REPRESENTATIONS OF GL_n DISTINGUISHED BY GL_{n-1} OVER A p-ADIC FIELD

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by

VENKETASUBRAMANIAN C G

Department of Mathematics and Statistics

School of Mathematics and Computer/Information Sciences

University of Hyderabad

Hyderabad - 500 046, INDIA

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DECLARATION

Department of Mathematics and Statistics
School of MCIS
University of Hyderabad
Hyderabad - 500 046
INDIA

I, **VENKETASUBRAMANIAN C G** hereby declare that the work embodied in the present thesis entitled **REPRESENTATIONS OF** GL_n **DISTINGUISHED BY** GL_{n-1} **OVER A** p**-ADIC FIELD** has been carried out by me under the supervision of Prof. Rajat Tandon, Department of Mathematics and Statistics, University of Hyderabad, Hyderabad - 500 046, India, as per the Ph.D ordinance of the university.

I declare that, to the best of my knowledge and belief, no part of this thesis was earlier submitted for the award of research degree of any university or institution.

Candidate
Venketasubramanian C G
Roll No: 04MMPP02

CERTIFICATE

Department of Mathematics and Statistics
School of MCIS
University of Hyderabad
Hyderabad - 500 046
INDIA

This is to certify that Mr. **VENKETASUBRAMANIAN C G** has carried out the research work embodied in the present thesis entitled **REPRESENTATIONS OF** GL_n **DISTINGUISHED BY** GL_{n-1} **OVER A** p**-ADIC FIELD** for the full period prescribed under the University rules. No part of this thesis was earlier submitted for the award of research degree of any University/Institute.

Supervisor and Head	Dean
Department of Mathematics and Statistics	School of MCIS
Prof. Rajat Tandon	**Prof. T. Amaranath**

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Contents

Chapter 1

Introduction and Statements of Results

1.1 Introduction

Let π be a representation of a topological group G and ρ a representation of a closed subgroup H of G. It is an important question in representation theory to understand whether ρ occurs as a quotient in the restriction of π to H, or more precisely, whether the space of H-equivariant maps $\mathrm{Hom_H}[\pi, \rho]$ is nonzero. In particular, for a character χ of H if $\mathrm{Hom_H}[\pi, \chi]$ is nonzero, π is said to be χ-distinguished with respect to H. A representation π of G is simply said to be H-*distinguished* if it is $\mathbb{1}$-distinguished with respect to H(or equivalently said to have a GL_{n-1}-invariant form), where $\mathbb{1}$ is the trivial character of H.

For a local field F, let $\mathrm{GL}_n := \mathrm{GL}_n(\mathrm{F})$. Let GL_{n-1} be embedded in GL_n via $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. Let $\mathbb{1}_n$ denote the trivial representation of GL_n. It is proved in [2] that for any irreducible admissible representation π of GL_n, $\dim_{\mathbb{C}}(\mathrm{Hom_{G_{n-1}}}[\pi, \mathbb{1}_{\mathrm{n-1}}]) \leq 1$. More generally, it is proved in [1] that for irreducible admissible representations π of GL_n and ρ of GL_{n-1}, ρ occurs as a quotient in the restriction of π to GL_{n-1} with multiplicity at most one, i.e., $\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL_{n-1}}}[\pi, \rho]) \leq 1$.

From now on we fix F to be a nonarchimedean local field. D. Prasad has proved in [14] that if π and ρ are irreducible generic representations of GL_n and GL_{n-1} respectively then $\mathrm{Hom}_{\mathrm{GL_{n-1}}}[\pi, \rho] \neq 0$. In the same paper (Theorem 1, [14]) Prasad has classified all irreducible admissible representations π of GL_3 which are GL_2-distinguished. For a general n, Y. Flicker [8] has classified all irreducible admissible unitary representations π of GL_n for which $\mathrm{Hom}_{\mathrm{GL_{n-1}}}[\pi, \chi] \neq 0$, where χ is a unitary character of GL_{n-1}.

D. Prasad formulated the following conjecture(Conjecture 1, [14]) describing those irreducible admissible representations π of GL_n which are GL_{n-1}-distinguished in terms of the Langlands parameter $\mathfrak{L}(\pi)$ of π which is an n-dimensional representation of the Weil-Deligne group W'_F of F.

Conjecture 1.1.1. (*Prasad*) *An irreducible admissible representation* π *of* GL_n *for* $n \geq 3$ *is* GL_{n-1}*-distinguished if and only if the Langlands parameter* $\mathfrak{L}(\pi)$ *associated to* π *by the Local Langlands correspondence has a subrepresentation* $\mathfrak{L}(\mathbb{1}_{n-2})$ *of dimension* $n-2$ *corresponding to the trivial representation* $\mathbb{1}_{n-2}$ *of* GL_{n-2} *such that the two dimensional quotient* $\mathfrak{L}(\pi)/\mathfrak{L}(\mathbb{1}_{n-2})$ *corresponds (under the Local Langlands correspondence) to an infinite dimensional representation of* GL_2.

However, it was observed by Prasad and the author that the Langlands parameters of the irreducible representations $\pi = \mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\nu^{\pm 1/2} \otimes 1)$ of GL_3 were not included in Theorem 1 in [14] even though these are GL_2-distinguished. Hence Theorem 1 in [14] and as a result Conjecture 1.1.1 needs modification.

In this thesis we have modified Prasad's Conjecture and proved the following

Theorem 1.1.2. *An irreducible admissible representation* π *of* GL_n *for* $n \geq 3$ *is* GL_{n-1}*-distinguished if and only if the Langlands parameter* $\mathfrak{L}(\pi)$ *associated to* π *by the Local Langlands correspondence has a subrepresentation* $\mathfrak{L}(\mathbb{1}_{n-2})$ *of dimension* $n-2$ *corresponding to the trivial representation* $\mathbb{1}_{n-2}$ *of* GL_{n-2} *such that the two di-*

mensional quotient $\mathfrak{L}(\pi)/\mathfrak{L}(\mathbb{1}_{n-2})$ corresponds (under the Local Langlands correspondence) either to an infinite dimensional representation of GL_2 *or the one dimensional representations $\nu^{\pm\frac{n-2}{2}}$ of* GL_2.

The essence of Theorem 1.1.2 is that it is the GL_2 part in the Langlands parameter $\mathfrak{L}(\pi)/\mathfrak{L}(\mathbb{1}_{n-2})$ which decides the GL_{n-1}-distinguishedness of π. In the unitary case, Flicker used Bernstein-Zelevinsky filtration and Tadic's classification[18] of unitary representations of GL_n to reduce the choice of π to essentially $\mathrm{ind}_{\mathrm{P_{n-2,2}}}^{\mathrm{GL_n}}(\mathbb{1}_{\mathrm{n-2}}\otimes\rho)$ where ρ is an irreducible admissible unitary representation of GL_2 and by Mackey theory showed that such representations are GL_{n-1}-distinguished whenever ρ is infinite dimensional.

It is a consequence of a theorem of Gelfand-Kazhdan that an irreducible supercuspidal representation of GL_n for $n \geq 3$ is not GL_{n-1}-distinguished. Since any non-supercuspidal irreducible admissible representation of GL_n is a quotient of a representation of the form $\xi = \mathrm{ind}_{\mathrm{P_{k,n-k}}}^{\mathrm{GL_n}}(\rho\otimes\tau)$, where ρ and τ are irreducible admissible representations of GL_k and GL_{n-k} respectively, it was a suggestion of Prasad to study GL_{n-1}-distinguishedness for ξ. Prasad used Mackey Theory in [14] to study the restriction of a representation of the type $\mathrm{ind}_{\mathrm{P_{2,1}}}^{\mathrm{GL_3}}(\rho\otimes\chi)$ to GL_2 for an irreducible admissible representation ρ of GL_2 and χ a character of GL_1. Our work is a generalization of this to GL_n where we study the restriction of a representation of the type $\mathrm{ind}_{\mathrm{P_{k,n-k}}}^{\mathrm{GL_n}}(\rho\otimes\tau)$ to GL_{n-1} where $P_{k,n-k}$ is a standard parabolic subgroup of GL_n of type $(k, n-k)$, ρ and τ are irreducible admissible representations of GL_k and GL_{n-k} respectively and "ind" denotes normalized parabolic induction.

There are several advantages of using Mackey theory to study our problem. Let ρ and τ be a smooth representations (not necessarily irreducible) of GL_k and GL_{n-k} respectively. Firstly, Mackey theory gives a complete picture of GL_{n-1}-distinguished representations of the form $\mathrm{ind}_{\mathrm{P_{k,n-k}}}^{\mathrm{GL_n}}(\rho\otimes\tau)$. In the process, it also gives the following recipe to construct GL_{n-1}-distinguished smooth representations of GL_n from representations of GL_m with $m < n$.

Theorem 1.1.3. *The following smooth representations of* GL_n *are* GL_{n-1}*-distinguished.*
(a) $\mathrm{ind}_{\mathrm{P_{k,n-k}}}^{\mathrm{GL_n}}(\rho\nu^{\frac{\mathrm{n-k-2}}{2}} \otimes \tau\nu^{\frac{-\mathrm{k}}{2}})$ *where* ρ *is a smooth representation of* GL_k *having* $\mathbb{1}_k$ *as a quotient and* τ *is a smooth representation of* GL_{n-k} *which is* GL_{n-k-1}*-distinguished.*
(b) $\mathrm{ind}_{\mathrm{P_{k,n-k}}}^{\mathrm{GL_n}}(\rho\nu^{\frac{\mathrm{n-k}}{2}} \otimes \tau\nu^{-(\frac{\mathrm{k-2}}{2})})$ *where* ρ *is a smooth representation of* GL_k *which is* GL_{k-1}*-distinguished and* τ *is a smooth representation of* GL_{n-k} *having* $\mathbb{1}_{n-k}$ *as a quotient.*

So most of the questions can be treated by induction. Finally, the theory can be used not only to study GL_{n-1}-distinguishedness but also to answer multiplicity questions for certain class of representations of GL_n. Since the GL_{n-1}-distinguishedness of π is dictated by the GL_2 part of the Langlands parameter of π obtained after going modulo the trivial representation of GL_{n-2} it was natural to observe that multiplicity i.e., $\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL_{n-1}}}[\pi, \mathbb{1}_{\mathrm{n-1}}])$ is also dictated by the same part.

For a smooth representation π of GL_n let $d_\pi = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL_{n-1}}}[\pi, \mathbb{1}_{\mathrm{n-1}}])$. We prove the following theorem of which part (b) is well known from [20]:

Theorem 1.1.4. (a) *Let* $\xi = \chi_1 \times \chi_2$ *where* χ_1 *and* χ_2 *are any two characters of* GL_1 *and* $\xi_0 = \nu^{-1/2} \times \nu^{1/2}$*. Then* $d_\xi = 1$ *for all* $\xi \neq \xi_0$ *and* $d_{\xi_0} = 2$*.*
(b) *Let* π *be an irreducible admissible representation of* GL_2*. Then* π *is* GL_1*-distinguished if and only if* $\pi = \mathbb{1}_2$ *or infinite dimensional. Moreover,* $\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL_1}}[\pi, \mathbb{1}]) = 1$.

As a consequence of Mackey theory and the fact that $d_\rho = 1$ for irreducible admissible representations ρ of GL_2, we deduce that $d_\pi \leq 1$ for irreducible admissible representations π of GL_n. At this juncture, it was natural to probe whether such a result would be true if one replaces π by a representation of GL_n parabolically induced from irreducible admissible representations of a Levi subgroup. We investigated $\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL_{n-1}}}[\xi, \mathbb{1}_{\mathrm{n-1}}])$ where $\xi = \mathrm{ind}_{\mathrm{P_{n_1,\ldots,n_r}}}^{\mathrm{GL_n}}(\rho_1 \otimes \cdots \otimes \rho_{\mathrm{r}})$ with ρ_i being irreducible admissible representations of GL_{n_i}.

It is a simple consequence of Mackey Theory that the cuspidal support of an irreducible admissible representation of $\mathrm{GL}_n, n \geq 3$ which is GL_{n-1}-distinguished is either $(2,1,...,1)$ or $(1,...,1)$. So we investigated d_ξ for such representations. The first theorem is the result for GL_3 which is used as a basis for the induction hypotheses

Theorem 1.1.5. *Let χ_i be characters of GL_1 for $i = 1,2,3$. Then the principal series representation $\xi = \chi_1 \times \chi_2 \times \chi_3$ of GL_3 is GL_2-distinguished if and only if one of the $\chi_i = 1$. Moreover, $d_\xi \leq 2$ and $d_\xi = 2$ if and only if $\xi = 1 \times \nu \times 1$ or $1 \times \nu^{-1} \times 1$.*

Observe that by Theorem 1.1.2 we know that the cuspidal support of any irreducible admissible GL_{n-1}-distinguished representation of $\mathrm{GL}_n, n \geq 3$ has its cuspidal support equal to either $\{\sigma, \nu^{\frac{n-3}{2}}, ..., \nu^{-(\frac{n-3}{2})}\}$ or $\{\chi_1, \chi_2, \nu^{\frac{n-3}{2}}, ..., \nu^{-(\frac{n-3}{2})}\}$ where σ is an irreducible supercuspidal representation of GL_2, χ_1, χ_2 and ν^j's are characters of GL_1. The next step is to generalize the above theorem to $n \geq 4$. For $n \geq 3$, let T_{n-2} denote the ordered set $\{\nu^{\frac{n-3}{2}}, ..., \nu^{-(\frac{n-3}{2})}\}$ corresponding to the trivial representation of GL_{n-2}, i.e., $T_1 = \{1\}, T_2 = \{\nu^{\frac{1}{2}}, \nu^{\frac{-1}{2}}\}$ and so on. We prove the following theorem.

Theorem 1.1.6. *Let $\xi = \chi_1 \times ... \times \chi_n$ for $n \geq 3$ and $d_\xi = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL_{n-1}}}[\xi, \mathbb{1}_{\mathrm{n-1}}])$. Let $[\chi_1, ..., \chi_n]$ denote the ordered set $\{\chi_1, ..., \chi_n\}$. Then $d_\xi \neq 0$ if and only if there exists χ_i, χ_j such that $[\chi_1, ..., \chi_n] \backslash \{\chi_i, \chi_j\}$ equals the ordered set T_{n-2}. For $k = 1, ..., n-1$ define $\xi_n(k) \in \mathrm{Alg(GL_n)}$ by*

$$
\begin{aligned}
\xi_n(1) \quad &= \nu^{\frac{n-3}{2}} \times \nu^{\frac{n-1}{2}} \times \nu^{\frac{n-3}{2}} \times ... \times \nu^{-(\frac{n-3}{2})}, \\
\xi_n(n-1) \quad &= \nu^{\frac{n-3}{2}} \times ... \times \nu^{-(\frac{n-3}{2})} \times \nu^{-(\frac{n-1}{2})} \times \nu^{-(\frac{n-3}{2})}, \\
\xi_n(k) \quad &= \nu^{\frac{n-3}{2}} \times ... \times \nu^{\frac{n-2k+1}{2}} \times \nu^{\frac{n-2k-1}{2}} \times \nu^{\frac{n-2k+1}{2}} \times \nu^{\frac{n-2k-1}{2}} \times ... \times \nu^{-(\frac{n-3}{2})}
\end{aligned}
$$

for $2 \leq k \leq n-2$. Then $d_\xi \leq 2$ and $d_\xi = 2$ if and only if $\xi = \xi_n(k)$ for some $k \in \{1, ..., n-1\}$.

This completes the case of cuspidal support $(1,...,1)$. The next result is for the cuspidal support of the type $(2,1,...,1)$.

Proposition 1.1.7. *Let σ be an irreducible supercuspidal representation of GL_2, $\chi_i, 1 \leq i \leq n-2$ characters of GL_1 and $\xi = \mathrm{ind}_{\mathrm{P_{2,1,...,1}}}^{\mathrm{GL_n}}(\sigma \otimes \chi_1 \otimes ... \otimes \chi_{\mathrm{n-2}})$. Then $d_\xi \leq 1$ and $d_\xi = 1$ if and only if the ordered set $[\chi_1, ..., \chi_{n-2}] = T_{n-2}$.*

We immediately have the following corollary:

Corollary 1.1.8. *Let* $n = n_1 + ... + n_r$, ρ_i *be irreducible admissible representations of* GL_{n_i} *and* $\xi = \mathrm{ind}_{\mathrm{P_{n_1,\ldots,n_r}}}^{\mathrm{GL_n}}(\rho_1 \otimes ... \otimes \rho_\mathrm{r})$. *Then* $d_\xi \leq 2$.

1.2 A Quick Tour of the thesis

We will now browse through the contents of this thesis. Chapter 2 is devoted to introducing notations and preliminaries which will be used in the sequel. The chapter begins with some generalities on representations of ℓ-groups and proceeds to the Bernstein-Zelevinsky and Langlands classifications of irreducible admissible representations of GL_n. Also, some results from Zelevinsky's theory of segments are stated. These are done in Sections 2.1, 2.2, 2.3 and 2.4 respectively. We have followed the survey articles [11], [15], [16] and the original papers of Bernstein and Zelevinsky [3], [22].

In the work that follows a crucial role is played by twists of a certain irreducible admissible GL_{n-1}-distinguished representation of $\mathrm{GL}_n, n \geq 3$, which we denote by L_n. For $n \geq 2$, the representation L_n may be defined recursively as the unique irreducible quotient of the length 2 representation $\mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\nu^{1/2} \otimes \nu^{\frac{\mathrm{n}+1}{2}})$. We can interpret L_n as the unique irreducible quotient of other length 2 representations as well which we will see later. In fact each one of these realizations is useful and serves different purposes. This is the content of Section 2.5 . We summarize in Section 2.6, some facts on Langlands parameters of irreducible admissible representations of GL_n which we will need in our work. Our principal references for this Section is [11], [15] and [21]. In Section 2.8, we have worked out some examples on product of characters which will be useful to us throughout the thesis, including the Proof of Theorem 1.1.2.

In Chapter 3 we describe the Mackey Theory for the restriction of a parabolically induced representation of GL_n to GL_{n-1} and derive some consequences. In Section 3.1, we study the action of GL_{n-1} on $\mathrm{Gr(k, n)}$, the space of all k-dimensional subspaces of the n-dimensional space F^n. Then the restriction of $\mathrm{ind}_{\mathrm{P_{k,n-k}}}^{\mathrm{GL_n}}(\rho \otimes \tau)$ to GL_{n-1} via Mackey theory is taken up, where ρ and τ are smooth representations of GL_k and GL_{n-k} respectively. There are three orbits for the GL_{n-1} action on $\mathrm{Gr(k,n)}$ out of which two are closed and the third one is the unique open orbit. This shows that if $\mathrm{ind}_{\mathrm{P_{k,n-k}}}^{\mathrm{GL_n}}(\rho \otimes \tau)$ is GL_{n-1}-distinguished, then at least one of the three conditions corresponding to each of the orbits, must be satisfied. The latter is in Sections 3.2 and 3.3. In Section 3.4 we first prove Theorem 1.1.4. We next move on to the GL_3 case and prove Theorem 1.1.2 for $n = 3$.

Chapter 4 is devoted mainly to prove Theorem 1.1.2 whose proof has two parts and we use Mackey theory for both parts. The 'if' part is direct whereas the 'only if' part is more difficult and is proved using induction on n. We begin the chapter with few basic results in Section 4.1. We study GL_{n-1}-distinguishedness for the product of two characters i.e., $\mathrm{ind}_{\mathrm{P_{k,n-k}}}^{\mathrm{GL_n}}(\chi \otimes \mu)$ where χ and μ are characters, in Section 4.2. After summarizing the idea of Proof of theorem 1.1.2 in Section 4.3 we prove Theorem 1.1.2 in Section 4.4.

In the final part of the thesis namely Chapter 5, we study the existence of GL_{n-1}-invariant forms for representations of GL_n which are parabolically induced from irreducible representations of a Levi subgroup. We show that 'multiplicity one' does not hold for these representations but is bounded by 2. We classify all principal series representations of the form $\mathrm{ind}_{\mathrm{P_{1,..,1}}}^{\mathrm{GL_n}}(\chi_1 \otimes ... \otimes \chi_\mathrm{n})$, where $\chi_i, (1 \leq i \leq n)$ is a character of GL_1, for which the space of GL_{n-1}-invariant forms has dimension 2; of course these are reducible principal series representations. This analysis of multiplicity for principal series representations which are not necessarily irreducible seems first general work in this direction and the results obtained seem to suggest that this question has a nice structure. We refer to the work of M. Harris and A. J. Scholl [10] which has partial results of this type in the case of triple products for GL_2. A curious feature of our proofs of multiplicity theorems is that for a statement about GL_n we go to GL_{n+1} or GL_{n+2}. We have used this in several places. We begin the Chapter with the motivation to Theorem 1.1.6 and Corollary 1.1.8. We discuss two examples in Section 5.2 which are typical of the principal series representations $\mathrm{ind}_{\mathrm{P}_{1,..,1}}^{\mathrm{GL_n}}(\chi_1 \otimes ... \otimes \chi_{\mathrm{n}})$ which have multiplicity 2. We also show that $d_\pi \leq 1$ for irreducible admissible representations of GL_n. In Section 5.3, we prove Theorem 1.1.6 and Corollary 1.1.8.

We end this introduction by mentioning that the classification of irreducible admissible representation of GL_n distinguished by GL_{n-1} achieved in this thesis may be useful in eventual understanding of complete branching from GL_n to GL_{n-1}(which is of multiplicity ≤ 1 by [1]), a problem which is of considerable interest in the subject.

Chapter 2

Preliminaries on Representations of GL_n

The aim of this chapter is to provide a quick introduction to facts about the Representation Theory of GL_n over a non-archimedean local field F which we will use in the thesis. The main reference for this topic would be indeed the classical papers ([3], [4],[22]) of Bernstein and Zelevinsky. We also make use of the survey articles [11], [15] and [16].

2.1 Generalities, Induction and Jacquet Functor

Let G be a locally compact totally disconnected topological group. Such a group is called an ℓ-group in the terminology of [3]. Such a G has a fundamental system of neighborhoods of the identity consisting of compact open subgroups. Let F be a non-archimedean local field. Then the group $\mathrm{GL}_n(\mathrm{F})$ of invertible $n \times n$ matrices with entries from F is an ℓ-group. A *representation* of G is a pair (π, V) where V is a complex vector space and π is a group homomorphism from G into the group $Aut_{\mathbb{C}}(V)$ of $\mathbb{C}$-linear automorphisms of V. We will call V a G-module. The vector space V is called the representation space of π. Whenever it is clear, we may drop V

and simply say that π is a representation of G.
A representation (π, V) of G is said to be *smooth* or *algebraic* if for every vector $v \in V$ the stabilizer $S_G(v) := \{g \in G | \pi(g)v = v\}$ of v is open in G. A representation (π, V) is said to be *admissible* if (π, V) is smooth and for any compact open subgroup K of G, the subspace $V^K := \{v \in V | \pi(k)v = v \text{ for all } k \in K\}$ of K-fixed vectors in V is finite dimensional. A subspace W of V is said to be G-*stable* if $\pi(g)W \subset W$ for all $g \in G$. A representation (π, V) is *irreducible* if V is non-zero and has no G-stable subspaces other than (0) and V.

Given two smooth representations (π, V) and (ρ, W) of G, we denote by $\mathrm{Hom_G}[\pi, \rho]$ the space of G-intertwining (or G-equivariant)operators from V to W i.e., the space of all linear maps $T : V \to W$ such that $T(\pi(g)v) = \rho(g)(T(v))$ for all $g \in G$ and for all $v \in V$. The representations (π, V) and (ρ, W) are said to be *equivalent* if there is a non-zero $T \in \mathrm{Hom_G}[\pi, \rho]$ such that T is a linear isomorphism from V to W. If π and ρ are equivalent we will write $\pi \cong \rho$. We will denote the category of all smooth representations and irreducible admissible representations of G by Alg(G) and Irr(G) respectively.

Let (π, V) be a smooth representation of G. Let $V^* = \mathrm{Hom}_{\mathbb{C}}[\mathrm{V}, \mathbb{C}]$. Let π^* denote the representation of G in $Aut(V^*)$ defined by $\pi^*(g)f(v) = f(\pi(g^{-1})v)$ for $g \in G, f \in V^*$ and $v \in V$. Let $\widetilde{V} := \{f \in V^* | S_G(f) \text{ is open in } G\}$. Then $\widetilde{V}$ is called the smooth (or algebraic) part of V^* under the G-action. Then $\widetilde{V}$ is G-stable and defines a smooth representation of G denoted by $(\widetilde{\pi}, \widetilde{V})$ and is called the *contragredient* representation of (π, V).

Induced Representations

Let G be an ℓ-group. There exists a left Haar measure $d_l(x)$ on G unique up to a positive real number. Since $d_l(xg)$ is a again a left Haar measure for a given $g \in G$, the uniqueness of the left Haar measure gives rise to a continuous homomorphism $\Delta_G : G \to \mathbb{R}_{>0}$ given by $d_l(xg) = \Delta_G(g)d_l(x)$. The homomorphism Δ_G is called the *modular character* of G.

In general, a *character* of G is a continuous group homomorphism of G into $\mathbb{C}^\times$.

Given the modular character Δ_G of G, $d_r(x) = \Delta_G^{-1} d_l(x)$ is a right Haar measure on G. A left Haar measure is right invariant if and only if the modular character is trivial. If $\Delta_G = 1$ then the group G is said to be *unimodular*. The group $\mathrm{GL}_n(F)$ is unimodular. For an ℓ-group G, we will denote by δ_G the inverse of the modular character i.e., $\delta_G = \Delta_G^{-1}$. Some authors define the modular character to be δ_G(for example see [5] 4.2.3).

Let H be a closed subgroup of G and (ρ, U) be a smooth representation of H. The method of induction is a fundamental process of constructing representations of a group from that of a subgroup. Let $I(G, H, U)$ denote the space of all functions

$$\left\{ f : G \to U \;\middle|\; \begin{array}{ll} (1) & f \text{ is locally constant} \\ (2) & f(hg) = \delta_G(h)^{-1/2}\delta_H(h)^{1/2}\rho(h)f(g),\ \forall h \in H,\ \forall g \in G \end{array} \right\}.$$

Let $(\pi, I(G, H, U))$ be the representation of G defined by $\pi(g)f(x) = f(xg)$ for $g, x \in G$ and $f \in I(G, H, U)$ i.e., π acts on $I(G, H, \rho)$ by right translation. This representation π is called the *(normalized) induced representation*, induced from the representation ρ of H to G and denoted by $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\rho)$. Let $I_c(G, H, U) := \{f \in I(G, H, U) | f \text{ has compact support modulo } H\}$. Then $I_c(G, H, U)$ is a G-stable subspace of $I(G, H, U)$ and we get a subrepresentation of $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\rho)$ in this space, which we denote by $\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}(\rho)$. We call this process normalized compact induction. If H is a closed subgroup of G such that G/H is compact then $Ind_H^G(\rho) = \mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}(\rho)$. We will deal only with normalized compact induction in our work. The basic properties of induction are summarized in the following theorem.

Theorem 2.1.1.

1. *Both* $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\rho)$ *and* $\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}(\rho)$ *are exact functors from the category of smooth representations of* H *to the category of smooth representations of* G.

2. *Both the induction functors are transitive i.e., if* K *is a closed subgroup of* H *then* $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\mathrm{Ind}_{\mathrm{K}}^{\mathrm{H}}) = \mathrm{Ind}_{\mathrm{K}}^{\mathrm{G}}$ *and* $\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}(\mathrm{ind}_{\mathrm{K}}^{\mathrm{H}}) = \mathrm{ind}_{\mathrm{K}}^{\mathrm{G}}$.

3. *For a smooth representation ρ of H, $\widetilde{\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}(\rho)} = \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\widetilde{\rho})$.*

4. *Let π be a smooth representation of G and ρ a smooth representation of H. Then*

$$\mathrm{Hom}_{\mathrm{G}}[\pi, \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\rho)] = \mathrm{Hom}_{\mathrm{H}}[\pi_{|\mathrm{H}}, ({\delta_{\mathrm{G}}^{-1/2}})_{|\mathrm{H}} {\delta_{\mathrm{H}}^{1/2}} \rho] \tag{2.1.1}$$

$$\mathrm{Hom}_{\mathrm{G}}[\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}(\rho), \widetilde{\pi}] = \mathrm{Hom}_{\mathrm{H}}[\rho, ({\delta_{\mathrm{G}}^{-1/2}})_{|\mathrm{H}} {\delta_{\mathrm{H}}^{1/2}} \widetilde{(\pi_{|\mathrm{H}})}] \tag{2.1.2}$$

Let $\mathrm{GL}_n := \mathrm{GL}_n(\mathrm{F})$. For a given $n \in \mathbb{N}$, g_n stands for an element of GL_n. In particular I_n will denote the identity element of GL_n. Let $(n_1, ..., n_r)$ be a partition of $n \in \mathbb{N}$. We denote the *standard parabolic subgroup* corresponding to the partition $n = n_1 + ... + n_r$ by $P_{n_1,...,n_r}$ which is the subgroup of all upper triangular block matrices where the diagonal blocks are $g_{n_i} \in \mathrm{GL}_{n_i}, i = 1, ..., r$ i.e.,

$$P_{n_1,...,n_r} = \left\{ \begin{pmatrix} g_{n_1} & * & * & * \\ & g_{n_2} & * & * \\ & & . & . \\ & & & g_{n_r} \end{pmatrix} : g_{n_i} \in GL_{n_i}(F) \right\}.$$

The unipotent radical of $P_{n_1,...,n_r}$ is the block upper triangular unipotent matrices given by :

$$N_{n_1,...,n_r} = \left\{ \begin{pmatrix} \mathbf{1}_{n_1} & * & * & * \\ & \mathbf{1}_{n_2} & * & * \\ & & . & . \\ & & & \mathbf{1}_{n_r} \end{pmatrix} \right\}$$

and the Levi subgroup of $P_{n_1,...,n_r}$ is the block diagonal subgroup :

$$M_{n_1,...,n_r} = \left\{ \begin{pmatrix} g_{n_1} & & & \\ & g_{n_2} & & \\ & & . & \\ & & & g_{n_r} \end{pmatrix} : g_i \in GL_{n_i}(F) \right\} \simeq \prod_{i=1}^{k} GL_{n_i}(F).$$

We also have the Levi decomposition $P_{n_1,...,n_r} = M_{n_1,...,n_r}N_{n_1,...,n_r}$. A parabolic subgroup of type $(n_1, ..., n_r)$ is a conjugate of $P_{n_1,...,n_r}$ in GL_n. The parabolic subgroup $P_{1,...,1}$ of GL_n is called the standard *Borel subgroup* and denoted by B_n.

We now come to the important concept of parabolic induction in GL_n. Let $P_{n_1,...,n_r}$ be the standard parabolic subgroup with Levi decomposition $P_{n_1,...,n_r} = M_{n_1,...,n_r}N_{n_1,...,n_r}$. Let ρ be a smooth representation of $M_{n_1,...,n_r}$. Extend ρ trivially across $N_{n_1,...,n_r}$ to get a smooth representation of $P_{n_1,...,n_r}$ again denoted by ρ. Now consider $\mathrm{Ind}_{\mathrm{P_{n_1,...,n_r}}}^{\mathrm{GL_n}}(\rho)$. We say that $\mathrm{Ind}_{\mathrm{P_{n_1,...,n_r}}}^{\mathrm{GL_n}}(\rho)$ is parabolically induced from $M_{n_1,...,n_r}$ to GL_n. In particular, let (ρ_i, V_i) be smooth representations of GL_{n_i}. Then $\rho_1 \otimes ... \otimes \rho_r$ defines a representation of $M_{n_1,...,n_r}$ in the space $V_1 \otimes ... \otimes V_r$. We get the parabolically induced representation $\mathrm{Ind}_{\mathrm{P_{n_1,...,n_r}}}^{\mathrm{GL_n}}(\rho_1 \otimes ... \otimes \rho_\mathrm{r})$.

