CHARACTERIZATION AND INFERENCE ASSOCIATED WITH DISCRETE AGEING CLASSES

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by

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To

The Unsung Heroes of Indian Army and Their Families

CERTIFICATE

Certified that the thesis entitled **"CHARACTERIZATION AND INFERENCE ASSOCIATED WITH DISCRETE AGEING CLASSES"** is a bonafide record of work done by Shri. Deemat C Mathew under my guidance in the Department of Mathematics and Statistics, University of Hyderabad and that no part of it has been included anywhere previously for the award of any degree or title.

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DECLARATION

This thesis contains no material which has been accepted for the award of any other Degree or Diploma in any University and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due references are made in the text of the thesis.

Hyderabad- 46, **Deemat C Mathew**

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Contents

Chapter 1

Introduction

The literature on reliability theory mainly deals with non-negative random variables with absolutely continuous life distributions. However, quite often we come across situations where the product life can be described by non-negative integer-valued random variable. For example, (i) a device can be monitored only once per time period and the random variable of interest is the number of time periods successfully completed by the device prior to its failure, (ii) a piece of equipment may operate in cycles and the random variable is the number of cycles completed prior to failure- as an illustration, one could be interested in the number of copies made by a photocopier before it fails and (iii) number of road accidents in a city in a given month. For recent discussions on various aspects of discrete reliability, one can refer Bracquemond and Gaudoin (2003), Kemp (2004), Nanda and Sengupta (2005), El- Arishi (2005), Chen and Manatunga (2007), Yu (2007), Goliforushani and Asadi (2008), Nawata et al. (2008, 2009), Nair and Kattumannil (2008), Kattumannil and Nair (2010), Dewan and Kattumannil (2011) and Khorashadizadeh et al. (2013). This necessitates the development of tools for studying the discrete failure time data analogous to the continuous case.

Let X be a discrete random variable with support $\mathbb{N} = \{1, 2, \ldots.\}$ or a subset thereof. Suppose $p(x) = P(X = x)$, $F(x) = P(X \leq x) = \sum_{j=1}^{x} p(j)$ and $R(x) = 1 - F(x) = P(X > x) = \sum_{j=x+1}^{\infty} p(j)$, denote respectively, the probability mass function, the distribution function and the reliability function of X at $X = x$. Note that $R(0) = 1$. Then the hazard rate and the reversed hazard rate functions of X denoted by $k(.)$ and $\lambda(.)$, respectively, are given by

$$k(x) = \frac{p(x)}{R(x-1)}, \quad x = 1, 2, \ldots \tag{1.1}$$

and

$$\lambda(x) = \frac{p(x)}{F(x)}, \quad x = 1, 2, \ldots. \tag{1.2}$$

It is interesting to note that in the discrete setup both the hazard rate and the reversed hazard rate can be interpreted as a probability which is not the case in the continuous case. If X represents the lifetime of a component then $k(x)$ is the probability that the component will fail at time $X = x$ given that it has survived up to the time x. Similarly, $\lambda(x)$ is the probability that the component will fail at time $X = x$, given that it has survived at most time x. Because of this nice interpretation of $k(x)$ and $\lambda(x)$ as a probability of an event, in the discrete case, it has received wide spread attention. The function $k(.)$ uniquely determines the distribution of X through

$$R(x) = \prod_{j=1}^{x}(1 - k(j)), \quad x = 1, 2, 3, \ldots. \tag{1.3}$$

For early discussions on discrete hazard rate see Salvia and Bollinger (1982), Padgett and Spurrier (1985) and Ebrahimi (1986). Xekalaki (1983) pointed out situations where the product life is discrete in nature and gave characterization results concerning the geometric, Waring and negative hyper-geometric distributions in terms of hazard rates. The topic of characterization based on discrete reliability concepts is discussed subsequently by Nair (1983), Guess and Perk (1988), Hitha and Nair (1989) and Nair and Hitha (1989). For surveys of discrete life distributions, one can refer to Bracquemond and Gaudoin (2003).

However, for many distributions the expressions for $k(.)$ and $\lambda(.)$ are not in simple closed forms and this has prompted reliability analysts to develop relationship between hazard rate and the conditional expectation that characterizes different lifetime models. Among them, the characterization for negative binomial was discussed by Osaki and Li (1988) and for the Poisson case by Ahmed (1991). Nair and Sankaran (1991) have obtained a similar characterization results for Ord families of distributions. Recently, Nair and Kattumannil (2008) have derived a general result in these direction for a class of discrete probability distributions. The results available in the literature in connection with characterization problems, based on relationship between hazard rate and the conditional expectation, became special cases of the results of Nair and Kattumannil (2008). Works of Nair and Kattumannil (2008) mainly focused on the mean residual life (or reversed mean residual life) and the hazard rate (or the reversed hazard rate).

Besides the mean, a second characteristic of the residual life, that plays a similar role in identifying life distributions and distinguishing them, is the variance residual life (or reversed variance residual life). The role of variance residual life in determining the life distribution has been discussed for continuous models by various authors, see Gupta (1987), Gupta et al. (1987), Gupta and Kirmani (2000, 2004), El- Arishi (2005) and Nair and Kattumannil (2010) etc. It remains an open problem to find similar characterizations for discrete distributions. In an attempt in this direction, Kattumannil and Nair (2010) discussed the characterization of a class of discrete distributions by properties of conditional variance. These properties include relationship between variance residual life, mean residual life and hazard rate. Motivated from Kattumannil and Nair (2010), in this thesis, we study the properties of reversed variance residual life and also we obtain the characterization results in terms of reversed variance residual life and reversed hazard rate.

Even though, the hazard rate and the reversed hazard rate defined in equations (1.1) and (1.2) have been widely used, many of the properties of the hazard rate and the reversed hazard rate which hold in the continuous case do not hold in the discrete case. For example, ageing classes such as IHRA and NBU have no unique definition in terms of $k(.)$ and $\lambda(.)$. Several problems with the definition of $k(.)$ were pointed out by Xie et al. (2002). They defined the hazard rate function for the discrete case in an alternate way (also see Roy and Gupta (1999)) as

$$k^*(x) = \ln \frac{R(x-1)}{R(x)}, \quad x = 1, 2, 3, \ldots . \tag{1.4}$$

Several advantages of the definition (1.4) were discussed in Xie et al. (2002). Exact expressions for hazard rate function $k^*(.)$ for some well known discrete distributions are available in Chapter 6 of Lai and Xie (2006). It is easy to verify that the functions $k(.)$ and $k^*(.)$ satisfy the relations of the form

$$k(x) = 1 - \exp(-k^*(x)), \quad or \quad k^*(x) = -\ln(1 - k(x)), \quad \forall x = 1, 2, \ldots . \tag{1.5}$$

Hence both $k(.)$ and $k^*(.)$ have the same monotonicity property, accordingly, any of these functions can be used to study the ageing properties.

Classification of lifetime distributions based on the notion of ageing helps in identifying the underlying model for lifetime data. Lai and Xie (2006) gave a comprehensive review of ageing concepts when the lifetimes are continuous random variables. A dedicated study on discrete ageing notion can be found in Bracquemond et al. (2001). For a recent survey of discrete reliability concepts and distributions, see Bracquemond and Gaudoin (2003). Kemp (2004) carried out an exhaustive study on the ageing behaviour of discrete life distributions and gave some new insight in this direction. Using the new definition of hazard rate we can define the discrete ageing classes uniquely. It leads to several research problems related to ageing classes in the discrete setup parallel to the continuous case. Making use of the definition (1.4), Dewan and Kattumannil (2011) have reviewed ageing concepts like ILR, IHR, IHRA, MRL, NBU and NBUE etc. for discrete random variables (details are given in Chapter 2) and looked at the stochastic orderings between two discrete random variables. In this thesis, we addressed the problem of testing constant hazard rate against different alternatives of ageing classes such as IHR, IHRA, NBU and NBUE.

The new hazard rates also helps to study the properties of the proportional hazards model. For instance, the prediction of the life of a die is very crucial for the success of the forging industry. This prediction helps to estimate the cost of the die and hence supply the die to the customer at a reasonably lower price. Reduced price can escalate the demand from the customer. Thus the study and analysis of lifetime of the die is important in this context. One can also estimate the number of components that can be produced using a particular die before it fails (Dewan and Kattumannil (2011)). Since the lifetime is computed as the number of metal components produced by the die, discrete ageing concepts are of interest. The performance of the die is affected by the stress applied and the environmental factors such as temperature and pressure. Proportional hazards model is a suitable model in this context. The new definition of hazard rate enables us to model the data using proportional hazards model. For more details, one can refer to Dewan and Kattumannil (2009). To study the proportional reversed hazards model we define a new reversed hazard rate in discrete setup and discuss the properties associated with it. This includes preservation of ageing classes and the testing problem related to proportionality constant. The above mentioned research problems will be considered in this thesis.

The thesis is organized into seven chapters. Basic concepts required for the development of the study are given in the second chapter. In Chapter 3, we obtain a lower bound for the conditional variance given the event $X \leq x$. In the process of deriving lower bound to the conditional variance, we obtain an identity connecting reversed variance residual life (RVRL) and reversed hazard rate. This enables us to characterize a class of discrete probability models satisfying specific conditions. Exact expression for RVRL is obtained for additive exponential dispersion family, modified power series family, Katz family and Ords family. The lower bound obtained is compared to the Cramer-Rao and Chapman-Robbins lower bounds.

The classes of increasing hazard rate (IHR) distributions and increasing hazard rate average (IHRA) distributions plays an important role in statistical theory of reliability. In Chapter 4, we develop a test criterion to test the constant hazard rate against IHR. We consider the testing problem

$$H_0 : X \text{ is geometric} \quad \text{against} \quad H_1 : X \text{ belongs to IHR but not geometric.}$$

The proposed measure of departure from the null hypothesis is estimated by a U-statistic. The consistency and asymptotic normality of the proposed test statistic is established. Test for IHRA alternative is also developed in this chapter. A Simulation study is carried out to assess the efficiency of the test statistics.

In Chapter 5, we develop a test procedure to test constant hazard rate against new better than used (NBU) alternatives. We say that X is NBU if $R(x + y) \leq R(x)R(y)$ for all $x, y \geq 0$. It is known that the NBU class includes IHR class as a subset. The measure of departure from the null hypothesis is obtained as

$$\Delta(F) = \mu^2 - \frac{1}{2}[\mu + E(X^2)], \tag{1.6}$$

where $\mu = E(X)$. Here also we obtain a non-parametric test statistic which is a U-statistic. The asymptotic theory of the test statistic is discussed in this chapter. We also discuss the test for constant hazard rate against the NBUE alternatives. A Simulation study is carried out to find the performance of various test statistics.

In Chapter 6, we provide a brief overview of the new definition of reversed hazard rate in discrete case defined by

$$\lambda^*(x) = \ln \frac{F(x)}{F(x-1)}, \quad x = 1, 2, \ldots. \tag{1.7}$$

We also study the properties of this new reversed hazard rate. This definition enables us to define the proportional reversed hazards model in discrete domain as parallel to continuous case. Then we discuss the properties of the proportional reversed hazards model. This includes preservation of ageing classes and stochastic comparisons under the proposed model. Also, we develop a test procedure to test the proportionality constant. That is, for the model

$$F^*(x) = (F(x))^\theta, \quad \theta > 0,$$

we consider the testing problem

$$H_0 : \theta = 1 \quad \text{against} \quad H_1 : \theta > 1.$$

A non-parametric test based on U-statistic is developed for the above testing problem. The asymptotic properties of the test statistic are studied.

Finally, Chapter 7 summarizes major contributions of the present study and discusses future works to be carried out in this area.

Chapter 2

Basic Concepts

2.1 Introduction

Many times a product lifetime can be described through a non-negative integer-valued random variable. For example, a piece of equipment operates in cycles and the experimenter observes the number of cycles successfully completed prior to failure. A frequently referred example is a photocopying machine whose life length would be the total number of copies it produces before failure. Another example is the length of the hospital stay of patients who were hospitalized due to an accident. It is important to develop parallel theory for discrete lifetime analogous to continuous setup.

2.2 Characterizations by relationships between reliability measures

In studying the lifetime of a device or an organism, the concepts based on remaining life based on the current age are effectively used in reliability and survival analysis to infer properties of the underlying life distribution. In this context, the characterizations of life distributions based on the properties of various functions of remaining life such as its mean, median, percentiles, variance etc. have been extensively studied in literature (see Nair and Kattumannil (2006, 2010) and references therein).

Let $h(.)$ be a Borel measurable function of the random variable X such that $E(h^2(X)) \leq \infty$. The mean and the variance of $h(X)$ is given by

$$\mu_h(x) = E(h(X)) \quad \text{and} \quad \sigma_h^2(x) = V(h(X)).$$

We denote the conditional expectations of $h(X)$ as

$$m_h(x) = E(h(X)|X > x) \tag{2.1}$$

and

$$r_h(x) = E(h(X)|X \leq x). \tag{2.2}$$

When $h(X) = X - x$, $m_h(x)$ gives the mean residual life $m(x)$ (MRL) and when $h(X) = x - X$, $r_h(x)$ gives the reversed mean residual life $r(x)$ (RMRL). Since $m_h(.), r_h(.), k(.)$ and $\lambda(.)$ uniquely determine the distribution of X, their functional forms are extensively used in modeling lifetime data. For many of the discrete distributions neither of the functions $m_h(.), r_h(.), k(.)$ and $\lambda(.)$ have simple tractable forms making analytic manipulations and comparisons of ageing behaviour based on them difficult. This has prompted reliability analysts to develop relationships between $k(.)$ and $m_h(.)$ or $\lambda(.)$ and $r_h(.)$ that characterize different lifetime models.

It seems that the relationships between conditional expectations and hazard rate was first derived in the case of the normal distribution (Kotz and Shanbhag (1980)) in the form

$$E(X|X > x) = \mu + \sigma^2 k(x), \tag{2.3}$$

where $\mu = E(X)$ and $\sigma^2 = V(X)$.

Since then similar results were proved for individual distributions and families in both discrete and continuous cases by various researchers. We use the same notations $k(.)$ and $\lambda(.)$ to denote the hazard rate and the reversed hazard rate, respectively in both continuous and discrete cases.

Nassar and Mahmoud (1985) characterized the mixture of exponential distributions each with means μ_1 and μ_2 by the relation

$$m_{(1)}(x) = x + (\mu_1 + \mu_2) - \mu_1 \mu_2 k(x), \tag{2.4}$$

where $m_{(1)}(x) = E[X|X > x]$ and $k(x)$ is the hazard rate. However the flow of the work in this area was spurred by Osaki and Li (1988) where they characterized the gamma distribution with parameters (α, β) by means of the identity

$$m_{(1)}(x) = \mu + \frac{x}{\alpha} k(x), \quad \forall x > 0 \tag{2.5}$$

and negative binomial distribution with parameters r and θ by

$$m_{(1)}(x) = \mu + \frac{(x + 1 - r)}{\theta} k(x + 1), \quad \forall x \geq r - 1, \tag{2.6}$$

where $\mu = E(X)$. Following Osaki and Li (1988) similar results were developed for other individual distributions as well as for classes of probability distributions and we discuss them in a chronological order. Adatia et al. (1991) gave a necessary and sufficient condition that a continuous, positive random variable following a gamma distribution with parameters (α, β) in terms of the conditional moments and an expression involving its hazard rate is in the form

$$
\begin{aligned}
m_{(k)}(x) \quad = \quad & \frac{(\beta+k-1)(\beta+k-2)\ldots(\beta+1)\beta}{\alpha^k} \\
& +\sum_{i=1}^{k-1} \frac{(\beta+k-1)(\beta+k-2)\ldots(\beta+i)\beta}{\alpha^{k+1-i}} x^i k(x) + \frac{1}{\alpha} x^k k(x),
\end{aligned}
\tag{2.7}
$$

where

$$
m_{(k)}(x) = E\left[X^k | X > x\right] \text{ for } k = 1, 2, \ldots \tag{2.8}
$$

The result was then used to develop a characterization of a mixture of two gamma distributions. Characterizations of beta, binomial, and Poisson distributions were presented by Ahmed (1991). He showed that for the Pearson type I distribution with probability density function

$$
f(x) = \frac{(x-a)^{\alpha-1}(b-x)^{\beta-1}}{B(\alpha,\beta)(b-a)^{\alpha+\beta-1}}, \quad a < x < b, \quad a \geq 0, b, \alpha, \beta > 0, \tag{2.9}
$$

satisfies

$$
m_{(1)}(x) = \mu\left(1 + \frac{(x-a)(b-x)k(x)}{a\beta + b\alpha}\right), \quad \forall a \leq x \leq b, \tag{2.10}
$$

and in particular the beta distribution is uniquely determined by the relationship

$$
m_{(1)}(x) = \mu\left(1 + \frac{x(1-x)k(x)}{\alpha}\right), \tag{2.11}
$$

obtained by setting $a = 0$ and $b = 1$ in (2.10). Further by setting $\beta = 1$, the result for the power distribution is as follows

$$m_{(1)}(x) = \mu \left(1 + \frac{x\ (1-x)\, k\,(x)}{m}\right), \tag{2.12}$$

with $\mu = \frac{\alpha}{(\alpha+1)}$. Ahmed (1991) also established that X is distributed as binomial distribution with probability mass function

$$p\,(x) = \binom{n}{x} \theta^x (1-\theta)^x,\ x = 0, 1, \ldots, n \tag{2.13}$$

if and only if

$$m_{(1)}(x) = \mu + (1+x)\,(1-\theta)\,k(x+1) \tag{2.14}$$

and X is Poisson distribution with probability mass function

$$p\,(x) = \frac{e^{-\eta}\eta^x}{x!},\ x = 0, 1, \ldots \tag{2.15}$$

if and only if

$$m_{(1)}(x) = \mu + (1+x)\,k(x+1)\ x = 0, 1, \ldots. \tag{2.16}$$

In a more general framework, Nair and Sankaran (1991) showed that the distribution of X belongs to the Pearson family specified by

$$\frac{f'(x)}{f(x)} = \frac{-(x+d)}{a_0 + a_1 x + a_2 x^2} \tag{2.17}$$

if and only if

$$m_{(1)}(x) = \mu + (b_0 + b_1 x + b_2 x^2)k(x), \tag{2.18}$$

where a_i's and b_i's, $i = 0, 1, 2$ are real constants with

$$b_i = a_i(1 - 2a_2)^{-1},\ i = 0, 1, 2 \text{ and } a_2 \neq \frac{1}{2},$$

provided that $\lim_{x \to a} x^2 f(x) = 0$. As a discrete analogue of (2.17), they also proved that in the support of the set of integers, X has distribution in the Ord family satisfying

$$\frac{p(x+1) - p(x)}{p(x)} = \frac{-(x+d)}{a_0 + a_1 x + a_2 x^2} \tag{2.19}$$

if and only if

$$m_{(1)}(x) = \mu + (c_0 + c_1 x + c_2 x^2)k(x+1), \tag{2.20}$$

where

$$c_i = (1 - 2a_2)^{-1} a_i,\ a_2 \neq \frac{1}{2},\ i = 0, 1, 2$$

and deduced the results available in literature in this contexts as particular cases. Another extension of (2.18) for the Pearson family by Glanzel (1991), involved higher order conditional moments resulting in the characteristic property

$$E\left(X^2 | X > x\right) = Q_1(x) E\left(X | X \geq x\right) + Q_2(x), \tag{2.21}$$

where $Q_1(.)$ and $Q_2(.)$ are polynomials of degree at most one with real coefficients. Characterization of Koicheva (1993) about gamma distribution also involved higher order moments and hazard rate.

Ruiz and Navarro (1994) established that the class of distributions satisfied by the differential equation of the form

$$\frac{f'(x)}{f(x)} = \frac{c - x - q'(x)}{q(x)}, \tag{2.22}$$

is characterized by the identity

$$m_{(1)}(x) = c + q(x)\,k(x), \tag{2.23}$$

where c is a constant, $q(.)$ is a function satisfying $\lim\limits_{x \to b} q(x)\,f(x) = 0$ with end points a and b in the support of X and $\int_a^b q'(x)\,f(x)dx < \infty$. The results of Shanbhag (1970), Osaki and Li (1988), Ahmed (1991), Nair and Sankaran (1991) are special cases of (2.22) and (2.23). Ruiz and Navarro (1994) also discussed the discrete analogues of the above results through illustrative examples.

Consul (1995) obtained a general theorem, based on conditional expectation, for the exponential class of distributions. The theorem is then applied to numerous discrete and continuous probability distributions of the exponential class providing specific characterizations for each one of them. He showed that the random variable X, continuous or discrete, has a distribution belonging to the exponential class of the form

$$f(x;\theta) = \exp\left[xQ(\theta) + T(x) + S(\theta)\right], \tag{2.24}$$

where $T(.)$ is a real-valued measurable function, $Q(.)$ and $S(.)$ are real functions on $\mathbb{R}$ with $Q(\theta)$ having continuous non-vanishing derivatives in $\mathbb{R}$, if and only if

$$m_{(1)}\left(x\right)=\mu+\left(Q'(\theta)\right)^{-1}\frac{\partial}{\partial\theta}\left(\log R\left(x\right)\right). \tag{2.25}$$

Note that the equation (2.25) can be written as

$$m_{(1)}\left(x\right)=\mu+\left(f\left(x\right)Q'(\theta)\right)^{-1}\frac{\partial R\left(x\right)}{\partial\theta}k\left(x\right), \tag{2.26}$$

so that the characterization of exponential family is in fact in terms of hazard rate and the conditional expectations. The characterizations for individual distributions including Lagrangian Poisson, generalized Poisson, Poisson, Borel-Tanner, generalized negative binomial, negative binomial, binomial, beta and gamma distributions were deduced from the above results. One can refer Osaki and Li (1988) and Ahmed (1991) respectively for the characterization of negative binomial and Poisson distributions.

Ghitany et al.(1995) established a characterization result for an absolutely continuous random variable whose density function is of the form

$$f\left(x\right)=\exp\left(-q\left(x,\theta\right)\right),\quad x\geq 0,\theta\in\Theta, \tag{2.27}$$

where $q(.,.)$ is a real-valued function with $q'(x,\theta)\neq 0$ and $q''(x,\theta)$, the derivatives of $q(x,\theta)$ w.r.t. θ, exists on $(0,\infty)$ for all $\theta\in\Theta$, by the identity

$$E\left[1+\frac{q\left(X,\theta\right)}{\left(q'\left(X,\theta\right)\right)^{2}}s\left(X,\theta\right)-\frac{s'\left(X,\theta\right)}{q'\left(X,\theta\right)}|X\geq x\right]=\frac{s\left(x,\theta\right)}{q'\left(x,\theta\right)}k\left(x\right), \tag{2.28}$$

with $s(x,\theta) \neq 0$ is a real-valued function and derivative w.r.t. θ, $s'(x,\theta)$ exists on $(0,\infty)$ for all $\theta \in \Theta$.

