

$$n^\mu \epsilon_{\mu\nu} = 0. \tag{2.21}$$

**The transverse-traceless gauge**

There is still some residual gauge freedom left: one can make further coordinate gauge transformations as long as the transverse condition eq.(2.21) is not violated. This requires that the associated gauge vector function $\chi_\mu$ be constrained by the condition:

$$\Box \chi_\mu = 0. \tag{2.22}$$

Such coordinate freedom can be used to simplify the polarization tensor, one can pick $\epsilon_{\mu\nu}$ to be traceless,

$$\epsilon^\mu_\mu = 0, \tag{2.23}$$

as well as

$$\epsilon_{\mu 0} = \epsilon_{0\mu} = 0. \tag{2.24}$$

This particular choice of the coordinates is called the transverse-traceless gauge, which is a subset of coordinates satisfying the Lorentz gauge condition.

The $4\times 4$ symmetric polarization matrix $\epsilon_{\mu\nu}$ has ten independent elements. Equations (2.21), (2.23), and (2.24) which superficially represent nine conditions actually fix only eight parameters because the condition $n^\mu \epsilon_{\mu 0} = 0$ is trivially satisfied by eq.(2.24). Thus $\epsilon_{\mu\nu}$ has only two independent elements and hence GW has only two independent polarization states. Consider a wave propagating in the $z$ direction $n^\alpha = (\omega, 0, 0, \omega)$, the transversality condition together with eq.(2.24) implies that $\epsilon_{3\nu} = \epsilon_{\nu 3} = 0$. Together with the conditions, eq.(2.23) and eq.(2.24), the metric perturbation has the form

$$h_{\mu\nu}(z,t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h^{+} & h^{\times} & 0 \\ 0 & h^{\times} & -h^{+} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i\omega(z-t)}. \tag{2.25}$$

The two polarization modes can be taken to be

$$\epsilon^{(+)}_{\mu\nu} = h^{+} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \epsilon^{(\times)}_{\mu\nu} = h^{\times} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.26}$$

with $h^{+}$ and $h^{\times}$ are the respectively known as "plus" and "cross" amplitudes.

## 2.2 Gravitational Waves in Expanding Universe

The perturbed metric for a homogeneous isotropic flat FLRW universe can be written as

$$ds^2 = S^2(\eta)(d\eta^2 - (\delta_{\mu\nu} + h_{\mu\nu})\, dx^{\mu}dx^{\nu}), \tag{2.27}$$

where $S(\eta)$ is the cosmological scale factor, $\eta$ is the conformal time defined by $d\eta = dt/S$ and $\delta_{\mu\nu}$ is the Kronecker delta symbol. The $h_{\mu\nu}$ are metric perturbations field containing only the pure GWs and is transverse-traceless i.e; $\nabla_{\mu}h^{\mu\nu} = 0, \delta^{\mu\nu}h_{\mu\nu} = 0$.

The present study mainly deals with the amplitude and spectral energy density of the relic GWs generated by the expanding space-time. Thus the perturbed matter source is therefore not taken into account in present work. Since the relic GWs are very weak one needs consider only the linearized field equation given by

$$\nabla_{\lambda}\left(\sqrt{-g}\ \nabla^{\lambda} h_{\mu\nu}(\mathbf{x},\eta)\right) = 0. \tag{2.28}$$

For a fixed wave number $n = |\mathbf{n}|$, here after $n$ is the comoving wave number unless it is mentioned, and a fixed polarization state $\mathbf{p}$ the linearized wave equation (2.28) gives [31]

$$h_n''(\eta) + 2\,\frac{S'}{S}\,h_n'(\eta) + n^2\,h_n(\eta) = 0, \tag{2.29}$$

where $' = d/d\eta$ means derivative with respect to the conformal time. The tensor perturbations have two independent physical degrees of freedom and are denoted as $h^+$ and $h^\times$, called polarization modes. Since each polarization state is same, here onwards we denote $h_n(\eta)$ without the polarization index. Next, we rescale the filed $h_n(\eta)$ by taking

$$h_n(\eta) = \frac{\mu_n(\eta)}{S(\eta)}, \tag{2.30}$$

where the mode functions $\mu_n(\eta)$ obey the minimally coupled Klein-Gordon equation

$$\mu_n'' + \Big(n^2 - U(\eta)\Big)\mu_n = 0. \tag{2.31}$$

where $U(\eta) = S''/S$.

The general solution of eq.(2.31) is a linear combination of Hankel's function, $H^{(1)}$ and $H^{(2)}$ with a generic power-law for the scale factor $S = \eta^q$, given by

$$\mu_n(\eta) = A_n\,\sqrt{n\eta}\;H^{(1)}_{(q-\frac{1}{2})}(n\eta) + B_n\,\sqrt{n\eta}\;H^{(2)}_{(q-\frac{1}{2})}(n\eta). \tag{2.32}$$

For a given model of the expansion of universe, consisting of a sequence of scale factors with different $q$, we can obtain a solution $\mu_n(\eta)$ by matching its value and derivative at successive stages.

## 2.3 Parametric Amplification of GWs

The main purpose of this section is to discuss the parametric (superadiabatic) amplification of relic GWs in flat FLRW universe [31, 32].

The equation (2.31) describes an oscillator with the varying frequency, known as parametrically excited oscillator. The external gravitational field is represented by the cosmological scale factor and it plays the role of a "pump" field supplying energy to the oscillator [33].

In the intervals of $\eta$-time such that $n^2 \gg |U(\eta)|$ the solutions of equation (2.31) have the form $\mu = e^{\pm in\eta}$, and are high-frequency waves with adiabatically changing amplitude $h = (1/S)\sin(n\eta + \varphi)$, where $\varphi$ is the phase of the wave. In an expanding universe, the amplitude decreases. The amplitudes of the waves with $n$ such that $n^2 \gg |U(\eta)|$ decrease adiabatically for all $\eta$.

If for a given $n$ there is an interval of time when $n^2 \ll |U(\eta)|$, the solutions to the second-order differential equation (2.31) are no longer oscillatory. In the case $U(\eta) = S''/S$ they are $\mu_1 = S$ and $\mu_2 = S\int S^{-2}d\eta$. The waves satisfying $n^2 \ll |U(\eta)|$ for some $\eta$ encounter the potential barrier[2] and are governed by the solutions $\mu_1$ and $\mu_2$ in the under-barrier region. The amplitude $\mu_f$ of the function $\mu(\eta)$ right after exit of the wave under the barrier depends on the initial phase $\varphi$ of the wave. The exiting amplitude $\mu_f$, can be larger or smaller than the entering amplitude $\mu_i$ defined right before the wave encounter the barrier. However, averaging $(\mu_f)^2$ over the initial phase $\varphi$ always leads to the dominant contribution from the solution $\mu_1$. This means that the adiabatic factor $1/S$ is cancelled out by $\mu_1 = S$ and the amplitude $h$ (with the factor $1/S$ taken into account) of a 'typical' wave can be regarded as remaining constant in the region occupied by the barrier. It stays constant instead of diminishing adiabatically, as the waves above the potential barrier do. Thus, the exiting amplitude $h_f$ of a 'typical' wave is equal to the entering amplitude $h_i$ and is larger than it would have been if the wave behaved adiabatically.

The amplification coefficient $R(n)$ for a given $n$ is the ratio $S(\eta_f)/S(\eta_i)$ where $S(\eta_i)$ is the value of the scale factor at the last oscillation of the wave

[2]The terminology 'barrier' is adapted for 'horizon' from [33].

before entering the under-barrier region, and $S(\eta_f)$ is the value of the scale factor at the first oscillation of the wave after leaving the under-barrier region. The waves with different wave numbers $n$, stay under the potential barrier for different intervals of time. This means that, in general, the amplification coefficient depends on $n$: $R(n) = 1$ for all $n$ above the top of the potential, and $R(n) \gg 1$ for smaller $n$. The initial spectrum of the waves $h(n) = A(n)/S$, defined at some $\eta$ well before the interaction began, transforms into the final spectrum $h(n) = B(n)/S$, defined at some $\eta$ well after the interaction is completed. The transformation occurs according to the rule: $B(n) = R(n)A(n)$. This is the essence of the mechanism of the superadiabatic (parametric) amplification of GWs and, in fact, of any other fluctuations obeying similar equations [33].

# Chapter 3

# Gravitational Waves Spectrum in Thermal Vacuum State

The spectrum of relic GWs depends not only on the details of expansion during the inflationary era but also on the subsequent stages, including the current epoch of the universe. There are variety of sources that responsible are for generation of these waves including from the dynamics of early universe to massive astrophysical objects such as neutron star binaries and black hole mergers etc. Thus the waves have a wide spectrum of frequencies i.e, the frequency vary from very low to high ($\mathcal{O}(10^{-19}$ -$10^{10})$ Hz). It is possible to discriminate the relic waves from other sources on the observational point of view also. It is believed that the relic GWs are mainly generated during inflationary epoch. And the waves that amplified during the inflation are low frequency only. Since the higher frequency waves are outside the barrier the corresponding amplitudes decreased during the evolution of the universe. But these high frequency waves can re-enter into the horizon of the universe again as the universe expanded further enough. Thus these waves can also contribute to the amplitude of the present GWs. The standard inflationary models are energetically not in favor of generating thermal GWs but the existence of GWs with thermal features is not ruled out completely. However, the inflationary stage probably generated the lower frequency thermal GWS due to the stimulated emission process [9]. The pre-inflationary period of the universe [10] and theories with higher dimensions [11] also predicts the existence of thermal GWs [12, 13]. There exist other alternative scenarios and candidate like the evaporation of mini black holes in the early universe, are also predict the existence of thermal GWs. In the present study, we mainly consider the thermal GWs that were probably generated in the pre-inflationary era, during the inflationary era due to stimulated emission mechanism and from the extra dimensional effects in the early universe. In this chapter, we consider the GWs in thermal vacuum states and study its contribution to the amplitude and energy density in the decelerated as well as accelerated FLRW universe. The contribution of the thermal GWs on its amplitude and spectral energy density due to lower and higher frequency is studied separately. A brief account of the plausible scenarios of generation of the thermal GWs and the properties of thermal state are discussed briefly. Computed contribution of the high frequency thermal GWs, due to the extra dimensional effect to its amplitude can be compared with the sensitivity of various missions to detect GWs.

## 3.1 GWs Spectrum in Expanding Universe

The approximate computation of the spectrum of the GWs is usually performed in two limiting cases depending up on the waves that are within or outside of the barrier. For the GWs outside barrier ($n^2 \gg S''/S$, short wave approximation) the corresponding amplitude decrease as $h_n \propto 1/S(\eta)$ while for the waves inside the barrier ($n^2 \ll S''/S$, long wave approximation), $h_n = C_n$ simply a constant. Thus these results can be used to estimate the spectrum for the present epoch of the universe.

The history of overall expansion of the universe is modeled as following sequence of successive epochs of power-law expansion [23].

The initial stage (inflationary)

$$S(\eta) = l_0 \, |\eta|^{1+\beta}, \qquad -\infty < \eta \leq \eta_1, \tag{3.1}$$

where $1 + \beta < 0$, $\eta < 0$ and $l_0$ is a constant.

The z-stage or reheating stage

$$S(\eta) = S_z(\eta - \eta_p)^{1+\beta_s}, \qquad \eta_1 < \eta \leq \eta_s, \tag{3.2}$$

where $1 + \beta_s > 0$. This z-stage is introduced to allow a general reheating epoch [34].

The radiation-dominated stage

$$S(\eta) = S_e(\eta - \eta_e), \qquad \eta_s \leq \eta \leq \eta_2, \tag{3.3}$$

The matter-dominated stage

$$S(\eta) = S_m(\eta - \eta_m)^2, \qquad \eta_2 \leq \eta \leq \eta_E, \tag{3.4}$$

where "$e$" and "$m$" subscripts are for the radiation and matter dominated stages and $\eta_E$ is the time when the dark energy density $\rho_\Lambda$ is equal to the matter energy density $\rho_m$. Before the discovery of accelerating expansion of the universe, the current expansion was taken to be as decelerating one because of the matter-domination. Thus, following the matter-dominated stage, it is reasonable to add an epoch of accelerating stage, which is probably driven by either the cosmological constant, or the quintessence, or some other kind of condensate [35]. The value of redshift $z_E$ at the time $\eta_E$ is $(1 + z_E) = S(\eta_0)/S(\eta_E)$, where $\eta_0$ is the present time. Since $\rho_\Lambda$ is constant and $\rho_m(\eta) \propto S^{-3}(\eta)$, we get

$$\frac{\rho_\Lambda}{\rho_m(\eta_E)} = \frac{\rho_\Lambda}{\rho_m(\eta_0)(1 + z_E)^3} = 1. \tag{3.5}$$

If the current value of $\Omega_\Lambda \sim 0.7$ and $\Omega_m \sim 0.3$, then it follows that

$$1 + z_E = \left(\frac{\Omega_\Lambda}{\Omega_m}\right)^{1/3} \sim 1.33. \tag{3.6}$$

The accelerating stage (up to the present)

$$S(\eta) = \ell_0 \, |\eta - \eta_a|^{-1}, \qquad \eta_E \leq \eta \leq \eta_0. \tag{3.7}$$

This stage describes the accelerating expansion of the universe, which is a new feature and hence its influence on the spectrum of relic GWs is of interest to study. It is be noted that the actual scale factor function $S(\eta)$ differs from equation (3.7), since the matter component exists in the current universe. However, the dark energy is dominant, therefore eq.(3.7) is an approximation to the current expansion behaviour.

Given $S(\eta)$ for the various epochs, the derivative $S' = dS/d\eta$ and the ratio $S'/S$ follow immediately. Except for $\beta_s$ which is imposed upon as the model parameter, there are ten constants in the expressions of $S(\eta)$. By the continuity conditions of $S(\eta)$ and $S'(\eta)$ at $\eta_1, \eta_s, \eta_2$, and $\eta_E$, one can fix only eight constants. The remaining two constants can be fixed by the overall normalization of $S$ and the observed Hubble constant as the expansion rate. Specifically, we put $|\eta_0 - \eta_a| = 1$ for the normalization of $S$, which fixes the $\eta_a$, and the constant $\ell_0$ is fixed by the following calculation,

$$\frac{1}{H} \equiv \left(\frac{S^2}{S'}\right)_{\eta_0} = \ell_0, \tag{3.8}$$

where $\ell_0$ is the Hubble radius at present.

In the expanding FLRW spacetime the physical wavelength is related to the comoving wave number as $\lambda \equiv 2\pi\, S(\eta)/n$, and the wave number $n_0$ corresponding to the present Hubble radius is $n_0 = 2\pi\, S(\eta_0)/\ell_0 = 2\pi$. And there is another wave number $n_E = 2\pi\, S(\eta_E)/(1/H) = n_0/(1 + z_E)$, whose corresponding wavelength at the time $\eta_E$ is the Hubble radius $1/H$.

By matching $S$ and $S'/S$ at the joint points of successive evolutionary stages of the universe, one gets [36]

$$l_0 = \ell_0\, b\, \zeta_E^{-(2+\beta)}\, \zeta_2^{\frac{\beta-1}{2}}\, \zeta_s^{\beta}\, \zeta_1^{\frac{\beta-\beta_s}{1-\beta_s}}, \tag{3.9}$$

where $b \equiv |1 + \beta|^{-(2+\beta)}$,

$$\zeta_E \equiv \frac{S(\eta_0)}{S(\eta_E)},\ \zeta_2 \equiv \frac{S(\eta_E)}{S(\eta_2)},\ \zeta_s \equiv \frac{S(\eta_2)}{S(\eta_s)},$$

and

$$\zeta_1 \equiv \frac{S(\eta_s)}{S(\eta_1)}. \tag{3.10}$$

With these specifications, the functions $S(\eta)$ and $S'(\eta)/S(\eta)$ are fully determined. In particular, $S'(\eta)/S(\eta)$ rises up during the accelerating stage, instead of decreasing as in the matter-dominated stage. This causes the modifications to the spectrum of relic GWs.

## 3.2 Thermal Background of Relic GWs

Recently, the thermal gravitational waves received much attention in gravitational waves astronomy and cosmology. There are theories such as extra dimensional scenario, universe without inflation, evaporation of primordial black holes and Dirac hypothesis, etc; predict the existence of thermal GWs. In this section, we provide a brief account of various scenarios of thermal GWs.

### 3.2.1 High energy scale and thermal GWs

One scenario is based on the assumption that the universe underwent a radiation dominated stage of expansion prior to the inflation-like acceleration phase [37]. Furthermore, it is assumed that, during this stage the temperature higher than $\sim 10^{19}$ GeV, a thermal equilibrium between the various components, including gravitons, is maintained through gravitational interaction. In this scenario, as the universe cooled down further and the gravitons decoupled, a background of thermal relic gravitons would be left behind [38, 39].

Assume that the universe was radiation dominated before the inflationary period, and that all the particle species were highly relativistic. The interaction rate $\Gamma$ for particles interacting solely through the gravitational force

$$\Gamma \simeq T^5/m_{pl}^4, \tag{3.11}$$

where $T$ is the physical temperature in the universe [38, 40]. If the gravitons be in thermal equilibrium with the other particles were possible only provided the interaction rate is large in comparison with the expansion rate of the universe characterized by Hubble parameter

$$H \simeq T^2/m_{pl}. \tag{3.12}$$

The gravitons would remain in thermal equilibrium with other particles, however, this equilibrium gets violated once $\Gamma \lesssim H$. Thus the gravitons would decouple from the other particle species, leaving behind a free-streaming thermal graviton background.