We recall that the group GL_n is unimodular. Let ν denote the character $|det(.)|$ of GL_n. For an element $p_{k,n-k} = \begin{pmatrix} g_k & * \\ 0 & g_{n-k} \end{pmatrix}$ in the parabolic subgroup $P_{k,n-k}$ of GL_n, $\delta_{P_{k,n-k}}(p_{k,n-k}) = \nu(g_k)^{n-k}\nu(g_{n-k})^{-k}$. For the standard Borel subgroup $B_n = P_{1,...,1}$ of GL_n, we have

$$\delta_{B_n}(diag(b_1, ..., b_n)) = \nu^{n-1}(b_1)\nu^{n-3}(b_2)......\nu^{-(n-3)}(b_{n-1})\nu^{-(n-1)}(b_n).$$

If χ is a character of GL_1, we will think of χ as a character of GL_n via $g \mapsto \chi(\mathrm{det(g)})$ and every character of GL_n is of this form. Given a character χ of GL_n and $\pi \in \mathrm{Alg(GL_n)}$ we will denote the *twist of* π *by* χ simply by $\pi\chi$ i.e., $\pi\chi(g) = \chi(\mathrm{det(g)})\pi(\mathrm{g})$.

We summarize some basic properties of parabolic induction in the following theorem.

Theorem 2.1.2. *Let $P = MN$ be a parabolic subgroup of GL_n. Let (ρ, W) be a smooth representation of M.*

1. *The functor $\rho \mapsto \mathrm{Ind}_{\mathrm{P}}^{\mathrm{GL_n}}(\rho)$ is an exact additive functor from the category of smooth representations of M to the category of smooth representations of GL_n.*

2. $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{GL_n}}(\rho) = \mathrm{ind}_{\mathrm{P}}^{\mathrm{GL_n}}(\rho)$ *(since G/P is compact).*

3. $\widetilde{\mathrm{Ind}_{\mathrm{P}}^{\mathrm{GL_n}}(\rho)} = \mathrm{Ind}_{\mathrm{P}}^{\mathrm{GL_n}}(\widetilde{\rho})$.

4. *If ρ is admissible (respectively unitary and finitely generated) then so is* $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{GL_n}}(\rho)$ *(respectively unitary and finitely generated)*

For smooth representations ρ of GL_k and τ of GL_{n-k} there is a product $\rho \times \tau$ attached to ρ and τ defined by

$$\rho \times \tau = \mathrm{ind}_{\mathrm{P_{k,n-k}}}^{\mathrm{GL_n}}(\rho \times \tau)$$

This product is associative in view of transitivity of induction i.e.,

$$(\rho_1 \times \rho_2) \times \rho_3 \cong \rho_1 \times (\rho_2 \times \rho_3).$$

Let $n = n_1 + ... + n_r$ and $\rho_1, ..., \rho_r$ be smooth representations of GL_{n_i} for $i = 1, ..., r$. We will denote by $\rho_1 \times \cdots \times \rho_r$ the parabolically induced representation $\mathrm{ind}_{\mathrm{P_{n_1,...,n_r}}}^{\mathrm{GL_n}}(\rho_1 \otimes \cdots \otimes \rho_\mathrm{r})$. The n_i's and the ρ_i's will be clear from the context. If $\xi = \mathrm{ind}_{\mathrm{P_{n_1,...,n_r}}}^{\mathrm{GL_n}}(\rho_1 \otimes \cdots \otimes \rho_\mathrm{r})$ we will denote the representation $\mathrm{ind}_{\mathrm{P_{n_r,...,n_1}}}^{\mathrm{GL_n}}(\widetilde{\rho}_\mathrm{r} \otimes \cdots \otimes \widetilde{\rho}_1)$ by $\xi^\vee$. If $\alpha_i \in \mathbb{R}$, $\nu^{\alpha_1} \times ... \times \nu^{\alpha_n}$ stands for $\mathrm{ind}_{\mathrm{P_{1,...,1}}}^{\mathrm{GL_n}}(\nu^{\alpha_1} \otimes ... \otimes \nu^{\alpha_\mathrm{n}})$.

Jacquet Functor

Let $P = MU$ be an ℓ-group where M, U are closed subgroups of P. Also let U be a normal subgroup which is the union of its compact open subgroups. Let θ be a character of U stable under the inner conjugation action of M on U and (π, V) a smooth representation of P. Let $V_{U,\theta} = V/V(U,\theta)$ where the subspace $V(U,\theta)$ is the linear span of vectors $\{\pi(u)x - \theta(u)x | u \in U, x \in V\}$. The representation π defines a representation $\pi_{U,\theta}$ of P in the space $V_{U,\theta}$. The representation $\delta_P^{-1/2}\delta_M^{1/2}\pi_{U,\theta}$ is called the (normalized) *Jacquet module* of π and its restriction to M is denoted by $r_{U,\theta}(\pi)$. Thus to a given representation π of P, the functor $r_{U,\theta}$ associates π to a representation of M. We call this functor the Jacquet functor. We abbreviate $r_{U,1}(\pi)$ to $r_U(\pi)$. We have (see [16])

$$\mathrm{Hom_P}[\rho, \pi] = \mathrm{Hom_M}[\mathrm{r_U}(\rho), \delta_\mathrm{P}^{-\frac{1}{2}}\delta_\mathrm{M}^{\frac{1}{2}}\pi] \tag{2.1.3}$$

Let $P = MN$ be the Levi decomposition of a parabolic subgroup P of GL_n. For a smooth representation ρ of P we have the Jacquet functor $r_N(\pi)$. If π is a smooth representation of GL_n then the Jacquet functor $r_N(\pi)$ of π is the Jacquet functor of π restricted to P. The Jacquet functor is an adjoint functor to parabolic induction. The Jacquet functor $\pi \mapsto r_N(\pi)$ is an exact additive functor from the category of smooth representations of GL_n to the category of smooth representations of M. We also have Frobenius Reciprocity: For a smooth representation ρ of M and a smooth representation π of GL_n(see [11])

$$\mathrm{Hom}_{\mathrm{GL_n}}[\pi, \mathrm{ind}_{\mathrm{P}}^{\mathrm{GL_n}}(\rho)] = \mathrm{Hom}_{\mathrm{P}}[\pi_{|\mathrm{P}}, \rho\delta_{\mathrm{P}}^{1/2}] = \mathrm{Hom}_{\mathrm{M}}[\mathrm{r_N}(\pi), \rho] \tag{2.1.4}$$

Moreover, if π is admissible and finitely generated respectively then so is $r_N(\pi)$. Also let, $Q \subset P$ be standard parabolic subgroups of GL_n with Levi decomposition $P = M_P N_P$ and $Q = M_Q N_Q$. Observe that $M_Q \subset M_P$, $N_P \subset N_Q$ and $M_Q(N_Q \cap M_P)$ is a parabolic subgroup of M_P with M_P as a Levi subgroup and $N_Q \cap M_P$ as unipotent radical. Then $r_{N_Q \cap M_P}[r_{N_P}(\rho)]$ is equivalent to $r_{N_Q}(\rho)$. From (2.1.4) above it is easy to see that if π occurs as a subrepresentation of $\mathrm{ind}_{\mathrm{P}}^{\mathrm{GL_n}}(\rho)$ then the Jacquet module of π is non-zero. In general, if π is a subquotient of $\mathrm{ind}_{\mathrm{P}}^{\mathrm{GL_n}}(\rho)$, then π can be realized as a subrepresentation of a Weyl group twist of the induced representation. For a discussion we refer to [15] Theorem 5.9, 5.10. This brings us to the notion of a supercuspidal representation.

An irreducible admissible representation π of GL_n is said to be *supercuspidal* if π does not occur as a subquotient of any representation parabolically induced from any proper parabolic subgroup. This is equivalent to saying that $r_N(\pi) = 0$ for all proper parabolic subgroups $P = MN$.

In what follows, we will see that the supercuspidal representations are building blocks for all irreducible admissible representations of GL_n.

2.2 Theory of Segments and Classification

In this section we describe the Bernstein-Zelevinsky and Langlands Classification for irreducible admissible representations of GL_n. We will also recall few other results which will be useful to us later. If $\pi \in \mathrm{Alg}(\mathrm{GL_n})$ is of finite length let $\mathrm{JH}^0(\pi)$ denote the set of all irreducible subquotients of π counted with multiplicity. The following theorem is the essence of Section 2 in [22].

Proposition 2.2.1. *Let $n = n_1 + ... + n_r$ and $\sigma_1, ..., \sigma_r$ be distinct classes of irreducible supercuspidal representations of GL_{n_i}. Let $\xi = \sigma_1 \times ... \times \sigma_r$. Then we have the following:*

1. *The representation ξ of GL_n has distinct irreducible subquotients i.e., each irreducible subquotient appears with multiplicity 1 in $\mathrm{JH}^0(\xi)$.*

2. *The representation ξ has a unique irreducible subrepresentation $Z(\xi)$ and a unique irreducible quotient $Q(\xi)$.*

3. *The length of ξ is 2^k where k is the number of pairs of the form $\{\sigma_i, \sigma_i\nu\}$ contained in $\{\sigma_1, ..., \sigma_r\}$.*

Let σ be an irreducible supercuspidal representations of GL_m. A *segment* is a set of classes of irreducible supercuspidal representations of the form $\{\sigma, \sigma\nu, ..., \sigma\nu^{k-1}\}$. A segment is usually denoted by the symbol Δ. A segment $\{\sigma, \sigma\nu..., \sigma\nu^{k-1}\}$ is usually written as $\Delta = [\sigma, \sigma\nu^{k-1}]$ thought of as the representation $\sigma \otimes \sigma\nu \otimes ... \otimes \sigma\nu^{k-1}$ of $\mathrm{GL}_m \times ... \times \mathrm{GL}_m(k$ times). Sometimes we will write a segment also by $(\sigma, ..., \sigma\nu^{k-1})$.

Corollary 2.2.2. *Let $n = km$. Then $\mathrm{ind}_{\mathrm{P_{m,...,m}}}^{\mathrm{GL_n}}(\sigma \otimes ... \otimes \sigma\nu^{k-1})$ has a unique irreducible subrepresentation denoted by $Z(\Delta)$ and a unique irreducible quotient denoted by $Q(\Delta)$.*

Example 2.2.1. Let $n > 1$ and σ be the character $\nu^{-(\frac{n-1}{2})}$ of GL_1. (So σ is a supercuspidal representation of GL_1.) Consider the segment $\Delta = [\sigma, \sigma\nu^{\frac{n-1}{2}}]$ i.e., $\Delta = (\nu^{-(\frac{n-1}{2})}, \nu^{-(\frac{n-3}{2})}, ..., \nu^{\frac{n-1}{2}})$. Then $Z(\Delta)$ is the trivial representation $\mathbb{1}_n$ of GL_n identified with space of constant functions in $I_{B_n}^{\mathrm{GL}_n}(\delta_{B_n}^{-1/2})$. The representation $Q(\Delta)$ is denoted by St_n and is called the *Steinberg* representation of GL_n. Similarly, a character χ of GL_n is thought of as $Z(\Delta')$ where $\Delta' = (\chi\nu^{-(\frac{n-1}{2})}, \chi\nu^{-(\frac{n-3}{2})}, ..., \chi\nu^{\frac{n-1}{2}})$. Here $Q(\Delta')$ is $St_n\chi$, namely, the twist of St_n by χ.

Remark 2.2.2. We remark here that the original notations adopted by Zelevinsky in [22] for $Z(\Delta)$ is $\langle\Delta\rangle$ and for $Q(\Delta)$ is $\langle\Delta\rangle^t$.

Two segments $\Delta_1 = [\sigma_1, \sigma_1\nu^{k-1}]$ and $\Delta_2 = [\sigma_2, \sigma_2\nu^{\ell-1}]$ are said to be *linked* if $\Delta_1 \not\subseteq \Delta_2, \Delta_2 \not\subseteq \Delta_1$ and $\Delta_1 \cup \Delta_2$ is a segment. If Δ_1 and Δ_2 are linked and $\Delta_1 \cap \Delta_2 = \emptyset$ then we say that Δ_1 and Δ_2 are *juxtaposed*. In this case, either $\sigma_2 = \sigma_1\nu^k$ or $\sigma_1 = \sigma_2\nu^\ell$. We say that Δ_1 *precedes* Δ_2 if Δ_1 and Δ_2 are linked and $\sigma_2 = \sigma_1\nu^r$ where $r > 0$. We now describe the classification of irreducible admissible representations of GL_n in terms of the irreducible supercuspidal representations of GL_m with $m \leq n$. In this sense supercuspidal representations are said to be the building blocks of irreducible admissible representations of GL_n. First we record the following result of Zelevinsky from [22].

Proposition 2.2.3. *Let $\Delta_1, ..., \Delta_r$ be segments. Then*

1. *The representation $Z(\Delta_1) \times ... \times Z(\Delta_r)$ is irreducible if and only if for each $i, j = 1, .., r$ the segments Δ_i and Δ_j are not linked.*

2. *The representation $Q(\Delta_1) \times ... \times Q(\Delta_r)$ is irreducible if and only if for each $i, j = 1, .., r$ the segments Δ_i and Δ_j are not linked.*

Theorem 2.2.4. (*First Form of Classification*) *Let $\Delta_1, ..., \Delta_r$ be segments. Suppose for each i, j such that $i < j$, Δ_i does not precede Δ_j. Then we have the following:*

1. *The representation $Z(\Delta_1) \times ... \times Z(\Delta_r)$ has a unique irreducible subrepresentation denoted by $Z(\Delta_1, ..., \Delta_r)$.*

2. *The representations $Z(\Delta_1, ..., \Delta_r)$ and $Z(\Delta'_1, ..., \Delta'_s)$ are equivalent if and only if $s = r$ and $\Delta'_i = \Delta_{\alpha(i)}$ for some permutation α of $\{1, ..., r\}$.(Here it is again assumed that Δ'_i does not precede Δ'_j for all $i' < j'$.)*

3. *Any irreducible admissible representation of* GL_n *is equivalent to some representation of the form* $Z(\Delta_1, ..., \Delta_r)$.

As noted earlier, the trivial representation $\mathbb{1}_n$ of GL_n may be thought of as $Z(\Delta)$ where $\Delta = [\nu^{-(\frac{n-1}{2})}, \nu^{\frac{n-1}{2})}]$. The Steinberg representation St_n of GL_n may be thought of as $Z(\Delta_1, ..., \Delta_n)$ where $\Delta_1 = \{\nu^{\frac{n-1}{2}}\}, ..., \Delta_n = \{\nu^{-(\frac{n-1}{2})}\}$ where now there are n segments. Observe that the 'does not precede condition' holds for the Δ_i.

We next present the second form of the Classification theorem. It is essentially replacing Z in the statements of Theorem 2.2.4 by Q and subrepresentation by quotient.

Theorem 2.2.5. (*Second Form of Classification*) *Let* $\Delta_1, ..., \Delta_r$ *be segments. Suppose for each* i, j *such that* $i < j$, Δ_i *does not precede* Δ_j. *Then we have the following*

1. *The representation* $Q(\Delta_1) \times ... \times Q(\Delta_r)$ *has a unique irreducible quotient denoted by* $Q(\Delta_1, ..., \Delta_r)$.

2. *The representations* $Q(\Delta_1, ..., \Delta_r)$ *and* $Q(\Delta'_1, ..., \Delta'_s)$ *are equivalent if and only if* $s = r$ *and* $\Delta'_i = \Delta_{\alpha(i)}$ *for some permutation* α *of* $\{1, ..., r\}$.*(Here it is again assumed that* Δ'_i *does not precede* Δ'_j *for* $i' < j'$.*)*

3. *Any irreducible admissible representation of* GL_n *is equivalent to some representation of the form* $Q(\Delta_1, ..., \Delta_r)$.

We next realize certain classes of representations of GL_n in terms of representations of the form $Q(\Delta_1, ..., \Delta_r)$. Let Z denote the center of GL_n. A smooth irreducible representation (π, V) is said to be *essentially square integrable* if there is a character $\chi : \mathrm{GL}_1 \to \mathbb{R}_{>0}$ such that $|f_{v,\tilde{v}}(g)|^2\chi(detg)$ is a function on $Z\backslash G$ for every matrix coefficient $f_{v,\tilde{v}}$ of π, and

$$\int_{Z\backslash G} |f_{v,\tilde{v}}(g)|^2\chi(detg)dg < \infty.$$

If χ can be taken to be trivial then π is said to be a *square integrable representation.* For any segment $\Delta = [\sigma, ..., \sigma\nu^{k-1}]$ the representation $Q(\Delta)$ is an essentially square integrable representation. Conversely, every essentially square integrable representation of GL_n is equivalent to some $Q(\Delta)$ for a uniquely determined Δ. A smooth representation (π, V) of GL_n is said to be *unitary* if V has an inner product $\langle ., . \rangle$ which is GL_n invariant, i.e., $\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle$ for all $g \in \mathrm{GL}_n$ and $v, w \in V$. The representation $Q(\Delta)$ is square integrable if and only if it is unitary which is so if and only if $\sigma\nu^{\frac{k-1}{2}}$ is unitary.

Let U_n be the subgroup of all upper triangular unipotent matrices and $\Psi : U_n \to \mathbb{C}$ be the character defined by $\Psi([u_{i,j}]) = \psi(u_{1,2} + ... + u_{n-1,n})$ for some choice of a nontrivial additive character $\psi : F \to \mathbb{C}$. A smooth representation (π, V) of GL_n is said to be *generic* if $\mathrm{Hom}_{\mathrm{GL_n}}(\pi, \mathrm{Ind}_{\mathrm{U_n}}^{\mathrm{GL_n}}\Psi) \neq 0$. An irreducible admissible generic representation of GL_n is equivalent to some $Q(\Delta_1) \times ... \times Q(\Delta_r)$ where no two of the Δ_i are linked. The final class of representations that we will consider is the class of tempered representations. An irreducible admissible representation (π, V) of GL_n is said to be *tempered* if the central character ω_π of π is unitary and if one (and equivalently every) matrix coefficient $f_{v,\widetilde{v}}$ is in $L^{2+\varepsilon}(Z\backslash G)$ for every $\varepsilon > 0$. An irreducible admissible representation $\pi = Q(\Delta_1, ..., \Delta_r)$ of GL_n is tempered if and only if $\pi = \mathrm{ind}_{\mathrm{P}}^{\mathrm{GL_n}}(\mathrm{Q}(\Delta_1) \otimes ... \otimes \mathrm{Q}(\Delta_r))$ with every $Q(\Delta_i)$ being square integrable. An irreducible admissible representation π of GL_n is said to be *essentially tempered* if it has the form $\pi = \tau\nu^t$ where τ is tempered and $t \in \mathbb{R}$.

Theorem 2.2.6. (*Langlands' Construction*) *Let* $n = \sum_{i=1}^r n_i$ *and* π_i *be an irreducible essentially tempered representation of* GL_{n_i} *for each* i. *Write* $\pi_i = \tau_i\nu^{x_i}$ *where the* τ_i *are tempered and* $x_i \in \mathbb{R}$. *Assume that* $x_1 > x_2 > ... > x_r$. *Then:*

1. *The representation* $\mathrm{ind}_{\mathrm{P_{n_1,...,n_r}}}^{\mathrm{GL_n}}(\pi_1 \otimes ... \otimes \pi_\mathrm{r})$ *has a unique irreducible quotient called a Langlands Quotient.*

2. *Every irreducible admissible representation of* GL_n *is a Langlands quotient where the parabolic* $P_{n_1,...,n_r}$ *and the* π_i *(equivalently, the* τ_i *and* $x_i \in \mathbb{R}$*) are uniquely determined.*

Remark 2.2.3. Let π be the Langlands quotient of $\tau_1\nu^{x_1} \times ... \times \tau_r\nu^{x_r}$. Since each τ_i is irreducible and tempered we may write

$$\tau_i = Q(\Delta_i^1) \times ... \times Q(\Delta_i^{m_i})$$

where the $Q(\Delta_i^k)'$s are square integrable. In particular, Δ_i^k and Δ_i^l are not linked for any k, l. Therefore, π is the unique irreducible quotient of

$$Q(\Delta_1^1)\nu^{x_1} \times ... \times Q(\Delta_1^{m_1})\nu^{x_1} \times ... \times Q(\Delta_r^1)\nu^{x_r} \times ... \times Q(\Delta_r^{m_r})\nu^{x_r}$$

Now twist the segments Δ_i^k (i.e., the representations constituting the segment) by the corresponding ν^{x_i} and rename the Δ_i^k in the following way:

$$\Delta_1^1 = \Delta_1', ..., \Delta_1^{m_1} = \Delta_{m_1}', \Delta_2^1 = \Delta_{m_1+1}', ..., \Delta_r^{m_r} = \Delta_{m_1+...+m_r}'.$$

Then π is the unique irreducible quotient of

$$Q(\Delta_1') \times ... \times Q(\Delta_{m_1+...+m_r}')$$

The condition $x_1 > ... > x_r$ forces the condition that Δ_i' and Δ_j' are not linked for $i' < j'$. Thus, the second form of our classification namely Theorem 2.2.5 is equivalent to Langlands' Construction and hence is also known as Langlands Classification.

The following lemma is obvious.

Lemma 2.2.7. *Let π be any irreducible admissible representation of GL_n which is not supercuspidal. Then π can be expressed as a quotient of $\mathrm{ind}_{\mathrm{P_{k,n-k}}}^{\mathrm{GL_n}}(\rho \otimes \tau)$ where $1 \leq k \leq n-1$ and ρ and τ are irreducible admissible representations of GL_k and GL_{n-k} respectively.*

2.3 Derivatives of Representations

The mirabolic subgroup of GL_n consisting of all $g \in \mathrm{GL}_n$ with the n^{th} row equal to $(0, ..., 0, 1)$ is denoted by P_n. The symbol p_n stands for an element of P_n for given

$n \in \mathbb{N}$. We denote the subgroup of unipotent upper triangular matrices in GL_n by U_n and the set of all $r \times s$ matrices with entries from F by $M_{r,s}$. For each $i \in \{1, .., n\}$, let V_n^i denote the subgroup of P_n consisting of all matrices of the type $\begin{pmatrix} I_{n-i} & z \\ 0 & v \end{pmatrix}$ where $z \in M_{n-i,i}, v \in U_i$. Note that $V_n^n = U_n$. If ψ is a nontrivial additive character of F we get a multiplicative character ψ_i of V_n^i defined by $\psi_i(v) = \psi(\sum_{j=1}^{i} v_{j,j+1})$.

If π is a smooth representation of P_n the i^{th} *derivative of* π $(1 \leq i \leq n)$ denoted by $\pi^{(i)}$ is a smooth representation of GL_{n-i} and is defined by $\pi^{(i)} = r_{V_n^i, \psi_i}(\pi)$. We also set $\pi^{(0)} = \pi$ and $\pi^{(i)} = 0$ for $i > n$. For a smooth representation π of GL_n, the i^{th} derivative is defined by $\pi^{(i)} = (\pi_{|P_n})^{(i)}$. Note that $\pi^{(1)} = r_{N_{n-1,1}}(\pi_{|P_n})$.

The following Lemma (Section 4.5,[4]) is called *Leibnitz rule.*

Lemma 2.3.1. *Let ρ and τ be smooth representations of GL_k and GL_{n-k} respectively. Then the r^{th}-derivative $(\rho \times \tau)^{(r)}$ of the representation $\rho \times \tau$ of GL_n has a filtration whose successive quotients are $\rho^{(i)} \times \tau^{(r-i)}$ for $i = 0, ..., r$.*

The following result on supercuspidal representations (cf. [3],[4],[9]) is the essence of Gelfand-Kazhdan Theory.

Theorem 2.3.2. (*Gelfand-Kazhdan*)

1. *Let σ be an irreducible supercuspidal representation of GL_n. Then the restriction of σ to P_n is $\mathrm{ind}_{\mathrm{U_n}}^{\mathrm{P_n}} \psi_{\mathrm{n}}$.*

2. *An irreducible admissible representation σ of GL_n is supercuspidal if and only if $\sigma^{(k)} = 0$ for $0 < k < n$ and $\sigma^{(n)} = 1$.*

For the empty segment $\emptyset$ we put $Z(\emptyset)$ and $Q(\emptyset)$ to be the identity representation of the trivial group $\mathrm{GL}_0 = \{1\}$.

Example 2.3.1. If Δ is any segment then it is proved in (3.5,[22]) that exactly one of the derivatives $Z(\Delta)^k$ is non-zero and this derivative is equal to $Z(\Delta^-)$ where Δ^- is the segment obtained by omitting the last representation in Δ. For example, the derivative of any character χ of GL_n is the character $\chi\nu^{-1/2}$ of GL_{n-1}. Any character χ of GL_n is viewed as $Z(\Delta)$ where $\Delta = [\chi\nu^{-(\frac{n-1}{2})}, ..., \nu^{\frac{n-1}{2}}]$. Then $\Delta^- = (\chi\nu^{-(\frac{n-1}{2})}, ..., \chi\nu^{\frac{n-3}{2}})$ and $Z(\Delta^-) = \chi\nu^{-1/2}$ which is a representation of GL_{n-1}. Hence the highest non-zero derivative is its first derivative and $\chi^{(1)} = \chi\nu^{-1/2.}$ Note that for a character χ of GL_1 the first derivative is the trivial representation of GL_0(the trivial group), since $\Delta^- = \emptyset$ in this case.

Example 2.3.2. For an irreducible supercuspidal representation σ of GL_m let the segment $\Delta = [\sigma, \sigma\nu^{k-1}]$. If $n = km$ then $Q(\Delta)$ is an irreducible representation of GL_n. It is a theorem in ([22],9.6) that if i is not divisible by m then the i^{th} derivative of $Q(\Delta)$ is zero i.e., $Q(\Delta)^{(i)} = 0$. Also, if $i = mp$ then for $0 \leq p \leq k-1$, $Q(\Delta)^{(i)} = Q(\Delta_i)$ where $\Delta_i = [\nu^p\sigma, \sigma\nu^{k-1}]$ and $Q(\Delta)^{(n)} = 1$ the trivial representation of the trivial group GL_0. For instance, if χ is a character of GL_1 and $\Delta = [\chi\nu^{-(\frac{n-1}{2})}, \chi\nu^{\frac{n-1}{2}}]$ then $Q(\Delta)$ is the twist of Steinberg representation of GL_n by the character χ i.e., $Q(\Delta) = St_n\chi$. $St_1 = 1$ is the trivial character of GL_1. Now by the above formula $(St_n\chi)^{(1)} = St_{n-1}\chi\nu^{\frac{1}{2}}$. We also have the second derivative $(St_n\chi)^{(2)} = St_{n-2}\chi\nu$. In general $(St_n\chi)^{(i)} = St_{n-i}\chi\nu^{\frac{i}{2}}$ and $(St_n\chi)^{(n)} = 1$, the trivial representation of GL_0, the trivial group.

For a representation $\pi \in \mathrm{Alg}(\mathrm{GL_n})$ of finite length, there is a $\mathbb{Z}$-linear operator $\mathfrak{D}$: $\mathfrak{R} \to \mathfrak{R}$ defined in ([4],4.5), where $\mathfrak{R} = \oplus\mathfrak{R}_n, (n = 0, 1, ..)$ and $\mathfrak{R}_\mathfrak{n}$ is the Grothendieck group of smooth GL_n modules of finite length. For a smooth GL_n module of finite length its image in $\mathfrak{R}_n$ is denoted by the same symbol. We have the multiplication map $\times : \mathfrak{R} \to \mathfrak{R}$ given by $(\pi_1, \pi_2) \mapsto \pi_1 \times \pi_2$ by which $\mathfrak{R}$ becomes a graded ring. The operator $\mathfrak{D}$ is defined by $\mathfrak{D}(\pi) = \sum_{k=0}^{n} \pi^{(k)}$. The map $\mathfrak{D}$ is extended to a $\mathbb{Z}$-linear operator. The 'Leibnitz rule' Lemma 2.3.1 implies that $\mathfrak{D}$ is a homomorphism of rings.

It follows from our Examples 2.3.1 and 2.3.2 that

$$\mathfrak{D}(\chi) = \chi + \chi\nu^{-1/2} \tag{2.3.1}$$

and

$$\mathfrak{D}(St_n\chi) = St_n\chi + St_{n-1}\chi\nu^{\frac{1}{2}} + ... + St_2\chi\nu^{\frac{n-2}{2}} + \chi\nu^{\frac{n-1}{2}} + 1. \qquad (2.3.2)$$

Moreover, if $\pi = \rho_1 \times ... \times \rho_r$ we have

$$\mathfrak{D}(\pi) = \mathfrak{D}(\rho_1) \times ... \times \mathfrak{D}(\rho_r). \qquad (2.3.3)$$

This is Corollary 4.6 in [4].

2.4 Some Theorems of Zelevinsky

We present in this section some results of Zelevinsky [22] that we will be using in the sequel. We will use the following result of Zelevinsky proved in [22], Section 4 repeatedly in our work, which describes the product $Z(\Delta_1) \times Z(\Delta_2)$, the first part of which follows from Theorem 2.2.3.

Lemma 2.4.1. *The representation $Z(\Delta_1) \times Z(\Delta_2)$ is irreducible if and only if Δ_1 and Δ_2 are not linked. If Δ_1 and Δ_2 are linked then $Z(\Delta_1) \times Z(\Delta_2)$ has length 2. If Δ_2 precedes Δ_1 then ξ has the unique irreducible quotient $Z(\Delta_1 \cup \Delta_2) \times Z(\Delta_1 \cap \Delta_2)$. If Δ_1 precedes Δ_2 then ξ has the unique irreducible subrepresentation $Z(\Delta_1 \cup \Delta_2) \times Z(\Delta_1 \cap \Delta_2)$.*

Note that if $\Delta_1 \cap \Delta_2 = \emptyset$ then $Z(\emptyset)$ is the trivial representation of the trivial group GL_0.

We now turn to the notion of *multiset* on a set, defined at the end of Introduction in [22]. Given a set Ω, a *multiset* on Ω is a function $\chi : \Omega \to \mathbb{Z}_{\geq 0}$, where $\mathbb{Z}_{\geq 0}$ is the set of non-negative integers. We usually write down a multiset $\chi : \Omega \to \mathbb{Z}_{\geq 0}$ as $\mathfrak{a} = \{..., x, ...x, y, ..., y, ...\}$ where each element $x \in \Omega$ is repeated $\chi(x)$ times. The function χ is called the characteristic function of $\mathfrak{a}$ and denoted by $\chi_{\mathfrak{a}}$. We write $x \in \mathfrak{a}$ if $\chi_{\mathfrak{a}}(x) > 0$. Given two multisets $\mathfrak{a}, \mathfrak{b}$ we write $\mathfrak{b} \subset \mathfrak{a}$ if $\chi_{\mathfrak{b}}(x) \leq \chi_{\mathfrak{a}}(x)$ for every $x \in \mathfrak{b}$. The empty multiset $\mathfrak{a} = \emptyset$ corresponds to $\chi = 0$. We will need only finite multisets i.e., the ones for which $\chi_{\mathfrak{a}}$ has finite support. The sum $\mathfrak{a}+\mathfrak{b}$ of two multisets $\mathfrak{a}$ and $\mathfrak{b}$ is defined by $\chi_{\mathfrak{a}+\mathfrak{b}} = \chi_{\mathfrak{a}} + \chi_{\mathfrak{b}}$.

We will deal with multisets on the set of segments. We will denote by $\mathfrak{O}$ the set of all finite multisets on segments. Given any multiset $\mathfrak{a} = \{\Delta_1, ..., \Delta_r\}$ of segments put $\pi(\mathfrak{a}) = Z(\Delta_1) \times ... \times Z(\Delta_r)$. Given a multiset $\mathfrak{a} = \{\Delta_1, ..., \Delta_r\}$ we say that $\mathfrak{a}$ satisfies the **does not precede condition** (abbreviated **DNP** hereafter) if for any $i < j$ Δ_i does not precede Δ_j. We now state a Proposition (Section 6.4,[22]) of Zelevinsky.

Proposition 2.4.2. *Let $(\Delta_1, ..., \Delta_r)$ and $(\Delta'_1, ..., \Delta'_r)$ be ordered sequences of segments. Suppose one of the following holds:*

1. *$(\Delta_1, ..., \Delta_r)$ is the same as $(\Delta'_1, ..., \Delta'_r)$ except that two non-linked consecutive segments (Δ_i, Δ_{i+1}) have been interchanged.*

2. *Both $(\Delta_1, ..., \Delta_r)$ and $(\Delta'_1, ..., \Delta'_r)$ satisfy **DNP** and one is a permutation of the other.*

Then $Z(\Delta_1) \times ... \times Z(\Delta_r)$ is equivalent to $Z(\Delta'_1) \times ... \times Z(\Delta'_r)$. Therefore if (2) holds then $Z(\Delta_1, ..., \Delta_r) \cong Z(\Delta'_1, ..., \Delta'_r)$.