Particularly for gamma distributions with parameters (λ,β)

$$E\left[X^{k-1}\left(X-\frac{\lambda+k-1}{\beta}\right)|X \geq x\right]=\frac{x^k}{\beta}k\left(x\right). \tag{2.29}$$

Putting $k=1$, (2.29) reduces to the result of Osaki and Li (1988). The distribution of X follows Weibull distribution with

$$f\left(x\right)=\lambda\beta x^{\beta-1}\exp\left(-\lambda x^{\beta}\right), \quad x \geq 0, \lambda, \beta > 0 \tag{2.30}$$

if and only if

$$E\left[X^{\beta}|X \geq x\right]=\frac{1}{\lambda\beta}\left[xk\left(x\right)+\beta\right], \tag{2.31}$$

holds for all $x \geq 0$. And for the Gompertz distribution with probability density function given by

$$f\left(x\right)=\lambda\exp\left(\beta x\right)\exp\left[-\frac{\lambda}{\beta}\left[\exp\left(\beta x\right)-1\right]\right] \quad x \geq 0, \lambda, \beta > 0, \tag{2.32}$$

the identity ((2.28)) reduces to

$$E\left[\exp\left(\beta X\right)|X \geq x\right]=\frac{1}{\lambda}\left[\left(1-\exp(\beta x)\right)k(x)+\lambda+\beta\right]. \tag{2.33}$$

The corresponding results for exponential and Rayleigh distributions were derived as special cases of (2.31). Navarro et al. (1998) gave a general method to obtain a distribution function

$F(.)$ through the moment of the residual life defined by $m^*_{(k)}(x) = E\left[(X-x)^k|X > x\right]$ and studied the characterization using the relation

$$m^*_{(k)}(x) = c + g\left(x\right)k\left(x\right), \tag{2.34}$$

where c is a constant. Extending the results given in Adatia et al. (1991), Koicheva (1993) and Ghitany et al. (1995) showed that the random variable X with differentiable density in its support (a, b) is of the form

$$\frac{f'\left(x\right)}{f\left(x\right)} = \frac{\left(c - g'\left(x\right) - x^k\right)}{g\left(x\right)} \tag{2.35}$$

if and only if

$$m_{(k)}(x) = c + g\left(x\right)k\left(x\right), \tag{2.36}$$

where c is a constant and $g(.)$ is a real function satisfying $\lim_{x\to b} g\left(x\right)f(x) = 0$.

For $k = 1$, Nair and Kattumannil (2006) obtained Theorem 3 given in Ruiz and Navarro (1994) for relations of the form (2.23) as well as particular characterizations given in Kotz and Shanbhag (1980), Osaki and Li (1988), Ahmed (1991) and Nair and Sankaran (1991) for some usual distributions. In a similar direction, Sankaran and Nair (2000) obtained a further extension of their (Nair and Sankaran (1991)) and Glanzel's (1991) characterizations for Pearson family. They delivered a necessary and sufficient condition for the distribution of X to belong to the Pearson system by the identity

$$m_{(k)}(x) = \left(a_{0,r} + a_{1,r}x + a_{2,r}x^2\right)x^{r-1}k\left(x\right) + d_r m_{(k)}(x) + a_{0,r}\left(r-1\right)m_{(k-2)}(x) \tag{2.37}$$

provided that $\lim_{x \to a} x^r f(x) = 0$, where

$$d_r = \frac{b_i r - d}{1 - (1+r)\, b_2},\ b_2 \neq \frac{1}{r+1},\ i = 0, 1, 2$$

and

$$a_{i,r} = b_i (1 - (1+r)\, b_2)^{-1}, i = 0, 1, 2.$$

Nair et al. (1999) obtained a relation of the form

$$m_{(1)}\,(x) = x + \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} \right) - \frac{1}{\theta_1 \theta_2} k(x+1),\ 0 < \theta_1, \theta_2 < 1 \tag{2.38}$$

to characterize the mixture of geometric law with probability mass function

$$p\,(x) = \alpha \theta_1 (1 - \theta_1)^x + (1 - \alpha)\, \theta_2 (1 - \theta_2)^x,\ 0 < \alpha < 1. \tag{2.39}$$

They also proved similar results for Waring distribution and extended this approach in several directions which include the higher order moments and factorial moments. Abraham and Nair (2001) established an identity connecting the hazard rate and the mean residual life to characterize a class of continuous distributions containing finite mixtures of exponential, Lomax and beta distributions. The identity

$$m_{(1)}(x) = x + (1 + ax)\,(\mu_1 + \mu_2 + a\mu_1\mu_2) - \mu_1\mu_2 (1 + ax)^2 k(x) \tag{2.40}$$

is satisfied for all x for a random variable X with density

$$f(x) = \theta f_1(x) + (1 - \theta)\, f_2(x), \quad x \in (a, b), a, b \in \mathbb{R}, \quad 0 < \theta < 1 \tag{2.41}$$

if and only if for $i = 1, 2$, the component densities are

$$f_i(x) = \lambda_i \exp(-\lambda_i x), \ \lambda_i > 0, \ x > 0, \ \text{for } a = 0$$

$$f_i(x) = \alpha_i \beta^{\alpha_i} (x + \beta)^{-(\alpha_i+1)}, \ \alpha_i, \ \beta > 0, x > 0, \ \text{for } a > 0$$

$$f_i(x) = \frac{C_i}{R}\left(1 - \frac{x}{R}\right)^{C_i - 1}, \ \mathrm{C}_i, \ R > 0, \ 0 < x < R, \ \text{for } a < 0, \tag{2.42}$$

where $\mu_i = \int_0^\infty x f_i(x) dx$.

Among the results discussed above, the exponential case as appeared in Nassar and Mahmoud (1985) is contained in the above formulation. A further generalization aimed at including more distributions than in (2.17) has appeared in Sankaran et al. (2003) that extended the Pearson family by replacing the linear functions in the numerator on the right of (2.17) with a quadratic function $b_0 + b_1 x + b_2 x^2$ to claim the characteristic property

$$E\left[\left(b_2 X^2 + (b_1 + 2a_2)X + b_0 + a_1\right) | X > x\right] + \left(a_0 + a_1 x + a_2 x^2\right) k(x) = 0, \tag{2.43}$$

provided that $\lim_{x \to b} x^r f(x) = 0, \ r = 0, 1, 2$ so that for a choice of

$$h(x) = \alpha x^2 + \beta x + \gamma$$

with

$$\alpha = b_2, \ \beta = b_1 + 2a_2 \ \text{ and } \ \gamma = b_0 + a_1 + \mu.$$

The above equation can be written as

$$m_h(x) = E(h(X) | X > x) = \mu + g_h(x) k(x), \tag{2.44}$$

where $g_h(x) = -\left(a_0 + a_1 x + a_2 x^2\right)$. Sindhu (2003) provided the discrete analogue of (2.44) which states that the random variable X belongs to the family of distributions satisfying

$$\frac{p(x+1) - p(x)}{p(x)} = \frac{-\left(b_0 + b_1 x + b_2 x^2\right)}{a_0 + a_1 x + a_2 x^2} \tag{2.45}$$

if and only if

$$m_{(1)}\left(x\right) = \mu + g_h\left(x\right) k\left(x+1\right), \tag{2.46}$$

where $g_h(x) = (a_0 + a_1 x + a_2 x^2)$ and $h(x) = Q_1 x^2 + Q_2 x + Q_3$, with $Q_1 = b_2,\ Q_2 = b_1 + 2a_2$ and $Q_3 = a_1 - a_2 + \mu$. More recently, Gupta and Bradley (2003) characterized the class of distributions satisfying

$$\frac{f'\left(x\right)}{f\left(x\right)} = \frac{\left(\mu - x\right)}{g\left(x\right)} - \frac{g'\left(x\right)}{g\left(x\right)} \tag{2.47}$$

with the identity given by

$$m_{(1)}\left(x\right) = \mu + g\left(x\right) k\left(x\right), \tag{2.48}$$

where μ is a constant and g satisfies the first-order linear differential equation

$$g'\left(x\right) + \frac{f'\left(x\right)}{f\left(x\right)} g\left(x\right) = \mu - x.$$

The characterizations based on (2.47) and (2.48) were discussed in Ruiz and Navarro (1994). But Gupta and Bradley (2003) also emphasized the ageing behaviour concerning the family (2.47). Nair et al. (2005) developed a characterization for continuous probability distributions using the relationship between the reversed hazard rate and the conditional expectations, which generalizes the results of Navarro and Ruiz (2004). A continuous random variable X in the support (a, b) has a probability density function of the form

$$f(x) = \frac{c}{g_h(x)} \exp\left[-\int_a^x \frac{h(t) - \mu}{\mu g_h(t)} dt\right], \tag{2.49}$$

where $h(.)$ is a function such that $E(h^2(X)) < \infty$ and $E(h(X)) = \mu$ and $g_h(.)$ is a real function and c is a constant which makes $f(x)$ a density, if and only if

$$r_h(x) = \mu\left(1 - g_h(x)\lambda(x)\right) \tag{2.50}$$

provided $\lim_{x\to\infty} g_h(x)f(x) = 0$.

The discrete analogue of the above work has been introduced by Gupta et al. (2006) and applied to Ord family and modified power series family for characterizing them. Let X be a discrete random variable defined on $\mathbb{N} = \{1, 2, ...\}$ with $\mu = E(h(X)) < \infty$, where $h(.)$ is a real function, then for a real function, $g_h(.)$ the condition

$$m_h(x) = \mu\left[1 - g_h(x)\lambda(x)\right] \tag{2.51}$$

is satisfied for all x in $\mathbb{N}$ if and only if the probability mass function $p(.)$ of X satisfies the difference equation

$$\frac{p(x+1)}{p(x)} = \frac{\mu g_h(x)}{h(x+1) - \mu\left(1 - g_h(x+1)\right)}. \tag{2.52}$$

The results of Nair et al. (2005) and Gupta et al. (2006) subsume all the available results in literature discussed in this context for specified values of $h(x)$ and $g_h(x)$. Further these papers enlightened the role of reversed hazard rate and the right truncated expectation of $h(X)$ in characterizing life distributions, which was not considered by the earlier authors.

Nair and Kattumannil (2006) consider the characterization problem associated with the class of continuous distribution specified by

$$\frac{f'(x)}{f'(x)} = \frac{-g_h'(x)}{g_h'(x)} + \frac{\mu - h(x)}{\sigma g_h(x)}, \tag{2.53}$$

in terms of variance bound and established the equivalence characterization in terms of the relationship between conditional expectation and hazard rate. Nair and Kattumannil (2008) characterize the class of discrete probability distributions specified by

$$\frac{p\,(x+1)}{p\,(x)} = \frac{\sigma g_h\,(x)}{h\,(x+1) - \sigma\,(1 - g_h\,(x+1))}, \tag{2.54}$$

by the identities

$$m_h\,(x) = \mu + \sigma g_h(x) k(x). \tag{2.55}$$

and

$$r_h\,(x) = \mu - \sigma g_h(x) \lambda(x). \tag{2.56}$$

Nair and Kattumannil (2008) also established a link between the characterization in terms of (2.55) and (2.56) and the characterization based on variance bound given by

$$V(c(X)) \geq E^2(c'(X) g_h(X)). \tag{2.57}$$

Motivated from Nair and Kattumannil (2008) we study the characterization problem associated with the truncated random variable which enables us to characterize the distribution based on reversed variance residual life. In the next section we discuss the characterization problem associated with variance residual life and reversed variance residual life.

2.3 Characterization by properties of conditional variance

Associated with a non-negative random variable X with distribution function $F(.)$, the MRL and RMRL are extensively used in modelling and analysis in reliability. Other than the mean, a second characteristic of the residual life that plays a similar role in identifying life distributions and distinguishing them is the variance residual life (VRL), defined as

$$v(x) = V(X - x|X > x) \tag{2.58}$$

and was first discussed by Launer (1984) while classifying life distributions based on the monotonic behaviour of VRL. Its relationships with the MRL, conditions under which increasing (decreasing) VRL classes exist, etc., are discussed in Gupta (1987) and Gupta et al. (1987). Mukherjee and Roy (1986) used relations of the form

$$v\left(x\right) = c\left(m_{(1)}(x) - x\right) \text{ and } h\left(x\right)\left(m_{(1)}(x) - x\right) = c,$$

where c a real constant, to characterize the exponential, Pearson type XI and finite range distributions. As a discrete analogue of the results of Mukherjee and Roy(1986), Hitha and Nair (1989) showed that the relations

$$v\left(x\right) = c\left(m_{(1)}(x) - x\right)\left(m_{(1)}(x) - x - 1\right) \text{ and } h(x)(m_{(1)}(x) - x) = c,$$

uniquely determine geometric, negative hyper-geometric and Waring distributions for $c = 1$, $c < 1$ and $c > 1$ respectively. Gupta et al.(1987) have studied conditions under which the variance residual life function is monotone and showed that the decreasing mean residual life (DMRL) class is contained in the decreasing variance residual life (DVRL) class. Gupta and Kirmani (1987) investigated the connection between MRL and VRL for the equilibrium distributions. Gupta (1987) studied the monotonic behavior of VRL in terms of the residual coefficient of variation defined by $s\left(x\right) = \sqrt{v\left(x\right)}/m\left(x\right)$, where $m(.)$ is the MRL, while the square of the residual coefficient of variation is the interest of Gupta and Kirmani (1988). Bounds on the residual moments and residual variance are obtained by Gupta and Kirmani (1990) and some examples were furnished as illustrations. Gupta and Kirmani (2000) showed that $v(x)$ and $s(x)$ characterize the life distributions in univariate as well as in bivariate case and proved that the constant $v(x)$ characterizes the univariate exponential distribution. They also studied the monotonic behaviour of $s(x)$ in terms of convex (concave) nature of mean residual life function. Gupta and Kirmani (2004) established that the ratio of hazard rate and mean residual life characterized the distributions and the result is then used to show that the second residual moment characterizes the distributions. They also discussed the application of the results to non-homogeneous Poisson process. Defining two classes, decreasing variance residual life (V_D) and increasing variance residual life (V_I), Stoyanov and Al-sadi (2004) discussed some ageing pattern of life distributions and then studied the properties of coherent system based on these classes. Kundu and Gupta (2003) provided two simple characterizations theorem on proportional (reversed) hazards model based on conditional variance. They proved that for any real number t such that

$F_X(t) > 0, \lambda_X(t) = \alpha\lambda_Y(t)$ with $\alpha > 0$ if and only if

$$V\left(-\ln F_X(Y)\,|Y < t\right) = \frac{1}{\alpha^2}, \tag{2.59}$$

and for any real number t such that $\bar{F}_X(t) > 0, k_X(t) = \alpha k_Y(t)$ with $\alpha > 0$ if and only if

$$V\left(-\ln \bar{F}_X(Y)\,|Y > t\right) = \frac{1}{\alpha^2}. \tag{2.60}$$

El-Arishi (2005) characterized the exponential family of distributions specified by

$$f(x,\theta) = \exp\left(T(x)Q(\theta) + H(x) + S(\theta)\right), \tag{2.61}$$

where $H(.)$ and $T(.)$ are real-valued Borel-measurable functions, $Q(.)$ and $S(.)$ are real functions on $\mathbb{R}$ with $Q(.)$ having continuous non-vanishing derivatives in $\mathbb{R}$, by the identity

$$\begin{aligned} V(X|X > x) &= \frac{Q''S' - Q'S''}{(Q')^3} - \frac{Q''}{(Q')^3}\frac{\partial logR(x-1)}{\partial\theta} \\ &\quad + \frac{1}{(Q')^2}\frac{\partial^2 logR(x-1)}{\partial\theta^2}, \end{aligned} \tag{2.62}$$

where S', Q', S'' and Q'' are respectively first and second derivatives of S and Q with respect to θ.

As an illustrative example, El-Arishi (2005) proved that the random variable X follows binomial distribution with parameters n and θ if and only if

$$\begin{aligned} V(X|X > x) &= (x+1)(1-\theta)(x+1-\theta(n+1))k(x+1) \\ &\quad -(x+1)^2k^2(x+1) + n\theta(1-\theta). \end{aligned} \tag{2.63}$$

Also the Poisson random variable with mean η is characterized by the identity

$$V(X|X > x) = \eta + (x+1)(x+1-\eta)k(x+1) \\ -(x+1)^2k^2(x+1). \tag{2.64}$$

The relationship between VRL, MRL and discrete coefficient of variation can be found in Khorashadizadeh et al. (2010).

In continuous setup, Nair and Kattumannil (2010) studied the properties of the left and right truncated variance of a function of a non-negative random variable, that characterize a class of continuous distributions. Those properties include characterizations by relationships the conditional variance has with truncated expectations and/or the hazard rate as well as lower bound to the conditional variance. Various results in literature become special cases of their formula and consequently they produce characteristic properties of families of distributions as well as individual models. Nair and Kattumannil (2010) showed that the characteristic properties are linked to those based on relationship between conditional means and hazard rate, discussed in the Section 2.2. As discrete analogue, Kattumannil and Nair (2010) discussed the properties of VRL for a non-negative integer-valued random variable.

2.4 Ageing concepts for discrete data

Classification of classes of lifetime distributions based on the notion of ageing has important role in reliability analysis as it helps in identification of the underlying model (see Barlow and Proschan (1981)). The ageing notions in continuous setup has been extensively studied in literature, see Lai and Xie (2006) and Shaked and Sahnthikumar (2007). Some work has been carried out when lifetimes are discrete random variables. When X is discrete, $-\ln R(x) \neq \sum_{i=1}^{x} k(i)$, that creates some difficulties while studying the discrete ageing notions. For example, the concepts like IFRA and NBU have no unique definition. Motivated from these problems Xie et al. (2002) gave a new definition of hazard rate $k^*(x)$ given in (1.4). Note that $k^*(x)$ can be considered as the discrete analogue of the hazard rate in the continuous setup. The new hazard rate is not bounded by one and additive for series system. In view of the new definition of hazard rate there is a need to re-look the ageing classes for discrete random variables. Recently, Dewan and Kattumannil (2011) have discussed certain ageing properties of the discrete hazard rate. They obtained several implications between various classes indicating discrete ageing. Stochastic ordering between two discrete random variables is also addressed in Dewan and Kattumannil (2011). In of view of the new definition of hazard rate, we have the following definition of the discrete ageing classes

Definition 2.1. *(Dewan and Kattumannil (2011)). Let X be a discrete random variable defined on* $\mathbb{N}$*, then*

(i) X *is said to be increasing (decreasing) in likelihood ratio (ILR (DLR)) if* $f(x)$ *is log concave (convex), ie. if* $p(x+1)p(x-1) \leq (\geq) p^2(x)$ *for all* x *in* $\mathbb{N}$.

(ii) X *is said to be increasing (decreasing) hazard rate (IHR (DHR)) if* $R(x)$ *is log concave (convex), ie. if* $R(x+1)R(x-1) \leq (\geq) R^2(x)$ *for all* x *in* $\mathbb{N}$.

(iii) X *is said to be increasing (decreasing) hazard rate average (IHRA (DHRA)) if* $\frac{1}{x}\sum_{y=1}^{x} k^*(y)$ *is increasing (decreasing) in* x *for all* x *in* $\mathbb{N}$.

(iv) X *is said to be decreasing (increasing) mean residual life (DMRL (IMRL)) if* $m(x) = E(X - x | X > x)$ *is decreasing (increasing) in* x *for all* x *in* $\mathbb{N}$.

(v) X *is said to be new better (worse) than used (NBU (NWU)) if* $R(x+k) \leq (\geq) R(x)R(k)$ *for all* x, k *in* $\mathbb{N}$.

(vi) X is said to be new better (worse) than used in expectation (NBUE (NWUE)) if $m(x) \leq (\geq)\mu$ for all x in $\mathbb{N}$, where μ is the mean of X.

Remark 2.1. *Most of the text books give two definitions for IHRA (DHRA) and NBU (NWU) notions in discrete setup. Here we stated a unique definition in view of the new hazard rate $k^*(x)$ given in (1.4). For recent discussions on these aspects see Lai and Xie (2006) and Dewan and Kattumannil (2011).*

Next we states some results, which give the chain of implications between these ageing concepts that will be useful for our discussions. For the proofs see Dewan and Kattumannil (2011).

Theorem 2.1. *The random variable X has ILR (DLR) implies that it has IHR (DHR).*

Remark 2.2. *The converse of the above theorem is not true.*

Lemma 2.1. *The random variable X is IHR (DHR) if and only if $\frac{R(x-1)}{R(x)}$ is increasing (decreasing) for all $x \in \mathbb{N}$.*

The following Lemma 2.2 helps to classify the distribution into DMRL (IMRL) classes based on the ratio of reliability function.

Lemma 2.2. *A sufficient condition that the random variable X has DMRL (IMRL) is that the sequence $\langle s(x) \rangle$ is decreasing(increasing) for all $x \in \mathbb{N}$, where $s(x) = R(x)/R(x-1)$.*

Theorem 2.2. *The random variable X has IHR (DHR) implies it has IHRA (DHRA).*

Theorem 2.3. *The random variable X has IHRA (DHRA) implies it has NBU (NWU).*

Theorem 2.4. *The random variable X has NBU (NWU) implies it has NBUE (NWUE).*

Theorem 2.5. *The random variable X has DMRL (IMRL) implies it has NBUE (NWUE).*

From above deliberation, as in the continuous case, we have the following chain of implications of the above mentioned classes.

$$ILR \Rightarrow IHR \Rightarrow IHRA \Rightarrow NBU \Rightarrow NBUE \tag{2.65}$$

and

$$ILR \Rightarrow IHR \Rightarrow DMRL \Rightarrow NBUE. \tag{2.66}$$

Now we discuss stochastic ordering of two discrete random variables.

Definition 2.2. *(Dewan and Kattumannil (2011)). Suppose that X_1 and X_2 are two random variables with corresponding probability mass functions $p_1(.)$ and $p_2(.)$ respectively, then*
(i) X_2 is said to be smaller than X_1 in the likelihood ratio order ($X_1 \geq_{lr} X_2$) if $p_1(x)/p_2(x)$ is increasing for all x in $\mathbb{N}$.
(ii) X_2 is said to be smaller than X_1 in the hazard rate order ($X_1 \geq_{hr} X_2$) if $R_1(x)/R_2(x)$ increases for all x in $\mathbb{N}$, where $R_1(.)$ and $R_2(.)$ are the reliability functions of X_1 and X_2 respectively.
(iii) X_2 is said to be smaller than X_1 in the reversed hazard rate order ($X_1 \geq_{rhr} X_2$) if $F_1(x)/F_2(x)$ increases for all x in $\mathbb{N}$, where $F_1(.)$ and $F_2(.)$ are the distribution functions of X_1 and X_2 respectively.
(iv) X_2 is said to be smaller than X_1 in the stochastic order ($X_1 \geq_{st} X_2$) if $R_1(x) \geq R_2(x)$ for all x in $\mathbb{N}$.
(v) X_2 is said to be smaller than X_1 in the mean residual life order ($X_1 \geq_{mrl} X_2$) if $E(X_1 - x|X_1 > x) \geq E(X_2 - x|X_2 > x)$ for all x in $\mathbb{N}$.
(vi) X_2 is said to be smaller than X_1 in the reversed mean residual life order ($X_1 \geq_{rmrl} X_2$)

if $E(x - X_1|X_1 \leq x) \leq E(x - X_2|X_2 \leq x)$ *for all* x *in* $\mathbb{N}$.

Next we state some results related to stochastic ordering between two random variables, the proof can be found in Dewan and Kattumannil (2011).

Theorem 2.6. *(i)* $X_1 \geq_{lr} X_2$ *implies* $X_1 \geq_{hr} X_2$.

(ii) $X_1 \geq_{lr} X_2$ *implies* $X_1 \geq_{rhr} X_2$.

