SOME RESULTS IN LINEAR, SYMBOLIC AND GENERAL TOPOLOGICAL DYNAMICS

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CERTIFICATE

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This is to certify that I, K. Ali Akbar, have carried out the research embodied in the present thesis entitled **SOME RESULTS IN LINEAR, SYMBOLIC AND GENERAL TOPOLOGICAL DYNAMICS** for the full period prescribed under Ph.D. ordinance of the University.

I declare that, to the best of my knowledge, no part of this thesis was earlier submitted for the award of research degree of any university.

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Dedicated to my parents

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- Ali Akbar K

Contents

List of Symbols

$\mathbb{C}$	The set of all complex numbers
$\mathbb{R}$	The set of all real numbers
$\mathbb{R}^+$	The set of all non-negative real numbers
$\mathbb{Q}$	The set of all rational numbers
$\mathbb{Z}$	The set of all integers
$\mathbb{N}$	The set of all natural numbers
$\mathbb{N}_0$	The set of all non-negative integers
$\subset\subset$	Finite subset
$\tilde{A}$	The smallest subset of $\mathbb{N}$ containing A and closed under the binary operation lcm whenever $A \subset \mathbb{N}$
$\bar{A}$	The closure of A whenever it is a subset of a topological space
$\bar{w}$	The infinite sequence defined $(\bar{w})_n = w_r$ whenever $n \equiv r(mod\ k)$ and w is a word of length k over an alphabet
$\bigcup$	Union
$\biguplus$	Disjoint union
$\mathfrak{F}_n$	$\{\{1\} \cup \tilde{A} : A \subset \mathbb{N}\ and\ \lvert A\rvert \leq \frac{n}{2}\} \cup \{\{1\} \cup \tilde{A} : A \subset \mathbb{N} \setminus \{1\}, 2 \in A\ and\ \lvert A\rvert = \frac{n+1}{2}\}$
I	A compact subinterval of $\mathbb{R}$
A^c	The complement of the set A
$\lvert A\rvert$	The cardinality of the set A
$P(f)$	The set of all periodic points of f
$Fiorb(f)$	$\{(m,n) \subset \mathbb{N}_0 \times \mathbb{N} :$ there exists $x \in X$ of type $(m,n)\}$
$Per(f)$	$\{n \in \mathbb{N}$ such that f has a point of period $n\}$
$Fix(f)$	The set of all fixed points of f

$\mathcal{PER}(X)$	$\{A \subset \mathbb{N} : A = Per(f)$ for some continuous self map f of $X\}$
$Det(A)$	The determinant of A
$\vee$	Least common multiple
gcd	Greatest common divisor
Max	Maximum
Min	Minimum
$\triangle$	Symmetric difference of sets
$\setminus$	Difference of sets
$N(f)$	The set of all non-ordinary points of f
$S(f)$	The set of all special points of f
G_f	The set of all self-conjugacies of f
$G_{f\uparrow}$	The set of all increasing self-conjugacies of f
S^1	The unite circle in the complex plane
$\mathbb{T}^2$	The two dimensional torus
T_A	The toral automorphism induced by the matrix A
$GL(2,\mathbb{Z})$	$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = \pm 1 \right\}$

.

Abstract

The thesis is conveniently divided into five chapters and an appendix. Among the chapters, one is on linear dynamics, one is on symbolic dynamics and other three are on general topological dynamical systems. Some of the main results of this thesis are combinatorial in nature; especially Chapter 2, some parts of Chapter 3 and some parts of Chapter 5. In the Appendix, we prove some more results on periods and orbits. The general mathematical setting is that of an abstract dynamical system with discrete time parameter, that is, a pair (X, f) where X is a topological space and f is a continuous mapping of X into itself. In some parts, we also consider non-continuous self maps. We are interested in the action of the iterates of f on X.

Chapter 1 is a general introduction. We explain the basic notions and examples of discrete dynamical systems, and some important results emphasizing the role of the set of periodic points and the set of periods in the theory of chaos. We discuss briefly about the definitions of chaos due to Devaney and Li-Yorke.

Chapter 2 is mainly devoted to the solution of a single problem: Determining the set of periods of a linear operator on a Hilbert space.

The main results are

(1) The family of period sets of linear operators on $\mathbb{C}^n$ is $\mathfrak{F}_{2n}$.

(2) The family of period sets of linear operators on $\mathbb{R}^n$ is $\mathfrak{F}_n$.

(3) The family of period sets of linear operators on l^2 having finite rank is $\bigcup_n \mathfrak{F}_n$.

(4) The family of period sets of isometries on l^2 is $\{A \subset \mathbb{N} : A = \tilde{A}$ and $1 \in A\}$.

(5) The family of period sets of linear operators on a vector space is $\{A \subset \mathbb{N} : A = \tilde{A}$ and $1 \in A\}$.

(6) There exists a Banach space X such that for every linear operator T on X, $Per(T)$ is very small.

In Chapter 3, we study subshifts as topological dynamical systems.

Some main results are

(1) The period set of an SFT (or a sofic shift) is a non-empty set of the form F $\mathbb{N}\Delta G$ where F and G are finite subsets of $\mathbb{N}$.

(2) The period set of a transitive SFT (or a transitive sofic shift) is either a singleton subset of $\mathbb{N}$ or a set of the form $k\mathbb{N} \setminus F$ for some finite subset F of $\mathbb{N}$ and for some $k \in \mathbb{N}$.

(3) The period set of a mixing SFT is a set of the form $\mathbb{N} \setminus F$ for some finite subset F of $\mathbb{N}$.

(4) For any $S \subset \mathbb{N}$, there exists a subshift X such that $Per(X) = S$.

(5) Let $(A^{\mathbb{Z}}, F)$ be any CA. Then $Fix(F)$ is an SFT or empty set. Conversely given any SFT X there exists a CA F such that $Fix(F) = X$.

(6) The following are equivalent for a non-singleton SFT X.

(i) X is transitive and $Per(X)$ is cofinite.

(ii) X is weak mixing.

(iii) X is mixing.

; and we prove that the above equivalence need not be true in the case of sofic shifts.

In Chapter 4, we study the transitive property of a continuous 2-dimensional toral automorphism and charaterize the set of all 2×2 integer matrices with determinant 1 and trace ± 2.

Some main results are

(1) There exist uncountably many non-empty open connected T-invariant sets,

(2) There exists a Lyapunov function for T, and

(3) The topological entropy, $h(T) = 0$,

where T is a non-hyperbolic toral automorphism.

In Chapter 5, we discuss the notion of *special points* and *non-ordinary points* of a dynamical system. These notions are relatively new to the literature, though they arise very naturally. By a special point we mean a point in the system which is unique by possessing some dynamical property. We call a point to be *ordinary* if points like it form a neighbourhood of it. The points which are not ordinary are called *non-ordinary.* It is observed that for systems with finitely many non-ordinary points, the idea of non-ordinary points and the idea of special points coincide. We call a system $(\mathbb{R}, f)$ to be *simple* if there are only finitely many kinds of orbits (upto order conjugacy). We classify completely, the class of simple systems. In particular, we prove that there are exactly 90 continuous self maps on $\mathbb{R}$ with exactly two non-ordinary points.

In the Appendix, we prove that

(1) There are only finitely many different kinds of finite orbits for every linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$.

(2) The family of period sets of endomorphisms on a torsion free abelian group is $\{A \subset \mathbb{N} : A = \tilde{A} \text{ and } 1 \in A\}$.

(3) For every sequence $(a_n) \in \mathbb{N} \cup \{0, \infty\}$ there exits a continuous map $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $Fiorb(f) = \bigcup_{n=1}^{\infty} \bigcup_{k \in K_{a_n}} \{(k, n)\}$ where $K_{a_n} = \{k \in \mathbb{N}_0 : k \leq a_n\}$.

(4) For the class, {linear operators on $\mathbb{R}^n$}, for all $n \in \mathbb{N}$, for every member f in this class the set $Fiorb(f)$ is finite.

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Chapter 1

Topological Dynamics: An Introduction

Topological dynamics is the study of the asymptotic behavior of the orbits of continuous self maps of topological spaces. The modern theory of dynamical systems originated at the end of the 19th century with fundamental questions concerning the stability and evolution of the solar system. Attempts to answer those questions led to the development of a rich and powerful field with applications to physics, biology, meteorology, astronomy, economics, and other areas. A number of themes that appear repeatedly in the study of dynamical systems are properties of individual orbits, periodic orbits, typical behavior of orbits, and chaotic behavior etc.

The purpose of this chapter is to provide a general introduction to the theory of topological dynamical systems, suitable for remaining chapters. We introduce some of the principal themes of topological dynamical systems both through examples and by explaining it.

1.1 Basic notions

1.1.1 Dynamical systems and Examples

A *topological dynamical system* (We call simply dynamical system) is a pair (X, f), where X is a topological space and $f : X \to X$ is a continuous self map of X. The space X can be thought of as the underlying set on which the motion takes place and f can be thought of as the rule according to which motion takes place. The continuous self maps of the interval $[0, 1]$ are called interval maps.

Let (X, f) be a dynamical system. For each positive integer n, we define the function f^n now. It is recursively defined by $f^1 = f$ and $f^{n+1} = f \circ f^n$ for all $n \in \mathbb{N}$ where $\circ$ denotes composition of functions. We also use the convention $f^0 =$ the identity function. It is easily seen that $f^m \circ f^n = f^{m+n}$ holds for all non-negative integers m, n.

For $x \in X$ the point $f(x) \in X$ is thought of as the position to which x moves (in one unit of time). If $f(x) = x$, we say that x is *fixed* (does not move), the element $f^2(x)$ is called the position to which x moves after two instants of time in the dynamical system (X, f). In general, if n is a positive integer then the element $f^n(x)$ is thought of as the position to which x moves after n instants of time. Thus in our study "time" is "discrete" and parametrised by the set $\mathbb{N}$ of natural numbers (the space X need not be discrete). This is the reason that it is called a *"discrete" dynamical system.*

The following examples will be frequently discussed in the thesis.

Example 1.1.1. (Tent map) Let $T : [0, 1] \to [0, 1]$ be the tent map: $T(x) = 2x$ for $0 \leq x \leq \frac{1}{2}$ and $T(x) = 2(1 - x)$ for $\frac{1}{2} < x \leq 1$. Then $([0, 1], T)$ is a dynamical system. There are two fixed points 0 and $\frac{2}{3}$.

Example 1.1.2. (Rotation) Let S^1 be the unit circle in the complex plane. i.e., $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, with complex multiplication as the group operation. Let

$a \in S^1$ be fixed. Let $\rho_a : S^1 \to S^1$ be defined by $\rho(z) = az$ for all $z \in S^1$. Then (S^1, ρ_a) is a dynamical system. Such maps ρ_a are called *rotations*. They describe motions on the circle with constant velocity. When $a \neq 1$, there are no fixed points.

We can also describe the circle as the set $S^1 = [0, 1]/\sim$, where $\sim$ indicates that 0 and 1 are identified. Addition *mod* 1 makes S^1 an abelian group. The natural distance on $[0, 1]$ induces a distance on S^1; specifically, $d(x, y) = min(|x - y|, 1 - |x - y|)$. For $\alpha \in \mathbb{R}$, let R_α be the rotation of S^1 by angle $2\pi\alpha$, i.e., $R_\alpha x = x + \alpha (mod\ 1)$. Then (S^1, R_α) is a dynamical system.

The two notations are related by $z = e^{2\pi i x}$, which is an isometry if we divide arc length on the multiplicative circle by 2π.

Example 1.1.3. (Logistic function) Let $r > 0$. The logistic function h_r on $[0, 1]$ is defined by the formula $h_r(x) = rx(1-x)$. One can prove that when $0 < r \leq 4$, h_r takes $[0, 1]$ into $[0, 1]$ and hence $([0, 1], h_r)$ is a dynamical system. When $r = 4$, the point $\frac{1}{2}$ goes to 0 after two instants of time. There are two fixed points 0 and $\frac{r-1}{r}$. But when $r > 4$, $([0, 1], h_r)$ is not a dynamical system.

Basic definitions

Let (X, f) be a dynamical system, and $x \in X$.

- The sequence x, $f(x)$, $f^2(x)$,..., $f^n(x)$,... is called the *forward f-trajectory* of x.
- The set $\{f^n(x) : n = 0, 1, 2, ...\}$ is called the *forward f-orbit* of x.
- x is a *periodic point* of f of period $p \in \mathbb{N}$ if $f^p(x) = x$ and $f^m(x) \neq x$ for all $1 \leq m \leq p - 1$. The set $\{x, f(x), f^2(x), \ldots, f^{p-1}(x)\}$ is called a *p-cycle*. Observe that periodic points of period 1 are exactly the *fixed* points.

- x is a *recurrent point* of f if there exists an increasing sequence $< n_k >$ of positive integers such that the sequence $< f^{n_k}(x) >$ converges to x as k tends to ∞.

- x is a *non-wandering point* of f if for every open set U containing x there exists $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \emptyset$.

- x is an *eventually periodic point* or *preperiodic point* of f if there exists $n \in \mathbb{N}$ such that $f^n(x)$ is a periodic point of f. Note that all eventually periodic points will reach a periodic point after finitely many iterations. The points which will reach a fixed point after finitely many iterations are called *eventually fixed points* .

- We say that f is *transitive* if for any two non-empty open sets $U, V \subset X$, there exists $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$.

- We say that f is *totally transitive* if f^n is transitive for all $n \in \mathbb{N}$.

- We say that f is *weak mixing* if $f \times f$ is transitive on $X \times X$.

- We say that f is *mixing*, if for any two non-empty open sets $U, V \subset X$, there exists $k \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq k$.

- A subset A of X is said to be *invariant* if $f(A) \subset A$. When A is an invariant set, $(A, f|_A)$ also becomes a dynamical system.

In dynamical systems, we are normally interested in studying the nature of the orbits of distinct points of the system. Suppose for some $x \in X$, the sequence $x, f(x), f^2(x), ...$ converges to some point say $x_0 \in X$, then we must have $f(x_0) = x_0$, because f is continuous. In dynamics we say that the point x is attracted by the fixed point x_0. The set of all points in X *attracted* by x_0 is called the *stable set* or the *basin of attraction* of the fixed point x_0 and is denoted by $W^s(x_0, f)$ or simply $W^s(x_0)$. A fixed point x_0 is said to be *attracting* if its stable set is a neighbourhood of it. Generalizing the notion of an attracting fixed point, we say that a compact set $C \subset X$ is an attractor if there is an open set U containing C, such that $f(\overline{U}) \subset U$ and $C = \bigcap_{n \geq 0} f^n(U)$. It follows that $f(C) = C$.

We denote the set of fixed, periodic, recurrent, non-wandering and preperiodic points of f by $Fix(f)$, $P(f)$, $R(f)$, $\Omega(f)$ and $E(f)$, respectively. Observe that, $Fix(f) \subseteq P(f) \subseteq R(f) \subseteq \Omega(f)$, $P(f) \subseteq E(f)$, and $P(f) = E(f) \cap R(f)$.

Let $Per(f)$= $\{n \in \mathbb{N}$ such that f has a point of period $n\}$ and we call this as the *set of periods* of f or simply the *period set* of f.

Example 1.1.4. 1. When f is the identity map or a constant map, all points are mapped to fixed points and hence $Per(f) = \{1\}$.

2. When f is the reflection map $x \mapsto -x$ on $\mathbb{R}$, the point 0 is fixed and all other points are periodic with period 2. Hence $Per(f) = \{1, 2\}$.

3. When f is the translation map $x \mapsto x + 1$ on $\mathbb{R}$, then $Per(f)$ is the empty set. Since every orbit is strictly monotone.

4. On the unit circle, if we consider the rotation by the angle $\frac{2\pi}{3}$, all points are periodic with period 3. In this case $Per(f) = \{3\}$.

5. For the shift map (defined in the last section of this chapter) on $\sum_2 = \{0, 1\}^{\mathbb{N}}$ the periodic points are of the form $x = \overline{w} = www\ldots$ for some word w over $\{0, 1\}$ and hence $Per(f) = \mathbb{N}$.

We now in the following proposition state ten important results that are easy to prove.

Proposition 1.1.5. *Let (X, f) be a dynamical system, where X is a Hausdorff space. Then the following hold:*

1. $\{x \in X : f^n(x) = x\}$ *is a closed subset of* X *for all* $n \in \mathbb{N}$*. In particular, the set of all fixed points is a closed subset of* X*.*

2. In any trajectory, either all terms are distinct, or only finitely many terms are distinct.

3. Orbits of any two periodic points are either identical or disjoint.

4. If a trajectory converges, it converges to a fixed point.

5. An element is eventually periodic if and only if it has a finite orbit.

6. Every orbit is an invariant set; the orbit of a periodic point is an invariant set, and it has no non-empty proper invariant subset.