During the expansion of universe, these GWs background would preserve its thermal spectrum, but redshifted to very low temperatures [39, 41]. To discuss further, it is convenient to consider history of the universe into three stages (a) initial radiation dominated stage ($i$), (b) inflationary stage ($inf$) and (c) post-inflationary stage ($p$). Thus the present temperature $T_0$ of the GWs background is

$$\frac{T_0}{m_{pl}} \simeq \frac{S_i}{S_{inf}} \times \frac{S_{inf}}{S_p} \times \frac{S_p}{S_0}, \tag{3.13}$$

where $S_i, S_{inf}, S_p$ and $S_0$ are the values of the scale factors at the time of graviton decoupling, the beginning of the inflationary stage, the end of inflation and the present universe, respectively.

The temperature of the GWs decreases with the expansion. On the other hand, at the end of inflation the temperature of the thermal bath, containing the rest of the particles is significantly boosted by the process of reheating. Assuming that the observed CMB is the relic of this thermal bath, its temperature $T_2$ at the beginning of the post-inflationary stage can be related to value of the scale factor at this stage through the relation given by [38]

$$S_p/S_0 = (2.37K_{elv}/T_2)(3.91/106.75)^{1/3}. \tag{3.14}$$

Using this result, and denoting the temperature of the thermal bath at the beginning of the inflationary stage as $T_1$, eq.(3.13) can be rewritten as

$$T_0 \simeq 8.0 \times 10^{-27} (T_1/T_2) e^{60-N} \; K_{elv}, \tag{3.15}$$

where $N \equiv \log(S_p/S_{inf})$ is number of e-folds during the inflationary stage. This spectrum is peaked at the frequency[1]

$$\nu \simeq 4.7 \times 10^{-16} (T_1/T_2) e^{60-N} \text{ Hz}. \tag{3.16}$$

The peak frequency depends on the value of the number of e-folds during inflation, and the ratio of the temperatures $T_1$ and $T_2$.

### 3.2.2 Extra dimensional scenario and thermal GWs

Cosmology with extra dimensions have been motivated since Kaluza and Klein (KK) showed that classical electromagnetism and general relativity could be combined in a five-dimensional framework [11]. The modern scenarios involving extra dimensions are being explored in particle physics, gravity and cosmology. Although there exist different models of extra dimensions, there are some general features and signals common to all of them.

In presence of $D$ extra spatial dimensions, the 3+D+1- dimensional action for gravity can be written as

$$S = \int d^4x \left[ \int d^D y \sqrt{-g_D} \frac{R_D}{16\pi G_D} + \sqrt{-g} L_m \right], \tag{3.17}$$

where

$$G_D = G \frac{m_{pl}^2}{m_D^{2+D}}, \tag{3.18}$$

and $g$ is the determinant of four dimensional metric, $G$ is Newton's constant, $g_D, G_D$ and $R_D$ denote the higher dimensional counter parts of the determinant of metric, Newton's constant, and the Ricci scalar, respectively. And $m_D$ is the fundamental scale of the extra dimension.

[1]Here onwards $\nu$ means frequency except in Chapter 4.

Since the gravitational interactions are not strong enough to produce a thermal gravitons at temperatures below the Planck scale ($m_{pl} \sim 1.22 \times 10^{19}$ GeV), the standard inflationary cosmology predicted the existence the cosmic GWs background which are non-thermal in nature. However if the universe contains extra dimensions which is favorable to generate thermal GWs, this can happen when energies in the universe are higher than the fundamental scale $m_D$, the gravitational coupling strength increases significantly, as the gravitational field spreads out into the full spatial volume. Instead of freezing out at $\sim \mathcal{O}(m_{pl})$, as in 3+1 dimensions, gravitational interactions freeze-out at $\sim \mathcal{O}(m_D)$. If the gravitational interactions become strong at an energy scale below the reheating temperature ($m_D < T_{RH}$), gravitons get the opportunity to thermalize, creating a thermal GWs background. The qualitative result, the creation of a thermal GWs background if $m_D < T_{RH}$, is unchanged by the type of extra dimensions chosen [13].

Thus, if extra dimensions do exist, and the fundamental scale of those dimensions is below the reheat temperature, a relic thermal GWs background ought to exist today. Compared to the relic thermal photon background, a thermal GWs background would have the same statistics, and high degree of isotropy and homogeneity.

The energy density ($\rho_g$) and fractional energy density ($\Omega_g$ ) of a thermal GWs background are

$$\rho_g = \frac{\pi^2}{15}\left(\frac{3.91}{g_\star}\right)^{4/3} T^4_{CMB}, \tag{3.19}$$

$$\Omega_g = \frac{\rho_g}{\rho_c} \simeq 3.1 \times 10^{-4}(g_\star)^{4/3}, \tag{3.20}$$

where $\rho_c$ is the critical energy density of the universe, $T_{CMB}$ is the present temperature of the CMB, and $g_\star$ is the number of relativistic degrees of freedom at the scale of $m_D$. The $g_\star$ is dependent on the particle content of the universe, i.e. whether (and at what scale) the universe is supersymmetric, has a KK tower, etc. Other quantities, such as the temperature ($T$), peak frequency ($\nu$), number density ($n$), and entropy density ($s$) of the thermal GWs background can be derived from the CMB if $g_\star$ is known, as

$$n_g = n_{CMB}\left(\frac{3.91}{g_\star}\right), \quad s_g = s_{CMB}\left(\frac{3.91}{g_\star}\right), \tag{3.21}$$

$$T_g = T_{CMB}\left(\frac{3.91}{g_\star}\right)^{1/3}, \ \nu_g = \nu_{CMB}\left(\frac{3.91}{g_\star}\right)^{1/3}. \tag{3.22}$$

These quantities are not dependent on the number of extra dimensions. But $m_D$ is just barely above the scale of the standard model, then $g_\star$= 106.75. Then the thermal GWs background has a temperature of 0.905 Kelvin, with a peak frequency of 19 GHz [13].

### 3.2.3 Alternative possibilities of thermal GWs

Currently, there are alternative scenarios that would also create a thermal cosmic GWs background and we here discuss some of them briefly.

It is believed that mini black holes existed in the early universe known as primordial black holes. The primordial black holes with masses less than $10^{15}$ g would have decayed by today, producing thermal photons, gravitons, and other forms of radiation. In order to create a large mass fraction of low-mass primordial black holes but not high-mass ones, the spectral index $n'$ of the density fluctuations in the early universe is less than or equal to 2/3 [42]. But the observed scale-invariant spectrum suggest that $n' \simeq 1$ for the density fluctuations [43]. Therefore the primordial black holes as a reasonable candidate for creating a thermal cosmic GWs background may be ruled out.

The other alternative scenario is due to the Dirac hypothesis. According to this hypothesis the difference in magnitude between the gravitational and electromagnetic coupling strengths arises due ttheir time dependent nature [44]. Thus, the gravitational coupling would have been stronger in the early universe and hence created a thermal GWs at that epoch. But the cosmological models based on these hypothesis are difficult to reconcile [45] and also constrained with the geophysical and astronomical observations [46]. The generation of thermal GWs subsequent to the end of inflation is difficult because allowed variation of the gravitational coupling constant is very small.

The difficulties faced by the alternative scenarios probably points towards the extra dimensions as the leading candidate for the existence of thermal GWs.

## 3.3 Thermal Vacuum State

An effective approach to deal with thermal vacuum states is the thermo-field dynamics (TFD)[47]. In this approach the expectation value of mixed state at non-zero temperature is obtained by an equivalent computation with a pure state. This is made by introducing a fictitious field which is the identical image of the original real field. Thus a temperature dependent vacuum for the expanded field can be obtained from the absolute vacuum by a Bogoliubov type of transformation. Therefore, in TFD a fictitious space called the tilde space also introduced besides the Hilbert space, and the direct product space is made up of above two spaces. Every operator and state in the Hilbert space has corresponding operator and state in the tilde space [47].

Based on the physical and tilde system, thermofield dynamics creates an operator called the thermal operator $\mathcal{T}(\theta)$ and is invariant under the tilde conjugation i.e., $\widetilde{\mathcal{T}}(\theta) = \mathcal{T}(\theta)$ [47]. According to the tilde conjugation $\widetilde{CO} \equiv C^*\widetilde{O}$, where $C$ is any coefficients appears in the expressions of quantities for a physical system, $O$ any operator, the superscript $*$ means complex conjugation, and $\widetilde{O}$ represents the corresponding operator for the tilde system.

Therefore a thermal vacuum state $(Tr)$ is defined as [48]

$$|Tr\rangle = \mathcal{T}(\theta_n)|0\;\tilde{0}\rangle, \tag{3.23}$$

where $|0\,\tilde{0}\rangle$ is the two mode vacuum state at zero temperature, and

$$\mathcal{T}(\theta_n) \;=\; \exp[-\theta_{\mathbf{n}}(a_{\mathbf{n}}\tilde{a}_{\mathbf{n}} - a^{\dagger}_{\mathbf{n}}\tilde{a}^{\dagger}_{\mathbf{n}})], \tag{3.24}$$

is the thermal operator. Where $\theta_n$ is related to the average number of the thermal particle,

$$\bar{n}_n = \sinh^2\theta_n. \tag{3.25}$$

For a given temperature $T$, $\bar{n}_n$ is provided by the Bose-Einstein distribution,

$$\bar{n}_n = [\exp(\hbar\omega_n/k_B T) - 1]^{-1}, \tag{3.26}$$

where $\omega_n$ is the resonance frequency of the field. The $a_{\mathbf{n}}$, $a^{\dagger}_{\mathbf{n}}$ and $\tilde{a}_{\mathbf{n}}$, $\tilde{a}^{\dagger}_{\mathbf{n}}$, are the annihilation and creation operators respectively in Hilbert and tilde space. These operators obey the usual commutation relations,

$$[a_{\mathbf{n}}, a^{\dagger}_{\mathbf{n}'}] = [\tilde{a}_{\mathbf{n}}, \tilde{a}^{\dagger}_{\mathbf{n}'}] = \delta^3(\mathbf{n} - \mathbf{n}'), \tag{3.27}$$

and all the other relations are zero. By the appropriate action of the thermal operators on $a_{\mathbf{n}}, a^{\dagger}_{\mathbf{n}}, \tilde{a}_{\mathbf{n}}, \tilde{a}^{\dagger}_{\mathbf{n}}$, we get [48]

$$\mathcal{T}^{\dagger} a_{\mathbf{n}} \mathcal{T} \;=\; a_{\mathbf{n}} \cosh\theta_n + \tilde{a}^{\dagger}_{\mathbf{n}} \sinh\theta_n, \tag{3.28}$$

$$\mathcal{T}^{\dagger} a^{\dagger}_{\mathbf{n}} \mathcal{T} \;=\; a^{\dagger}_{\mathbf{n}} \cosh\theta_n + \tilde{a}_{\mathbf{n}} \sinh\theta_n. \tag{3.29}$$

Thus the occupation number in the thermal vacuum state can be written as

$$\langle a^{\dagger}_{\mathbf{n}} a_{\mathbf{n}'} \rangle = \left(\frac{1}{e^{n/T} - 1}\right) \delta^3(\mathbf{n} - \mathbf{n}'). \tag{3.30}$$

This important result in the thermal vacuum state is very useful for the further study.

## 3.4 Gravitational Waves Spectrum

In quantum theory of GWs, the field $h_{\mu\nu}$ is a field operator, which is written as a sum of the plane wave Fourier modes. The tensor perturbations have two independent physical degrees of freedom and are denoted as $h^+$ and $h^\times$. Thus to compute the spectrum of GWs, we express $h^+$ and $h^\times$ in terms of the creation ($a^\dagger$) and annihilation ($a$) operators,

$$\begin{aligned} h_{\mu\nu}(\mathbf{x},\eta) &= \frac{\sqrt{16\pi} l_{pl}}{(2\pi)^{3/2}} \sum_{\mathbf{p}} \int d^3n\, \epsilon^{\mathbf{p}}_{\mu\nu}(\mathbf{n}) \\ &\quad \times \frac{1}{\sqrt{2n}} \Big[ a^{\mathbf{p}}_{\mathbf{n}}\, h^{\mathbf{p}}_{\mathbf{n}}(\eta)\, e^{i\mathbf{n}.\mathbf{x}} + a^{\dagger\,\mathbf{p}}_{\mathbf{n}}\, h^{*\,\mathbf{p}}_{\mathbf{n}}(\eta)\, e^{-i\mathbf{n}.\mathbf{x}} \Big], \end{aligned} \tag{3.31}$$

where $l_{pl}$ is the Planck length.

The polarization tensor $\epsilon^{\mathbf{p}}_{\mu\nu}(\mathbf{n})$ with $\mathbf{p} = +, \times$, is symmetric and transverse-traceless $n^\mu \epsilon^{\mathbf{p}}_{\mu\nu}(\mathbf{n}) = 0$, $\delta^{\mu\nu}\epsilon^{\mathbf{p}}_{\mu\nu}(\mathbf{n}) = 0$ and satisfy the conditions $\epsilon^{\mu\nu\mathbf{p}}(\mathbf{n})\epsilon^{\mathbf{p}'}_{\mu\nu}(\mathbf{n}) = 2\delta_{\mathbf{p}\mathbf{p}'}$ and $\epsilon^{\mathbf{p}}_{\mu\nu}(-\mathbf{n}) = \epsilon^{\mathbf{p}}_{\mu\nu}(\mathbf{n})$, the creation and annihilation operators satisfy $[a^{\mathbf{p}}_{\mathbf{n}}, a^{\dagger}_{\mathbf{n}'}{}^{\mathbf{p}'}] = \delta_{\mathbf{p}\mathbf{p}'}\delta^3(\mathbf{n}-\mathbf{n}')$ and the initial vacuum state is defined as

$$a^{\mathbf{p}}_{\mathbf{n}}|0\rangle = 0, \tag{3.32}$$

for each $\mathbf{n}$ and $\mathbf{p}$. The energy density of the GWs in vacuum state is $t_{00} = \frac{1}{32\pi l_{pl}^2}\frac{\partial h_{\mu\nu}}{\partial x_0}\frac{\partial h^{\mu\nu}}{\partial x_0}$.

The power spectrum of GWs is defined as

$$\int_0^\infty h^2(n,\eta)\frac{dn}{n} = \langle 0|h^{\mu\nu}(\mathbf{x},\eta)\, h_{\mu\nu}(\mathbf{x},\eta)|0\rangle, \tag{3.33}$$

where right hand side is the vacuum expectation value of the operator $h^{\mu\nu}h_{\mu\nu}$. Substituting equation (3.31) in eq.(3.33) and taking the contribution from each polarization is same, we get

$$h(n,\eta) = \frac{4 l_{pl}}{\sqrt{\pi}}\, n \mid h(\eta) \mid . \tag{3.34}$$

Thus once the mode function $h(\eta)$ is known, the spectrum $h(n,\eta)$ follows.

The spectrum at the present time $h(n,\eta_0)$ can be obtained, provided the initial spectrum is specified. The initial condition is taken to be the inflationary stage. Thus the initial amplitude of the spectrum is given by

$$h(n,\eta_i) = A\left(\frac{n}{n_0}\right)^{2+\beta}, \tag{3.35}$$

where $A = 8\sqrt{\pi}\frac{l_{pl}}{l_0}$ is a constant. The power spectrum for the primordial perturbation of energy density is $P(n) \propto \mid h(n, \eta_0) \mid^2$ and in terms of initial spectral index $n'$, it is defined as $P(n) \propto n^{n'-1}$. Thus the scale invariant spectral index $n' = 1$ for the pure de-Sitter expansion can be obtained with the relation $n' = 2\beta + 5$ for $\beta = -2$. Here onwards $\beta$ is treated as a number.

## 3.5 Low Frequency Contribution to the Spectrum

In this section, we consider the contribution of low frequency GWs to its spectrum in thermal vacuum state. Using eqs.(3.23), (3.28) and (3.29) in eq.(3.33) the power spectrum in thermal vacuum state is obtained as

$$h_T^2(n, \eta) = \frac{16 l_{pl}^2}{\pi} n^2 \mid h(\eta) \mid^2 \coth\Big[\frac{n}{2T}\Big], \tag{3.36}$$

Thus in comparison with eq.(3.35), the spectrum in thermal vacuum state is

$$h_T(n, \eta_i) = A\Big(\frac{n}{n_0}\Big)^{2+\beta} \coth^{1/2}\Big[\frac{n}{2T}\Big]. \tag{3.37}$$

The last term becomes significant when the ratio $n/(2T)$ is less than unity. The wave number $n$ and temperature $T$ are comoving quantities which are related to the physical parameters at the time of inflation [10, 13]. Thus it is expected an enhancement of the spectrum by a factor $\coth^{1/2}[n/2T] = \coth^{1/2}[HS_i/2T_i]$, here $i$ means inflationary [10].

It is convenient to consider the amplitude of waves in different range of wave numbers. Thus the amplitude of the spectrum in thermal vacuum state for different ranges are given by:

(i) when $n \leq n_E$, the corresponding wavelength is greater than the present Hubble radius. Thus the amplitude remain as the initial one and can be written as

$$h_T(n, \eta_0) = A\Big(\frac{n}{n_0}\Big)^{2+\beta} \coth^{1/2}\Big[\frac{n}{2T}\Big], \tag{3.38}$$

(ii) the amplitude remains approximately same as long as the wave in inside the barrier but begin to decrease when it leaves the barrier by a factor $1/S(\eta)$, depending on the value of scale factor at that time. This process continues until the barrier becomes higher than $n$ at a time $\eta$ earlier than $\eta_0$, so the amplitude has decreased by the ratio of the scale factor at the time of leaving the barrier $S_b$ to its value at $\eta$, $S(\eta)$. This is in the range $n_E \leq n \leq n_0$.

$$h_T(n, \eta_0) = A\Big(\frac{n}{n_0}\Big)^{\beta-1} \coth^{1/2}\Big[\frac{n}{2T}\Big] \frac{1}{(1+z_E)^3}. \tag{3.39}$$

Note that this range is a new feature on account of the current acceleration of the universe which is absent in the decelerating model as pointed out in [23]. The amplitude of the waves that left the barrier at $S_b$ with waves numbers $n > n_0$ has decreased up to the present time by a factor $S_b/S(\eta_0)$. This affect the amplitude of the present spectrum and is obtained as

$$h_T(n, \eta_0) = A\Big(\frac{n}{n_0}\Big)^{2+\beta} \coth^{1/2}\Big[\frac{n}{2T}\Big] \frac{S_b}{S(\eta_0)}. \tag{3.40}$$

This result can be used to obtain the spectrum of the GWs in the remaining range of wave numbers.