If $\mathfrak{a} = \{\Delta_1, ..., \Delta_r\}$ satisfies **DNP** then $\pi(\mathfrak{a})$ has a unique irreducible submodule by Theorem 2.2.4 which will be denoted by $Z(\mathfrak{a})$. Given any multiset $\mathfrak{a} = \{\Delta_1, ..., \Delta_r\}$ in $\mathfrak{O}$ one can choose an ordering of $\mathfrak{a}$ such that $\mathfrak{a}$ satisfies **DNP**. Then $\pi(\mathfrak{a})$ and $Z(\mathfrak{a})$ depend only on $\mathfrak{a}$ by Proposition 2.4.2.

Some Results on the product $Z(\Delta_1) \times ... \times Z(\Delta_r)$

We record here three results of Zelevinsky in [22]: Theorem 7.1, Proposition 8.4 and Theorem 9.13 which we will need in the sequel. We state them one by one after introducing the notations used.

For each $\mathfrak{a}, \mathfrak{b} \in \mathfrak{O}$ the multiplicity with which $Z(\mathfrak{b})$ occurs in $\mathrm{JH}^0(\pi(\mathfrak{a}))$ is denoted by $m(\mathfrak{b}, \mathfrak{a})$. An *elementary operation* on a multiset $\mathfrak{a} = \{\Delta_1, ..., \Delta_r\}$ is the replacement of two linked segments $\{\Delta_1, \Delta_2\}$ in $\mathfrak{a}$ by the pair $\{\Delta_1 \cup \Delta_2, \Delta_1 \cap \Delta_2\}$. We put a partial order on $\mathfrak{O}$ as follows: define $\mathfrak{b} < \mathfrak{a}$ if $\mathfrak{b}$ can be obtained from $\mathfrak{a}$ by a sequence of elementary operations. We have the following theorem which describes all possible irreducible subquotients of the representation $Z(\Delta_1) \times ... \times Z(\Delta_r)$.

Theorem 2.4.3. *For $\mathfrak{a}, \mathfrak{b}$ in $\mathfrak{O}$, the multiplicity $m(\mathfrak{b}, \mathfrak{a}) \neq 0$ if and only if $\mathfrak{b} \leq \mathfrak{a}$. Moreover, $m(\mathfrak{a}, \mathfrak{a}) = 1$ for any $\mathfrak{a} \in \mathfrak{O}$.*

Recall that given two multisets $\mathfrak{a}$ and $\mathfrak{b}$ we had defined $\mathfrak{a} + \mathfrak{b}$. The following proposition says that there exists an irreducible subquotient of $Z(\mathfrak{a}_1) \times ... \times Z(\mathfrak{a}_r)$ which appears with multiplicity one.

Proposition 2.4.4. *For each $\mathfrak{a}_1, ..., \mathfrak{a}_r$ in $\mathfrak{O}$ the representation $Z(\mathfrak{a}_1 + ... + \mathfrak{a}_r)$ occurs in $\mathrm{JH}^0(\mathrm{Z}(\mathfrak{a}_1) \times ... \times \mathrm{Z}(\mathfrak{a}_\mathrm{r}))$ with multiplicity 1.*

We conclude this section with the following theorem which may be deemed to be the generalization of Proposition 2.2.1.

Theorem 2.4.5. *Let $\mathfrak{a} = \{\Delta_1, ..., \Delta_r\}$ be a multiset of segments in $\mathfrak{O}$. If $\mathfrak{a}$ is such that any two of its segments have an empty intersection then the irreducible subquotients of the representation $\pi(\mathfrak{a}) = Z(\Delta_1) \times ... \times Z(\Delta_r)$ are all distinct. The set of irreducible subquotients (counted with multiplicity) of $\pi(\mathfrak{a})$ i.e., $\mathrm{JH}^0(\pi(\mathfrak{a})) = \{\mathrm{Z}(\mathfrak{b}) : \mathfrak{b} \leq \mathfrak{a}\}$.*

2.5 The representation L_n

In this section we single out the infinite dimensional representation L_n of GL_n of central character ν^n which plays an important role in this work. This representation also appeared in a prominent way in [14]. What is rather important for us is that the representation L_n arises in many principal series representations of GL_n, and these various realizations are useful to us. We begin with the following definition. Note that $\nu^{\frac{1}{2}}$ as a representation of GL_{n-1} is $Z([\nu^{-(\frac{n-3}{2})}, ..., \nu^{\frac{n-1}{2}}])$.

Definition 2.5.1. For $n \geq 2$, define L_n to be the unique irreducible quotient of $\mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\nu^{\frac{1}{2}} \otimes \nu^{\frac{n+1}{2}})$.

By Lemma 2.4.1, L_n is well defined and sits in the following exact sequences of GL_n-modules

$$0 \to \nu \to \mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\nu^{\frac{1}{2}} \otimes \nu^{\frac{n+1}{2}}) \to \mathrm{L_n} \to 0 \tag{2.5.1}$$

L_n as a quotient of $L_{k+2}\nu^{\frac{n-k-2}{2}} \times \nu^{-\frac{k}{2}}$

For $n \geq 2$ let $\Psi_n = \nu^{\frac{n-1}{2}} \times \nu^{\frac{n-3}{2}} \times ... \times \nu^{-(\frac{n-3}{2})} \times \nu^{\frac{n+1}{2}} \in \mathrm{Alg(GL_n)}$. To avoid any ambiguity in our notation, we emphasize that $\Psi_2 = \nu^{\frac{1}{2}} \times \nu^{\frac{3}{2}}$ and $\Psi_3 = \nu \times 1 \times \nu^2$. Observe that $L_2 = St_2\nu$.

Let $n \geq 3$. Note that $\nu^{\frac{1}{2}}$ is the unique irreducible quotient of $\nu^{\frac{n-1}{2}} \times \nu^{\frac{n-3}{2}} \times ... \times \nu^{-(\frac{n-3}{2})} \in \mathrm{Alg(GL_{n-1})}$ and $\mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\nu^{\frac{1}{2}} \otimes \nu^{\frac{n+1}{2}})$ is a quotient of Ψ_n. Therefore L_n is an irreducible quotient of Ψ_n. By Theorem 2.2.1, since Ψ_n has a unique irreducible quotient, that must be L_n. Among the characters appearing in Ψ_n, the character $\nu^{\frac{n+1}{2}}$ is linked only to $\nu^{\frac{n-1}{2}}$ and hence we can "move" it to the left till it is to the right of $\nu^{\frac{n-1}{2}}$. By Proposition 2.4.2 all the representations so obtained have the same orientation, are equivalent and hence have the unique irreducible quotient L_n. We are going to realize L_n as the unique irreducible of quotient of $n-2$ distinct length 2 representations different from the one in (2.5.1).

To this end, for $k = 0, ..., n-3$ put $\tau_{n-k-2} = \nu^{\frac{n-2k+1}{2}} \times ... \times \nu^{-(\frac{n-3}{2})} \in \mathrm{Alg(GL_{n-k-2})}$ and look at $\Pi_k = \Psi_{k+2}\nu^{\frac{n-k-2}{2}} \times \tau_{n-k-2} \in \mathrm{Alg(GL_n)}$. Then by the previous paragraph, each Π_k has the unique irreducible quotient L_n. In Π_k, the GL_{k+2} component $\Psi_{k+2}\nu^{\frac{n-k-2}{2}}$ has the unique irreducible quotient $L_{k+2}\nu^{\frac{n-k-2}{2}}$. On the other hand the GL_{n-k-2} part τ_{n-k} has the character $\nu^{-\frac{k}{2}}$ as the unique irreducible quotient. We claim that the representation $\xi_k = L_{k+2}\nu^{\frac{n-k-2}{2}} \times \nu^{-\frac{k}{2}} \in \mathrm{Alg(GL_n)}$ has length 2 as a GL_n-module. If the claim is true then ξ_k has the unique irreducible quotient L_n for every k.

Let $k > 0$. Put $\Delta_1 = [\nu^{\frac{n-2k-1}{2}}, \nu^{\frac{n-1}{2}}], \Delta_2 = \{\nu^{\frac{n+1}{2}}\}$ and $\Delta_3 = [\nu^{-(\frac{n-3}{2})}, \nu^{\frac{n-2k-3}{2}}]$. Then $Z(\Delta_1) = \nu^{\frac{n-k-1}{2}} \in \mathrm{Irr}(\mathrm{GL_{k+1}})$ and $Z(\Delta_3) = \nu^{\frac{-k}{2}} \in \mathrm{Irr}(\mathrm{GL_{n-k-2}})$. Put $n = k+2$ in (2.5.1), twist by $\nu^{\frac{n-k-2}{2}}$ and take product with $Z(\Delta_3) = \nu^{-\frac{k}{2}}$ to get the following exact sequence of GL_n modules namely

$$0 \to \mathrm{ind}_{\mathrm{P_{k+2,n-k-2}}}^{\mathrm{GL_n}}(\nu^{\frac{\mathrm{n-k}}{2}} \otimes \nu^{\frac{-\mathrm{k}}{2}}) \to \mathrm{Z}(\Delta_1) \times \mathrm{Z}(\Delta_2) \times \mathrm{Z}(\Delta_3) \to \xi_\mathrm{k} \to 0$$

By Theorem 2.4.5, we conclude $Z(\Delta_1) \times Z(\Delta_2) \times Z(\Delta_3)$ has length 4 with distinct irreducible subquotients and $\mathrm{ind}_{\mathrm{P_{k+2,n-k-2}}}^{\mathrm{GL_n}}(\nu^{\frac{\mathrm{n-k}}{2}} \otimes \nu^{\frac{-\mathrm{k}}{2}})$ has length 2. Therefore ξ_k has length 2.

If $k = 0$, our representation is $\xi_0 = St_2\nu^{\frac{n}{2}} \times \mathbb{1}_{n-2}$ which we show to be of length 2 in the following Lemma. We introduce some notation which we will be using in the proof. Let $\mathfrak{a}_1 = \{\Delta_1, \mu\nu^{1/2}, \mu\nu^{-1/2}\}$, $\mathfrak{a}_2 = \{\Delta_1, \Delta_2\}$ be multisets of segments where $\Delta_1 = [\nu^{-(\frac{n-3}{2})}, \nu^{\frac{n-3}{2}}]$ and $\Delta_2 = [\mu\nu^{-1/2}, \mu\nu^{1/2}]$. If $\mu = \nu^{\pm\frac{n-2}{2}}$, let $\Delta_3 = \Delta_1 \cup \Delta_2$ and $\mathfrak{a}_3 = \{\Delta_3, \nu^{\pm\frac{n-3}{2}}\}$. We have $Z(\Delta_1) = \mathbb{1}_{n-2}, Z(\Delta_2) = \mu \in \mathrm{Irr}(\mathrm{GL}_2)$ and $Z(\Delta_3) = \nu^{\pm 1/2} \in \mathrm{Irr}(\mathrm{GL_{n-1}})$.

Lemma 2.5.1. *For $n \geq 3$, $\xi = \mathrm{ind}_{\mathrm{P_{2,n-2}}}^{\mathrm{GL_n}}(\mathrm{St}_2\mu \otimes \mathbb{1}_{\mathrm{n-2}})$ is irreducible for all μ except for $\mu = \nu^{\pm n/2}$. If $\mu = \nu^{\frac{n}{2}}$, ξ has length 2 and sits in the exact sequence*

$$0 \to Z_n \to \mathrm{ind}_{\mathrm{P_{2,n-2}}}^{\mathrm{GL_n}}(\mathrm{St}_2\nu^{\frac{\mathrm{n}}{2}} \otimes \mathbb{1}_{\mathrm{n-2}}) \to \mathrm{L_n} \to 0 \qquad (2.5.2)$$

Proof. Assume $\mu \neq \nu^{\pm n/2}$. We show that there is only one possible Jordan-Holder constituent for ξ. By Proposition 2.4.4, $Z(\mathfrak{a}_1)$ occurs in ξ with multiplicity 1. By Theorem 2.4.3, $Z(\mathfrak{a}_2)$ or $Z(\mathfrak{a}_3)$ are the only possible Jordan-Holder constituents of ξ other than $Z(\mathfrak{a}_1)$.

If $\mu \neq \nu^{\pm\frac{n-2}{2}}$ then $\mathfrak{a}_3$ does not exist and therefore the only possible Jordan-Holder factor of ξ is the irreducible $Z(\mathfrak{a}_2) = \mathrm{ind}_{\mathrm{P_{n-2,2}}}^{\mathrm{GL_n}}(\mathbb{1}_{\mathrm{n-2}} \otimes \mu)$. But since $\mathfrak{D}(\mu)$ is not contained in $\mathfrak{D}(St_2\mu)$ it can be seen that $Z(\mathfrak{a}_2)$ is not a factor of ξ and ξ is irreducible. Assume that $\mu = \nu^{\frac{n-2}{2}}$. We claim that (cf. [22], Example 11.4) that neither $\mathfrak{D}(Z(\mathfrak{a}_2))$ nor $\mathfrak{D}(Z(\mathfrak{a}_3))$ is contained in $\mathfrak{D}(\xi)$. This will show that $\xi = Z(\mathfrak{a}_1)$ and hence is irreducible.

Proof of Claim: It is enough to exhibit one factor each in $\mathfrak{D}(Z(\mathfrak{a}_2))$ and $\mathfrak{D}(Z(\mathfrak{a}_3))$ which is not contained in $\mathfrak{D}(\xi)$. By Lemma 2.4.1, we have $\mathrm{ind}_{\mathrm{P_{n-2,2}}}^{\mathrm{GL_n}}(\mathbb{1}_{\mathrm{n-2}} \times \nu^{\frac{\mathrm{n-2}}{2}})$ has $Z(\mathfrak{a}_2)$ and $Z(\mathfrak{a}_3) = \mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\nu^{\frac{1}{2}} \otimes \nu^{\frac{\mathrm{n-3}}{2}})$ as Jordan-Holder factors each appearing with multiplicity one. The first derivative of $Z(a_3)$ has the irreducible $\mathrm{ind}_{\mathrm{P_{n-2,1}}}^{\mathrm{GL_n}}(\mathbb{1}_{\mathrm{n-2}} \otimes \nu^{\frac{\mathrm{n-3}}{2}})$ as a component which does not appear in the first derivative of ξ(which is glued from $\mathrm{ind}_{\mathrm{P_{n-2,1}}}^{\mathrm{GL_{n-1}}}(\mathbb{1}_{\mathrm{n-2}} \otimes \nu^{\frac{\mathrm{n-1}}{2}})$ and $\mathrm{ind}_{\mathrm{P_{n-3,2}}}^{\mathrm{GL_{n-1}}}(\nu^{-\frac{1}{2}} \otimes \mathrm{St}_2\nu^{\frac{\mathrm{n-2}}{2}})$. Similarly using the second derivative of $\mathrm{ind}_{\mathrm{P_{n-2,2}}}^{\mathrm{GL_n}}(\mathbb{1}_{\mathrm{n-2}} \otimes \nu^{\frac{\mathrm{n-2}}{2}})$ we find that the irreducible $L_{n-2}\nu^{-1}$ is the second derivative of $Z(\mathfrak{a}_3)$, but the second derivative of ξ is composed of $\mathbb{1}_{n-2}$ and the irreducible $\mathrm{ind}_{\mathrm{P_{n-3,1}}}^{\mathrm{GL_{n-2}}}(\nu^{-\frac{1}{2}} \otimes \nu^{\frac{\mathrm{n-1}}{2}})$. This proves the claim.

If $\mu = \nu^{n/2}$ we look at the exact sequence

$$0 \to \xi \to \mathrm{ind}_{\mathrm{P_{1,1,n-2}}}^{\mathrm{GL_n}}(\nu^{\frac{\mathrm{n+1}}{2}} \otimes \nu^{\frac{\mathrm{n-1}}{2}} \otimes \mathbb{1}_{\mathrm{n-2}}) \to \mathrm{ind}_{\mathrm{P_{2,n-2}}}^{\mathrm{GL_n}}(\nu^{\mathrm{n}/2} \otimes \mathbb{1}_{\mathrm{n-2}}) \to 0$$

and observe that $\mathrm{ind}_{\mathrm{P_{1,1,n-2}}}^{\mathrm{GL_n}}(\nu^{\frac{\mathrm{n-1}}{2}} \otimes \nu^{\frac{\mathrm{n+1}}{2}} \otimes \mathbb{1}_{\mathrm{n-2}})$ has length 4 and distinct irreducible subquotients by Theorem 2.4.5. By the same theorem, $\mathrm{ind}_{\mathrm{P_{2,n-2}}}^{\mathrm{GL_n}}(\nu^{\frac{\mathrm{n}}{2}} \otimes \mathbb{1}_{\mathrm{n-2}})$ has length 2. Therefore, ξ has length 2. Note that ξ is a quotient of Ψ_n and L_n is the unique irreducible quotient of Ψ_n. Therefore the exact sequence (2.5.2) follows. □

The importance of the exact sequence (2.5.2) is that it helps us to realize L_n as a Langlands quotient. We also have to realize many other representations as Langlands quotient as well, which we do next.

2.6 Summary on Langlands parameter

In this section we summarize facts (c.f. [11] or [21]) about Langlands parameters of certain irreducible admissible representations of GL_n which we will be using to prove Theorem 1.1.2.

The Local Langlands Correspondence

We will follow the exposition in [15] closely. More detailed expositions can be found in [11] and [21].

Let W_{F} denote the Weil group of F. By local class field theory we may identify the characters of $\mathrm{F}^* = \mathrm{GL}_1$ and the characters of W_{F}. The local Langlands Correspondence generalizes this to a bijection between irreducible admissible representations of GL_n and certain n-dimensional representations of the Weil-Deligne group W'_{F} of F. The Weil- Deligne group may be defined to be $W_{\mathrm{F}} \times SL(2, \mathbb{C})$.

An n-dimensional representation Π of the Weil-Deleigne group W'_{F} which is semi-simple when restricted to W_{F} and algebraic when restricted to $SL(2, \mathbb{C})$ is of the form $\Pi = \sum_{i=1}^{r} \Pi_i \otimes Sp(m_i)$ where Π_i are irreducible representations of W_{F} of dimension n_i and $Sp(m_i)$ is the unique m_i dimensional irreducible representation of $SL(2, \mathbb{C})$.

The following version of the Local Langlands Correspondence is from [15].

Theorem 2.6.1. (***Local Langlands Correspondence***) *There exists a natural bijective correspondence between irreducible admissible representations of* GL_n *and* n*-dimensional representations of the Weil-Deligne group* W'_{F} *of* F *which are semi-simple when restricted to* W_{F} *and algebraic when restricted to* $SL(2, \mathbb{C})$*. The correspondence reduces to class field theory for* $n = 1$*, and is equivariant under twisting and taking contragredients.*

Remark 2.6.1. To any pair of representations π_1 and π_2 of GL_n and GL_m respectively one can attach 'L-functions' and 'ε-factors' which we will not define here since we do not need it in the sequel. Similarly, to any pair of representations of W'_{F} one attaches 'L-functions' and 'ε-factors'. The Local Langlands correspondence is supposed to be natural in the sense that it is the unique correspondence which preserves the 'L and ε-factors'.

We denote the underlying map of the Local Langlands Correspondence by $\mathfrak{L}$. Then for a given $\pi \in \mathrm{Irr}(\mathrm{GL_n})$ the n-dimensional representation $\mathfrak{L}(\pi)$ of W'_F is called the

Langlands parameter of π. The Bernstein-Zelevinsky Classification reduces the local Langlands Correspondence to one between irreducible supercuspidal representations of GL_n and irreducible representations of W_{F}.

The Langlands parameters can be described via Langlands classification theorem if we know the Langlands parameter of supercuspidal representations. We discuss this here. Let a non-supercuspidal irreducible admissible representation π of GL_n be given. Recall that the Langlands classification (Theorem 2.2.6) states that a representation of the form $\xi = \pi_1\nu^{x_1} \times ... \times \pi_r\nu^{x_r}$, where $n = \sum n_i$, $\pi_i \in \mathrm{Irr}(\mathrm{GL_{n_i}})$ are tempered and $x_i \in \mathbb{R}$ are such that $x_1 > x_2 > ... > x_r$ has a unique irreducible quotient, which is called a *Langlands quotient*. Conversely any $\pi \in \mathrm{Irr}(\mathrm{GL_n})$ can be expressed uniquely as a Langlands quotient. In particular, if π is the Langlands quotient of ξ then $\xi^{\vee} = \widetilde{\pi}_r\nu^{-x_r} \times ... \times \widetilde{\pi}_1\nu^{-x_1}$ has a Langlands quotient (since $\widetilde{\pi}_i$ are tempered and $-x_r > ... > -x_1$) which is nothing but $\widetilde{\pi}$. Also recall that Langlands Classification is equivalent to saying that π is the unique irreducible quotient of $Q(\Delta_1) \times ... \times Q(\Delta_k)$, where Δ_k's are segments and Δ_i does not precede Δ_j for $i < j$. Let Sp_n denote the unique n-dimensional irreducible representation of $SL(2, \mathbb{C})$. If $\Delta_i = [\sigma_i, ..., \sigma_i\nu^{\ell_i - 1}]$ for a supercuspidal $\sigma_i \in \mathrm{Irr}(\mathrm{GL_{m_i}})$ we have $\mathfrak{L}(\pi) = \oplus_{i=1}^{k}\mathfrak{L}(\sigma_i) \otimes Sp(l_i)$.

Our aim is to understand $\pi \in \mathrm{Irr}(\mathrm{GL_n})$ for which $\mathfrak{L}(\pi)$ has an $n-2$ dimensional subrepresentation corresponding to the trivial representation $\mathbb{1}_{n-2}$ of GL_{n-2} such that the two-dimensional quotient representation $\mathfrak{L}(\pi)/\mathfrak{L}(\mathbb{1}_{n-2})$ is $\mathfrak{L}(\tau)$ where $\tau \in \mathrm{Irr}(\mathrm{GL}_2)$ is either infinite dimensional or one of the characters $\nu^{\pm\frac{n-2}{2}}$.

The following remark may be useful in some situations.

Remark 2.6.2. Let $\rho \in \mathrm{Irr}(\mathrm{GL_k})$ and $\tau \in \mathrm{Irr}(\mathrm{GL_k})$ are such that $\rho \times \tau$ is irreducible. Write $\rho = Z(\Delta_1, ..., \Delta_r)$ and $\tau = Z(\Delta'_1, ..., \Delta'_s)$ in the Zelevinsky classification. Put $\mathfrak{a} = \{\Delta_1, ..., \Delta_r\}$ and $\mathfrak{b} = \{\Delta'_1, ..., \Delta'_s\}$. Then $\rho\times\tau = Z(\mathfrak{a})\times Z(\mathfrak{b})$. By Proposition 2.4.4, $Z(\mathfrak{a} + \mathfrak{b})$ occurs in $\mathrm{JH}^0(\rho \times \tau)$ and by the irreducibility of $\rho \times \tau$ implies $Z(\mathfrak{a} + \mathfrak{b}) = \rho \times \tau$. Similar consideration using $Q(\Delta)$ which directly corresponds to Langlands parametrization instead of $Z(\Delta)$, suggests that $\mathfrak{L}(\rho \times \tau) = \mathfrak{L}(\rho) \oplus \mathfrak{L}(\tau)$.

If χ is a character of GL_1, we will denote $\mathfrak{L}(\chi)$ by the same symbol χ. For any character χ of GL_1 viewed as the character $g \mapsto \chi(\det(g))$ of GL_n, $\mathfrak{L}(\chi)$ is $\chi\nu^{\frac{n-1}{2}} \oplus \cdots \oplus \chi\nu^{-(\frac{n-1}{2})}$. Note that $\mathfrak{L}(1\!\!1_{n-2}) = \nu^{\frac{n-3}{2}} \oplus \nu^{\frac{n-5}{2}} \oplus \ldots \oplus \nu^{-(\frac{n-3}{2})}$. For a supercuspidal $\sigma \in \mathrm{Irr}(\mathrm{GL}_2)$, $\mathrm{ind}_{\mathrm{P_{2,n-2}}}^{\mathrm{GL_n}}(\sigma \otimes 1\!\!1_{\mathrm{n-2}})$ is irreducible by Lemma 2.4.1 and has Langlands parameter $\mathfrak{L}(\sigma) \oplus \nu^{\frac{n-3}{2}} \oplus \nu^{\frac{n-5}{2}} \oplus \ldots \oplus \nu^{-(\frac{n-3}{2})}$. If $\chi \neq \nu^{\frac{n-1}{2}}, \nu^{-(\frac{n+1}{2})}$, the representation $\mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\nu^{-1/2} \otimes \chi)$ is irreducible by Lemma 2.4.1, and has Langlands parameter $\nu^{\frac{n-3}{2}} \oplus \nu^{\frac{n-5}{2}} + \ldots \oplus \nu^{-(\frac{n-3}{2})} \oplus \nu^{-(\frac{n-1}{2})} \oplus \chi$. Hence, $\mathfrak{L}(\mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\nu^{-1/2} \otimes \nu^{-(\frac{n-3}{2})}))/\mathfrak{L}(1\!\!1_{\mathrm{n-2}})$ is $\nu^{-(\frac{n-3}{2})} \oplus \nu^{-(\frac{n-1}{2})}$ which corresponds to the character $\nu^{-(\frac{n-2}{2})}$ of GL_2. If $\chi \neq \nu^{-(\frac{n-1}{2})}, \nu^{\frac{n+1}{2}}$, then $\mathfrak{L}(\mathrm{ind}_{\mathrm{P_{1,n-1}}}^{\mathrm{GL_n}}(\chi \otimes \nu^{1/2})) = \chi \oplus \nu^{\frac{\mathrm{n-1}}{2}} \oplus \nu^{\frac{\mathrm{n-3}}{2}} \oplus \ldots \oplus \nu^{-(\frac{\mathrm{n-5}}{2})} \oplus \nu^{-(\frac{\mathrm{n-3}}{2})}$. Consequently, $\mathfrak{L}(\mathrm{ind}_{\mathrm{P_{1,n-1}}}^{\mathrm{GL_n}}(\nu^{\frac{\mathrm{n-3}}{2}} \otimes \nu^{1/2}))/\mathfrak{L}(1\!\!1_{\mathrm{n-2}})$ is $\nu^{\frac{n-3}{2}} \oplus \nu^{\frac{n-1}{2}}$ which corresponds to the character $\nu^{\frac{n-2}{2}}$ of GL_2. If $\chi_1 \neq \chi_2\nu^{\pm 1}$ and neither χ_1 nor χ_2 equals $\nu^{\pm(\frac{n-1}{2})}$, the representation $\mathrm{ind}_{\mathrm{P_{1,1,n-2}}}^{\mathrm{GL_n}}(\chi_1 \otimes \chi_2 \otimes 1\!\!1_{\mathrm{n-2}})$ is irreducible by Theorem 2.2.3(i) and has the Langlands parameter $\chi_1 \oplus \chi_2 \oplus \nu^{\frac{n-3}{2}} \oplus \nu^{\frac{n-5}{2}} \oplus \ldots \oplus \nu^{-(\frac{n-3}{2})}$.

In the previous paragraph, we have analyzed all those representations $\pi \in \mathrm{Irr}(\mathrm{GL_n})$ for which $\mathfrak{L}(\pi)$ has the subrepresentation $\mathfrak{L}(1\!\!1_{n-2})$ and $\mathfrak{L}(\pi)/\mathfrak{L}(1\!\!1_{n-2})$ corresponds either to $\nu^{\pm\frac{n-2}{2}}$ or an infinite dimensional of GL_2 other than a twist $St_2\chi$ of the Steinberg representation of GL_2. It now remains to consider such representations namely for which $\mathfrak{L}(\pi)/\mathfrak{L}(1\!\!1_{n-2})$ is $\mathfrak{L}(St_2\chi)$.

Lemma 2.6.2. *For $2 \leq k \leq n-2$, $\xi_k = \nu^{\frac{n-3}{2}} \times \ldots \times \nu^{\frac{n-2k+1}{2}} \times St_2\nu^{\frac{n-2k}{2}} \times \nu^{\frac{n-2k-1}{2}} \times \ldots \times \nu^{-(\frac{n-3}{2})} \in Alg(\mathrm{GL}_n)$ has the Langlands quotient $1\!\!1_{n-2} \times St_2\nu^{\frac{n-2k}{2}}$. Also the representation $\xi_1 = St_2\nu^{\frac{n-2}{2}} \times \nu^{\frac{n-3}{2}} \times \ldots \times \nu^{-(\frac{n-3}{2})}$ has the Langlands quotient $St_2\nu^{\frac{n-2}{2}} \times 1\!\!1_{n-2}$. The representation $\xi_0 = St_2\nu^{\frac{n}{2}} \times \nu^{\frac{n-3}{2}} \times \ldots \times \nu^{-(\frac{n-3}{2})}$ has the Langlands quotient L_n. Consequently the representations $\xi_{n-1} := \xi_1^\vee$ and $\xi_n := \xi_0^\vee$ have $St_2\nu^{-(\frac{n-2}{2})} \times 1\!\!1_{n-2}$ and $\widetilde{L_n}$ as their Langlands quotients respectively.*

Proof. We already know that ξ_k has a Langlands quotient for every k. Note that $\nu^{\frac{n-3}{2}} \times \ldots \times \nu^{\frac{n-2k+1}{2}} \in \mathrm{Alg}(\mathrm{GL_{k-1}})$ has the character $\nu^{\frac{n-k-1}{2}}$ of GL_{k-1} and $\nu^{\frac{n-2k-1}{2}} \times \ldots \times \nu^{-(\frac{n-3}{2})} \in \mathrm{Alg}(\mathrm{GL_{n-k-1}})$ has the character $\nu^{-(\frac{k-1}{2})}$ of GL_{n-k-1} as quotients. Therefore ξ_k has $\eta := \mathrm{ind}_{\mathrm{P_{k-1,2,n-k-1}}}^{\mathrm{GL_n}}(\nu^{\frac{\mathrm{n-k-1}}{2}} \times \mathrm{St}_2\nu^{\frac{\mathrm{n-2k}}{2}} \times \nu^{-(\frac{\mathrm{k-1}}{2})})$ as a quotient. By

Lemma 2.5.1, $\nu^{\frac{n-k-1}{2}} \times St_2\nu^{\frac{n-2k}{2}}$ is irreducible and hence $\eta \cong St_2\nu^{\frac{n-2k}{2}} \times \nu^{\frac{n-k-1}{2}} \times \nu^{-(\frac{k-1}{2})}$. By Lemma 2.4.1, $\mathrm{ind}_{\mathrm{P_{k-1,n-k-1}}}^{\mathrm{GL_{n-1}}}(\nu^{\frac{\mathrm{n-k-1}}{2}} \times \nu^{-(\frac{\mathrm{k-1}}{2})})$ has the unique irreducible quotient $\mathbb{1}_{n-2}$. It is readily seen that η has $\mathrm{ind}_{\mathrm{P_{2,n-2}}}^{\mathrm{GL_n}}(\mathrm{St}_2\nu^{\frac{\mathrm{n-2k}}{2}} \otimes \mathbb{1}_{\mathrm{n-2}})$ as a quotient. By Lemma 2.5.1, this quotient is irreducible, which proves the first assertion. Since ξ_0 has $\mathrm{ind}_{\mathrm{P_{2,n-2}}}^{\mathrm{GL_n}}(\mathrm{St}_2\nu^{\frac{\mathrm{n}}{2}} \otimes \mathbb{1}_{\mathrm{n-2}})$ as a quotient it follows from Lemma 2.5.1 that ξ_0 has the Langlands quotient L_n. The other cases are similar. □

To conclude the summary on Langlands parameters we note the following. It is well known that the Langlands parameter of the Steinberg representation St_2 is $\nu^{-\frac{1}{2}}Sp_2$ where Sp_2 is the unique 2-dimensional irreducible representation of $SL(2,\mathbb{C})$. By Lemma 2.5.1 and Lemma 2.6.2, $Sp_2\chi\nu^{\frac{-1}{2}} \oplus \nu^{\frac{n-3}{2}} \oplus \nu^{\frac{n-5}{2}} \oplus ... \oplus \nu^{-(\frac{n-3}{2})}$ is the Langlands parameter of $\mathrm{ind}_{\mathrm{P_{2,n-2}}}^{\mathrm{GL_n}}(\mathrm{St}_2\chi \otimes \mathbb{1}_{\mathrm{n-2}})$ when $\chi \neq \nu^{\pm n/2}$, of L_n when $\chi = \nu^{n/2}$ and of $\widetilde{L_n}$ when $\chi = \nu^{-n/2}$. In what follows we will have occasion to use two results on extensions of GL_n modules which we quote for easy reference. The first one is a general result and well-known.