7. A subset of X *is invariant if and only if it is a union of orbits.*

8. The closure of an invariant set is also invariant.

9. The set of all periodic points is an invariant set.

10. For each subset A *of* X*, the set* $\bigcup_{n=0}^{\infty} f^n(A)$ *is the smallest invariant set containing* A*.*

The following theorem gives equivalent conditions for transitivity.

Theorem 1.1.6. *[1], [15], [33] Let* (X, f) *be a dynamical system. Then the following are equivalent.*

(1) f *is transitive.*

(2) *for every non-empty open set* $U \subset X$, $\bigcup_{n=0}^{\infty} f^n(U)$ *is dense in* X*.*

(3) *for every pair of non-empty open sets* U *and* V *in* X*, there is a positive integer* n *such that* $f^{-n}(U) \cap V \neq \emptyset$*.*

(4) *for every non-empty open set* $U \subset X$, $\bigcup_{n=1}^{\infty} f^n(U)$ *is dense in* X*.*

(5) *if* $E \subset X$ *is closed and* $f(E) \subset E$*, then* $E = X$ *or* E *is nowhere dense in* X*.*

(6) *if* $U \subset X$ *is open and* $f^{-1}(U) \subset U$*, then* $U = \emptyset$ *or* U *is dense in* X*.*

The following theorem brings out the relation between transitivity and denseness of forward orbit of a point.

Theorem 1.1.7. *[18] (Birkhoff's transitivity theorem) A continuous map f of a complete separable metric space X without isolated points has a dense forward orbit if and only if for every pair U, V of non-empty open subsets of X there is a non-negative integer n such that $f^n(U) \cap V \neq \emptyset$.*

The following proposition links the transitivity and weak mixing.

Proposition 1.1.8. *[26] Let (X, f) be a dynamical system. If f is weak mixing then f^2 is transitive.*

Proof. Let U, V be non-empty open sets in X. Consider the open sets $U \times U$, $V \times f^{-1}(V)$ in $X \times X$. Choose $n > 1$ such that $(f \times f)^{-n}(U \times U) \cap (V \times f^{-1}(V)) \neq \emptyset$. Which implies $f^{-n}(U) \cap V$ and $f^{-(n-1)}(U) \cap V$ are non-empty. Hence f^2 is transitive since either n or $n-1$ is even. □

Remark 1.1.9. The implications f is mixing $\implies$ f is weak mixing $\implies$ f is transitive easily follows in a dynamical system (X, f) because of Proposition 1.1.8.

The following theorem gives some equivalent statements regarding transitivity for continuous interval maps (see Chapter 3 for other equivalent statements).

Theorem 1.1.10. *[1], [15], [21] For the system $([0, 1], f)$, the following are equivalent.*

(1) *f^2 is transitive.*

(2) *f^n is transitive for every $n > 0$.*

(3) *f is transitive and has a periodic point of period $2n + 1$ for some $n \in \mathbb{N}$.*

(4) *f is mixing.*

(5) *f^n is mixing for all $n > 0$.*

The following theorem says that any continuous interval map can be decomposed into transitive components.

Theorem 1.1.11. *[43] (Blokh, Barge-Martin)*

Let $f : I \to I$ be an interval map such that the periodic points are dense in I. Then the interval I decomposes into transitive components C_n in the following way.

(1) C_n is a closed non-degenerate interval or C_n is the union of two disjoint closed non-degenerate intervals,

(2) $f|_{C_n}$ is transitive,

(3) the complement set of $\bigcup C_n$ is included in $\{x \in X : f^2(x) = x\}$.

In addition, the number of transitive components C_n is finite or countable and their interiors are pairwise disjoint.

1.2 Period set and Chaos

We ask:

Q_1**:** Given $A \subset \mathbb{N}$, *Can we find* a *continuous map* f *from* $\mathbb{R}$ *to* $\mathbb{R}$ *such that* $Per(f) = A$?

Q_2**:** Given $A \subset \mathbb{N}$, *Can we find a continuous toral automorphism* T *from* $\mathbb{T}^2$ *to* $\mathbb{T}^2$ *such that* $Per(T) = A$?

We first consider the question Q_1.

Theorem 1.2.1. ***(Li and Yorke)***

Let f be a continuous map from $\mathbb{R}$ to $\mathbb{R}$. If $3 \in Per(f)$, then $Per(f) = \mathbb{N}$.

This theorem is difficult to prove. But here is an easy observation. If in the dynamical system $(\mathbb{R}, f)$ there are two points x and y such that $x < f(x)$ and $f(y) < y$, then there exists z between x and y such that $f(z) = z$. This is proved by applying intermediate value theorem to the function $f(x) - x$. This implies that if $1 \notin Per(f)$, then the motion is uni-directional and so no point can be periodic. In other words, $Per(f) \neq \emptyset \implies 1 \in Per(f)$.

This elementary result, in combination with the result of Li and Yorke, exhibit the numbers 3 and 1 in the two extremes of an order. If a 3-cycle is there, all n-cycles have to be there. If no 1-cycle is there then no n-cycles can be there.

This leads to a search of pairs of positive integers (m, n) such that if an m-cycle is there, n-cycle has to be there. What are all such pairs?

Sharkovskii's theorem provides a complete answer to question Q_1. We discuss this in Chapter 2 in detail.

Next consider the question Q_2. It is easy to see that the number 1 should belong to such a set. In [31], we give a complete answer to the question Q_2, and we mentioned this result in Chapter 2.

In the next section, we explain the importance of the set of periods and periodic points in the theory of chaos.

1.2.1 Chaos

The presence or the lack of chaotic behaviour is one of the most prominent traits of a dynamical system. Through there are many definitions for chaos in the literature on dynamical systems, we discuss mainly two notions namely Li-Yorke chaos and Devaney chaos.

The expression "chaos" became popular through the paper of Li and Yorke [36], "Period three implies chaos". This notion is commonly known as Li-Yorke chaos, for interval maps. We believe this was the first definition for chaos. This notion can be extended to any metric space with a small modification. Later, Devaney introduced his notion of chaos in [23] known as Devaney chaos. Positive topological entropy is another definition of chaos (see [50]). It was shown that both positive topological entropy and Devaney chaos imply Li-Yorke chaos. Thus, in a certain sense Li-Yorke chaos is the weakest notion of chaos. Chaotic systems share the property of having a high degree of sensitivity to initial conditions. In other words, a very small change in initial values will multiply in such a way that the new computed system bears no resemblance to the one predicted.

Definition 1.2.2. We say that the system (X, f) (where X is a metric space with metric d) is *sensitive to the initial conditions* if there exists $\delta > 0$ such that for any $x \in X$ and for any $\epsilon > 0$ there exists a point $y \in X$ with $d(x, y) < \epsilon$ and $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) > \delta$. This $\delta > 0$ is called a *sensitivity constant*.

It has been recognized by Sharkovskii (see [48]), Li and Yorke (see [36]) and many others that there is a hidden, self-organizing order in chaotic systems. A certain degree of order in chaotic systems has led to various definitions of chaos in the literature.

1.2.2 Devaney's definition of chaos

Definition 1.2.3. According to Devaney a dynamical system (X, f) is said to be *chaotic* if

1. f is transitive.
2. f has a dense set of periodic points.
3. f is sensitive to the initial conditions.

Theorem 1.2.4. *[11]*

Let $f : X \to X$ be a continuous map on an infinite metric space (X, d). If f is transitive and its set of periodic points is dense, then f possesses sensitive dependence *on initial conditions, i.e., f is chaotic.*

Remark 1.2.5. Theorem 1.2.4 says that, when X is an infinite metric space, the conditions (1) and (2) in the above definition imply the condition (3). However, no other condition is implied by the other two.

Remark 1.2.6. In fact, for continuous maps on intervals in $\mathbb{R}$, transitivity implies that the set of periodic points is dense (see [55]). Hence it follows from Theorem 1.2.4 that in this case *transitivity implies chaos*. The period sets of an interval map and its transitivity property are closely related (see Chapter 2).

1.2.3 Li-Yorke chaos

In a dynamical system (X, f), let $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ be the respective orbits of two distinct points $x_0, y_0 \in X$. We ask two questions:

Q_1**:** Do x_n and y_n come arbitrarily close to each other?

Q_2**:** Do x_n and y_n keep a minimum positive distance from each other for infinitely many n?

We can generate examples which gives affirmative answer to one question and not to other (or to both questions). In the study of *scrambled sets* we are interested in pairs of points for which both the two questions have affirmative answers.

Definition 1.2.7. Let (X, f) be a dynamical system and let $S \subset X$ be a set with atleast two points. Then, S is a *scrambled set* for f if for any two distinct points $x, y \in S$

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$$

and

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0.$$

From the famous paper of Li and Yorke [36] there ensued a definition of chaos as follows:

Definition 1.2.8. A continuous self map of an interval is said to be Li-Yorke chaotic if it has an uncountable scrambled set.

Li and Yorke proved that the occurrence of a period-3 point forces chaos for interval maps.

Theorem 1.2.9. *If an interval map has a point of period three, then it is Li-Yorke chaotic.*

Here is a theorem connecting the set $Per(f)$ and Li-Yorke chaos.

Theorem 1.2.10. *[51] Let f be a self map of an interval I in $\mathbb{R}$. Then*

(i) If $Per(f)$ properly contains the set $\{1, 2, 2^2, 2^3, \ldots\}$, then f is Li-Yorke chaotic.

(ii) If $Per(f)$ is properly contained in $\{1, 2, 2^2, 2^3, \ldots\}$, then f is not Li-Yorke chaotic.

Corollary 1.2.11. *For any interval map f, Devaney's chaos implies Li-Yorke chaos.*

Remark 1.2.12. For an interval map f, Devaney's chaos implies a point of period six. And f has positive entropy if and only if $Per(f)$ properly contains $\{1, 2, 2^2, 2^3, \ldots\}$. Therefore positive entropy implies Li-Yorke chaos.

Remark 1.2.13. We cannot say anything if $Per(f) = \{1, 2, 2^2, 2^3, \ldots\}$. In [49] it was shown that there are continuous interval maps f, g such that $Per(f) = Per(g) = \{1, 2, 2^2, 2^3, \ldots\}$, f is Li-Yorke chaotic but g is not Li-Yorke chaotic.

Theorem 1.2.14. *[50] An interval map f fails to be Li-Yorke chaotic if and only if every orbit of f is approximable by periodic points; ie., given a point x and any $\epsilon > 0$, there exists a periodic point p such that $limsup_{n\to\infty}|f^n(x) - f^n(p)| < \epsilon$.*

1.3 Topological conjugacies

In order to classify dynamical systems we need a notion of equivalence. The notion of topological conjugacy in dynamical systems is analogous to the notion of "isomorphism" among groups and to "homeomorphisms" among topological spaces. i.e., we say that two dynamical systems are "having the same dynamical properties" or "dynamically same" if they are topologically conjugate.

Roughly speaking, by saying (X, f) and (Y, g) are topologically conjugate, we mean:

(1) X and Y have the same kind of topology.

(2) f and g have the same kind of dynamics.

Definition 1.3.1. Two dynamical systems (X, f) and (Y, g) are said to be *topologically conjugate* (or simply *conjugate*) if there exists a homeomorphism $h : X \to Y$ (called topological conjugacy) such that $h \circ f = g \circ h$. We say simply, f is conjugate to g, and we write it as $f \sim g$. The case when h happens to be an increasing homeomorphism (For example, when $X = \mathbb{R}$ or an interval) we say that f and g are *increasingly conjugate* or *order conjugate* .

Remark 1.3.2. When $Y = X$ and $g = f$, we say that h is a self conjugacy of f. Being conjugate (and as well as order conjugate) is an equivalence relation among dynamical systems.

Let (X, f) and (Y, g) be two dynamical systems. Then a topological conjugacy from f to g carries orbits of f to "similar " g-orbits. Said precisely,

Theorem 1.3.3. *[30]*

Let (X, f) and (Y, g) be two dynamical systems and let $h : X \to Y$ be a topological conjugacy from f to g. Then

1. $h^{-1} : Y \to X$ is a topological conjugacy from g to f.

2. $h \circ f^n = g^n \circ h$ for all $n \in \mathbb{N}$.

3. $x \in X$ is a periodic point of f of period p if and only if $h(x)$ is a periodic point of g of period p.

4. If x is a periodic point of f with stable set $W^s(x)$, then the stable set of $h(x)$ is $h(W^s(x))$.

5. $(x_0, x_1, ...)$ is an orbit of f then $(h(x_0), h(x_1), ...)$ is an orbit of g.

6. The periodic points of f are dense in X if and only if the periodic points of g are dense in Y.

7. f is transitive on X if and only if g is transitive on Y.

8. f is mixing on X if and only if g is mixing on Y.

9. f is weak mixing on X if and only if g is weak mixing on Y.

10. f is chaotic on X if and only if g is chaotic on Y.

1.3.1 Dynamical properties

Definition 1.3.4. Properties preserved by topological conjugacy are called *dynamical properties*. Many kinds of examples are provided below.

Example 1.3.5. (Dynamical properties of a point)

(a) Periodic point.

(b) Periodic point of period n_0 for some $n_0 \in \mathbb{N}$.

(c) Eventually fixed point.

(d) Point whose orbit has exactly n elements.

(e) Non-ordinary point (see Chapter 5).

Example 1.3.6. (Dynamical properties of subsets)

(a) Invariant subset.

(b) The property that $f(A) = A$.

(c) Dense set.

(d) Maximal dynamically independent set (see Chapter 5).

(e) Finite set.

Example 1.3.7. (Dynamical properties of dynamical systems)

(a) Having no fixed point.

(b) Having no invariant sets, except the whole set and the empty set.

(c) Having dense set of periodic points.

(d) Surjectivity.

(e) Transitivity.

1.3.2 The shift map- An example

Let $\sum_2 = \{s_0s_1s_2... : \; s_i = 0 \text{ or } 1 \text{ for all } i \in \mathbb{N}_0\}$ be the set of all sequences of 0s and 1s. If $s = s_0s_1s_2...$ and $t = t_0t_1t_2...$, then define $d(s,t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$. Note that $(\sum_2, d)$ is a metric space and $d(s,t) \leq 2$ for all $s, t \in \sum_2$. We will discuss more on shifts in Chapter 3.

The following proposition which can be proved easily, asserts that, if we start with any sequence from $\sum_2$, by keep on changing the initial terms we can go arbitrarily close to any point in the space $\sum_2$.

Proposition 1.3.8. *[30]*

Let $s, t \in \sum_2$.

1. *If the first* $n+1$ *digits in* s *and* t *are identical, then* $d(s,t) \leq \frac{1}{2^n}$.

2. *If* $d(s,t) \leq \frac{1}{2^n}$, *then the first* n *digits in* s *and* t *are identical.*

Definition 1.3.9. The shift map $\sigma : \sum_2 \to \sum_2$ is defined by,

$$\sigma(s_0 s_1 s_2....) = s_1 s_2 s_3 \ldots .$$

In other words, the shift map forgets the first term.

It follows from the above proposition that the shift map is continuous.

Theorem 1.3.10. *[30]*

The shift map has the following properties.

1. *The set of periodic points of the shift map is dense in* $\sum_2$.

2. *The shift map has* 2^n *periodic points whose period divides* n.

3. *The set of eventually periodic points of the shift map that are not periodic is dense in* $\sum_2$.

4. *There is an element of* $\sum_2$ *whose orbit is dense in* $\sum_2$.

5. *The set of points that are neither periodic nor eventually periodic is dense in* $\sum_2$.

Proof. 1. Suppose $s = s_0 s_1 s_2....$ is a periodic point of σ with period k. Then $\sigma^k(s) = s$ and hence $\sigma^n(\sigma^k(s)) = \sigma^n(s)$ for all $n \in \mathbb{N}$. This implies that $s_{n+k} = s_n$ for all n. That is s is a periodic point with period k if and only if s is a sequence formed by repeating the k-digits $s_0 s_1 s_2 ... s_{k-1}$ infinitely often.

To prove that the periodic points of σ are dense in $\sum_2$, we must show that for all points $t \in \sum_2$ and all $\epsilon > 0$, there is a periodic point s of σ such that $d(t,s) < \epsilon$. Let $t \in \sum_2$. If $t = t_0 t_1 t_2...$ then choose n such that $\frac{1}{2^n} < \epsilon$. Let $s = t_0 t_1 t_2 ... t_n t_0 t_1 t_2 ... t_n t_0 t_1 t_2 ... t_n ...$. As t and s agree on the first $n+1$ digits, by Proposition 1.3.8, we get $d(s,t) \leq \frac{1}{2^n} < \epsilon$.

Proof of 2 is easy and 3 is similar to that of 1.

4. The sequence which begins with 0 1 00 01 10 11 and then includes all possible blocks of 0 and 1 with three digits, followed by all possible blocks of 0 and 1 with four digits, and so forth - called the *Morse sequence* - has dense orbit since it contains blocks of 0 and 1 with all possible lengths.