(iii) the wave number that does not hit the barrier in the range $n_0 \leq n \leq n_2$ gives the amplitude as follows

$$h_T(n, \eta_0) = A\Big(\frac{n}{n_0}\Big)^{\beta} \coth^{1/2}\Big[\frac{n}{2T}\Big] \frac{1}{(1+z_E)^3}, \tag{3.41}$$

the spectrum in this interval is different from that of the matter dominated case by a factor $1/(1+z_E)^3$. The wave lengths of the spectrum in the range are long but smaller than the present Hubble radius.

(iv) in the range of wave number $n_2 \leq n \leq n_s$ the amplitude is

$$h(n, \eta_0) = A\Big(\frac{n}{n_0}\Big)^{1+\beta} \Big(\frac{n_0}{n_2}\Big) \frac{1}{(1+z_E)^3}. \tag{3.42}$$

This is the interesting range on the observational point of view of Adv.LIGO, ET and LISA missions [2]-[4].

(v) for the wave number range $n_s \le n \le n_1$ which is in the high frequency case and gives the corresponding amplitude as

$$h(n, \eta_0) = A\Big(\frac{n}{n_0}\Big)^{1+\beta-\beta_s}\Big(\frac{n_s}{n_0}\Big)^{\beta_s}\Big(\frac{n_0}{n_2}\Big)\frac{1}{(1+z_E)^3}. \tag{3.43}$$

Note that the temperature dependent factor in the high frequency range is negligible, hence the term is dropped in the expression (iv) and (v) and the effect of temperature dependent factor is discussed in section.(3.7).

### 3.5.1 Normalization of the spectrum

The overall multiplication factor $A$ in all the spectra is determined in absence of the temperature dependent term with the CMB data of WMAP [23]. This is based on the assumption that the contribution from GWs and the density perturbations are the same order of magnitude at low multipole moments, $l$. Therefore it is possible to write $\Delta T/T \simeq h(n, \eta_0)$. The observed CMB anisotropies [49] at lower multipoles is $\Delta T/T \simeq 0.37 \times 10^{-5}$ at $l \sim 2$ which corresponds to the largest scale anisotropies that have been observed so far. Thus taking this to be the perturbations at the Hubble radius gives

$$h(n_0, \eta_0) = A\frac{1}{(1+z_E)^3} = 0.37 \times 10^{-5} \times r^{1/2}, \tag{3.44}$$

where $r$ is the tensor-to-scalar ratio and it is taken as unity for normalizing the spectrum in the present work. However, there is a subtlety here in the interpretation of $\Delta T/T$ at low multipoles, whose corresponding scale is very large $\sim \ell_0$. At present the Hubble radius is $\ell_0$, and the Hubble diameter is $2\ell_0$. On the other hand, the smallest characteristic wave number is $n_E$, whose corresponding physical wave length at present is $2\pi S(\eta_0)/n_E = \ell_0(1+z_E) \simeq 1.32\ell_0$, which is within the Hubble diameter $2\ell_0$, and is theoretically observable. So, instead of eq.(3.44), if $\Delta T/T \simeq 0.37 \times 10^{-5}$ at $l \sim 2$ were taken as the amplitude of the spectrum at frequency $\nu_E$, one would have $h_T(n_E, \eta_0) = A/(1+z_E)^{2+\beta} = 0.37 \times 10^{-5}$, yielding a smaller $A$ than that in eq.(3.44) by a factor $(1+zE)^{1-\beta} \sim 2.3$ [23].

### 3.5.2 Allowed range of $\beta$ and the spectrum

Next objective is to check the allowed range of $\beta$. During the inflationary expansion, when the $n$-mode wave enters the barrier with $\lambda_i = 1/H(\eta_i)$, it follows that $\lambda_i = \frac{l_0}{b}\left(\frac{n_0}{n}\right)^{2+\beta}$. For the classical treatment of the background gravitational field to be valid, this wavelength should be greater than the Planck length, $\lambda_i > l_{pl}$, so

$$\left(\frac{\nu}{\nu_0}\right)^{2+\beta} < \frac{8\sqrt{\pi}}{A}. \tag{3.45}$$

At the highest frequency $\nu = \nu_1$, this gives the following constraint

$$\beta < -2 + \ln\left(\frac{8\sqrt{\pi}}{A}\right) \Big/ \ln\left(\frac{\nu_1}{\nu_0}\right), \tag{3.46}$$

which depends on $A$. Thus, for given $A$ in (3.44), one obtains the upper limit $\beta < -1.78$. Plugging $b/l_0$ given by eq.(3.9) into $A$, using $\nu_2/\nu_0 = 58.8$ and $\ell_{0/l_{pl}} = 1.238 \times 10^{61}$, get [23]

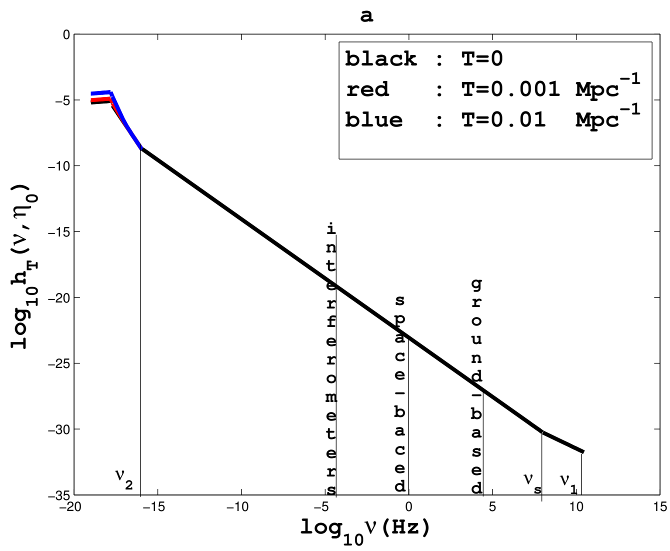
$$1.484 \times 10^{58} \times \frac{A}{(1+z_E)^3} = \left(\frac{\nu_1}{\nu_0}\right)^{-\beta}\left(\frac{\nu_1}{\nu_s}\right)^{\beta_s}. \tag{3.47}$$

For, given $A$ in (3.44), and $\beta = -1.9$, then $\beta_s = -0.552$.

Next, we obtain the spectrum in the thermal vacuum state with the following parameters. By taking $n = 2\pi\nu$, $\nu_E = 1.5 \times 10^{-18}$ Hz, $\nu_0 = 2 \times 10^{-18}$ Hz, $\nu_2 = 117 \times 10^{-18}$ Hz, $\nu_s = 10^8$ Hz, $\nu_1 = 3 \times 10^{10}$ Hz, the value of $\nu_1$ is taken such a way that spectral energy density does not exceed the level of $10^{-6}$, as required by the nucleosyntheis bound. The range of frequency is chosen in accordance with generation of GWs that vary from early universe to various astrophysical sources. The range of frequency is matching with the interest of CMB, Adv.LIGO, ET and LISA operations for detection of the GWs. The spectrum is computed in the thermal vacuum state with the chosen values of the parameters for the accelerated as well as decelerated FLRW universe with comoving temperatures $T = 0.001\,\mathrm{Mpc}^{-1}$ and $T = 0.01\,\mathrm{Mpc}^{-1}$. The selected comoving temperatures are considered in the context of tensor mode study of CMB [10]. Since we use the natural unit, the wave number and temperature that appear in the temperature dependent term of the spectrum is computed numerically in the $\mathrm{Mpc}^{-1}$ unit. And henceforth the unit of comoving temperature is taken as $\mathrm{Mpc}^{-1}$.

The obtained GWs spectra in thermal state are normalized with WMAP 7-year data. The amplitude of GWs spectrum in the thermal state is found enhanced in comparison to its zero temperature case (vacuum case). It is observed that the spectrum, for $T = 0.001\,\mathrm{Mpc}^{-1}$ get maximum enhancement $\sim 1.51$ times than the vacuum case, at $l$=2 and $\nu = \nu_E$, and it is $\sim 4.6$ times for $T = 0.01$ $\mathrm{Mpc}^{-1}$. The plots for the amplitude of spectrum $h_T(n, \eta_0)$ versus the frequency $\nu$ for $\beta = -1.9$ and $\beta_s = -0.552$ are given in Fig.[3.1]. The amplitude of the spectrum get enhanced in the frequency range, $10^{-19}$ Hz $\leqslant \nu \leqslant 1.49 \times 10^{-17}$ Hz, due to the thermal effect of GWs but not for the frequency range $1.49 \times 10^{-17}\,\mathrm{Hz} \leqslant \nu \leqslant 3 \times 10^{10}$ Hz because there is a suppression in the high frequency range due to the $\coth^{1/2}[n/2T]$ term. For comparison, the amplitude of the spectra are plotted for the decelerated and accelerated FLRW universe, panel (b), Fig.[3.1] for the frequency range $\nu_* \leqslant \nu \leqslant \nu_2$ where $\nu_*= 10^{-19}$ Hz. The results of study show that the amount of enhancement of the amplitude of the spectrum is independent of the model of expansion of the universe viz accelerated or decelerated and hence it is due to the thermal nature of the GWs. The obtained amplitude of the spectrum with enhancement in thermal vacuum state for both models of expansion of the universe is less than the upper bound of the WMAP 7-year data (panel (b), Fig.[3.1] the pink line is indicating the upper bound).

Further it is observed that position of the peak remains at $\nu_E$ for the accelerated universe (as pointed out in [23] for the vacuum case) but get enhanced due to the thermal GWs. This enhancement is the new feature of the spectrum of the relic GWs in the lower frequency range $\nu_* \leq \nu \leq \nu_2$ (panel(b), Fig.[3.1]).



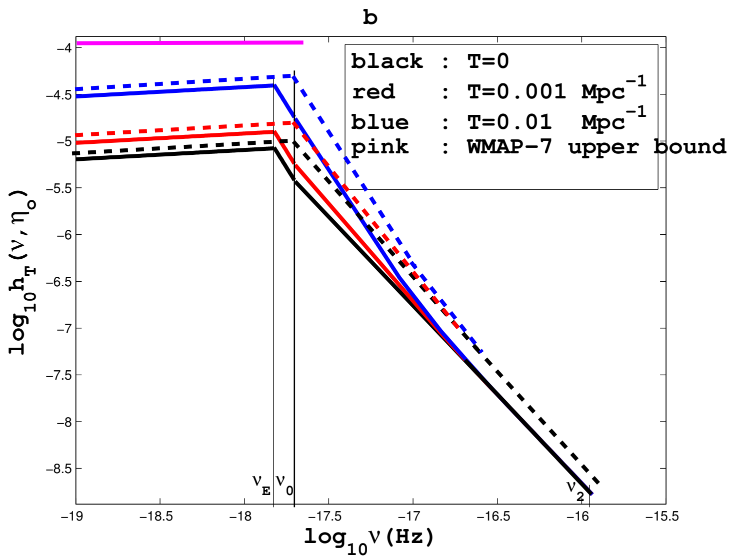




Figure 3.1: The amplitude of the GWs for the accelerated (solid lines) and decelerated (dashed lines) universe.

## 3.6 Spectral Energy Density in Thermal Vacuum State

In this section, we compute and study the spectral energy density of the GWs in thermal state for the flat FLRW universe. The spectral energy density parameter $\Omega_g(\nu)$ of the GWs is defined through the relation

$$\frac{\rho_{\mathbf{g}}}{\rho_{\mathbf{c}}} = \int \Omega_g(\nu) \frac{d\nu}{\nu}, \tag{3.48}$$

where $\rho_{\mathbf{g}}$ is the energy density of the GWs and $\rho_{\mathbf{c}}$ is the critical energy density of the universe. Thus

$$\Omega_g(\nu) = \frac{\pi^2}{3} h_T^2(k, \eta_0) \left(\frac{\nu}{\nu_0}\right)^2. \tag{3.49}$$

There is one more consistency condition to be satisfied. Since the spacetime is assumed to be spatially flat $K = 0$ with $\Omega = 1$, the fraction density of relic GWs should be less than unity, i.e; $\rho_{\mathbf{g}}/\rho_{\mathbf{c}} < 1$. After normalizing the obtained spectrum with WMAP 7 year data, we integrate $\int \Omega_g(\nu)\, d\nu/\nu$ from the $\nu_* = 10^{-19}$ Hz up to the frequency $\nu_1 = 3 \times 10^{10}$ Hz, with $\beta = -1.9$ and $\beta_s = -0.552$, to get the total spectral energy density of GWs.

The spectral energy density is evaluated, by splitting the full range of frequency into five, for the thermal and zero temperature cases and the obtained results, in the accelerated flat FLRW universe, are:

(a) $\nu_* \le \nu \le \nu_E$,

$$\begin{aligned}
\frac{\rho_g}{\rho_c} &= 5.8 \times 10^{-11}, \quad T = 0, \\
\frac{\rho_g}{\rho_c} &= 8.8 \times 10^{-11}, \quad T = 0.001 \text{ Mpc}^{-1}, \\
\frac{\rho_g}{\rho_c} &= 2.6 \times 10^{-10}, \quad T = 0.01 \text{ Mpc}^{-1},
\end{aligned}$$

(b) $\nu_E \leq \nu \leq \nu_H$,

$$\begin{aligned}
\frac{\rho_g}{\rho_c} &= 2.3 \times 10^{-11}, \quad T = 0, \\
\frac{\rho_g}{\rho_c} &= 3.5 \times 10^{-11}, \quad T = 0.001 \text{ Mpc}^{-1}, \\
\frac{\rho_g}{\rho_c} &= 1.1 \times 10^{-10}, \quad T = 0.01 \text{ Mpc}^{-1},
\end{aligned}$$

(c) $\nu_H \leq \nu \leq \nu_2$,

$$\begin{aligned}
\frac{\rho_g}{\rho_c} &= 2.4 \times 10^{-11}, \quad T = 0, \\
\frac{\rho_g}{\rho_c} &= 3.7 \times 10^{-11}, \quad T = 0.001 \text{ Mpc}^{-1}, \\
\frac{\rho_g}{\rho_c} &= 1.2 \times 10^{-10}, \quad T = 0.01 \text{ Mpc}^{-1},
\end{aligned}$$

(d) $\nu_2 \leq \nu \leq \nu_s$,

$$\frac{\rho_g}{\rho_c} = 8.97 \times 10^{-9}, \quad T = 0,$$

(e) $\nu_s \leq \nu \leq \nu_1$,

$$\frac{\rho_g}{\rho_c} = 2.7 \times 10^{-6}, \quad T = 0.$$

It is to be noted that in the frequency range of (d) and (e) the thermal cases are not shown because the thermal contribution in the high frequency range is negligible, due to the temperature dependent term. The combined results are plotted in Fig.[3.2]. Further, the contribution to $\rho_\mathbf{g}/\rho_\mathbf{c}$ from the low frequency range is $\mathcal{O}(10^{-11} - 10^{-10})$ while from the higher frequency range it is $\mathcal{O}(10^{-6})$. Since the order of contribution to the total spectral energy density $\rho_\mathbf{g}/\rho_\mathbf{c}$ from the lower frequency side is very small in contrast with the higher frequency side, we get

$$\frac{\rho_\mathbf{g}}{\rho_\mathbf{c}} = 2.7 \times 10^{-6}, \quad \nu_* \leq \nu \leq \nu_1, \tag{3.50}$$

and is the same as that of the non-zero temperature case of high frequency. However $\rho_g/\rho_c$ of the GWs with $T \neq 0$ is higher than the zero temperature

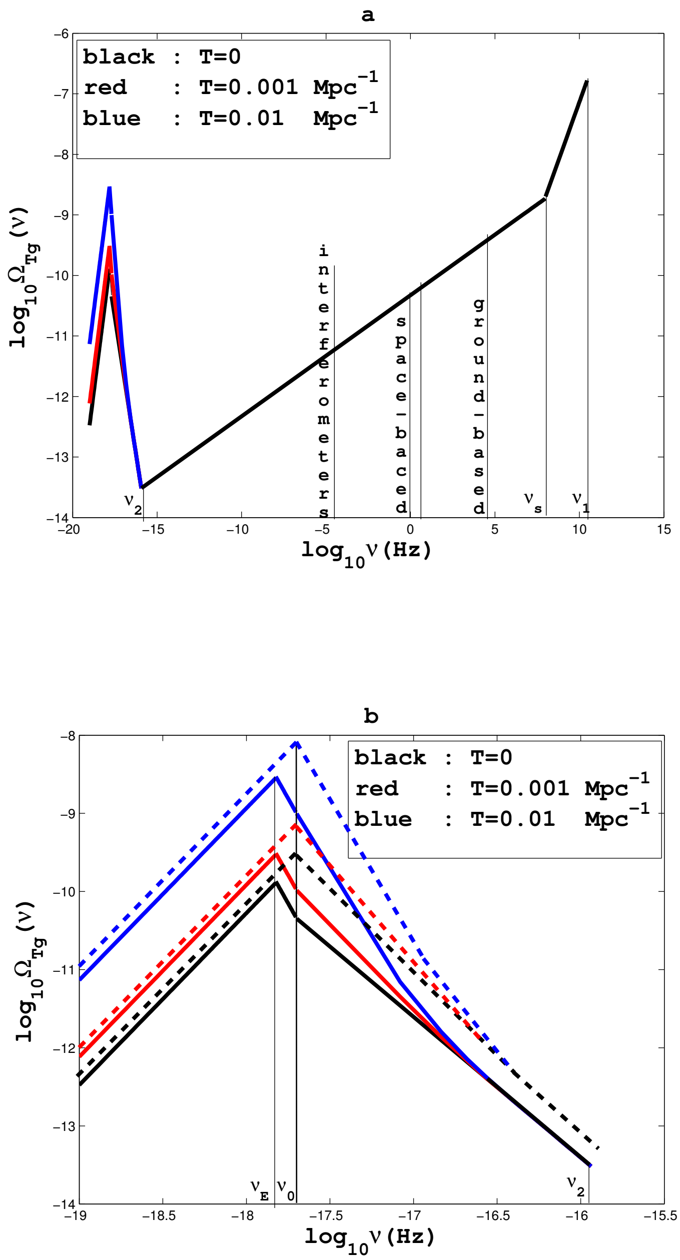



Figure 3.2: The spectral energy density of the GWs for the accelerated (solid lines) and decelerated (dashed lines) universe.

case at lower frequency range $\nu_* \le \nu \le \nu_2$. Therefore it is expected an enhancement for the spectral energy density in the thermal vacuum state in the frequency range $\nu_* \le \nu \le \nu_2$ only (panel (b), Fig.[3.2]). And it is the range of interest on the observational point of view of the relic GWs. Hence to observe the thermal effect on the relic GWs, in this range, the spectrum of GWS must be analysed separately otherwise the signature of the thermal effect gets suppressed due to the contributions from the higher frequency range.