Lemma 2.6.3. *Let* π_1 *and* π_2 *be representations of* GL_n *with central characters* ω_1 *and* ω_2 *respectively. If* $\omega_1 \neq \omega_2$ *then* $\mathrm{Ext}^1_{\mathrm{GL_n}}[\pi_1, \pi_2] = 0$.

The next Lemma is a special case of a Lemma of D. Prasad (Lemma 6 in [14])

Lemma 2.6.4. (*Prasad*) *For any principal series representation* ρ *of* GL_2 (*not necessarily irreducible*), $\mathrm{Ext}^1_{\mathrm{GL}_2}[\rho, \mathbb{1}_2] = 0$ *if and only if* $\mathrm{Hom}_{\mathrm{GL}_2}[\rho, \mathbb{1}_2] = 0$.

2.7 Preliminaries on GL_{n-1}-distinguished representations of GL_n

Given a representation π of GL_n, by $\mathrm{Hom}_{\mathrm{GL_{n-1}}}[\pi, \mathbb{1}_{\mathrm{n-1}}]$, we mean $\mathrm{Hom}_{\mathrm{GL_{n-1}}}[\pi_{|\mathrm{GL_{n-1}}}, \mathbb{1}_{\mathrm{n-1}}]$. We begin by observing that a character χ of GL_n is GL_{n-1}-distinguished if and only if χ is the trivial representation $\mathbb{1}_n$. The next lemma follows from a well known theorem due to Gelfand-Kazhdan [3] which says that the outer automorphism $g \mapsto {}^tg^{-1}$

(which preserves GL_{n-1}) takes an irreducible admissible representation of GL_n to $\widetilde{\pi}$. Therefore we have the following consequence

Lemma 2.7.1. *Let* $\pi \in \mathrm{Irr}(\mathrm{GL_n})$. *Then* π *is* GL_{n-1}*-distinguished if and only if its contragredient* $\widetilde{\pi}$ *is* GL_{n-1}*-distinguished.*

The above Lemma 2.7.1 has the following generalization when one has product of irreducible representations.

Lemma 2.7.2. (***Duality Lemma***) *Let* $n = \sum_{i=1}^{r} n_i$, $\xi = \mathrm{ind}_{\mathrm{P_{n_1,\ldots,n_r}}}^{\mathrm{GL_n}}(\rho_1 \otimes \ldots \otimes \rho_\mathrm{r})$ *and* $\xi^\vee = \mathrm{ind}_{\mathrm{P_{n_r,\ldots,n_1}}}^{\mathrm{GL_n}}(\widetilde{\rho}_\mathrm{r} \otimes \ldots \otimes \widetilde{\rho}_1)$ *where* $\rho_i \in \mathrm{Irr}(\mathrm{GL_{n_i}})$. *Then* $\mathrm{Hom}_{\mathrm{GL_{n-1}}}[\xi, \mathbb{1}_{\mathrm{n-1}}] = \mathrm{Hom}_{\mathrm{GL_{n-1}}}[\xi^\vee, \mathbb{1}_{\mathrm{n-1}}]$. *Moreover, if* ξ *has the unique irreducible quotient* π *then* $\xi^\vee$ *has the unique irreducible quotient* $\widetilde{\pi}$.

Proof. Fix $(s_n)_{i,j} := (-1)^i \delta_{i,n+1-j} \in \mathrm{GL}_n$. The automorphism $s : \mathrm{GL}_n \to \mathrm{GL}_n$ defined by $s(g) = s_n{}^t g^{-1} s_n^{-1}$ induces a functor $S : \mathrm{Alg}(\mathrm{GL_n}) \to \mathrm{Alg}(\mathrm{GL_n})$. The automorphism s maps $P_{n_1,\ldots,n_r}$ to $P_{n_r,\ldots,n_1}$ and GL_{n-1} to a conjugate of GL_{n-1}. It is clear from the above mentioned theorem of Gelfand-Kazhdan that S maps ξ to $\xi^\vee$. Our first assertion follows. The last assertion also follows from the same theorem. □

It follows from Theorem 2.3.2 that the restriction of an irreducible supercuspidal representation σ of GL_n to GL_{n-1} is $\mathrm{ind}_{\mathrm{U_{n-1}}}^{\mathrm{GL_{n-1}}} \psi_{\mathrm{n-1}}$. From this theorem and using Frobenius Reciprocity it was noted by Prasad in [14] that supercuspidal representations do not have a GL_{n-1}-invariant form for $n > 2$ whereas for $n = 2$ this representation is in fact GL_1-distinguished. We record this in the following lemma.

Lemma 2.7.3. *If* $\sigma_n \in Irr(\mathrm{GL}_n)$ *is supercuspidal then* $\mathrm{Hom}_{\mathrm{GL_{n-1}}}[\sigma_\mathrm{n}, \mathbb{1}_{\mathrm{n-1}}] = 0$ *for* $n \geq 3$ *whereas* $\mathrm{Hom}_{\mathrm{GL_1}}[\sigma_2, 1] \neq 0$.

2.8 Some Examples on product of characters

We now present few examples which will describe how to work with product of characters. This is just an application of the Zelevinsky Theory we have discussed so far.

These examples will be useful in understanding quotients of parabolically induced representations which are important from our point of view. We start with the most basic case i.e., that of GL_2. Recall that given segments $\Delta_1, ..., \Delta_r$ the representation $Z(\Delta_1) \times ... \times Z(\Delta_r)$ is irreducible if and only if Δ_i and Δ_j are not linked for all $i, j = 1, ...r$. Therefore to understand the reducibility of a product we must know when two segments are linked and in that case how the representation decomposes.

Example 2.8.1. (Segment Theory for GL_2) Let χ_1 and χ_2 be two characters of GL_1. Recall that $\chi_1 \times \chi_2$ denotes $\mathrm{ind}_{\mathrm{P}_{1,1}}^{\mathrm{GL}_2}(\chi_1 \otimes \chi_2)$. We think of χ_1 and χ_2 as two segments. By Lemma 2.4.1, $\chi_1 \times \chi_2$ is reducible if and only if the segments (χ_1) and (χ_2) are linked i.e., if and only if $\chi_2 = \chi_1 \nu^{\pm 1}$. Fix $\chi_1 = \nu^{\frac{-1}{2}}$. Then $\nu^{-\frac{1}{2}} \times \chi_2$ is linked if and only if $\chi_2 = \nu^{\frac{1}{2}}$ or $\chi_2 = \nu^{\frac{-3}{2}}$.

If $\chi_2 = \nu^{\frac{1}{2}}$ observe that $\nu^{-\frac{1}{2}}$ precedes $\nu^{\frac{1}{2}}$ and therefore by Lemma 2.4.1, $\nu^{-\frac{1}{2}} \times \nu^{\frac{1}{2}}$ has length 2 and has the unique irreducible subrepresentation $Z([\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}]) = \mathbb{1}_2$, the trivial representation of GL_2(cf. Example 2.2.1). The quotient is infinite dimensional and is the Steinberg representation St_2. We have the following exact sequence of GL_2-modules which is of fundamental importance.

$$0 \to \mathbb{1}_2 \to \nu^{-\frac{1}{2}} \times \nu^{\frac{1}{2}} \to St_2 \to 0 \tag{2.8.1}$$

Twisting the above sequence by a character μ, we get

$$0 \to \mu \to \mu\nu^{-\frac{1}{2}} \times \mu\nu^{\frac{1}{2}} \to St_2\mu \to 0 \tag{2.8.2}$$

Taking contragredient of the above and replacing μ^{-1} by λ we get

$$0 \to St_2\lambda \to \lambda\nu^{\frac{1}{2}} \times \lambda\nu^{-\frac{1}{2}} \to \lambda \to 0 \tag{2.8.3}$$

These exact sequences contains the complete picture of a reducible principal series of GL_2. Note that in the third exact sequence $\lambda\nu^{-\frac{1}{2}}$ precedes $\lambda\nu^{\frac{1}{2}}$ and therefore the character λ sits as a quotient in the product.

Remark 2.8.2. Note from the above example that the only principal series $\chi_1 \times \chi_2$ of GL_2 which has the trivial representation of GL_2 as a quotient is the principal series $\nu^{\frac{1}{2}} \times \nu^{\frac{-1}{2}}$.

We next consider $\mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\nu^{\frac{1}{2}} \otimes \mu)$ where μ is a character of GL_1.

Example 2.8.3. Let $\xi = \mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\nu^{\frac{1}{2}} \otimes \mu)$. Recall from Example 2.2.1 that $1\!\!1_{n-1}$ as a character of GL_{n-1} is viewed as $Z((\nu^{-(\frac{n-2}{2})}, ..., \nu^{\frac{n-2}{2}}))$. Let $\Delta = (\nu^{\frac{n-3}{2}}, ..., \nu^{\frac{n-1}{2}})$. Then $Z(\Delta) = \nu^{\frac{1}{2}}$. Let Δ' be the (singleton) segment $[\mu]$. Then Δ and Δ' are linked if and only if either $\mu = \nu^{\frac{n+1}{2}}$ or $\mu = \nu^{-(\frac{n-1}{2})}$. Fix $\mu = \nu^{\frac{n+1}{2}}$ then Δ precedes Δ' and therefore by Lemma 2.4.1, $Z(\Delta \cup \Delta')$ is a subrepresentation of ξ. (Note that the intersection $\Delta \cap \Delta' = \emptyset$.) Now

$$\Delta \cup \Delta' = (\nu^{-(\frac{n-3}{2})}, ..., \nu^{\frac{n-1}{2}}, \nu^{\frac{n+1}{2}}).$$

Observe that it is the segment obtained from $\Delta'' = (\nu^{-(\frac{n-1}{2})}, ..., \nu^{\frac{n-1}{2}})$ by twisting throughout by ν. Since $Z(\Delta'') = 1\!\!1_n$, we conclude that $Z(\Delta \cup \Delta') = \nu$. Therefore, $\mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\nu^{\frac{1}{2}} \otimes \nu^{\frac{n+1}{2}}$ has the unique irreducible subrepresentation ν. It has length 2 and therefore has a unique irreducible quotient. This irreducible quotient was defined to be L_n in Section 2.5. We recall the exact sequence (2.5.1) in which L_n sits namely

$$0 \to \nu \to \mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\nu^{\frac{1}{2}} \otimes \nu^{\frac{n+1}{2}}) \to \mathrm{L_n} \to 0 \tag{2.8.4}$$

Twisting (2.8.4) by $\nu^{-\frac{1}{2}}$ we get

$$0 \to \nu^{\frac{1}{2}} \to \mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(1\!\!1_{\mathrm{n-1}} \otimes \nu^{\frac{n}{2}}) \to \mathrm{L_n}\nu^{-\frac{1}{2}} \to 0. \tag{2.8.5}$$

We have the dual exact sequence

$$0 \to \widetilde{L_n}\nu^{\frac{1}{2}} \to \mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(1\!\!1_{\mathrm{n-1}} \otimes \nu^{-\frac{n}{2}}) \to \nu^{-\frac{1}{2}} \to 0. \tag{2.8.6}$$

If we twist (2.8.4) by ν^{-1} then we have

$$0 \to 1\!\!1_n \to \mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\nu^{-\frac{1}{2}} \otimes \nu^{\frac{n-1}{2}}) \to \mathrm{L_n}\nu^{-1} \to 0 \tag{2.8.7}$$

All these exact sequences are crucial to us and will be used in the sequel.

Recall from Proposition 2.4.2 that if $(\Delta_1, ..., \Delta_r)$ are segments and $(\Delta'_1, ..., \Delta'_r)$ is same as $(\Delta_1, ..., \Delta_r)$ except that two non-linked consecutive segments (Δ_i, Δ_{i+1}) have been interchanged then

$$Z(\Delta_1) \times ... \times Z(\Delta_r) \cong Z(\Delta'_1) \times ... \times Z(\Delta'_r).$$

Example 2.8.4. Consider $\xi = \mathrm{ind}_{\mathrm{P_{n-2,1,1}}}^{\mathrm{GL_n}}(\mathbb{1}_{\mathrm{n-2}} \otimes \mu \otimes \chi)$ where $\mathbb{1}_2 \times \mu$ is irreducible. We look at the irreducible quotients of ξ.

Let us write $\xi = \mathbb{1}_{n-2} \times \mu \times \chi$. Recall that the segment for $\mathbb{1}_{n-2}$ is $(\nu^{-(\frac{n-3}{2})}, ..., \nu^{\frac{n-3}{2}})$. Note that by our assumption $\mu \neq \nu^{\pm\frac{n-1}{2}}$. By Proposition 2.2.3(1), ξ is irreducible if and only if $\chi \neq \mu\nu^{\pm 1}$ and $\chi \neq \nu^{\pm\frac{n-1}{2}}$.

(a) If χ and μ are linked, let $\chi = \mu\nu^{\pm 1}$. Then by Example 2.8.1, the quotient of $\mu \times \mu\nu$ is $St_2\mu\nu^{\frac{1}{2}}$ and the quotient of $\mu \times \mu\nu^{-1}$ is the character $\mu\nu^{-\frac{1}{2}}$ of GL_2. The irreducible quotients of ξ are quotients of $\mathrm{ind}_{\mathrm{P_{n-2,2}}}^{\mathrm{GL_n}}(\mathbb{1}_{\mathrm{n-2}} \otimes \mathrm{St}_2\mu\nu^{\frac{1}{2}})$ where $\mu\nu^{\frac{1}{2}} \neq \nu^{\frac{n}{2}}, \nu^{-(\frac{n-2}{2})}$ in the first case. In the second case they are quotients of $\mathrm{ind}_{\mathrm{P_{n-2,2}}}^{\mathrm{GL_n}}(\mathbb{1}_{\mathrm{n-2}} \otimes \mu\nu^{-\frac{1}{2}})$, where $\mu\nu^{-\frac{1}{2}} \neq \nu^{\frac{n-2}{2}}, \nu^{-\frac{n}{2}}$.

(b) Let $\xi = \mathbb{1}_{n-2} \times \mu \times \chi$ be such that $\mu \neq \chi\nu^{\pm 1}$. That is both $\mu \times \chi$ and $\mathbb{1}_{n-2} \times \mu$ are irreducible. Suppose $\mathbb{1}_{n-2} \times \chi$ is reducible. Then we may write $\xi = \mu \times \mathbb{1}_{n-2} \times \chi$ and $\chi = \nu^{\pm\frac{n-1}{2}}$. We look at the irreducible quotients of $\mathbb{1}_{n-2} \times \chi$. Let $\chi = \nu^{\frac{n-1}{2}}$. Then, by (2.8.5) $\mathbb{1}_{n-2} \times \nu^{\frac{n-1}{2}}$ has the unique irreducible quotient $L_{n-1}\nu^{-\frac{1}{2}}$. Therefore any irreducible quotient of ξ is a quotient of $\mu \times L_{n-1}\nu^{-\frac{1}{2}}$. If $\chi = \nu^{-(\frac{n-1}{2})}$, by appealing to (2.8.6) we may conclude that any irreducible quotient of ξ is a quotient of $\mathrm{ind}_{\mathrm{P_{1,n-1}}}^{\mathrm{GL_n}}(\mu \otimes \nu^{-\frac{1}{2}})$

Example 2.8.5. We look at $\xi = \mathrm{ind}_{\mathrm{P_{n-2,1,1}}}^{\mathrm{GL_n}}(\mathbb{1}_{\mathrm{n-2}} \otimes \lambda \otimes \nu^{-(\frac{n-3}{2})})$. Assume that $\lambda \neq \nu^{\pm\frac{n-1}{2}}$. Then λ is not linked to $(\nu^{-\frac{n-3}{2}}, ..., \nu^{\frac{n-3}{2}})$ and hence $\mathbb{1}_{n-2} \times \lambda$ is irreducible. Also since $\nu^{-(\frac{n-3}{2})}$ is a part of the segment for $\mathbb{1}_{n-2}$ the product $\mathbb{1}_{n-2} \times \nu^{-(\frac{n-3}{2})}$ is also irreducible. Therefore, ξ is reducible if and only if λ is linked to $\nu^{-(\frac{n-3}{2})}$. The only possible choice for λ is $\nu^{-(\frac{n-5}{2})}$ whence $\lambda \times \nu^{-(\frac{n-3}{2})}$ has the unique irreducible quotient

$\nu^{-(\frac{n-4}{2})} \in \mathrm{Irr}(\mathrm{GL}_2)$ by Example 2.8.1. Therefore ξ has the unique irreducible quotient $\mathrm{ind}_{\mathrm{P_{n-2,2}}}^{\mathrm{GL_n}}(\mathbb{1}_{\mathrm{n-2}} \otimes \nu^{-(\frac{\mathrm{n-4}}{2})})$.

Example 2.8.6. Consider the representation $\xi = \mathrm{ind}_{\mathrm{P_{n-2,1,1}}}^{\mathrm{GL_n}}(\nu \otimes \lambda \otimes \nu^{-(\frac{\mathrm{n-3}}{2})})$ where $\lambda \neq \nu^{-(\frac{n-3}{2})}, \nu^{-(\frac{n+1}{2})}$. The segment for the character ν of GL_{n-2} is $(\nu^{-(\frac{n-5}{2})}, ..., \nu^{\frac{n-1}{2}})$. The condition on λ means that λ is not linked to this segment and hence $\nu \times \lambda$ is irreducible. Write $\xi = \lambda \times \nu \times \nu^{-(\frac{n-3}{2})}$. Observe that the segment $\nu^{-(\frac{n-3}{2})}$ precedes the segment $(\nu^{-(\frac{n-5}{2})}, ..., \nu^{\frac{n-1}{2}})$. By Lemma 2.4.1, $\nu \times \nu^{-(\frac{n-3}{2})}$ has the unique irreducible quotient $Z((\nu^{-(\frac{n-3}{2})}, ..., \nu^{\frac{n-1}{2}})) = \nu^{\frac{1}{2}}$. Therefore any irreducible quotient of ξ is a quotient of $\eta = \mathrm{ind}_{\mathrm{P_{1,n-1}}}^{\mathrm{GL_n}}(\lambda \otimes \nu^{\frac{1}{2}})$. Observe that η is irreducible if and only if $\lambda \neq \nu^{-(\frac{n-1}{2})}$.(because, by our assumption already $\lambda \neq \nu^{\frac{n+1}{2}}$.) Therefore the only choice of λ for which η is reducible is $\nu^{-(\frac{n-1}{2})}$. Look at (2.8.7). Applying Duality Lemma to $\nu^{\frac{1}{2}} \times \nu^{\frac{n-1}{2}}$ we find η and hence ξ has the unique irreducible quotient $\widetilde{L_n}\nu$.

Example 2.8.7. We next consider the representation $\xi = \mathrm{ind}_{\mathrm{P_{n-3,1,1,1}}}^{\mathrm{GL_n}}(\nu^{\frac{1}{2}} \otimes \chi_1 \otimes \chi_2 \otimes \nu^{-(\frac{\mathrm{n-3}}{2})})$. Assume that $\chi_2 \neq \chi_1\nu^{\pm 1}$ and both χ_1, χ_2 are not equal to either $\nu^{\frac{n-1}{2}}, \nu^{-(\frac{n-3}{2})}$. We look at the irreducible quotients of ξ.

The character $\nu^{\frac{1}{2}}$ of GL_{n-3} is $Z((\nu^{-(\frac{n-5}{2})}, ..., \nu^{\frac{n-3}{2}}))$. The assumption on χ_1 and χ_2 implies that $\nu^{\frac{1}{2}} \times \chi_1 \times \chi_2$ is irreducible. Write $\xi = \chi_1 \times \chi_2 \times \nu^{\frac{1}{2}} \times \nu^{-(\frac{n-3}{2})}$. Observe that $\nu^{-(\frac{n-3}{2})}$ precedes $(\nu^{-(\frac{n-5}{2})}, ..., \nu^{\frac{n-3}{2}})$. Therefore, $\nu^{\frac{1}{2}} \times \nu^{-(\frac{n-3}{2})}$ has the unique irreducible quotient $Z((\nu^{-(\frac{n-3}{2})}, \nu^{-(\frac{n-5}{2})}, ..., \nu^{\frac{n-3}{2}})) = \mathbb{1}_{n-2}$. Therefore any irreducible quotient of ξ is a quotient of $\eta = \chi_1 \times \chi_2 \times \mathbb{1}_{n-2}$. Now η is irreducible if and only if both χ_1 and χ_2 are not equal to $\nu^{-(\frac{n-1}{2})}$. (By choice χ_1 and χ_2 are not linked and neither of them is equal to $\nu^{\frac{n-1}{2}}$.)

Assume without loss of generality that $\chi_2 = \nu^{-(\frac{n-1}{2})}$. Replace n by $n-1$ in (2.8.5) and applying the Duality Lemma to the representation in the middle we show that $\nu^{-(\frac{n-1}{2})} \times \mathbb{1}_{n-2}$ has the unique irreducible quotient $\widetilde{L_{n-1}}\nu^{\frac{1}{2}}$. Therefore, any quotient of ξ is a quotient of $\chi_1 \times \widetilde{L_{n-1}}\nu^{\frac{1}{2}}$.

We finally consider three pairs of characters and their quotient which will be relevant to our discussion.

Example 2.8.8. Let $n \geq 4$ and $2 \leq k \leq n-2$.

(a). Consider $\mathrm{ind}_{\mathrm{P_{k,n-k}}}^{\mathrm{GL_n}}(\nu^{\frac{n-k}{2}} \otimes \nu^{-\frac{k}{2}})$. Let

$$\Delta_1 = (\nu^{\frac{n-2k-1}{2}}, ..., \nu^{\frac{n-3}{2}}), \Delta_2 = (\nu^{-(\frac{n-1}{2})}, ..., \nu^{\frac{n-2k-1}{2}}).$$

Then $\nu^{\frac{n-k-2}{2}} = Z(\Delta_1)$ and $\nu^{-\frac{k}{2}} = Z(\Delta_2)$. Also Δ_2 precedes Δ_1. We have

$$\Delta_1 \cup \Delta_2 = (\nu^{-(\frac{n-1}{2})}, ..., \nu^{\frac{n-k-1}{2}}), ..., \nu^{\frac{n-3}{2}}), \Delta_1 \cap \Delta_2 = (\nu^{\frac{n-2k-1}{2}})$$

and $Z(\Delta_1 \cup \Delta_2) = \nu^{-\frac{1}{2}} \in \mathrm{Irr}(\mathrm{GL_{n-1}})$ and $Z(\Delta_1 \cap \Delta_2) = \nu^{\frac{n-2k-1}{2}} \in \mathrm{Irr}(\mathrm{GL}_1)$. Therefore by Lemma 2.4.1, $\mathrm{ind}_{\mathrm{P_{k,n-k}}}^{\mathrm{GL_n}}(\nu^{\frac{n-k}{2}} \otimes \nu^{-\frac{k}{2}})$ has length 2 and has the unique irreducible quotient $\mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\nu^{-\frac{1}{2}} \otimes \nu^{\frac{n-2k-1}{2}})$.

(b). Consider $\mathrm{ind}_{\mathrm{P_{k,n-k}}}^{\mathrm{GL_n}}(\nu^{\frac{n-k}{2}} \otimes \nu^{-(\frac{k-2}{2})})$. By similar analysis as in (a), we conclude that this representation has length 2 and has the unique irreducible quotient $\mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\nu^{\frac{1}{2}} \otimes \nu^{\frac{n-2k+1}{2}})$.

(c). Consider $\mathrm{ind}_{\mathrm{P_{k,n-k}}}^{\mathrm{GL_n}}(\nu^{\frac{n-k}{2}}) \otimes \nu^{-\frac{k}{2}}$. The segments corresponding to $\nu^{\frac{n-k}{2}}$ and $\nu^{-\frac{k}{2}}$ are

$$(\nu^{\frac{n-2k+1}{2}}, ..., \nu^{\frac{n-1}{2}}) \quad \text{and} \quad (\nu^{-(\frac{n-1}{2})}, ..., \nu^{\frac{n-2k-1}{2}})$$

respectively, whence there is no intersection amongst the segments and again by Lemma 2.4.1 we conclude that $\mathrm{ind}_{\mathrm{P_{k,n-k}}}^{\mathrm{GL_n}}(\nu^{\frac{n-k}{2}}) \otimes \nu^{-\frac{k}{2}}$ has $\mathbb{1}_n$ as the unique irreducible quotient.

Chapter 3

Mackey theory and its Consequences

Let G be a finite group and H, K be subgroup of G. Then we know that finite dimensional complex representations of G are semisimple. If (ρ, V) is a representation of H then we have the following theorem [17] to determine the restriction of the induced representation $\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}(\rho)$ to K namely;
Let $G = \cup_{i=1}^{s} Hg_iK$ be a double coset decomposition. Then

$$\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}(\rho)_{|\mathrm{K}} = \oplus_{\mathrm{i=1}}^{\mathrm{s}} \mathrm{ind}_{\mathrm{g_i^{-1}Hg_i \cap K}}^{\mathrm{K}}(\rho^{\mathrm{g_i}})$$

where $\rho^{g_i}(g_i^{-1}hg_i) = \rho(h)$.

Observe that $\mathrm{ind}_{\mathrm{g_i^{-1}Hg_i \cap K}}^{\mathrm{K}}(\rho^{\mathrm{g_i}})$ comes from those functions in the space of $\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}(\rho)$ which have their support in the double coset Hg_iK. This method of studying a restriction of an induced representation of a group to a subgroup is what is known as *Mackey Theory.*

If the finite group G is replaced by an ℓ-group and H, K are closed subgroups of G then the above theorem is true up to semi-simplification. More precisely, let X be an ℓ-space, Y a closed subspace and E be an ℓ-sheaf (see. [3],[4]) defined on X. Let $\Gamma_c(X, E)$ denote the space of all locally constant and compactly supported sections of E. Then by [3] we have the following exact sequence:

$$0 \to \Gamma_c(X \setminus Y, E_{|X \setminus Y}) \to \Gamma_c(X, E) \to \Gamma_c(Y, E_{|Y}) \to 0 \qquad (3.0.1)$$

Let G be an ℓ-group and H a closed subgroup of G. Let (ρ, U) be a smooth representation of H. Let E_ρ be the ℓ-sheaf associated to the representation ρ. Then by [3] $\Gamma_c(H\backslash G, E_\rho)$ can be naturally realized as $\mathrm{ind}_{\mathrm{H}}^{\mathrm{G}}(\rho)$.

We now explain how (3.0.1) can be applied in our context. Let $X = P_{k,n-k}\backslash \mathrm{GL}_n$. Then X can be identified with $\mathrm{Gr(k,n)}$, the space of all k-dimensional subspaces of F^n. We let GL_{n-1}-act on X. For this action we show that there are three orbits say O_1, O_2, O_3 out of which the first two are closed orbits and third one is the unique open orbit. We apply the above exact sequence with $Y = O_1 \cup O_2$. Then the RHS of the exact sequence is $\Gamma_c(O_1, E_{\rho_{|O_1}}) \oplus \Gamma_c(O_2, E_{\rho_{|O_2}})$ and the LHS is $\Gamma_c(O_3, E_{\rho_{|O_3}})$. Since the orbits can be identified with GL_{n-1} modulo the Stabilizer, the space of sections corresponding to O_i can be realized as a suitable induced representation of GL_{n-1}. Let $\mathrm{GL}_n = \cup_{i=1}^3 P_{k,n-k} g_i \mathrm{GL}_{n-1}$. Then $\Gamma_c(O_i, E_{\rho_{|O_i}})$ is equivalent to $\mathrm{ind}_{\mathrm{S_i}}^{\mathrm{GL_{n-1}}}(\rho_{\mathrm{i}})$ where $\rho_i(h) = \delta_H^{1/2}(h)\delta_{S_i}^{-1/2}(h)\rho(h)$ for $h \in S_i$ where S_i is identified with a subgroup of $P_{k,n-k}$ by $g_i S_i g_i^{-1}$. The appearance of the characters is in order to take care of normalized induction.

3.1 GL_{n-1}-action on $\mathrm{Gr(k,n)}$

Let $n > 1$ be a positive integer, k be a positive integer such that $1 \leq k \leq n-1$ and $X = \mathrm{Gr(k,n)}$ be the space of all k dimensional subspaces of the vector space F^n. Then the usual action of GL_n on X is transitive. Let $Z_0 = \langle e_1, ..., e_k \rangle$, $Z_1 = \langle e_1, ..., e_{k-1}, e_n \rangle$, $Z_2 = \langle e_1, ..., e_{k-1}, e_k + e_n \rangle$ be fixed k dimensional subspaces in F^n. We may identify X as the orbit under GL_n of the point $Z_0 \in X$. It is easy to see that the stabilizer of Z_0 in GL_n equals $P_{k,n-k}$ and hence $X = \mathrm{GL}_n / P_{k,n-k}$. Let $h_0 = I_n$, the identity of GL_n. Define $h_1, h_2 \in \mathrm{GL}_n$ as follows: $h_1(e_i) = e_i$ for all $i \neq k, n$, $h_1(e_k) = e_n$

and $h_1(e_n) = e_k$. $h_2(e_i) = e_i$ for all $i \neq k$ and $h_2(e_k) = e_k + e_n$. We begin with the following lemma:

Lemma 3.1.1. *Let Y^k be a linear subspace of F^n of dimension k, W_1 the subspace generated by $\{e_1, \cdots, e_{n-1}\}$ and W_2 the subspace generated by $\{e_n\}$. Then exactly one of the following holds:*
(i) $\dim(Y^k \cap W_1) = k$ *and* $\dim(Y^k \cap W_2) = 0$
(ii) $\dim(Y^k \cap W_1) = k - 1$ *and* $\dim(Y^k \cap W_2) = 1$
(iii)$\dim(Y^k \cap W_1) = k - 1$ *and* $\dim(Y^k \cap W_2) = 0$

Proof. For $i = 1, 2$ we have $\dim(Y^k \cap W_i) = \dim\ Y^k + \dim(W_i) - \dim(Y^k + W_i)$. The lemma follows from this easily. □

We now let GL_{n-1} act on X. We denote the stabilizer and orbit of a point $x \in X$ for the GL_{n-1} action by S_x and O_x respectively.

Proposition 3.1.2. *For the action of* GL_{n-1} *on X there are precisely three orbits, namely O_{Z_0}, O_{Z_1} and O_{Z_2}.*

Proof. If Y^k is a point in X then it satisfies precisely one condition in Lemma 3.1.1. If it satisfies condition (i) in Lemma 3.1.1, $Y^k \subset W_1$ and has a basis $\{y_1, \cdots, , y_k\}$. Then clearly there exists a $g \in \mathrm{GL}_{n-1}$ such that $g.Z_0 = Y^k$. If Y^k satisfies (ii) in Lemma 3.1.1, Y^k has a basis $\{y_1, \cdots, , y_k\}$ where $y_i \in W_1$ for $1 \leq i \leq k-1$ and without loss of generality $y_k = e_n$. Define $g \in \mathrm{GL}_n$ such that g maps e_i to y_i for $1 \leq i \leq k-1$, and remaining e_i's to themselves. Then we have $g.Z_1 = Y^k$ and $g \in \mathrm{GL}_{n-1}$. Finally if Y^k satisfies (iii) Y^k has a basis $= \{y_1, \cdots, , y_k\}$ where $y_i \in W_1$ for $i = 1, ...k-1$ and $y_k = x_{1k}e_1 + \cdots + x_{n-1,k}e_{n-1} + e_n$. Define $g \in \mathrm{GL}_n$ such that g maps e_i to y_i for $1 \leq i \leq k-1$, $e_k + e_n$ to y_k and e_i to e_i for $k+1 \leq i \leq n$. Then $g \in \mathrm{GL}_{n-1}$ and $g.Z_2 = Y^k$. □

We identify S_{Z_i} with a subgroup of $P_{k,n-k}$ which is given by conjugation by h_i i.e., $h_i^{-1} S_{Z_i} h_i$. The orbits, stabilizers and the corresponding subgroups in $P_{k,n-k}$ for the

GL_{n-1} action is listed in Table 1 below. Note also that the orbits O_{Z_0} and O_{Z_1} are closed whereas the orbit O_{Z_2} is open.