(iii) $X_1 \geq_{hr} X_2$ *implies* $X_1 \geq_{st} X_2$.

(iv) $X_1 \geq_{rhr} X_2$ *implies* $X_1 \geq_{st} X_2$.

(v) $X_1 \geq_{hr} X_2$ *implies* $X_1 \geq_{mrl} X_2$.

(vi) $X_1 \geq_{rhr} X_2$ *implies* $X_1 \geq_{rmrl} X_2$.

Hence we have the following chain of implications in the discrete domain which is well-known in the continuous case.

$$X_1 \geq_{lr} X_2 \quad \text{implies} \quad X_1 \geq_{hr} X_2 \quad \text{implies} \quad X_1 \geq_{st} X_2$$

$$X_1 \geq_{hr} X_2 \quad \text{implies} \quad X_1 \geq_{mrl} X_2.$$

$$X_1 \geq_{lr} X_2 \quad \text{implies} \quad X_1 \geq_{rhr} X_2 \quad \text{implies} \quad X_1 \geq_{st} X_2$$

$$X_1 \geq_{rhr} X_2 \quad \text{implies} \quad X_1 \geq_{rmrl} X_2.$$

While studying the ageing notions discussed above, geometric distribution has special characteristics. The geometric distribution is a common discrete distribution used to model the lifetime of a device. Lui (1997) deliberated both the point and interval estimation of the reliability function of the geometric distribution. Testing the equality of failure probability per time unit among several comparison groups for the geometric distribution is considered by Salvia (1984) and Vit (1974). Recently, Sarhan et al. (2007) proposed the estimators of reliability measures in geometric distribution model using dependent masked system life test data.

A discrete random variable X is said to have a geometric distribution if the probability mass function is given by

$$P(X = x) = p(1-p)^{x-1},\ x \geq 1, \tag{2.67}$$

for some $p \in (0,1)$. Let $q = 1 - p$. The cumulative distribution function $F(.)$ and reliability fucntion $R(.)$ are

$$F(x) = (1 - q^x), \quad x = 1, 2, ..., \text{ and } R(x) = q^x, \quad x = 1, 2, ...$$

The moments of the geometric random variable can be obtained as (Casella and Berger (1990))

$$E(X) = \frac{1}{p} \tag{2.68}$$

$$E(X^2) = \frac{1+q}{p^2} \tag{2.69}$$

$$E(X^3) = \frac{6q + p^2}{p^3} \tag{2.70}$$

$$E(X^4) = \frac{(1+q)(12q + p^2)}{p^4}. \tag{2.71}$$

Now from (1.4), we can obtain the hazard rate

$$k^*(x) = \ln \frac{R(x-1)}{R(x)} = -\ln q. \tag{2.72}$$

The mean residual life for geometric random variable is given by

$$m(x) = E(X - x|X > x) = \frac{\sum_{y=x}^{\infty} R(y)}{R(x)} = \frac{1}{p}. \tag{2.73}$$

Note that geometric distribution has constant hazard rate and MRL and constant hazard rate characterizes geometric distribution.

Now for the testing problem discussed in Chapters 4 and 5, we need the knowledge of powerful technique of non-parametric inference known as U-statistic. Now we discuss this technique.

2.5 U-statistics

Definition 2.3. *Let $\mathcal{F}$ be a family of distribution and let $X_1, X_2, ..., X_n$ be a sample from the distribution $F \in \mathcal{F}$. A parameter θ is said to be estimable of degree m for the family of distributions $\mathcal{F}$ if m is the smallest sample size for which there exists a function $h(x_1, x_2, ..., x_m)$ such that*

$$E_F(h(X_1, X_2, ..., X_m)) = \theta \tag{2.74}$$

for every $F \in \mathcal{F}$.

The function $h(.)$ defined in (2.74) is known as the kernel for the parameter θ.

For any kernel $h(.)$, we can always create one that is symmetric in its argument and unbiased for θ by using

$$h^*(X_1, X_2, ..., X_m) = \frac{1}{r!} \sum_{\mathcal{A}} h(X_{\alpha_1}, X_{\alpha_2}, ..., X_{\alpha_m}), \tag{2.75}$$

where $\alpha_1, \alpha_2, ..., \alpha_m$ is a permutation of the numbers $1, 2, ..., m$ and $\mathcal{A}$ is the set of all permutations $(\alpha_1, \alpha_2, ..., \alpha_m)$ of the integers $(1, 2, ..., m)$. Hence, without loss of generality, we shall assume that the kernel $h(.)$ is symmetric.

A U-statistic for the estimable function θ is constructed with symmetric kernel $h(.)$ by forming

$$U(X_1, ..., X_n) = \frac{1}{\binom{n}{m}} \sum_{\underline{\beta} \in \mathcal{B}} h(X_{\beta_1}, X_{\beta_2}, ..., X_{\beta_m}), \tag{2.76}$$

where $\underline{\beta} = (\beta_1, \beta_2, ..., \beta_m)$ is a combination of m integers from $(1, 2, ..., n)$ and $\mathcal{B}$ is the set of all such combinations.

Example 2.1. *Let $\mathcal{F}$ denote the collection of all distributions with finite variance σ^2. Then*

$$E_F(X_1^2 - X_1X_2) = \sigma^2 \ \ \forall F \in \mathcal{F}.$$

Thus variance is estimable and of degree 2. The associated symmetric kernel is

$$\begin{aligned} h(x_1, x_2) &= \frac{1}{2}[(x_1^2 - x_1x_2) + (x_2^2 - x_1x_2)] \\ &= \frac{1}{2}(x_1 - x_2)^2. \end{aligned} \tag{2.77}$$

The U-statistic use the symmetric kernel (2.77) to form

$$\begin{aligned} U(X_1, ..., X_n) &= \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} \frac{(X_i - X_j)^2}{2} \\ &= \frac{1}{n-1}[\sum_{i=1}^{n} X_i^2 - n\bar{X}^2] = s^2. \end{aligned} \tag{2.78}$$

2.5.1 Variance of U-statistics

For $c = 0, 1, ..., m$, let

$$h_c(x_1, ..., x_c) = Eh(x_1, ..., x_c, X_{c+1}, ..., X_m), \tag{2.79}$$

where $X_{c+1}, ..., X_m$ are independent and identically distributed with distribution function F. Then $h_0 = \theta$ and $h_m(x_1, ..., x_m) = h(x_1, ..., x_m)$. Let

$$\sigma_c^2 = V(h_c(X_1, ..., X_c)).$$

Lemma 2.1.

$$\sigma_c^2 = Cov(h(X_{\beta_1}, X_{\beta_2}, ..., X_{\beta_m}), h(X_{\beta'_1}, X_{\beta'_2}, ..., X_{\beta'_m})),$$

where $(\beta_1, \beta_2, ..., \beta_m)$ and $(\beta'_1, \beta'_2, ..., \beta'_m)$ are subsets of integers $\{1, 2, ..., n\}$ having exactly c integers in common. If $c = 0$ then the kernels based on β and β' are independent and hence covariance is 0.

Proof. See Hoeffding (1948). □

Now the variance of U-statistic is

$$\begin{aligned} Var(U) &= E\left(\frac{1}{\binom{n}{m}} \sum_{\beta \in \mathcal{B}} h(X_{\beta_1}, X_{\beta_2}, ..., X_{\beta_m}) - \theta)^2\right) \\ &= \frac{1}{\binom{n}{m}^2} \sum_{\beta} \sum_{\beta'} Cov(h(X_{\beta_1}, X_{\beta_2}, ..., X_{\beta_m}), h(X_{\beta'_1}, X_{\beta'_2}, ..., X_{\beta'_m})). \end{aligned} \tag{2.80}$$

All the terms in (2.80) for which β and β' have exactly c integers in common have the same covariance σ_c^2. The number of such terms is $\binom{n}{m}\binom{m}{c}\binom{n-m}{m-c}$. Hence

$$Var(U) = \frac{1}{\binom{n}{m}} \sum_{c=1}^{m} \binom{m}{c}\binom{n-m}{m-c}\sigma_c^2. \tag{2.81}$$

Example 2.2. *Consider the above example of estimating variance with kernel* $h(x_1, x_2) = \frac{1}{2}(x_1 - x_2)^2$. *Then the corresponding U-statistic is*

$$U = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{1}{2}(X_i - X_j)^2.$$

Let

$$\begin{aligned} h_1(x_1) &= E(h(x_1, X_2)) \\ &= E(X - x_1)^2/2 = \frac{\sigma^2}{2} + \frac{(x_1 - \mu)^2}{2}. \end{aligned}$$

Then

$$\begin{aligned} \sigma_1^2 &= V(h_1(X_1)) \\ &= V((X - \mu)^2/2) = \frac{1}{4}(\mu_4 - \sigma^4) \end{aligned}$$

and

$$\sigma_2^2 = V((X_1 - X_2)^2/4) = \frac{\mu_4 - \sigma^4}{2}.$$

The variance of U-statistic is then

$$Var(U) = \frac{1}{\binom{n}{2}}\left[\binom{2}{1}\binom{n-2}{1}\sigma_1 + \binom{2}{2}\binom{n-2}{0}\sigma_2\right]$$

$$= \frac{\mu_4 - \sigma^4}{n}.$$

Theorem 2.7. *Let $U(X_1, ..., X_n)$ be the U-statistic for a symmetric kernel $h(X_1, ..., X_m)$. If $E(h^2(X_1, ..., X_m)) < \infty$, then*

$$\lim_{n\to\infty} nV(U(X_1, ..., X_n)) = m^2\sigma_1^2.$$

Proof. See Deshpande and Purohit (2005). □

Theorem 2.8. *[Hoeffding 1948] Let $X_1, X_2 ..., X_n$ be a random sample from distribution $F \in \mathcal{F}$ and θ be an estimable parameter of degree m with symmetric kernel $h(X_1, ..., X_m)$. Define by forming*

$$U(X_1, X_2, ..., X_n) = \frac{1}{\binom{n}{m}} \sum_{\underline{\beta}\in\mathcal{B}} h(X_{\beta_1}, X_{\beta_2}, ..., X_{\beta_m})$$

where $\underline{\beta} = (\beta_1, \beta_2, ..., \beta_m)$ is a combination of m integers from $(1, 2, ..., n)$ and $\mathcal{B}$ is the set of all such combinations. Also let $\sigma_1^2 = V(h_1(X_1))$.
If $\sigma_1^2 > 0$ then

$$\sqrt{n}(U - \theta) \xrightarrow{\mathcal{L}} N(0, m^2\sigma_1^2). \tag{2.82}$$

The above results can be generalized to the case of s U-statistics. Let $U_i,\ i = 1, 2, ..., s$ be U- statistics based on symmetric kernels $h_i(.)$ of degree m_i, for $i = 1, 2, ..., s$. If $E[h_i(X_1, ..., X_{m_i})]^2$ exists, then, as $n \to \infty$, the joint distribution of

$$[\sqrt{n}(U_1 - \theta_1), \sqrt{n}(U_2 - \theta_2), ..., \sqrt{n}(U_s - \theta_s)] \tag{2.83}$$

converges to the multivariate normal distribution with mean zero and variance covariance matrix $((m_i m_j \sigma_{ij}))$ with $i, j = 1, 2, ..., s$ and

$$\sigma_{ij} = E[h_i(X_1, ..., X_{m_i}) h_j(X_1, ..., X_{m_i,} X_{m_j+1}, ..., X_{2m_j - m_i})] - \theta_i \theta_j. \tag{2.84}$$

For details, one can refer to Xu (2007).

Two-Sample Problems. The important extension to k-sample problems for $k \geq 2$ has been made by Lehmann (1951). The basic ideas are contained in the two-sample case which is discussed here. Here P is a family of pairs of probability measures, (F, G). Consider independent samples $X_1, ..., X_{n_1}$ from F and $Y_1, ..., Y_{n_2}$ from G.

Let $h(x_1, ..., x_{m_1}, y_1, ..., y_{m_2})$ be a kernel and let P be the set of all pairs such that the expectation

$$\theta = \theta(F, G) = E_{F,G}(h(X_1, ..., X_{m_1}, Y_1, ..., Y_{m_2})),$$

is finite. We may assume without loss of generality that h is symmetric under independent permutations of $x_1, ..., x_{m_1}$ and $y_1, ..., y_{m_2}$. The corresponding U-statistic is

$$U_{n_1,n_2} = \frac{1}{\binom{n_1}{m_1}\binom{n_2}{m_2}} \sum h(X_{i_1}, X_{i_2}, ..., X_{i_{m_1}}, Y_{j_1}, Y_{j_2}, ..., Y_{j_{m_2}}), \tag{2.85}$$

where the sum is over all $\binom{n_1}{m_1}$ $\binom{n_2}{m_2}$ sets of subscripts such that $1 \leq i_1 < ... < i_{m_1} \leq n_1$ and $1 \leq j_1 < ... < j_{m_2} \leq n_2$. Again it is clear that U is an unbiased estimator of θ. This idea can be generalized to the k sample problems ($k \geq 2$). The following theorem gives the asymptotic distribution of U-statistic based on two samples. Let

$$\sigma^2_{(ij)} = Cov[h(X_1, ..., X_i, X_{i+1}, ..., X_{m_1}, Y_1, ..., Y_j, Y_{i+1}, ..., X_{m_2}),$$

$$h(X_1, ..., X_i, X'_{i+1}, ..., X'_{m_1}, Y_1, ..., Y_j, Y'_{i+1}, ..., X'_{m_2}], \qquad (2.86)$$

where X's and Y's are independently distributed according to F and G respectively. Next we state the theorem which gives the asymptotic normality of two sample U-statistic.

Theorem 2.9. *For $F, G \in \mathcal{F}$,*

$$V(U_{n_1,n_2}) = \frac{1}{\binom{n_1}{m_1}\binom{n_2}{m_2}} \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} [\binom{m_1}{i}\binom{n_1 - m_1}{m_1 - i}\binom{m_2}{j}\binom{n_2 - m_2}{m_2 - j}]\sigma^2_{(ij)}. \qquad (2.87)$$

Moreover, if $\sigma^2_{(m_1 m_2)}$ is finite, and if $\frac{n_1}{N} \to p \in (0,1)$ as $N = (n_1 + n_2) \to \infty$, then

$$\sqrt{N}(U_{n_1,n_2} - \theta) \xrightarrow{\mathcal{L}} N(0, \sigma^2), \text{ where } \sigma^2 = \frac{m_1^2}{p}\sigma^2_{(10)} + \frac{m_2^2}{1-p}\sigma^2_{(01)}. \qquad (2.88)$$

Proof. See Ferguson (2005). □

The research problems discussed in Chapter 1 will be addressed for possible solutions in the subsequent chapters taking advantage of the above results as background materials.

Chapter 3

Characterizing Discrete Life Distributions by Properties of Reversed Variance Residual Life [1]

3.1 Introduction

In this chapter, we discuss the characterization of a non-negative integer-valued random variable by properties of conditional variance. First we establish a relationship between RVRL and conditional expectations or failure rate to characterize a general class of discrete life distributions and the results are then applied to families of distributions. It is shown that the work by El-Arishi (2005) on binomial and Poisson random variables are special cases of our findings. A lower bound to the conditional variances is obtained and compared with the minimum variance unbiased estimators obtained by the well known classical theorems on inference, when the functions appearing in the bounds are chosen as estimators of the desired parametric functions. Nair and Kattumannil (2008) showed that the probability mass function $p(.)$ satisfies the difference equation

$$\frac{p(x+1)}{p(x)} = \frac{\sigma_h g_h(x)}{\sigma_h g_h(x+1) - \mu_h + h(x+1)}, \tag{3.1}$$

[1] The results in this chapter have been accepted for publication in Communications in Statistics- Theory and Methods (*see* Mathew and Beg, 2013).

if and only if

$$r_h(x) = \mu_h - \sigma_h g_h(x)\lambda(x), \tag{3.2}$$

where μ_h and σ_h are the mean and standard deviation of $h(X)$ and $g_h(.)$ is a positive real function on $\mathbb{N}$.

Utilizing the identity (3.2), we establish a relationship between RVRL and reversed hazard rate to characterize the distributions that satisfy (3.1). A lower bound to the conditional variance is also calculated and compared with the minimum variance unbiased estimators obtained by the Cramer-Rao and Chapman-Robbins inequalities.

The chapter is organized as follows. In Section 3.2, we derive an expression for the covariance between a real-valued functions $c(.)$ and $h(.)$ on the condition that $X \leq x$ and deduce a characterization for the model represented by (3.1) in terms of RVRL. Consequently, it produces characteristic properties of families of distributions as well as individual models. Exact expression of RVRL for the additive exponential dispersion family (AEDF, Jorgensen (1987)) with non-negative integer support, the modified power series family, the Ord family and the Katz family (Johnson et al. (1992)) are obtained. Following the methodology by Cacoullos and Papathanasiou (1997) a lower bound for the conditional variance is obtained in Section 3.3. Section 3.4 provides major findings of this study.

3.2 Conditional covariance identity

Let $h(.)$ be a real-valued non-constant function of a random variable X defined on $\mathbb{N} \cup \{0\}$ such that $E(h^2(X)) < \infty$. To obtain the conditional variance expression for $h(X)$, we obtain a conditional covariance identity for a real-valued functions $c(.)$ and $h(.)$ in terms of $\lambda(.)$ and

$g_h(.)$, where $g_h(.)$ is a positive real function on $\mathbb{N}$ satisfying the condition

$$\sigma_h p(x) g_h(x) = \sum_{y=0}^{x} (\mu_h - h(y)) p(y), \tag{3.3}$$

with $\sigma_h g_h(0) = \mu_h - h(0)$. The identity (3.3) is equivalent to (3.2) (see Nair and Kattumannil (2008)). Hence, the associated random variable has the probability mass function of the form (3.1).

Remark 3.1. *For a given $h(.)$ the value of $g_h(.)$ characterizes the distribution of X. For example, when $h(x) = x$, the random variable X has Poisson distribution with mean λ in the class of discrete probability distributions supported by the set of non-negative integers if and only if $g_h(x) = \lambda^{\frac{1}{2}}$.*

Defining $\mathfrak{B}$ to be the class of all real-valued functions defined on the range of X, we have the following theorem.

Theorem 3.1. *Let X be a discrete random variable supported on $\mathbb{N} \cup \{0\}$ or a subset of it, $c(.), h(.)$ be functions in $\mathfrak{B}$ and $g_h(.)$ be a positive function such that $E(c^2(X)) < \infty$, $E(|\Delta c(X)| g_h(X)) < \infty$, $E(|c(X)h(X)|) < \infty$ and $E(h^2(X)) < \infty$. Then for every non-constant function $c(.) \in \mathfrak{B}$, the covariance expression satisfies*

$$\begin{aligned} Cov(c(X), h(X) | X \leq x) &= \sigma_h E[\Delta c(X) g_h(X) | X \leq x] \\ &\quad + (\mu_h - r_h(x))(a(x) - c(x+1)), \end{aligned} \tag{3.4}$$

where $a(x) = E(c(X)|X \leq x)$ and $\Delta c(x) = c(x+1) - c(x)$ if and only if the probability mass function satisfies the equation (3.1) with $\sigma_h g_h(0) = \mu_h - h(0)$ and $p(0) > 0$ is evaluated using $\sum_{x=0}^{\infty} p(x) = 1$.

Proof. To prove the if part, we consider

$$\begin{aligned} E[c(X)h(X)-\mu_h)|X\leq x] &= [F(x)]^{-1}\sum_{k=0}^{x}c(k)(h(k)-\mu_h)p(k) \\ &= [F(x)]^{-1}\big[\sum_{y=0}^{x}c(y+1)\sum_{k=0}^{y}(\mu_h-h(k))p(k) \\ &\quad -\sum_{y=0}^{x}c(y)\sum_{k=0}^{y}(\mu_h-h(k))p(k) \\ &\quad -c(x+1)\sum_{k=0}^{x}(\mu_h-h(k))p(k)\big] \\ &= [F(x)]^{-1}\big[\sum_{y=0}^{x}\Delta c(y)\sum_{k=0}^{y}(\mu_h-h(k))p(k)\big] \\ &\quad -c(x+1)(\mu_h-r_h(x)). \end{aligned} \tag{3.5}$$

Suppose (3.1) holds. Then, we have (3.3) and substituting (3.3) in (3.5), we obtain

$$\begin{aligned} E[c(X)(h(X)-\mu_h)|X\leq x] &= [F(x)]^{-1}\big[\sum_{y=0}^{x}\sigma_h\Delta c(y)g_h(y)p(y)\big]-c(x+1)(\mu_h-r_h(x)) \\ &= \sigma_h E[\Delta c(X)g_h(X)|X\leq x]-c(x+1)(\mu_h-r_h(x)). \end{aligned} \tag{3.6}$$

Or

$$E[c(X)h(X)|X\leq x]=\sigma_h E[\Delta c(X)g_h(X)|X\leq x]-c(x+1)(\mu_h-r_h(x))+\mu_h a(x),$$

Now

$$\begin{aligned} Cov(c(X),h(X)|X\leq x) &= \sigma_h E[\Delta c(X)g_h(X)|X\leq x]-c(x+1)(\mu_h-r_h(x)) \\ &\quad +\mu_h a(x)-a(x)r_h(x) \end{aligned}$$

$$= \sigma_h E(\Delta c(X) g_h(X)|X \leq x) + (\mu_h - r_h(x))(a(x) - c(x+1))$$

which is the same as (3.4).

Conversely, assume

$$Cov(c(X), h(X)|X \leq x) = \sigma_h E(\Delta c(X) g_h(X)|X \leq x) + (\mu_h - r_h(x))(a(x) - c(x+1)).$$

Then

$$\begin{aligned} E[c(X)h(X)|X \leq x] &= Cov(c(X), h(X)|X \leq x) \\ &\quad + E[c(X)|X \leq x]E[h(X)|X \leq x] \\ &= \sigma_h E(\Delta c(X) g_h(X)|X \leq x) + a(x) r_h(x) \\ &\quad + (\mu_h - r_h(x))(a(x) - c(x+1)) \\ &= \sigma_h E[\Delta c(X) g_h(X)|X \leq x] - c(x+1)(\mu_h - r_h(x)) \\ &\quad + \mu_h a(x) \\ &= [F(x)]^{-1} \Big[\sum_{y=0}^{x} \sigma_h \Delta c(y) g_h(y) p(y) \Big] - c(x+1)(\mu_h - r_h(x)) \end{aligned} \tag{3.7}$$

Since the left hand side of (3.7) and (3.6) are the same, we have

$$[F(x)]^{-1} \Big[\sum_{y=0}^{x} \Delta c(y) \sum_{k=0}^{y} (\mu_h - h(k)) p(k) \Big] = [F(x)]^{-1} \Big[\sum_{y=0}^{x} \sigma_h \Delta c(y) g_h(y) p(y) \Big].$$

To arrive at (3.1), we follow the methodology of Srivastava and Sreehari (1991).