## 3.7 High Frequency Contribution to the Spectrum

In this section, we consider contribution of very high frequency thermal GWs ($\nu_s \le \nu \le \nu_1$) to its spectrum and spectral energy density for the decelerated as well as accelerated flat FLRW universe. The origin of these very high frequency thermal GWs is assumed to be due to the extra dimensional effects in the early universe[13]. As studied in section 3.5, the computation of GWs spectrum again consider in the same five frequency ranges. Since this section mainly deals with the contribution of very high frequency range, the amplitude of GWs spectrum in the other frequency ranges, i.e; (i) up to (v) are not computing again which are obtained in section 3.5. The spectrum GWs in the higher frequency range is given in (v) section 3.5 and hence using eq.(3.43) the amplitude of the high thermal GWs can be expressed as

$$h_T(n,\eta_0) = A\Big(\frac{n}{n_0}\Big)^{1+\beta-\beta_s}\Big(\frac{n_s}{n_0}\Big)^{\beta_s}\Big(\frac{n_0}{n_2}\Big)\coth^{1/2}\Big[\frac{n}{2T}\Big]\frac{1}{(1+z_E)^3}. \tag{3.51}$$

As discussed in section 3.5 the temperature dependent term can be neglected, however the inclusion of high frequency thermal GWs the term becomes significant once again in the range $\nu_s \le \nu \le \nu_1$.

It is already observed that the contribution of thermal GWs in the ranges $n_2 \le n \le n_s$ and $n_s \le n \le n_1$ is insignificant. But taking into account of the extra dimensional effect, the spectrum of GWs is peaked with a temperature $T_*$=1.19 × $10^{25}$ Mpc$^{-1}$ [13]. Therefore it is expected an enhancement for the amplitude of spectrum (orange lines, Figs.[3.3] and [3.6]) in the range $n_s \leq n \leq n_1$ compared to $T = 0$ case, for the accelerated as well as decelerated universe. But at the same time, ignoring the thermal contribution to the amplitude of spectrum in the range $n_2 \leq n \leq n_s$ leads to a discontinuity at $n_s$, see Fig.[3.3]. This problem is evaded by fitting a new line in the range $n_2 \leq n \leq n_s$ for the amplitude $h$ of eq.(3.42) as follows:

Let the amplitude of GWs in the range $n_0 \leq n \leq n_2$ is given by eq.(3.41) as

$$h_{1T}(n,\eta_0) = A\Big(\frac{n}{n_0}\Big)^{\beta}\coth^{1/2}\Big[\frac{n}{2T}\Big]\frac{1}{(1+z_E)^3}, \tag{3.52}$$

and the amplitude in the $n_s \leq n \leq n_1$ by eq.(3.51) as

$$h_{2T}(n,\eta_0) = A\Big(\frac{n}{n_0}\Big)^{1+\beta-\beta_s}\Big(\frac{n_s}{n_0}\Big)^{\beta_s}\Big(\frac{n_0}{n_2}\Big)\coth^{1/2}\Big[\frac{n}{2T_*}\Big]\frac{1}{(1+z_E)^3}. \tag{3.53}$$

Thus the new slope for eq.(3.42), in the range $n_2 \leq n \leq n_s$, is obtained by taking $y \equiv \log_{10}(h)$ and $x \equiv \log_{10}(n)$, then

$$\log_{10}(h) - \log_{10}(h)_i = \frac{\log_{10}(h)_f - \log_{10}(h)_i}{\log_{10}(n_f) - \log_{10}(n_i)}(\log_{10}(n) - \log_{10}(n_i)), \tag{3.54}$$

where the subscripts $i$ and $f$ are respectively indicating the first and last points of the straight line. By putting $n_i \equiv n_2$ from eq.(3.52) and $n_f \equiv n_s$ from eq.(3.53) in eq.(3.54), we get [2]

$$h = (h_{1T})_{n_2} g(n), \tag{3.55}$$

where

$$g(n) = \Big(\frac{n}{n_2}\Big)^{\gamma}, \tag{3.56}$$

and

$$\gamma = \frac{\log_{10}(h_{2T})_{n_s} - \log_{10}(h_{1T})_{n_2}}{\log_{10}(n_s) - \log_{10}(n_2)} = \frac{\log_{10}\left(\left(\frac{n_s}{n_2}\right)^{1+\beta}\coth^{1/2}\left[\frac{n_s}{2T_*}\right]\right)}{\log_{10}\left(\frac{n_s}{n_2}\right)}, \tag{3.57}$$

[2]here, $\coth^{1/2}\left[\frac{n_2}{2T}\right] = 1$.

is the slope of the line and thus we find the amplitude,

$$h(n,\eta_0) = A\Big(\frac{n_2}{n_0}\Big)^{\beta}\frac{1}{(1+z_E)^3}\Big(\frac{n}{n_2}\Big)^{\gamma}, \tag{3.58}$$

and for the convenience we designate the obtained amplitude with the thermal GWs as 'modified amplitude'. It can be seen that, when $T_*$ becomes zero eq.(3.57) leads to $\gamma = 1+\beta$, and hence eq.(3.42) is recovered from eq.(3.58) in the range $n_2 \leq n \leq n_s$.

The overall multiplication factor $A$ in all the spectra is determined as described in the section 3.5. The allowed values of $\beta$, are $\beta_s$ also obtained and are respectively given by $\beta = -1.9$, and $\beta_s = -0.552$ [23].

Next, we obtain the spectrum in the thermal vacuum state with the following parameters. By taking $n = 2\pi\nu$, $\nu_E = 1.5\times10^{-18}$ Hz, $\nu_0 = 2\times10^{-18}$ Hz, $\nu_2 = 117\times10^{-18}$ Hz, $\nu_s = 10^8$ Hz, $\nu_1 = 3\times10^{10}$ Hz. The value of $\nu_1$ is again taken such a way that the spectral energy density does not exceed the level of $10^{-6}$ , as required by the rate of primordial nucleosynthesis calculation. The range of frequency is chosen in accordance with generation of GWs that vary from early universe to various astrophysical sources. The range is matching with the interest of CMB, Adv.LIGO, ET and LISA operations for detection of the GWs. The spectrum is computed in the thermal vacuum state with the chosen values of the parameters for the accelerated as well as decelerated models with $T = 0.001\,\mathrm{Mpc}^{-1}$ in the low frequency range $n < n_2$ (and similar behavior for $T = 0.01\,\mathrm{Mpc}^{-1}$ with this range, see subsection.(3.5.2)). This temperature is considered in the context of $B$ mode angular spectrum of CMB spectrum in thermal state [10]. And $T_*$ $=1.19\times10^{25}\,\mathrm{Mpc}^{-1}$ (†) [3] the high range $n_s \leqslant n \leqslant n_1$ which is from the extra dimensional scenario [13]. The obtained spectra are normalized with the CMB anisotropy spectrum of WMAP 7-year data. The amplitude of the spectrum of the thermal GWs is enhanced compared to its zero temperature case (vacuum case). It is observed that the spectrum for $T = 0.001\,\mathrm{Mpc}^{-1}$

[3](†) here, $T_* = 0.905$ $\mathrm{K}_{elv}$= $1.19\times10^{25}\,\mathrm{Mpc}^{-1}$.

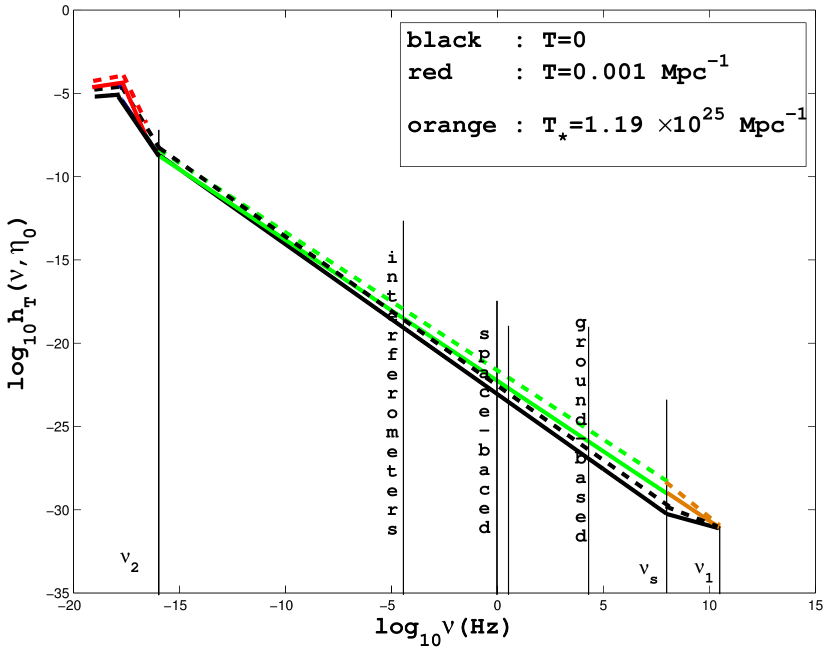



Figure 3.3: The amplitude of the GWs for the accelerated (solid lines) and decelerated (dashed lines) universe.

get maximum enhancement $\sim 1.51$ times than the vacuum case, at $\nu = \nu_E$, and it is $\sim 20$ times for $T_*$=$1.19\times10^{25}$ Mpc$^{-1}$ at $\nu = \nu_s$.

The plots for the amplitude of spectrum $h_T(n, \eta_0)$ versus the frequency $\nu$ for $\beta = -1.9$ and $\beta_s = -0.552$ are given in Fig.[3.3]. The amplitude of the spectrum is found enhanced in the frequency ranges, $10^{-19}$ Hz $\leq \nu < 1.49 \times 10^{-17}$ Hz, and $\nu_s \leq \nu \leq \nu_1$, the lower value of this range is selected such way that the spectral density does not exceed the upper bound of nucleosynthesis. But there is a suppression of the contribution of thermal GWs in the range $1.49 \times 10^{-17}$ Hz $\leq \nu < \nu_s$ due to the $\coth^{1/2}[n/2T]$ term. For comparison, amplitude of the spectra are plotted for the decelerated and accelerated FLRW universe in Fig.[3.3].

### 3.7.1 Comparison with the sensitivity of Adv.LIGO, ET and LISA

The new enhancement of the amplitude of GWs spectrum due to the high frequency GWs from extra dimensional effect (the modified amplitude, Fig.[3.3], green lines) can be compared with the sensitivity of Adv.LIGO [2], ET [3] and LISA [4]. For Adv.LIGO and ET cases, consider the root mean square amplitude per root Hz which equal to

$$\frac{h(\nu)}{\sqrt{\nu}}. \tag{3.59}$$

The comparison of the sensitivity (10 Hz - $10^4$ Hz) curve of the ground based interferometer Adv.LIGO and ET with the GW spectra of $\beta = -1.9$ for the accelerated and decelerated flat FLRW universe are given in Fig.[3.4]. This shows that the Adv.LIGO is unlikely to detect the enhancement of the spectrum due to the extra dimensional effect with it current stands but be possible with the sensitivity of ET.

Next, we compare the enhancement of the spectrum with the sensitivity ($10^{-7}$ Hz$-10^0$ Hz) of space based detector LISA. It is assumed that LISA has one year observation time which corresponds to frequency bin $\Delta\nu = 3 \times 10^{-8}$Hz ( one cycle/year) around each frequency. Hence to make a comparison with the sensitivity curve, a rescaling of the spectrum $h(\nu)$ is required in eq.(3.34) into the root mean square spectrum $h(\nu, \Delta\nu)$ in the band $\Delta\nu$, given by

$$h(\nu, \Delta\nu) = h(\nu)\sqrt{\frac{\Delta\nu}{\nu}}. \tag{3.60}$$

The plots of the LISA sensitivity with the modified amplitude of the spectrum are given in Fig.[3.5] for the accelerated and decelerated flat FLRW universe. This show that the LISA is unlikely to detect the spectrum with the new enhancement feature of the GWs of the present study.

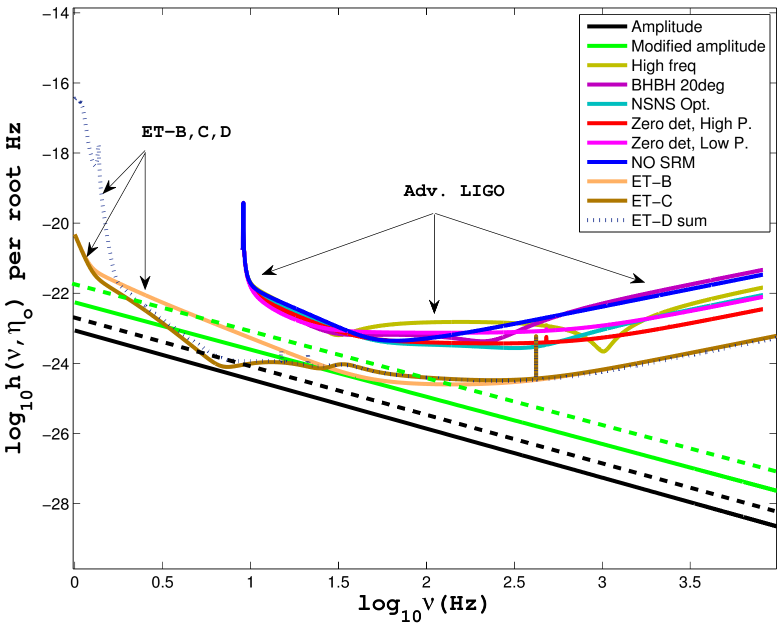



Figure 3.4: Comparison of the modified amplitude of the spectrum for the accelerated (solid black and green lines) and decelerated (dashed black and green lines) universe with the sensitivity curves of Adv.LIGO [2] and ET [3].

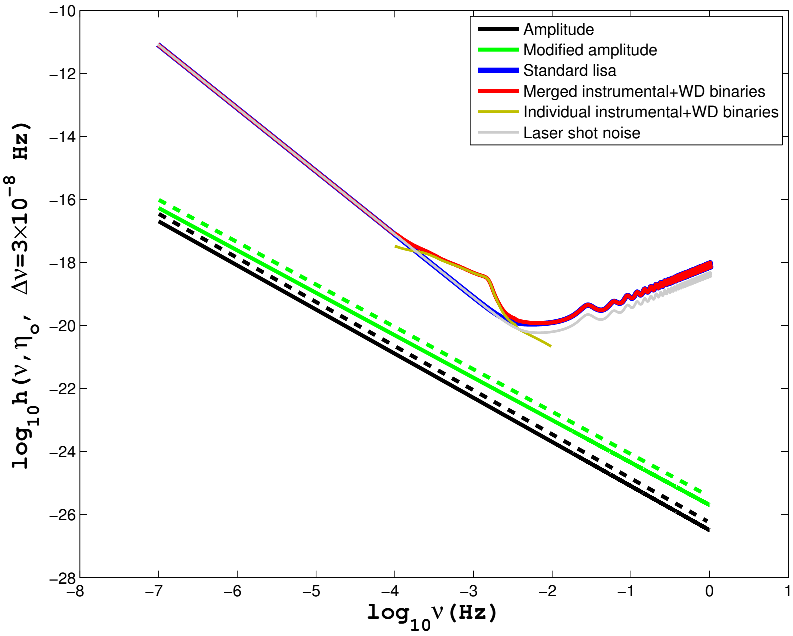



Figure 3.5: Comparison of the modified amplitude of the spectrum for the accelerated (solid black and green lines) and decelerated (dashed black and green lines) universe with the LISA [4] sensitivity curve.