Table 1

Orbit	Stabilizer in GL_{n-1}	Corresponding subgroup of $P_{k,n-k}$
O_{Z_0}	$P_{k,n-k-1} = \left\{ \begin{pmatrix} g_k & X \\ 0 & g_{n-k-1} \end{pmatrix} \right\}$	$\left\{ \begin{pmatrix} g_k & X^0 \\ 0 & \begin{bmatrix} g_{n-k-1} & 0 \\ 0 & 1 \end{bmatrix} \end{pmatrix} \right\}$
O_{Z_1}	$P_{k-1,n-k} = \left\{ \begin{pmatrix} g_{k-1} & Y \\ 0 & g_{n-k} \end{pmatrix} \right\}$	$\left\{ \begin{pmatrix} \begin{bmatrix} g_{k-1} & 0 \\ 0 & 1 \end{bmatrix} & Y_0 \\ 0 & g_{n-k} \end{pmatrix} \right\}$
O_{Z_2}	$S_{Z_2} = \left\{ \begin{pmatrix} p_k & X \\ 0 & g_{n-k-1} \end{pmatrix} \right\}$	$\left\{ \begin{pmatrix} p_k & X^0 \\ 0 & \begin{pmatrix} g_{n-k-1} & 0 \\ -X_k & 1 \end{pmatrix} \end{pmatrix} \right\}$

A word on Notation: In Table 1 above, $g_0 = 1$. Also for $X = (x_{ij}) \in M_{k,n-k-1}$, X_k is the k^{th} row of the matrix X and $X^0 = (X\ 0) \in M_{k,n-k}$. For the matrix $Y \in M_{k-1,n-k}$, the matrix $Y_0 = \begin{pmatrix} \mathrm{Y} \\ 0 \end{pmatrix} \in M_{k,n-k}$.

The idea of Mackey theory is to reduce the problem of studying the restriction to GL_{n-1} of an induced representation π of GL_n to studying three induced representations of GL_{n-1}. The question of existence of GL_{n-1} invariant forms for π can now be addressed by studying GL_{n-1} invariant forms for induced representations of GL_{n-1} itself. To this end, for $n > 2$ let $\rho \in \mathrm{Alg(G_k)}$ and $\tau \in \mathrm{Alg(GL_{n-k})}$ be smooth representations for $1 \leq k \leq n-1$. We study the existence of GL_{n-1} invariant forms for the parabolically induced representation $\mathrm{ind}_{\mathrm{P_{k,n-k}}}^{\mathrm{GL_n}}(\rho \otimes \tau)$.

Let $\rho \in \mathrm{Alg(GL_k)}$ and $\tau \in \mathrm{Alg(GL_{n-k})}$. By Mackey theory we get an exact sequence of GL_{n-1} modules

$$0 \to \mathrm{ind}_{\mathrm{S_{Z_2}}}^{\mathrm{GL_{n-1}}}(\rho_\tau)_2 \to (\mathrm{ind}_{\mathrm{P_{k,n-k}}}^{\mathrm{GL_n}}(\rho \otimes \tau))_{|\mathrm{GL_{n-1}}} \to \mathrm{ind}_{\mathrm{S_{Z_0}}}^{\mathrm{GL_{n-1}}}(\rho_\tau)_0 \oplus \mathrm{ind}_{\mathrm{SZ_1}}^{\mathrm{GL_{n-1}}}(\rho_\tau)_1 \to 0 \tag{3.1.1}$$

where the actions of $(\rho_\tau)_j$ on S_{z_j} are given by:

$$(\rho_\tau)_0 \begin{pmatrix} g_k & * \\ 0 & g_{n-k-1} \end{pmatrix} = \nu^{1/2}\rho(g_k) \otimes \tau \begin{pmatrix} g_{n-k-1} & 0 \\ 0 & 1 \end{pmatrix}$$

$$(\rho_\tau)_1 \begin{pmatrix} g_{k-1} & * \\ 0 & g_{n-k} \end{pmatrix} = \rho \begin{pmatrix} g_{k-1} & 0 \\ 0 & 1 \end{pmatrix} \otimes \nu^{-1/2}\tau(g_{n-k})$$

$$(\rho_\tau)_2 \begin{pmatrix} p_k & X \\ 0 & g_{n-k-1} \end{pmatrix} = \rho(p_k) \otimes \tau \begin{pmatrix} g_{n-k-1} & 0 \\ -X_k & 1 \end{pmatrix}$$

Note: For the action of $(\rho_\tau)_i$ above that there is no term involving ρ when $k = 1$ since the action of ρ is trivial for $i = 1, 2$ and similarly for τ when $k = n-1$ for $i = 0, 2$.

3.2 Representations on the Orbits

From now on we will denote the closed orbits O_{Z_0} by $\mathcal{C}_1$, O_{Z_1} by $\mathcal{C}_2$ and the open orbit O_{Z_2} by $\mathcal{O}$ respectively. *By saying that a representation of the form $\rho \times \tau$ of GL_n has a G_{n-1} invariant form on a particular orbit, we mean the constituent representation* $\mathrm{ind}_{\mathrm{S_{Z_i}}}^{\mathrm{GL_{n-1}}}(\rho_\tau)_\mathrm{i}$ *in the corresponding Mackey exact sequence has a GL_{n-1} invariant form.* We observe that if any one of the representations which occur on the right hand side i.e., corresponding to the closed orbits, of the exact sequence (3.1.1) has a GL_{n-1} invariant form then they induce a GL_{n-1} invariant form on $\rho \times \tau$. But a GL_{n-1}-invariant form on the left i.e., on the open orbit may or may not extend to an GL_{n-1}-invariant form for $\rho \times \tau$. In any case we have to investigate when does

$\mathrm{ind}_{\mathrm{S_{Z_i}}}^{\mathrm{GL_{n-1}}}(\rho_\tau)_{\mathrm{i}}$ have a GL_{n-1}-invariant form.

By (2.1.2), on $\mathcal{C}_1$, $\mathrm{ind}_{\mathrm{P_{k,n-k-1}}}^{\mathrm{GL_{n-1}}}(\rho_\tau)_0$ has a nonzero GL_{n-1}-invariant form if and only if

$$\mathrm{Hom}_{\mathrm{GL_k}}[\rho\nu^{-(\frac{\mathrm{n-k}}{2}-1)}, \mathbb{1}_{\mathrm{k}}] \neq 0 \quad \text{and } \mathrm{Hom}_{\mathrm{GL_{n-k-1}}}[\tau_{|\mathrm{GL_{n-k-1}}}\nu^{\mathrm{k}/2}, \mathbb{1}_{\mathrm{n-k-1}}] \neq 0$$

Similarly on $\mathcal{C}_2$, $\mathrm{ind}_{\mathrm{Pk-1,n-k}}^{\mathrm{GL_{n-1}}}(\rho_\tau)_1$ has a nonzero GL_{n-1}-invariant form if and only if

$$\mathrm{Hom}_{\mathrm{GL_{k-1}}}[\rho_{|\mathrm{GL_{k-1}}}\nu^{-(\frac{\mathrm{n-k}}{2})}, \mathbb{C}] \neq 0 \quad \text{and } \mathrm{Hom}_{\mathrm{GL_{n-k}}}[\tau\nu^{\frac{\mathrm{k}}{2}-1}, \mathbb{C}] \neq 0$$

On the open orbit $\mathcal{O}$, we have

$$\mathrm{Hom}_{\mathrm{GL_{n-1}}}[\mathrm{ind}_{\mathrm{S_{Z_2}}}^{\mathrm{GL_{n-1}}}(\rho_\tau)_2, \mathbb{1}_{\mathrm{n-1}}] = \mathrm{Hom}_{\mathrm{S_{Z_2}}}[(\rho_\tau)_2, \delta_{\mathrm{S_{z_2}}}^{1/2}]$$

The latter space is nonzero if and only if

$$\text{(I) } \mathrm{Hom}_{\mathrm{P_k}}[\rho_{|\mathrm{P_k}}, \nu^{\frac{\mathrm{n-k}}{2}}] \neq 0 \text{ and} \quad \text{(II) } \mathrm{Hom}_{\mathrm{P^t_{n-k}}}[\tau_{|\mathrm{P^t_{n-k}}}, \nu^{-\mathrm{k}/2}] \neq 0.$$

By (2.1.3)

$$\mathrm{Hom}_{\mathrm{P_k}}[\rho_{|\mathrm{P_k}}\nu^{-(\frac{\mathrm{n-k}}{2})}, \mathbb{1}_{\mathrm{k}}] = \mathrm{Hom}_{\mathrm{GL_{k-1}}}[\rho^{(1)}\nu^{-(\frac{\mathrm{n-k-1}}{2})}, \mathbb{1}_{\mathrm{k-1}}].$$

If τ is irreducible, since the automorphism $g \mapsto {}^t g^{-1}$ of GL_{n-k} maps P^t_{n-k} to P_{n-k} we may rewrite the condition (II) and apply Frobenius Reciprocity to get

$$\mathrm{Hom}_{\mathrm{P_{n-k}}}[\widetilde{\tau}_{|\mathrm{P_{n-k}}}\nu^{\frac{-\mathrm{k}}{2}}, \mathbb{1}_{\mathrm{n-k}}] = \mathrm{Hom}_{\mathrm{GL_{n-k-1}}}[\widetilde{\tau}^{(1)}\nu^{\frac{-\mathrm{k}+1}{2}}, \mathbb{1}_{\mathrm{n-k-1}}] \neq 0$$

So the condition for $\mathrm{ind}_{\mathrm{S_{Z_2}}}^{\mathrm{GL_{n-1}}}(\rho_\tau)_2$ to have a GL_{n-1} invariant form when τ is irreducible is

$$\mathrm{Hom}_{\mathrm{GL_{k-1}}}[\rho^{(1)}\nu^{-(\frac{\mathrm{n-k-1}}{2})}, \mathbb{1}_{\mathrm{k-1}}] \neq 0 \text{ and} \quad \mathrm{Hom}_{\mathrm{GL_{n-k-1}}}[\widetilde{\tau}^{(1)}\nu^{\frac{-\mathrm{k}+1}{2}}, \mathbb{1}_{\mathrm{n-k-1}}] \neq 0.$$

However, we emphasize that even when we have a nontrivial form on $\mathrm{ind}_{\mathrm{S_{Z_2}}}^{\mathrm{GL_{n-1}}}(\rho_\tau)_2$ it may not always extend to $\mathrm{ind}_{\mathrm{P_{k,n-k}}}^{\mathrm{GL_n}}(\rho\otimes\tau)$. We will show later in some special situations that the GL_{n-1}-invariant form existing on the open orbit indeed extends to $\mathrm{ind}_{\mathrm{P_{k,n-k}}}^{\mathrm{GL_n}}(\rho\otimes\tau)$.

3.3 Summary of the analysis

We summarize below the conclusions obtained from Mackey theory. Let $\rho \in \mathrm{Alg}(\mathrm{G_k})$ and $\tau \in \mathrm{Alg}(\mathrm{GL_{n-k}})$. If $\rho \times \tau$ is GL_{n-1}-distinguished then at least one of the set of following conditions 3.3.1, 3.3.2, 3.3.3 must hold:

$$\text{(a)}\quad \mathrm{Hom}_{\mathrm{GL_k}}[\rho\nu^{-(\frac{n-k-2}{2})}, 1\!\!1_{\mathrm{k}}] \neq 0 \quad\text{and}\quad \text{(b)}\quad \mathrm{Hom}_{\mathrm{GL_{n-k-1}}}[\tau\nu^{\mathrm{k}/2}, 1\!\!1_{\mathrm{n-k-1}}] \neq 0 \qquad (3.3.1)$$

$$\text{(a)}\quad \mathrm{Hom}_{\mathrm{GL_{k-1}}}[\rho\nu^{-(\frac{n-k}{2})}, 1\!\!1_{\mathrm{k-1}}] \neq 0 \quad\text{and}\quad \text{(b)}\quad \mathrm{Hom}_{\mathrm{GL_{n-k}}}[\tau\nu^{\frac{k-2}{2}}, 1\!\!1_{\mathrm{n-k}}] \neq 0 \qquad (3.3.2)$$

$$\text{(a)}\ \mathrm{Hom}_{\mathrm{GL_{k-1}}}[\rho^{(1)}\nu^{-(\frac{n-k-1}{2})}, 1\!\!1_{\mathrm{k-1}}] \neq 0 \quad\text{and}\quad \text{(b)}\quad \mathrm{Hom}_{\mathrm{P^t_{n-k}}}[\tau_{|\mathrm{P^t_{n-k}}}, \nu^{-\mathrm{k}/2}] \neq 0 \qquad (3.3.3)$$

Moreover on the open orbit $\mathcal{O}$, if $\tau \in \mathrm{Irr}(\mathrm{GL_{n-k}})$ we may write

$$(3.3.3)\text{(b) as } \mathrm{Hom}_{\mathrm{GL_{n-k-1}}}[\widetilde{\tau}^{(1)}\nu^{\frac{-k+1}{2}}, 1\!\!1_{\mathrm{n-k-1}}] \neq 0.$$

Note that by saying (3.3.1) holds we mean both conditions (a) and (b) in (3.3.1) hold. Similar statement applies to (3.3.2) and (3.3.3). Since (3.3.1) and (3.3.2) are conditions on the closed orbits if either (3.3.1) or (3.3.2) holds then $\rho \times \tau$ is GL_{n-1}-distinguished. To show that a representation is not GL_{n-1}-distinguished we will usually show that one of the conditions in each orbit fails. **If $k = 1$ observe that** (3.3.2)**(a) and** (3.3.3)**(a) are automatic. Similarly if $k = n - 1$** (3.3.1)**(b) and** (3.3.3)**(b) are automatic.**

Assume that (3.3.3) does not hold and precisely one of (3.3.1) or (3.3.2) hold. It follows from the exact sequence (3.1.1) that

$$\mathrm{Hom}_{\mathrm{GL_{n-1}}}[\mathrm{ind}^{\mathrm{GL_n}}_{\mathrm{P_{k,n-k}}}(\rho \times \tau), 1\!\!1_{\mathrm{n-1}}] = \mathrm{Hom}_{\mathrm{GL_{n-1}}}[\mathrm{ind}^{\mathrm{GL_{n-1}}}_{\mathrm{S_{Z_i}}}(\rho_\tau)_{\mathrm{i}}, 1\!\!1_{\mathrm{n-1}}] \qquad (3.3.4)$$

where $i = 0$ or $i = 1$ according to whether (3.3.1) or (3.3.2) holds. If only (3.3.1) holds and $\rho = \nu^{\frac{n-k-2}{2}}$ then

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL_{n-1}}}[\rho \times \tau, \mathbb{1}_{\mathrm{n-1}}]) = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL_{n-k-1}}}[\tau\nu^{\frac{\mathrm{k}}{2}}, \mathbb{1}_{\mathrm{n-k-1}}]). \tag{3.3.5}$$

Similarly if only (3.3.2) holds and $\tau = \nu^{-(\frac{k-2}{2})}$ then

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL_{n-1}}}[\rho \times \tau, \mathbb{1}_{\mathrm{n-1}}]) = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL_{k-1}}}[\rho\nu^{-(\frac{\mathrm{n-k}}{2})}, \mathbb{1}_{\mathrm{k-1}}]). \tag{3.3.6}$$

The next result which is an obvious restatement of the conditions (3.3.1) and (3.3.2) provides a recipe to construct GL_{n-1}-distinguished representations of GL_n inductively from representations of GL_m with $m < n$.

Theorem 3.3.1. *The following smooth representations of* GL_n *have a* GL_{n-1} *invariant form*
(a) $\mathrm{ind}_{\mathrm{P_{k,n-k}}}^{\mathrm{GL_n}}(\rho\nu^{\frac{\mathrm{n-k-2}}{2}} \otimes \tau\nu^{\frac{\mathrm{-k}}{2}})$ *where* $\rho \in \mathrm{Alg(GL_k)}$ *has* $\mathbb{1}_k$ *as a quotient and* $\tau \in \mathrm{Alg(GL_{n-k})}$ *is* GL_{n-k-1}*-distinguished.*
(b) $\mathrm{ind}_{\mathrm{P_{k,n-k}}}^{\mathrm{GL_n}}(\rho\nu^{\frac{\mathrm{n-k}}{2}} \otimes \tau\nu^{-(\frac{\mathrm{k-2}}{2})})$ *where* $\rho \in \mathrm{Alg(GL_k)}$ *is* GL_{k-1}*-distinguished and* $\tau \in \mathrm{Alg(GL_{n-k})}$ *has* $\mathbb{1}_{n-k}$ *as a quotient.*

Proof. The proof follows from our conditions (3.3.1) and (3.3.2). □

3.4 A prologue to the general case: $n = 2, 3$

We conclude this chapter by describing the theory of GL_{n-1}-distinguished representations of GL_n for $n = 2$ and 3. The case of $n = 2$ is a consequence of a Lemma of a lemma of Waldspurger [20].

This is rather the beginning point of the theory and we will of course need it in the sequel. So we take this occasion to give a proof of this fact using Mackey theory. For $n = 3$, Prasad [14] obtained the classification of GL_2-distinguished representations of GL_3. As we have pointed out in Chapter 1, our method of approach to determining GL_{n-1}-distinguished irreducible admissible representations of GL_n is a generalization of methods in [14] and the case $n = 3$ best illustrates the nuances of the arguments in general. Prasad used the conditions (3.3.1),(3.3.2) and (3.3.3) for $n = 3, k = 2$ and Theorem 3.4.1(b). However, one of the crucial ingredient of the proof was to show that no twist $St_3\lambda$ is GL_2-distinguished for which he used the Bernstein-Zelevinsky filtration of a representation of GL_n restricted to P_n. Our proof will use Mackey Theory and a trick which is a feature of several of our proofs. We use it to prove both non-distinguishedness and also calculate multiplicity. We shall illustrate it below by using it to show

(i) $\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL}_1}[\mathrm{St}_2, \mathbb{1}]) = 1$ and

(ii) $St_3, St_3\nu^{-1}$ are not GL_2-distinguished.

The GL_2 case

Let us make the following definition: For any smooth representation π of GL_n let $d_\pi = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL}_{n-1}}[\pi, \mathbb{1}_{n-1}])$. Let $\xi = \chi_1 \times \chi_2$ where χ_1 and χ_2 are characters of GL_1. Let us apply Mackey Theory to ξ with $n = 2$ and $k = 1$. Note from Section 3.1 that $S_{Z_2} = \{1\} \subset \mathrm{GL}_1$. It is then easy to see that the exact sequence (3.1.1) holds for $n = 2$ as well and is precisely

$$0 \to C_c^\infty(\mathrm{GL}_1) \to \mathrm{ind}_{\mathrm{P}_{1,1}}^{\mathrm{GL}_2}(\chi_1 \otimes \chi_2)_{|\mathrm{GL}_1} \to \chi_1\nu^{1/2} \oplus \chi_2\nu^{-1/2} \to 0 \qquad (3.4.1)$$

where $C_c^\infty(\mathrm{GL}_1)$ is the right regular representation of GL_1. It follows that for any character χ_2 we obtain $\mathrm{Hom}_{\mathrm{GL}_1}[\nu^{-1/2} \times \chi_2, 1] \neq 0$. Similarly for any character χ_1 we obtain $\mathrm{Hom}_{\mathrm{GL}_1}[\chi_1 \times \nu^{1/2}, 1] \neq 0$. By the uniqueness of Haar measure $\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL}_1}[\mathrm{C}_\mathrm{c}^\infty(\mathrm{GL}_1), 1]) = 1$. By Lemma 2.6.3 and the exact sequence (3.4.1) it follows that for $\chi_1 \times \chi_2$ with $\chi_1 \neq \nu^{-1/2}$ and $\chi_2 \neq \nu^{1/2}$

$$\mathrm{Hom}_{\mathrm{GL}_1}[\chi_1 \times \chi_2, 1] \cong \mathrm{Hom}_{\mathrm{GL}_1}[\mathrm{C}_\mathrm{c}^\infty(\mathrm{GL}_1), \mathbb{1}] \qquad (3.4.2)$$

Therefore for all χ_1, χ_2 we have $d_{\chi_1 \times \chi_2} \neq 0$.

If $\chi_1 \times \chi_2$ is such that $\chi_1 \neq \nu^{-1/2}$ and $\chi_2 \neq \nu^{1/2}$ then by (3.4.2) $d_{\chi_1 \times \chi_2} = 1$.(This holds irrespective of whether $\chi_1 \times \chi_2$ is irreducible or not.) Now if $\xi = \nu^{-1/2} \times \chi_2$ is irreducible and then $\chi_2 \neq \nu^{1/2}$. Then by Theorem 2.2.4 ξ is equivalent to $\chi_2 \times \nu^{-1/2}$ and (3.4.2) holds for ξ, whence $d_\xi = 1$. The case when $\chi_2 = \nu^{1/2}$ is similar. This shows that for all irreducible principal series $\chi_1 \times \chi_2$ we have $d_{\chi_1 \times \chi_2} = 1$.

We next consider a reducible principal series $\xi = \chi_1 \times \chi_2$ with $\chi_1 = \nu^{-1/2}$. Then χ_2 is either $\nu^{-3/2}$ or $\nu^{1/2}$. First let $\chi_2 = \nu^{-3/2}$. Then $\mathrm{JH}^0(\xi) = \{\nu^{-1}, \mathrm{St}_2\nu^{-1}\}$. We look at $\zeta = \nu^{-3/2} \times \nu^{-1/2}$. Then $\mathrm{JH}^0(\zeta) = \{\nu^{-1}, \mathrm{St}_2\nu^{-1}\}$ and by the previous paragraph, we have $d_\zeta = 1$ which implies $d_{St_2\nu^{-1}} = 1$. This in turn implies that $d_{\nu^{-1/2} \times \nu^{-3/2}} = 1$. By the Duality Lemma we conclude that $d_{\nu^{3/2} \times \nu^{1/2}} = 1$. Therefore, for all principal series $\xi = \chi_1 \times \chi_2$ such that $\xi \neq \nu^{-1/2} \times \nu^{1/2}$ we have $d_\xi = 1$. This coupled with the fact that for a reducible ξ, $\mathrm{JH}^0(\xi) = \{\chi, \mathrm{St}_2\chi\}$ for some character χ implies that, $d_{St_2\chi} = 1$ all $\chi \neq 1$.

It follows from the exact sequence (3.4.1) that $d_{\nu^{-1/2} \times \nu^{1/2}} \geq 2$. Since $\mathrm{JH}^0(\nu^{-1/2} \times \nu^{1/2}) = \{1\!\!1_2, \mathrm{St}_2\}$ and $d_{1\!\!1_2} = 1$, we may conclude that $d_{\nu^{-1/2} \times \nu^{1/2}} = 2$ if $d_{St_2} = 1$. We claim that $d_{St_2} = 1$.(Observe that already we know $d_{St_2} \neq 0$.)

If possible let $m = d_{St_2} > 1$. Consider the representation $\eta = 1\!\!1_2 \times St_2\nu^{-1}$ of GL_4. By Lemma 2.5.1 this representation is irreducible. Note that (3.3.5) holds for η and yields $d_\eta = m$. By Lemma 2.7.1, $d_{\widetilde{\eta}} = m$ where $\widetilde{\eta} = 1\!\!1_2 \times St_2\nu$. Now again note that (3.3.5) holds for $\widetilde{\eta}$ and yields $m = d_{\widetilde{\eta}} = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL}_1}[\mathrm{St}_2\nu^2, 1]) = 1$, a contradiction. This proves that $d_{St_2} = 1$. Recall that the restriction of an irreducible supercuspidal representation σ of GL_2 to GL_1 is the right regular representation $C_c^\infty(\mathrm{GL}_1)$ and hence $\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL}_1}[\sigma, 1\!\!1]) = 1$. Therefore by the classification theorem for GL_2 we have proved the following theorem:

Theorem 3.4.1. (*a*) *Let $\xi = \chi_1 \times \chi_2$ where χ_1 and χ_2 are any two characters of* GL_1

except $\chi_1 = \nu^{-\frac{1}{2}}, \chi_2 = \nu^{\frac{1}{2}}$ and $\xi_0 = \nu^{-1/2} \times \nu^{1/2}$. Then $d_\xi = 1$ and $d_{\xi_0} = 2$.

(*b*) *Let π be an irreducible admissible representation of GL_2. Then π is GL_1-distinguished if and only if $\pi = \mathbb{1}_2$ or infinite dimensional. Moreover $\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL}_1}[\pi, \mathbb{1}]) = 1$.*

Remark 3.4.1. The statement (b) in the above theorem can be obtained as a consequence of Lemma 8 and 9 in [20].

GL_3 case

Let us begin by recalling the Jordan-Holder factors for some representations of GL_3. We have $\mathrm{JH}^0(\nu^{-1} \times 1 \times \nu) = \{\mathbb{1}_3, \mathrm{St}_3, \mathrm{L}_3\nu^{-1}, \widetilde{\mathrm{L}_3}\nu\}$ where L_3 is the representation defined in Section 2.5. Therefore the irreducible subquotients of $\nu^{-2} \times \nu^{-1} \times 1$ are twists by ν^{-1} of members of $\mathrm{JH}^0(\nu^{-1} \times 1 \times \nu)$. A similar statement holds for $1 \times \nu \times \nu^2$. Next we have,(See [22], Example 11.1)

$$\mathrm{JH}^0(\nu \times \nu \times 1) = \{\mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\mathrm{St}_2\nu^{1/2} \otimes \nu), \mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\nu^{1/2} \otimes \nu)\}$$

The contragredients of these subquotients constitute $\mathrm{JH}^0(\nu^{-1} \times \nu^{-1} \times 1)$. Prasad proved the following theorem in [14]. The proof which we give below is almost Prasad's original proof except for minor variations.

Theorem 3.4.2. (*Prasad*) *An irreducible admissible representation π of GL_3 is GL_2-distinguished if and only if π is one of the following:*

1. $\mathbb{1}_3$
2. $\mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\nu^{1/2} \otimes \chi)$ *where $\chi \neq \nu^2, \nu^{-1}$ and its contragredient.*
3. $\mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\rho \otimes 1)$ *such that ρ is either an irreducible supercuspidal or $St_2\mu$, or* $\mathrm{ind}_{\mathrm{P}_{1,1}}^{\mathrm{GL}_2}(\chi_1 \otimes \chi_2)$ *where $\mu \neq \nu^{\pm 3/2}$, $\chi_2 \neq \chi_1\nu^{\pm 1}$ and both $\chi_1, \chi_2 \neq \nu^{\pm 1}$*
4. *The representation L_3 and its contragredient.*

Before we begin the proof let us recall that for a smooth representation ρ of GL_2 and a character χ of GL_1, the representation $\mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\rho \otimes \chi)$ is GL_2-distinguished implies that at least one of the following conditions holds.(put $n = 3$ and $k = 2$ in (3.3.1)-(3.3.3).)

(a) $\mathrm{Hom}_{\mathrm{GL}_2}[\rho\nu^{1/2}, \mathbb{1}_2] \neq 0$

(b) (i) $\mathrm{Hom}_{\mathrm{GL}_1}[\rho\nu^{-1/2}, 1] \neq 0$ and (ii) $\chi = 1$.

(c) $\mathrm{Hom}_{\mathrm{GL}_1}[\rho^{(1)}, 1] \neq 0$.

Moreover, if (a) or (b) holds then $\rho \times \chi$ is GL_2-distinguished. Let us also recall the exact sequences in which the representation L_3 (see. Section 2.5) sits as a quotient namely

$$0 \to \nu \to \mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\nu^{1/2} \otimes \nu^2) \to \mathrm{L}_3 \to 0 \tag{3.4.3}$$

and

$$0 \to St_3\nu \to \mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\mathrm{St}_2\nu^{3/2} \otimes 1) \to \mathrm{L}_3 \to 0 \tag{3.4.4}$$

Proof. Assume that π is GL_2-distinguished. By Lemma 2.7.3 π is not supercuspidal. Then such a π can be expressed as a quotient of a representation of the form $\mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\rho \otimes \chi)$ where ρ is an irreducible admissible representation of GL_2 and χ is a character of GL_1. Write $\xi = \mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\rho \otimes \chi)$. One of (a),(b),(c) holds for ξ. Since ρ is irreducible, ξ satisfies (a) if and only if ρ is equal to the character $\nu^{-1/2}$. By Theorem 3.4.1, the condition (b)(i) holds for ρ if and only if $\rho = \nu^{1/2}$ or ρ is infinite dimensional. Therefore, we have shown,

(e): $\mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\nu^{-1/2} \otimes \chi)$ is GL_2-distinguished for all characters χ of GL_1

(f): $\mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\rho \otimes 1)$ is GL_2-distinguished for all irreducible infinite dimensional representations of GL_2 and the character $\nu^{1/2}$.

Finally look at (c). If ρ is supercuspidal $\rho^{(1)} = 0$. For a character ρ of GL_2, $\rho^{(1)}$ is equal to the character $\rho\nu^{-1/2}$. Therefore $\rho^{(1)}$ has a trivial quotient if and only if ρ equals to the character $\nu^{1/2}$ of GL_2. For $\rho = St_2\mu$, we have $\rho^{(1)} = \mu\nu^{1/2}$ and this equals one if and only if $\mu = \nu^{-1/2}$. If $\rho = \chi_1 \times \chi_2$ is an irreducible principal series, then $\rho^{(1)}$ has a trivial quotient if and only if at least one of χ_1 or χ_2 equals 1.

Therefore, π is a quotient of one of the following representations.
(g) $\mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\nu^{-1/2} \otimes \chi)$.
(h) $\mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\rho \otimes \chi)$ where $\rho = \nu^{1/2}, St_2\nu^{-1/2}$ or an irreducible $1 \times \mu$.
(i) $\mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\rho \otimes 1)$ where ρ is infinite dimensional or $\nu^{1/2}$.
We analyze each of these below. Note that if ρ is one dimensional and $\mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\rho \otimes \chi)$ is GL_2-distinguished then the only choice for ρ is $\nu^{\pm 1/2}$.
Recall that for a representation π of GL_n, $d_\pi = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL}_{n-1}}[\pi, \mathbb{1}_{n-1}])$. We require the following two Lemmas:

Lemma 3.4.3. *The representation* $\mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\chi \otimes \mu)$ *is* GL_2*-distinguished if and only if* $\chi = \nu^{\pm 1/2}$*. The representation* L_3 *is* GL_2*-distinguished and for* $\chi \neq 1$*, the representation* $L_3\chi$ *is not* GL_2*-distinguished.*

Proof. By Lemma 2.4.1, $\mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\nu^{-1/2} \times \chi)$ is reducible if and only if $\chi = \nu^{-2}$ or ν. If $\chi = \nu^{-2}$, by taking contragredient of (3.4.3), one has the exact sequence

$$0 \to \widetilde{L_3} \to \mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\nu^{-1/2} \otimes \nu^{-2}) \to \nu^{-1} \to 0 \tag{3.4.5}$$

By (e), $\mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\nu^{-1/2} \otimes \nu^{-2})$ is GL_2-distinguished, the character ν^{-1} is not and hence $\widetilde{L_3}$ must be GL_2-distinguished. By Lemma 2.7.1, L_3 is GL_2-distinguished. We also observe that for any χ, (3.3.5) holds for the representation $\mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\nu^{-1/2} \otimes \chi)$, since it has a GL_2-invariant form only on the orbit $\mathcal{C}_1$, whence $d_{\nu^{-1/2} \times \chi} = 1$. This together with Lemma 2.7.1 implies that $d_{L_3} = d_{\widetilde{L_3}} = 1$. Also, for $\chi \neq 1$ it follows from (3.4.3)[by twisting (3.4.3) by χ and observing from (a),(b),(c) that ρ is necessarily $\nu^{\pm 1/2}$)] that if $L_3\chi$ is GL_2-distinguished then $\chi = \nu$. Then we have the exact sequence

$$0 \to \mathbb{1}_3 \to \mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\nu^{-1/2} \otimes \nu) \to \mathrm{L}_3\nu^{-1} \to 0 \tag{3.4.6}$$

Let $V = \mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\nu^{-1/2} \otimes \nu)$.(the representation space). Define $T : V \to \mathbb{C}$ by $T(f) = f(I_n)$. Then $0 \neq T \in \mathrm{Hom}_{\mathrm{GL}_2}[\mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\nu^{-1/2} \otimes \nu), \mathbb{1}_2]$ and T is nontrivial on $\mathbb{1}_3$. Since $d_{\nu^{-1/2}\times\chi} = 1$ it follows that $L_3\nu^{-1}$ is not GL_2-distinguished.