5. Since the set of non-periodic points includes as a subset the orbit of the Morse sequence, proof follows from 4. □

Theorem 1.3.11. *The shift map σ is chaotic on $\sum_2$.*

Proof. Proof follows from Theorems 1.2.4 and 1.3.10. □

Remark 1.3.12. We can generalize Theorems 1.3.10 and 1.3.11 in the case of shifts generated by more than two symbols.

1.3.3 A chaotic linear operator- An example

If X is Banach space and if $T : X \to X$ is a compact linear operator, then T is not transitive. This is an unpublished result from Kitai's thesis (see [47]). Because of this result, none of linear operators are transitive in finite dimensional spaces. But there exists a chaotic linear operator on l^2 (see Theorem 1.3.13).

Let $l^2 = \{x = (x_1, x_2, ..., x_n, ...) : ||x||^2 = \sum_{n=0}^{\infty} |x_n|^2 < \infty, x_i \in \mathbb{C}\}$. Then with respect to the inner product $< x, y >= \sum_{n=0}^{\infty} x_n y'_n$, where x_n, y_n denote the nth coordinate of x, y respectively, and y'_n here denotes the complex conjugate of y_n. Define $B(x_0, x_1, x_2, ...) = (x_1, x_2, x_3, ...)$. Then $||B(x)|| \leq ||x||$ for all $x \in l^2$. Hence $B : l^2 \to l^2$ is continuous. If $x \in l^2$, then $||B^n x||^2 = \sum_{k=n}^{\infty} |x_k|^2$. Which converge to zero as n tends to infinity. Therefore all the orbits converges to the fixed point zero. But the following theorem says the dynamics of $2B$ is not so simple. The core part 'transitivity' of the following theorem is proved by S. Rolewicz in [41].

Theorem 1.3.13. *[26] $2B$ is chaotic on l^2.*

Proof. Let $T = 2B$. Then $T^n(x_0, x_1, ...) = 2^n(x_n, x_{n+1}, ...)$. Let $y \in l^2$. If $y = (y_0, y_1, y_2, ...)$ then define vectors x_n as:

$x_n = (y_0, ..., y_{n-1}, \frac{y_0}{2^n}, ..., \frac{y_{n-1}}{2^n}, \frac{y_0}{2^{2n}}, ..., \frac{y_{n-1}}{2^{2n}}, \frac{y_0}{2^{3n}}, ..., \frac{y_{n-1}}{2^{3n}}, ...)$. Then $x_n \in l^2$, and it has T-period n. Observe that x_n converges to y.

Let U, V be two non-empty open sets in l^2. Choose vectors $x \in U$ and $y \in V$. Now let $z_n = (x_0, ..., x_{n-1}, \frac{y_0}{2^n}, ..., \frac{y_{n-1}}{2^n}, 0, 0, 0, ...)$. Then z_n converges to x, and $T^n(z_n) = (y_0, y_1, ..., y_{n-1}, 0, 0, 0, ...)$ converges to y. Hence for all large n, $z_n \in U$ and $T^n(z_n) \in V$. Then T is transitive. Hence T is chaotic by Theorem 1.2.4. □

Chapter 2

Set of Periods of a Linear Operator

One important class of dynamical systems that has been well-studied is the class of linear operators on a Hilbert space. In this chapter our main results characterize the sets of periods for isometries of Hilbert spaces, linear operators on a vectorspace, linear operators on the vector spaces $\mathbb{C}^n$, $\mathbb{R}^n$, and the Hilbert space l^2. Reader may refer [42] for some ideas regarding vector spaces, normed spaces, Banach-spaces and Hilbert spaces. For a topological space X and $A \subset \mathbb{N}$, we write $A \in \mathcal{PER}(X)$ if there exists a continuous map $f : X \to X$ such that $Per(f) = A$.

We ask: What is $\{\text{Per}(T) : T \text{ is a linear operator}\}$ on the spaces mentioned above. To answer this question, we first introduce some simple notations. For $A \subset \mathbb{N}$, $|A|$ denotes the cardinality of A, and $\tilde{A}$ denotes the smallest subset of $\mathbb{N}$ containing A and closed under the binary operation least common multiple (lcm). A subset $A \subset \mathbb{N}$ is said to be closed under lcm if $m, n \in A$ then the lcm of m and n belongs to A. For each $n \in \mathbb{N}$, let $\mathfrak{F}_n$ denote $\{\{1\} \cup \tilde{A} : A \subset \mathbb{N} \textit{ and } |A| \leq \frac{n}{2}\} \cup \{\{1\} \cup \tilde{A} : A \subset \mathbb{N} \setminus \{1\}, 2 \in A \textit{ and } |A| = \frac{n+1}{2}\}$. Note that $\mathfrak{F}_{2m} = \{\{1\} \cup \tilde{A} : A \subset \mathbb{N} \text{ and } |A| \leq m\}$ for all $m \in \mathbb{N}$.

2.1 Motivation

There have been a lot of papers that characterize the sets of periods, for various classes of self maps, like (i) continuous self maps of the real line $\mathbb{R}$ (see Sec 2.1.1), (ii) polynomials on $\mathbb{C}$ (see Sec 2.1.2), (iii) toral automorphisms (see Sec 2.1.5), (iv) totally transitive maps on I (see Sec 2.1.4), and (v) degree one maps on S^1 (see Sec 2.1.3).

2.1.1 Sharkovskii's theorem

The following total order on $\mathbb{N}$ is called the Sharkovskii's ordering:

$3 \succ 5 \succ 7 \succ 9 \succ ... \succ 2 \times 3 \succ 2 \times 5 \succ 2 \times 7 \succ ...$

$\succ 2^n \times 3 \succ 2^n \times 5 \succ 2^n \times 7 \succ ...$

$...2^n \succ .... \succ 2^2 \succ 2 \succ 1$

We write $m \succ n$ if m precedes n (not necessarily immediately) in this order. In what follows, n-cycle means a cycle of length n.

Theorem 2.1.1. *[48] Let $m \succ n$ in the Sharkovskii's ordering. For every continuous self map of $\mathbb{R}$, if there is an m-cycle, then there is an n-cycle.*

A converse of Sharkovskii's theorem : [23]

Let m and n be distinct positive integers. Let m not precede n in the above ordering. Then there is a continuous map f from $\mathbb{R}$ to $\mathbb{R}$, where there is an m-cycle but no n-cycle.

A combined statement:[24], [25]

$m \succeq n$ in the Sharkovskii's ordering if and only if for every continuous self map of $\mathbb{R}$, the existence of an m-cycle forces that of an n-cycle.

It is sometimes convenient to work with the reverse order $\prec$, instead of $\succ$.

A subset S of $\mathbb{N}$ is called an initial segment in this ordering $\prec$, if the following holds:

$m \in S$ and $n \prec m$ imply $n \in S$.

This theorem can be reformulated as follows:

Theorem 2.1.2. *(a) Initial segments in the ordering $\prec$, are precisely the sets of periods, for continuous self maps of $\mathbb{R}$.*

(b) Non-empty ones among them, are precisely the sets of periods of interval maps.

We denote by $\mathcal{S}$ the family mentioned in (b) above. Accordingly, we have: $\mathcal{PER}(I) = \mathcal{S}$ and $\mathcal{PER}(\mathbb{R}) = \mathcal{S} \cup \{\emptyset\}$ where I is a compact subinterval of $\mathbb{R}$.

2.1.2 Baker's theorem

Theorem 2.1.3. *[9] Let p be a complex polynomial. Then the set of periods of p has to be one of the following subsets of $\mathbb{N}$.*

1. *The whole set $\mathbb{N}$.*
2. $\mathbb{N} \setminus \{2\}$.
3. $\{1, n\}$ *for* $n \in \mathbb{N} \setminus \{1\}$.
4. $\{1\}$.
5. *Empty set.*

Moreover, the following hold:

(a) *Any polynomial p such that $Per(p) = \mathbb{N} \setminus \{2\}$ has to be topologically conjugate to $z^2 - z$.*

(b) *For all polynomials p of degree ≥ 2, $Per(p) \supset \mathbb{N} \setminus \{2\}$.*

The following table gives some examples:

If Per(p) is	then an example of p is
Empty set	$z+2$
$\{1\}$	z
$\{1,2\}$	$-z$
$\mathbb{N}\setminus\{2\}$	z^2-z
$\mathbb{N}$	z^2

It is proved that the sets of all periods of a real polynomial is an infinite proper subfamily of $\mathcal{S}\cup\{\emptyset\}$. This implies there is a subset of $\mathbb{N}$, occurring as Per(f) for a continuous self map of $\mathbb{R}$, but not as Per(p) for a real polynomial. Explicitly, $\{2^k : k \in \mathbb{N}_0\}$ is one such set.

2.1.3 Circle maps

Let S^1 be the unit circle. The family $\mathcal{PER}(S^1)$ has been completely described by Block and Coppel (See [15], [13]).

Theorem 2.1.4. *The following are equivalent for a subset S of $\mathbb{N}$.*

(1) $1 \in S \in \mathcal{PER}(S^1)$.

(2) *If $n \in S$ for some $n > 1$, (at least) one of the following should hold:*

(i) Every integer greater than n belongs to S.

(ii) Every integer that comes later than n in the Sharkovskii's ordering, belongs to S.

Corollary 2.1.5. *If $\{1,2,3\} \subset Per(f)$ for a circle map f, then $Per(f) = \mathbb{N}$. Conversely, if $S \subset \mathbb{N}$ has the property that for any $f \in C(S^1, S^1)$, $S \subset Per(f)$ implies*

$Per(f) = \mathbb{N}$ *then* $\{1, 2, 3\} \subset S$.

Contrast this with the following consequence of Sharkovskii's theorem, proved independently in [36]:

If $3 \in Per(f)$, then $Per(f) = \mathbb{N}$. Moreover 3 is the only number with this property.

Theorem 2.1.6. *([14]) Let* $f \in C(S^1, S^1)$ *and suppose that* $Per(f)$ *is finite. Then there are integers* m *and* n *(with* $m \geq 1$ *and* $n \geq 0$*) such that*

$Per(f) = \{m, 2.m, 2^2.m, ..., 2^n.m\}$.

Compare this with a corresponding result for interval maps, where $\mathrm{Per}(f)$ has to be $\{1, 2, 2^2, ..., 2^n\}$ for some $n \in \mathbb{N}$.

If $\mathcal{F}$ is the family of degree one maps of the circle, then $\mathcal{PER}(\mathcal{F})$ has been calculated in [38].

2.1.4 Transitive maps on the interval

An important subclass of this class is that of transitive interval maps. When X is a compact metric space without isolated points, this transitivity, is equivalent to the existence of a dense orbit (see Theorem 1.1.7). Now we seek to find the family $\{\mathrm{Per}(f) : f$ is a transitive interval map$\}$. Its importance is evident from the following two reformulations:

(a) Which lengths of cycles should coexist with a dense orbit?

(b) Which lengths of cycles are available in all chaotic systems?

We have:

Theorem 2.1.7. *[39] (a) Every transitive interval map must have a cycle of length* 6 *(and therefore cycles of length* n *for all* n *with* $6 \succ n$ *in the Sharkovskii's order).*

(b) Conversely if $n \in \mathbb{N}$ has the property that every transitive interval map must have a cycle of length n, then $6 \succ n$ in the Sharkovskii's order.

We are not saying that if $6 \in \text{Per}(f)$, then f is transitive. See next theorem for a complete answer to our question.

Theorem 2.1.8. *The following are equivalent for a subset S of $\mathbb{N}$.*

(a) $S = Per(f)$ for some transitive interval map.

(b) S has the following two properties:

(i) $n \in S \setminus \{1\}$ implies $n+2 \in S$.

(ii)1 and $2 \in S$.

(c) $6 \in S$ and $S = Per(g)$ for some interval map g.

The formulation (b) is as given in [5]. This can be deduced from the following theorem.

Theorem 2.1.9. *[20], [39] Given any odd integer $k > 4$, there exists a transitive map f on $[0,1]$ such that $k \in Per(f)$ but $k-2 \notin Per(f)$.*

One can construct a transitive map whose set of periods is $2\mathbb{N} \cup \{1\}$.

$$f(x) = \begin{cases} 2x + \frac{1}{2} & \text{if } \ 0 \leq x < \frac{1}{4} \\ \frac{3}{2} - 2x & \text{if } \frac{1}{4} \leq x < \frac{3}{4} \\ 2x - \frac{3}{2} & \text{if } \ \frac{3}{4} \leq x < 1 \end{cases},$$

is one such example (see [5]).

Here the subintervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ are mapped to each other. It follows that these are invariant under $f \circ f$. Therefore $f \circ f$ is not transitive, though f is. This example leads us to another important dynamical property called total transitivity. Since the transitivity of $f \circ f$ is equivalent, among interval maps, to some well-known properties like total transitivity, weak mixing and mixing, we now consider this class of interval maps. We ask for a description of the family $\{\text{Per}(f) : f \text{ is totally transitive}\}$. Obviously, this is contained in the family of all sets S satisfying the conditions of Theorem 2.1.8.

Actually we have:

Theorem 2.1.10. *[51]* $2\mathbb{N} \cup \{1\}$ *is the only subset of* $\mathbb{N}$ *that arises as Per*(f) *for some transitive interval map* f*, but does not arise as Per*(g) *for any totally transitive interval map* g*.*

Theorem 2.1.11. *[51] The following are equivalent for a subset of* $\mathbb{N}$*.*

(a) It is the set of periods of a totally transitive interval map.

(b) It is either $\mathbb{N}$ *or* $\mathbb{N} \setminus \{3, 5, ..., 2n+1\}$ *for some* $n \in \mathbb{N}$*.*

The following following theorem immediately follows from above theorem.

Theorem 2.1.12. *The following are equivalent for an interval map* f*.*

(a) f *is totally transitive.*

(b) f *is transitive and the complement of Per*(f) *is finite.*

Thus a knowledge of Per(f) is enough to distinguish totally transitive systems among transitive systems. The above theorem is true in the more general setting of graph maps, (that include interval maps as a particular case).

Take a connected planar graph, with a finite set of vertices and edges. Provide it with the relative topology from the plane. Any continuous self map of it, is called a graph map (when there are only two vertices and one edge, these are nothing but interval maps)

Theorem 2.1.13. *[4] A transitive graph map is totally transitive if and only if its set of periods has a finite complement.*

2.1.5 Toral automorphisms

The class of toral automorphisms, induced by 2×2 integer matrices of determinant ± 1, is an important class of dynamical systems, studied extensively (see [18],[23]).

Now we take up the natural question: Which subsets of $\mathbb{N}$ arise as the set of periods of a continuous toral automorphism? We answer this question in [31]. The following theorem is surprising because it gives a short list of five finite subsets and three infinite subsets and asserts that there are no others.

Theorem 2.1.14. *[31] Let T_A be the toral automorphism induced by a 2×2 integer matrix A. Then $Per(T_A)$ is one of the following eight subsets of $\mathbb{N}$.*

(1) $\{1\}$

(2) $\{1, 2\}$

(3) $\{1, 3\}$

(4) $\{1, 2, 4\}$

(5) $\{1, 2, 3, 6\}$

(6) $2\mathbb{N} \cup \{1\}$

(7) $\mathbb{N} \setminus \{2\}$

(8) $\mathbb{N}$.

Remark 2.1.15. It is also noteworthy that for a hyperbolic toral automorphism, the period set has only two possibilities, namely $\mathbb{N} \setminus \{2\}$ and $\mathbb{N}$; where as for non-hyperbolic toral automorphisms, there are seven possibilities; and there is an overlap because $\mathbb{N}$ can arise as the period set, in both the hyperbolic and non-hyperbolic cases.

The following table gives some examples.

If $T(x, y)$ is	then $Per(T)$ is
$(x, y)(mod\ 1)$	{1}
$(x, x - y)(mod\ 1)$	{1,2}
$(x - 2y, x - y)(mod\ 1)$	{1,2,4}
$(-x + y, -x)(mod\ 1)$	{1,3}
$(x - y, x)(mod\ 1)$	{1,2,3,6}
$(-x + y, -y)(mod\ 1)$	$2\mathbb{N} \cup \{1\}$
$(x + y, x)(mod\ 1)$	$\mathbb{N} \setminus \{2\}$
$(x + 2y, x + y)(mod\ 1)$	$\mathbb{N}$

.

The following chart shows that any two matrices having the same minimal polynomial, should also have the same period sets for their induced toral automorphisms. Note that for all non-hyperbolic automorphisms, the trace of the matrix has absolute value at most 2.

Minimal polynomial of A	$Per(T_A)$
$x^2 - 1, x + 1$	{1,2}
$x^2 + 1$	{1,2,4}
$x^2 + x + 1$	{1,3}
$x^2 - x + 1$	{1,2,3,6}
$x^2 - 2x + 1$	$\mathbb{N}$
$x^2 + 2x + 1$	$2\mathbb{N} \cup \{1\}$
$x - 1$	{1}

Question: What is the analogue of Theorem 2.1.14 for higher dimensional toral automorphisms?