### 3.7.2 Spectral energy density

Since we obtained the contribution of the very high frequency thermal GWs, due to the extra dimensional effect, to the amplitude of GWs it is appropriate to study the corresponding spectral energy also. The computation of the spectral energy density by including the very high thermal GWs can be achieved through the spectral energy density parameter $\Omega_g(\nu)$ of GWs is defined in section 3.6. Thus, after the normalization of the obtained spectrum of thermal GWs, we integrate $\int \Omega_g(\nu)\, d\nu/\nu$ from $\nu_* = 10^{-19}$ Hz up to $\nu_1 = 3 \times 10^{10}$ Hz, with $\beta = -1.9$ and $\beta_s = -0.552$. The spectral energy density of GWs is recomputed for the thermal and zero temperature cases again by splitting the total frequency into five ranges, and the obtained results for the accelerated FLRW universe are:

(a) $\nu_* \leq \nu \leq \nu_E$,

$$\frac{\rho_g}{\rho_c} = 5.8 \times 10^{-11}, \quad T = 0,$$
$$\frac{\rho_g}{\rho_c} = 8.8 \times 10^{-11}, \quad T = 0.001 \text{ Mpc}^{-1},$$

(b) $\nu_E \leq \nu \leq \nu_H$,

$$\frac{\rho_g}{\rho_c} = 2.3 \times 10^{-11}, \quad T = 0,$$
$$\frac{\rho_g}{\rho_c} = 3.5 \times 10^{-11}, \quad T = 0.001 \text{ Mpc}^{-1},$$

(c) $\nu_H \leq \nu \leq \nu_2$,

$$\frac{\rho_g}{\rho_c} = 2.4 \times 10^{-11}, \quad T = 0,$$
$$\frac{\rho_g}{\rho_c} = 3.7 \times 10^{-11}, \quad T = 0.001 \text{ Mpc}^{-1},$$

(d) $\nu_2 \leq \nu \leq \nu_s$,

$$\frac{\rho_g}{\rho_c} = 8.97 \times 10^{-9}, \quad T = 0,$$

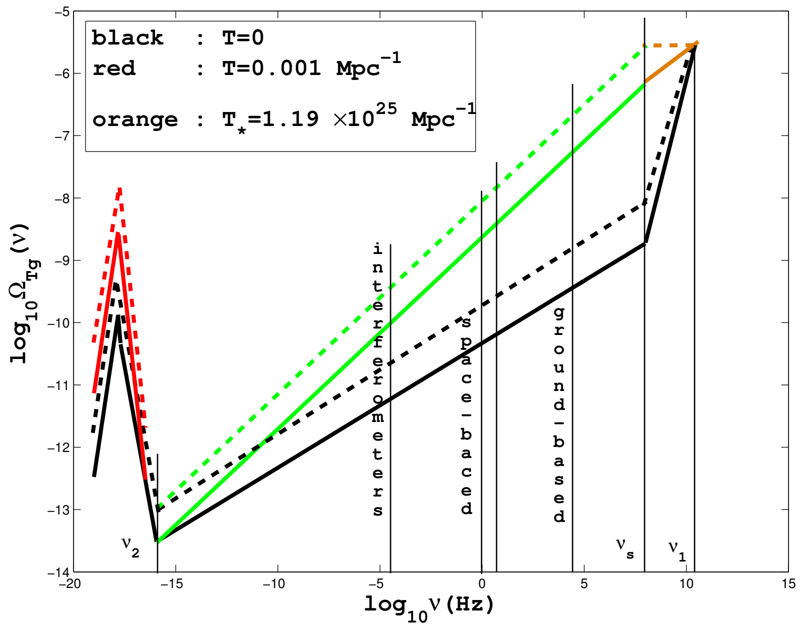



Figure 3.6: The spectral energy density of the GWs for the accelerated (solid lines) and decelerated (dashed lines) universe.

(e) $\nu_s \le \nu \le \nu_1$,

$$\begin{aligned}\frac{\rho_g}{\rho_c} &= 2.7\times10^{-6}, \quad T=0,\\ \frac{\rho_g}{\rho_c} &= 6.67\times10^{-6}, \quad T_* = 1.19\times10^{25}\,\mathrm{Mpc}^{-1}.\end{aligned}$$

It is to be noted that in the frequency range of (d) the thermal case is not shown because the thermal contribution in this frequency range is still negligible due to the temperature dependent term. However by taking into account the extra dimensional effect, the upper limit of temperature of the relic waves is $T_*$=1.19×10$^{25}$ Mpc$^{-1}$. Thus it is expected an enhancement of the spectral energy density in range $\nu_s \le \nu \le \nu_1$ compared to $T = 0$ case for the accelerated as well as decelerated universe. But at the same time ignoring the thermal contribution to the spectral density in the range $\nu_2 \le \nu \le \nu_s$ leads to a discontinuity at $\nu_s$, see Fig.[3.6]. This problem is solved by fitting a new line as discussed in the context of estimation of the

Table 3.1: Comparison of the estimated upper bound of spectral energy density of various studies with the present work. Here $\Omega_g^{(dec)}$ and $\Omega_g^{(acc)}$ are respectively the spectral energy density of the relic GWs in the decelerated and accelerated universe of the present study and $\Omega_g^{(est)}$ is the estimated upper bound of various studies.

| Frequency($\nu$) Hz | $\Omega_g^{(dec)}(\nu)$ | $\Omega_g^{(acc)}(\nu)$ | $\Omega_g^{(est)}(\nu)$ |
|---|---|---|---|
| $10^{-9} - 10^{-7}$ | $4.98 \times 10^{-9}$ | $1.03 \times 10^{-9}$ | $2 \times 10^{-8}$ [50] |
| $69 - 156$ | $34.84 \times 10^{-8}$ | $7.2 \times 10^{-8}$ | $8.4 \times 10^{-4}$[51] |
| $41.5 - 169.25$ | $4.93 \times 10^{-7}$ | $1.02 \times 10^{-7}$ | $6.9 \times 10^{-6}$[52] |

amplitude of the spectrum and hence recomputed the spectral density in the range $\nu_2 \leq \nu \leq \nu_s$ which gives the new value for $\rho_g/\rho_c = 8.21\times10^{-7}$. This changes the slope indicating enhancement of the spectral energy density of the GWs in the range $\nu_2 \leq \nu \leq \nu_s$, green lines, Fig.[3.6].

The enhancement of spectral energy density $\Omega_g(\nu)$ in the frequency range (d) can be compared with the estimated upper bound of various studies and are given in Tab.[3.1]. Thus $\Omega_g^{(dec)}$ and $\Omega_g^{(acc)}(\nu)$ are less than the upper bound of the estimated values of the respective frequency range.

Further see that the contribution to $\rho_\mathbf{g}/\rho_\mathbf{c}$ from the low frequency range is $\mathcal{O}(10^{-11} - 10^{-10})$ while from the higher frequency range it is $\mathcal{O}(10^{-6})$. Since the order of contribution to the total $\rho_\mathbf{g}/\rho_\mathbf{c}$ from the lower frequency side is very small in contrast with higher frequency side, we get for the accelerated universe as

$$\frac{\rho_\mathbf{g}}{\rho_\mathbf{c}} \simeq 6.67 \times 10^{-6} \quad \nu_* \leq \nu \leq \nu_1, \tag{3.61}$$

and is of the same order as that of the zero temperature case. However $\rho_g/\rho_c$ of the GWs with $T \neq 0$ is higher than the zero temperature case at lower frequency range $\nu_* \leq \nu \leq \nu_2$. Therefore it is expected an enhancement for the spectral energy density in the thermal vacuum state in the frequency range $\nu_* \leq \nu \leq \nu_2$.

## 3.8 Discussion and Conclusion

In this chapter, we considered thermal GWs with frequency range $10^{-19}$ Hz to $10^{10}$ Hz and obtained its spectrum and spectral energy density for the accelerated as well as decelerated flat FLRW universe. It is found that the spectrum gets enhanced due to the thermal effects on the GWs. This enhancement is the new feature of the spectrum if the relic GWs existed in thermal vacuum states. It is observed that the inclusion of the very high relic thermal GWs leads to a discontinuity in the amplitude of the spectrum at $\nu_s$. This is due to the fact that the temperature dependent term is insignificant in the higher frequency side $\nu_2 \leq \nu \leq \nu_s$. To evade this problem a new equation of line is derived and thus the amplitude get enhanced in the range $\nu_2 \leq \nu \leq \nu_s$. This is the new feature of the spectrum and we designates it as the 'modified amplitude' of the spectrum. The modified amplitude of the spectrum is compared with the sensitivity of the Adv.LIGO, ET and LISA missions. The comparison of the Adv.LIGO and LISA sensitivity shows that the modified amplitude is unlikely to be detected with its current stands but be possible with the sensitivity of ET.

The spectral energy density of the GWs is estimated in thermal vacuum state for the accelerated and decelerated flat FLRW universe. It is observed that the total spectral energy density get enhanced in the lower frequency range $\mathcal{O}(10^{-11} - 10^{-10})$ and from the higher frequency range it is $\mathcal{O}(10^{-6})$. A comparison of the estimated upper bound of spectral energy density of various studies with the present work is done. It shows that $\Omega_g^{(dec)}$ and $\Omega_g^{(acc)}$ are less than the estimated upper bound of various studies. The total estimated value of $\rho_{\mathbf{g}}/\rho_{\mathbf{c}}$ by including the very high frequency thermal relic GWs does not alter the upper bound of the nucleosynthesis rate. Thus the relic thermal GWs with very high frequency range are not ruled out and is testable with the upcoming data of various missions especially with ET for detecting GWs.

# Chapter 4

# Gravitational Waves Amplitude and Reheating Stage

The inflationary period played a crucial role on the further evolution stages of the universe. At the end of inflation the universe was devoid of particles and particles were created during the oscillatory phase of inflaton, the field that responsible for the inflation. It is believed that the created particles collided each other and hence raised the temperature of the universe known as reheating stage of the universe. The reheating was essential for the nucleosynthesis process since the inflation brought temperature of the universe below the required for thermo nuclear reactions. Towards the end of inflation, during the reheating, the equation of state of the universe is considered quite complicated and also model-dependent [34]. Thus a new stage called z-stage [1] is introduced to allow a general reheating epoch [17]. The different evolution stages of the universe have been affecting the relic GWs, it is also interesting to study the effect of z-stage on its amplitude.

To study the amplitude of GWs, during the inflationary stage and z-stage of the universe it is usually described with a power law of expansion with index of power law respectively denoted as $\beta$ and $\beta_s$. It is possible to estimate the index of the power law of expansion of z-stage, $\beta_s$, provided the reheating

[1] z from Zeldovich

temperature $T_{rh}$ of the universe is known. The range of reheating temperature must be larger than a few MeV [53], for the creation of light elements, but less than the energy scale at the end of inflation, that is $T_{rh} \lesssim 10^{15}$ GeV. This constrain leads to restriction on the allowed values of the index of power law of the z-stage. Thus the present chapter is mainly to get an estimates for the allowed values of $\beta_s$ and hence $\beta$ with non-thermal GWs. Therefore to obtain the corresponding amplitude and spectral energy density. In earlier studies, $\beta_s$ is estimated from the spectral energy density of GWs by taking into account the bound of nucleosynthesis rate which corresponds to a reheating temperature, $T_{rh} \lesssim 10^{14}$ GeV [23]. However, recently it is discussed that even a lower value for the reheating temperature i.e, $\mathcal{O}(10^6 - 10^9)$ GeV [54, 55, 56]. Therefore it has been proposed to determine the reheating temperature from GWS with space base laser interferometer experiments, the Big Bang Observer (BBO)[57] and the ultimate DECI-hertz Interferometer Gravitational wave Observatory (DECIGO) [58, 59] etc,. Recently, the reheating temperature of the universe is also estimated from WMAP-7 year data and it gives value as $T_{rh} \sim 10^6$ GeV [60]. In the present study the reheating temperature range is taken as $\mathcal{O}(10^6 - 10^9)$ GeV to get an estimate of $\beta_s$ for the non-thermal GWs in higher frequency range. Therefore the amplitude and spectral energy density of the non-thermal GWs with the new estimated $\beta_s$ is studied for the accelerated as well as decelerated flat FLRW universe. Thus it is also possible to estimate an upper bound of $\beta$, for the decelerated and accelerated universe, with the new obtained values of $\beta_s$.

## 4.1 Estimation of Upper Bound of $\beta$

The inflationary stage lasts up to $\eta = \eta_1, \eta_1 < 0$ (see, eq.(3.1)). To make the present study more general, assume that inflationary stage was followed by the interval of z-stage. It is known that an interval of evolution governed by the most stiff matter and leads to a relative increase of the amplitude of the wave [61]. Also the requirement of consistency for the GWs production with the observational restrictions does not allow the stiff matter interval too much long [61], [62]. The z-stage of expansion that includes in the present study is treated as quite general in nature. The evolution of the universe at this stage can be governed by a stiffer than radiation matter, as well as by softer than radiation matter [63]. It can also be simply a part of the radiation-dominated era. Thus, consider $S(\eta)$ at the interval of time from $\eta_1$ to some $\eta_s$ in the form

$$S(\eta) = S_z(\eta - \eta_p)^{1+\beta_s}, \qquad \eta_1 < \eta \leq \eta_s, \tag{4.1}$$

where $1 + \beta_s > 0$. For the particular choices $\beta_s = 0, 1$, the z-stage reduces to an interval of expansion governed by the radiation-dominated and matter-dominated respectively. Hence introduction of the z-stage is quite useful to consider a general reheating epoch [34].

The upper bound of $\beta$ considered in Chapter 3 is based on the spectral energy density of GWs constrained with the nucleosynthesis rate. In this section, we first give an estimate $\beta_s$ on the consideration of the reheating temperature with a range $\mathcal{O}(10^6 - 10^8)$ GeV and therefore determine the upper bound of $\beta$. To proceed further, the frequency of GWs and reheating temperature of $z$-stage are to be connected by considering end of the inflationary era and the reheating stage as follow.

Let $S(\eta_1)$ and $S(\eta_s)$ be the scale factors respectively at the end of inflationary stage and reheating stage, then using eq.(3.10) we get [60]

$$\frac{S(\eta_s)}{S(\eta_1)} = \frac{1}{\pi}\Big(\frac{5}{4}\frac{m^2 m_{pl}^2}{g_* T_{rh}^4}\Big)^{1/3}, \tag{4.2}$$

where $m$ is mass of the inflaton field and $g_* = 106.75$ is number of relativistic species produced during the reheating stage. It is to be noted that for deriving the expression (4.2) the inflationary potential, for the massive inflaton, is taken as $V = \frac{1}{2}m^2\phi^2$, in the flat FLRW universe.

The energy density at the reheating and inflationary stages satisfy the following condition [60]

$$\rho(S(\eta_s)) \leq \rho(S(\eta_1)). \tag{4.3}$$

Thus by taking $m = 1.7 \pm 0.6 \times 10^{13}$ GeV from the estimate of WMAP 7-year data, and using eqs.(4.2), eq.(4.3) with the numerical value of $g_*$, gives [60]

$$T_{rh} \leq 2 \times 10^{15} \ \text{GeV}. \tag{4.4}$$

This corresponds to the inflationary upper bound on the reheating temperature.

As mentioned earlier, towards the end of inflation, during the reheating, the equation of state of the universe can be quite complicated and is rather model-dependent. Hence introduction of the z-stage is useful to consider a general reheating epoch [34]. In the present study, we consider the z-stage evolution as a general form to obtain an estimate of $\beta_s$. Further it is assumed that the inflationary model with the potential $V = \frac{1}{2}m^2\phi^2$. Let $S(\eta_1)$, $S(\eta_s)$ be the scale factors respectively at the end of the inflationary stage and reheating stage. And let $\nu_1, \nu_s$ be their corresponding frequency of the non-thermal GWs, then with the help of equation (3.10) [36], we obtain

$$\frac{S(\eta_s)}{S(\eta_1)} = \left(\frac{\nu_1}{\nu_s}\right)^{1+\beta_s}. \tag{4.5}$$

Therefore substituting eq.(4.5) in eq.(4.2), we get

$$1 + \beta_s = \frac{\log\left[\frac{1}{\pi}\left(\frac{5}{4}\frac{m^2 m_{pl}^2}{g_* T_{rh}^4}\right)^{1/3}\right]}{\log\left(\frac{\nu_1}{\nu_s}\right)}. \tag{4.6}$$

Thus, for the range of reheating temperature, $\mathcal{O}(10^6 - 10^8)$ GeV, value of $\beta_s$ with tensor to scale ratio $r = 0.14$, is obtained as $2.6 < \beta_s < 3.9$.

With the new estimated range of $\beta_s$ an upper bound on $\Omega_g$ is obtained which is same as $10^{-6}$. Therefore using eqs.(3.49), (3.51) the estimate corresponds to the upper bound on $\beta$ for the non-thermal GWs in the decelerated

Table 4.1: Upper bound of $\beta$ and $\beta_s$ for the non-thermal GWs in the decelerated and accelerated FLRW universe. Here $\beta_d$ and $\beta_a$ stands for the value of $\beta$ corresponds to the decelerated and accelerated universe respectively.

| $r$ | $T_{rh}(GeV)$ | $\beta_s$ | $\beta_d$ | $\beta_a$ |
|---|---|---|---|---|
| 0.14 | $\sim 10^6$ | $\sim 3.9$ | $< -1.82$ | $< -1.80$ |
| 0.14 | $\sim 10^8$ | $\sim 2.6$ | $< -1.82$ | $< -1.80$ |

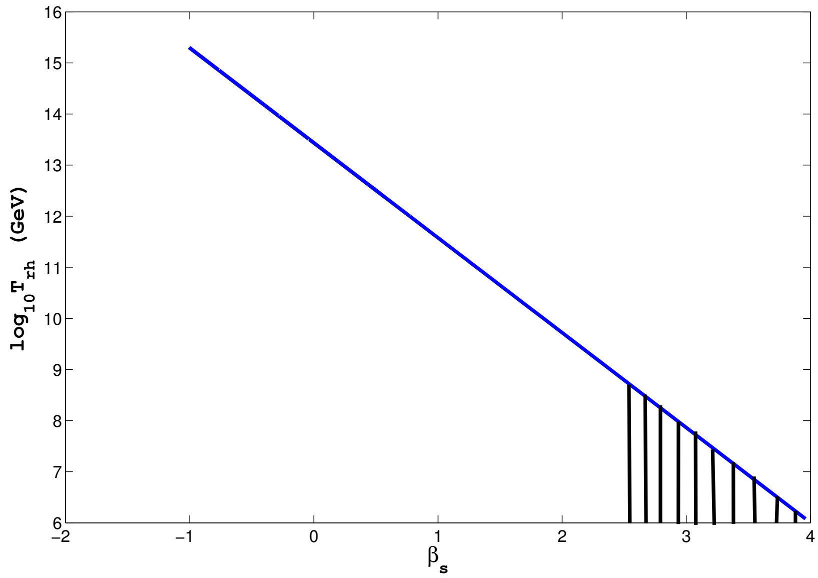



Figure 4.1: Reheating temperature for various $\beta_s$. The shaded black region indicates the new allowed range of $\beta_s$ values for the reheating temperature range $\mathcal{O}(10^6 - 10^8)$ GeV.

and accelerated flat FLRW universe are obtained. And the estimated results are presented in Tab.[4.1].