Let $c(x) = \theta s^x$, for some real θ and $0 < s < 1,$ then the above equation can be written as

$$\sum_{y=0}^{x} s^y \sum_{k=0}^{y} (\mu_h - h(k))p(k) = \sigma_h \sum_{y=0}^{x} s^y g_h(y)p(y),$$

or

$$\sum_{y=0}^{x} s^y \frac{1}{p(y)} \big[\sum_{k=0}^{y} (\mu_h - h(k))p(k) \big] p(y) = \sigma_h \sum_{y=0}^{x} s^y g_h(y)p(y). \tag{3.8}$$

Using the uniqueness of generating function, from (3.7) we can write

$$\frac{1}{p(y)} \sum_{k=0}^{y} (\mu_h - h(k))p(k) = \sigma_h g_h(y),$$

for every y, or

$$\sigma_h g_h(x)p(x) = \sum_{k=0}^{x} (\mu_h - h(k))p(k),$$

which is equivalent to (3.1). This completes the proof. □

Corollary 3.1. *When* $h(X) \in \mathfrak{B}$,

$$V(h(X)|X \leq x) = \sigma_h E(\Delta h(X) g_h(X)|X \leq x) - (h(x+1)\mu_h)\sigma_h g(x)\lambda(x)$$
$$-\sigma_h^2 g_h^2(x)\lambda^2(x), \tag{3.9}$$

if and only if (3.1) holds.

Proof. We have the identity (3.2)

$$r_h(x) = \mu_h - \sigma_h g_h(x)\lambda(x)$$

Then,

$$\begin{aligned}(\mu_h - r_h(x))(r_h(x) - h(x+1)) &= \sigma_h g_h(x)\lambda(x)(\mu_h - \sigma_h g_h(x)\lambda(x) - h(x+1)) \\ &= -(h(x+1) - \mu_h)\sigma_h g_h(x)\lambda(x) - \sigma_h^2 g_h^2(x)\lambda^2(x).\end{aligned}$$

Now the necessary part follows from Theorem 3.1 by taking $c(x) = h(x)$. To prove the converse, let

$$\begin{aligned}V(h(X)|X \leq x) = \sigma_h E(\Delta h(X) g_h(X)|X \leq x) - (h(x+1)\mu_h)\sigma_h g(x)\lambda(x) \\ -\sigma_h^2 g_h^2(x)\lambda^2(x),\end{aligned} \tag{3.10}$$

Consider,

$$\begin{aligned}E(h(X)^2|X \leq x) &= V(h(X)|X \leq x) + (E(h(X)|X \leq x))^2 \\ &= \sigma_h E[\Delta h(X) g_h(X)|X \leq x] - h(x+1)(\mu_h - r_h(x)) + \mu_h r_h(x) \\ &= [F(x)]^{-1}\big[\sum_{y=0}^{x} \sigma_h \Delta h(y) g_h(y) p(y)\big] \\ &\quad -h(x+1)(\mu_h - r_h(x)) + E(\mu_h h(X)|X \leq x).\end{aligned}$$

Then

$$\begin{aligned}E(h(X)(h(X) - \mu_h)|X \leq x) &= [F(x)]^{-1}\big[\sum_{y=0}^{x} \sigma_h \Delta h(y) g_h(y) p(y)\big] \\ &\quad -h(x+1)(\mu_h - r_h(x)).\end{aligned} \tag{3.11}$$

Now substituting $c(x) = h(x)$ in (3.5), we obtain

$$E[h(X)(h(X) - \mu_h)|X \le x] = [F(x)]^{-1}\big[\sum_{y=0}^{x} \Delta h(y) \sum_{k=0}^{y}(\mu_h - h(k))p(k)\big] - h(x+1)(\mu_h - r_h(x)). \quad (3.12)$$

From (3.11) and 3.12), we can write

$$\sum_{y=0}^{x} \Delta h(y) \sum_{k=0}^{y}(\mu_h - h(k))p(k) = \sum_{y=0}^{x} \sigma_h \Delta h(y) g_h(y) p(y). \quad (3.13)$$

Changing x to $x - 1$ we get

$$\sum_{y=0}^{x-1} \Delta h(y) \sum_{k=0}^{y}(\mu_h - h(k))p(k) = \sum_{y=0}^{x-1} \sigma_h \Delta h(y) g_h(y) p(y). \quad (3.14)$$

Subtracting (3.14) from (3.13) we obtain

$$\sigma_h g_h(x) p(x) = \sum_{k=0}^{x}(\mu_h - h(k))p(k),$$

which is equivalent to (3.2), hence X belongs to the class specified by (3.1). □

Remark 3.2. *Characterization by the relationships between RVRL and reversed hazard rate is a special case of (3.9) when $h(x) = x$ and in that case we ignore the suffix h.*

Next we look at the characterization results in terms of (3.9) for the AEDF, the modified power series family, the Ord family and the Katz family of distributions which together cover most of the discrete probability distributions used in reliability modeling.

Corollary 3.2. *The distribution of X belongs to the AEDF with probability mass function*

$$p(x|\theta,\phi) = q(x|\phi)exp[x\theta - \phi b(\theta)], \tag{3.15}$$

with parameters ϕ and θ for all x in support of X if and only if

$$V(X|X \le x) = \frac{\partial r_{(1)}(x)}{\partial \theta}, \tag{3.16}$$

where $r_{(1)}(x) = E(X|X \le x)$

Proof. For AEDF we have

$$r_{(1)}(x) = \mu - \frac{R(x)}{F(x)}h^*, \tag{3.17}$$

where $\mu = E(X) = \phi b'(\theta)$ and $h^* = \frac{\partial}{\partial\theta}\ln R(x)$. From (3.17) and (3.2), we can say that X belongs to the family specified by (3.1) with $g(x)$ given by

$$\begin{aligned} g(x) &= \frac{R(x)h^*}{F(x)\sigma\lambda(x)} \\ &= \frac{R'(x)}{\sigma p(x)} = \frac{-F'(x)}{\sigma p(x)}, \end{aligned} \tag{3.18}$$

where $R'(x) = \frac{\partial R(x)}{\partial\theta} = -\frac{\partial F(x)}{\partial\theta} = -F'(x)$.

Suppose $h(x) = x$. Using (3.18) we have

$$\begin{aligned} \sigma E\big[\Delta h(X)g(X)|X \le x\big] &= \sigma E\big[((X+1) - X)g(X)|X \le x\big] \\ &= -\frac{1}{F(x)}\sum_{y=0}^{x}\frac{\partial F(y)}{\partial\theta} \end{aligned}$$

$$= -\frac{1}{F(x)}\frac{\partial}{\partial\theta}\big[\sum_{y=0}^{x} F(y)\big]. \tag{3.19}$$

Again, consider

$$\begin{aligned} -\sum_{y=0}^{x} F(y) &= \sum_{y=0}^{x} yF(y) - \sum_{y=0}^{x}(y+1)F(y) \\ &= -(x+1)F(x) + \sum_{y=0}^{x} y(F(y) - F(y-1)) \\ &= r_{(1)}(x)F(x) - (x+1)F(x). \end{aligned} \tag{3.20}$$

Hence

$$\begin{aligned} -\frac{\partial}{\partial\theta}\sum_{y=0}^{x} F(y) &= \frac{\partial}{\partial\theta}\big[r_{(1)}(x)F(x) - (x+1)F(x)\big] \\ &= F(x)\frac{\partial r_{(1)}(x)}{\partial\theta} + r_{(1)}(x)\frac{\partial F(x)}{\partial\theta} - (x+1)\frac{\partial F(x)}{\partial\theta} \\ &= F(x)\frac{\partial r_{(1)}(x)}{\partial\theta} + (r_{(1)}(x) - (x+1))\frac{\partial F(x)}{\partial\theta}. \end{aligned} \tag{3.21}$$

Substituting (3.21) in (3.19) and using (3.2) and (3.18), we obtain

$$\begin{aligned} \sigma E\big[g(X)|X \le x\big] &= \frac{1}{F(x)}\big[F(x)\frac{\partial r_{(1)}(x)}{\partial\theta} + (r_{(1)}(x) - (x+1))\frac{\partial F(x)}{\partial\theta}\big] \\ &= \frac{\partial r_{(1)}(x)}{\partial\theta} + (r_{(1)}(x) - (x+1))\frac{-\sigma g(x)p(x)}{F(x)} \\ &= \frac{\partial r_{(1)}(x)}{\partial\theta} + ((x+1) - r_{(1)}(x))(\mu - r_{(1)}(x)). \end{aligned} \tag{3.22}$$

Substituting (3.22) in (3.9), we get final form of $V(X|X \le x)$ as

$$V(X|X \le x) = \frac{\partial r_{(1)}(x)}{\partial\theta} + ((x+1) - r_{(1)}(x))(\mu - r_{(1)}(x)) + (r_{(1)}(x) - (x+1))(\mu - r_{(1)}(x))$$

$$= \frac{\partial r_{(1)}(x)}{\partial \theta}. \quad (3.23)$$

The converse is obtained by retracing the above steps to arrive at (3.19) and hence $g(x)$ given in (3.18). Hence the distribution of X belongs to (3.15). □

Corollary 3.3. *The distribution of X belongs to Ord's family of distributions specified by the difference equation*

$$\frac{p(x+1)-p(x)}{p(x)} = \frac{-(x+d)}{a_0+a_1x+a_2x^2}, \quad (3.24)$$

if and only if

$$\begin{aligned} V(X|X \leq x) &= E\big[(b_0+b_1X+b_2X^2)|X \leq x\big]-(x+1-\mu)\sigma g(x)\lambda(x) \\ &\quad -\sigma^2 g^2(x)\lambda^2(x), \end{aligned} \quad (3.25)$$

where $b_0 = \mu + \frac{a_0-a_1+a_2}{1-2a_2}, b_1 = \frac{a_1-1}{1-2a_2}, b_2 = \frac{a_2}{1-2a_2}$ *and* $a_2 \neq \frac{1}{2}$.

Proof. In analogy with (3.1) the equation (3.24) can be written as

$$\begin{aligned} \frac{p(x+1)}{p(x)} &= \frac{(a_0-d)+a_1-1)x+a_2x^2}{a_0+a_1x+a_2x^2} \\ &= \frac{\sigma g(x)}{\sigma g(x+1)-\mu+h(x+1)}. \end{aligned}$$

For $h(x) = x$, $g(.)$ must be a quadratic function of x of the form $g(x) = b_0+b_1x+b_2x^2$. Substituting in the equation (3.26) and identifying the coefficient of x we get $b_0 = \mu + \frac{a_0-a_1+a_2}{1-2a_2}, b_1 = \frac{a_1-1}{1-2a_2}, b_2 = \frac{a_2}{1-2a_2}$. Now applying Corollary 2.1 we have the result (3.25). To prove the converse suppose that (3.25) holds. Using the identity (3.9) we have

$$\sigma E\big[g(X)|X \leq x\big] = E\big[(b_0 + b_1 X + b_2 X^2)|X \leq x\big].$$

Or

$$\sigma \sum_{k=0}^{x} g(k)p(k) = \sum_{k=0}^{x}(b_0 + b_1 k + b_2 k^2)p(k). \tag{3.26}$$

Changing x to $x-1$ in (3.26) and then subtracting (3.26) we obtain

$$\sigma g(x) = (b_0 + b_1 x + b_2 x^2).$$

Substituting the value of $g(x)$ in (3.1) we obtain

$$\begin{aligned}\frac{p(x+1)}{p(x)} &= \frac{b_0 + b_1 x + b_2 x^2}{(b_0 + b_1(x+1) + b_2(x+1)^2) - \mu + x + 1} \\ &= \frac{b_0 + b_1 x + b_2 x^2}{(1 + b_0 + b_1 + b_2 - \mu) + (1 + b_1 + 2b_2)x + b_2 x^2}.\end{aligned}$$

Now consider the identity (Nair and Kattumannil (2008))

$$\begin{aligned}E(X|X > x) &= \mu + \frac{\sigma g(x)k(x)}{1 - k(x)} \\ &= \mu + \sigma g(x)\frac{p(x)}{p(x+1)}k(x+1) \\ &= \mu + \big((1 + b_0 + b_1 + b_2 - \mu) + (1 + b_1 + 2b_2)x + b_2 x^2\big)k(x+1) \\ &= \mu + (c_0 + c_1 x + c_2 x^2)k(x+1).\end{aligned}$$

where $c_i = a_i(1-2a_2)^{-1}, a_2 \neq \frac{1}{2}$ and $i = 0, 1, 2$. Hence the distribution of X belongs to the Ord family (see Nair and Kattumannil(2008)). $\square$

Corollary 3.4. *The distribution of X follows the Katz family specified by*

$$\frac{p(x+1)}{p(x)} = \frac{\alpha + \beta x}{1+x}, \tag{3.27}$$

if and only if

$$V(X|X \le x) = (1-\beta)^{-1}(\alpha + \beta m_1(x)) - (x+1-\mu)\sigma g(x)\lambda(x) \\ -\sigma^2 g^2(x)\lambda^2(x). \tag{3.28}$$

Proof. when $h(x) = x$, comparing (3.27) with (3.1) we have

$$\frac{\sigma g(x)}{\sigma g(x+1) - \mu + (1+x)} = \frac{\alpha + \beta x}{1+x}. \tag{3.29}$$

Clearly $g(x)$ is linear in x, substitute $g(x) = a_0 + a_1 x$ in equation (3.29) we have

$$\frac{\sigma(a_0 + a_1 x)}{\sigma(a_0 + a_1(x+1)) - \mu + (1+x)} = \frac{\alpha + \beta x}{1+x}.$$

Comparing the coefficients we obtain $a_0 = \frac{\alpha}{\sigma(1-\beta)}$ and $a_1 = \frac{\beta}{\sigma(1-\beta)}$, and hence $\sigma g(x) = (1-\beta)^{-1}(\alpha + \beta x)$.

Substituting the value of $g(x)$ in (3.9) the identity (3.28) follows.

Conversely, suppose that (3.28) holds, using (3.9) we obtain

$$\sigma E\big[g(X)|X \le x\big] = E\Big[\big((1-\beta)^{-1}(\alpha + \beta m_1(x))\big)|X \le x\Big],$$

or

$$\sigma \sum_{k=x+1}^{\infty} g(k)p(k) = (1-\beta)^{-1} \sum_{k=x+1}^{\infty} (\alpha + \beta x)p(k). \tag{3.30}$$

Changing x to $x-1$ in (3.30) we get

$$\sigma \sum_{k=x}^{\infty} g(k)p(k) = (1-\beta)^{-1} \sum_{k=x}^{\infty} (\alpha + \beta x)p(k).$$

Subtracting (3.30) from the above equation we have

$$\sigma g(x) = (1-\beta)^{-1}(\alpha + \beta x).$$

Now using the identity (3.1) we can write

$$\begin{aligned} \frac{p(x+1)}{p(x)} &= \frac{\alpha + \beta x}{(\alpha\beta(x+1)) + (1+x-\mu)(1-\beta)} \\ &= \frac{\alpha + \beta x}{x+1+\alpha-\mu+\mu\beta}, \end{aligned} \tag{3.31}$$

and putting $x = 0$ in (3.31), we get

$$\frac{p(1)}{p(0)} = \frac{\alpha}{1+\alpha-\mu+\mu\beta}. \tag{3.32}$$

Again using the value of $g(0)$ as stated in Theorem 3.1, from (3.1) we have

$$\frac{p(1)}{p(0)} = \frac{\mu(1-\beta)}{(\alpha+\beta)+(1-\mu)(1-\beta)}. \tag{3.33}$$

Equating (3.32) and (3.33) and then rearranging the terms we have

$$(1-\beta)\mu^2 - (1+2\alpha)\mu + \frac{(\alpha+\alpha^2)}{1-\beta} = 0.$$

This is a quadratic equation in μ giving solutions
$\mu = \frac{\alpha}{(1-\beta)}$ or $\mu = \frac{1+\alpha}{(1-\beta)}$. Second solution is inadmissible as it gives

$$\frac{p(x+1)}{p(x)} = \frac{\alpha+\beta x}{x}, \text{ for } \quad x = 0, 1, 2, ...$$

which is not defined at $x = 0$. The first solution gives

$$\alpha - \mu + \mu\beta = 0.$$

Then (3.31) reduces to (3.27) so that the distribution of X belongs to the Katz family. □

Corollary 3.5. *The distribution of X belongs to modified power series family with probability mass function*

$$p(x,\theta) = \frac{a(x)(u(\theta))^x}{A(\theta)}, \tag{3.34}$$

where $x \in \mathbb{N} \cup \{0\}, a(x) \geq 0, u(\theta)$ and $A(\theta)$ are positive, finite and differentiable if and only if

$$V(X|X \leq x) = \frac{u(\theta)}{u'(\theta)} \frac{\partial r_{(1)}(x)}{\partial \theta}. \tag{3.35}$$

Proof. From (3.34)

$$A(\theta)F(x,\theta) = \sum_{y=0}^{x} a(y)\big(u(\theta)\big)^{y}.$$

Differentiating w.r.t. θ gives

$$E(X|X \le x) = \mu\big[1 + (\ln A(\theta))'\big(\frac{\partial \ln F(x,\theta)}{\partial\theta}\big)\big]. \tag{3.36}$$

When $h(x) = x$, using the expressions for $E(X)$ and the recurrence relations for moments in Johnson et al. (1992) and comparing (3.36) with (3.2) we get

$$g(x) = -\frac{\mu}{\sigma}\frac{A(\theta)}{A'(\theta)}\frac{1}{p(x)}\frac{\partial F(x,\theta)}{\partial\theta} = -\frac{u(\theta)}{u'(\theta)}\frac{1}{\sigma p(x)}\frac{\partial F(x,\theta)}{\partial\theta}.$$

Now consider

$$\begin{aligned}\sigma E\big[g(X)|X \le x\big] &= -\frac{u(\theta)}{u'(\theta)F(x)}\sum_{y=0}^{x}\frac{\partial F(y)}{\partial\theta} \\ &= -\frac{u(\theta)}{u'(\theta)F(x)}\frac{\partial}{\partial\theta}\big[\sum_{y=0}^{x}F(y)\big].\end{aligned} \tag{3.37}$$

Again

$$\begin{aligned}-\sum_{y=0}^{x}F(y) &= \sum_{y=0}^{x}yF(y) - \sum_{y=0}^{x}(y+1)F(y) \\ &= \sum_{y=0}^{x}yf(y) + (x+1)F(x) \\ &= r_{(1)}(x)F(x) - (x+1)F(x).\end{aligned}$$

Hence

$$\begin{aligned}
-\frac{\partial}{\partial\theta}\sum_{y=0}^{x}F(y) &= \frac{\partial}{\partial\theta}\big[r_{(1)}(x)F(x)-(x+1)F(x)\big] \\
&= F(x)\frac{\partial r_{(1)}(x)}{\partial\theta}+r_{(1)}(x)\frac{\partial F(x)}{\partial\theta}-(x+1)\frac{\partial F(x)}{\partial\theta} \\
&= F(x)\frac{\partial r_{(1)}(x)}{\partial\theta}+(r_{(1)}(x)-(x+1))\frac{\partial F(x)}{\partial\theta}. \qquad (3.38)
\end{aligned}$$

Substituting (3.38) in (3.37) and using (3.2) gives

$$\begin{aligned}
\sigma E\big[\Delta h(X)g(X)|X\le x\big] &= \frac{u(\theta)}{u'(\theta)F(x)}\big[F(x)\frac{\partial r_{(1)}(x)}{\partial\theta}+(r_{(1)}(x)-(x+1))\frac{\partial F(x)}{\partial\theta}\big] \\
&= \frac{u(\theta)}{u'(\theta)}\frac{\partial r_{(1)}(x)}{\partial\theta}+\frac{u(\theta)}{u'(\theta)F(x)}\big((r_{(1)}(x)-(x+1))\frac{\partial F(x)}{\partial\theta}\big) \\
&= \frac{u(\theta)}{u'(\theta)}\frac{\partial r_{(1)}(x)}{\partial\theta}+(r_{(1)}(x)-(x+1))\frac{-\sigma g(x)p(x)}{F(x)} \\
&= \frac{u(\theta)}{u'(\theta)}\frac{\partial r_{(1)}(x)}{\partial\theta}+((x+1)-r_{(1)}(x))(\mu-r_{(1)}(x)). \qquad (3.39)
\end{aligned}$$

Substituting in (3.9) we get final form of $V(X|X\le x)$ as

$$\begin{aligned}
V(X|X\le x) &= \sigma E\big[g(X)|X\le x\big]+(r_{(1)}(x)-(x+1))(\mu-r_{(1)}(x)) \\
&= \frac{u(\theta)}{u'(\theta)}\frac{\partial r_{(1)}(x)}{\partial\theta}+((x+1)-r_{(1)}(x))(\mu-r_{(1)}(x))+(r_{(1)}(x)-(x+1))(\mu-r_{(1)}(x)) \\
&= \frac{u(\theta)}{u'(\theta)}\frac{\partial r_{(1)}(x)}{\partial\theta} \qquad (3.40)
\end{aligned}$$

The converse part is obtained by retracing the above steps to arrive at (3.36). Hence the family of distributions belongs to the family specified by (3.34) (see Nair and Kattumannil (2008)). □

Remark 3.3. *When $u(\theta)=\theta$ in the above equations, the results for the sub-class of gener-*

alized power series distributions can be obtained.

Example 3.1. *For the binomial random variable with probability mass function*

$$p(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x},\ x = 0, 1, 2, ..., n$$

the value of $g(x)$ *is given by*

$$\sigma g(x) = \theta(n-x),$$

so that

$$\mu - r_{(1)}(x) = (n-x)\theta\lambda(x) \tag{3.41}$$

and the variance expression using (3.9) is given by

$$V(X|X \le x) = n\theta(1-\theta) - \theta(n-x)(x+1-\theta(n+1))\lambda(x) \\ -\theta^2(n-x)^2\lambda^2(x). \tag{3.42}$$

Example 3.2. *For the Poisson distribution with probability mass function*

$$p(x) = \frac{e^{-\eta}\eta^x}{x!},\ x = 0, 1, 2, ...,$$

the value of g(x) is given by

$$\sigma g(x) = \eta.$$

Hence by using (3.2)

$$r_{(1)}(x) = \eta(1-\lambda(x)) \tag{3.43}$$

and substituting in equation (3.9) gives

$$V(X|X \le x) = \eta - (x+1-\eta)\lambda(x) - \eta^2\lambda^2(x). \tag{3.44}$$

Remark 3.4. *The characterizations by relationship between reversed hazard rate and conditional variance for the binomial and the Poisson random variables discussed in above examples are similar to those respective identities given by El-Arishi (2005) which is stated in Chapter 2.*

Example 3.3. *For the negative binomial distribution with probability mass function*

$$p(x) = \binom{k+x-1}{k-1}\theta^k(1-\theta)^x,\ x = 0, 1, 2, ... \tag{3.45}$$

The value of g(x) is given by

$$\sigma g(x) = \frac{1-\theta}{\theta}(x+k).$$

Now

$$r_{(1)}(x) = \mu - (x+1)\theta k(x), \tag{3.46}$$

Hence the conditional variance identity is valid and can be written as

$$V(X|X \le x) = \frac{\mu}{p} + \frac{(x+1)}{p}k(x+1) + \frac{(x+1-\mu)}{p}k(x+1) - \frac{(x+1)^2}{p^2}k^2(x+1).$$

A special case of (3.45) is the geometric distribution with probability mass function

$$p(x) = \theta(1-\theta)^x,\ x = 0, 1, 2, ...$$

and

$$\sigma g(x) = \mu(x+1),$$

hence we have

$$r_{(1)}(x) = \mu(1-(x+1)\lambda(x)).$$

3.3 Lower bound to the conditional variance

The covariance identity in Section 2 provides scope for introducing bounds on right truncated variance of a function of non-negative integer-valued random variable. Next we obtained a lower bound for the variance of a function of right truncated random variable and compare our bound with that provided by Cramer-Rao and Chapman-Robbins inequalities.