The representation $\mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\nu^{1/2} \otimes \chi)$ is contragredient to say $\eta = \mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\nu^{-1/2} \otimes \chi^{-1})$. If it is irreducible then by Lemma 2.7.1 and previous paragraph it is GL_2-distinguished. If it is reducible then ξ is reducible and it follows from our analysis of η in previous paragraph that its irreducible quotients(they are duals of subrepresentations of η in two cases when η is reducible) are either $\mathbb{1}_3$ or L_3 and both are GL_2 distinguished. □

Lemma 3.4.4. *Let* $\xi = \mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\mathrm{St}_2\mu \times \chi)$. *Then* ξ *is* GL_2*-distinguished if and only if* $\chi = 1$. *Moreover,* $d_\xi \leq 1$ *and* $St_3\mu$ *is not* GL_2*-distinguished for any character* μ.

Proof. It follows from (f) above that if $\chi = 1$ then ξ is GL_2-distinguished. Conversely assume that ξ is GL_2-distinguished. Then only either (c) or (b) holds for ξ. If (b) holds $\chi = 1$ and if (c) holds $\mu = \nu^{-1/2}$. Assume for $\chi \neq 1$ that $St_2\nu^{-1/2} \times \chi$ is irreducible, $\chi \neq 1$. If $St_2\nu^{-1/2} \times \chi$ is GL_2-distinguished then so would be its contragredient $St_2\nu^{1/2} \times \chi^{-1}$ by Lemma 2.7.1. But then this contragredient does not satisfy any of the three conditions (a),(b),(c), a contradiction. This forces $\xi_0 = St_2\nu^{-1/2} \times \chi$ to be reducible. We know from Lemma 2.5.1 that ξ_0 is reducible if and only if $\chi = \nu$ or $\chi = \nu^{-2}$ whence it has the unique irreducible quotient St_3 and $L_3\nu^{-2}$ respectively. But $L_3\nu^{-2}$ is not GL_2-distinguished. If St_3 is GL_2-distinguished then the representation $St_3\nu^{\frac{1}{2}} \times \nu^{-\frac{1}{2}}$ of GL_4 must be GL_3-distinguished by Theorem 3.3.1. Note that this representation is irreducible by Proposition 2.2.3(2) and hence its contragredient $St_3\nu^{-\frac{1}{2}} \times \nu^{\frac{1}{2}}$ must be GL_3-distinguished by Lemma 2.7.1. But it is easy to see that $St_3\nu^{-\frac{1}{2}} \times \nu^{\frac{1}{2}}$ fails to satisfy the conditions (3.3.1),(3.3.2) and (3.3.3), a contradiction. We have shown that $d_\xi \neq 0$ if and only if $\chi = 1$.

Since St_3 is a quotient of $\mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\mathrm{St}_2\nu^{-1/2} \otimes \nu)$, it follows from the first part of this lemma that the only twist of St_3 which may be GL_2-distinguished is $St_3\nu^{-1}$. By applying the same argument used to show that St_3 is not GL_2-distinguished, it follows that $St_3\nu^{-1}$ is not GL_2-distinguished. To prove the $d_\xi \leq 1$ part, let $d_\xi \neq 0$. Then $\xi = St_2\mu \otimes 1$. Observe that (3.3.6) holds for all representations ξ except $\xi_0 = St_2\nu^{-1/2} \times 1$ and hence $d_\xi = 1$ for such ξ by Theorem 3.4.1. Now $St_2\nu^{-1/2} \times 1$ is irreducible by Proposition 2.2.3 (2) and therefore by Lemma 2.7.1, $d_{St_2\nu^{-1/2}\times 1} = d_{St_2\nu^{1/2}\times 1} = 1$. □

Proof of Theorem 3.4.2 (cont):

Observe that if π is one of the representations in the statement of Theorem 3.4.2 then π is indeed GL_2-distinguished by our preceding discussions. Conversely assume that π is GL_2-distinguished then π is a quotient of a representation as in (g),(h) or (i) above.

Quotients from (g):If $\mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\nu^{-1/2} \times \chi)$ is irreducible then π is one such. It follows from Proof of Lemma 3.4.3 that there are no GL_2-distinguished irreducible quotients for a reducible $\mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\nu^{-1/2} \times \chi)$.

Quotients from (h): The case when $\rho = \nu^{1/2}$ or $\rho = St_2\nu^{-1/2}$ is complete by Lemmas 3.4.3 and 3.4.4. It remains to consider $\rho = 1 \times \mu$ where $\mu \neq \nu^{\pm 1}$. We have by Lemma 2.4.1 that $1 \times \mu \times \chi$ is reducible if and only if either $\chi = \nu^{\pm 1}$ or $\chi = \mu\nu^{\pm 1}$. We have to look at irreducible quotients of $1 \times \mu \times \chi$ when it is not reducible. It follows from the examples that we have considered at the beginning of the GL_3 case that such a representation has irreducible subquotients of the following form: a character of GL_3, a twist of L_3 or its contragredient, a twist of St_3, a quotient of $\mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\mathrm{St}_2\mu' \otimes \chi'$ or $\mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\chi' \otimes \mu')$ for some characters χ', μ'. But we have already considered all these representations and their quotients.

Quotients from (i): Finally let $\rho \times 1 := \mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\rho \otimes 1)$ where ρ is irreducible and infinite dimensional. If ρ is supercuspidal then by Lemma 2.4.1, $\rho \times 1$ is irreducible. If $\rho = St_2\mu$, by Lemma 2.5.1 $\rho \times 1$ is irreducible except when $\mu = \nu^{\pm 3/2}$. If $\mu = \nu^{3/2}$, it follows from (3.4.4) that it has the unique irreducible quotient L_3. If $\rho = St_3\nu^{-3/2}$ then $\rho \times 1$ has the unique irreducible quotient $St_3\nu^{-1}$, which is again not GL_2-distinguished.

If $\rho = \chi_1 \times \chi_2$ is an irreducible principal series, we have already analyzed the quotients of such representations. □

We have the following theorem which summarizes Theorem 3.4.2 in a nice way. This is Theorem 1 in [14] except for the twist $\nu^{\pm 1/2}$ which we have pointed out earlier.

Theorem 3.4.5. *An irreducible admissible representation π of GL_3 is GL_2-distinguished if and only if the Langlands parameter $\mathfrak{L}(\pi)$ has a one dimensional subrepresentation $\mathfrak{L}(1)$ corresponding to the trivial character of GL_1 such that the two dimensional quotient $\mathfrak{L}(\pi)/\mathfrak{L}(1)$ corresponds to either an infinite dimensional representation of GL_2 or the characters $\nu^{\pm 1/2}$.*

Proof. If π is GL_2-distinguished then π is one of the representations in Theorem 3.4.2. Then by Section 2.6, π has Langlands parameter of the prescribed form. □

We end this chapter by mentioning that the proof of Theorem 1.1.2 in the case when $n \geq 4$ is along similar lines of the proof of Theorem 3.4.2.

Chapter 4

Proof of the main theorem

4.1 Few Basic Results

In this chapter we prove some results leading to the theorem on the determination of irreducible admissible representations of GL_n which are GL_{n-1}-distinguished. As an easy consequence of Mackey theory, we first classify representations parabolically induced from two characters which are GL_{n-1}-distinguished. After proving some necessary Lemmas, we prove the Main Theorem. We start with a lemma which is key to analyzing representations which may have an invariant form on the open orbit.

Lemma 4.1.1. *If $\rho \in \mathrm{Irr}(\mathrm{GL_m})$ satisfies $\mathrm{Hom}_{\mathrm{GL_{m-1}}}[\rho^{(1)}, \mathbb{1}_{\mathrm{m-1}}] \neq 0$ then ρ is one of the following*
(a) an irreducible representation of the form $\mathrm{ind}_{\mathrm{P_{m-1,1}}}^{\mathrm{GL_m}}(\mathbb{1}_{\mathrm{m-1}} \otimes \chi)$ for some character χ of GL_1
(b) the character $\nu^{1/2}$
(c) the representation $\widetilde{L_m}\nu^{1/2}$.

Proof. Let ω_ρ denote the central character of ρ. Note that

$$\mathrm{Hom}_{\mathrm{GL_{m-1}}}[\rho^{(1)}, \mathbb{1}_{\mathrm{m-1}}] = \mathrm{Hom}_{\mathrm{M_{m-1,1}}}[\mathrm{r}_{\mathrm{N_{m-1,1}}}(\rho), \mathbb{1}_{\mathrm{m-1}} \otimes \omega_\rho]$$

where $M_{m-1,1}$ is the Levi subgroup of $P_{m-1,1}$. By (2.1.4), the latter space is equal to

$$\mathrm{Hom}_{\mathrm{GL_m}}[\rho, \mathrm{ind}_{\mathrm{P_{m-1,1}}}^{\mathrm{GL_m}}(\mathbb{1}_{\mathrm{m-1}} \otimes \omega_\rho)].$$

If $\omega_\rho \neq \nu^{\frac{m}{2}}$ or $\nu^{\frac{-m}{2}}$, $\mathrm{ind}_{\mathrm{P_{m-1,1}}}^{\mathrm{GL_m}}(\mathbb{1}_{\mathrm{m-1}} \otimes \omega_\rho)$ is irreducible (see. Example 2.8.3) and ρ is of type (a) in the statement of the Lemma. If $\omega_\rho = \nu^{m/2}$, ξ has the unique irreducible subrepresentation $\nu^{1/2}$ by (2.8.5). On the other hand if $\omega_\rho = \nu^{\frac{-m}{2}}$ by (2.8.6) we get that $\rho = \widetilde{L_m}\nu^{1/2}$. □

Let $\rho \in \mathrm{Irr}(\mathrm{GL_k})$ and $\tau \in \mathrm{Irr}(\mathrm{GL_{n-k}})$ and assume $\rho \times \tau$ is GL_{n-1}-distinguished. We will apply Mackey theory to $\rho \times \tau$. By (3.3.1) it has a GL_{n-1}-invariant form on the closed orbit $\mathcal{C}_1$ if and only if $\rho = \nu^{\frac{n-k-2}{2}}$ and $\tau\nu^{\frac{k}{2}}$ is GL_{n-k-1}-distinguished. Similarly by (3.3.2) it has a GL_{n-1}-invariant form on $\mathcal{C}_2$ if and only if $\rho\nu^{-(\frac{n-k}{2})}$ is GL_{k-1}-distinguished and $\tau = \nu^{-(\frac{k-2}{2})}$.
For $2 \leq k \leq n-2$ define two sets $\mathcal{OP}_1(k)$ and $\mathcal{OP}_2(n-k)$ by

$$\mathcal{OP}_1(k) = \{\nu^{\frac{n-k}{2}}, \widetilde{L_k}\nu^{\frac{n-k}{2}}, \mathrm{ind}_{\mathrm{P_{k-1,1}}}^{\mathrm{GL_k}}(\nu^{\frac{\mathrm{n-k-1}}{2}} \otimes \chi) : \chi \neq \nu^{\frac{\mathrm{n-1}}{2}}, \nu^{\frac{\mathrm{n-2k-1}}{2}}\}$$

and

$$\mathcal{OP}_2(n-k) = \{\nu^{\frac{-k}{2}}, L_{n-k}\nu^{-k/2}, \mathrm{ind}_{\mathrm{P_{n-k-1,1}}}^{\mathrm{GL_{n-k}}}(\nu^{-(\frac{\mathrm{k-1}}{2})} \otimes \mu) : \mu \neq \nu^{-(\frac{\mathrm{n-1}}{2})}, \nu^{\frac{\mathrm{n-2k+1}}{2}}.\}$$

Also fix $\mathcal{OP}_1(1) = \mathcal{OP}_2(1) = \mathrm{Irr}(\mathrm{GL_1})$. Then by Lemma 4.1.1 and (3.3.3) we see that $\rho \times \tau$ has a GL_{n-1}-invariant form on the open orbit if and only if $\rho \in \mathcal{OP}_1(k)$ and $\tau \in \mathcal{OP}_2(n-k)$. However, we once again note that when $k = n-1$, τ can be any arbitrary character and when $k = 1$, ρ can be any arbitrary character.

We record the conclusions in the following Lemma for further reference.

Lemma 4.1.2. *Let* $\rho \in \mathrm{Irr}(\mathrm{GL_k})$ *and* $\tau \in \mathrm{Irr}(\mathrm{GL_{n-k}})$. *Then if* $\rho \times \tau$ *is* GL_{n-1}-*distinguished then at least one of the following must hold.*
(I) On $\mathcal{C}_1$, $\rho = \nu^{\frac{n-k-2}{2}}$ *and* $\tau\nu^{k/2}$ *is* GL_{n-k-1}-*distinguished*
(II) On $\mathcal{C}_2$, $\rho\nu^{-(\frac{n-k}{2})}$ *is* GL_{k-1}-*distinguished and* $\tau = \nu^{-(\frac{k-2}{2})}$

(III) On $\mathcal{O}$, $\rho \in \mathcal{OP}_1(k)$ and $\tau \in \mathcal{OP}_2(n-k)$

Moreover, if (I) or (II) holds then $\rho \times \tau$ is GL_{n-1}-distinguished.

We may apply these conditions even if only one(not necessarily both) of the representations ρ or τ is irreducible as that irreducible representation have to be necessarily of the types in (I),(II) or (III).

One other conclusion from the above Lemma is that if both ρ and τ are irreducible and neither of them is a character $\rho \times \tau$ does not have a GL_{n-1}-invariant form on the closed orbits.

Proposition 4.1.3. *Let $n \geq 3$. Then L_n is GL_{n-1}-distinguished and for any $\chi \neq 1$, $L_n\chi$ is not GL_{n-1}-distinguished.*

Proof. We first note that for a representation of the form $\mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\mu \otimes \chi)$ to be GL_{n-1}-distinguished it is necessary by Lemma 4.1.2 that $\mu = \nu^{\pm 1/2}$. For a character χ of GL_1, let $\xi_\chi = \mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\nu^{-1/2} \otimes \chi)$ and $d_{\xi_\chi} = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL_{n-1}}}[\xi_\chi, \mathbb{1}_{\mathrm{n-1}}])$. Also fix $\chi_0 = \nu^{-(\frac{n+1}{2})}$ and $\chi_1 = \nu^{\frac{n-1}{2}}$. Then ξ_χ is reducible by Example 2.8.3(see (2.8.7) and take contragredient of (2.5.1)) if and only if $\chi = \chi_0$ or χ_1. By Lemma 4.1.2, ξ_χ has a GL_{n-1}-invariant form only on the closed orbit $\mathcal{C}_1$ and therefore $d_{\xi_\chi} = 1$ for all χ. Take contragredient of the exact sequence (2.5.1) to get

$$0 \to \widetilde{L_n} \to \xi_{\chi_0} \to \nu^{-1} \to 0.$$

Since ν^{-1} is not GL_{n-1}-distinguished we must have $\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL_{n-1}}}[\widetilde{\mathrm{L_n}}, \mathbb{1}_{\mathrm{n-1}}]) = 1$. By Lemma 2.7.1 the same is true for L_n.

Suppose for some $\chi \neq 1$ that $L_n\chi$ is GL_{n-1}-distinguished. From the beginning of the previous paragraph and (2.5.1) it is necessary that $\chi = \nu^{-1}$. Look at the exact sequence (2.8.7). We may define $T : \xi_{\chi_1} \to \mathbb{C}$ by $T(f) = f(I_n)$ for f in the representation space of ξ_{χ_1}. Then $T \in \mathrm{Hom}_{\mathrm{GL_{n-1}}}[\xi_{\chi_1}, \mathbb{1}_{\mathrm{n-1}}]$ because

$$T\left(\xi_{\chi_1}\begin{bmatrix} g_{n-1} & 0 \\ 0 & 1 \end{bmatrix} f\right) = \left(\xi_{\chi_1}\begin{bmatrix} g_{n-1} & 0 \\ 0 & 1 \end{bmatrix} f\right)(I_n) = f\left(\begin{bmatrix} g_{n-1} & 0 \\ 0 & 1 \end{bmatrix}\right)$$

But the last term in the previous equality is just $f(I_n)$ because GL_{n-1} is embedded in $P_{n-1,1}$ and $\xi_{\chi_1} = \delta_{P_{n-1,1}}^{-\frac{1}{2}}$. Note that T is nonzero on the submodule $\mathbb{1}_n$. Therefore, $d_{\xi_{\chi_1}} = 1$ forces $L_n\nu^{-1}$ is not GL_{n-1}-distinguished. □

Remark 4.1.1. We have proved something stronger in the course of proving Proposition 4.1.3, namely for any character χ of GL_1, $\xi = \mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\nu^{-(\frac{1}{2})} \otimes \chi)$ has $d_\xi = 1$ and $d_{L_n} = 1$.

The next two lemma's are the generalizations of Lemma 3.4.3 and Lemma 3.4.4 to GL_n.

Lemma 4.1.4. *If $n \geq 3$ and χ is a character of GL_1 then $\mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\widetilde{\mathrm{L_{n-1}}}\nu^{\frac{1}{2}} \otimes \chi)$ is GL_{n-1}-distinguished if and only if $\chi = \nu^{-(\frac{n-3}{2})}$, whence it has the unique irreducible quotient $\mathbb{1}_{n-2} \times St_2\nu^{-(\frac{n-2}{2})}$.*

Proof. Let $\xi = \mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\widetilde{\mathrm{L_{n-1}}}\nu^{\frac{1}{2}} \otimes \chi)$. It is obvious by Lemma 4.1.2(II) that if $\chi = \nu^{-(\frac{n-3}{2})}$, then ξ is GL_{n-1}-distinguished. For the converse we use induction. The statement is true for $n = 3$ by Lemma 3.4.4. Let $n = 4$. Note that $\xi^\vee = \chi^{-1} \times L_3\nu^{-1/2}$ is a quotient of $\eta = [\chi^{-1} \times St_2\nu] \times \nu^{-1/2}$. Apply Mackey theory to η with $k = 3$. It is easy to see that (3.3.1)(i) hold for η since it cannot have a one dimensional quotient and (3.3.3)(i) does not for η since its derivative is glued from $St_2\nu$ and $\chi^{-1} \times \nu^{3/2}$(neither has $\mathbb{1}_2$ as a quotient); On $\mathcal{C}_2$, (3.3.2)(i) holds for η only if $\chi = \nu^{-1/2}$. Our claim follows from the Duality Lemma.

Let $n > 4$. From Section 2.5, recall that L_n is a quotient of $L_{k+2}\nu^{\frac{n-k-2}{2}} \times \nu^{-\frac{k}{2}}$. Using this and applying the Duality Lemma, ξ is a quotient of

$$\eta = \mathrm{ind}_{\mathrm{P_{2,n-2}}}^{\mathrm{GL_n}}(\nu^{\frac{n-4}{2}} \otimes [\widetilde{\mathrm{L_{n-3}}}\nu^{-\frac{1}{2}} \times \chi])$$

By Lemma 4.1.2 (II) and (III), η does not have any invariant form on the open orbit $\mathcal{O}$ as well as on the closed orbit $\mathcal{C}_2$. On the closed orbit $\mathcal{C}_1$, (3.3.2)(i) holds if and only if $[\widetilde{L_{n-3}}\nu^{-\frac{1}{2}} \times \chi].\nu$ is GL_{n-3}-distinguished. By induction this is true if and only if $\chi\nu = \nu^{-(\frac{n-5}{2})}$ i.e., if and only if $\chi = \nu^{-(\frac{n-3}{2})}$. Finally, if $\chi = \nu^{-(\frac{n-3}{2})}$, ξ is a quotient of the representation ξ_{n-1} in Lemma 2.6.2, whence the unique irreducible quotient of ξ is $1\!\!1_{n-2} \times St_2\nu^{-(\frac{n-2}{2})}$. $\square$

Lemma 4.1.5. *The representation $Z_n \in \mathrm{Irr}(\mathrm{GL_n})$ in (2.5.2) is not GL_{n-1}-distinguished.*

Proof. Note that $Z_3 = St_3.\nu$ is not GL_2-distinguished by Theorem 3.4.2. Let $n \geq 4$. By definition, Z_n is the unique irreducible submodule of

$$\nu^{\frac{n+1}{2}} \times \nu^{\frac{n-1}{2}} \times \nu^{-(\frac{n-3}{2})} \times \nu^{-(\frac{n-5}{2})} \times ... \times \nu^{\frac{n-3}{2}}.$$

Then, $\widetilde{Z_n}$ is the unique irreducible quotient of $\nu^{-(\frac{n+1}{2})} \times \nu^{-(\frac{n-1}{2})} \times \nu^{\frac{n-3}{2}} \times \nu^{\frac{n-5}{2}} \times ... \times \nu^{-(\frac{n-3}{2})}$). By Proposition 2.4.2 (1), this principal series is equivalent to $\rho \times \tau$ where $\rho = \nu^{\frac{n-3}{2}} \times \nu^{\frac{n-5}{2}} \times ... \times \nu^{-(\frac{n-5}{2})} \in \mathrm{Alg}(\mathrm{GL_{n-3}})$, $\tau = \nu^{-(\frac{n+1}{2})} \times \nu^{-(\frac{n-1}{2})} \times \nu^{-(\frac{n-3}{2})} \in \mathrm{Alg}(\mathrm{GL}_3)$ and $\rho \times \tau$ also has $\widetilde{Z_n}$ as the unique irreducible quotient. It is now easy to see that $\rho \times \tau$ has the quotient $\eta = \mathrm{ind}_{\mathrm{P_{n-3,3}}}^{\mathrm{GL_n}}(\nu^{1/2} \otimes \mathrm{St}_3\nu^{-(\frac{n-1}{2})})$. The conditions (I),(II) and (III) in Lemma 4.1.2 does not hold for $St_3\nu^{-(\frac{n-1}{2})}$ and hence η is not GL_{n-1}-distinguished. This completes the proof. $\square$

4.2 Product of two characters

In the next two Propositions we complete the picture of GL_{n-1}-distinguishedness of $\mathrm{ind}_{\mathrm{P_{k,n-k}}}^{\mathrm{GL_n}}(\rho \otimes \tau)$ where both $\rho \in \mathrm{Irr}(\mathrm{GL_k})$ and $\tau \in \mathrm{Irr}(\mathrm{GL_{n-k}})$ are characters.

Proposition 4.2.1. *Let $n > 2$ and $\xi = \mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\chi \otimes \mu)$ or $\mathrm{ind}_{\mathrm{P_{1,n-1}}}^{\mathrm{GL_n}}(\mu \otimes \chi)$ where χ and μ are characters of GL_{n-1} and GL_1 respectively. Let $d_\xi := \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL_{n-1}}}[\xi, 1\!\!1_{\mathrm{n-1}}])$. Then $d_\xi \neq 0$ if and only if $\chi = \nu^{\pm 1/2}$ in which case $d_\xi = 1$.*

Proof. Let $\xi = \mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\chi \otimes \mu)$. By Remark 4.1.1 we need only consider the case when $\xi = \nu^{1/2} \times \chi$ is reducible.(Otherwise we may consider $\widetilde{\xi}$ whence we will be reduced to the $\chi = \nu^{-1/2}$ and apply Lemma 2.7.1.) Now ξ is reducible if and only if $\mu = \nu^{\frac{n+1}{2}}$ or $\nu^{-(\frac{n-1}{2})}$(by Example 2.8.3 (2.8.4) and contragredient of (2.8.7)). In the first case, it follows from (2.8.4) and Remark 4.1.1) that ξ is GL_{n-1}-distinguished with $d_\xi = 1$.

In the second case ξ has the quotient $\mathbb{1}_n$ and subrepresentation $\widetilde{L_n\nu}$ whence again the same conclusion is obtained. The result holds for $\mathrm{ind}_{\mathrm{P_{1,n-1}}}^{\mathrm{GL_n}}(\mu\otimes\chi)$ by the Duality Lemma. □

The next Proposition is the analogue of previous one which is the case $k = 1, n-1$.

Proposition 4.2.2. *For $2 \leq k \leq n-2$ let $\xi = \mathrm{ind}_{\mathrm{P_{k,n-k}}}^{\mathrm{GL_n}}(\chi\otimes\mu)$ where χ and μ are characters of GL_k and GL_{n-k} respectively. Also let $d_\xi := \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL_{n-1}}}[\xi, \mathbb{1}_{\mathrm{n-1}}])$. Then $d_\xi \neq 0$ if and only if one of (a),(b),(c) below holds, in which case $d_\xi = 1$.*
(a) $\chi = \nu^{\frac{n-k-2}{2}}$ and $\mu = \nu^{\frac{-k}{2}}$
(b) $\chi = \nu^{\frac{n-k}{2}}$ and $\mu = \nu^{-(\frac{k-2}{2})}$
(c) $\chi = \nu^{\frac{n-k}{2}}$ and $\mu = \nu^{\frac{-k}{2}}$.

Proof. By Lemma 4.1.2 I,II and III for ξ to be GL_{n-1}-distinguished one of (a),(b) or (c) must hold. In the case of (a) and (b), by Theorem 3.3.1, ξ is indeed GL_{n-1} distinguished and has $d_\xi = 1$ by (3.3.5) and (3.3.6) respectively. In the case (c), ξ has length 2 and has the trivial representation $\mathbb{1}_n$ as the quotient by Example 2.8.8(c). It follows that it is GL_{n-1} distinguished. Also $\widetilde{\xi}$ is not GL_{n-1}-distinguished by Lemma 4.1.2 which proves that there is no nontrivial form on the subrepresentation of ξ. This shows that $d_\xi = 1$ in this case. □

Though Propositions 4.2.1 and 4.2.2 are obvious applications of Mackey theory they have interesting consequences which we record in the following Remarks.

Remark 4.2.1. All the representations ξ in Proposition 4.2.2 which have $d_\xi = 1$ are reducible and are GL_n modules of length 2 by Lemma 2.4.1. The GL_{n-1}-invariant form on such ξ is trivial on the submodule and nontrivial on the quotient. It follows that $d_{\widetilde{\xi}} = 0$.

Remark 4.2.2. The assumption "irreducible" cannot be dropped from Lemma 2.7.1 if $n > 2$. If π is an admissible reducible representation of $\mathrm{GL}_n, n > 2$ having a GL_{n-1} invariant form then $\widetilde{\pi}$ need not have a GL_{n-1} invariant form. By Remark 4.2.1, all the ξ in Proposition 4.2.2 for which $d_\xi = 1$ are indeed admissible and GL_{n-1}-distinguished but $\widetilde{\xi}$ is not GL_{n-1} distinguished. In particular, $\mathrm{ind}_{\mathrm{P_{n-2,2}}}^{\mathrm{GL_n}}(\mathbb{1}_{\mathrm{n-2}} \otimes \nu^{\frac{\mathrm{n-2}}{2}})$ is not GL_{n-1}-distinguished.

4.3 Idea of the Proof of Theorem 1.1.2

One important step to reach our goal is the following Lemma.

Lemma 4.3.1. *If* $\pi \in \mathrm{Irr}(\mathrm{GL_n})$ *is* GL_{n-1}*-distinguished for* $n \geq 3$*, then* π *can be expressed as a quotient of* $\mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\rho \otimes \chi)$ *where* $\rho \in \mathrm{Irr}(\mathrm{GL_{n-1}})$ *and* χ *is a character of* GL_1.

Proof. Let π be a quotient of $\xi = \sigma_1 \times ... \times \sigma_r$ where $\sigma_i \in \mathrm{Irr}(\mathrm{GL_{n_i}})$ are supercuspidal. We claim that $n_i \leq 2$ for all i and there may exist at most one $n_i = 2$. If the claim is true it will prove that π is a quotient of either $\chi_1 \times ... \times \chi_n$ or $\sigma \times \chi_1 \times ... \times \chi_{n-2}$ where $\sigma \in \mathrm{Irr}(\mathrm{GL}_2)$ is supercuspidal and χ_i are characters of GL_1. From the Claim, applied to $\widetilde{\pi}$, it follows that the Jacquet module of $\widetilde{\pi}$ with respect to either the group of upper triangular matrices, or the 2,1,...1 parabolic is nonzero. It follows, in particular, that the Jacquet module of $\widetilde{\pi}$ with respect to the $(n-1,1)$ parabolic is nonzero. Taking an irreducible quotient of the Jacquet module of $\widetilde{\pi}$ and using Frobenius reciprocity we find that $\widetilde{\pi}$ is a submodule of the desired principal series representation. By taking contragredients, the assertion of the Lemma follows.

Proof of Claim: By Proposition 2.4.2, we may assume that $n_1 \geq n_2 \geq ... \geq n_r$. We show that $n_1 \leq 2$ and $n_i = 1$ for all $i > 1$ which will complete the proof. Write $\xi = \sigma_1 \times \tau$ where $\tau = \sigma_2 \times ... \times \sigma_r$. If $n_1 > 1$, σ_1 does not satisfy (I) and (III) in Lemma 4.1.2. By Lemma 2.7.3, σ_1 fails to satisfy (II) if $n_1 \geq 3$. This shows that $n_1 \leq 2$ and in that case ξ may have a GL_{n-1}-invariant form only on the closed orbit $\mathcal{C}_2$. We apply (3.3.2). The condition (3.3.2)(ii) holds for τ only if $n_i = 1$ for all $i > 1$. □

Let $n \geq 3$. Suppose $\pi \in \mathrm{Irr}(\mathrm{GL_n})$ is GL_{n-1}-distinguished. We have to show that the Langlands parameter $\mathfrak{L}(\pi)$ has the form described in Theorem 1.1.2. We proceed by induction on n. The theorem is true for $n = 3$ by Theorem 3.4.2. Therefore assume that the theorem is true for $n-1$. By Lemma 4.3.1, if $\pi \in \mathrm{Irr}(\mathrm{GL_n})$ is GL_{n-1}-distinguished for $n \geq 3$ then π can be expressed as a quotient of $\mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\rho \otimes \chi)$ where $\rho \in \mathrm{Irr}(\mathrm{GL_{n-1}})$ and χ is a character of $\mathrm{GL_1}$. On the open orbit, by Lemma 4.1.2 (III), $\rho \in \mathcal{OP}_1(n-1)$ and χ is a character of $\mathrm{GL_1}$. On the closed orbit $\mathcal{C}_1$, by Lemma 4.1.2 (I), $\rho = \nu^{-\frac{1}{2}}$ and χ is any character of $\mathrm{GL_1}$. On the closed orbit $\mathcal{C}_2$, by Lemma 4.1.2 (II), $\rho\nu^{-\frac{1}{2}}$ is GL_{n-2}-distinguished and $\chi = \nu^{-(\frac{n-3}{2})}$. Therefore our π is a quotient of either

(a) $\mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\nu^{-\frac{1}{2}} \otimes \chi)$ or

(b) $\mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\rho \otimes \chi)$ where $\rho \in \mathcal{OP}_1(n-1)$ or

(c) $\mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\rho \otimes \nu^{-(\frac{n-3}{2})})$ where $\rho\nu^{-\frac{1}{2}}$ is GL_{n-2}-distinguished.

The quotients arising from (a) and (b) can be found directly. To get the quotients arising from (c) we need to know explicitly what the representation $\rho\nu^{-\frac{1}{2}}$ equals to. This is achieved by the induction hypotheses. We may thus recover all $\pi \in \mathrm{Irr}(\mathrm{GL_n})$ which are GL_{n-1}-distinguished.

Conversely suppose $\mathfrak{L}(\pi)$ is as in Theorem 1.1.2. Then we explicitly know from Section 2.6, what π is. The fact that such a π is GL_{n-1}-distinguished is a consequence of Theorem 3.3.1 and Proposition 4.1.3. Let $\rho \in \mathrm{Irr}(\mathrm{GL_{n-1}})$ and $\chi \in \mathrm{Irr}(\mathrm{GL_1})$. In the following proof if $\rho \times \chi$ has a unique irreducible quotient, we will denote it by $\mathrm{U}(\rho, \chi)$. We note by Lemma 2.4.1 that $\mathrm{U}(\mathrm{ind}_{\mathrm{P_{n-2,1}}}^{\mathrm{GL_{n-1}}}(\nu \otimes \nu^{-(\frac{n-3}{2})})) = \nu^{\frac{1}{2}} \in \mathrm{Irr}(\mathrm{GL_{n-1}})$.