In earlier study, for $\beta_s = -0.552$ the corresponding reheating temperature obtained as $T_{rh} \sim 10^{14}$ Gev and is based on the basis of nucleosynthesis rate calculation result [23]. But, in the present section, we use the range of reheating temperature as $T_{rh} \sim (10^6 - 10^8)$ GeV to obtain the new estimate of $\beta_s$. A plot for the reheating temperature verses $\beta_s$ is shown in Fig.[4.1]. The shaded region with black lines is range of interest of $\beta_s$ and hence the corresponding reheating temperature of the present study.

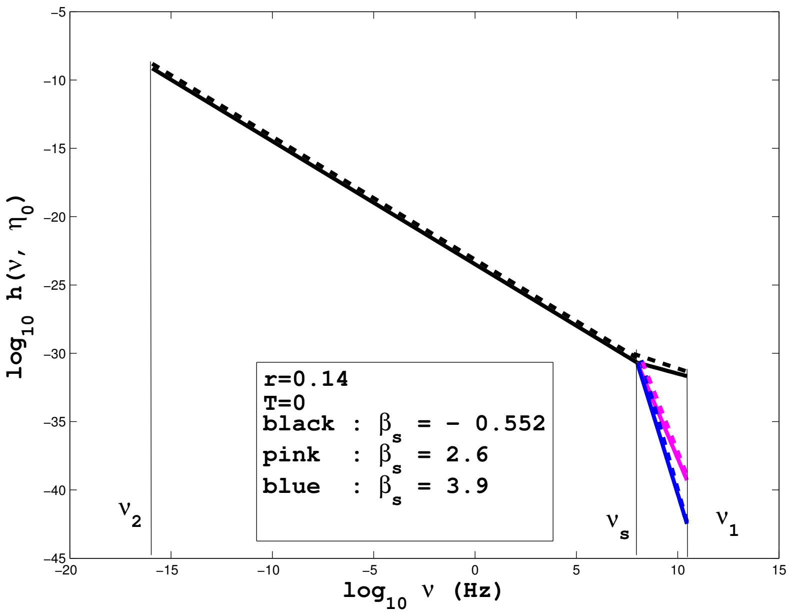



Figure 4.2: The amplitude of the GWs in the vacuum state for the accelerated (solid line) and decelerated (dashed line) universe with $\beta = -1.9, r = 0.14$. The frequency range $\nu_s \leq \nu \leq \nu_1$ is the interesting range for $\beta_s$.

## 4.2 Amplitude and Spectral Energy Density

The objective of this section is to study the amplitude and spectral energy density for the non-thermal GWs in the FLRW universe with the new estimated values of $\beta_s$, so that it reflects the effect of reheating stage on the amplitude and spectrum. Therefore the interesting wave number range is $n_s \leq n \leq n_1$, and the corresponding amplitude is obtained with eq.(3.43), given by

$$h(n, \eta_0) = A\Big(\frac{n}{n_0}\Big)^{1+\beta-\beta_s}\Big(\frac{n_s}{n_0}\Big)^{\beta_s}\Big(\frac{n_0}{n_2}\Big)\frac{1}{(1+z_E)^3}. \qquad (4.7)$$

The normalization factor $A$ is same as used in subsection 3.5.1. The amplitude of the GWs for the z-stage is computed by using the estimated values of $\beta_s, \beta$ and eq.(4.7) and the obtained spectrum is normalized with respect to WMAP 7-year data of CMB. Plots of the amplitude of the accelerated and

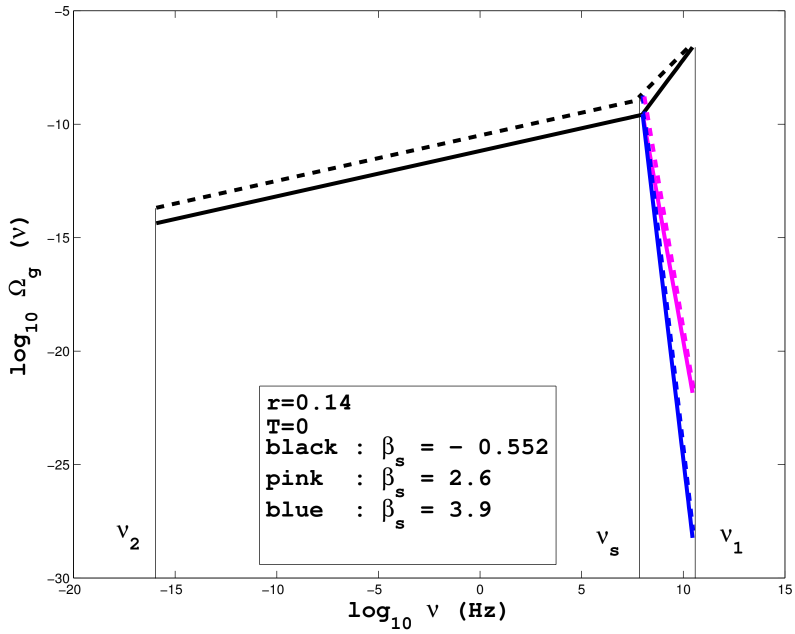



Figure 4.3: The spectral energy density of the GWs in the vacuum state for the accelerated (solid line) and decelerated (dashed line) universe with $\beta = -1.9, r = 0.14$. The frequency range $\nu_s \leq \nu \leq \nu_1$ is the interesting range for $\beta_s$.

decelerated universe for $\beta_s = 2.6, 3.9$ (selected from the estimated valid range of $\beta_s$) with $r = 0.14$ are shown in Fig.[4.2] with pink and blue colors respectively. Plot for the amplitude with $\beta_s = -0.552$, which is used in [23], with $r = 0.14$ are shown in black color. And for all $\beta_s$ the corresponding value of $\beta = -1.9$ is used (this value is for comparing with the earlier work [23]). It can be seen that the amplitude of the GWs spectrum with the estimated $\beta_s$ of the present study for the accelerated and decelerated universe cases exhibit a dip in the frequency range $\nu_s \leq \nu \leq \nu_1$ compared to $\beta_s = -0.552$ case.

The spectral energy density is computed with new estimated values of $\beta_s$. The calculation shows that $\Omega_g$ decreased around $10^{-4}$ times for the $\beta_s = 2.6 - 3.9$ due to the reheating temperature range (indicated with pink and blue colors, Fig.[4.3]) compared to $\beta_s = -0.552$ (black color, Fig.[4.3]). This leads to a dip of spectral energy in the higher range of frequency ($\nu_s \leq \nu \leq \nu_1$) compared with $\beta_s = -0.552$ case and is shown in Fig.[4.3]. The origin of this dip is due to behavior of the amplitude of the waves that we noted already in the same range of frequency. It shows that the upper bound of total spectral energy density shifts from $\Omega_g < \mathcal{O}(10^{-7}_{r=0.14})$ with $\beta_s = -0.552$ to the new upper bound $\Omega_g < \mathcal{O}(10^{-11}_{r=0.14})$ with $\beta_s = 2.6 - 3.9$, and with $\beta = -1.9$ for both accelerated as well as decelerated FLRW flat universe.

## 4.3 Discussions and Conclusions

Using the reheating temperature range $(10^6 - 10^8)$ GeV in the z-stage, we estimated $\beta_s$ for the GWs in vacuum state. The amplitude of the spectrum is obtained. It is observed that the amplitude of the GWs spectrum with the estimated $\beta_s$ for the accelerated and decelerated universe exhibit a dip in the higher frequency range $\nu_s \leq \nu \leq \nu_1$ compared with $\beta_s = -0.552$ case. The dip of amplitude in the frequency range $\nu_s \leq \nu \leq \nu_1$ is the new feature of the spectrum of GWs due to the reheating stage.

With the new estimated range of $\beta_s$ and the upper bound on $\Omega_g < \mathcal{O}(10^{-6})$ subjected to the nucleosynthesis rate, we obtained the corresponding upper bound on $\beta$ for the decelerated and accelerated FLRW universe. The present study shows that $\Omega_g$ decreased around $10^{-4}$ times for the $\beta_s = 2.6 - 3.9$ compared to $\beta_s = -0.552$ case. This leads to a dip of spectral energy in the higher range of frequency ($\nu_s \leq \nu \leq \nu_1$) compared with $\beta_s = -0.552$ case. The origin of this dip on the amplitude of the waves is the signature of the reheating stage. The spectral energy density is computed with new estimated values of $\beta_s$. It shows that the upper bound of the total spectral energy density shift to the new upper bound with estimated values of $\beta_s$. In the present work value of $\beta = -1.9$ is used but the calculations can be repeated up to corresponding upper bound of $\beta$'s. It can be seen that the trend of lowering the spectral energy density remains unaltered. Thus the trend of shifting of the spectral energy towards the lower order is due to the effect of $\beta_s$ and further it dependents on the reheating temperature. Therefore the dip of the amplitude and hence the spectral energy density of GWs in the higher frequency range supports lower value of the reheating temperature.

# Chapter 5

# Thermal Squeezing and $B$ Mode Correlation of CMB

The inflationary scenario [8] predicts a stochastic cosmic background of GWs [14] with a nearly scale invariant spectrum. These waves are not directly detected yet but its existence can be realized through the CMB. The origin of CMB is the one the predictions of the standard cosmology and its existence is observationally confirmed [64]. Though the distribution of CMB is isotropic, a small level of temperature fluctuations, $10^{-5}$ , also observed [65] known as CMB anisotropy and is the signature of the large scale structure formation in the early universe. Later it is observed that CMB show polarization effects also. It is believed that the generated GWs background could have left an observable imprint on the temperature and polarization anisotropies of the CMB [66]. The GWs mode of the CMB is called the $B$ mode angular power spectrum denoted by $C_l^{BB}$. Thus the detection of $B$ mode polarization of the CMB is definitely a verification, not only on the classical predictions of general relativity but also the inflationary scenario itself. The $B$ mode polarization may be observable with the Planck [5] or other missions.

Though the $B$ mode polarization is not detected directly, for example, the recent Wilkinson Microwave Anisotropy Probe (WMAP) 7-year result gives an upper bound on the $B$ mode angular power spectrum, $\frac{l(l+1)}{2\pi}C_{l=(2-7)}^{BB} < 0.055(\mu K)^2$ (here onwards $\mu$K means micro kelvin)[67]. And the other observations are giving even higher upper bound on the power spectrum, for examples, DASI $\frac{l(l+1)}{2\pi}C_l^{BB} < 50(\mu K)^2$ [68], Boomerang $\frac{l(l+1)}{2\pi}C_l^{BB} < 8.6(\mu K)^2$ [69], Maxipol $\frac{l(l+1)}{2\pi}C_l^{BB} < 112.3(\mu K)^2$ [70], QUaD $\frac{l(l+1)}{2\pi}C_l^{BB} < 10(\mu K)^2$ [6], CBI $\frac{l(l+1)}{2\pi}C_l^{BB} < 3.76(\mu K)^2$ [71], Capmap $\frac{l(l+1)}{2\pi}C_l^{BB} < 4.8(\mu K)^2$ [72]. The pivot frequency of these experiments varies from 23-145 GHz. If this is the case then there exist discrepancy between the standard theoretical and observed power spectra of the $B$ mode polarization. It is shown that the $B$ mode correlation gets enhanced due to thermal gravitons for the lower multipole moments [10], after the release of WMAP 3-year data, but is inadequate to match with the current WMAP 7-year data. Thus an additional physical mechanism is required to explain and increase the power level of the spectrum for compatibility with the current WMAP 7-year data. In this chapter, we study by considering the GWs in the thermal squeezed vacuum state and hence examine whether the $B$ mode of angular power spectrum get enhanced.

It is believed that the gravitons are generated during the inflation via the parametric amplification process or squeezing phenomena. This is purely a quantum phenomena with no classical analogue. Therefore the created gravitons in the early universe are placed in a specific quantum state called the squeezed vacuum state [7], more specifically in the zero temperature squeezed vacuum, a well known state in the quantum optics [73]. Meanwhile the gravitons created during the inflation with thermal features received much attention on the CMB anisotropy [14, 10, 13]. Therefore it is reasonable to consider that the created gravitons during the tensor perturbations existed in the squeezed vacuum state with thermal features. Thus the thermal squeezed vacuum state is a natural choice for the gravitons to exhibit both the quantum as well as thermal features [9]. If the GWs existed in the thermal squeezed vacuum state then it is possible to investigate the observational consequence of these effects on the $B$ mode correlation of CMB anisotropy.

## 5.1 GWs in Thermal Squeezed Vacuum State

It is believed that the variable gravitational field of early universe generated the relic GWs. The generating mechanism of these waves is inevitable quantum mechanical in nature [33] known as parametric amplification or squeezing phenomena. And hence the generated perturbations are placed in the squeezed vacuum quantum states [66]. This means that different modes of the created field are not totally independent, as is often assumed in the literature on inflation, but on the contrary some of them are highly correlated which leads to the picture of standing waves and modulated spectra.

The perturbed metric for a homogeneous isotropic flat FRW universe can be written in the form

$$ds^2 = S^2(\eta)(d\eta^2 - (\delta_{ij} + h_{ij})dx^i dx^j), \tag{5.1}$$

where $S(\eta)$ is the cosmological scale factor, $\eta$ is the conformal time defined by $d\eta = dt/S$ and $\delta_{ij}$ is the Kronecker delta symbol. Since the GWs field $h_{ij}$ have two states of polarization, which can be expanded over the spatial Fourier harmonics as

$$\begin{aligned} h_{ij}(\eta, x) &= \frac{\sqrt{16\pi}}{S(\eta)m_{pl}} \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3\mathbf{n} \sum_{\mathbf{p}=1}^{2} \epsilon_{ij}^{\mathbf{p}}(\mathbf{n}) \frac{1}{\sqrt{2n}} \\ &\quad \times [a_{\mathbf{n}}^{\mathbf{p}}(\eta)\, e^{i\,\mathbf{n}\,.\,\mathbf{x}} + a_{\mathbf{n}}^{\mathbf{p}\dagger}(\eta)\, e^{-i\,\mathbf{n}\,.\,\mathbf{x}}], \end{aligned} \tag{5.2}$$

where $m_{pl}$ [1] is the Planck mass, $\epsilon_{ij}^{\mathbf{p}}(\mathbf{n})$ is polarization tensor with two states $\mathbf{p} = +, \times$ for mode $\mathbf{n}$ and $n = |\mathbf{n}|$ is the comoving wave number. The polarization tensors $\epsilon_{ij}^{\mathbf{p}}(\mathbf{n})$ satisfy the transverse-traceless conditions $\epsilon_{ij}^{\mathbf{p}} n^j = 0$, $\epsilon_{ij}^{\mathbf{p}} \delta^{ij} = 0$ and $\epsilon_{ij}^{\mathbf{p}} \epsilon^{ij\acute{\mathbf{p}}} = 2\delta_{\mathbf{p}\acute{\mathbf{p}}}$, and leave independent only two components of $h_{ij}$ for mode $\mathbf{n}$ of the field.

In quantum mechanical language $a_{\mathbf{n}}^{\mathbf{p}}(\eta)$ and $a_{\mathbf{n}}^{\mathbf{p}\dagger}(\eta)$ are called the creation and annihilation operators. The evolution of these operators are governed

[1] $m_{pl} = l_{pl}^{-1}$ in natural unit.

by the Heisenberg equations of motion

$$\frac{da_{\mathbf{n}}(\eta)}{d\eta} = -i[a_{\mathbf{n}}(\eta), \mathcal{H}], \qquad \frac{da^{\dagger}_{\mathbf{n}}(\eta)}{d\eta} = -i[a^{\dagger}_{\mathbf{n}}(\eta), \mathcal{H}]. \tag{5.3}$$

And dynamical content is determined by the Hamiltonian $\mathcal{H}$. Assume that the GWs interact with background gravitational field only and there is no anisotropic material sources. Therefore the Hamiltonian for each polarization component takes the following form

$$\mathcal{H} = n\, a^{\dagger}_{\mathbf{n}} a_{\mathbf{n}} + n\, a^{\dagger}_{\text{-}\mathbf{n}} a_{\text{-}\mathbf{n}} + 2\sigma(\eta)\, a^{\dagger}_{\mathbf{n}} a^{\dagger}_{\text{-}\mathbf{n}} + 2\sigma^{*}(\eta)\, a_{\mathbf{n}} a_{\text{-}\mathbf{n}}, \tag{5.4}$$

where the coupling function $\sigma(\eta) = \frac{i}{2}\frac{S'}{S}$ and $' = d/d\eta$. The solution to the Heisenberg equations of motion (5.3) can be written as

$$a_{\mathbf{n}}(\eta) \;=\; u_n(\eta) a_{\mathbf{n}}(0) + v_n a^{\dagger}_{\text{-}\mathbf{n}}(0), \tag{5.5}$$

$$a^{\dagger}_{\mathbf{n}}(\eta) \;=\; u^{*}_n(\eta) a^{\dagger}_{\mathbf{n}}(0) + v^{*}_n a_{\text{-}\mathbf{n}}(0), \tag{5.6}$$

where $a_{\mathbf{n}}(0)$, $a^{\dagger}_{\mathbf{n}}(0)$ are the initial values of the operators taken long before the coupling became significant and $a_{\mathbf{n}}(\eta)$, $a^{\dagger}_{\mathbf{n}}(\eta)$ are at some later times $\eta$. The operators $a_{\mathbf{n}}(0), a^{\dagger}_{\mathbf{n}}(0)$ and $a_{\mathbf{n}}(\eta), a^{\dagger}_{\mathbf{n}}(\eta)$ satisfy their usual corresponding commutation relations.