2.2 Set of periods of linear operators on a vector space

Let $\mathbb{V}$ be a vector space over a scalar field $\mathbb{K}$. A map $T : \mathbb{V} \to \mathbb{V}$ is said to be linear if $F(kx + y) = kF(x) + F(y)$ for $k \in \mathbb{K}, x, y \in \mathbb{V}$. Linear maps are also called linear operators or linear transformations. A linear operator from a normed space X to a normed space Y is continuous if and only if it maps bounded sets in X onto bounded sets in Y (see [42]). Hence such a map is known as a bounded linear map. A subset $\mathbb{W}$ of $\mathbb{V}$ is said to be a subspace if $kw_1 + w_2 \in \mathbb{W}$ for $k \in \mathbb{K}, w_1, w_2 \in \mathbb{W}$. If S is a subset of $\mathbb{V}$ then the smallest subspace of $\mathbb{V}$ containing S is called the span of S. We say that the vectors $x_1, x_2, ..., x_n$ are linearly independent if the relation $k_1x_1 + ... + k_nx_n = 0$, $k_i \in \mathbb{K}$ holds only if each k_i is 0. A set $S \subset \mathbb{V}$ is said to be linearly independent if every finite subset of S is linearly independent. A linear independent set S, whose span is equal to $\mathbb{V}$, is called a basis. Let $\mathbb{W}_1, \mathbb{W}_2, ..., \mathbb{W}_k$ be subspaces of $\mathbb{V}$. Let $\mathbb{W} = \mathbb{W}_1 + \mathbb{W}_2 + ... + \mathbb{W}_k = \{w_1 + w_2 + ... + w_k : w_i \in \mathbb{W}_i, 1 \leq i \leq k\}$. Then we say that $\mathbb{W}$ is the direct sum of the $\mathbb{W}_i$s if for each j, $1 \leq j \leq k$, $\mathbb{W}_j \cap (\mathbb{W}_1 + ... + \mathbb{W}_{j-1} + \mathbb{W}_{j+1} + ... + \mathbb{W}_k) = \{0\}$. We write $\mathbb{W} = \mathbb{W}_1 \bigoplus ... \bigoplus \mathbb{W}_k$.

Let $\mathfrak{F}$ and $\mathfrak{G}$ be two families of subsets of $\mathbb{N}$. Define $\mathfrak{F} \bigvee \mathfrak{G} = \{A \bigvee B : A \in \mathfrak{F}, B \in \mathfrak{G}\}$ where $A \bigvee B = \{a \vee b : a \in A, b \in B\}$ and $a \vee b = \text{lcm}\{a, b\}$. Note that $\vee_{i=1}^{n} k_i = \text{lcm}\{k_1, k_2, ..., k_n\}$ for $k_i \in \mathbb{N}$. A triple $(k_1, k_2, k_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ satisfies property 'P' if each number divides the *lcm* of the other two.

Theorem 2.2.1. *The following are equivalent for a subset A of $\mathbb{N}$.*

(1) $1 \in A$ and A is closed under lcm.

(2) There is a vector space $\mathbb{V}$ over a scalar $\mathbb{K}$ and a linear operator $T : \mathbb{V} \to \mathbb{V}$ such that $Per(T) = A$.

Proof. For $1 \implies 2$ see Appendix A.

$2 \implies 1$

Without loss of generality we can assume that $\mathbb{K} = \mathbb{R}$ (similar proof can apply in the case of general scalar filed). Note that $T(0) = 0$. Hence $1 \in Per(T)$.

Next, let $m, n \in Per(T)$.

Claim: $m \vee n \in Per(T)$.

Step 1: If x has T-period m, y has T-period n and $x + y$ has T-period k, then (m, n, k) satisfies property 'P'.

This is because, $T^{m\vee n}(x + y) = x + y$; and write $x = (x + y) - y$, $y = (x + y) - x$.

Step 2: If $\lambda \neq 0$ is a non-zero real number, then x and λx have the same T-period.

This is because, if $T^p(x) = x$ then $T^p(\lambda x) = \lambda x$. Conversely, if $T^q(\lambda x) = \lambda x$ then $\lambda(T^q(x) - x) = 0$. Which implies $T^q(x) = x$.

Step 3: If x has T-period $p^r m$, p does not divide m and y has T-period $p^r n$, p does not divide n and $\lambda, \mu \neq 0$ then either $x + \lambda y$ or $x + (\lambda + \mu)y$ has T-period $p^r k$ for some k.

Let $x + \lambda y$ and $x + (\lambda + \mu)y$ have T-periods k_1, k_2 respectively. By Step 1, it follows that $p^r n$ divides $k_1 \vee k_2$ since cy has T-period $p^r n$ for all $c \neq 0$ because of Step 2. Which implies either p^r divides k_1 or p^r divides k_2.

Step 4: There exists $\lambda_0 \in \mathbb{R}$ such that p^r divides the T-period of $x + \lambda y$ for all $\lambda > \lambda_0$; and hence on the line $\{x + \lambda y : \lambda \in \mathbb{R}\}$, every element has T-period as multiple of p^r, except possibly one element whenever x has T-period $p^r m$, y has T-period $p^r n$ and p does not divide m and n.

If $\lambda, \mu \neq 0$ then either p^r divides the T-period of $x + \lambda y$ or p^r divides the T-period of $x + (\mu + \lambda)y$ by Step 3. Consider the line $\{x + \lambda y : \lambda \in \mathbb{R}\}$. Suppose there exists $\lambda_0 \in \mathbb{R}$ such that p^r does not divide T-period of $x + \lambda_0 y$. Then for any $\lambda_1 \in \mathbb{R} \setminus \{0\}$, p^r divides T-period of $x + (\lambda_0 + \lambda_1)y$.

Step 5: If $m, n \in Per(T)$ then $m \vee n \in Per(T)$.

Let $m, n \in Per(T)$. Then there exist x, y such that x has T-period m and y has T-period n. Let $m = p_1^{r_1} p_2^{r_2} ... p_t^{r_t}$, $n = p_1^{s_1} p_2^{s_2} ... p_t^{s_t}$ and $r_i, s_i \in \{0, 1, 2, ...\}$, p_is are distinct primes.

If $r_1 = s_1$ then apply Step 4. If $r_1 \neq s_1$ then $p_1^{Max(r_1,s_1)}$ divides T-period of $x + y$ by Step 1. Which implies if $r_1 \neq s_1$ then $p_1^{Max(r_1,s_1)}$ divides T-period of $x + \lambda y$ except possibly one point. This is true for all r_i, s_i except possibly one point. Therefore $m \vee n$ divides T-period of $x + \lambda y$ except possibly t points. Hence $m \vee n \in Per(T)$.

This is more than we claimed. □

2.3 Set of periods of linear operators on a Hilbert space

Unless stated otherwise $\mathbb{V}$ denotes a vector space over a scalar field $\mathbb{K}$.

2.3.1 Basic results

Lemma 2.3.1. The set P of periodic points of a linear operator $T : \mathbb{V} \to \mathbb{V}$ forms a subspace of $\mathbb{V}$. This subspace is T-invariant, and hence $\mathrm{Per}(T) = \mathrm{Per}(T|_{\mathrm{P}})$.

Proof. Let P be the set of periodic points of a linear operator $T : \mathbb{V} \to \mathbb{V}$. Let $x, y \in$ P. Then there exist $m, n \in \mathbb{N}$ such that T^mx=x and T^ny=y. This implies $T^{mn}(x + y) = x + y$. Hence $x + y \in$ P.

Next suppose that $x \in \mathrm{P}$ and $k \in \mathbb{K}$.

There exists $m \in \mathbb{N}$ such that $T^m x = x$ since $x \in \mathrm{P}$. Then $T^m(kx) = kx$. This implies $kx \in \mathrm{P}$. Hence P is a subspace of $\mathbb{V}$. Lastly, if x is a periodic point, then so is Tx. Thus P is T-invariant. □

Lemma 2.3.2. Let $T : \mathbb{V} \to \mathbb{V}$ be a linear operator and $\mathbb{V} = \mathbb{W}_1 \bigoplus \mathbb{W}_2$ such that $\mathbb{W}_1$ and $\mathbb{W}_2$ are T-invariant subspaces of $\mathbb{V}$. Then $\mathrm{Per}(T) = \mathrm{Per}(T|_{\mathbb{W}_1}) \bigvee \mathrm{Per}(T|_{\mathbb{W}_2})$.

Proof. Let $n \in \mathrm{Per}(T)$. Then there exists $x \in \mathbb{V}$ having T- period n and $x = w_1 + w_2$ for some $w_1 \in \mathbb{W}_1$ and $w_2 \in \mathbb{W}_2$. This implies $T^n w_1 - w_1 = T^n w_2 - w_2 \in \mathbb{W}_1 \cap \mathbb{W}_2 = \{0\}$. Then w_1 and w_2 are periodic, with T-periods say r_1 and r_2 respectively. Let $l = r_1 \vee r_2$. Then both r_1 and r_2 divide n. Therefore l divides n. But $T^l x = T^l w_1 + T^l w_2 = w_1 + w_2 = x$. Therefore n divides l, and hence $l = n$. Thus $n \in \mathrm{Per}(T|_{\mathbb{W}_1}) \bigvee \mathrm{Per}(T|_{\mathbb{W}_2})$.

Next assume that $n \in \mathrm{Per}(T|_{\mathbb{W}_1}) \bigvee \mathrm{Per}(T|_{\mathbb{W}_2})$. Then there exist $w_1 \in \mathbb{W}_1$ and $w_2 \in \mathbb{W}_2$ such that w_1 is having $T|_{\mathbb{W}_1}$-period r and w_2 is having $T|_{\mathbb{W}_2}$-period s where $n = r \vee s$. By a similar argument, we can show that T- period of $w_1 + w_2$ is $r \vee s$. This implies $n \in \mathrm{Per}(T)$. Hence the proof. □

For later use, we state the following known theorem (see [29]).

Theorem 2.3.3. ***(Primary decomposition theorem)***

Let T be a linear operator on the finite-dimensional vector space $\mathbb{V}$ over the field $\mathbb{K}$. Let p be the minimal polynomial for T, $p = {p_1}^{r_1} ... {p_m}^{r_m}$, where the p_is are distinct irreducible monic polynomials over $\mathbb{K}$ and the r_is are positive integers. Let $\mathbb{W}_i$ be the null space of $p_i(T)^{r_i}$, $i = 1, ..., m$. Then

(i) $V = W_1 \bigoplus ... \bigoplus W_m$;

(ii) each W_i is invariant under T;

(iii) if T_i is the operator induced on W_i by T, then the minimal polynomial for T_i is $p_i^{r_i}$.

Jordan canonical form and Jordanizing matrix (see [28])

Let A be an $n \times n$ matrix over $\mathbb{R}$ and $p_A(x)$ denotes the characteristic polynomial of A. Write $p_A(x) = \prod(x - a_i)^{b_i} \prod((x - \alpha_i)(x - \alpha_i')^{\beta_i}$ with the α_i being the complex non-real roots, α_i' denotes the complex conjugate of α_i and the a_i the real roots. The different Jordan blocks composing the matrix are either

$$\begin{bmatrix} a_i & 1 & & \\ & a_i & 1 & \\ & & \cdot & \\ & & & a_i \end{bmatrix} \text{ or } \begin{bmatrix} B & I_2 & & \\ & B & I_2 & \\ & & \cdot & \\ & & & B \end{bmatrix}$$

with $B = \begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ for $\alpha_i = p + iq$ and $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Note that an eigenvalue can be responsible for more than one block. This matrix is similar to the original matrix A.

Remark 2.3.4. Let A be an $n \times n$ matrix over $\mathbb{K}$ where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. Assume that $p_A(x)$ has only real roots whenever A has real entries. Then A is similar to a matrix having different Jordan blocks composing the matrices of the following form

$$\begin{bmatrix} \alpha_i & 1 & & \\ & \alpha_i & 1 & \\ & & \cdot & \\ & & & \alpha_i \end{bmatrix}$$ for $\alpha_i \in \mathbb{K}$. This is called *Jordan canonical form*.

Theorem 2.3.5. *If A is a $n \times n$ matrix of complex numbers such that some power of A is identity, then A is diagonalizable over $\mathbb{C}$.*

Proof. This follows from Jordan canonical form. □

A simple observation shows that $\mathfrak{F}_{2m} = \{\{1\} \cup \tilde{A} : A \subset \mathbb{N} \text{ and } |A| \leq m\}$, and $\mathfrak{F}_{2m+1} = \mathfrak{F}_{2m} \cup \{\{1\} \cup \tilde{B} : B \subset \mathbb{N} \setminus \{1\}, 2 \in B \text{ and } |B| = m+1\}$ for all $m \in \mathbb{N}$.

Lemma 2.3.6. $\mathfrak{F}_m \bigvee \mathfrak{F}_p \subset \mathfrak{F}_{m+p}$ for all $m, p \in \mathbb{N}$.

Proof. Let $A \in \mathfrak{F}_m \bigvee \mathfrak{F}_p$. Then $A = B \bigvee C$ for some $B \in \mathfrak{F}_m$, $C \in \mathfrak{F}_p$. Let $r \in B$ and $s \in C$. Then there exist $T : \mathbb{R}^m \to \mathbb{R}^m$ and $x \in \mathbb{R}^m$ such that $T^r x = x$ and $T^i x \neq x$ for all $i < r$; and $S : \mathbb{R}^p \to \mathbb{R}^p$ and $y \in \mathbb{R}^p$ such that $S^s x = x$ and $S^j x \neq x$ for all $j < s$ (see the sufficient part of the proof of Theorem 2.3.12). Define $V : \mathbb{R}^{m+p} \to \mathbb{R}^{m+p}$ by $V(a, b) = (Ta, Sb)$ where $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^p$. Then $r \vee s \in Per(V)$. Therefore $B \bigvee C \subset Per(V)$. Now let $t \in Per(V)$. Then there exist $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ such that $V^t(x, y) = (x, y)$ and $V^q(x, y) \neq (x, y)$ for all $q < t$. Then $T^t x = x$ and $V^t y = y$. This implies x and y are periodic. If $r = T$-period of x and $s = S$-period of y then $r \vee s$ divides t. Also $V^{r \vee s}(x, y) = (x, y)$. Therefore t divides $r \vee s$. Therefore $t = r \vee s \in B \bigvee C$. Therefore $B \bigvee C = Per(V) \in \mathfrak{F}_{m+p}$. Hence $\mathfrak{F}_m \bigvee \mathfrak{F}_p \subset \mathfrak{F}_{m+p}$. □

2.3.2 Set of periods of linear operators on $\mathbb{C}^n$

Let $\text{diag}(d_1, d_2, ..., d_n)$ denotes the diagonal matrix having diagonal entries $d_1, d_2, ..., d_n$.

Theorem 2.3.7. *Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a linear operator. Then $Per(T) \in \mathfrak{F}_{2n}$. Conversely for every $A \in \mathfrak{F}_{2n}$, there is a linear operator $T : \mathbb{C}^n \to \mathbb{C}^n$ such that $Per(T) = A$.*

Proof. Let $A = \text{Per}(T)$ for some linear operator $T : \mathbb{C}^n \to \mathbb{C}^n$, and let M denote the matrix associated to T.

Case 1. M is similar to some diagonal matrix D.

Let $D = \text{diag}(d_1, d_2, ..., d_n)$ and $X = (x_1, x_2, ..., x_n) \in \mathbb{C}^n$ be such that $D^p X = X$. Then we have $d_i^p x_i = x_i$ for $1 \leq i \leq n$.

Let $B = \{k_i \in \mathbb{N} : d_i^{k_i} = 1 \neq d_i^j \forall j < k_i$, for some $1 \leq i \leq n\}$ and $J = \{i : k_i \in B\}$. Then the element $\sum_{i \in I \subset J} x_i \delta_i$ where δ_i is the i^{th} coordinate vector in $\mathbb{C}^n$ has T- period $\vee_{i \in I} k_i$ by Lemma 2.3.2 and Theorem 2.3.3. Therefore $\text{Per}(T) = \{\vee_{i \in I} k_i : I \subset J\} \cup \{1\}$. Therefore $\text{Per}(T) \in \mathfrak{F}_{2n}$.

Case 2. M is not diagonalizable.

Note that $(T|_P)^n = I$ for some n, where P denote the set of all periodic points of T. Then the result follows from Lemma 2.3.1, Lemma 2.3.2 and Theorem 2.3.5. Hence for any linear operator $T : \mathbb{C}^n \to \mathbb{C}^n$, $\text{Per}(T) \in \mathfrak{F}_{2n}$.