4.4 Proof of Theorem 1.1.2

We recall Theorem 1.1.2 from Chapter 1 below.

Theorem 1.1.2 *An irreducible admissible representation* π *of* GL_n *for* $n \geq 3$ *is* GL_{n-1}*-distinguished if and only if the Langlands parameter* $\mathfrak{L}(\pi)$ *associated to* π *by* *the Local Langlands correspondence has a subrepresentation $\mathfrak{L}(\mathbb{1}_{n-2})$ of dimension $n-2$ corresponding to the trivial representation $\mathbb{1}_{n-2}$ of GL_{n-2} such that the two dimensional quotient $\mathfrak{L}(\pi)/\mathfrak{L}(\mathbb{1}_{n-2})$ corresponds (under the Local Langlands correspondence) either to an infinite dimensional representation of GL_2 or the one dimensional representations $\nu^{\pm\frac{n-2}{2}}$ of GL_2.*

Proof. Assume $\pi \in \mathrm{Irr}(\mathrm{GL_n})$ is GL_{n-1}-distinguished. We have to consider the representations of the type (a),(b),(c) in 4.3. The theorem is true for $n = 3$ by Theorem 3.4.2 and therefore we assume that $n \geq 4$.

(a): Quotients on the Closed orbit $\mathcal{C}_1$; ξ has a GL_{n-1}-invariant form on $\mathcal{C}_1$ if and only if $\rho = \nu^{-\frac{1}{2}}$. We have shown in Proposition 4.1.3 that ξ is reducible if and only if $\chi = \nu^{\frac{n-1}{2}}$ or $\nu^{-(\frac{n+1}{2})}$ whence $\mathrm{U}(\rho,\chi)$ exists and is either ν^{-1} and $L_n\nu^{-1}$. Neither of these are GL_{n-1}-distinguished. Otherwise $\rho \times \chi$ is $\mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\nu^{-\frac{1}{2}} \otimes \chi)$ where $\chi \neq \nu^{\frac{n-1}{2}}, \nu^{-(\frac{n+1}{2})}$ and irreducible by Lemma 2.4.1.

(b): Quotients on the Open Orbit $\mathcal{O}$; We have to look at the quotients of representations of the type (b) in 4.3.

(1): If $\nu^{\frac{1}{2}} \times \chi$ is irreducible we know that it is GL_{n-1}-distinguished by Proposition 4.2.1. If $\nu^{\frac{1}{2}} \times \chi$ is reducible, its irreducible quotients have already been shown (in Proof of Proposition 4.2.1) to be L_n and $\mathbb{1}_n$.

(2): By Lemma 4.1.4, $\widetilde{L_{n-1}}\nu^{\frac{1}{2}} \otimes \chi$ is GL_{n-1}-distinguished if and only if $\chi = \nu^{-(\frac{n-3}{2})}$ and has the unique irreducible quotient $\mathbb{1}_{n-2} \times St_2\nu^{-(\frac{n-2}{2})}$.

(3): It now remains to consider the irreducible $\rho = \mathrm{ind}_{\mathrm{P_{n-2,1}}}^{\mathrm{GL_{n-1}}}(\mathbb{1}_{\mathrm{n-2}} \otimes \mu)$. Put $\xi = \rho \times \chi$. We have to pick the irreducible quotients of $\xi = \mathbb{1}_{n-2} \times \mu \times \chi$. Remember that $\mu \neq \nu^{\pm\frac{n-1}{2}}$. By Example 2.8.4 (a) and (b) the irreducible quotients of ξ are quotients of

(i) $\mathbb{1}_{n-2} \times St_2\lambda$ where $\lambda \neq \nu^{\frac{n}{2}}, \nu^{-(\frac{n-2}{2})}$

(ii) $\mathbb{1}_{n-2} \times \beta$ where β is a character of GL_2 not equal $\nu^{-\frac{n}{2}}, \nu^{\frac{n-2}{2}}$

(iii) $\mathrm{ind}_{\mathrm{P_{1,n-1}}}^{\mathrm{GL_n}}(\mu \otimes \mathrm{L_{n-1}}\nu^{-\frac{1}{2}})$
(iv) $\mathrm{ind}_{\mathrm{P_{1,n-1}}}^{\mathrm{GL_n}}(\mu \otimes \nu^{\frac{-1}{2}})$
By Lemma 2.5.1, $1\!\!1_{n-2} \times St_2\lambda$ is either irreducible or has the unique irreducible quotient $\widetilde{L_n}$. By Proposition 4.2.2, $1\!\!1_{n-2} \times \lambda$ is GL_{n-1}-distinguished if and only if $\lambda = \nu^{-(\frac{n-2}{2})}$ whence by Example 2.8.8(a) it has the unique irreducible quotient $\mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\nu^{-\frac{1}{2}} \otimes \nu^{-(\frac{n-3}{2})})$. By Lemma 4.1.4 and the Duality Lemma $\mathrm{ind}_{\mathrm{P_{1,n-1}}}^{\mathrm{GL_n}}(\mu \otimes \mathrm{L_{n-1}}\nu^{-\frac{1}{2}})$ is GL_{n-1}-distinguished if and only if $\mu = \nu^{\frac{n-3}{2}}$, whence it has the unique irreducible quotient $St_2\nu^{\frac{n-2}{2}} \times 1\!\!1_{n-2}$. Finally, by Example 2.8.3, $\mathrm{ind}_{\mathrm{P_{1,n-1}}}^{\mathrm{GL_n}}(\mu \otimes \nu^{-\frac{1}{2}})$ is irreducible if $\mu \neq \nu^{-(\frac{n+1}{2})}$ and and hence GL_{n-1}-distinguished by Proposition 4.2.1. It has the quotient $\widetilde{L_n}$ if $\mu = \nu^{-(\frac{n+1}{2})}$ by (2.5.1) and Duality Lemma.

(c): Quotients on the Closed orbit $\mathcal{C}_2$; By 4.3 (c), our $\chi = \nu^{-(\frac{n-3}{2})}$ and $\rho\nu^{-\frac{1}{2}} \in \mathrm{Irr}(\mathrm{GL_{n-1}})$ is a GL_{n-2}-distinguished representation. Since Theorem 1.1.2 is true for $n = 3$ we may assume by induction that the Langlands parameter $\mathfrak{L}(\rho\nu^{-\frac{1}{2}})$ is of the form $\mathfrak{L}(1\!\!1_{n-3}) \oplus \mathfrak{L}(\tau_1)$ where $\tau_1 \in \mathrm{Irr}(\mathrm{GL}_2)$ is either infinite dimensional or the characters $\nu^{\pm\frac{n-3}{2}}$. Then, by the Langlands Correspondence, the Langlands parameter of ρ has the form $\mathfrak{L}(\nu^{\frac{1}{2}}) \oplus \mathfrak{L}(\tau)$ where $\tau \in \mathrm{Irr}(\mathrm{GL}_2)$ is either infinite dimensional, $\nu^{\frac{n-2}{2}}$ or $\nu^{-(\frac{n-4}{2})}$ and $\nu^{\frac{1}{2}}$ is the character of GL_{n-3}. By Section 2.6, such a ρ is one of the following:
(i) the character $\nu^{\frac{1}{2}}$
(ii) an irreducible $\mathrm{ind}_{\mathrm{P_{1,n-2}}}^{\mathrm{GL_{n-1}}}(\gamma \otimes \nu)$
(iii) an irreducible $\mathrm{ind}_{\mathrm{P_{1,n-2}}}^{\mathrm{GL_{n-1}}}(\lambda \otimes 1\!\!1_{\mathrm{n-2}})$
(iv) $L_{n-1}\nu^{\frac{1}{2}}$, $\widetilde{L_{n-1}}\nu^{\frac{1}{2}}$ and
(v) $\mathrm{ind}_{\mathrm{P_{n-3,2}}}^{\mathrm{GL_{n-1}}}(\nu^{\frac{1}{2}} \otimes \tau)$ where $\tau \in \mathrm{Irr}(\mathrm{GL}_2)$ is either a supercuspidal σ or a $St_2\mu$, $\mu \neq \nu^{\frac{n}{2}}, \nu^{-(\frac{n-2}{2})}$ or $\chi_1 \times \chi_2$ where both $\chi_1, \chi_2 \neq \nu^{\frac{n-1}{2}}, \nu^{-(\frac{n-3}{2})}$. So for every such ρ we now look at the irreducible quotients of $\rho \times \nu^{-(\frac{n-3}{2})}$.

If $\rho = \nu^{\frac{1}{2}}$ then by Lemma 2.4.1, $\nu^{\frac{1}{2}} \times \nu^{-(\frac{n-3}{2})}$ is itself irreducible.

If $\rho = L_{n-1}\nu^{\frac{1}{2}}$ or $\widetilde{L_{n-1}}\nu^{\frac{1}{2}}$ we have already shown that $\mathrm{U}(\rho,\chi)$ equals L_n (Section 2.5) and $\mathbb{1}_{n-2} \times St_2\nu^{-(\frac{n-2}{2})}$(Lemma 4.1.4) respectively.
For ρ as in (ii), by Example 2.8.6 $\mathrm{U}(\rho,\chi)$ exists in all cases. If $\gamma \neq \nu^{-(\frac{n-1}{2})}$, $\mathrm{U}(\rho,\chi)$ is equal to the irreducible $\mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\nu^{\frac{1}{2}} \otimes \gamma)$. If $\gamma = \nu^{-(\frac{n-1}{2})}$ the unique irreducible quotient of this representation is $\widetilde{L_n}\nu$, which we know is not GL_{n-1}-distinguished.

If ρ is as in (iii), then by Example 2.8.5 $\mathrm{U}(\rho,\chi)$ is the irreducible $\mathrm{ind}_{\mathrm{P_{n-2,2}}}^{\mathrm{GL_n}}(\mathbb{1}_{\mathrm{n-2}} \otimes [\lambda \times \nu^{-(\frac{n-3}{2})}])$ if $\lambda \neq \nu^{-(\frac{n-5}{2})}$. If $\lambda = \nu^{-(\frac{n-5}{2})}$ then $\mathrm{U}(\rho,\chi)$ is a quotient of $\mathbb{1}_{n-2} \times \nu^{-(\frac{n-4}{2})}$ which is not GL_{n-1}-distinguished by Proposition 4.2.2.

Let ρ be as in (v). If $\tau = \sigma$ is supercuspidal then $\mathrm{U}(\rho,\chi)$ is by the irreducible $\mathbb{1}_{n-2} \times \sigma$. If $\tau = St_2\mu$ note that $\mathrm{U}(\rho,\chi)$ is $\mathbb{1}_{n-2} \times St_2\mu$ if $\mu \neq \nu^{-\frac{n}{2}}$.($St_2\mu$ is linked with $\nu^{-(\frac{n-3}{2})}$ only for two choices $\mu = \nu^{-\frac{n}{2}}, \nu^{-(\frac{n-6}{2})}$. But if $\mu = \nu^{-(\frac{n-6}{2})}$ we may appeal to Lemma 2.6.2.) If $\mu = \nu^{-(\frac{n}{2})}$ our $\mathrm{U}(\rho,\chi)$ equals $\widetilde{Z_n}$, where Z_n is the representation in (2.5.2). By Lemma 4.1.5, Z_n is not GL_{n-1}-distinguished. If $\tau = \chi_1 \times \chi_2$ by Example 2.8.7 the irreducible quotients arising from our representation are either an irreducible $\mathbb{1}_{n-2} \times \chi_1 \times \chi_2$ or a quotient of $\chi_1 \times \widetilde{L_{n-1}}\nu^{\frac{1}{2}}$. By Lemma 4.1.2, $\chi_1 \times \widetilde{L_{n-1}}\nu^{\frac{1}{2}}$ is not GL_{n-1}-distinguished for any choice of χ_1.

To summarize, we collect all the GL_{n-1}-distinguished irreducible quotients of $\rho \times \chi$ which we have obtained from all the three orbits. By Section 2.6, this collection is precisely those $\pi \in \mathrm{Irr}(\mathrm{GL_n})$ for which there is a subrepresentation $\mathfrak{L}(\mathbb{1}_{n-2})$ such that $\mathfrak{L}(\pi)/\mathfrak{L}(\mathbb{1}_{n-2})$ either corresponds to an infinite dimensional representation in $\mathrm{Irr}(\mathrm{GL}_2)$ or $\nu^{\pm\frac{n-2}{2}}$. This completes the proof along one direction.

Conversely, let $\mathfrak{L}(\pi)$ be as in the statement of Theorem 1.1.2. We show that π is GL_{n-1}-distinguished. It follows from Section 2.6 that if $\mathfrak{L}(\pi)/\mathfrak{L}(\mathbb{1}_{n-2})$ corresponds to an irreducible infinite dimensional representation of GL_2 then it is one of the following:

(a) $\mathbb{1}_n$

(b) $\mathrm{ind}_{\mathrm{P_{2,n-2}}}^{\mathrm{GL_n}}(\tau \otimes \mathbb{1}_{\mathrm{n-2}})$ where $\tau \in \mathrm{Irr}(\mathrm{GL}_2)$ is a supercuspidal or $St_2\chi$ or $\chi_1 \times \chi_2$ where $\chi \neq \nu^{\pm n/2}$, $\chi_2 \neq \chi_1\nu^{\pm 1}$ and $\chi_1, \chi_2 \neq \nu^{\pm \frac{n-1}{2}}$

(c) L_n and $\widetilde{L_n}$

(d) an irreducible $\mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\nu^{-\frac{1}{2}} \otimes \chi)$ where $\chi \neq \nu^{\frac{n-3}{2}}$ and its contragredient.

We have already observed in Section 2.6 that representations π for which $\mathfrak{L}(\pi)/\mathfrak{L}(\mathbb{1}_{n-2})$ corresponds to the character $\nu^{\pm \frac{n-2}{2}}$ of GL_2 are respectively $\mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\nu^{\nu^{1/2} \otimes \frac{n-3}{2}})$ and its contragredient $\mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\nu^{-1/2} \otimes \nu^{-(\frac{n-3}{2})})$. The trivial representation $\mathbb{1}_n$ is obviously GL_{n-1} distinguished. By Proposition 4.1.3, L_n and $\widetilde{L_n}$ are GL_{n-1} distinguished. It follows from Theorem 3.3.1 with $k = 2, n-1$ that representations other than $\mathbb{1}_n, L_n, \widetilde{L_n}$ are GL_{n-1}-distinguished. □

The following corollary is the generalization of Theorem 3.4.2, the proof of which follows from Theorem 1.1.2 and (2.6).

Corollary 4.4.1. *The following is a complete list of irreducible admissible representations of* GL_n *which are* GL_{n-1}*-distinguished.*

(1) the trivial representation

(2) $\mathrm{ind}_{\mathrm{P_{n-1,1}}}^{\mathrm{GL_n}}(\nu^{-1/2} \otimes \chi)$ *where* $\chi \neq \nu^{\frac{n-1}{2}}, \nu^{-(\frac{n+1}{2})}$ *and its contragredient*

(3) $\mathrm{ind}_{\mathrm{P_{n-2,2}}}^{\mathrm{GL_n}}(\mathbb{1}_{\mathrm{n-2}} \otimes \eta)$ *where* $\eta \in \mathrm{Irr}(\mathrm{GL}_2)$ *is either a supercuspidal or* $St_2\mu$ *or* $\chi_1 \times \chi_2$ *where* $\mu \neq \nu^{\pm \frac{n}{2}}$ *and both* $\chi_1, \chi_2 \neq \nu^{\pm \frac{n-1}{2}}$

(4) the representation L_n *and its contragredient* $\widetilde{L_n}$*.*

Chapter 5

Some Theorems on Multiplicity

Let $n = \sum n_i$, $\rho_i \in \mathrm{Irr}(\mathrm{GL_{n_i}})$ and $\xi = \mathrm{ind}_{\mathrm{P_{n_1,\ldots,n_r}}}^{\mathrm{GL_n}}(\rho_1 \otimes \ldots \otimes \rho_\mathrm{r})$. We prove that $\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL_{n-1}}}[\xi, \mathbb{1}_{\mathrm{n-1}}]) \leq 2$. In the process we classify all principal series representations of the form $\mathrm{ind}_{\mathrm{P_{1,\ldots,1}}}^{\mathrm{GL_n}}(\chi_1 \otimes \ldots \otimes \chi_\mathrm{n})$ (where χ_i are characters of $\mathrm{GL_1}$) whose space of GL_{n-1}-invariant forms have dimension equal to 2.

5.1 Motivation for the Theorem

Recall that for $\xi \in \mathrm{Alg}(\mathrm{GL_n})$

$$d_\xi = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL_{n-1}}}[\xi, \mathbb{1}_{\mathrm{n-1}}]).$$

If $\rho_i \in \mathrm{Irr}(\mathrm{GL_{n_i}})$ and $\xi = \mathrm{ind}_{\mathrm{P_{n_1,\ldots,n_r}}}^{\mathrm{GL_n}}(\rho_1 \otimes \ldots \otimes \rho_\mathrm{r})$ we prove in this section that $d_\xi \leq 2$. For $\pi \in \mathrm{Irr}(\mathrm{GL_n})$ it is proved in [2] that $d_\pi \leq 1$. Our Theorem 1.1.2 describes precisely for which $\pi \in \mathrm{Irr}(\mathrm{GL_n})$ one has $d_\pi \neq 0$. Since the GL_{n-1}-distinguishedness is dictated by the $\mathrm{GL_2}$ part of $\mathfrak{L}(\pi)/\mathfrak{L}(\mathbb{1}_{n-2})$, the multiplicity d_π is also dictated by d_ρ where $\rho \in \mathrm{GL_2}$ corresponds to a twist of $\mathfrak{L}(\pi)/\mathfrak{L}(\mathbb{1}_{n-2})$(see (3.3.5) or (3.3.6)). Since $d_\rho \leq 1$ for $\rho \in \mathrm{Irr}(\mathrm{GL_2})$ by Theorem 3.4.1, we must have $d_\pi \leq 1$. This is the guiding philosophy. Therefore we may say that "Multiplicity one for $\mathrm{GL_2}$" implies "Multiplicity one for GL_n". A proof of this is contained in Remark 5.1.3 below. We now ask the same question for a representation of the type ξ. We have shown in Theorem 3.4.1 that there is a principal series representation ξ_0 of GL_2 which has $d_{\xi_0} = 2$. We show that $d_\xi = 2$, whenever a copy of this contained in ξ in a prescribed fashion. We also show that our guiding philosophy remains true in this case as well i.e., if $d_{\xi_2} \leq 2$ for a principal series representation ξ_2 of GL_2 then $d_{\xi_n} \leq 2$ for a principal series representation ξ_n of GL_n.

It follows from the proof of Theorem 1.1.2 that the cuspidal support of any irreducible admissible representation of GL_n which is GL_{n-1}-distinguished is either

$$\{\sigma, \nu^{\frac{n-3}{2}}, \nu^{\frac{n-5}{2}}, ..., \nu^{-(\frac{n-3}{2})}\} \quad \text{or} \quad \{\chi_1, \chi_2, \nu^{\frac{n-3}{2}}, \nu^{\frac{n-5}{2}} ..., \nu^{-(\frac{n-3}{2})}\}$$

where σ is an irreducible supercuspidal representation of GL_2 and χ_1, χ_2 are arbitrary characters of GL_1. For $n \geq 3$, let T_{n-2} denote the ordered set $\{\nu^{\frac{n-3}{2}}, ..., \nu^{-(\frac{n-3}{2})}\}$ corresponding to the trivial representation $\mathbb{1}_{n-2}$ of GL_{n-2}. Then $T_1 = \{1\}, T_2 = \{\nu^{\frac{1}{2}}, \nu^{-\frac{1}{2}}\}, T_3 = \{\nu, 1, \nu^{-1}\}$ and so on.

Remark 5.1.1. Note that $\mathbb{1}_{n-2} \times \chi \times \mu$ and $\chi \times \mu \times \mathbb{1}_{n-2}$ is GL_{n-1}-distinguished for any two characters χ, μ of GL_1. This follows from Theorem 3.4.1 and Theorem 3.3.1(applied with $k = n-2$ or $k = 2$). We will use this in what follows without further reference.

Example 5.1.2. Using Theorem 3.4.1 and the recipe (Theorem 3.3.1) given by Mackey theory, given any $n > 2$, starting with $\xi_0 = \nu^{-1/2} \times \nu^{1/2}$, we can construct a representation π of GL_n with $d_\pi = 2$. For example for $n \geq 4$, let $\pi = \mathrm{ind}_{\mathrm{P_{2,n-2}}}^{\mathrm{GL_n}}([\nu^{\frac{n-3}{2}} \times \nu^{\frac{n-1}{2}}] \otimes \mathbb{1}_{\mathrm{n-2}})$. It is easy to check that (3.3.1)(ii) and (3.3.3)(ii) fail(applied with k=2), but (3.3.2)(i) and (3.3.2)(ii) hold whence ξ has GL_{n-1}-invariant form coming only from $\mathcal{C}_2$. Since (3.3.2)(i) yields $d_{\xi_0} = 2$ it follows from (3.3.6) that $d_\pi = 2$.

Remark 5.1.3. For $n > 2$ and a nontrivial $\pi \in \mathrm{Irr(G_n)}$ we have not used the well-known fact ([1], [2]) that $d_\pi \leq 1$ up to this point. We may deduce this using Mackey theory and Theorem 3.4.1. Let $\pi \in \mathrm{Irr}(\mathrm{GL_n})$ be infinite dimensional and GL_{n-1}-distinguished. All such π are precisely the ones in Corollary 4.4.1 except $\mathbb{1}_n$. If π is one of (2) or (4) of Corollary 4.4.1 then by Proposition 4.2.1, $d_\pi = 1$. For $n \geq 4$ if π is of type (3) in Corollary 4.4.1, then by Corollary 4.2.1 (b),(c) it follows that π has a GL_{n-1}-invariant form only on the closed orbit $\mathcal{C}_1$. It follows from (3.3.5) and Theorem 3.4.1 that $d_\pi = 1$. For $n = 3$ and π an irreducible $\chi_1 \times \chi_2 \times \chi_3$ a proof that $d_\pi = 1$ is contained in Theorem 5.2.1 below. For an irreducible $\pi = St_2\chi \times 1$ it follows from Lemma 3.4.4 that $d_\pi = 1$.

If $\xi = \chi_1 \times ... \times \chi_n$ put $\rho = \chi_1 \times ... \times \chi_{n-1}$ and $\tau = \chi_n$ so that $\xi = \rho \times \tau$. We recall our conditions (3.3.1), (3.3.2) and (3.3.3) in this particular case when $k = n-1$, namely

$$\mathrm{Hom}_{\mathrm{GL_{n-1}}}[\rho\nu^{\frac{1}{2}}, \mathbb{1}_{\mathrm{n-1}}] \neq 0 \tag{5.1.1}$$

$$\mathrm{Hom}_{\mathrm{GL_{n-1}}}[\rho\nu^{-\frac{1}{2}}, \mathbb{1}_{\mathrm{n-2}}] \neq 0 \text{ and } \chi_{\mathrm{n}} = \nu^{-(\frac{\mathrm{n-3}}{2})} \tag{5.1.2}$$

$$\mathrm{Hom}_{\mathrm{GL_{n-2}}}[\rho^{(1)}, \mathbb{1}_{\mathrm{n-2}}] \neq 0 \tag{5.1.3}$$

We also recall that if either (5.1.1)or (5.1.2) holds then ξ is GL_{n-1}-distinguished. In this case assume that (5.1.2) holds. Then obviously

$$d_\xi \geq d_{(\rho\nu^{-\frac{1}{2}})}. \tag{5.1.4}$$

Moreover, if (5.1.1) and (5.1.3) does not hold but (5.1.2) holds, then by (3.3.6) we have

$$d_\xi = d_{(\rho\nu^{-\frac{1}{2}})}. \tag{5.1.5}$$

We recall that for $\rho = \chi_1 \times ... \times \chi_{n-1}$ the representation $\rho^{(1)}$ is glued from(=has a filtration $0 = V_0 \subset V_1 \subset V_2 \subset ... \subset V_{n-1} = \rho^{(1)}$ such that the quotients are isomorphic after a permutation to) the product of χ_i's in the same order with one of the χ_i dropped. This is a consequence of the Leibnitz rule. For instance, $(\chi_1 \times \chi_2)^{(1)}$ is glued from χ_1 and χ_2. Similarly, $(\chi_1 \times \chi_2 \times \chi_3)^{(1)}$ is glued from $\chi_1 \times \chi_2$, $\chi_2 \times \chi_3$ and $\chi_1 \times \chi_3$. In general, for $\rho = \chi_1 \times ... \times \chi_{n-1}$ we say that the derivative $\rho^{(1)}$ is glued from representations of the form $\lambda_1 \times ... \times \lambda_{n-2}$ where the λ_j's are χ_i's occurring in the same order as in ρ with one of the χ_i's dropped. Therefore, for $\rho^{(1)}$ to have a trivial quotient it is necessary that at least one $\lambda_1 \times ... \times \lambda_{n-2}$ is the representation $\nu^{\frac{n-3}{2}} \times ... \times \nu^{-(\frac{n-3}{2})}$.

5.2 Two Examples and the result for $n = 3$

In this section we present two examples: one in the case of GL_3 and another in the case of GL_4. We start with an example of a representation ξ of GL_3 for which $d_\pi = 2$. The interesting fact about this representation is that it is a principal series which is also a direct sum of two irreducible representations of GL_3 both of which are GL_2-distinguished.

Example 5.2.1. Let η denote the principal series $1 \times \nu \times 1$ of GL_3 and $\eta_1 = \mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\mathrm{St}_2\nu^{1/2} \otimes 1)$ and $\eta_2 = \mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\nu^{1/2} \otimes 1)$. Observe that both η_1 and η_2 are irreducible and by Theorem 1.1.2 are GL_2-distinguished. We apply Mackey theory with $k = 2$ to η. Note that (5.1.2) holds. Therefore by (5.1.4), $d_\xi \geq d_{\nu^{-1/2} \times \nu^{1/2}} = 2$ by Theorem 3.4.1. Though there is a GL_2-invariant form on the open orbit also, we show that it does not extend to $\xi_{|\mathrm{GL}_2}$ as follows. By [22] 11.1, the representation η is a direct sum of the irreducible representations η_1 and η_2. Hence $\mathrm{Hom}_{\mathrm{GL}_2}[\eta, \mathbb{1}_2] = \mathrm{Hom}_{\mathrm{GL}_2}[\eta_1, \mathbb{1}_2] \oplus \mathrm{Hom}_{\mathrm{GL}_2}[\eta_2, \mathbb{1}_2]$ and $\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL}_2}[\eta, \mathbb{1}_2]) = 2$. The same conclusions hold for $\eta^\vee = 1 \times \nu^{-1} \times 1$. The next theorem shows that in fact these are the only $\chi_1 \times \chi_2 \times \chi_3$ for which $d_\xi = 2$.

Next we consider $\chi_1 \times ... \times \chi_n$ where $n \geq 3$. Our approach is via induction on n and Mackey theory. To this end, we begin with the result for $n = 3$, which forms the basis for induction.

Theorem 5.2.1. *Let χ_i be characters of GL_1 for $i = 1, 2, 3$. Then the principal series representation $\xi = \chi_1 \times \chi_2 \times \chi_3$ of GL_3 is GL_2-distinguished if and only if one of the $\chi_i = 1$. Moreover, $d_\xi \leq 2$ and $d_\xi = 2$ if and only if $\xi = 1 \times \nu \times 1$ or $1 \times \nu^{-1} \times 1$.*

Proof. Suppose ξ is GL_2-distinguished. Apply Mackey theory with $k = 2$. By (5.1.1), if ξ has a GL_2-invariant form on $\mathcal{C}_1$ then $\xi = 1 \times \nu^{-1} \times \chi_3$, where χ_3 is arbitrary. For ξ to have a GL_2-invariant form on $\mathcal{C}_2$, we must have $\chi_3 = 1$ by (5.1.2). Finally, it is necessary by (5.1.3) that one of χ_1 or χ_2 is 1 for ξ to have a GL_2-invariant form on the open orbit.

Conversely suppose ξ is such that one of the χ_i is 1. Consider $\xi = \chi \times 1 \times 1$ for any character χ. Observe that (5.1.2) holds for ξ and hence $d_\xi \neq 0$. We claim that $d_\xi = 1$ and prove it by contradiction. Assume $d_\xi > 1$. By (5.1.2) and (5.1.5) $\eta = \chi\nu^{1/2} \times \nu^{1/2} \times \nu^{1/2} \times \nu^{-1/2} \in \mathrm{Alg}(\mathrm{GL}_4)$ should have $d_\eta \geq 2$. We have the following exact sequence of GL_4-modules

$$0 \to \chi\nu^{1/2} \times \nu^{1/2} \times St_2 \to \eta \to \chi\nu^{1/2} \times \nu^{1/2} \times 1\!\!1_2 \to 0$$

Apply Lemma 4.1.2 to the submodule with $n = 4, k = 1$. Then the submodule is not GL_3-distinguished, because $\nu^{1/2} \times St_2$ fails to satisfy each condition. But the quotient is GL_3-distinguished by (3.3.6) with

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL}_3}[\nu^{1/2} \times \nu^{1/2} \times 1\!\!1_2, 1\!\!1_3]) = \mathrm{d}_{\chi\nu^{-3/2} \times \nu^{-1/2}} = 1$$

which implies $d_\eta = 1$, a contradiction. By Duality Lemma, our statement is true for all $\xi = 1 \times 1 \times \chi$. Next let $\xi = 1 \times \chi \times 1$ with $\chi = \nu^{\pm 1}$. Then by Example 5.2.1 we have $d_\xi = 2$. So it now remains to consider the case when the multiplicity of the trivial character is 1.

If $\chi_3 = 1$ then ξ is GL_2-distinguished since (5.1.2) holds. Observe that since $\chi_1, \chi_2 \neq 1$ (5.1.1) and (5.1.3) fail. Therefore, by (5.1.5) $d_\xi = d_{\chi_1\nu^{-1/2} \times \chi_2\nu^{-1/2}}$. The last quantity in the equality is less than or equal to 2 and equals two if and only if $\chi_1 = 1$ and $\chi_2 = \nu$ whence $\xi = 1 \times \nu \times 1$. By the Duality Lemma the result holds when $\chi_1 = 1$. Finally let $\chi_2 = 1$. We need only treat the cases when both χ_1 and χ_3 are linked with 1, otherwise it reduces to one of the cases above. Therefore, we may assume $\xi = \mu \times 1 \times \chi$ where $\mu, \chi \in \{\nu, \nu^{-1}\}$. Write $\xi = \rho \times \chi$ where $\rho = \mu \times 1$ and (3.1.1) in this case is just

$$0 \to \mathrm{ind}_{\mathrm{P}_2}^{\mathrm{GL}_2}(\rho_{|\mathrm{P}_2}) \to \mathrm{ind}_{\mathrm{P}_{2,1}}^{\mathrm{GL}_3}(\rho \otimes \chi) \to \rho.\nu^{1/2} \oplus \mathrm{ind}_{\mathrm{P}_{1,1}}^{\mathrm{GL}_2}(\rho_\chi) \to 0 \qquad (5.2.1)$$

where the actions of the representations are as defined in Chapter 3. Then (5.1.1) does not hold for ξ and (5.1.2) does not hold since $\chi \neq 1$. Therefore ξ does not have GL_2-invariant form on both closed orbits. Since $\mu \neq 1$, $\rho^{(1)} = \mu \oplus 1$ and therefore $\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL}_1}[\rho^{(1)}, 1]) = 1$. By Lemma 2.6.4, $\mathrm{Ext}^1_{\mathrm{GL}_2}[\rho\nu^{1/2}, \mathbb{1}_2] = 0$ and $\mathrm{Ext}^1_{\mathrm{GL}_2}[\mathrm{ind}_{\mathrm{P}_{1,1}}^{\mathrm{GL}_2}(\rho_\chi), \mathbb{1}_2] = 0$. It follows from (5.2.1) that $\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL}_2}[\mu \times 1 \times \chi, \mathbb{1}_2]) = 1$ for $\chi, \mu \in \{\nu, \nu^{-1}\}$ □

The point to observe is that two GL_{n-1}-distinguished principal series $\xi = \chi_1 \times \ldots \times \chi_n$ may have the same $\mathrm{JH}^0(\xi)$ but may have different d_ξ. We have already shown this in the case of GL_2 and GL_3. For example, $\xi_1 = 1 \times \nu \times 1$ and $\xi_2 = \nu \times 1 \times 1$ have $\mathrm{JH}^0(\xi_1) = \mathrm{JH}^0(\xi_2)$. By Theorem 5.2.1, $d_{\xi_1} = 2$ whereas $d_{\xi_2} = 1$.