The complex functions $u_n$ and $v_n$ are related by the equations

$$iu'_n = n\, u_n + i(S'/S)\, v^{*}_n, \qquad iv'_n = n\, v_n + i(S'/S)\, u^{*}_n, \tag{5.7}$$

and satisfy the conditions

$$\begin{aligned} |u_n|^2 - |v_n|^2 &= 1, \\ u_n(0) = 1, v_n(0) &= 0. \end{aligned} \tag{5.8}$$

It can be shown that theses conditions follow from the requirement that the function $\mu_n \equiv u_n + v^{*}_n$ which obeys the equation [33]

$$\mu''_n + (n^2 - S''/S)\mu_n = 0, \tag{5.9}$$

and is precisely the equation for the classical complex amplitude $\mu$ of the GWs (2.31). The solutions $u_n(\eta)$, $v_n(\eta)$ to equation (5.7) depend only on the $n = (n_1^2 + n_2^2 + n_3^2)^{1/2}$ not its direction. Also, these solutions are identical for both polarizations and they obey the same equations and initial conditions.

Let the field be in a quantum state and the vacuum state is defined by the requirement that $a_{\mathbf{n}}(0)|0\rangle = 0$ for all the modes $\mathbf{n}$ and both $\mathbf{p}$ states. According to the Heisenberg picture, the state of the field does not change in time but the operators do. The operators $a_{\mathbf{n}}(\eta)$ and $a^\dagger_{\mathbf{n}}(\eta)$ determine all the statistical properties of the field at the later times. It follows from eqs.(5.5),(5.6) that the mean value of $a_{\mathbf{n}}(\eta)$ and $a^\dagger_{\mathbf{n}}(\eta)$ are respectively $\langle 0|a_{\mathbf{n}}(\eta)|0\rangle = 0$ and $\langle 0|a^\dagger_{\mathbf{n}}(\eta)|0\rangle = 0$, but the variances of the quadratic combinations of $a_{\mathbf{n}}(\eta)$, $a^\dagger_{\mathbf{n}}(\eta)$ are not, given by

$$\begin{aligned}
\langle 0|a_{\mathbf{n}}(\eta)a_{\mathbf{n}'}(\eta)|0\rangle &= u_n(\eta)v_{n'}(\eta)\delta^3(\mathbf{n}+\mathbf{n}'),\\
\langle 0|a^\dagger_{\mathbf{n}}(\eta)a^\dagger_{\mathbf{n}'}(\eta)|0\rangle &= v^*_n(\eta)u^*_{n'}(\eta)\delta^3(\mathbf{n}+\mathbf{n}'),\\
\langle 0|a_{\mathbf{n}}(\eta)a^\dagger_{\mathbf{n}'}(\eta)|0\rangle &= u_n(\eta)u^*_{n'}(\eta)\delta^3(\mathbf{n}-\mathbf{n}'),\\
\langle 0|a^\dagger_{\mathbf{n}}(\eta)a_{\mathbf{n}'}(\eta)|0\rangle &= v^*_n(\eta)v_{n'}(\eta)\delta^3(\mathbf{n}-\mathbf{n}').
\end{aligned} \tag{5.10}$$

The two complex functions $u_n$ and $v_n$ are restricted by the constraint, eq.(5.8), therefore these are parameterized by the three real functions $r_n(\eta)$, $\gamma_n(\eta), \varepsilon_n(\eta)$ given by

$$u_n = e^{i\varepsilon_n}\cosh r_n, \qquad v_n = e^{-i(\varepsilon_n - 2\gamma_n)}\sinh r_n, \tag{5.11}$$

and the real functions obey the equations

$$\begin{aligned}
r'_n &= (S'/S)\cos 2\gamma_n,\ \gamma'_n = -n - (S'/S)\sin 2\gamma_n \coth 2r_n,\\
\varepsilon'_n &= -n - (S'/S)\sin 2\gamma_n \tanh r_n.
\end{aligned} \tag{5.12}$$

These equations are used for an explicit calculation of $r, \gamma, \varepsilon$ if a time-dependent scale factor $S(\eta)$ is given.

By using eq.(5.11) the Bogoliubov transformation of eqs.(5.5) and (5.6) are casted in the form

$$a_{\mathbf{n}}(\eta) = \mathcal{R}^{\dagger} Z^{\dagger} a_{\mathbf{n}}(0) Z \mathcal{R}, \quad a^{\dagger}_{\mathbf{n}}(\eta) = \mathcal{R}^{\dagger} Z^{\dagger} a^{\dagger}_{\mathbf{n}}(0) Z \mathcal{R}, \tag{5.13}$$

where

$$\begin{aligned} Z(r_n, \gamma_n) &= \exp\left[\xi_n^* a_{\mathbf{n}}(0) a_{\textbf{-n}}(0) - \xi_n a^{\dagger}_{\mathbf{n}}(0) a^{\dagger}_{\textbf{-n}}(0)\right], && (5.14)\\ \xi_n &= r_n e^{2i\gamma_n}, \\ \mathcal{R}(\varepsilon_n) &= \exp\Big[\zeta_n \Big(a^{\dagger}_{\mathbf{n}}(0) a_{\mathbf{n}}(0) + a^{\dagger}_{\textbf{-n}}(0) a_{\textbf{-n}}(0)\Big)\Big], && (5.15)\\ \zeta_n &= -i\varepsilon_n, \end{aligned}$$

are known as the two mode squeezed operator $Z$, which appears naturally for the complex solutions to eq.(5.9), and the two mode rotation operator $\mathcal{R}$ respectively. The functions $r_n, \gamma_n, \varepsilon_n$ are called squeezing parameter, squeezing angle and rotation angle respectively [74]. The eqs.(5.5),(5.6) and (5.13) demonstrate explicitly inevitable appearance of the squeezing phenomena in the case of GWs and physical situations of similar kind.

Let $|\phi_{in}\rangle$ be a quantum state and $F(a_{\mathbf{n}}(\eta), a^{\dagger}_{\mathbf{n}}(\eta); a_{\textbf{-n}}(\eta), a^{\dagger}_{\textbf{-n}}(\eta))$ a function of operator arguments. Then the expectation value of $F$ in the aforementioned quantum state can be obtained with the help of eq.(5.13) as

$$\begin{aligned} &\langle\phi_{in}|F(a_{\mathbf{n}}(\eta), a^{\dagger}_{\mathbf{n}}(\eta); a_{\textbf{-n}}(\eta), a^{\dagger}_{\textbf{-n}}(\eta))|\phi_{in}\rangle \\ &= \langle\phi_{in}|\mathcal{R}^{\dagger} Z^{\dagger} F(a_{\mathbf{n}}(0), a^{\dagger}_{\mathbf{n}}(0); a_{\textbf{-n}}(0), a^{\dagger}_{\textbf{-n}}(0)) Z\mathcal{R}|\phi_{in}\rangle \\ &= \langle\phi_{out}|F(a_{\mathbf{n}}(0), a^{\dagger}_{\mathbf{n}}(0); a_{\textbf{-n}}(0) a^{\dagger}_{\textbf{-n}}(0))|\phi_{out}\rangle, \end{aligned} \tag{5.16}$$

where

$$|\phi_{out}\rangle = Z(r_n, \gamma_n)\mathcal{R}(\varepsilon_n)|\phi_{in}\rangle. \tag{5.17}$$

Since the " *in* " particles are indistinguishable from the " *out* " particles, one can view eq.(5.17) as a result of transformation from an initial quantum state into a final quantum state during the process of evolution from in region to out region. In particular, the initial vacuum state, $|00\rangle$ for two particles, transforms into a two-mode squeezed vacuum state $|SS\rangle_2$:

$$|SS\rangle_2 = Z(r_n, \gamma_n)|00\rangle, \qquad \varepsilon = 0, \tag{5.18}$$

where $Z$ is given by eq.(5.14).

The thermal counter parts of the squeezed vacuum states are known as the thermal squeezed vacuum. The squeezed vacuum states and its thermal counter parts are used to study the particle creation in the early universe [75]. Based on the details given in section 3.3, Chapter 3 in analogous to the zero temperature squeezed vacuum states, a thermal squeezed vacuum state, $(Tsv)$, [9] is defined as

$$|Tsv\rangle = \mathcal{T}(\theta_n)\mathcal{Z}(r,\gamma)\tilde{\mathcal{Z}}(\tilde{r},\tilde{\gamma})|0\ \tilde{0}\rangle, \tag{5.19}$$

where the thermal operator $\mathcal{T}(\theta_n)$ is defined in eq.(3.24) and $\mathcal{Z}, \tilde{\mathcal{Z}}$ are the single mode squeezing operators respectively in the Hilbert and tilde space, given by

$$\mathcal{Z}(r,\gamma) = \exp\frac{1}{2}\Big[ r\left(e^{-2i\gamma}\, a_{\mathbf{n}}a_{\mathbf{n}} - e^{2i\gamma}\, a^{\dagger}_{\mathbf{n}}a^{\dagger}_{\mathbf{n}}\right)\Big], \tag{5.20}$$

$$\tilde{\mathcal{Z}}(\tilde{r},\tilde{\gamma}) = \exp\frac{1}{2}\Big[ \tilde{r}\left(e^{-2i\tilde{\gamma}}\, \tilde{a}_{\mathbf{n}}\tilde{a}_{\mathbf{n}} - e^{2i\tilde{\gamma}}\, \tilde{a}^{\dagger}_{\mathbf{n}}\tilde{a}^{\dagger}_{\mathbf{n}}\right)\Big], \tag{5.21}$$

where $r$ $(\tilde{r})$ and $\gamma$ $(\tilde{\gamma})$ are the squeezing parameter and squeezing angle. These parameters respectively represent the strength and direction of the squeezing with $0 \leqslant r\ (\tilde{r}) < \infty$ and $-\pi/2 \leqslant \gamma\ (\tilde{\gamma}) \leqslant \pi/2$. In the present study it is assumed that, numerically, $\tilde{r} = r$ and $\tilde{\gamma} = \gamma$.

The $a_{\mathbf{n}}$, $a^{\dagger}_{\mathbf{n}}$ and $\tilde{a}_{\mathbf{n}}$, $\tilde{a}^{\dagger}_{\mathbf{n}}$, are the annihilation and creation operators respectively in Hilbert and tilde space. These operators are obeying the usual commutation relations, $[a_{\mathbf{n}}, a^{\dagger}_{\mathbf{n}'}] = [\tilde{a}_{\mathbf{n}}, \tilde{a}^{\dagger}_{\mathbf{n}'}] = \delta^3(\mathbf{n} - \mathbf{n}')$ and remaining relations are zero.

By the appropriate action of the operators i.e; eqs.(3.24),(5.20), and (5.21) on $a_{\mathbf{n}}, a_{\mathbf{n}}^{\dagger}, \tilde{a}_{\mathbf{n}}, \tilde{a}_{\mathbf{n}}^{\dagger}$, we get [9]:

$$\begin{aligned}
\mathcal{T}^{\dagger} a_{\mathbf{n}} \mathcal{T} &= a_{\mathbf{n}} \cosh \theta_n + \tilde{a}_{\mathbf{n}}^{\dagger} \sinh \theta_n, & (5.22)\\
\mathcal{T}^{\dagger} a_{\mathbf{n}}^{\dagger} \mathcal{T} &= a_{\mathbf{n}}^{\dagger} \cosh \theta_n + \tilde{a}_{\mathbf{n}} \sinh \theta_n, & (5.23)\\
\mathcal{Z}^{\dagger} a_{\mathbf{n}} \mathcal{Z} &= a_{\mathbf{n}} \cosh r - a_{\mathbf{n}}^{\dagger} \exp(i\gamma) \sinh r, & (5.24)\\
\mathcal{Z}^{\dagger} a_{\mathbf{n}}^{\dagger} \mathcal{Z} &= a_{\mathbf{n}}^{\dagger} \cosh r - a_{\mathbf{n}} \exp(-i\gamma) \sinh r, & (5.25)\\
\tilde{\mathcal{Z}}^{\dagger} \tilde{a}_{\mathbf{n}} \tilde{\mathcal{Z}} &= \tilde{a}_{\mathbf{n}} \cosh \tilde{r} - \tilde{a}_{\mathbf{n}}^{\dagger} \exp(-i\tilde{\gamma}) \sinh \tilde{r}, & (5.26)\\
\tilde{\mathcal{Z}}^{\dagger} \tilde{a}_{\mathbf{n}}^{\dagger} \tilde{\mathcal{Z}} &= \tilde{a}_{\mathbf{n}}^{\dagger} \cosh \tilde{r} - \tilde{a}_{\mathbf{n}} \exp(i\tilde{\gamma}) \sinh \tilde{r}. & (5.27)
\end{aligned}$$

These obtained result are useful to compute the power spectrum of the GW in the thermal squeezed vacuum state.

## 5.2 GWs Power Spectrum in Thermal Squeezed Vacuum State

The tensor perturbations have two independent degrees of freedom which are taken as $h^{+}$ and $h^{\times}$ polarization modes. To compute the spectrum of GWs during inflation, we express $h^{+}$ and $h^{\times}$ in terms of the creation ($a^{\dagger}$) and annihilation ($a$) operators,

$$\begin{aligned}
h^{(\mathbf{p})}(\mathbf{x}, \eta) &= \frac{\sqrt{16\pi}}{S(\eta) m_{pl}} \int \frac{d^3 n}{(2\pi)^{3/2}} [a_{\mathbf{n}}\, \mu_n(\eta) + a_{-\mathbf{n}}^{\dagger}\, \mu_n^{*}(\eta)] e^{i\mathbf{n}.\mathbf{x}} \\
&= \int \frac{d^3 n}{(2\pi)^{3/2}} h_{\mathbf{n}}(\eta) e^{i\mathbf{n}.\mathbf{x}}, & (5.28)
\end{aligned}$$

The power spectrum of the tensor perturbations is defined as

$$\langle h_{\mathbf{n}} h_{\mathbf{n}'} \rangle = \frac{2\pi^2}{k^3} P_T\, \delta^3(\mathbf{n} - \mathbf{n}'). \quad (5.29)$$

The usual quantization condition between the fields and their canonical momenta yield $[a_{\mathbf{n}}, a_{\mathbf{n}'}^{\dagger}] = \delta^3(\mathbf{n} - \mathbf{n}')$. If the graviton field had zero occupation prior to inflation then the vacuum satisfies $a_{\mathbf{n}}(0)|0\rangle = 0$, $\langle a_{\mathbf{n}}^{\dagger}(0) a_{\mathbf{n}}(0)\rangle = 0$ and we would obtain a correlation function $\sim |\mu_n(\eta)|^2$. However, if the GWs extisted in thermal squeezed vacuum state at some earlier epoch then it is expected that the field retain its thermal and quantum features even after decoupling from the other radiation fields.

Using eqs.(5.19) and (5.22 - 5.27) in (5.29), we get

$$\begin{aligned}\langle h_{\mathbf{n}} h_{\mathbf{n}'}\rangle &= \frac{16\pi}{S^2(\eta) m_{pl}^2}\Big[(1+2\sinh^2 r)|\mu_n|^2 + \frac{1}{2}\sinh 2r\,(\mu_n^2\,\mathrm{e}^{i\gamma} + \mu_n^{*2}\,\mathrm{e}^{-i\gamma})\Big] \\ &\quad \times \coth\Big[\frac{n}{2T}\Big]\,\delta^3(\mathbf{n}-\mathbf{n}'). \end{aligned} \tag{5.30}$$

Therefore in comparison with eq.(5.29) the power spectrum in the thermal squeezed vacuum state is obtained as

$$\begin{aligned} P_T(n) &= \frac{8n^3}{\pi m_{pl}^2 S^2(\eta)}\Big[(1+2\sinh^2 r)|\mu_n|^2 \\ &\quad + \frac{1}{2}\sinh 2r\,(\mu_n^2\,\mathrm{e}^{i\gamma} + \mu_n^{*2}\,\mathrm{e}^{-i\gamma})\Big]\coth\Big[\frac{n}{2T}\Big], \end{aligned} \tag{5.31}$$

where the mode functions $\mu_n(\eta)$ obey the minimally coupled Klein-Gordon equation:

$$\mu_n'' + \Big(n^2 - \frac{S''}{S}\Big)\mu_n = 0.$$

In a quasi de-Sitter universe during inflation, conformal time $\eta$ and the scale factor during inflation $S(\eta)$ are related by

$$S(\eta) = -1/H\eta(1-\epsilon), \tag{5.32}$$

where $\epsilon = \frac{m_{pl}^2}{16\pi}\Big(\frac{V'}{V}\Big)^2$ and $V$ is the potential of the inflaton field.