Conversely suppose that $A \in \mathfrak{F}_{2n}$. If $A = \{a_1, a_2, ..., a_m\}$ with $m \leq n$ then let $d_j = e^{\frac{2\pi i}{a_j}}$ for all $1 \leq j \leq m$ and $D = \text{diag}(d_1, d_2, ..., d_m, 1, 1, ..., 1)$. Then $\text{Per}(T_D) = \{1\} \cup \tilde{A}$ where T_D denotes the linear operator on $\mathbb{C}^n$ associated to D. Hence the proof. □

Theorem 2.3.7 says that, the family of period sets of linear operators on $\mathbb{C}^n$ is $\mathfrak{F}_{2n}$.

2.3.3 Set of periods of linear operators on $\mathbb{R}^n$

In general, the conclusion of Theorem 2.3.5 may not be true for real vector spaces. That is, if A is an $n \times n$ matrix of real numbers such that some power of A is identity then A need not be diagonalizable over $\mathbb{R}$. For example, consider $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Therefore a similar proof as in previous section does not work in the case of $\mathbb{R}^n$.

Lemma 2.3.8. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator. If $\{1, 2\} \subset Per(T)$ then $Per(T) = \{1, 2\}$.

Proof. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator. For a base $\mathcal{B}$, let $[T]_{\mathcal{B}}$ denotes the matrix of T with respect to $\mathcal{B}$.

Suppose there exists an element X of T-period 2.

Case 1. $\mathcal{B} = \{X, TX\}$ is a basis.

In this case, $[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, say C. Therefore $C^2 = I$. Thus every element in $\mathbb{R}^2$ is of T-period 2 except $(0,0)$. Hence $Per(T) = \{1,2\}$.

Case 2. $\{X, TX\}$ is not linearly independent.

In this case, there exists $\lambda \in \mathbb{R}$ such that $TX = \lambda X$. ie., X is an eigen vector with eigen value λ. If $\lambda \neq 1$ then $T^2X = \lambda^2 X \neq X$. Which is impossible since $T^2X = X$. If $\lambda = 1$ then X is a fixed point. This is also impossible. Therefore $\lambda = -1$. Which implies $TX = -X$.

Take a basis $\mathcal{B} = \{X, Y\}$. Then $[T]_{\mathcal{B}} = \begin{bmatrix} -1 & b \\ 0 & d \end{bmatrix}$ for some $b, d \in \mathbb{R}$. Let $B = [T]_{\mathcal{B}}$. Then by induction we can prove that

$$B^{2m} = \begin{bmatrix} 1 & b(d-1)(1 + d^2 + ... + d^{2m-2}) \\ 0 & d^{2m} \end{bmatrix} \text{ for all } m \in \mathbb{N}.$$

Note that, if $B^{2m} = I$ then $B^2 = I$.

Suppose some point outside the line through the origin and X is a periodic point of T-period n.

Here all points are periodic. This is because, the set $\{V \in \mathbb{R}^2 : T^{2n}V = V\}$ is a subspace of $\mathbb{R}^2$. Therefore $B^{2n} = I$. Which implies $B^2 = I$. Then $Per(T) = \{1,2\}$ since $T^2 = I$.

Hence the proof. □

Corollary 2.3.9. *If T admits a real eigen value then $Per(T) = \{1,2\}$.*

Proposition 2.3.10. *The following are true.*

(i) $\mathfrak{F}_1 = \{A \subset \mathbb{N} : A$ is the set of periods of a linear operator $T : \mathbb{R} \to \mathbb{R}\}$, and

(ii) $\mathfrak{F}_2 = \{A \subset \mathbb{N} : A$ is the set of periods of a linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2\}$, where $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are as in page 19.

Proof. (i) Any linear operator from $\mathbb{R}$ to $\mathbb{R}$ is either an identity map or a reflection map or a contraction map or an expansion map. Hence $\mathfrak{F}_1 = \{A \subset \mathbb{N} : A$ is the set of periods for some linear operator $T : \mathbb{R} \to \mathbb{R}\} = \{\{1\}, \{1, 2\}\}$.

(ii) Let $X \in \mathbb{R}^2$ be a periodic point of a linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ having period n.

Case 1: If X and TX are linearly independent.

Here X is an eigen vector and hence $TX = \pm X$. Therefore X has period 1 or 2.

Case 2: If X and TX are not linearly independent.

Here all points are periodic (see the proof of Lemma 2.3.8 Case 2). Therefore $T^n Y = Y$ for all Y where n is the T-period of X. If for some Y with period m; $2 < m < n$ then Y and TY are not linearly independent. Which implies $T^m = I$ and hence $T^m X = X$, a contradiction (see the proof of Lemma 2.3.8).

Therefore $Per(T) \subset \{1, 2, n\}$. But $n \in Per(T)$. Then by Lemma 2.3.8 we have $Per(T) = \{1, n\}$.

Hence $\mathfrak{F}_2 = \{A \subset \mathbb{N} : A$ is the set of periods of a linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2\}$. □

Remark 2.3.11. Proposition 2.3.10 can be obtained from Jordan canonical form and Jordanizing matrix.

Proof. Let A be a 2×2 matrix over $\mathbb{R}$. Then A is similar to either D_1 or D_2 or D_3, where

$D_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, $D_2 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, and $D_3 = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ for some $\lambda, \lambda_1, \lambda_2, \alpha, \beta \in \mathbb{R}$; by Jordan canonical form and Jordanizing matrix. Note that $D_1^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}$ and $D_2^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}$.

Now consider D_3, and let $w = tan^{-1}(\beta/\alpha)$. Then $cos\ w = \alpha/|\gamma|$, $sin\ w = \beta/|\gamma|$ where $\gamma = \alpha + i\beta$. Then $D_3 = |\gamma| \begin{bmatrix} cos\ w & sin\ w \\ -sin\ w & cos\ w \end{bmatrix}$, and hence $D_3^n = |\gamma|^n \begin{bmatrix} cos\ nw & sin\ nw \\ -sin\ nw & cos\ nw \end{bmatrix}$.

Hence the proof follows. □

In general, we have the following theorem.

Theorem 2.3.12. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Then $Per(T) \in \mathfrak{F}_n$. Conversely for every $A \in \mathfrak{F}_n$, there is a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $Per(T) = A$.*

Proof. First part is proved by induction on n, as under.

By Proposition 2.3.10, $\text{Per}(T) \in \mathfrak{F}_1 = \{\{1\}, \{1, 2\}\}$ for a linear operator $T : \mathbb{R} \to \mathbb{R}$. Hence the result is true for $n = 1$.

Assume that the result is true for $n = m$. i.e., $\text{Per}(T) \in \mathfrak{F}_m$ for all linear operator $T : \mathbb{R}^m \to \mathbb{R}^m$. Now we have to prove that Per $(T) \in \mathfrak{F}_{m+1}$ for all linear operators $T : \mathbb{R}^{m+1} \longrightarrow \mathbb{R}^{m+1}$.

For $m = 1$, $\text{Per}(T) \in \mathfrak{F}_2$ by Proposition 2.3.10. Suppose $m \geq 2$. If the minimal polynomial of T has at least two distinct irreducible monic polynomial factors then by Theorem 2.3.3, $\mathbb{R}^{m+1} = \mathbb{W}_1 \bigoplus \mathbb{W}_2$ for some non-trivial proper T-invariant subspaces $\mathbb{W}_1$ and $\mathbb{W}_2$. By Lemma 2.3.2, $\text{Per}(T) = \text{Per}(T|_{\mathbb{W}_1}) \bigvee \text{Per}(T|_{\mathbb{W}_2})$. By induction hypothesis, $\text{Per}(T|_{\mathbb{W}_1})$ and $\text{Per}(T|_{\mathbb{W}_2})$ belong to $\mathfrak{F}_r$ for some $r \leq m$. Suppose $\text{Per}(T|_{\mathbb{W}_1}) \in \mathfrak{F}_{r_1}$ and $\text{Per}(T|_{\mathbb{W}_2}) \in \mathfrak{F}_{r_2}$ with $r_1 + r_2 = m + 1$. Then by Lemma 2.3.6, $\text{Per}(T) \in \mathfrak{F}_{r_1} \bigvee \mathfrak{F}_{r_2} \subset \mathfrak{F}_{m+1}$. Next suppose the minimal polynomial of T has only one irreducible monic polynomial factor. If the minimal polynomial has a real root then this factor is of the form $(x - \alpha)$ for some real number α. Then by Jordan canonical form $\text{Per}(T) \in \mathfrak{F}_1$. If the minimal polynomial has no real root then we get a matrix for T in which the blocks $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ appear on the diagonal, where $\alpha \pm i\beta$ are the only eigen values of T by Jordanizing matrix. Then $\text{Per}(T) \in \mathfrak{F}_2$ (see the proof of Remark 2.3.11). Hence by induction hypothesis, $\text{Per}(T) \in \mathfrak{F}_n$ for all linear operators $T : \mathbb{R}^n \to \mathbb{R}^n$.

We next show that for $A \in \mathfrak{F}_n$ there exists a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $\text{Per}(T) = A$.

Case 1. When n is even.

Then $A = \{1\} \bigcup \tilde{B}$ for some $B \subset \mathbb{N}$ such that $|B| \leq \frac{n}{2}$. Let $B = \{b_1, b_2, ..., b_k\}, k \leq \frac{n}{2}$. Define $T_A : \mathbb{R}^n \to \mathbb{R}^n$ as $T_A(x_1, x_2, ..., x_n) = (y_1, y_2, ..., y_n)$, where $(y_{2i-1}, y_{2i}) = \rho_{\frac{2\pi}{b_i}}(x_{2i-1}, x_{2i}), 1 \leq i \leq k$ and $y_i = x_i$ for all $i > 2k$ and $\rho_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the rotation map by an angle θ. Then by Lemma 2.3.2, $\text{Per}(T_A) = \text{Per}(\rho_{\frac{2\pi}{b_1}}) \bigvee \text{Per}(\rho_{\frac{2\pi}{b_2}}) \bigvee ... \bigvee \text{Per}(\rho_{\frac{2\pi}{b_k}}) \bigvee \{1\} = A$.

Case 2. When n is odd.

Then $A \in \mathfrak{F}_{n-1} \bigcup \{\{1\} \cup \tilde{B} : B \subset \mathbb{N} \setminus \{1\},\ 2 \in B \text{ and } |B| = \frac{n+1}{2}\}$. If $A \in \mathfrak{F}_{n-1}$ then as in the previous case there exists $T : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that $\text{Per}(T) = A$. Define $T_A = T \times I : \mathbb{R}^n \to \mathbb{R}^n$, where I is the identity operator on $\mathbb{R}$. Then $\text{Per}(T_A) = A$. Next suppose that $A = \{1\} \bigcup \tilde{B}$ for some $B \subset \mathbb{N} \setminus \{1\}$ such that $2 \in B$ and $|B| = \frac{n+1}{2}$. Let $B \setminus \{2\} = \{b_1, b_2, ..., b_{\frac{n-1}{2}}\}$. Define $T_A(x_1, x_2, ..., x_n) = (y_1, y_2, ..., y_n)$, where $(y_{2l-1}, y_{2l}) = \rho_{\frac{2\pi}{b_l}}(x_{2l-1}, x_{2l}),\ 1 \leq l \leq \frac{n-1}{2}$, $y_n = -x_n$. Hence $\text{Per}(T_A) = \tilde{A}$. □

Corollary 2.3.13. *For every linear operator T on $\mathbb{R}^n$, $|Per(T)| \leq 2^{[\frac{n+1}{2}]}$ where $[\ \ ]$ denotes the greatest integer function. This estimate is sharp.*

Proof. For every $A \subset \mathbb{N}$, $|\tilde{A}| \leq 2^{|A|} - 1$ and equality holds when A consists of distinct primes. Hence the proof follows from Theorem 2.3.12. □

Theorem 2.3.12 says that, the family of period sets of linear operators on $\mathbb{R}^n$ is $\mathfrak{F}_n$, and the above corollary says that, as T varies over all linear operators on $\mathbb{R}^n$, though there is no common bound to the length of T-cycles, there is a common bound to the number of T-cycle lengths.

Remark 2.3.14. The conclusion of Theorem 2.3.12 is true in the case of linear operators on a finite dimensional vector space over $\mathbb{R}$ and the conclusion of Theorem 2.3.7 is true in the case of linear operators on a finite dimensional vector space over $\mathbb{C}$ since any two finite dimensional vector spaces over a scalar field $\mathbb{K}$ of the same dimension are isomorphic.

Remark 2.3.15. For every subset A of $\mathbb{N}$ (A may not be closed under lcm) we can find a map $T : \mathbb{R}^n \to \mathbb{R}^n$ (T may not be linear) such that $\text{Per}(T) = A$. If $n \geq 2$ then we can assume that T has to be continuous also. Contrast this with our Theorem 2.3.12.

Remark 2.3.16. By Jordanizing the matrix of linear operators we can say that any linear operator on $\mathbb{R}^n$ can be written in terms of identity map, reflection map, contraction map, expansion map on $\mathbb{R}$; and a rotation or a constant times rotation on $\mathbb{R}^2$ up to conjugacy. This representation gives another proof for the necessary part of Theorem 2.3.12.

2.3.4 Set of periods of linear operators on l^2

The ideas involved in the following theorem is different from the ideas involved in Theorem 2.2.1. But the Theorem 2.2.1 has its own merit, because we can use same ideas in the case of endomorphism of torsion free abelian group (see Appendix A), and the scalar field involved in it is a general scalar field.

Theorem 2.3.17. *Let $\mathbb{V}$ be an infinite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. Then the following are equivalent for a subset A of $\mathbb{N}$.*

(i) $1 \in A$ and A is closed under lcm.

(ii) $A = Per(T)$ for some linear operator $T : \mathbb{V} \to \mathbb{V}$.

Proof. Suppose that $A \subset \mathbb{N}$ satisfies condition (i). Let S be an infinite linearly independent set in $\mathbb{V}$. Define a map $\phi : S \to S$ which is a bijection such that, $\text{Per}(\phi) = A \setminus \{1\}$. Extend ϕ to a linear operator T on $\mathbb{V}$ such that it is identity on a complementary subspace of span S. Then $\text{Per}(T) = \text{Per}(\phi) = A$.

Next claim $\text{Per}(T)$ is closed under lcm for every linear operator $T : \mathbb{V} \to \mathbb{V}$.

Let $\mathbb{V}$ be an infinite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. Let $m, n \in \text{Per}(T)$. Let $x, y \in \mathbb{V}$ be such that the T- period of x is m, and the T-period of y is n. Let $\mathbb{W}$ be the linear span of $\{x, Tx, T^2x, ..., T^{m-1}x\} \cup \{y, Ty, T^2y, .., T^{n-1}y\}$. Then $\mathbb{W}$ is finite dimensional and T- invariant. Therefore $\text{Per}(T|_{\mathbb{W}}) \in \mathfrak{F}_k$ for some $k \in \mathbb{N}$ by Theorem 2.3.12. Note that $m, n \in \text{Per}(T|_{\mathbb{W}})$ and hence $m \vee n \in \text{Per}(T|_{\mathbb{W}})$ because every member of $\mathfrak{F}_k$ is closed under lcm. Therefore $m \vee n \in \text{Per}(T)$. □

Remark 2.3.18. Let $\mathbb{V}$ be a vector space. Then the following are equivalent for a subset A of $\mathbb{N}$.

(i) $1 \in A$ and A is closed under lcm.

(ii) $A = \text{Per}(T)$ for some linear operator $T : \mathbb{V} \to \mathbb{V}$.

Proof. By Theorem 2.3.17 and Remark 2.3.14 proof follows. □

Let $l^2 = \{x = (x_1, x_2, ..., x_n, ...) : ||x||^2 = \sum_{n=0}^{\infty} |x_n|^2 < \infty, x_i \in \mathbb{K}\}$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Theorem 2.3.19. *The following are equivalent for a subset A of $\mathbb{N}$.*

(i) $1 \in A$ and A is closed under lcm.

(ii) $A = Per(T)$ *for some bounded linear operator* $T : l^2 \to l^2$.

(iii) $A = Per(T)$ *for some linear operator* $T : l^2 \to l^2$.

(iv) $A = Per(T)$ *for some linear isometry* $T : l^2 \to l^2$.

Proof. Let $T : l^2 \to l^2$ be a linear isometry. Then Per(T) is closed under lcm by Theorem 2.3.17. Next suppose that $A \subset \mathbb{N}$ which satisfies condition (i). Let $\mathbb{R}^{\mathbb{N}}$ be the set of all sequences in $\mathbb{R}$. If A is infinite, say $\{a_1, a_2, a_3, ...\}$, then define $T_A : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ such that $T_A((x_n)) = ((y_n))$ where $(y_{2n-1}, y_{2n}) = \rho_{\frac{2\pi}{a_n}}(x_{2n-1}, x_{2n}), x_n \in \mathbb{R}$ for all $n \in \mathbb{N}$ and $\rho_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the rotation map by an angle θ. If A is finite, take same type of rotations for all elements of A and define T_A as above, with the modification $y_m = x_m$ for all $m > 2|A|$. Then l^2 is a T_A-invariant subspace of $\mathbb{R}^{\mathbb{N}}$ and $T_A = \prod_n \rho_{\frac{2\pi}{a_n}}$. Hence Per($T_A$) $= A$. All other implications also follows from Theorem 2.3.17. □

Theorem 2.3.20. *The following are equivalent for a finite subset* A *of* $\mathbb{N}$.