Theorem 3.2. *Under the condition of the Theorem 3.1, for every $c(.)$ in $\mathfrak{B}$,*

$$V(c(X)|X\leq x) \geq \big[V(h(X)|X\leq x)\big]^{-1}\big[\sigma_h E\big[\Delta c(X) g_h(X)|X\leq x\big] + (\mu_h - r_h(x))(a(x)-c(x+1))\big]^2 \tag{3.47}$$

if (3.1) holds, with equality whenever $c(.)$ is a linear function of $h(.)$. Conversely, if the conditional variance is given by (3.47), then (3.1) is true provided $V(h(X)|X\leq x)$ is as in (3.9).

Proof. From Cauchy- Schwartz inequality and using (3.4) we have (3.47).

To prove the converse of the theorem, assume

$$c(x) = h(x) + \theta l(x)$$

for some arbitrary real θ. Then

$$a(x) = r_h(x) + \theta b(x),\ b(x) = E(l(X)|X \leq x)$$

so that (3.47) becomes

$$\begin{aligned} V(h(X)|X \leq x)[V(h(X)|X \leq x) + \theta^2 V(l(X)|X \leq x) \\ +2\theta Cov(l(X), h(X)|X \leq x)] \geq [\sigma_h E\big[\Delta h(X) g_h(X)|X \leq x\big] \\ +\sigma_h \theta E\big[\Delta l(X) g_h(X)|X \leq x\big] \\ +\big[r_h(x) + \theta b(x) - h(x+1) \\ -\theta l(x+1)\big](\mu_h - r_h(x))\big]^2. \end{aligned} \tag{3.48}$$

Using (3.9), inequality (3.48) simplifies to

$$\begin{aligned} \theta^2\big[V(h(X)|X \leq x)V(l(X)|X \leq x) - [\sigma_h E\big[\Delta l(X) g_h(X)|X \leq x\big] \\ +(\mu_h - r_h(x))(b(x) - l(x+1))]\big]^2\big] + 2\theta V(h(X)|X \leq x) \\ \big[Cov(l(X), h(X)|X \leq x) - [\sigma_h E\big[\Delta l(X) g_h(X)|X \leq x\big] \\ +(\mu_h - r_h(x))(b(x) - l(x+1))]\big] \geq 0. \end{aligned} \tag{3.49}$$

For (3.49) to be true for all θ, the coefficient of θ must vanish and therefore,

$$\begin{aligned} Cov(l(X), h(X)|X \leq x) &= \sigma_h E\big[\Delta l(X) g_h(X)|X \leq x\big] \\ &\quad +(\mu_h - r_h(x))(b(x) - l(x+1)), \end{aligned}$$

which on simplification reduces to

$$E(l(X),(h(X)-\mu_h)|X\leq x) \quad = \quad E\big(\Delta l(X)g_h(X)|X\leq x\big) \\ +l(x)E((h(X)-\mu_h)|X\leq x), \qquad (3.50)$$

which is similar to the expression (3.6), hence rest of the proof follows from Theorem 3.1.

□

Now we examine how our inequality compares with the lower bound of the variance in the Cramer- Rao and the Chapman- Robbins inequalities.

Let X belong to the family specified by (3.34) which is also known as discrete exponential family (Nair and Kattumannil (2008)), then the random variable $(X|X\leq t)$ has density

$$f(x)=\frac{a(x)\big[u(\theta)\big]^x}{F(t)A(\theta)},\ X\leq t.$$

For the above density the Cramer-Rao lower bound is attained for the unbiased estimator of $r_h(t)=E(h(X)|X\leq t)$ and this bound is

$$V_{lb}(T))=\frac{[r_h'(t)]^2}{E\big[\frac{\partial\ln f(t)}{\partial\theta}\big]^2},$$

where $r_h'(t)=\frac{\partial r_h(t)}{\partial\theta}$. For the attainment of (3.47) a necessary and sufficient condition is that (3.2) holds, consequently

$$r_h(x)=\mu_h-\sigma_h\lambda(x)g_h(x)$$

which is equivalent to

$$\sum_{t=0}^{x} h(t)f(t) = \mu_h \sum_{t=0}^{x} f(t) + \mu_h \frac{A(\theta)}{A'(\theta)} \frac{\partial}{\partial\theta} \sum_{t=0}^{x} f(t), \tag{3.51}$$

using the value of $g_h(x)$. From (3.51)

$$h(x)f(x) = \mu_h f(x) + \mu_h \frac{A(\theta)}{A'(\theta)} \frac{\partial f}{\partial\theta}$$

or

$$\frac{\partial \ln f(x)}{\partial\theta} = \frac{u'(\theta)}{u(\theta)}(h(x) - \mu),$$

and

$$E\big[\frac{\partial \ln f(x)}{\partial\theta}\big]^2 = \big[\frac{u'(\theta)}{u(\theta)}\big]^2 V(h(X)|X \leq t).$$

Hence the Cramer-Rao lower bound for an unbiased estimator of $r_h(t)$ using $h(x)$ is

$$\begin{aligned} V_{lb}(T) &= \frac{[r_h'(t)]^2}{\big[\frac{u'(\theta)}{u(\theta)}\big]^2 V(h(X)|X \leq t)} \\ &= \frac{[r_h'(t)]^2}{\big[\frac{u'(\theta)}{u(\theta)}\big]^2 \frac{u(\theta)}{u'(\theta)} r_h'(t)} \\ &= \frac{u(\theta)}{u'(\theta)} r_h'(t), \end{aligned} \tag{3.52}$$

which is also the bound obtained in (3.35). Hence in regular case the two bounds are equal. Note that Theorem 3.2 does not require the regularity conditions of the Cramer-Rao theorem and accordingly it is applicable in non-regular cases also. We illustrate this with the following example. Note that in the discrete uniform distribution over $\{1, 2, ..., N\}$, the random variable $(X|X \leq y),\ y < N$ has the probability mass function f^* given by

$$f^*(x) = \frac{f(x)}{F(y)} = \frac{1}{N}\frac{N}{y} = \frac{1}{y}; \quad x = 1, 2, ..., y.$$

Taking $(h\,(X)\,|X \leq y) = 2X - 1$, we have

$$\begin{aligned} E_{f^*}(2X - 1) &= \frac{1}{y}\sum_{t=1}^{y}(2t - 1) \\ &= \frac{1}{y}\left(\frac{2y(y+1)}{2} - y\right) = y. \end{aligned}$$

That is $(h(X)|X \leq y)$ is unbiased for y. The variance is given by

$$V(h(X)|X \leq y) = E(4X^2|X \leq y) - E^2(2X|X \leq y).$$

Consider

$$E^2(X|X \leq y) = \frac{(y+1)^2}{4}$$

and

$$\begin{aligned} E(4X^2|X \leq y) &= 4\sum_{x=1}^{y} x^2\frac{1}{y} \\ &= \frac{4}{y}\frac{y(y+1)(2y+1)}{6} = \frac{4(y+1)(2y+1)}{6}. \end{aligned}$$

Thus

$$\begin{aligned} V(h(X)|X \leq y) &= \frac{4(y+1)(2y+1)}{6} - (y+1)^2 \\ &= \frac{y^2 - 1}{3}. \end{aligned} \tag{3.53}$$

For the random variable $C(X) = 2X$, we consider the bound obtained by (3.47). Using the the identity (3.6) we have

$$\begin{aligned}
V(h(X)|X \leq y) &= \sigma E[\Delta c(X) g_h(X)|X \leq y] - c(x+1)(\mu_h - r_h(x)) \\
&= E[c(X)(h(X) - \mu_h)|X \leq y] \\
&= E[2X(2X - 1 - y)|X \leq y] \\
&= E[4X^2 - 2X - 2Xy|X \leq y] \\
&= \sum_{x=1}^{y} 4x^2 \frac{1}{y} - \sum_{x=1}^{y} 2x \frac{1}{y} - \sum_{x=1}^{y} 2xy \frac{1}{y} \\
&= \frac{1}{y} \frac{4y(y+1)(2y+1)}{6} - \frac{1}{y} \frac{2y(y+1)}{2} - \frac{1}{y} \frac{2y^2(y+1)}{2} \\
&= \frac{y^2 - 1}{3}.
\end{aligned}$$

Hence the random variable $C(X) = 2X$ attains the bound in (3.47) which is in fact the actual conditional variance of $h(X)$ given in (3.53).

Now we compare our bound with that provided by Chapman- Robbins inequality. For $\theta, \varphi \in \Theta \subset \mathbb{R}$, $f_\theta(x)$ and $f_\varphi(x)$ are different satisfying $\{f_\theta(x) > 0\} \supset \{f_\varphi(x) > 0\}$. Let $E_\theta(h(X)|X \leq x) = r_{(h,\theta)}(x)$, choosing $c(x) = \left[\frac{f_\varphi(x)}{f_\theta(x)} - 1\right]$ and using the identity (3.6)

$$\begin{aligned}
\sigma E[\Delta c(X) g(X)|X \leq x] - c(x+1)(\mu - r_h(x)) &= E[c(X)(h(X) - \mu)|X \leq x] \\
&= E\Big[\Big[\frac{f_\varphi(X)}{f_\theta(X)} - 1\Big](h(X) - \mu)|X \leq x\Big] \\
&= E\Big[\Big[\frac{f_\varphi(X)}{f_\theta(X)} - 1\Big] h(X)|X \leq x\Big] \\
&= E\Big[\frac{f_\varphi(X)}{f_\theta(X)} h(X)|X \leq x\Big] - E\big[h(X)|X \leq x\big] \\
&= [r_{(h,\varphi)}(x) - r_{(h,\theta)}(x)]. \qquad (3.54)
\end{aligned}$$

Now the inequality (3.47) reduces to

$$V(c(X)|X \leq x) \geq \frac{[r_{(h,\varphi)}(x) - r_{(h,\theta)}(x)]^2}{V(h(X)|X \leq x)}$$

or

$$V(h(X)|X \leq x) \geq \frac{[r_{(h,\varphi)}(x) - r_{(h,\theta)}(x)]^2}{V(\frac{f_\varphi(X)}{f_\theta(X)}|X \leq x)}, \tag{3.55}$$

which is the Chapman-Robbins inequality. It is well known that this bound does not require the regularity conditions of the Cramer- Rao inequality.

3.4 Conclusion

In this chapter, we discussed the characterization of non-negative integer-valued random variable using reversed variance residual life. A special attention was given to the characterizations by relationship between conditional variance and the reversed hazard rate. A lower bound to the conditional variance is also established. Our bound is compared to the Cramer-Rao and Chapman-Robbins lower bounds so that construction of minimum variance unbiased estimators of relevant parametric functions in truncated distributions can be possible.

Chapter 4

A Non-parametric Test for Constant Hazard Rate against IHR and IHRA Alternatives [1]

4.1 Introduction

The characterization of classes of lifetime distributions based on notions of ageing, as proposed in Barlow and Proschan (1981), has been a very fruitful research field in reliability theory. As discussed in Chapter 2, an important problem of discrete distributions is that $\sum_{i=1}^{x} k(i) \neq -\ln R(x)$. So an alternative definition of discrete hazard rate could be the sequence $\{k^*(x)\}_{x\geq 1}$ such that

$$H(x) = \sum_{i=1}^{x} k^*(i) = -\ln R(x)$$

or

$$R(x) = \exp[-\sum_{i=1}^{x} k^*(i)]. \tag{4.1}$$

[1] The results in this chapter have been communicated as entitled (*see* Kattumannil et al., 2013a).

Then the modified hazard rate is given by

$$k^*(x) = H(x) - H(x-1) = \ln \frac{R(x-1)}{R(x)}. \tag{4.2}$$

With the above definition, we have seen that the hazard rate $k^*(.)$ and $k(.)$ are linked by the relationship

$$k^*(x) = -\ln \frac{R(x)}{R(x-1)} = -\ln[1-k(x)] \tag{4.3}$$

So both the functions have the same monotonicity property : $k^*(.)$ is increasing/decreasing if and only if $k(.)$ is increasing/decreasing. The study of length of life of human beings, organisms, structures, materials, etc., is of great importance in the actuarial, biological, engineering and medical sciences. It is clear that research on ageing properties (univariate, bivariate, and multivariate) is currently being vigorously pursued. Many of the univariate definitions do have physical interpretations such as arising from shock models. The simple IHR, IHRA, NBU, NBUE, DMRL etc. have been shown to be very useful in reliability related decision making, such as replacement and maintenance studies.

The IHR phenomenon is well understood and needs no further elaboration. To put in non-technical terms, a device having IHR lifetime deteriorates with age, i.e., the age has an adverse effect on the device if it has an IHR lifetime distribution.

The IHRA class is perhaps one of the more important ageing classes in reliability analysis. Curiously, interest about IHRA has seemed to wane in the recent time. It has been shown that a device subject to shocks governed by a Poisson process, which fails when the accumulated damage exceeds a fixed threshold, has an IHRA distribution (Esary et al., 1973). One of the attractive properties that an IHRA (DHRA) distribution enjoys is that its reliability bound can be obtained in terms of its known quantile. It is the smallest class containing the exponential distribution which is closed under the formation of coherent systems as well as under convolution.

This chapter is dedicated to the study the testing problem associated with IHR and IHRA classes. The chapter is organized as follows. In Section 3.2 , we develop test statistics for testing the constant hazard rate against IHR and IHRA ageing classes. Section 3.3 provides a simulation study to assess the performance of the tests. Finally, Section 3.4 concludes the study.

4.2 The proposed non-parametric test

Let X be a non-negative integer-valued random variable with support $\mathbb{N}$. Recall the following definitions of IHR and IHRA.

Definition 4.1. *Let X be a random variable defined on $\mathbb{N}$.*
(i) X is said to be IHR (DHR) if $k^*(x)$ is increasing (decreasing) in x. That is,

$$k^*(x) \geq (\leq) k^*(x-1), \quad \forall x > 1. \tag{4.4}$$

(ii)X is said to have IHRA (DHRA) if $\frac{1}{x}\sum_{y=1}^{x} k^*(y)$ is an increasing (decreasing) sequence in x.

When the distribution of X is geometric with mean $\frac{1}{p}$, then the hazard rate is $\log \frac{1}{1-p}$. and constant hazard rate characterizes the geometric distribution. Next we develop tests based on U-statistic for testing geometric against IHR and IHRA alternatives.

4.2.1 Testing against IHR alternatives

Let $X_1, X_2, ..., X_n$ be a random sample from discrete distribution. We are interested in testing the hypothesis

$$H_{10} : F \quad \text{is geometric (constant hazard rate)}$$

against

$$H_{11} : F \quad \text{is IHR but not geometric.}$$

From Definition 4.1, if F is IFR then

$$\ln \frac{R(x)}{R(x+1)} \geq \ln \frac{R(x-1)}{R(x)}$$

or

$$R^2(x) \geq R(x+1)R(x-1), \quad \forall x. \tag{4.5}$$

Hence under the null hypothesis, $R^2(x) - R(x-1)R(x+1) = 0$, for all x and under the alternative hypothesis, it can be seen that

$$R^2(x) - R(x-1)R(x+1) > 0 \text{ for some } x.$$

Hence the quantity defined by

$$\delta(F) = R^2(x) - R(x-1)R(x+1)$$

is a measure of departure from the null hypothesis H_{10} towards the alternative hypothesis H_{11}. Now consider

$$\begin{aligned}
\Delta(F) &= \sum_{x=1}^{\infty}[R^2(x) - R(x-1)R(x+1)] \\
&= \sum_{x=1}^{\infty} R^2(x) - \sum_{x=1}^{\infty}[R(x-1)R(x+1)] \\
&= \sum_{x=1}^{\infty} R^2(x) - \sum_{x=1}^{\infty}[(R(x)+p(x))(R(x)-p(x+1))] \\
&= \sum_{x=1}^{\infty} R^2(x) - \sum_{x=1}^{\infty} R^2(x) - \sum_{x=1}^{\infty}[R(x)p(x)] \\
&\quad + \sum_{x=1}^{\infty}[R(x)p(x+1)] + \sum_{1}^{\infty}[p(x)p(x+1)] \\
&= -\sum_{x=1}^{\infty}[R(x)p(x)] + \sum_{x=1}^{\infty}[R(x)p(x+1)] + \sum_{x=1}^{\infty}[p(x)p(x+1)] \\
&= -\sum_{x=1}^{\infty}[R(x)p(x)] + \sum_{x=1}^{\infty}[R(x-1)p(x)] - p(1) + \sum_{x=1}^{\infty}[p(x)p(x+1)] \\
&= -P(X_2 > X_1) + P(X_2 \geq X_1) - P(X_1 = 1) + P(X_2 = X_1 + 1) \\
&= P(X_2 = X_1) + P(X_2 = X_1 + 1) - P(X_1 = 1). \qquad (4.6)
\end{aligned}$$

Now, define a function

$$h^*(X_1, X_2) = I(X_2 = X_1) + I(X_2 = X_1 + 1) - I(X_1 = 1),$$

where $I(.)$ is the indicator function. Then it can be seen that $E(h^*(X_1, X_2)) = \Delta(F)$. Define a symmetric kernel

$$h(X_1, X_2) = \frac{1}{2}[2I(X_2 = X_1) + I(X_2 = X_1 + 1) + I(X_2 = X_1 - 1) - I(X_1 = 1) - I(X_2 = 1)].$$

Then a U-statistic based on $h(.)$ is given by

$$\tilde{\Delta}(F) = \frac{1}{\binom{n}{2}} \sum_{i=1}^{n} \sum_{j<i} h(X_i, X_j). \tag{4.7}$$

Then the statistic $\tilde{\Delta}(F)$ is unbiased for $\Delta(F)$. Hence, for large values of $\tilde{\Delta}(F)$, we reject the null hypothesis H_{10} against the alternatives H_{11}.

4.2.2 Asymptotic properties

In this section, we obtain the variance and asymptotic distribution of the above test statistic. We have

$$h(X_1, X_2) = \frac{1}{2}[2I(X_2 = X_1)+I(X_2 = X_1+1)+I(X_2 = X_1-1)-I(X_1 = 1)-I(X_2 = 1)].$$

Now to find the variance of $\Delta(F)$, consider

$$\begin{aligned} h_1(x_1) &= E(h(x_1, X_2)) \\ &= \frac{1}{2}E(2I(X_2 = x_1) + I(X_2 = x_1 + 1) + I(X_2 = x_1 - 1) - I(x_1 = 1) - I(X_2 = 1)) \\ &= \frac{1}{2}[2P(X_2 = x_1) + P(X_2 = x_1 + 1) + P(X_2 = x_1 - 1) - I(x_1 = 1) - P(X_2 = 1)] \\ &= \frac{1}{2}[2p(x_1) + p(x_1 + 1) + p(x_1 - 1) - I(x_1 = 1) - p(1)] \end{aligned} \tag{4.8}$$

where $p(x) = P(X = x)$.

Then we obtain

$$\sigma_1^2 = V(h_1(X_1)) = \frac{1}{4}V\left[2p(X_1) + p(X_1 + 1) + p(X_1 - 1) - I(X_1 = 1)\right] \tag{4.9}$$

Again,

$$\begin{aligned} h_2(x_1, x_2) &= E(h(x_1, x_2)) \\ &= h(x_1, x_2) \end{aligned}$$

Then

$$\begin{aligned} \sigma_2^2 &= V(h_2(X_1, X_2)) \\ &= \frac{1}{4} V[2I(X_2 = X_1) + I(X_2 = X_1 + 1) + I(X_2 = X_1 - 1) \\ &\quad -I(X_1 = 1) - I(X_2 = 1)]. \end{aligned}$$

Hence

$$V(\tilde{\Delta}(F)) = \frac{1}{\binom{n}{2}}[2(n-2)\sigma_1^2 + \sigma_2^2]. \tag{4.10}$$

Theorem 4.1. *As $n \to \infty$, $\sqrt{n}(\tilde{\Delta}(F) - \Delta(F))$ is asymptotically normal with mean 0 and variance $\sigma^2 = V[2p(X) + p(X-1) + p(X+1) - I(X=1)]$.*
Under H_{10}, the variance is given by

$$\sigma_0^2 = [\frac{4(q+1)^2 p^2}{q(1+q+q^2)} + pq - 2(q+1)p^2].$$

Proof. Using the results of Hoeffding (1948), we have the asymptotic normality. Since the kernel has degree 2, the asymptotic variance is given by

$$\sigma^2 = 4\sigma_1^2.$$

Using (4.9) we have

$$\sigma^2 = V[2P(X) + P(X-1) + P(X+1) - I(X=1)].$$

Under H_{10},

$$\begin{aligned}
\sigma_0^2 &= V([2pq^{X-1} + pq^X + pq^{X-2} - I(X=1)]) \\
&= V[(2q + q^2 + 1)pq^{X-2} - I(X=1)] \\
&= (q+1)^4 p^2 V(q^{X-2}) + V(I(X=1)) - 2(q+1)^2 pCov(q^{X-2}, I(X=1)).
\end{aligned} \tag{4.11}$$

Now,

$$\begin{aligned}
E(q^{X-2}) &= \sum_{x=1}^{\infty} q^{x-2} pq^{x-1} \\
&= \frac{p}{q^3} \frac{q^2}{1-q^2} \\
&= \frac{1}{q(1+q)}.
\end{aligned}$$

$$\begin{aligned}
E(q^{2X-4}) &= \sum_{x=1}^{\infty} q^{2x-4} pq^{x-1} \\
&= \frac{1}{q^2} \frac{1}{1+q+q^2}.
\end{aligned}$$

$$E(q^{X-2} I(X=1)) = \sum_{x=1}^{\infty} q^{x-2} pq^{x-1} I(x=1)$$

$$= \frac{p}{q}.$$

$$Cov(q^{X-2}, I(X=1)) = \frac{p}{q} - \frac{1}{q(1+q)}p.$$

Then under the null hypothesis, the asymptotic variance is given by

$$\begin{aligned}
\sigma_0^2 &= [4(q+1)^4p^2(\frac{1}{q^2(1+q+q^2)} - \frac{1}{q^2(1+q)^2}) \\
&\quad +pq - 2(q+1)^2p(\frac{p}{q} - \frac{1}{q(1+q)}p)] \\
&= [\frac{4(q+1)^2p^2}{q(1+q+q^2)} + pq - 2(q+1)p^2]. \qquad (4.12)
\end{aligned}$$

□

Now the test procedure is, for large values of n, we reject the null hypothesis if

$$\frac{\sqrt{n}\tilde{\Delta}(F)}{\sigma_0} > Z_\alpha,$$

where Z_α is the upper α-percentile of $N(0,1)$.