Observe that in the Example 5.2.1, $d_\pi = 2$ is attributed to each irreducible subquotient contributing one each. We next present an example in the GL_4 case which is a prototype of a different phenomena.

Example 5.2.2. Let $\xi_1 = \nu^{\frac{1}{2}} \times \nu^{\frac{1}{2}} \times \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}$ and $\xi_2 = \nu^{\frac{1}{2}} \times \nu^{-\frac{1}{2}} \times \nu^{\frac{1}{2}} \times \nu^{-\frac{1}{2}}$. We show that $d_{\xi_1} = 1$ and $d_{\xi_2} = 2$. It is easy to see that both the ξ_i are GL_3-distinguished since ξ_i have the GL_3-distinguished $\mathbb{1}_2 \times \nu^{\frac{1}{2}} \times \nu^{-\frac{1}{2}}$ as a quotient. Also by [22](11.3), $\mathrm{JH}^0(\xi_1) = \mathrm{JH}^0(\xi_2) = \{\mathrm{St}_2 \times \mathrm{St}_2, \mathrm{St}_2 \times \mathbb{1}_2, \mathrm{St}_2 \times \mathbb{1}_2, \mathbb{1}_2 \times \mathbb{1}_2\}$. Therefore by Theorem 1.1.2 and Remark 5.1.3, we have $d_{\xi_i} \leq 2$. Put $\rho = \nu^{\frac{1}{2}} \times \nu^{-\frac{1}{2}} \times \nu^{\frac{1}{2}}$ and $\chi = \nu^{-\frac{1}{2}}$ so that $\xi_2 = \rho \times \chi$. Apply Mackey theory to ξ_2 and note that (5.1.2) holds for ξ_2 and therefore d_{ξ_2} is at least $d_{\rho\nu^{\frac{-1}{2}}}$, which is 2 by Example 5.2.1. It follows that $d_{\xi_2} = 2$. We claim that $d_{\xi_1} = 1$. Otherwise, $d_{\xi_1} \geq 2$ and (3.3.2) holds for $\eta = \xi_1\nu^{\frac{1}{2}} \times \nu^{-1} \in \mathrm{Alg}(\mathrm{GL}_5)$ whence $d_\eta = d_{\xi_1} \geq 2$. Now $\eta = \nu \times \nu \times 1 \times 1 \times \nu^{-1}$ and sits in the following exact sequence:

$$0 \to \mathrm{ind}_{\mathrm{P}_{2,3}}^{\mathrm{GL}_5}([\nu \times \nu] \times [1 \times \mathrm{St}_2\nu^{-\frac{1}{2}}]) \to \eta \to \mathrm{ind}_{\mathrm{P}_{3,2}}^{\mathrm{GL}_5}([\nu \times \nu \times 1] \otimes \nu^{-\frac{1}{2}}) \to 0$$

By Lemma 4.1.2,(with $n = 5, k = 2$) it follows that the subrepresentation in the above exact sequence is not GL_4-distinguished. On the other hand, apply Mackey theory with $n = 5, k = 3$ to $\mathrm{ind}_{\mathrm{P}_{3,2}}^{\mathrm{GL}_5}([\nu \times \nu \times 1] \otimes \nu^{-\frac{1}{2}})$. By Lemma 4.1.2, (3.3.1) and (3.3.3) does not hold for $\nu^{-\frac{1}{2}} \in \mathrm{Irr}(\mathrm{GL}_2)$. But (3.3.2) holds and (3.3.6) holds whence $d_\eta = d_{1\times 1\times \nu^{-1}} = 1$, a contradiction.

Remark 5.2.3. The difference in philosophy of Example 5.2.1 and Example 5.2.2 is that the representations in the former has multiplicity 2 coming from two distinct GL_2-distinguished representations in its Jordan-Holder series whereas ξ_2 in Example 5.2.2 has multiplicity 2 coming from $St_2 \times \mathbb{1}_2$ which appears twice in $\mathrm{JH}^{(0)}(\xi_2)$. We show in the next theorem that these two are the only two types of principal series which have $d_\xi = 2$.

5.3 Statement of the theorem and Proof for $n \geq 4$.

The following theorem is the generalization of Theorems 3.4.1 and 5.2.1 to $n > 3$. Observe that we classify all $\xi = \chi_1 \times ... \times \chi_n$ which have $d_\xi = 2$. As in the case of GL_3 it is easy to see that the condition on ξ is necessary. We use induction on n and Mackey theory for the proof. We recall that T_{n-2} is the ordered set $\{\nu^{\frac{n-3}{2}}, ..., \nu^{-(\frac{n-3}{2})}\}$ corresponding to the trivial character of GL_{n-2}.

Theorem 5.3.1. *Let $\xi = \chi_1 \times ... \times \chi_n$ for $n \geq 3$ and $d_\xi = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GL_{n-1}}}[\xi, \mathbb{1}_{\mathrm{n-1}}])$. Let $[\chi_1, ..., \chi_n]$ denote the ordered set $\{\chi_1, ..., \chi_n\}$. Then $d_\xi \neq 0$ if and only if there exists χ_i, χ_j such that $[\chi_1, ..., \chi_n] \backslash \{\chi_i, \chi_j\}$ equals the ordered set T_{n-2}. For $k = 1, ..., n-1$ define $\xi_n(k) \in \mathrm{Alg}(\mathrm{GL_n})$ by*

$$\begin{aligned}
\xi_n(1) \quad &= \nu^{\frac{n-3}{2}} \times \nu^{\frac{n-1}{2}} \times \nu^{\frac{n-3}{2}} \times ... \times \nu^{-(\frac{n-3}{2})},\\
\xi_n(n-1) \quad &= \nu^{\frac{n-3}{2}} \times ... \times \nu^{-(\frac{n-3}{2})} \times \nu^{-(\frac{n-1}{2})} \times \nu^{-(\frac{n-3}{2})},\\
\xi_n(k) \quad &= \nu^{\frac{n-3}{2}} \times ... \times \nu^{\frac{n-2k+1}{2}} \times \nu^{\frac{n-2k-1}{2}} \times \nu^{\frac{n-2k+1}{2}} \times \nu^{\frac{n-2k-1}{2}} \times ... \times \nu^{-(\frac{n-3}{2})}
\end{aligned}$$

for $2 \leq k \leq n-2$. Then $d_\xi \leq 2$ and $d_\xi = 2$ if and only if $\xi = \xi_n(k)$ for some $k \in \{1, ..., n-1\}$.

Note: For $n = 3$ the condition $2 \leq k \leq n-2$ is void. There does not exist such a k and hence the statement is true by Theorem 5.2.1.

Proof. Assume that ξ is GL_{n-1} distinguished. The claim is true for $n = 3$. We apply Mackey theory to ξ with $k = n-1$. If (5.1.1) holds, $\xi = \nu^{\frac{n-3}{2}} \times ... \times \nu^{-(\frac{n-3}{2})} \times \nu^{-(\frac{n-1}{2})} \times \chi_n$ where χ_n can be arbitrary. Suppose (5.1.2) holds. The condition (ii) gives $\chi_n = \nu^{-(\frac{n-3}{2})}$. By (i) and induction, there exists χ_i, χ_j such that

$$[\chi_1 \nu^{-1/2}, ..., \chi_{n-1}\nu^{-1/2}] \setminus \{\chi_i \nu^{-1/2}, \chi_j \nu^{-1/2}\} = \{\nu^{\frac{n-4}{2}}, \nu^{\frac{n-6}{2}} ..., \nu^{-(\frac{n-4}{2})}\} = T_{n-3}.$$

Therefore, $[\chi_1, ..., \chi_n] \setminus \{\chi_i, \chi_j\} = T_{n-2}$. If (5.1.3) holds, it is necessary that there exists a χ_i such that $[\chi_1, ..., \chi_{n-1}] \setminus \{\chi_i\} = T_{n-2}$. We choose $j = n$ which completes the proof along one direction.

Conversely, let $\xi = \chi_1 \times ... \times \chi_n$ and suppose there exists χ_i, χ_j such that $[\chi_1, ..., \chi_n] \setminus \{\chi_i, \chi_j\} = T_{n-2}$. We know that the theorem is true for $n = 3$. Assume therefore that $n \geq 4$ and the converse is true for $n-1$. We prove the converse by a sequence of Lemmas. We recall that if $\rho = \chi_1 \times ... \times \chi_{n-1}$, for $\rho^{(1)}$ to have $\mathbb{1}_{n-2}$ as a quotient, it is necessary that there exists a χ_i such that $[\chi_1, ..., \chi_{n-1}] \setminus \{\chi_i\} = T_{n-2}$.

Lemma 5.3.2. *If $[\chi_3, ..., \chi_n] = T_{n-2}$, then $0 \neq d_\xi \leq 2$ and $d_\xi = 2$ if and only if $\xi = \xi_n(1)$ or $n = 4$ and $\xi = \xi_4(2)$.*

Proof. Put $\rho = \chi_1 \times ... \times \chi_{n-1}$. Observe that $\chi_n = \nu^{-(\frac{n-3}{2})}$. Note that $\rho^{(1)}$ is glued from representations of the form $\lambda_1 \times ... \times \lambda_{n-2}$(i.e., recall that λ_t's are χ_ℓ's occurring in the same order as χ_ℓ's with one of the χ_ℓ's dropped) with either $\lambda_{n-2} = \chi_{n-1} = \nu^{-(\frac{n-5}{2})}$ or $\chi_1 \times ... \times \chi_{n-2}$. Then for the first type $[\lambda_1, ..., \lambda_{n-2}] \neq T_{n-2}$. Now, since $[\chi_3, ..., \chi_n] = T_{n-2}$, the set $[\chi_1, ..., \chi_{n-2}] = T_{n-2}$ if and only if $n = 4$ whence $\xi = \xi_4(2)$. We have already shown in Example 5.2.2 that $d_{\xi_4(2)} = 2$. Therefore, for $\xi \neq \xi_4(2)$ we have $[\chi_1, ..., \chi_{n-2}] \neq T_{n-2}$. This shows that $\rho^{(1)}$ does not have $\mathbb{1}_{n-2}$ as a quotient and therefore (5.1.3) does not hold. Since $\chi_{n-1} \neq \nu^{-(\frac{n-1}{2})}$, (5.1.1) also fails to hold. But

(5.1.2) holds and by (5.1.5) we have $0 \neq d_\xi = d_{\rho\nu^{-1/2}} \leq 2$. Moreover, by induction, $d_\xi = 2$ if and only if $\xi = \xi_n(1)$ or $\xi_4(2)$. □

Corollary 5.3.3. *If $[\chi_1, ..., \chi_{n-2}] = T_{n-2}$, then $0 \neq d_\xi \leq 2$ and $d_\xi = 2$ if and only if $\xi = \xi_n(n-1)$ or $\xi_4(2)$ for $n = 4$.*

Proof. Apply Lemma 5.3.2 to $\xi^\vee = \chi_n^{-1} \times ... \times \chi_1^{-1}$ and invoke the Duality Lemma. □

Lemma 5.3.4. *Assume that $i = n-1, j < n-1, \chi_{n-1} \neq \nu^{-(\frac{n-3}{2})}$ and $\{\chi_1, ..., \chi_{n-2}\} \neq T_{n-2}$. Then $0 \neq d_\xi \leq 2$ and $d_\xi = 2$ if and only if $\xi = \xi_n(k)$ for some $k \in \{1, .., n-2\}$.*

Proof. Put $\rho = \chi_1 \times ... \times \chi_{n-1}$. By hypotheses $\chi_n = \nu^{-(\frac{n-3}{2})}$. Note that $\rho^{(1)}$ is glued from representations of the form $\lambda_1 \times ... \times \lambda_{n-2}$ with either $\lambda_{n-2} = \chi_{n-1} \neq \nu^{-(\frac{n-3}{2})}$ or $\chi_1 \times ... \times \chi_{n-2}$. In both cases $[\lambda_1, ..., \lambda_{n-2}] \neq T_{n-2}$ and hence (5.1.3) fails. Also by our hypotheses (5.1.1) does not hold. By induction, (5.1.2) holds and we apply (5.1.5) to get $0 \neq d_\xi \leq 2$ and $d_\xi = 2$ if and only if $\xi = \xi_n(k)$ for $k \in \{1, ..., n-2\}$. □

Lemma 5.3.5. *Assume that $i = n-1, j < n-1, \chi_{n-1} = \nu^{-(\frac{n-3}{2})}$ and $[\chi_1, ..., \chi_{n-2}] \neq T_{n-2}$. Then $d_\xi = 1$.*

Proof. Observe that by our covering hypotheses $\chi_n = \nu^{-(\frac{n-3}{2})}$. Therefore, by induction (5.1.2) holds for ξ and by (5.1.4), $d_\xi \neq 0$. If possible, let $d_\xi > 1$. If $n = 4$, $\xi = \chi_1 \times \chi_2 \times \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}$ with either χ_1 or χ_2 equal to $\nu^{\frac{1}{2}}$. If both χ_1 and χ_2 are equal to $\nu^{\frac{1}{2}}$ our ξ is the representation ξ_1 in Example 5.2.2 for which $d_\xi = 1$. If $\chi_2 \neq \nu^{\frac{1}{2}}$ consider $\xi^\vee$ and we may apply Lemma 5.3.4 to get $d_{\xi^\vee} = 1$. If $\chi_1 \neq \nu^{\frac{1}{2}}$, we look at the exact sequence of GL_4-modules

$$0 \to \chi_1 \times \nu^{-\frac{1}{2}} \times St_2 \to \xi \to \chi_1 \times \nu^{-\frac{1}{2}} \times \mathbb{1}_2 \to 0$$

By Lemma 4.1.2(with $n = 4$ and $k = 1$) the submodule is not GL_3-distinguished. For the quotient, apply Mackey Theory with $n = 4, k = 2$ and note that only (3.3.2) holds for $\mathbb{1}_2$. By (3.3.6), $d_{(\chi_1 \times \nu^{-\frac{1}{2}} \times \mathbb{1}_2)} = d_{(\chi_1 \nu^{-1} \times \nu^{\frac{-3}{2}})} = 1$. It follows from the exact sequence that $d_\xi = 1$.

If $n > 4$, consider $\eta = \xi\nu^{\frac{1}{2}} \times \nu^{-(\frac{n-2}{2})} \in \mathrm{Alg}(\mathrm{GL_{n+1}})$. Then (5.1.2) holds for η as well and therefore by (5.1.4), $d_\eta \geq d_\xi > 1$. But $\eta = \chi_1\nu^{\frac{1}{2}} \times ... \times \chi_n\nu^{\frac{1}{2}} \times \nu^{-(\frac{n-2}{2})}$ with the character in its last but one position as $\chi_n\nu^{\frac{1}{2}} = \nu^{-(\frac{n-4}{2})}$. It is easy to observe that the hypotheses in Lemma 5.3.4 holds for the representation η of GL_{n+1}. Therefore by Lemma 5.3.4, $d_\eta = 1$, a contradiction. □

Lemma 5.3.6. *Assume that $2 \leq i, j \leq n-1$. Then $0 \neq d_\xi \leq 2$ and $d_\xi = 2$ if and only if $\xi = \xi_n(k)$ for some k.*

Proof. Put $\rho = \chi_1 \times ... \times \chi_{n-1}$. Without loss of generality we may assume that $j < i$.(Otherwise, replace ξ by $\xi^\vee$.) By hypotheses, $\chi_1 = \nu^{\frac{n-3}{2}}$ and $\chi_n = \nu^{-(\frac{n-3}{2})}$. We may also assume that $[\chi_1, ..., \chi_{n-2}] \neq T_{n-2}$ for if it is, then we are reduced to Corollary 5.3.3. In this case $\rho^{(1)}$ is glued from representations of the form $\lambda_1 \times ... \times \lambda_{n-2}$ such that $\lambda_{n-2} = \chi_{n-1}$ or $\chi_1 \times ... \times \chi_{n-2}$. Since $[\chi_1, ..., \chi_{n-2}] \neq T_{n-2}$, for (5.1.3) to hold it is necessary that there exists a $\lambda_1 \times ... \times \lambda_{n-2}$ such that $[\lambda_1, ..., \lambda_{n-2}] = T_{n-2}$, which implies $\chi_{n-1} = \nu^{-(\frac{n-3}{2})}$. We claim that $i = n-1$. If the claim is true then we may now apply Lemma 5.3.5 and get $d_\xi = 1$.

If possible let $i < n-1$. Observe that by our hypotheses, χ_{i+1} equals $\chi_{i-1}\nu^{-1}$ if $j \neq i-1$ and $\chi_{i-2}\nu^{-1}$ if $j = i-1$. Now amongst the $\lambda_1 \times ... \times \lambda_{n-2}$ either both χ_i and χ_j occur or exactly one of them occurs. If both of them occur then $[\lambda_1, ..., \lambda_{n-2}] \neq T_{n-2}$ and we have a contradiction. If exactly one of them occurs, say χ_i then χ_{i+1} also occurs. If $j \neq i-1$ we get from above that $\chi_i = \chi_{i-1}$ and both of them occur in T_{n-2}, a contradiction. If $j = i-1$ then $\chi_i = \chi_{i-2}$ and again one gets a contradiction. One can treat the case when χ_j occurs amongst $[\lambda_1, ..., \lambda_{n-2}]$ similarly. Therefore our claim is proved.

We may now assume that (5.1.3) and (5.1.1) does not hold for our ξ. Now (5.1.2) holds for ξ by induction and since $\chi_n = \nu^{-\frac{n-3}{2}}$. By induction and (5.1.5) we conclude that $0 \leq d_\xi \leq 2$ and $d_\xi = 2$ if and only if $\xi = \xi_n(k)$ for some k. □

Lemma 5.3.7. *Let either*
(a) $i = 1, j > 1$ *and* $\chi_1 = \nu^{\frac{n-3}{2}}$ *or*
(b) $i = n, j < n$ *and* $\chi_n = \nu^{-(\frac{n-3}{2})}$ *hold. Then* $d_\xi = 1$.

Proof. In view of the Duality Lemma, it is enough to consider $i = n, j < n$. If $j = n-1$ then we are done by Corollary 5.3.3. So assume $i = n$ and $j < n-1$. By hypotheses $[\chi_1, ...\chi_{j-1}, \chi_{j+1}, ..., \chi_{n-1}] = T_{n-2}$. Therefore $\chi_{n-1} = \nu^{-(\frac{n-3}{2})}$ and since $j < n-1$, $[\chi_1, ..., \chi_{n-2}] \neq T_{n-2}$. Choose $i_0 = n-1$ and $j_0 = j$. Then $[\chi_1, ..., \chi_n] \backslash \{\chi_{i_0}, \chi_{j_0}\} = T_{n-2}$, $\chi_{i_0} = \nu^{-(\frac{n-3}{2})}$. We may now apply Lemma 5.3.5 to ξ get $d_\xi = 1$. □

Proof of Converse(continued): We now complete the proof of the converse. Our assumption on $\xi = \chi_1 \times ...\chi_n$ is that there exist χ_i, χ_j such that $[\chi_1, ..., \chi_n] \setminus \{\chi_i, \chi_j\} = T_{n-2}$. If $2 \leq i, j \leq n-1$ we are done by Lemma 5.3.6. So assume that $i = n$ and $j < n$. If $\chi_n = \nu^{-(\frac{n-3}{2})}$ we are done by Lemma 5.3.7. If $\chi_n \neq \nu^{-(\frac{n-3}{2})}$ look at j.

Case 1: If $j \neq 1$ then $\chi_1 = \nu^{\frac{n-3}{2}}$ and we look at $\xi^\vee = \chi_n^{-1} \times ... \times \chi_1^{-1}$. Put $\rho = \chi_n^{-1} \times ... \times \chi_2^{-1}$ and apply Mackey theory to $\xi^\vee = \rho \times \chi_1^{-1}$. Then $\rho^{(1)}$ is glued from representations of the form $\lambda_1 \times ... \times \lambda_{n-2}$ where either $\lambda_1 = \chi_n^{-1}$ or $\chi_{n-1}^{-1} \times ... \times \chi_2^{-1}$. Since $\chi_n \neq \nu^{-(\frac{n-3}{2})}$, $[\lambda_1, ..., \lambda_{n-2}] \neq T_{n-2}$ for the first type of factor. Therefore, (5.1.3) holds for $\xi^\vee$ only if $[\chi_2, ..., \chi_{n-1}] = T_{n-2}$. In this case, choose $i_0 = 1$ and $j_0 = n$ to get $[\chi_1, ..., \chi_n] \setminus \{\chi_{i_0}, \chi_{j_0}\} = T_{n-2}$ and apply Lemma 5.3.7 (a) to ξ. Otherwise, (5.1.3) and (5.1.1) does not hold for $\xi^\vee$. By induction, (5.1.2) holds for $\xi^\vee$, $0 \neq d_{\xi^\vee} \leq 2$ and $d_{\xi^\vee} = 2$ if and only if $\xi^\vee = \xi_n(k)$ for some k.

Case 2: If $j = 1$, $[\chi_2, ..., \chi_{n-1}] = T_{n-2}$. We need only consider the case when $\chi_1 \in \{\nu^{\frac{n-1}{2}}, \nu^{\frac{n-5}{2}}\}$ and $\chi_n \in \{\nu^{-(\frac{n-1}{2})}, \nu^{-(\frac{n-5}{2})}\}$(Otherwise either χ_1 is not linked to $\nu^{\frac{n-3}{2}}$ or χ_n is not linked to $\nu^{-(\frac{n-3}{2})}$ and we can reduce to Lemma 5.3.7). Among these, if $\chi_1 = \nu^{\frac{n-1}{2}}$ and $\chi_n = \nu^{-(\frac{n-1}{2})}$, ξ has the unique irreducible quotient $\mathbb{1}_n$ and by Theorem 1.1.2, has no other GL_{n-1}-distinguished irreducible subquotient whence $d_\xi = 1$. So now assume that either $\chi_1 \neq \nu^{\frac{n-1}{2}}$ or $\chi_n \neq \nu^{-(\frac{n-1}{2})}$. Say $\chi_n = \nu^{-(\frac{n-5}{2})}$. It is easy to see that $d_\xi \neq 0$ since it has the GL_{n-1}-distinguished quotient $\chi_1 \times \nu^{-(\frac{n-5}{2})} \times \mathbb{1}_{n-2} \in \mathrm{Alg(GL_n)}$. We claim that $d_\xi = 1$. If possible, let $d_\xi > 1$. By (5.1.4) and Theorem 3.3.1, $\eta = \xi\nu^{\frac{1}{2}} \times \nu^{-(\frac{n-2}{2})} \in \mathrm{Alg(GL_{n+1})}$ has $d_\eta > 1$. By the same argument $\zeta = \nu^{\frac{n-1}{2}} \times \eta\nu^{-\frac{1}{2}} \in \mathrm{Alg(GL_{n+2})}$ has $d_\zeta > 1$. Our representation ζ is then

$$\zeta = \nu^{\frac{n-1}{2}} \times \chi_1 \times \nu^{\frac{n-3}{2}} \times ... \times \nu^{-(\frac{n-3}{2})} \times \nu^{-(\frac{n-5}{2})} \times \nu^{-(\frac{n-1}{2})} \in \mathrm{Alg(GL_{n+2})}$$

which we think of as $\mu_1 \times \mu_2 \times ... \times \mu_{n+1} \times \mu_{n+2}$ with $\mu_2 = \chi_1$ and $\mu_{n+1} = \chi_n = \nu^{-(\frac{n-5}{2})}$. If $\chi_1 = \nu^{\frac{n-1}{2}} = \nu^{\frac{[n+2]-3}{2}}$ we may choose $i_0 = 1, j_0 = n$ and apply Lemma 5.3.7 (a) to ζ to get $d_\zeta = 1$, a contradiction. If $\chi_1 = \nu^{\frac{n-5}{2}}$, we can choose $i_0 = 2$ and $j_0 = n$ and apply Lemma 5.3.6 to ζ. Then $d_\zeta = 2$ if and only if $\zeta = \xi_{n+2}(k)$ for some $k \in \{1, ..., n+1\}$. But our ζ is not equal to any $\xi_{n+2}(k)$, again showing $d_\zeta = 1$, a contradiction. □

Remark 5.3.1. We have just counted the number of principal series representations $\xi = \chi_1 \times ... \times \chi_n$ of GL_n which have $d_\xi = 2$. For $n = 1$ there are none. If $n = 2$ there is only one namely $\nu^{-\frac{1}{2}} \times \nu^{\frac{1}{2}}$. If $n \geq 3$ they are $\xi_n(k)$ for $k = 1, ..., n-1$ in Theorem 5.3.1 and therefore there are precisely $n-1$ of them for any n.

Remark 5.3.2. Among the representations $\xi_n(k)$ in Theorem 5.3.1, $\xi_n(1)$ has two irreducible quotients both of which are GL_{n-1}-distinguished, namely $\pi_1 = \mathrm{ind}_{\mathrm{P_{1,n-1}}}^{\mathrm{GL_n}}(\nu^{\frac{n-3}{2}} \otimes \nu^{\frac{1}{2}})$ and $\pi_2 = \mathrm{ind}_{\mathrm{P_{2,n-2}}}^{\mathrm{GL_n}}(\mathrm{St}_2\nu^{\frac{n-2}{2}} \otimes \mathbb{1}_{n-2})$. Since $\xi_n(n-1) = \xi_1^\vee$, the irreducible $\widetilde{\pi_1}$ and $\widetilde{\pi_2}$ are quotients of ξ_{n-1} and they account for $d_{\xi_n(n-1)} = 2$, each one contributing 1. This was the point of Example 5.2.1 in the case $n = 3$. It is not difficult to see that the representations $\xi_n(k)$ in Theorem 5.3.1 for $2 \leq k \leq n-2$ has the GL_{n-1}-distinguished irreducible $\mathrm{ind}_{\mathrm{P_{2,n-2}}}^{\mathrm{GL_n}}(\mathrm{St}_2\nu^{\frac{n-2k}{2}} \otimes \mathbb{1}_{n-2})$ as a subquotient appearing with multiplicity 2 in $\mathrm{JH^0(\xi_k)}$. In contrast,

$$\xi = \nu^{\frac{n-3}{2}} \times ... \times \nu^{\frac{n-2k+1}{2}} \times \nu^{\frac{n-2k+1}{2}} \times \nu^{\frac{n-2k-1}{2}} \times \nu^{\frac{n-2k-1}{2}} \times ... \times \nu^{-(\frac{n-3}{2})} \in \mathrm{Alg(GL_n)}$$

also has $\mathrm{ind}_{\mathrm{P_{2,n-2}}}^{\mathrm{GL_n}}(\mathrm{St}_2\nu^{\frac{n-2k}{2}} \otimes \mathbb{1}_{n-2})$ appearing with multiplicity 2 in $\mathrm{JH}^0(\xi)$ but by Theorem 5.3.1, $d_\xi = 1$.

We next consider the case when there is a supercuspidal of GL_2 in the support.

Proposition 5.3.8. *Let $\sigma \in \mathrm{Irr}(\mathrm{GL}_2)$ be supercuspidal, $\chi_i, 1 \leq i \leq n-2$ characters of GL_1 and $\xi = \mathrm{ind}_{\mathrm{P}_{2,1,\ldots,1}}^{\mathrm{GL_n}}(\sigma \otimes \chi_1 \otimes \ldots \otimes \chi_{\mathrm{n}-2})$. Then $d_\xi \leq 1$ and $d_\xi = 1$ if and only if $[\chi_1, \ldots, \chi_{n-2}] = T_{n-2}$.*

Proof. Our proof is by induction on n. The result is true for $n = 3$ by Theorem 3.4.2 and (3.3.6). For $n \geq 4$ put $\rho = \sigma \times \chi_1 \times \ldots \times \chi_{n-3}$ so that $\xi = \rho \times \chi_{n-2}$. Apply Mackey theory with $k = n-1$. It is easy to see that (3.3.1) and (3.3.3) does not hold for ξ. On $\mathcal{C}_2$, (3.3.2) holds if and only if $\rho\nu^{-\frac{1}{2}}$ is GL_{n-2}-distinguished, which by induction yields $[\chi_1, \ldots, \chi_{n-3}] = [\nu^{\frac{n-3}{2}}, \ldots, \nu^{-(\frac{n-5}{2})}]$. (3.3.2)(ii) holds if and only if $\chi_{n-2} = \nu^{-(\frac{n-3}{2})}$. This means that $d_\xi \neq 0$ if and only if $[\chi_1, \ldots, \chi_{n-2}] = T_{n-2}$. We apply (5.1.4) and it follows by induction that $d_\xi \leq 1$. □

We have the following Corollary which is now an easy consequence of Proposition 5.3.8 and Theorem 5.3.1.

Corollary 5.3.9. *Let $n = n_1 + \ldots + n_r$, $\rho_i \in \mathrm{Irr}(\mathrm{GL}_{\mathrm{n_i}})$ and $\xi = \mathrm{ind}_{\mathrm{P}_{\mathrm{n}_1,\ldots,\mathrm{n}_\mathrm{r}}}^{\mathrm{GL_n}}(\rho_1 \otimes \ldots \otimes \rho_\mathrm{r})$. Then $d_\xi \leq 2$.*

We give an application based on Theorems 1.1.2 and 5.3.1.

Example 5.3.3. Let $n \geq 4$ and $\pi = \mathrm{ind}_{\mathrm{P}_{1,\mathrm{n}-2,1}}^{\mathrm{GL_n}}(\nu^{-(\frac{\mathrm{n}-1}{2})} \otimes \mathbb{1}_{\mathrm{n}-2} \otimes \nu^{\frac{\mathrm{n}-1}{2}})$. One may apply Mackey theory with $k = n-1$ to π by choosing $\rho = \mathrm{ind}_{\mathrm{P}_{1,\mathrm{n}-2}}^{\mathrm{GL}_{\mathrm{n}-1}}(\nu^{-(\frac{\mathrm{n}-1}{2})} \otimes \mathbb{1}_{\mathrm{n}-2})$ and $\tau = \nu^{\frac{n-1}{2}}$. Since ρ has the unique irreducible quotient $\widetilde{L_{n-1}}\nu^{\frac{1}{2}}$ and $\chi \neq \nu^{-\frac{n-3}{2}}$ we can see that (3.3.1) and (3.3.2) fail. Since $\rho^{(1)}$ has the quotient $(\widetilde{L_{n-1}}\nu^{1/2})^{(1)}$ which by Lemma 4.1.2 has $\mathbb{1}_{n-2}$ as a quotient the condition (3.3.3) holds. We claim that π is GL_{n-1}-distinguished. The crucial point is that π does not have any quotient which is GL_{n-1}-distinguished but has the unique irreducible submodule $\mathbb{1}_n$. The aim of the example is to emphasize that we can avoid the extension problem by appealing to Theorem 1.1.2 and 5.3.1. Put

$$\xi = \nu^{-(\frac{n-1}{2})} \times \nu^{\frac{n-3}{2}} \times \ldots \times \nu^{-(\frac{n-3}{2})} \times \nu^{\frac{n-1}{2}} \in \mathrm{Alg}(\mathrm{GL_n}).$$

Then π is a quotient of ξ. By Theorem 1.1.2 the only irreducible subquotient of ξ which is GL_{n-1}-distinguished is $\mathbb{1}_n$ and $\mathbb{1}_n$ occurs with multiplicity one in $\mathrm{JH}^0(\xi)$. By Theorem 5.3.1 ξ is GL_{n-1}-distinguished and has $d_\xi = 1$ whence the GL_{n-1}-invariant form on ξ must come from $\mathbb{1}_n$. Since $\mathbb{1}_n$ is a subquotient of π we conclude that π must be GL_{n-1}-distinguished.

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