(i) $1 \in A$, A *closed under lcm.*

(ii) $A = Per(T)$ *for some linear operator* $T : l^2 \to l^2$ *having finite rank.*

Proof. Proof follows from Theorem 2.3.12 since the set of all periodic points for linear operators having finite rank is finite dimensional. □

Theorem 2.3.20 says that, the family of period sets of linear operators on l^2 having finite rank is $\mathfrak{F} = \bigcup_n \mathfrak{F}_n$, and Theorem 2.3.19 says that, the family of period sets of isometries on l^2 is $\{A \subset \mathbb{N} : A = \tilde{A}$ and $1 \in A\}$.

Remark 2.3.21. Let T be a bounded linear operator on a complex Hilbert space. Then a vector is a period point for T if and only if it is a finite linear combination of eigen vectors of T where the eigen values are nth roots of unity (see [26]). This is because, let $Tv_i = \lambda_i v_i$ for $i \in \{1, 2, ..., m\}$; and for each i, there exists an $n_i \geq 1$ such that $\lambda_i^{n_i} = 1$ (take the least n_i with this property). If $x = c_1 v_1 + c_2 v_2 + ... + c_m v_m$

for scalars c_i, then x is a periodic point with period $n = n_1 \vee n_2 \vee ... \vee n_m$ by Lemma 2.3.2. For the converse, if x is a periodic point for T then $T^n x = x$ for some $n \geq 1$. Hence $x \in$ Kernel of $(T^n - I)$. Write $z^n - 1 = (z - \lambda_1)(z - \lambda_2)...(z - \lambda_n)$, λ_i is nth root of unity and $\lambda_i \neq \lambda_j$ for all $i \neq j$. Then $x \in$ Kernel of $(T^n - I) =$ Span of {Kernel of $(T - \lambda_i) : 1 \leq i \leq n$}. Therefore the periodic points are the vectors of the form $x = c_1 v_1 + c_2 v_2 + ... + c_m v_m$ for scalars $\{c_i\}$, c_is are n_ith roots of unity, v_is are distinct, then period of x is an element of $\{1 \cup \tilde{A} : A \subset \{n_1, n_2, ..., n_m\}\}$. Therefore $Per(T)$ is in $\mathfrak{F}_m$. Therefore, if T is a linear operator on l^2 having finite rank then $Per(T) = \bigcup_n \mathfrak{F}_n$. This remark characterize not only period sets of bounded linear operators on a complex Hilbert space but also its periodic points. A similar proof is not applicable in the case of real Hilbert space because of lack of linear factorization.

2.4 Set of periods of linear operators on Banach Spaces

In this section, we prove that there exist Banach spaces having $Per(T)$ is very small.

Theorem 2.4.1. *(Banach-Stone theorem) [35]*

(1) Let $T : C(X, \mathbb{R}) \to C(X, \mathbb{R})$ be an isometry where $C(X, \mathbb{R})$ denotes the set of all continuous real valued maps on a compact T_2 space X. Then there exist a homeomorphism $h : X \to X$, and a continuous map $\phi : X \to \mathbb{R}$ such that $|\phi(x)| = 1$ and for $f \in C(X, \mathbb{R})$ we have $Tf(x) = \phi(x) f(h(x))$ for all $x \in X$.

(2) Let $T : C(X, \mathbb{C}) \to C(X, \mathbb{C})$ be an isometry where $C(X, \mathbb{C})$ denotes the set of all continuous complex valued maps on a compact T_2 space X. Then there exists a homeomorphism $h : X \to X$, and a continuous map $\phi : X \to S^1$ such that for $f \in C(X, \mathbb{C})$ we have $Tf(x) = \phi(x) f(h(x))$ for all $x \in X$.

Definition 2.4.2. A topological space is said to be rigid if the identity map is the only homeomorphism on it.

Theorem 2.4.3. *Let X be a compact connected rigid T_2 space and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then for every linear isometry $T : C(X, \mathbb{K}) \to C(X, \mathbb{K})$, $Per(T)$ is in $\mathfrak{F}_2$ whenever all points of $C(X, \mathbb{K})$ are T-periodic and $Per(T)$ is finite.*

Proof. Let $\phi : X \to S$ be a continuous map, and $h : X \to X$ be a homeomorphism where $S = S^1$ whenever $K = \mathbb{C}$, and $S = \mathbb{R}$ whenever $K = \mathbb{R}$. Define $T_h(f) = f \circ h$ and $M_\phi(f) = \phi f$ for all $f \in C(X, \mathbb{K})$. Then T_h and M_ϕ are linear isometries on $C(X, \mathbb{K})$. Because, for $x \in X, \alpha, \beta \in \mathbb{K}, f, g \in C(X, \mathbb{K})$; we have

$M_\phi(\alpha f + \beta g)(x) = \phi(\alpha f + \beta g)(x) = \alpha\phi f(x) + \beta\phi f(x) = (\alpha\phi f + \beta\phi g)(x)$ and

$||\phi f|| = Sup_{x\in X}|\phi(x)f(x)| = Sup_{x\in X}|f(x)| = ||f||$.

$T_h(\alpha f + \beta g) = (\alpha f + \beta g)h = \alpha(f \circ h) + \beta(f \circ h) = \alpha T_h(f) + \beta T_h(f)$ and

$||f \circ h|| = Sup_{x\in X}|f(h(x))|$. As x varies in X, $h(x)$ varies in throughout X since h is onto. Therefore $Sup_{x\in X}|f(h(x))| = Sup_{y\in X}|f(y)| = ||f||$.

Now consider the set $\mathcal{C} = \{M_\phi \circ T_h : h : X \to X$ homeomorphism, $\phi : X \to S$ is continuous$\}$. Note that composition of two isometries are isometries. Then by Banach-stone theorem $\mathcal{C}$ is the collection of all isometries on $C(X, \mathbb{K})$ up to homeomorphism.

Here $H(X) = \{I\}$ since X is rigid. Suppose that $Per(T)$ is finite and all points are periodic (for all real linear isometries this happen, see Remark 2.4.4). Then $T^n = I$ for some $n \in \mathbb{N}$. Then $(M_\phi)^n(f) = f$ for all f since $(M_\phi)^n = M_{\phi^n}$. Therefore $\phi^n f = f$ for all f. Which implies $(\phi(x))^n = 1$ for all x. ie., $\phi(x)$ is an nth root of unity. Then ϕ is constant since the range of ϕ is countable and X is connected. Hence $Per(T) \in \mathfrak{F}_2$. □

Remark 2.4.4. Let X be a rigid space, and $T : C(X, \mathbb{R}) \to C(X, \mathbb{R})$ be a real isometry then Per(T) is finite and all points are periodic.

Proof. By Theorem 2.4.1(1), for every linear isometry $T : C(X,\mathbb{R}) \to C(X,\mathbb{R})$ there exists a homeomorphism $h : X \to X$ and a continuous function $\phi : X \to \mathbb{R}$ such that $|\phi(x)| = 1$ and for $f \in C(X,\mathbb{R})$ we have $Tf(x) = \phi(x)f(h(x))$ for all $x \in X$. Let $T : C(X,\mathbb{R}) \to C(X,\mathbb{R})$ be a linear isometry. Note that T^n is also a linear isometry for all $n \in \mathbb{N}$. Again by Theorem 2.4.1(1), we have $T^n = T^m$ for some $m \neq n$. Hence the proof. □

For each self map f on a set X, we associate a subset Per(f) of $\mathbb{N}$. If f belongs to a certain nice class of function, then not all subsets of $\mathbb{N}$ may arise as the set of periods.

It is natural to ask: Which subsets of $\mathbb{N}$ arise as $Per(f)$, for some f in that class? We answer this question, for some classes of linear operators, as shown in the following chart.

The class of maps, $n \in \mathbb{N}$	**Period sets**
Linear operators on $\mathbb{C}^n$	$\mathfrak{F}_{2n} = \{\{1\} \cup \tilde{A} : A \subset \mathbb{N}\ and\ \|A\| \leq n\}$
Linear operators on $\mathbb{R}^{2n}$	same as above
Linear operators on $\mathbb{R}^{2n+1}$	$\mathfrak{F}_{2n} \cup \{\{1\} \cup \tilde{A} : A \subset \mathbb{N} \setminus \{1\}, 2 \in A$ and $\|A\| = n+1\}$
Linear operators on l^2 having finite rank	$\mathfrak{F} = \bigcup_n \mathfrak{F}_n$, where $\mathfrak{F}_n s$ are as in introduction
Isometries of l^2	$\{A \subset \mathbb{N} :\ A = \tilde{A}$ and $1 \in A\}$
Linear operators on a vector space	$\{A \subset \mathbb{N} :\ A = \tilde{A}$ and $1 \in A\}$
Linear operators on an infinite dimensional vector space	$\{A \subset \mathbb{N} :\ A = \tilde{A}$ and $1 \in A\}$

Chapter 3

Dynamics of Subshifts

The subshifts form a very important class of dynamical systems. This is a dynamically rich class. Most of the dynamical properties such as transitivity, sensitivity, chaos, recurrence, limit sets, etc. can be easily understood in this class.

Milnor and Thurston have proved that if f is a piecewise monotonic map on the closed interval I, then there is a countable subset C of I such that $I\backslash C$ is f-invariant and such that $f|_{I\backslash C}$ is topologically conjugate to some subshift. Therefore, if we understand the dynamics of subshifts, then we can hope to understand the dynamics of large class of maps on I. There are plenty of books that explain how their study would throw light on still larger classes of dynamical systems (see [18], [23] and [37]). In this chapter, we study the subshifts as topological dynamical systems.

3.1 Introduction

In this section, we introduce some preliminaries on symbolic dynamics. We concentrate mainly on two-sided shifts. The case of one-sided shifts is similar. Let $\mathcal{A}$ be a non-empty finite set with discrete topology. We refer $\mathcal{A}$ to an alphabet. Next consider the set $\mathcal{A}^{\mathbb{Z}}$, which denotes the set of doubly-infinite sequences $(x_i)_{i\in\mathbb{Z}}$ where each $x_i \in \mathcal{A}$, with product topology. It is compact, metrizable, totally disconnected space without isolated points, and homeomorphic the Cantor set. There is a natural countable base of clopen sets, called **cylinders**, for the product topology on $\mathcal{A}^{\mathbb{Z}}$. A cylinder is a set of the form $C^{n_1,n_2,...,n_k}_{j_1,j_2,...,j_k} = \{(x_l) : x_{n_i} = j_i, i = 1, 2, ..., k\}$, where $n_1, n_2, ..., n_k$ are indices in $\mathbb{Z}$, and $j_i \in \mathbb{Z}$. The shift is the homeomorphism $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ given by $\sigma(x)_i = x_{i+1}$ for all $i \in \mathbb{Z}$. The pair $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ forms a dynamical system called a full shift (two-sided). A subshift is a σ-invariant non-empty closed subset X of a full shift, together with the restriction of σ to X (we call simply X as a subshift without referring σ). We denote the set of all periodic points of σ in X by $P(X)$, and the set of periods of all periodic points of σ in X by $\mathrm{Per}(X)$. A word w of length k on $\mathcal{A}$ is a concatenation $w_1w_2...w_k$, where each $w_i \in \mathcal{A}$, $\overline{w}$ is defined by $(\overline{w})_n = w_r$ whenever $n \equiv r(\textbf{mod}\ k)$. If $|\mathcal{A}| = k$ then choosing a word $x = x_1x_2...x_n$ is same as filling n blanks with k elements of $\mathcal{A}$, which can be done in k^n ways where $|\mathcal{A}|$ denotes the cardinality of $\mathcal{A}$. Hence there are only finitely many such x. A subshift X is said to be a subshift of finite type (we call simply SFT) if $X = X_{\mathcal{F}} = \{x \in \mathcal{A}^{\mathbb{Z}} : \text{no word in } \mathcal{F} \text{ appears in } x\}$ for some finite set of words $\mathcal{F}$. An SFT X is said to be a k-step SFT if there exists a finite set of words $\mathcal{F}$ having length atmost k such that $X = X_{\mathcal{F}}$. A subshift is said to be sofic if there exists a continuous surjection from an SFT to it. Note that all SFTs are sofic shifts. For $X \subset A^{\mathbb{Z}}$, we denote $W_k(X)$ the set of words of length k that occur in X. For a finite subset A of $\mathbb{N}$, we denote $A \subset\subset \mathbb{N}$. An admissible metric on $\mathcal{A}^{\mathbb{Z}}$ is given by $d_1(x, y) = \sum_{i\in\mathbb{Z}} \frac{\rho(x_i,y_i)}{2^{|i|}}$, ρ is the discrete metric on $\mathcal{A}$. Another metric on $\mathcal{A}^{\mathbb{Z}}$ is given by $d_2(x, y) = 2^{-l}$, where $l = min\{|i| : x_i \neq y_i\}$. Both the metrics d_1, d_2 generates the product topology on $\mathcal{A}^{\mathbb{Z}}$, and note that for each $x \in \mathcal{A}^{\mathbb{Z}}$ the open d_2-ball with radius 2^{-l} is the symmetric cylinder $C^{-l,-l+1,...,l}_{x_{-l},x_{-l+1},...,x_l}$. Since the alphabet set $\mathcal{A}$ is finite, we may assume that $\mathcal{A} = \{0, 1, ..., m-1\}$ for some $m \geq 2$. Then $\mathcal{A}^{\mathbb{Z}}$ becomes an abelian group with coordinatewise addition modulo m.

Now we consider the following well known examples.

Example 3.1.1. Even shift:

Let X be the set of sequences of 0s and 1s in which two successive appearances of 1 are separated by a block of consecutive 0s of even length (which may be the empty block, of length zero). Here $\mathrm{Per}(X)$ is $\mathbb{N}$.

Example 3.1.2. Square-Free Sequences:

This subshift is defined by forbidding any subword to immediately follow a copy of itself. This is a more complicated subshift. Here $\mathrm{Per}(X)$ is $\emptyset$.

Now we take up the natural question: Which subsets of $\mathbb{N}$ arises as the set of periods of Subshifts? The above two examples show that $\emptyset$ and $\mathbb{N}$ should come in such a collection of subsets of $\mathbb{N}$. We answer this question in Sec. 3.2.2.

3.2 Set of periods of subshifts

Definition 3.2.1. A graph G is a set V of vertices and a set E of edges (both sets finite unless declared otherwise) such that all the endpoints of edges in E are contained in V. It is often denoted $G = (V, E)$, or (V_G, E_G), or $(V(G), E(G))$. Sometimes, each edge is regarded as a pair of vertices.

Definition 3.2.2. Let A be a $k \times k$ matrix with entries 0 or 1, called adjacency matrix. We call the matrix irreducible if for all $1 \leq i, j \leq k$, there exists $N \in \mathbb{N}$ such that $A^N(i, j) > 0$.

Vertex shift

Let k be a fixed positive integer and A be an adjacency matrix of order k. If i and j are integers between 1 and k, we denote by $A(i,j)$ the entry in the i^{th} row and j^th column of A. We take the full shift $\sum_k = \{1, 2, ..., k\}^{\mathbb{Z}}$ (where the suffix k is same as the number of rows in the given matrix A) and then define a subset X_A^v of $\sum_k$ as follows:

$X_A^v = \{x \in \sum_k : A(x_i, x_{i+1}) = 1$ for all integers $i \in \mathbb{Z}\}$. Here, x_i is the i^{th} term of x, and it is an integer between 1 and k; similarly x_{i+1} is also an integer between 1 and k. Therefore $A(x_i, x_{i+1})$ makes sense, and it is either 0 or 1. If it is 1, and if the same happens for all $i \in \mathbb{Z}$, then we take x in the subset X_A^v. If $A(x_i, x_{i+1}) = 0$ for some $i \in \mathbb{Z}$, then x is not in X_A^v. Two consecutive terms of x are said to form a 2-block in x. It is of the form $x_i x_{i+1}$ for some $i \in \mathbb{Z}$. The matrix A is used as follows:

If $A(i,j) = 0$, then the 2-block ij is forbidden in the sense that for all elements $x \in X_A^v$ and for all $m \in \mathbb{Z}$ the 2-block $x_m x_{m+1}$ will never be the 2-block ij. Thus A decides which 2-blocks are forbidden for X_A^v. This gives rise to another description of X_A^v as follows:

$X_A^v = \{x \in \sum_k :$ if $A(i,j) = 0$, then the 2-block ij is not a 2-block in $x\}$. The subshift X_A^v (see the Remark 3.2.3) induced by the matrix A is called the vertex shift.