Remark 4.1. *If* $\frac{\sqrt{n}\tilde{\Delta}(F)}{\sigma_0} < -Z_\alpha$, *we reject the null hypothesis in favour of DHR alternatives.*

4.2.3 Testing against IHRA alternatives

Let $X_1, X_2, ..., X_n$ be a random sample from a discrete distribution. We consider the testing problem to test the hypothesis

$$H_{20} : F \quad \text{is geometric}$$

against

$$H_{21} : F \quad \text{is IHRA but not geometric.}$$

Consider

$$\begin{aligned}
\frac{1}{x}\sum_{y=1}^{x} k^*(y) &= \frac{1}{x}\sum_{y=1}^{x} \ln \frac{R(x-1)}{R(x)} \\
&= \frac{1}{x}\sum_{y=1}^{x} \ln R(x-1) - \frac{1}{x}\sum_{y=1}^{x} \ln R(x) \\
&= \frac{-1}{x} \ln R(x) = -\ln(R(x))^{1/x}. \qquad (4.13)
\end{aligned}$$

Hence the random variable X is IHRA if and only if $R(x)^{1/x}$ is decreasing in x.

Under the alternative hypothesis, it can be seen that $R(x)^{1/x}$ is decreasing in x. That is, for any integer $b > 1$,

$$R(x)^{1/x} \quad \geq \quad R(x+b)^{1/(x+b)}$$

or

$$[R(k)]^b \geq [\frac{R(k+b)}{R(k)}]^k, \qquad (4.14)$$

for $k = 1, 2, 3, ..., b = 1, 2, 3, ...$

Hence the hypothesis may also be written as

$$H_{20} : [R(k)]^b = [\frac{R(k+b)}{R(k)}]^k$$

against

$$H_{21} : [R(k)]^b \geq [\frac{R(k+b)}{R(k)}]^k,$$

for $k = 1, 2, 3, ...$ and $b = 1, 2, 3, ...$

Now define

$$\Delta(F) = \sum_{x=0}^{\infty} [R(x)]^b. \tag{4.15}$$

Under null hypothesis,

$$\Delta(F) = \frac{1}{1-q^b}.$$

So $\Delta(F) - \frac{1}{1-q^b}$ can be considered as a measure of departure from the null hypothesis. For testing against IHRA in continuous case, one can refer to Deshpande (1983).

Consider

$$\begin{aligned} \Delta(F) - \frac{1}{1-q^b} &= \sum_{x=0}^{\infty} [R(x)]^b - \frac{1}{1-q^b} \\ &= \sum_{x=0}^{\infty} P(\min(X_1, X_2, ..., X_b) > x) - \frac{1}{1-q^b} \\ &= E(\min(X_1, X_2, ..., X_b)) - \frac{1}{1-q^b}, \end{aligned} \tag{4.16}$$

where $X_1, X_2, ..., X_b$ are independent and identically distributed random variables with distribution function F.

Consider the kernel function

$$h(X_1, X_2, ..., X_b) = \min(X_1, X_2, ..., X_b).$$

Then a U-statistic based on the kernel $h(.)$ is given by

$$\tilde{\Delta}(F) = \frac{1}{\binom{n}{b}} \sum_{\mathbb{I}_b} h(x_{i1}, x_{i2}, ...x_{ib}), \tag{4.17}$$

where $\mathbb{I}_b$ is the set of all permutations $(i_1, i_2, ..., i_b)$ of the integers $(1, 2, ..., n)$. Hence, for large values of $\tilde{\Delta}(F)$, we reject the null hypothesis H_{10} against the alternatives H_{11}.

4.2.4 Asymptotic properties

In this section, we derive the variance and asymptotic distribution of the test statistic. We have $h(X_1, X_2, ..., X_b) = \min(X_1, X_2, ..., X_b)$. To find the variance of $\Delta(F)$, consider

$$\begin{aligned} h_1(x) &= E(h(x, X_2, ..., X_b)) \\ &= E(xI(Z > x) + ZI(Z \leq x)) \\ &= x[R(x)]^{b-1} + E(ZI(Z \leq x)), \end{aligned} \tag{4.18}$$

where $Z = \min(X_2, ..., X_b)$.

Now

$$\begin{aligned} E(ZI(Z \leq x)) &= \sum_{y=1}^{x} z f_{\min}(z) \\ &= \sum_{u=1}^{x} R_{\min}(u-1) - xR_{\min}(x), \end{aligned}$$

where $f_{\min}$ and $R_{\min}$ denote the probability mass function and reliability function of Z.

Then we get

$$h_1(x) = \sum_{u=1}^{x} R_{\min}(u-1).$$

Therefore

$$\begin{aligned}\sigma_1^2 &= V(h_1(X_1)) \\ &= V(\sum_{u=1}^{X} R_{\min}(u-1)).\end{aligned}$$

If $b = 2$, $h(X_1, X_2) = \min(X_1, X_2)$ and

$$\tilde{\Delta}(F) = \frac{1}{\binom{n}{2}} \sum_{i<j} h(X_i, X_j) \tag{4.19}$$

so that $\sigma_2^2 = V(h_2(X_1, X_2)) = V(\min(X_1, X_2))$ and in this case we obtain the variance of the test statistic as

$$V(\tilde{\Delta}(F)) = \frac{1}{\binom{n}{2}}[2(n-2)\sigma_1^2 + \sigma_2^2]. \tag{4.20}$$

Theorem 4.2. *As $n \to \infty$, $\sqrt{n}(\tilde{\Delta}(F) - \Delta(F))$ is asymptotically normal with mean 0 and variance $\sigma^2 = b^2 V[\sum_{u=1}^{X} R(u-1)^{b-1}]$. Under H_{20}, the variance is given by*

$$\sigma_0^2 = b^2 \left[\frac{pq^{2(b-1)}}{(1-q^{2b-1})} - \frac{p^2 q^{2(b-1)}}{(1-q^b)^2} \right].$$

Proof. Using the results of Hoeffding (1948), we have the asymptotic normality. Here, we obtain the asymptotic variance σ^2 as

$$\begin{aligned}\sigma^2 &= b^2 V(h_1(X) \\ &= b^2 V(\sum_{u=1}^{X} R_{\min}(u-1))\end{aligned}$$

Then, under H_{20},

$$
\begin{aligned}
\sigma_0^2 &= b^2 V(h_1(X)) \\
&= b^2 V\left[\sum_{u=1}^{X} (q^{u-1})^{b-1}\right] \\
&= b^2 V\left[\mu_{min} - \sum_{u=X+1}^{\infty} (q^{u-1})^{b-1}\right] \\
&= b^2 V\left[\mu_{min} - \frac{[q^{b-1}]^X}{1-q^{b-1}}\right] \\
&= b^2 \left[\frac{pq^{2(b-1)}}{(1-q^{2b-1})} - \frac{p^2 q^{2(b-1)}}{(1-q^b)^2}\right],
\end{aligned}
$$

where $\mu_{min} = E(\min(X_2, ..., X_b))$. □

Hence, for large values of n, we reject the null hypothesis if

$$\frac{\sqrt{n}\tilde{\Delta}(F)}{\sigma_0} > Z_\alpha,$$

where Z_α is the upper α-percentile of $N(0,1)$.

Remark 4.2. *If $\frac{\sqrt{n}\tilde{\Delta}(F)}{\sigma_0} < -Z_\alpha$, we reject the null hypothesis in favour of DHRA alternatives.*

4.3 A simulation study

In this section, we carry out a simulation study to evaluate the performance of our tests against various alternatives. The simulation is done using R program.

To calculate the empirical power, we simulate observations from discrete Weibull distribution where the survival function of the Type I discrete Weibull distribution is given by

$$R(k) = q^{k^\beta} \tag{4.21}$$

and the hazard rate is given by

$$r(k) = 1 - q^{k^\beta - (k-1)^\beta}, \tag{4.22}$$

where q is the probability of surviving the first demand. The distribution is IHR for $\beta > 1$, DHR for $0 < \beta < 1$ and for $\beta = 1$, it reduces to the geometric distribution.

We calculate the test statistic for different sample sizes and check whether the test statistic accepts or rejects the null hypothesis. Then we repeat the whole procedure ten thousand times and we calculate empirical type I error and empirical power of the test. Table 4.1 gives the empirical type I error and power of the test when the alternative hypothesis is IHR. From Table 4.1, we can see that power of the test approaches one as the sample size increases. Similarly, Table 4.2 gives the empirical type I error and power of the test when the alternative hypothesis is IHRA. Here also, we can see that power of the test approaches one as the sample size increases.

Table 4.1 : Empirical Type I Error and Power (IHR)

q	n	Type I Error		Power	
		5% level	1% level	5% level	1% level
0.5	20	0.048	0.0164	0.4218	0.1224
	40	0.0407	0.0096	0.7969	0.4477
	60	0.0369	0.0092	0.9778	0.7417
	80	0.0350	0.0090	0.9973	0.9312
	100	0.0343	0.0078	0.9998	0.9905
0.7	20	0.0650	0.0105	0.7215	0.5916
	40	0.0599	0.0090	0.9581	0.774
	60	0.0571	0.0080	0.9991	0.9948
	80	0.0554	0.0076	1.000	0.9920
	100	0.0518	0.0070	1.000	0.9998
0.9	20	0.0450	0.0100	0.7313	0.5913
	40	0.0409	0.0080	0.9068	0.8291
	60	0.0390	0.0074	0.9905	0.9743
	80	0.0359	0.0069	0.9992	0.9976
	100	0.0345	0.0068	0.9996	0.9983

Table 4.2 : Empirical Type I Error and Power (IHRA)

q	n	Type I Error		Power	
		5% level	1% level	5% level	1% level
0.5	20	0.057	0.012	0.844	0.642
	40	0.051	0.0098	0.984	0.939
	60	0.049	0.0095	0.811	0.598
	80	0.0402	0.0093	0.893	0.764
	100	0.0393	0.0088	0.942	0.831
0.7	20	0.0601	0.0108	0.823	0.687
	40	0.0572	0.0092	0.924	0.827
	60	0.0522	0.0085	0.965	0.894
	80	0.0501	0.0079	0.981	0.927
	100	0.0498	0.0072	0.999	0.934
0.9	20	0.046	0.0101	0.553	0.452
	40	0.0422	0.0086	0.689	0.593
	60	0.0375	0.0079	0.797	0.708
	80	0.0332	0.0067	0.840	0.768
	100	0.0315	0.0064	0.895	0.840

4.4 Conclusion

In this Chapter, we developed a simple test for testing geometric against two ageing classes of life distributions namely, IHR and IHRA using U-statistics. The test statistics provided here are easily computable. Asymptotic properties of the test statistics were studied. Simulation study showed that the proposed tests perform well.

Chapter 5

A Non-parametric Test for Geometric Distribution against NBU and NBUE Alternatives [1]

5.1 Introduction

In Chapter 4, we discussed the testing problems associated with IHR and IHRA ageing classes. In studying the ageing behaviour, apart from hazard rate, the notions based on residual life also have an important role. The NBU and NBUE classes are the most popular notions based on the properties of residual life. In this chapter, we provide a new simple approach for testing geometric distribution against the above mentioned two classes. The chapter is organized as follows. In Section 5.2, we develop test statistics for testing the constant hazard rate against NBU and NBUE ageing classes for discrete lifetime data using U-statistics. Section 5.3 provides a simulation study to assess the performance of the tests. Finally, Section 5.4 concludes the study.

[1]The results in this chapter have been communicated as entitled (*see* Kattumannil et al., 2013b).

5.2 A non-parametric test for constant hazard rate

Let X be a non-negative integer-valued random variable with support $N = \{1, 2, ...\}$ or a subset thereof. We recall the following definitions of NBU and NBUE in discrete setup.

Definition 5.1. *Let X be a discrete random variable defined on $\mathbb{N}$, then*

(i) X is said to be NBU (NWU) if $R(x + y) \leq (\geq)R(x)R(y)$ for all x and y in $\mathbb{N}$.

(ii) X is said to be NBUE (NWUE) if $m(x) \leq (\geq)\mu$ for all x in $\mathbb{N}$, where $\mu = E(X)$ and $m(x) = E(X - x|X > x)$.

Next we develop tests based on U-statistics for testing geometric against NBU and NBUE alternatives. It is well known that the constant hazard rate and the constant mean residual life characterizes the geometric distribution. We will make use of this property to develop a non-parametric test.

5.2.1 Testing against NBU alternatives

Let $X_1, X_2, ..., X_n$ be a random sample from F. We are interested to test the hypothesis

$$H_{10} : F \quad \text{is geometric}$$

against

$$H_{11} : F \quad \text{is NBU but not geometric.}$$

Under alternative hypothesis, it can be seen that $R(x)R(y) - R(x+y)$ is non-negative and zero under null hypothesis . Hence the quantity defined by

$$\Delta(F) = \sum_{x=1}^{\infty}\sum_{y=1}^{\infty}[R(x)R(y) - R(x+y)] \tag{5.1}$$

is a measure of departure from the null hypothesis towards the alternative hypothesis.

Note that $E(X) = \sum_{x=1}^{\infty} P(X \geq x) = \sum_{x=0}^{\infty} R(x)$.

Consider

$$\begin{aligned}
\sum_{x=1}^{\infty}\sum_{y=1}^{\infty}[R(x)R(y) - R(x+y)] &= \sum_{x=0}^{\infty}\sum_{y=0}^{\infty}[R(x)R(y) - R(x+y)] \\
&= \sum_{x=0}^{\infty} R(x) \sum_{y=0}^{\infty} R(y) - \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} R(x+y) \\
&= \mu^2 - \sum_{x=0}\sum_{y=0} P(X > x+y) \\
&= \mu^2 - E[\sum_{x=0}\sum_{y=0} I(X > x+y)] \\
&= \mu^2 - E[\sum_{x=0}^{\infty}\sum_{u=x}^{\infty} I(X > u)] \\
&= \mu^2 - E[\sum_{u=0}^{\infty}(u+1)I(X > u)] \\
&= \mu^2 - E[\sum_{u=0}^{X-1}(u+1)] \\
&= \mu^2 - \frac{1}{2}E[X^2 + X]. \qquad (5.2)
\end{aligned}$$

Under H_{10}, $\Delta(F) = 0$ and $\Delta(F) > 0$ under H_{11}.

Consider $h^*(X_1, X_2) = 2X_1X_2 - X_1^2 - X_1$, then it can be seen that $E(h^*(X_1, X_2)) = 2\Delta(F)$.

Defining symmetric kernel

$$h(x_1, x_2) = \frac{1}{4}(4x_1x_2 - x_1^2 - x_2^2 - x_1 - x_2),$$

an unbiased estimate of $\Delta(F)$ is given by

$$\tilde{\Delta}(F) = \frac{1}{\binom{n}{2}} \sum_{i=1}^{n} \sum_{j<i} h(x_i, x_j). \tag{5.3}$$

For large values of $\tilde{\Delta}(F)$, we reject the null hypothesis H_{10} against the alternative hypothesis H_{11}. Note that $\tilde{\Delta}(F)$ is a U-statistic for $\Delta(F)$ based on $h(.)$.

5.2.2 Asymptotic properties

In this section, we derive the variance and asymptotic distribution of the above statistic. We have $h(X_1, X_2) = \frac{1}{4}(4X_1X_2 - X_1^2 - X_2^2 - X_1 - X_2)$. Now, to find the variance of $\tilde{\Delta}(F)$, we consider the following.

$$\begin{aligned} h_1(x_1) &= E(h(x_1, X_2)) \\ &= \frac{1}{4}E(4x_1X_2 - x_1^2 - x_1 - X_2^2 - X_2) \\ &= \frac{1}{4}[4x_1\mu - x_1^2 - x_1 - \mu_2' - \mu], \end{aligned}$$

where $\mu = E(X_2)$ and $\mu_2' = E(X_2^2)$.

$$\begin{aligned} \sigma_1^2 &= V(h_1(X_1)) \\ &= \frac{1}{16}V((4\mu - 1)X_1 - X_1^2). \end{aligned} \tag{5.4}$$

Again,

$$\begin{aligned} h_2(x_1, x_2) &= E(h(x_1, x_2)) \\ &= h(x_1, x_2). \end{aligned}$$

Therefore

$$\begin{aligned} \sigma_2^2 &= V(h_2(X_1, X_2)) \\ &= \frac{1}{16} V(4X_1X_2 - X_1^2 - X_2^2 - X_1 - X_2) \\ &= \frac{1}{16} V(4X_1X_2 - X_1(X_1+1) - X_2(X_2+1)) \\ &= \frac{1}{16}(V_1 + 2V_2 - 4C_1), \qquad (5.5) \end{aligned}$$

where $V_1 = V(4X_1X_2)$, $V_2 = V(X_1(X_1+1)) = V(X_2(X_2+1))$, and $C_1 = Cov(4X_1X_2, X_1(X_1+1)) = Cov(4X_1X_2, X_2(X_2+1))$. Hence, we obtain the variance of $\tilde{\Delta}(F)$ as

$$V(\tilde{\Delta}(F)) = \frac{1}{\binom{n}{2}}[2(n-2)\sigma_1^2 + \sigma_2^2]. \qquad (5.6)$$

Next, we discuss the asymptotic distribution of the test statistic.

Theorem 5.1. *As* $n \to \infty$, $\sqrt{n}(\tilde{\Delta}(F) - \Delta(F))$ *is asymptotically normal with mean* 0 *and variance* $\sigma^2 = \frac{1}{4}V((4\mu - 1)X - X^2)$. *Under* H_{10}, $\sigma_0^2 = \frac{q^2}{p^4}$.

Proof. Using the results of Hoeffding (1948), we have the asymptotic normality. Now to find the asymptotic variance,

$$\sigma^2 = m^2\sigma_1^2.$$

Therefore

$$\begin{aligned} \sigma^2 &= 4V(h_1(X)) \\ &= \frac{1}{4}V(4\mu X - X^2 - X). \end{aligned}$$

Under H_{10},

$$V(X^2) = \frac{4q(2q+3) + pq(p-4)}{p^4}$$

and

$$Cov(X, X^2) = \frac{q^2 + 3q}{p^3}.$$

Therefore

$$\begin{aligned} \sigma_0^2 &= \frac{1}{4}V((4\mu - 1)X - X^2) \\ &= \frac{1}{4}(4\mu - 1)^2 V(X) + \frac{1}{4}V(X^2) - \frac{1}{4}2(4\mu - 1)Cov(X, X^2) \\ &= \frac{q^2}{p^4}. \end{aligned}$$

□

Hence, for large values of n, we reject the null hypothesis H_{10} in favour of H_{11} if

$$\sqrt{n}\tilde{\Delta}(F)\frac{p^2}{q} > Z_\alpha,$$

where Z_α is the upper α-percentile of $N(0,1)$.

Remark 5.1. *If $\sqrt{n}\tilde{\Delta}(F)\frac{p^2}{q} < -Z_\alpha$, we reject the null hypothesis H_{10} in favour of NWU alternatives.*

5.2.3 Testing against NBUE alternatives

Let $X_1, X_2, ..., X_n$ be a random sample from a discrete distribution F. Consider the testing problem

$$H_{20} : F \quad \text{is geometric}$$

against

$$H_{21} : F \quad \text{is NBUE but not geometric.}$$

We know that $m(x) = \mu$, characterizes the geometric distribution specified in (2.67). Hence using the Definition 5.1, the above testing problem can be written as

$$H_{20} : \mu = m(x) \quad \forall x$$

against

$$H_{21} : \mu > m(x) \quad for some x.$$

Under the alternative hypothesis, it can be seen that $\mu - m(x) > 0,$ for some x. We define

$$\Delta^*(F) = \sum_{x=0}^{\infty}[\mu - m(x)]p(x)R(x). \tag{5.7}$$

Under H_{20}, $\Delta^*(F) = 0$ and $\Delta^*(F)$ is positive under H_{21}. So $\Delta^*(F)$ can be considered as a measure of departure from the null hypothesis. Consider

$$\begin{aligned}\Delta^*(F) &= \sum_{x=0}^{\infty}[\mu - m(x)]p(x)R(x) \\ &= \mu\sum_{x=0}^{\infty}p(x)R(x) - \sum_{x=0}^{\infty}p(x)\sum_{y=x}^{\infty}R(y)\end{aligned}$$

$$\begin{aligned} &= \mu P(X_1 > X_2) - \sum_{y=0}^{\infty} R(y)(1 - R(y)) \\ &= \mu[P(X_1 > X_2) - 1] + E(\min(X_1, X_2)) \\ &= E(\min(X_1, X_2)) - \mu P(X_1 \leq X_2). \end{aligned} \tag{5.8}$$

where X_1 and X_2 are independent and identically distributed random variables with distribution function F.

Let

$$\tilde{\Delta}^*(F) = U_1 - U_2 U_3 \tag{5.9}$$

where $U_1 = \frac{1}{\binom{n}{2}} \sum_{i=1}^{n} \sum_{i<j} \min(x_i, x_j)$, $U_2 = \frac{1}{n} \sum_i x_i$ and $U_3 = \frac{1}{\binom{n}{2}} \sum_{i=1}^{n} \sum_{i<j} I(x_i \leq x_j)$. Clearly $\tilde{\Delta}^*(F)$ is an estimator of $\Delta^*(F)$. We reject H_{20} for large values of $\tilde{\Delta}^*(F)$. The following theorem provides the asymptotic normality of the test statistic.

Theorem 5.2. *As $n \to \infty$, $\sqrt{n}(\tilde{\Delta}^*(F) - \Delta^*(F))$ is asymptotically normal with mean 0 and variance $\sigma^2 = \sigma^2_{(1)} + \mu^2 \sigma^2_{(3)} + [P(X_1 \leq X_2)]^2 \sigma^2_{(2)} - 2\mu\sigma_{13} - 2P(X_1 \leq X_2)\sigma_{12} + 2\mu\sigma_{23}$ where $\sigma^2_{(1)}$, $\sigma^2_{(2)}$ and $\sigma^2_{(3)}$ are the asymptotic variances of U_1, U_2 and U_3 respectively and $\sigma_{ij} = Cov(U_i, U_j)$.*

Proof. Using the results of Hoeffding (1948), we have

$$\sqrt{n}(U_1 - E(\min(X_1, X_2))) \sim N(0, \sigma^2_{(1)})$$

$$\sqrt{n}(U_2 - \mu) \sim N(0, \sigma^2_{(2)})$$

and

$$\sqrt{n}(U_3 - P(X_1 \leq X_2)) \sim N(0, \sigma^2_{(3)}).$$

Now,

$$\begin{aligned}\sqrt{n}(\tilde{\Delta}^*(F) - \Delta^*(F)) &= \sqrt{n}(U_1 - U_2U_3 - E(\min(X_1, X_2)) - \mu P(X_1 \leq X_2)) \\ &= \sqrt{n}(U_1 - E(\min(X_1, X_2))) - \sqrt{n}(U_2U_3 - U_2P(X_1 \leq X_2) \\ &\quad +U_2P(X_1 \leq X_2) - \mu P(X_1 \leq X_2)) \\ &= \sqrt{n}(U_1 - E(\min(X_1, X_2))) - U_2\sqrt{n}(U_3 - P(X_1 \leq X_2)) \\ &\quad +P(X_1 \leq X_2)(U_2 - \mu). \qquad (5.10)\end{aligned}$$

Let

$$\begin{aligned}h_1(x) &= E(\min(x, X_2)) \\ &= xR(x-1) + E(X_2I(X_2 < x))\end{aligned}$$

and

$$\begin{aligned}h_1'(x) &= P(x \leq X_2) \\ &= R(x-1).\end{aligned}$$

Hence

$$\sigma_{(1)}^2 = 4Var(h_1(X)),$$

$$\sigma_{(2)}^2 = Var(X),$$

$$\sigma_{(3)}^2 = 4Var(h_1'(X)),$$

$$\begin{aligned}\sigma_{12} &= Cov(E[min(X, X_2)], X) \\ &= Cov(XR(X), X) + Cov(X, E(X_2 I(X_2 \leq X))),\end{aligned}$$

$$\begin{aligned}\sigma_{13} &= Cov(E[min(X, X_2)], R(X-1)) \\ &= Cov(XR(X), R(X-1)) + Cov(E(X_2 I(X_2 \leq X)), R(X-1)),\end{aligned}$$

and

$$\sigma_{23} = Cov(X, R(X-1)).$$

Under H_{20},

$$\sigma^2_{(1)} = \frac{1}{p^2}\frac{q^3}{(1+q)^2(1+q+q^2)},$$

$$\sigma^2_{(2)} = \frac{q}{p^2},$$

$$\sigma^2_{(3)} = \frac{q}{(1+q)^2(1+q+q^2)},$$

$$\sigma_{12} = \frac{q}{p(1-q^2)} - \frac{pq}{1-q^2}[\frac{1+q^2}{(1-q^2)^2}],$$

$$\sigma_{13} = \frac{q}{p}\frac{1-q^2-q^3}{(1+q)^3(1+q+q^2)^2} - \frac{q^2}{p}\frac{2+2q+q^2}{(1+q)^3(1+q+q^2)^2},$$

and

$$\sigma_{23} = \frac{p-(1-q^2)}{(1-q^2)^2}.$$

Then the asymptotic variance of the test statistic under H_{20} is given by

$$\sigma_0^2 = \frac{q^2 + 6q^3 + 8q^4 + 5q^5 + 2q^6}{p^2(1+q)^3(1+q+q^2)^2}. \tag{5.11}$$

□

Hence, for large values of n, we reject the null hypothesis H_{20} in favour of H_{22} if

$$\frac{\sqrt{n}\tilde{\Delta}^*(F)}{\sigma_0} > Z_\alpha,$$

where Z_α is the upper α-percentile of $N(0,1)$.