For constant $\epsilon$ the mode functions $\mu_k(\eta)$ obey the minimally coupled Klein-Gordon equation [76],

$$\mu_n'' + \Big[n^2 - \frac{1}{\eta^2}\Big(\nu^2 - \frac{1}{4}\Big)\Big]\mu_n = 0, \tag{5.33}$$

where, for small $\epsilon$, $\nu = \frac{3}{2} + \epsilon$. Equation (5.33) has the general solution given by,

$$\mu_n(\eta) = \sqrt{-\eta}\,[c_1(n)\,H_\nu^{(1)}(-n\eta) + c_2(n)\,H_\nu^{(2)}(-n\eta)], \tag{5.34}$$

where $H_\nu^{(1)}, H_\nu^{(2)}$ are the Hankel functions. When the modes are well within the horizon then the mode function can be approximated by the flat spacetime solutions $\mu_n^0(\eta) = (1/\sqrt{2n})\, e^{-in\eta}$, $(n \gg SH)$. Matching the general solution in eq.(5.34) with the solution in the high frequency (flat spacetime) limit gives the value of the constants of integration $c_1(n) = (\sqrt{\pi}/2)\, e^{i(\nu+1/2)(\pi/2)}$ and $c_2(n) = 0$. Equation (5.34) then implies that for $-n\eta \gg 1$ or $n \ll SH$,

$$\mu_n(\eta) = e^{i(\nu-1/2)(\pi/2)}\, 2^{\nu-3/2} \frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} \frac{1}{\sqrt{2n}} (-n\eta)^{1/2-\nu}. \tag{5.35}$$

Substituting eq.(5.35) in eq.(5.31), for the superhorizon modes $(n \ll SH)$ for the tensor power spectrum, we obtain

$$\begin{aligned} P_T(n) \;&=\; \frac{16\pi}{m_{pl}^2}\Big(\frac{H}{2\pi}\Big)^2\Big(\frac{n}{SH}\Big)^{n_T}\Big[1 + 2\,\sinh^2 r \\ &\quad + \sinh 2r\,\cos(\gamma + (\nu - 1/2)\pi)\Big]\coth\Big[\frac{n}{2T}\Big], \end{aligned} \tag{5.36}$$

with $n_T = 3 - 2\nu = -2\epsilon$. Thus the power spectrum can be rewritten as

$$\begin{aligned} P_T(n) \;&=\; \mathcal{A}_T(n_0)\Big(\frac{n}{n_0}\Big)^{n_T}\Big[1 + 2\,\sinh^2 r \\ &\quad + \sinh 2r\,\cos(\gamma + (2 - n_T)\frac{\pi}{2})\Big]\coth\Big[\frac{n}{2T}\Big], \end{aligned} \tag{5.37}$$

where $n_0$ is the pivot point, $\mathcal{A}_T(n_0) = \frac{16\pi}{m_{pl}^2}\Big(\frac{H_{n_0}}{2\pi}\Big)^2$ and $H_{n_0}$ is the Hubble parameter evaluated when $SH = n_0$ during the inflation.

## 5.3 B Mode Angular Power Spectrum of CMB in Thermal Squeezed Vacuum State

In this section we compute the $B$ mode angular power of the CMB in thermal squeezed vacuum state. The angular power spectrum of $BB$ mode correlation of CMB for the multipole moments, $l$, is derived based on the physical conditions of the early universe and physics of CMB and is denoted as $C_l^{BB}$ [76] - [87].

Since, we obtained the power spectrum of the GWs, $P_T(n)$, in thermal squeezed vacuum state, the $B$ mode angular power spectrum is computed as follows. The angular power spectrum of the $B$ mode polarization of the CMB is given by [77],

$$\begin{aligned} \frac{C_l^{BB}}{(4\pi)^2} &= \int dn\, n^2 P_T(n) \\ &\times \Big| \int_0^{\eta_0} d\eta\, g(\eta)\, \psi(n,\eta) \Big[ 2j_l'(x) + \frac{4j_l(x)}{x} \Big] \Big|^2, \end{aligned} \quad (5.38)$$

where $g(\eta) = \dot{\kappa}\, e^{-\kappa}$ is the visibility function, $\dot{\kappa}$ is the differential optical depth for the Thomson scattering, $\psi(n,\eta)$ is the source term, $j_l(x)$ is the spherical Bessel function of order $l$, $j_l'(x)$ is derivative with respect to the $x$ and $x = n(\eta_0 - \eta)$. Thus the $B$ mode correlation of the CMB in the thermal squeezed vacuum state is obtained by substituting eq.(5.37) in eq.(5.38).

We have added the unlensed tensor contributions to generate the $B$ mode correlation spectrum. The plots are obtained by running the new updated version of CAMB code after the WMAP 7-year data with the following parameters: $\Omega_b$= 0.05, $\Omega_c = 0.25$, and $\Omega_\nu = 0.70$. Optical depth $\kappa = 0.08$ and Hubble parameter $h = 0.7$. The value of tensor spectral index is taken $n_T =$ - 0.01. Tensor-to-scalar ratio $r$ is taken as 0.1 at $n_0 = 0.002$ Mpc$^{-1}$.

We computed the $B$ mode angular power spectrum in the thermal squeezed vacuum state with a range of squeezing parameter, angle and for different comoving temperatures. The obtained results are plotted and given in Fig.(5.1).

## 5.4 Discussions and Conclusions

The obtained $B$ mode angular power spectra show that the power spectrum get enhanced in the case of thermal squeezed vacuum state (in Fig.(5.1), dash black, green and blue colors are respectively with different choice of the squeezing parameter and angle) compared to the spectrum at $T = 0$ case (gray color with zero squeezing). The spectra for the temperature $T = 0.0001$

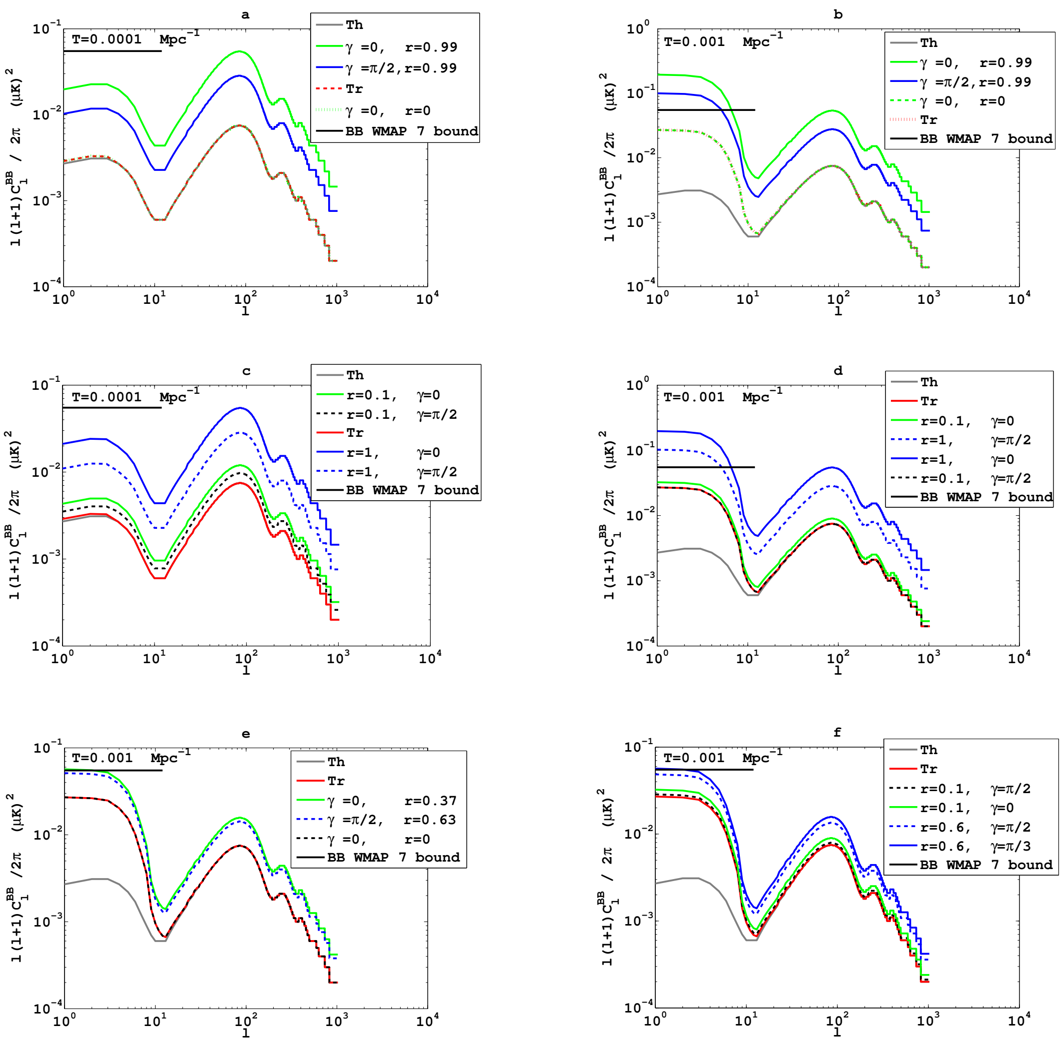



Figure 5.1: The $B$ mode angular power spectrum for a range of squeezing parameter ($r$) and angle ($\gamma$) with different comoving temperatures as a function of multipole moments ($l$). In all panels gray color represents the theoretical power spectrum (Th) with $T = 0$ and black thick line is the bound from WMAP 7-year data [79]. The red, blue, dashed black and green colors are respectively the power spectrum for the thermal vacuum state (Tr) and thermal squeezed vacuum state ($Tsv$).

Mpc$^{-1}$ lies within the upper bound of the WMAP 7-year data (panels (a) and (c)) but for $T = 0.001$ Mpc$^{-1}$ are out of the bound (panels (b) and (d)), for the same set of the squeezing parameters and angles. This indicates that the $B$ mode correlation to be within the WMAP 7-year bound for the temperature $T = 0.001$ Mpc$^{-1}$, then the squeezing strength and angle are to be tuned in the range as shown in panels (e) and (f). The power spectra in all the panels are reduced to that of the thermal case in absence of squeezing. The standard angular power spectrum (zero temperature case) is recovered in absence of temperature and squeezing.

The angular power spectrum of the $B$ mode in the thermal state is considered in [10] and showed that the power spectrum gets enhanced for $T = 0.0001$ Mpc$^{-1}$ up to $l = 2 - 4$, $T = 0.001$ Mpc$^{-1}$ up to $l = 2 - 30$. Further it discussed that the thermal effect is negligible for the multipole moments $l > 100$. In the present study, it is observed that the $B$ mode power spectrum get enhanced for all the multipole moments ($l \geqslant 2$), GWs in the thermal squeezed vacuum state. The tensor mode power spectrum depends on competition between the squeezing and thermal effects in a manner which dominates over the other. The present study shows that the enhancement of the $B$ mode correlation, due to the GWs in the thermal squeezed vacuum state, lies well within in the current observational data of WMAP 7-year.

It is to be noted that the upper bound on the $B$ mode angular power spectrum of the CMB provided by WMAP 7-year data not from the direct measurement of $B$ mode polarization but is estimated from the $TT$ and $TE$ power spectra [80], respectively known as temperature-temperature and temperature-electric correlation polarization spectra of CMB. Therefore the enhancement of the $B$ mode angular power spectrum of the CMB is either observed or ruled out with the Planck and other mission data. The influence of the thermal squeezed vacuum gravitons on the various power spectra of the CMB anisotropy like $TT, TE$, etc, can also be studied with the upcoming data of various experiments which is beyond scope of the present work.

# Chapter 6

# Summary and Conclusions

Gravitational waves are the classical predictions of Einstein's general theory of relativity. The existence of GWs are due the dynamics of very early universe and the various astrophysical objects. Therefore the GWs supposed to exhibit a wide spectrum of frequency $10^{-19}$ Hz to $10^{10}$ Hz. Since the relic GWs were generated in the early universe it can provide valuable information in understanding the early physical conditions of the universe impartially. Therefore investigation of the relic gravitational waves is paramount relevance in cosmology. At a glance, the relic GWs were mainly generated during the inflationary period of the early universe. Therefore the detection of the GWs not only verify the classical prediction of general relativity but also the inflationary scenario itself. There are several on going missions to detect the GWs but not realized its existence directly yet, however it is promising in near by future.

The GWs generated during the inflationary period are believed to be non thermal in nature as the models are energetically not in favour to create them. However the existence of GWs with thermal features is not ruled out completely due to several scenarios such as pre-inflationary period, extra dimensions, evaporation of mini black holes, Dirac hypothesis, etc;. Even, due to the stimulated emission process during the inflationary period is also not ruled out the creation of thermal GWs. The thermal GWs can contribute to its amplitude and hence the spectral energy density.

The amplitude of these waves to be observed today depends on the evolution history of the universe because the universe dominated with different form of energy at its various epoch. Therefore it is interesting to study the contribution of the thermal gravitational waves to its spectrum in the present universe. The inflation era and subsequent stages, including the reheating stage, of the universe played an important role on the amplitude and hence spectral energy density of the relic GWs. The other reason to consider the GWs in thermal state is due to result of the various missions that measured the CMB anisotropy show that there exist a discrepancy between the theoretical and estimated $B$ mode angular power spectrum.

The GWs are considered in thermal vacuum state and its amplitude and spectral energy density are computed for the accelerated as well as decelerated flat FLRW universe. It is found the amplitude of spectrum get enhanced, in general, due to the contribution of thermal waves irrespective of the accelerated or decelerated model. This enhancement is the new feature of the spectrum GWs. It is observed that the inclusion of the very high relic thermal GWs due to extra dimensional effect leads to a discontinuity in the amplitude of the spectrum, at $\nu_s$. This problem is solved by deriving a new equation of line and consequence is the enhancement of the amplitude in the high frequency range ($\nu_2 \leq \nu \leq \nu_s$). This is the new feature of the spectrum and designated it as the 'modified amplitude'. The modified amplitude of the spectrum is compared with the sensitivity of the gravitational waves detectors, Adv.LIGO, ET and LISA and shows that the modified amplitude is unlikely to detect with its current stands of Adv.LIGO and LISA but be possible with ET. Thus the existence of the thermal GWs may be verified in future and therefore some light on the extra dimensional scenario also.

The spectral energy density of the GWs is estimated in thermal vacuum state for the accelerated and decelerated flat FLRW universe. It is observed that the spectral energy density gets enhanced in the lower as well as higher frequency ranges. A comparative study of the estimated upper bound of spectral energy density of various earlier studies with the present work is carried out. It shows that spectral energy density for the accelerated and decelerated flat universe are less than the estimated upper bound of the pervious studies. The total estimated value of GWs by including the very high frequency thermal relic GWs does not alter the upper bound of the nucleosynthesis rate. Thus the relic thermal gravitons with very high frequency range are not ruled out and is testable with the upcoming data of various missions to detect GWs.

Recently, discussed that it is possible to estimate the reheating temperature of the universe with GWs. It is shown that the reheating temperature is much lower than the estimated from inflation. The scenario of the lower reheating temperature of the universe is currently under the investigation with GWs detectors BBO, DECIGO and etc. Thus, an estimate of the spectral index of reheating (z-stage) with the low reheating temperature for the non-thermal GWs in vacuum state is obtained. Hence, obtained the corresponding upper bound on the spectral index of inflation for the decelerated and accelerated flat FLRW universe and thus the amplitude of the GWs spectrum. The computed GW spectrum exhibit a dip in the higher frequency range ($\nu_s \leq \nu \leq \nu_1$) and is the feature of the reheating stage on the spectrum. The spectral energy density is also computed with new estimated values of spectral index of the reheating stage and shows that the upper bound of total spectral energy density shift to the new upper bound.

The angular power spectrum of $B$ mode polarization of the CMB is obtained in thermal squeezed vacuum state. The obtained power spectrum is found enhanced in the thermal squeezed vacuum state compared to the theoretical zero temperature case. The spectra for the two comoving temperatures are obtained, and observed that the $B$ mode correlation to be within the WMAP 7-year bound then the associated squeezing parameter and angle are to be tuned accordingly. The $B$ mode angular power of the zero temperature case is recovered in absence of temperature and squeezing. In the present study, it is observed that the $B$ mode angular power spectrum of CMB get enhanced for all the multipole moments, higher than or equal two, in the thermal squeezed vacuum state than the previous study where the enhancement shown only for few multipole moments. The nature of the $B$ mode power spectrum depends on competition between the squeezing and thermal effects in a manner which dominates over the other.

The present study shows that enhancement of the $B$ mode correlation, due to the GWs in the thermal squeezed vacuum, lies well within in the current observational data of the WMAP 7-year. It is to be noted that the upper bound on the $B$ mode angular power spectrum of the CMB provided by the WMAP 7-year data not from the direct measurement of $B$ mode polarization but estimated it from the $TT$ and $TE$ power spectra. Therefore the enhancement of the $B$ mode angular power spectrum of the CMB is either observed or ruled out with the Planck and other mission data. And hence it may throw some light in understanding the existence of the GWS in thermal squeezed vacuum state also. The influence of the thermal squeezed vacuum state GWs on the various power spectra of the CMB anisotropy like $TT, TE$, etc, are also can be studied with the upcoming data of various experiments and is beyond the scope of present study.

From this present study it can be concluded that the existence thermal GWs is not ruled out completely. The detection of GWs with thermal features may provide in understanding the extra dimensional scenario also. The thermal GWs may explain the enhancement of $B$ mode angular power spectrum of CMB so that it matches with current observational results. However the existence of the thermal GWs are to be verified in future with the progressive missions to detect them.

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# List of Publications

1. **Modified Gravitational Wave Spectrum** (with P K Suresh), Class.Quantum Grav. **29** 175009 (2012)
2. **B Mode Correlation Enhancement of CMB from Thermal Squeezed Vacuum state** (with P K Suresh), Int.J.Mod.Phys.**D** (2012)
3. **Thermal Gravitational Waves in Accelerating Universe** (with P K Suresh), communicated.
4. **Reheating Effect on the Spectral energy density of Gravitational Waves** (with P K Suresh), communicated.