Digraphs:

A directed graph or digraph is a graph in which every edge is directed. A digraph G has a finite set V of vertices and a subset E of $V \times V$ called the set of edges. If $(x, y) \in E$, we say that there is an edge from the vertex x to the vertex y. For every digraph G, there is an associated matrix A_G, called its adjacency matrix. Its size is $k \times k$ where $k = |V|$. We index the elements of V as $\{v_1, v_2, ..., v_k\}$. The $(i,j)^{th}$ entry of the adjacency matrix is 1 if $(v_i, v_j) \in E$ and 0 otherwise. In the reverse direction, if A is any $k \times k$ matrix of zeros and ones, there is a digraph G_A whose vertex set is $V = \{1, 2, ..., k\}$ and the edge set is $E = \{(i, j) : A(i, j) = 1\}$. We say that a digraph $G = (V, E)$ is strongly connected if given any two vertices u and v, there is a direct path from u to v, that is there is a finite sequence $\{u = v_0, v_1, ..., v_r = v\}$ such that $(v_i, v_{i+1}) \in E$ for all $i = 0, 1, ..., r - 1$.

Remark 3.2.3. X_A^v is the set of all doubly infinite paths in the digraph G_A specified by a sequence of vertices. A directed doubly infinite path in G_A is defined as a sequence $(v_n)_{n\in\mathbb{Z}}$ in V such that $(v_n, v_{n+1}) \in E$ for all $n \in \mathbb{Z}$. The set of all directed paths in G is $\bigcap_{n\in\mathbb{Z}} \bigcup_{(v,w)\in E} C_{n,n+1}^{v,w}$. Hence this set is either an empty set or an SFT.

Edge Shift

Let A be an adjacency matrix. An infinite path in the digraph G_A can also be specified by a sequence of edges (rather than vertices). This gives a subshift X_A whose alphabet is the set of edges in G_A. More generally, a finite directed graph G, possibly with multiple directed edges connecting pairs of vertices, corresponds to a matrix A whose $(i, j)^{th}$ entry is a non-negative integer specifying the number of directed edges in G from the i^{th} vertex to the j^{th} vertex. The set X_A^e of infinite directed paths in G_A, labeled by the edges, is a subshift, and is called the edge shift determined by A. Any edge shift is a subshift of finite type.

3.2.1 Set of periods of subshifts of finite type

Now we take up the natural question: Which subsets of $\mathbb{N}$ arise as the set of periods of SFTs? To answer this question, we introduce some preliminaries from graph theory.

The notion of strongly connected digraph is well known. For every subshift of finite type, there is an associated digraph and an associated matrix as described above (see [18], [37]). This may or may not be strongly connected. But, the period set of a strongly connected simple digraph can be described easily.

Let $G = (V, E)$ be any digraph with vertex set V and edge set E. A subgraph $G' = (V', E')$ of G is said to be a full subgraph if $E' = E \cap \{(v_1, v_2) : v_1, v_2 \in V'\}$. A digraph is said to be simple if from every vertex v to a vertex w there is at most one edge and it is said to be strongly connected if between any two vertices there exists a directed path. It is to be noted that a connected digraph (in the corresponding undirected graph there exists a path between any two vertices) may not be strongly connected. The subshift of finite type associated with a simple digraph G is denoted as X_G. Note that $\mathrm{Per}(X_G) = \mathrm{Per}(X_{G'})$, G' is a finite union of such strongly connected simple digraphs (see the proof of Theorem 3.2.16). A cycle (closed directed path) C is said to be simple if X_C contains a periodic point whose period is exactly the length of C. (Here the terminology is different from the terminology in graph theory. In graph theory, a closed directed path with no repeated vertices other than the starting and ending vertices are usually called simple cycles.)

Proposition 3.2.4. *[18] Every SFT is conjugate to one in which every forbidden word has length 2.*

Proof. Let X be a k-step SFT with $k > 0$. Let Γ be the directed graph whose set of vertices is $W_k(X)$; a vertex $x_1...x_k$ is connected to a vertex $x'_1...x'_k$ by a directed edge if $x_1...x_kx'_k = x_1x'_1...x'_k \in W_{k+1}(X)$. Let A be the adjacency matrix of Γ. The continuous map $c(x)_i = x_i...x_{i+k-1}$ commutes with shifts and gives a conjugacy from X to X^e_A; and the map uv to e, where e is the edge from u to v, defines a 2-block conjugacy from X^e_A to X^v_A. Conversely, any edge shift is naturally conjugate to a vertex shift. □

For $a, b \in \mathbb{N}$, let $\gcd(a, b)$ denote the greatest common divisor (gcd) of a and b.

Lemma 3.2.5. [8] Let $a, b \in \mathbb{N}$.

(i) If $\gcd(a,b) = 1$ then for every $n > ab$ there exist positive integers x and y such that $n = ax + by$.

(ii) If $\gcd(a,b) = 1$ and $n = ab$ then there are no positive integers x and y such that $n = ax + by$.

Proof. If $\gcd(a,b) = 1$ there exist positive integers x and y such that $ax - by = 1$. For every $n > ab$, write $n = m(ab) + k$, $1 \leq k \leq ab$. Hence the lemma follows. □

The following remarks easily follows from the Lemma 3.2.5.

Remark 3.2.6. If $a_i \in \mathbb{N}$, $i = 1, 2, ..., n$, and $\gcd(a_1, ..., a_n) = 1$ then for every positive integer $n > a_1 a_2 ... a_n$ there exist positive integers, x_is such that $n = a_1 x_1 + a_2 x_2 + ... + a_n x_n$.

Remark 3.2.7. If $a_i \in \mathbb{N}$, $i = 1, 2, ..., n$, and $(a_1, a_2, ..., a_n) = k$ then for every $n > \frac{a_1 a_2 ... a_n}{k^n}$ there exist positive integers, x_is such that $kn = a_1 x_1 + a_2 x_2 + ... + a_n x_n$.

Theorem 3.2.8. *The following are equivalent for a subset S of $\mathbb{N}$.*

(1) *$S = Per(X_G)$ for some strongly connected simple digraph G.*

(2) *Either S is singleton or $S = k\mathbb{N} \setminus F$ for some $k \in \mathbb{N}$ and for some $F \subset\subset \mathbb{N}$.*

Proof. $1 \implies 2$

Assume (1) and S is not singleton. Let $k = \gcd(S)$.

Claim: $k\mathbb{N} \setminus S$ is finite.

Case 1. When $k \in S$.

If m and l are distinct and are lengths of two cycles having one vertex common, then $al + bm \in S$ for all $a, b \in \mathbb{N}$. This is because, consider the cycle starting and ending at the above common vertex such that which wind the first cycle exactly a times and the second cycle exactly b times. If G is a strongly connected digraph then for all $m \in S$

there exists $l \in S$ (since S is not singleton) such that $al + bm \in S$ for all $a, b \geq 1$. Let $k' = \gcd(l, m)$. Then all but finitely many elements of $k'\mathbb{N}$ are in S by Lemma 3.2.5 and hence $m\mathbb{N} \setminus S$ is finite for all $m \in S$. In particular, $k\mathbb{N} \setminus S$ is finite.

Case 2. When $k \notin S$.

We first claim that there exists a finite subset S' of S such that k is equal to the gcd of all its elements. Note that $k < s$ for all $s \in S$. Let $s_1 \in S$. Since $k = \gcd(S)$ there exists $s_2 \in S$ such that $k \leq \gcd(s_1, s_2) < s_1$. If $\gcd(s_1, s_2) = k$ then take $S' = \{s_1, s_2\}$. Else there exists $s_3 \in S$ such that $k \leq \gcd(s_1, s_2, s_3) < \gcd(s_1, s_2)$. If $k = \gcd(s_1, s_2, s_3)$ then take $S' = \{s_1, s_2, s_3\}$. After a finite number of steps we will get S', say $\{s_1, s_2, ..., s_r\}$, as we claimed. Let C_i be a simple cycle having length s_i and $x_i \in C_i$ for $i = 1, 2, ..., r$. Then there exists a directed path from $x_{i (mod\ r)}$ to $x_{i+1 (mod\ r)}$ which doesn't contain any cycle. Let C be a simple cycle of length s obtained from the cycles C_is and the above paths. Then $s_i\mathbb{N} + s\mathbb{N} \subset S$ for $i = 1, 2, ..., r$. Hence $s_1\mathbb{N} + s\mathbb{N} + s_2\mathbb{N} + s\mathbb{N} + ... + s_r\mathbb{N} + s\mathbb{N} \subset S$. ie., $s_1\mathbb{N} + s_2\mathbb{N} + ... + s_n\mathbb{N} + rs\mathbb{N} \subset S$. But $s_1\mathbb{N} + s_2\mathbb{N} + ... + s_n\mathbb{N} + rs\mathbb{N}$ contains all but finitely many elements of $k\mathbb{N}$ by Lemma 3.2.5. Therefore $k\mathbb{N} \setminus S$ is finite. Hence $S = k\mathbb{N} \setminus F$ where $F = k\mathbb{N} \setminus S$.

$2 \implies 1$

Case 1. When $S = \{k\}$ for some $k \in \mathbb{N}$.

Consider a strongly connected simple digraph with exactly one cycle of length k.

Case 2. When $S = k\mathbb{N} \setminus F$ for some $k \in \mathbb{N}$ and for some $F \subset\subset \mathbb{N}$.

Subcase 1. When $F = \emptyset$.

If $k = 1$ then consider a strongly connected simple digraph having at least two vertices with one loop. Otherwise, consider a strongly connected simple digraph with two cycles of length k having exactly one vertex in common.

Subcase 2. When $F \neq \emptyset$.

Without loss of generality, we can assume that $F \subset\subset k\mathbb{N}$. Let $F = \{k_1, k_2, ..., k_n\}$. Let $F' = \{l_i : l_i = \frac{k_i}{k}\}$ and $max(F') = m$. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $n \in \mathbb{N} \setminus F'$ there exist $a, b \in \mathbb{N}$ such that $n = a(m+1) + b(m+2)$. Let $\{p_1, p_2, ..., p_r\} = \{p \in \mathbb{N} \setminus F' : p \leq n_0\}$. Now consider a strongly connected simple digraph made of cycles of length $k(m+2), k(m+1), kp_1, k(m+1), kp_2, k(m+1), ..., k(m+1), kp_r$ arranged cyclically with exactly one vertex common to all pairs of adjacent cycles. Then $\text{Per}(X_G) = k\mathbb{N} \setminus F$. □

Now we have an immediate corollary to the Theorem 3.2.8.

Corollary 3.2.9. *The following are equivalent for a subset S of $\mathbb{N}$.*

(1) $S = Per(X_G)$ *for some simple digraph G.*

(2) $S = Per(X)$ *for some SFT X.*

(3) $S = \bigcup_{i=1}^{n}(k_i\mathbb{N} \setminus F_{k_i}) \cup F$ *for some $k_i, n \in \mathbb{N}$ and for some $F_{k_i}, F \subset\subset \mathbb{N}$.*

Remark 3.2.10. Let A be an $m \times m$ adjacency matrix with non-zero rows and columns. Then the following are equivalent.

(i) A is irreducible.

(ii) The digraph induced by A is strongly connected.

(iii) The subshift X_A is transitive.

Proof. The equivalence of (i) and (iii) is a well known result (see [18]), and the equivalence of (i) and (ii) easily follows from the following known result.

The number of allowed words of length $n+1$ beginning with the symbol i and ending with the symbol j is the ij^{th} entry of A^n (see [18]). □

Remark 3.2.11. Suppose some row of A or column of A is of full of zeros, say i^{th} row. Then remove i^{th} row and i^{th} column. Doing this for all such i we obtain another matrix $\tilde{A}$ of smaller size. Then X_A and $X_{\tilde{A}}$ are in a sense one and the same. Therefore the equivalence of (ii) and (iii) is true for all subshifts induced by adjacency matrices.

Now we have:

Theorem 3.2.12. *The following are equivalent for an SFT X.*

(i) *There exists $x \in X$ such that $X = X_{\mathcal{F}}$ where $\mathcal{F} = \{$words not occurring in $x\}$.*

(ii) *There exists $x \in X$ such that the closure of σ-orbit of $x = X$.*

(iii) *X is transitive.*

(iv) *The graph of X is strongly connected.*

Proof. The equivalence (ii) and (iii) follows from Theorem 1.1.7.

$(ii) \Rightarrow (i)$

Let $w = w_1 ... w_n$ be a word not occurring in x. Then w does not occur in $\sigma^n(x)$ for all $n \in \mathbb{N}$. If $y \in$ Closure of σ-orbit of x, then w does not occur in y. Because $\{z : w \text{ occur in } z\} = \bigcup_i C^{i+1,i+2,\ldots,i+|w|}_{w_1,w_2,\ldots,w_n}$ is open in Σ_A. Therefore the closure of σ-orbit of $x \subset X_{\mathcal{F}}$.

Let $\mathcal{F} = \{$words not occurring in $x\}$, $z \in X_{\mathcal{F}}$ and C be any cylinder containing z. Any word occur in z occur in x also. Therefore $\sigma^n(x) \in C$ for some $n \in \mathbb{Z}$. Hence $X_{\mathcal{F}} \subset$ Closure of σ-orbit of x. Hence the proof.

All other implications follows from Remarks 3.2.10 and 3.2.11. □

Remark 3.2.13. The period set of a transitive subshift of finite type is either a singleton subset of $\mathbb{N}$ or a set of the form $k\mathbb{N} \setminus F$ for some $k \in \mathbb{N}$ and for some $F \subset\subset \mathbb{N}$.

Proof. This remark follows from Theorem 3.2.8, and Remarks 3.2.11 and 3.2.10. □

Remark 3.2.14. A subset of $\mathbb{N}$ arises as the set of all periods of a Devaney chaotic non-singleton SFT if and only if it is of the form $k\mathbb{N} \setminus F$ for some positive integer k and for some $F \subset\subset \mathbb{N}$.

Proof. This remark follows from the Remark 3.2.13 and Theorem 1.2.4. □

In our next theorem, we give different proof for Corollary 3.2.9. The proof seems to be nice and different even though it is lengthy. By comparing the proof of Corollary 3.2.9 and Theorem 3.2.15, we can understand the use of graph theory in topological dynamics.

Theorem 3.2.15. *The following are equivalent for a subset S of $\mathbb{N}$.*

(1) $S = Per(X_{\mathcal{F}})$ *for some SFT* $X_{\mathcal{F}}$.

(2) $S = \bigcup_{i=1}^{l}(m_i\mathbb{N} \setminus F_{m_i}) \cup F$ *for some* $m_i, l \in \mathbb{N}$ *and for some* $F, F_{m_i} \subset\subset \mathbb{N}$.

Proof. $1 \implies 2$

Let $S = \text{Per}(X_{\mathcal{F}})$ for some SFT $X_{\mathcal{F}}$. Assume without loss of generality that every member of $\mathcal{F}$ has length 2 because of Proposition 3.2.4.

First we claim that $\text{Per}(X_{\mathfrak{F}}) \neq \emptyset$. Let $y \in X_{\mathfrak{F}}$. Since $\mathcal{A}$ is finite, there exist $i < j$ such that $y_i = y_j$. Then $\bar{y}_{[i,j)}$ where $y_{[i,j)} = y_i y_{i+1} ... y_{j-1}$ is periodic point in $X_{\mathfrak{F}}$ since every subword of length two occurs in y.

Take some $x \in X_{\mathcal{F}}$ that is periodic. Then $x = \overline{x_1 x_2 ... x_n}$ for some word $x_1 x_2 ... x_n$ on $\mathcal{A}$ where $\mathcal{A} \supset \{x_1, x_2, ..., x_n\}$. We may assume that the period of x is n.

Case 1. When $n \leq |\mathcal{A}|$ ($|\mathcal{A}|$ denotes the cardinality of $\mathcal{A}$).

There are only finitely many such x. Their periods will be taken as elements of the finite set F that we are constructing.

Case 2. When $n > |\mathcal{A}|$.