Remark 5.2. *If $\frac{\sqrt{n}\tilde{\Delta}^*(F)}{\sigma_0} < -Z_\alpha$, we reject the null hypothesis in favour of NWUE alternatives.*

5.3 A simulation study

In this section, we carry out a simulation study to evaluate the performance of our tests against various alternatives. The simulation was done using R program.

To calculate the empirical power, we simulate observations from discrete Weibull distribution where the survival function of the Type I discrete Weibull distribution is given by

$$R(k) = q^{k^\beta}, \quad k = 1, 2, 3, \ldots.$$

and the hazard rate is given by

$$r(k) = 1 - q^{k^\beta - (k-1)^\beta},$$

where q is the probability of surviving the first demand. The distribution is IHR for $\beta > 1$, DHR for $0 < \beta < 1$ and for $\beta = 1$, it reduces to the geometric distribution.

We calculate the test statistic for different sample sizes. We check whether the test statistic accepts or rejects the null hypothesis. Then we repeat the whole procedure ten thousand times and we calculate empirical type I error and empirical power of the test. Table 5.1 gives the empirical type I error and power of the test when the alternative hypothesis is NBU. From Table 5.1, we can see that power of the test approaches one as the sample size increases. Similarly, Table 5.2 gives the empirical type I error and power of the test when the alternative hypothesis is NBUE. Here also, we can see that power of the test approaches one as the sample size increases.

5.4 Conclusion

In this chapter, we developed a simple test for testing geometric against the two ageing classes of life distributions namely, NBU and NBUE using U-statistics. The test statistics provided here are easily computable. Asymptotic properties of the test statistics were studied. Simulation study showed that the proposed tests perform well.

Table 5.1 : Empirical Type I Error and Power (NBU)

q	n	Type I Error		Power	
		5% level	1% level	5% level	1% level
0.5	20	0.0440	0.0100	0.3264	0.2448
	40	0.0432	0.0090	0.5626	0.4130
	60	0.0399	0.0089	0.6339	0.4938
	80	0.0372	0.0078	0.8436	0.7564
	100	0.0354	0.0075	0.9511	0.9103
0.7	20	0.0480	0.0105	0.4561	0.2938
	40	0.0425	0.0090	0.5626	0.3800
	60	0.0397	0.0080	0.5591	0.4130
	80	0.0361	0.0076	0.8436	0.7564
	100	0.0350	0.0070	0.9695	0.9283
0.9	20	0.0450	0.0100	0.2870	0.3306
	40	0.0409	0.0080	0.3905	0.3778
	60	0.0390	0.0074	0.4631	0.4309
	80	0.0359	0.0069	0.7908	0.6525
	100	0.0345	0.0068	0.9513	0.9105

Table 5.2 : Empirical Type I Error and Power (NBUE)

q	n	Type I Error		Power	
		5% level	1% level	5% level	1% level
0.5	20	0.0770	0.0260	0.7203	0.5922
	40	0.0670	0.0200	0.8170	0.6040
	60	0.0590	0.0150	0.9880	0.9090
	80	0.0530	0.0130	0.9990	0.9920
	100	0.0490	0.0110	1.0000	0.9998
0.7	20	0.0600	0.0130	0.7445	0.5216
	40	0.0540	0.0090	0.8560	0.6350
	60	0.0480	0.0080	0.9930	0.9400
	80	0.0440	0.0070	0.9999	0.9990
	100	0.0360	0.0050	1.0000	1.0000
0.9	20	0.0490	0.0150	0.6980	0.5923
	40	0.0450	0.0112	0.8770	0.6730
	60	0.0400	0.0080	0.9980	0.9560
	80	0.0350	0.0070	0.9997	0.9980
	100	0.0300	0.0060	1.0000	1.0000

Chapter 6

Proportional Reversed Hazards Model for Discrete Data [1]

6.1 Introduction

In lifetime data analysis, the concept of reversed hazard rate has potential application when the time elapsed since failure is a quantity of interest in order to predict the actual time of failure. The reversed hazard rate is more useful in estimating reliability function when the data are left censored or right truncated. For recent discussion on reversed hazard rate in continuous domain see Block et al. (1998), Gupta and Nanda (2001), Finkelstein (2002), Chandra and Roy (2001, 2005), Kundu and Gupta (2003), Gupta et al. (2006), Gupta and Gupta (2007) and Nanda and Das (2011). A little work has been carried out in discrete setup in connection with reversed hazard rate. Recently, Nanda and Sengupta (2005) have discussed reversed hazard rate in discrete setup and obtained several interesting results. Gupta et al. (2006) characterized certain class of discrete life distributions by means of a relationship between reversed hazard rate and right truncated expectation. The results by Gupta et al. (2006) enlightened the role of reversed hazard rate and the right truncated expectation in characterizing discrete life distributions which was not covered by the earlier authors.

[1] The results in this chapter have been communicated as entitled (*see* Mathew et al., 2013).

Recently, Nawata et al. (2008) analyzed the length of the hospital stay of patients hospitalized for cataract and related diseases. This will help the government to make future medical policies, such as medical care payments, medical insurance for poor people etc. The length of stay may depend on several factors like sex, age, prevalence of other diseases like diabetes, mental strength etc. Nawata et al. (2008) considered a proportional hazards model (PHM) in discrete setup (also see Nawata et al. (2009)) to incorporate such factors. However, the model described by them is not suitable as it does not account for all factors and as the hazard rate $k(x)$ is bounded by one. Chen and Manatunga (2007) gave a detailed discussion on the problems related to the proportional hazards model in discrete setup. Making use of the new definition of hazard rate $k^*(x)$, Dewan and Kattumannil (2009) proposed a proportional hazards model for discrete data as parallel to continuous case. The model proposed by Dewan and Kattumannil (2009) can also incorporate continuous covariates. For a discussion on proportional odds model, one can refer to Khorashadizadeh et al. (2013).

As parallel to PHM model, the proportional reversed hazards model (PRHM) has become popular in recent times. For survey and some properties of PRHM in continuous domain, see Gupta and Gupta (2007) and references therein. Nanda and Paul (2003) developed a test for reversed hazard rates under the assumption of proportionality of reversed hazard rates, Recent work by Nanda and Das (2011) enlightened the role of PRHM models in modeling lifetime distributions. The PRHM in discrete setup has potential application in system reliability. Consider a parallel system with n independent components. If the lifetime of each component has distribution function $F(.)$, then the lifetime of the system $Z = max[X_1, X_2, ..., X_n]$ has the distribution given by

$$F_{(n)}(x) = [F(x)]^n \tag{6.1}$$

In continuous domain, the model (6.1) is called the PRHM (see Gupta et al. (1998)). The model (6.1) does not follow a PRHM in discrete setup when we use the reverse hazard rate $\lambda(.)$ defined in (1.2). This motivate us to develop a PRHM in the discrete data which is analogous to the continuous case.

In this chapter, we introduce a new definition for reversed hazard rate and study its properties. We also propose a proportional reversed hazards model for discrete data parallel to the version for continuous data. We discuss the preservation of ageing properties under the model. Stochastic comparisons are also given. Finally, we develop a non-parametric test for testing the proportionality constant of the proposed model.

6.2 Modified reversed hazard rate

As seen, there are some difference between reversed hazard rate in discrete and continuous cases. That is, in discrete case, reversed hazard rate is conditional probability, it is bounded by one. The reversed hazard rate is not additive in the case of parallel system. Hence, we define a new reversed hazard rate.

Definition 6.1. *Let X be a discrete random variable with support $\mathbb{N}$ or a subset of it. Let $p(.)$ and $F(.)$ be the probability mass function and distribution function of X, respectively. The modified reversed hazard rate is defined as*

$$\lambda^*(x) = \ln \frac{F(x)}{F(x-1)}. \tag{6.2}$$

This definition can be treated as the discrete version of reversed hazard rate in the continuous domain. In the continuous case, the reversed hazard rate is defined as

$$\lambda^*(x) = \frac{F^{'}(x)}{F(x)} = \frac{\partial \ln F(x)}{\partial x}.$$

In the above definition, if we take $[\ln F(x) - \ln F(x-1)]$ for $\frac{\partial \ln F(x)}{\partial x}$ so that (6.1) follows. Note that $\lambda^*(x)$ is not a probability, not bounded by one and is additive for parallel system as in the continuous case. The cumulative reversed hazard rate is given by

$$\Lambda^*(x) = \sum_{k=x}^{\infty} \lambda^*(k) = -\ln F(x).$$

Hence the function $\lambda^*(.)$ determines the distribution of X uniquely by the relation

$$F(x) = \exp(\sum_{k=x}^{\infty} -\lambda^*(k)). \tag{6.3}$$

The functions $\lambda(.)$ and $\lambda^*(.)$ are related through

$$\lambda(x) = 1 - \exp(-\lambda^*(x)),$$

so that both $\lambda(.)$ and $\lambda^*(.)$ have same monotonicity properties. That is $\lambda(.)$ is increasing (decreasing) if and only if $\lambda^*(.)$ is increasing (decreasing) in x. Hence any of these functions can be used to study the monotonicity properties.

Remark 6.1. *Non-negative random variables with support* $[0, \infty)$ *cannot have increasing reversed hazard rates. (see Block et al. (1998)).*

Now we give the expressions for reverse hazard rate of some discrete distributions.

Example 6.1. *Consider the discrete Pareto distribution with distribution function given by*

$$F(x) = 1 - (\frac{d}{x+d})^c, \quad x = 1, 2, ..., \ c, d > 0.$$

The reversed hazard rate $\lambda^*(x)$ *is given by*

$$\begin{aligned} \lambda^*(x) &= \ln \frac{F(x)}{F(x-1)} \\ &= \ln \frac{1-(\frac{d}{x+d})^c}{1-(\frac{d}{x+d-1})^c}, \ d, c > 0. \end{aligned}$$

Example 6.2. *Consider the discrete logistic distribution with distribution function given by*

$$F(x) = 1 - \frac{\exp(-\frac{x-c}{d}) + \exp(-\frac{x}{d})}{1 + \exp(-\frac{x-c}{d})}, \quad x = 1, 2, ..., \ c, d > 0.$$

The reversed hazard rate $\lambda^*(x)$ *is given by*

$$\lambda^*(x) = \ln \frac{(1-\exp(-\frac{x}{d}))(1+\exp(-\frac{x-c-1}{d}))}{(1+\exp(-\frac{x-c}{d}))(1-\exp(-\frac{x-1}{d}))}, d, c > 0.$$

Example 6.3. *Consider the discrete Weibull distribution (Nakagawa-Osaki Model, Bracquemond and Gaudoin (2003)) with distribution function given by*

$$F(x) = 1 - (q^x)^\beta, \quad x = 1, 2, ..., \ \beta > 0, \ 0 < q < 1.$$

The reversed hazard rate $\lambda^*(x)$ *is given by*

$$\lambda^*(x) = \log \frac{1-(q^x)^\beta}{1-(q^{x-1})^\beta}, \ \beta > 0, \ 0 < q < 1.$$

The following theorem gives a sufficient condition to check the monotonicity of the reversed hazard rate.

Theorem 6.1. *Let X be a random variable with probability mass function $p(.)$. Let $l(x) = \frac{p(x)}{p(x+1)}$. If $l(x)$ is increasing in x then X has decreasing reversed hazard rate (DRHR).*

Proof. Suppose $l(x)$ is increasing in x, then $l(x) - l(x-1) > 0$. Consider,

$$\begin{aligned}
\frac{1}{\lambda(x+1)} - \frac{1}{\lambda(x)} &= \frac{F(x+1)}{p(x+1)} - \frac{F(x)}{p(x)} \\
&= \frac{\sum_{j=1}^{x+1} p(j)}{p(x+1)} - \frac{\sum_{j=1}^{x} p(j)}{p(x)} \\
&= \frac{p(x)}{p(x+1)} + \frac{p(x-1)}{p(x+1)} + ... + \frac{p(1)}{p(x+1)} \\
&\quad - \frac{p(x-1)}{p(x)} - \frac{p(x-2)}{p(x)} - ... - \frac{p(1)}{p(x)}.
\end{aligned}$$

Writing $\frac{p(x-j)}{p(x+1)} = \frac{p(x-j)}{p(x-j+1)} \frac{p(x-j+1)}{p(x-j+2)} ... \frac{p(x)}{p(x+1)}$, $j = 1, 2, ...x$, we obtain

$$\begin{aligned}
\frac{1}{\lambda(x+1)} - \frac{1}{\lambda(x)} &= l(x-1)[l(x) - l(x-2)] + l(x-1)l(x-2)[l(x) - l(x-3)] + ... \\
&\quad + \prod_{k}^{x-3} l(x-k)l(x) - l(x-k+1) \\
&\quad + [l(x) - l(x-1)] + \frac{p(1)}{p(x+1)} \\
&= \sum_{j=1}^{x-3} \{(\prod_{k=1}^{j} l(x-k))[l(x) - l(x-(j+1))]\} \\
&\quad + [l(x) - l(x-1)] + \frac{p(1)}{p(x+1)}.
\end{aligned}$$

When $l(x)$ is increasing in x, each term in the above sum is greater than zero, which implies that $\lambda(x)$ is decreasing in x. Since $\lambda(.)$ and $\lambda^*(.)$ have same monotonicity properties, $\lambda^*(.)$

is also decreasing. Hence the theorem. □

Next we give some illustrative examples.

Example 6.4. *Let X be a binomial random variable with parameters n and θ. Then*

$$p(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}\ x = 0, 1, 2, ...$$

Consider

$$\begin{aligned} l(x) &= \frac{p(x)}{p(x+1)} \\ &= \frac{\binom{n}{x}\theta^x(1-\theta)^{n-x})}{\binom{n}{x+1}\theta^{x+1}(1-\theta)^{n-x-1})} \\ &= \frac{x+1}{n-x}\frac{1-\theta}{\theta}. \end{aligned}$$

Clearly $l(x)$ is an increasing function in x, hence binomial distribution has DRHR.

Example 6.5. *Let X be a Poisson random variable with parameters η. Then*

$$p(x) = \frac{e^{-\eta}\eta^x}{x!},\ x = 0, 1, 2, ...$$

Consider

$$\begin{aligned} l(x) &= \frac{p(x)}{p(x+1)} \\ &= \frac{e^{-\eta}\eta^x}{x!}\frac{(x+1)!}{e^{-\eta}\eta^{x+1}} \\ &= \frac{x+1}{\eta}. \end{aligned}$$

This is an increasing function in x, accordingly Poisson distribution has DRHR.

The converse of Theorem 6.1 need not be true. The following example illustrates this fact.

Example 6.6. *Consider the discrete random variable X with probability distribution*

X	1	2	3	4	5
$p(x)$	0.1	0.4	0.3	0.1	0.1

The reversed hazard rate $\lambda(x)$ and $l(x)$ are given by

X	1	2	3	4	5
$F(x)$	0.1	0.5	0.8	0.9	1
$\lambda(x)$	1	0.8	0.37	0.11	0.1
$l(x)$	0.25	1.3	3	1	-

Here $\lambda(x)$ is decreasing, but $l(x)$ is not increasing.

6.3 Proportional reversed hazards model

The PRHM in discrete setup has potential application in system reliability. Consider a parallel system with n independent components. If the lifetime of each component has distribution function $F(.)$ then the lifetime of the system $Z = max[X_1, X_2, ..., X_n]$ has the distribution given by

$$F_{(n)}(x) = [F(x)]^n. \tag{6.4}$$

The reversed hazard rate corresponding to Z, is given by

$$\begin{aligned}\lambda^*_{F_n}(x) &= \log\big[\frac{F_{(n)}(x)}{F_{(n)}(x-1)}\big]\\ &= \log\big[\frac{F(x)}{F(x-1)}\big]^n\\ &= n\log\big[\frac{F(x)}{F(x-1)}\big]\\ &= n\lambda^*(x).\end{aligned}$$

Clearly $\lambda^*_{F_n}(x)$ is proportional to $\lambda^*(x)$. Hence, we can define a PRHM in discrete setup as follows. Let $G(.)$ be the distribution function of the random variable Y and denote $\bar{G}(y) = 1 - G(y)$. Suppose that $G(x)$ is related through $F(x)$ as

$$G(x) = [F(x)]^\theta,\ \theta > 0. \tag{6.5}$$

Then the model (6.5) is called the PRHM as the reversed hazard rate of Y is proportional to that of X. The probability mass function of Y can be expressed as

$$\begin{aligned}g(y) &= G(y) - G(y-1)\\ &= [F(y)]^\theta - [F(y-1)]^\theta\\ &= [F(y)]^\theta\left[1 - \frac{F(y-1)^\theta}{F(y)^\theta}\right]\\ &= [F(y)]^\theta\left[1 - e^{-\theta\lambda^*(y)}\right].\end{aligned} \tag{6.6}$$

The r^{th} moment of Y is given by

$$E(Y^r) = \sum_{k=1}^{\infty} k^r g(k)$$

$$= \sum_{k=1}^{\infty} k^r([F(k)]^\theta - [F(k-1)]^\theta).$$

In particular,

$$E(Y) = \sum_{k=0}^{\infty} \bar{G}(k) = \sum_{k=0}^{\infty}(1 - [F(k)]^\theta)$$

$$\begin{aligned} E(Y^2) &= \sum_{k=0}^{\infty}(2k+1)\bar{G}(k) \\ &= \sum_{k=0}^{\infty}(2k+1)(1 - [F(k)]^\theta). \end{aligned}$$

Hence, the variance is given by

$$V(Y) = \sum_{k=0}^{\infty}(2k+1)(1 - [F(k)]^\theta) - (\sum_{k=0}^{\infty}(1 - [F(k)]^\theta))^2.$$

Next we prove certain ageing properties for the model (6.5).

Theorem 6.2. *The random variable X is IRHR(DRHR) and the reversed hazard rate of Y is proportional to that of X, then Y is IRHR(DRHR).*

Proof: By definition

$$\lambda_Y^*(x) = \theta \lambda_X^*(x), \quad \theta > 0.$$

Hence, the random variable X has IRHR (DRHR) if and only if Y has IRHR (DRHR).

Lemma 6.1. *Let ϕ be a real function on R. Let x_1, x_2, y_1 and y_2 be such that $x_1 \leq y_1 \leq y_2$ and $x_1 \leq x_2 \leq y_2$.*

(i) If ϕ is convex , then

$$\frac{\phi(y_1)-\phi(x_1)}{y_1-x_1} \leq \frac{\phi(y_2)-\phi(x_2)}{y_2-x_2}.$$

(ii) If ϕ is concave , then

$$\frac{\phi(y_1)-\phi(x_1)}{y_1-x_1} \geq \frac{\phi(y_2)-\phi(x_2)}{y_2-x_2}.$$

For the proof see Finkelstein (2002).

Theorem 6.3. *The random variable X is NBU(NWU) and the reversed hazard rate of Y is proportional to that of X, $\theta > 1$ ($\theta \lneq 1$) then Y is NBU (NWU).*

Proof. Suppose that X has NBU then for all $x, t \geq 0$ we have

$$R(x+t) \leq R(x)R(t).$$

That is

$$1-F(x+t) \leq (1-F(x))(1-F(t))$$

which implies

$$1-F(x+t) \leq 1-F(x)-F(t)+F(x)F(t)$$

or

$$F(x+t)-F(x) \geq F(t)-F(x)F(t). \tag{6.7}$$

Let $x_1 = F(x)F(t)$, $x_2 = F(x)$, $y_1 = F(t)$ and $y_2 = F(x+t)$. Then, we have $x_1 \leq y_1 \leq y_2$

and $x_1 \leq x_2 \leq y_2$. Hence by Lemma 6.1, for $\phi(x) = x^\theta$ and $\theta > 1$ we have the inequality

$$\frac{[F(t)]^\theta - [F(x)F(t)]^\theta}{F(t) - F(x)F(t)} \leq \frac{[F(x+t)]^\theta - [F(x)]^\theta}{F(x+t) - F(x)}.$$

That is

$$(F(x+t) - F(x))\left([F(t)]^\theta - [F(x)]^\theta [F(t)]^\theta\right) \leq (F(t) - F(x)F(t))\left([F(x+t)]^\theta - [F(x)]^\theta\right). \tag{6.8}$$

By (6.7), we have

$$(F(t) - F(x)F(t))\left([F(t)]^\theta - [F(x)]^\theta [F(t)]^\theta\right) \leq (F(t) - F(x)F(t))\left([F(x+t)]^\theta - [F(x)]^\theta\right),$$

which gives

$$[F(t)]^\theta - [F(x)]^\theta [F(t)]^\theta \leq [F(x+t)]^\theta - [F(x)]^\theta.$$

Or

$$F_Y(t) - F_Y(x)F_Y(t) \leq F_Y(x+t) - F_Y(x). \tag{6.9}$$

Rewriting this in terms of $R_Y(x)$, we get

$$R_Y(x+t) \leq R_Y(x)R_Y(t). \tag{6.10}$$

Hence, Y is NBU for $\theta > 1$. By using the second part of the Lemma 6.1, proof for NWU case follows. □

Theorem 6.4. *A sufficient condition that the random variable X has IRMRL is the sequence $\{u(x)\}$ is increasing for all $x \in \mathbb{N}$, where $u(x) = \frac{F(x)}{F(x+1)}$.*

Proof. The RMRL is given by

$$\begin{aligned} r(x) &= E(x - X | X \leq x) \\ &= \frac{1}{F(x)} \sum_{k=1}^{x} (x-k) p(k) \\ &= \frac{1}{F(x)} [(x-1)p(1) + (x-2)p(2) + ... + p(x-1)] \\ &= \frac{1}{F(x)} \sum_{k=1}^{x-1} F(k). \end{aligned}$$

Consider

$$\begin{aligned} r(x+1) - r(x) &= \frac{1}{F(x+1)} \sum_{k=1}^{x} F(k) - \frac{1}{F(x)} \sum_{k=1}^{x-1} F(k) \\ &= \frac{F(1)}{F(x+1)} + (u(x) - u(x-1)) \\ &\quad + \sum_{k=1}^{x-2} \{ (\prod_{t=1}^{k} u(x-t))(u(x) - u(x-(k+1)) \}. \end{aligned}$$

When $u(x)$ is increasing in x, each term in the above finite sum is greater than zero. Hence

$$r(x+1) - r(x) > 0.$$

□

Theorem 6.5. *Let X_1 and X_2 be discrete random variables with distribution functions $F_1(.)$*

and $F_2(.)$ respectively. Suppose Y_1 and Y_2 are two random variables such that the reversed hazard rate of Y_i is proportional to that of X_i, $i = 1, 2$, with same proportionality constant θ. Then $X_1 \geq_{lr} X_2$, implies $Y_1 \geq_{lr} Y_2$, provided $\langle F_1(x)\rangle$ and $\langle F_2(x)\rangle$ are log convex and log concave sequences respectively.

Proof. Let $g_1(.)$ and $g_2(.)$ be the density functions of Y_1 and Y_2 respectively. Also suppose that $\lambda_1^*(.)$ and $\lambda_2^*(.)$ are reverse hazard rate functions of X_1 and X_2 respectively. Then, using (6.6), we obtain

$$\frac{g_1(x)}{g_2(x)} = \frac{(F_1(x))^\theta exp(\theta\lambda_1^*(x) - 1)}{(F_2(x))^\theta exp(\theta\lambda_2^*(x) - 1)} \tag{6.11}$$

Now let $X_1 \geq_{lr} X_2$, then by Dewan and Kattumannil (2011)

$$X_1 \geq_{rhr} X_2,$$

which implies

$$\begin{aligned} &\frac{F_1(x)}{F_2(x)} \uparrow \text{ in } x \\ \Rightarrow &\left(\frac{F_1(x)}{F_2(x)}\right)^\theta \uparrow \text{ in } x \text{ for } \theta > 0. \end{aligned} \tag{6.12}$$

Since $\langle F_1(x)\rangle$ is log convex we have

$$\begin{aligned} F_1(x+1)F_1(x-1) \geq (F_1(x))^2 & \\ \Rightarrow \quad & \frac{F_1(x+1)}{F_1(x)} \geq \frac{F_1(x)}{F_1(x-1)} \\ \Rightarrow \quad & \log\frac{F_1(x+1)}{F_1(x)} \geq \log\frac{F_1(x)}{F_1(x-1)} \\ \Rightarrow \quad & \lambda_1^*(x+1) \geq \lambda_1^*(x). \end{aligned} \tag{6.13}$$

Again, since $\langle F_2(x)\rangle$ is log concave,

$$\begin{aligned} F_2(x+1)F_2(x-1) \leq (F_2(x))^2 & \\ & \Rightarrow \frac{F_2(x+1)}{F_2(x)} \leq \frac{F_2(x)}{F_2(x-1)} \\ & \Rightarrow \log\frac{F_2(x+1)}{F_2(x)} \leq \log\frac{F_2(x)}{F_2(x-1)} \\ & \Rightarrow \lambda_2^*(x+1) \leq \lambda_2^*(x). \end{aligned} \tag{6.14}$$

From (6.13) and (6.14) we have

$$\begin{aligned} (\lambda_1^*(x+1)-\lambda_2^*(x+1)) & - (\lambda_1^*(x)-\lambda_2^*(x)) \geq 0 \\ & \Rightarrow (\lambda_1^*(x)-\lambda_2^*(x)) \uparrow \text{ in } x \\ & \Rightarrow e^{\theta(\lambda_1(x)-\lambda_2(x)} \uparrow \text{ in } x. \end{aligned} \tag{6.15}$$

Hence by (6.12) and (6.15)

$$\frac{g_1(x)}{g_2(x)} \uparrow \text{ in } x.$$

That is, $Y_1 \geq_{lr} Y_2$. □

Theorem 6.6. *Let X_1 and X_2 be discrete random variables with distribution functions $F_1(.)$ and $F_2(.)$ respectively. Suppose Y_1 and Y_2 are two random variables such that the reversed hazard rate of Y_i is proportional to that of X_i, $i = 1, 2$, with same proportionality constant θ. Then $X_1 \geq_{rhr} X_2$, implies $Y_1 \geq_{rhr} Y_2$.*

Proof. Suppose $X_1 \geq_{rhr} X_2$. Then

$$\ln\frac{F_{X_1}(x)}{F_{X_1}(x-1)} \geq \ln\frac{F_{X_2}(x)}{F_{X_2}(x-1)}$$

$$\begin{aligned} &\Rightarrow \quad \frac{F_{X_1}(x)}{F_{X_1}(x-1)} \geq \frac{F_{X_2}(x)}{F_{X_2}(x-1)} \\ &\Rightarrow \quad \frac{[F_{X_1}(x)]^\theta}{[F_{X_1}(x-1)]^\theta} \geq \frac{[F_{X_2}(x)]^\theta}{[F_{X_2}(x-1)]^\theta} \end{aligned}$$

$$\Rightarrow \tfrac{F_{Y_1}(x)}{F_{Y_1}(x-1)} \geq \tfrac{F_{Y_2}(x)}{F_{Y_2}(x-1)} \Rightarrow Y_1 \geq_{rhr} Y_2.$$

□

Theorem 6.7. *Let X_1 and X_2 be discrete random variables with distribution functions $F_1(.)$ and $F_2(.)$ respectively. Suppose Y_1 and Y_2 are two random variables such that the reversed hazard rate of Y_i is proportional to that of X_i, $i = 1, 2$, with same proportionality constant θ. Then $X_1 \geq_{st} X_2$, implies $Y_1 \geq_{st} Y_2$.*

Proof. Suppose $X_1 \geq_{st} X_2$. Then

$$\begin{aligned} F_{X_1}(x) \quad &\leq \quad F_{X_2}(x) \\ &\Rightarrow \quad [F_{X_1}(x)]^\theta \leq [F_{X_2}(x)]^\theta \\ &\Rightarrow \quad F_{Y_1}(x) \leq F_{Y_2}(x) \\ &\Rightarrow \quad Y_1 \geq_{st} Y_2. \end{aligned}$$

□

6.4 Test for equality of reverse hazard rate

In this section, we will develop a non-parametric test for comparing lifetimes of two systems. Let X and Y be random variables denoting the lifetimes of two systems with distribution functions $F(.)$ and $G(.)$ respectively. Let $\lambda_F^*(.)$ and $\lambda_G^*(.)$ be the corresponding reversed hazard rates. Note that the distribution functions $F(.)$ and $G(.)$ are related through (6.5). We consider the problem of testing the null hypothesis

$$H_0 : \theta = 1$$

against the alternative

$$H_1 : \theta > 1$$

We are interested in one way alternatives as the case with $\theta < 1$ can be solved with the old definition of reversed hazard rate. In the context of proportional reversed hazards model, the above testing problem can be restated as

$$H_0 : \lambda_G^*(x) = \lambda_F^*(x), \quad \forall x$$

against the alternative

$$H_1 : \lambda_G^*(x) > \lambda_F^*(x), \quad \text{for some } x.$$

Note that even though the null hypothesis is similar, the alternative hypothesis we consider is a larger class. We know that $\lambda_G^*(x) = \theta\lambda_F^*(x)$ if and only if $G(x) = [F(x)]^\theta$ so that under H_1,

$$\begin{aligned}
&\lambda_G^*(x) > \lambda_F^*(x) \\
&\qquad \Rightarrow \quad \log \frac{G(x)}{G(x-1)} > \log \frac{F(x)}{F(x-1)} \\
&\qquad \Rightarrow \quad \frac{G(x)}{G(x-1)} > \frac{F(x)}{F(x-1)}
\end{aligned}$$

$$\Rightarrow \quad \frac{G(x)}{F(x)} > \frac{G(x-1)}{F(x-1)}.$$

Hence under H_1, $\frac{G(x)}{F(x)}$ is non-decreasing in x for all $x > 0$. That is , H_1 holds if and only if, for every $x > y \geq 1$,

$$\delta(x,y) = G(x)F(y) - G(y)F(x) \geq 0.$$

Note that under the null hypothesis, $\delta(x,y) = 0$. Let X_1, X_2 and Y_1, Y_2 be independent samples from F and G respectively. Define

$$\begin{aligned}\Delta(F,G) &= \sum_{y=1}^{\infty}\sum_{x=y+1}^{\infty}(G(x)F(y) - G(y)F(x))f(x)g(y) \\ &= \sum_{y=1}^{\infty}\sum_{x=y}^{\infty}G(x)F(y)f(x)g(y) - \sum_{y=1}^{\infty}\sum_{x=y}^{\infty}G(y)F(x)f(x)g(y) \\ &= P[X_1 \leq Y_1 < Y_2 \leq X_2] - P[Y_1 \leq Y_2 < X_1 \leq X_2].\end{aligned}$$

The last expression is obtained as follows. Consider

$$\begin{aligned}P[X_1 \leq Y_1 < Y_2 \leq X_2] &= \sum_{y=1}^{\infty}\sum_{x=y+1}^{\infty} P[X_1 \leq y < Y_2 \leq x]P(Y_1 = y)P(X_2 = x) \\ &= \sum_{x=1}^{\infty}\sum_{y=x+1}^{\infty} P[X_1 \leq y]P[y < Y_2 \leq x]f(x)g(y) \\ &= \sum_{x=1}^{\infty}\sum_{y=x+1}^{\infty} F(y)[G(x) - G(y)]f(x)g(y) \\ &= \sum_{x=1}^{\infty}\sum_{y=x+1}^{\infty} F(y)G(x)f(x)g(y) \\ &\quad -\sum_{x=1}^{\infty}\sum_{y=x+1}^{\infty} F(y)G(y)f(x)g(y).\end{aligned}$$

In similar lines, we obtain

$$\begin{aligned} P[Y_1 \leq Y_2 < X_1 \leq X_2] &= \sum_{x=1}^{\infty}\sum_{y=x+1}^{\infty} F(x)G(y)f(x)g(y) \\ &\quad -\sum_{x=1}^{\infty}\sum_{y=x+1}^{\infty} F(y)G(y)f(x)g(y). \end{aligned}$$

Hence

$$\begin{aligned} P[X_1 \leq Y_1 < Y_2 \leq X_2] &- P[Y_1 \leq Y_2 < X_1 \leq X_2] \\ &= \sum_{x=1}^{\infty}\sum_{y=x+1}^{\infty} F(y)G(x)f(x)g(y) \\ &\quad -\sum_{x=1}^{\infty}\sum_{y=x+1}^{\infty} F(y)G(y)f(x)g(y) \\ &\quad -\sum_{x=1}^{\infty}\sum_{y=x+1}^{\infty} F(x)G(y)f(x)g(y) \\ &\quad +\sum_{x=1}^{\infty}\sum_{y=x+1}^{\infty} F(y)G(y)f(x)g(y) \\ &= \sum_{y=1}^{\infty}\sum_{x=y}^{\infty} G(x)F(y)f(x)g(y) \\ &\quad -\sum_{y=1}^{\infty}\sum_{x=y}^{\infty} G(y)F(x)f(x)g(y). \end{aligned}$$

It is clear that under H_0, $\Delta(F,G) = 0$ and under H_1, $\Delta(F,G)$ is positive. Hence, $\Delta(F,G)$ is a measure to detect any departure from the null hypothesis towards the alternative hypothesis. Next, we find an estimator of $\Delta(F,G)$. Let x_1, x_2, and y_1, y_2 be samples from two independent populations having distribution functions F and G respectively. Consider the kernel function $h(.)$ defined by

$$h(x_1, x_2, y_1, y_2) = \begin{cases} 1 & \text{if } xyyx \\ -1 & \text{if } yyxx \\ 0 & \text{otherwise,} \end{cases}$$

where the arrangement $xyyx$ represents $\{X_1 \leq Y_1 < Y_2 \leq X_2\} \cup \{X_1 \leq Y_2 < Y_1 \leq X_2\} \cup \{X_2 \leq Y_1 < Y_2 \leq X_1\} \cup \{X_2 \leq Y_2 < Y_1 \leq X_1\}$. Similarly, we represent $yyxx$, $yxxy$, $xyxy$, $yxyx$ and $xxyy$. Consider,

$$\begin{aligned} E(h(X_1, X_2, Y_1, Y_2)) &= E(I(XYYX)) - E(I(YYXX)) \\ &= P(XYYX) - P(YYXX). \end{aligned}$$

Since X_1 and X_2 are i.i.d. with distribution $F(.)$ and Y_1 and Y_2 are i.i.d. with distribution $G(.)$, we have

$$\begin{aligned} P(XYYX) &= P(\{X_1 \leq Y_1 < Y_2 \leq X_2\} \cup \{X_1 \leq Y_2 < Y_1 \leq X_2\} \\ &\quad \cup \{X_2 \leq Y_1 < Y_2 \leq X_1\} \cup \{X_2 \leq Y_2 < Y_1 \leq X_1\}) \\ &= P(X_1 \leq Y_1 < Y_2 \leq X_2) + P(X_1 \leq Y_2 < Y_1 \leq X_2) \\ &\quad + P(X_2 \leq Y_1 < Y_2 \leq X_1) + P(X_2 \leq Y_2 < Y_1 \leq X_1) \\ &= 4P(X_1 \leq Y_1 < Y_2 \leq X_2). \end{aligned}$$

In similar lines, we obtain

$$P(YYXX) = 4P(Y_1 \leq Y_2 < X_1 \leq X_2).$$

Hence

$$\begin{aligned} E(h(X_1, X_2, Y_1, Y_2)) &= 4P(X_1 \le Y_1 < Y_2 \le X_2) - 4P(Y_1 \le Y_2 < X_1 \le X_2) \\ &= 4\Delta(F, G). \end{aligned}$$

Let $X_1, X_2, ...X_{n_1}$ and $Y_1, Y_2, ...Y_{n_2}$ are samples of size n_1 and n_2 from F and G respectively. A U-statistic for $4\Delta(F, G)$ based on the kernel $h(x_1, x_2, y_1, y_2)$ is given by

$$\tilde{\Delta}(F, G) = \frac{1}{\binom{n_1}{2}\binom{n_2}{2}} \sum\sum h(x_{i_1}, x_{i_2}, y_{j_1}, y_{j_2}), \tag{6.16}$$

where summation is over $1 \le i_1 < i_2 \le n_1$ and $1 \le j_1 < j_2 \le n_2$. Next we find the asymptotic variance of $\tilde{\Delta}(F, G)$. The following lemma is useful in this direction.

Lemma 6.2. *Let X and V be independent random variables where $V \sim U(0, 1)$, if we can write*

$$F(x, v) = P\{X < x\} + vP\{X = x\} \tag{6.17}$$

then

$$F(X, V) \sim U(0, 1). \tag{6.18}$$

Proof. Note that $F(x, 1)$ is the distribution function. To make $F(x, v)$, a non-decreasing function, we assume that $(x, v) \le (x', v')$ if and only if $x < x'$ or $x = x'$ and $v \le v'$. For any $0 \le u \le 1$, find $x(u)$ and $v(u)$ such that

$$F(x(u), v(u)) = u.$$

Consider

$$\begin{aligned} P[F(X(u),V(u)) \leq u] &= P[(X(u),V(u)) \leq (x(u),v(u))] \\ &= P[X(u) < x(u)] + P[V(u) \leq v(u)]P[X(u) = x(u)] \\ &= F(x(u),v(u)) = u. \end{aligned}$$

Hence the result. □

Consider a discrete random variable X with distribution function $F(.)$. Then

$$\begin{aligned} F(x) &= P[X < x] + P[X = x] \\ &= P[X < x] + P[X = x] + vP[X = x] - vP[X = x] \\ &= F(x,v) + (1-v)p(x). \end{aligned}$$

Since X and V are independent,

$$\begin{aligned} E(F(X)) &= E(F(X,V)) + E\left((1-V)p(X)\right) \\ &= \frac{1}{2} + \frac{1}{2}E(p(X)). \end{aligned}$$

6.4.1 Asymptotic properties

In this section, we derive the variance and asymptotic distribution of the above statistic under the null hypothesis.

$$h(x_1, x_2, y_1, y_2) = \begin{cases} 1 & \text{if } xyyx \\ -1 & \text{if } yyxx \\ 0 & \text{otherwise.} \end{cases}$$

Now, to find the variance of $\tilde{\Delta}(F, G)$, we consider the following.

$$\begin{aligned} h_{10}(x_1) &= E(h(x_1, X_2, Y_1, Y_2)) \\ &= 2P[x \leq Y_1 < Y_2 \leq X_2] - 2P[Y_1 \leq Y_2 < x \leq X_2], \end{aligned}$$

since under H_0, $P[X_2 \leq Y_2 < Y_1 \leq x] = P[Y_2 \leq Y_1 < X_2 \leq x]$ and $P[X_2 \leq Y_1 < Y_2 \leq x] = P[Y_1 \leq Y_2 < X_2 \leq x]$. Consider

$$\begin{aligned} h_{10}(x_1) &= 2P[x \leq Y_1 < Y_2 \leq X_2] - 2P[Y_1 \leq Y_2 < x \leq X_2] \\ &= 2\sum_{y=x+1}^{\infty} P[x \leq Y_1 < y \leq X_2]g(y) \\ &\quad -2\sum_{y=1}^{x-1} P[y \leq Y_2 < x \leq X_2]g(y) \\ &= 2\sum_{y=x+1}^{\infty} \bar{F}(y-1)[G(y-1) - G(x-1)]g(y) \\ &\quad -2\sum_{y=1}^{x-1}[G(x-1) - G(y-1)]\bar{F}(x-1)g(y) \\ &= 2\sum_{y=x+1}^{\infty} (1 - F(y-1))[G(y-1) - G(x-1)]g(y) \\ &\quad -2\sum_{y=1}^{x-1}[G(x-1) - G(y-1)](1 - F(x-1))g(y) \\ &= 2\sum_{y=x+1}^{\infty} [(1 - F(y-1))F(y-1) - (1 - F(y-1))F(x-1)]f(y) \end{aligned}$$

$$\begin{aligned} & -2\sum_{y=1}^{x-1}[F(x-1)(1-F(x-1))-F(y-1)(1-F(x-1))]f(y) \\ = \quad & C+F(x-1)\sum_{y=x}^{\infty}[F(y-1)]f(y)-F(x-1)-F^3(x-1), \end{aligned}$$

since X and Y have same distribution under the null hypothesis, where C is a constant independent of x. Hence

$$\begin{aligned} \sigma_{10}^2 &= V(h_{10}(X_1)) \\ &= V(F(X-1)\sum_{y=X}^{\infty}[F(y-1)]f(y)-F(X-1)-F^3(X-1)). \end{aligned} \tag{6.19}$$

Similarly, when

$$h_{01}(y_1) = E(h(X_1,X_2,y_1,Y_2)),$$

under the null hypothesis, we obtain

$$\begin{aligned} \sigma_{01}^2 &= V(h_{01}(Y_1)) \\ &= \sigma_{10}^2. \end{aligned} \tag{6.20}$$

Next, we discuss the asymptotic property of the test statistic.

Let $\sigma_{22}^2 = V(h(X_1,X_2,Y_1,Y_2))$.

Theorem 6.8. *If* $\frac{n_1}{N} \to p \in (0,1)$ *as* $N = (n_1+n_2) \to \infty$ *and if* $\sigma_{22}^2 < \infty$ *then*

$\sqrt{N}(\tilde{\Delta}(F,G) - \Delta(F,G))$ *is asymptotically normal with mean* 0 *and variance*

$$\sigma^2 = \frac{4}{p(1-p)} V\Big(F(X-1)\sum_{y=X}^{\infty}[F(y-1)f(y)] - F(X-1) - F^3(X-1)\Big).$$

Proof. The normality follows from Theorem 2.9. Now the asymptotic variance is given by

$$\begin{aligned}
\sigma^2 &= \frac{4}{p}\sigma_{10}^2 + \frac{4}{1-p}\sigma_{01}^2 \\
&= \left(\frac{4}{p} + \frac{4}{1-p}\right)\sigma_{10}^2 \\
&= \frac{4}{p(1-p)} V\Big(F(X-1)\sum_{y=X}^{\infty} F(y-1)f(y) - F(X-1) - F^3(X-1)\Big).
\end{aligned}$$

□

6.5 Conclusion

In this chapter we defined modified reversed hazard rate which can be considered as the discrete analogue of reversed hazard rate in continuous setup. We proposed a proportional reversed hazards model for discrete data and studied its properties. We developed a non-parametric test for testing the proportionality constant in the proposed model.

Chapter 7

Conclusions

7.1 Conclusions and future work

The research problems discussed in Chapter 1 in connection with discrete reliability analysis have been solved in this thesis. In Chapter 3, first we established a relation between VRL and conditional expectations or reversed hazard rate to characterize a general class of discrete life distributions and the results are then applied to families of distributions. A lower bound to the conditional variances is obtained and compared with the minimum variance unbiased estimators obtained by the well known classical theorems on inference, when the function $c(.)$ appearing in the bounds are chosen as estimators of the desired parametric functions.

The present thesis is concentrated on variance bounds in the discrete case with respect to functions of a single random variable. As a natural extension, the possibility of similar results for the multivariate case is an open problem. Since definitions of hazard rate and mean residual life in multivariate case exist in more than one way, each case has to be dealt with separately. An important aspect of modelling using multivariate distributions is the dependency structure in the data. Hence covariance between residual lives also can play a significant role in the process. These and other concepts for the study of bounds for multi dimensional random variables will be taken up in a future work.

As pointed out earlier, the new definition of the hazard rate and the reversed hazard rate in the discrete setup gives several open problem in the area of discrete reliability. The testing constant hazard rate against IHR and IHRA has significant role in identifying the underline model which generates the data. These problems were solved in Chapter 4. Non-parametric test statistic based on U-statistics was developed for solving the same. The asymptotic properties of the test statistics were studied. The simulation results showed that the proposed test perform well in terms of empirical type 1 error and empirical power. In Chapter 5, we considered the problem of testing constant hazard rate against NBU and NBUE alternatives. It is shown that the test based on U-statistics performs well in this case also. Testing constant hazard rate against other ageing classes will be addressed in future.

The proportional hazards and proportional reversed hazards models are very common in reliability analysis while dealing with series and parallel systems. In Chapter 6, we introduced reversed hazards model relevant to situations when lifetimes are discrete and studied their ageing properties. The main advantage of the proposed model is that it can incorporate continuous covariates without requiring them to be categorized. We studied the properties of the proposed model. We discussed the problem of testing the proportionality parameter introduced in the model.

The concept of frailty model is becoming popular in literature if the covariates presented in the proportional hazards model is random. Accordingly, there is a scope for studying the frailty model in discrete setup. Investigating the negative dependence in frailty models in line of the results of Xu and Li (2007) is an another problem of interest. In light of our proposed model, one can also consider the problems related to dynamic proportional hazard rate and reversed hazard rate models introduced by Nanda and Das (2011).

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