Then the symbols x_i cannot be all distinct for $1 \leq i \leq n$. Let $\{j - i : x_i = x_j \neq x_k, 1 \leq i < k < j \leq n\} = \{a_1, a_2, ..., a_r\}$. Here $\bar{x}_{[i,j)} \in X_{\mathcal{F}}$ since every subword of length two occurs in x. Write $x = x_1 x_2 ... x_{|\mathcal{A}|} x_{|\mathcal{A}|+1} ... x_n$. Therefore $x_i = x_j$ for some $i \neq j$, $1 \leq i, j \leq |\mathcal{A}| + 1$. Therefore one of these a_is, say a is less than or equal to $|\mathcal{A}|$. Hence the numbers of the form $ra + sn$ are in $\text{Per}(X_{\mathcal{F}})$ for $r, s \in \mathbb{N}$ since $\bar{x}_{[i,j)}, \bar{x} \in X_{\mathcal{F}}$. By Lemma 3.2.5, $\{ra + sn : r, s \in \mathbb{N}\}$ contains $k\mathbb{N} \setminus F$ for some $F \subset\subset k\mathbb{N}$ where $k = \gcd(a, n)$. Clearly $k \leq |\mathcal{A}|$. Therefore for all periodic points $x \in X_{\mathcal{F}}$ except finitely many, there exists $k_x \in \mathbb{N}$, $k_x \leq |\mathcal{A}|$ such that the period of x is in $k_x\mathbb{N}$ and $k_x\mathbb{N} \setminus F_{k_x} \subset \text{Per}(X_{\mathcal{F}})$ for some $F_{k_x} \subset\subset \mathbb{N} \setminus \{\text{the period of } x\}$. Therefore $\text{Per}(X_{\mathcal{F}}) = F \cup \bigcup_{x \in P(X_{\mathcal{F}})}(k_x\mathbb{N} \setminus F_{k_x}) = F \cup \bigcup_{m \in B \subset \{1,2,...,|\mathcal{A}|\}}(m\mathbb{N} \setminus F_m)$ for some $F, F_m \subset\subset \mathbb{N}$.

$2 \implies 1$

Assume (2). ie., $S = \bigcup_{i=1}^{l}(m_i\mathbb{N} \setminus F_{m_i}) \cup F$ for some $m_i, l \in \mathbb{N}$ and for some $F, F_{m_i} \subset\subset \mathbb{N}$.

Case 1. When S is finite.

Let $S = \{k_1, k_2, ..., k_n\}$ and $\mathcal{A}_i = \{x_{i1}, x_{i2}, ..., x_{ik_i}\}$ be a set of k_i distinct symbols for $1 \leq i \leq n$ such that $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ whenever $i \neq j$. Let $\mathcal{F}_i$ be equal to the set of all words on $\mathcal{A}_i$ of length two not occurring in $x_{i1}x_{i2}...x_{ik_i}x_{i1}$ for $i = 1, 2, ..., n$ and $\mathcal{F}' = \{xy : x \in \mathcal{A}_i, y \in \mathcal{A}_j, i \neq j\}$. Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup ... \cup \mathcal{F}_n \cup \mathcal{F}'$. Then $\text{Per}(X_{\mathcal{F}}) = S$.

Case 2. When $S = k\mathbb{N} \setminus F$ for some $F \subset\subset \mathbb{N}$.

Subcase 1. When $F = \emptyset$.

If k=1 then consider an alphabet with at least two distinct elements and take $\mathcal{F} = \phi$. Then $\text{Per}(X_{\mathcal{F}}) = \mathbb{N}$.

Otherwise, consider an alphabet $\mathcal{A} = \{1, u_1, u_2, ..., u_{k-1}, v_1, v_2, ..., v_{k-1}\}$. Let $\mathcal{F}$ be equal to the set of all words on $\mathcal{A}$ of length two not occurring in $1u1v1$ where $u = u_1u_2...u_{k-1}$ and $v = v_1v_2...v_{k-1}$. Then $\text{Per}(X_{\mathcal{F}}) = k\mathbb{N}$.

Subcase 2. When $F \neq \emptyset$.

Without loss of generality, we can assume that $F \subset\subset k\mathbb{N}$. Let $F = \{k_1, k_2, ..., k_n\}$, $F' = \{l_i : l_i = \frac{k_i}{k}\}$ and $max(F') = m$. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $n \in \mathbb{N} \setminus F'$ there exist $a, b \in \mathbb{N}$ such that $n = a(m+1) + b(m+2)$. Let $\{p_1, p_2, ..., p_r\} = \{p \in \mathbb{N} \setminus F' : p \leq n_0\}$. Let $\mathcal{A}_i = \{x_{i1}, x_{i2}, ..., x_{in_i}\}$ be a set of n_i distinct symbols for $1 \leq i \leq r+2$ such that $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ whenever $1 \leq i < j \leq r+2,\ i \neq r+1$; and $\mathcal{A}_{r+1} \cap \mathcal{A}_{r+2}$ is singleton, say $\{x_0\}$ where $n_i = kp_i$ for $i = 1, 2, ..., r$; $n_{r+1} = k(m+1)$ and $n_{r+2} = k(m+2)$. Let $\mathcal{F}_i$ be equal to the set of all words on $\mathcal{A}_i$ of length two not occurring in $x_{i1}x_{i2}...x_{in_i}x_{i1}$ for $i = 1, 2, ..., r+2$ and $\mathcal{F}' = \{xy : x \in \mathcal{A}_i, y \in \mathcal{A}_j, 1 \leq i, j \leq r+2, i \neq j$ and $i + j \neq 2r+3\} \cup \{xy : x \in \mathcal{A}_{r+1} \setminus \{x_0\}, y \in \mathcal{A}_{r+2} \setminus \{\ x_0\}\}$. Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup ... \cup \mathcal{F}_{r+2} \cup \mathcal{F}'$. Then $\text{Per}(X_{\mathcal{F}}) = k\mathbb{N} \setminus F$.

For $S_1, S_2 \subset \mathbb{N}$, let $\text{Per}(X_{\mathcal{F}_1}) = S_1$ and $\text{Per}(X_{\mathcal{F}_2}) = S_2$. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two disjoint alphabets such that the symbols occurring in $X_{\mathcal{F}_1}$ and $X_{\mathcal{F}_2}$ are from $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. Let $\mathcal{F}' = \{xy : x \in \mathcal{A}_1$ and $y \in \mathcal{A}_2\}$, and $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}'$. Then $\text{Per}(X_{\mathcal{F}}) = S_1 \cup S_2$.

Hence the proof. □

Theorem 3.2.16. *Let $X_{\mathcal{F}}$ be an SFT for some finite set of words $\mathcal{F}$ over an alphabet $\mathcal{A}$ with dense set of periodic points. Then there exists some finite set of words $\mathcal{G} \supset \mathcal{F}$ and an SFT $X_{\mathcal{G}}$ with dense set of periodic points such that $X_{\mathcal{G}}$ is a finite union of transitive SFTs, and Per$(X_{\mathcal{F}}) =$ Per$(X_{\mathcal{G}})$.*

Proof. Let X^G denotes the subshift of finite type associated for a simple digraph G such that $X^G = X_{\mathcal{F}}$. Let V be the set of all vertices of G. For $v_1, v_2 \in V$, we say that $v_1 \sim v_2$ whenever there is a directed path from v_1 to v_2 and vice-versa. Then $\sim$ forms an equivalence relation on V. Each equivalence class corresponds to a strongly connected simple digraph. Let G' be the union of all such strongly connected simple digraphs. Observe that $Per(X^G) = Per(X^{G'})$. Now consider $\mathcal{G} \supset \mathcal{F}$ such that $X_{\mathcal{G}} = X^{G'}$. Hence the proof. □

Remark 3.2.17. Theorem 3.2.16 describes the structure for SFTs similar to that for interval maps proved by Blokh et al. (see Theorem 1.1.11).

Multi-dimensional SFTs

Let $d \geq 1$, $\mathcal{A}$ be an alphabet, and let $\mathcal{A}^{\mathbb{Z}^d}$ be the set of all maps from $\mathbb{Z}^d$ to $\mathcal{A}$. For every non-empty subset $X \subset \mathbb{Z}^d$, the map $\pi_X : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^X$ is the projection which restricts each $x \in \mathcal{A}^{\mathbb{Z}^d}$ to X. For every $\mathbf{n} \in \mathbb{Z}^d$, we define $\sigma_{\mathbf{n}} : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ such that $(\sigma_{\mathbf{n}}(x))_{\mathbf{m}} = x_{\mathbf{n+m}}$ where $x = (x_{\mathbf{m}})$. Then the map $\sigma_{\mathbf{n}}$ is a homeomorphism of the compact metric space $\mathcal{A}^{\mathbb{Z}^d}$.

A non-empty closed $\sigma_{\mathbf{n}}$-invariant subset $X \subset \mathbb{Z}^d$ is called a multi-dimensional subshift. A multi-dimensional subshift is said to be a multi-dimensional subshift of finite type if there exists a finite set $F \subset \mathbb{Z}^d$ and a subset $P \subset \mathcal{A}^F$ such that $X = X(F,P) = \{x \in \mathcal{A}^{\mathbb{Z}^d} : \pi_F \circ \sigma_{\mathbf{n}}(x) \in P \ \forall \ \mathbf{n} \in \mathbb{Z}^d\}$. This is only when $X = \{x \in \mathcal{A}^{\mathbb{Z}^d} : \pi_F \circ \sigma_{\mathbf{n}}(x) \in \pi_F(X) \ \forall \ \mathbf{n} \in \mathbb{Z}^d\}$.

Remark 3.2.18. [37] There are multi-dimensional SFTs without any periodic points.

Question: Characterize the set of periods of multi-dimensional SFTs.

3.2.2 Set of periods of a general subshift

Lemma 3.2.19. There exist subshifts without any periodic points (See [34], Sturmian shifts).

Proof. Let ρ_θ be the irrational rotation on the circle S^1 given by $\rho_\theta(z) = e^{i\theta}z$ where $0 < \theta < \frac{\pi}{8}$. Let V be the minor open arc with end points 1 and $e^{i\frac{\pi}{8}}$. Define a sequence $x = (x_n) \in \{0,1\}^{\mathbb{Z}}$ as follows

$$x_n = \begin{cases} 1 & \text{if } \rho_\theta^n(1) \in V \\ 0 & \text{if } \rho_\theta^n(1) \notin V \end{cases}.$$

This sequence $x = (x_n)$ generates a subshift X of the shift space $\{0,1\}^{\mathbb{Z}}$, where $X =$ Closure of $\{\sigma^n(x) : n \in \mathbb{Z}\}$.

Claim: X does not contain any periodic point.

Define $N(1, V) = \{n \in \mathbb{Z} : \rho_\theta^n(1) \in V\}$. Given $d \in \mathbb{N}$, consider $a, a+d, a+2d, ...$ for $a \in \mathbb{N}$ and write $\rho_\theta^{a+nd}(z) = \rho_{d\theta}^n(\rho_\theta^a(z))$. Then there exists $n_0 \in \mathbb{N}$ such that $a + n_0 d \notin N(1, V)$ since the $\rho_{d\theta}$-orbit of any point is dense in S^1. Hence $N(1, V)$ does not contain arbitrarily large arithmetic progressions with a given common difference.

Suppose w is a word containing 1, say $w_i = 1$. Let $d = |w|$. If w^n occurs in x for every $n \in \mathbb{N}$, then $i + nd \in N(1, V)$, which is a contradiction. Therefore $V_{w^m} \bigcap \{\sigma^n(x) : n \in \mathbb{Z}\} = \emptyset$, for some $m \in \mathbb{N}$ and hence $\bar{w} \notin X$. If w does not contain 1 then consider arbitrary large $n_0 \in \mathbb{N}$. Then $V_{w^{n_0}} \cap \{\sigma^n(x) : n \in \mathbb{Z}\} = \emptyset$ and hence $\bar{0} \notin X$.

Therefore X does not contain any periodic point. □

The Theorem 3.2.20 says that, if we relax the SFT condition in Theorem 3.2.15 then every subset of $\mathbb{N}$ will arise as a period set.

We first discuss the ideas of construction.

Given a non-empty subset $A \subset \mathbb{N}$, we want to construct a subshift X such that $\mathrm{Per}(X) = A$. For every $n \in A$, let $F_n = \{\sigma^i(x) : i \in \mathbb{N}_0\}$ for some element $x \in \sum_2 = \{0,1\}^{\mathbb{Z}}$ of σ-period n. We expect $Y =$ Closure of $(\bigcup_{n \in A} F_n)$ to be a subshift as required. Clearly Y is a subshift such that $\mathrm{Per}(Y) \supset A$. But we have to choose F_ns more carefully to ensure that there are no other periods for elements in Y. Because, if $A = \mathbb{N} \backslash \{1\}$ and if $F_n = \sigma$-orbit of $\overline{0^{n-1}1}$. Then $\overline{0} \in Y$ where $Y =$ Closure of $(\bigcup_{n \in A} F_n)$. Therefore $\mathrm{Per}(Y) = \mathbb{N}$, and hence it properly contains A.

Theorem 3.2.20. *Let $S \subset \mathbb{N}$. Then there exists a subshift X such that Per(X) is equal to S.*

Proof. We prove this theorem in the case of one-sided subshifts. A similar proof will work in the case of two-sided subshifts.

If S is empty then the proof follows from Lemma 3.2.19. If $S = \{n_1, n_2, ..., n_k\}$ then consider $x^{(i)} \in \mathcal{A}^{\mathbb{N}}$ such that σ-period of $x^{(i)}$ is n_i for $1 \leq i \leq k$. Let $X = \{\sigma^{m_i}(x^{(i)}) : 0 \leq m_i < n_i, 1 \leq i \leq k\}$.

Let $S \subset \mathbb{N}$ be an infinite set, l be the smallest element in S, and $\mathcal{A} = \{0, 1\}$.

First we prove for $l = 1$.

Let $X =$ Closure of B, where $B = \{\sigma^n(\overline{10^k}) : \ k+1 \in S, 1 \leq n \leq k+1\}$. Then Per$(X)$ contains $S \setminus \{1\}$, since B contains periodic points of period k for all $k \in S$. But every neighbouhood of $\overline{0}$ meets X since S is infinite. Hence Per$(X) \supset S$.

Next we prove that there is no other periodic point in X.

Let $y = \overline{y_1 y_2 .. y_r}$ be in X such that y does not belongs to B. Then the neighbourhood $V = V_{y_1 y_2 ... y_r} = \{(x_l)_{l \in \mathbb{N}} :\in X : x_1 x_2 ... x_r = y_1 y_2 ... y_r\}$ of y meets B. Therefore $y_1 y_2 ... y_r$ occurs in $\overline{10^k}$ for infinitely many ks. Suppose $y_1 y_2 ... y_r \neq 0^r$.

Hence $y_1 y_2 ... y_r$ is a subword of $10^r 10^r$ and contains exactly one 1......(1).

Similarly $y_1 y_2 ... y_r y_1 y_2 ... y_r$ occurs in $\overline{10^k}$ for infinitely many k since $V_{y_1 y_2 ... y_r y_1 y_2 ... y_r}$ is a neighbourhood of y. This is impossible for all large k because of (1).

Hence Per(X) is equal to S.

Next we generalize the proof for all $l \geq 1$.

Let $u = 0$ if $l = 1$, $u = 10$ if $l = 2$ and $u = 10...01$ ($l - 2$ zeros) if $l \geq 3$. Let $X = \overline{B}$ where $B = \{\sigma^n(\overline{u}) : 0 \leq n \leq l\} \bigcup \{\sigma^n(\overline{v_k u^{[\frac{k}{l}]}}) : 0 \leq n \leq k, k+1 \in S \setminus \{l\}\}$, and $v_k = 0$ or 00...01 such that $\overline{v_k u^{[\frac{k}{l}]}}$ has period $k+1$. Note that $\{v_k : k+1 \in S \setminus \{l\}\}$ is finite since $|v_k| \leq l + 1$.

Next we prove that there is no other periodic point in X.

Let $y = \overline{y_1 y_2 .. y_r}$ be in X such that y does not belong to $\{\sigma^n(\overline{v_k u^{[\frac{k}{l}]}}) : 1 \leq n \leq$

$k, k = l$ or $k+1 \in S \setminus \{l\}\}$. Then the neighbourhood $V = V_{y_1 y_2 ... y_r}$ of y meets $\{\sigma^n(\overline{v_k u^{[\frac{k}{l}]}}) : 1 \leq n \leq k, k = l$ or $k+1 \in S \setminus \{l\}\}$ where v_k is an empty word or $v_k = 0$ or $v_k = 00...01$. Therefore $y_1 y_2 ... y_r$ occurs in $\overline{v_k u^{[\frac{k}{l}]}}$ for infinitely many ks.

Hence $y_1 y_2 ... y_r$ is a subword of $u^r v_i u^r$ for some i (i varies over a finite set).

Similarly $y_1 y_2 ... y_r y_1 y_2 ... y_r ... y_1 y_2 ... y_r$ occurs in $\overline{v_k u^{[\frac{k}{l}]}}$ for infinitely many k since $V_{y_1 y_2 ... y_r y_1 y_2 ... y_r ... y_1 y_2 ... y_r}$ is a neighbourhood of y. This is impossible for all large k.

Hence $\mathrm{Per}(X)$ is equal to S. □

The following corollary immediately follows from Theorem 3.2.20.

Corollary 3.2.21. *There are uncountably many conjugacy classes of subshifts.*

Contrast the above corollary with the case of SFTs.

Next we generalize Theorem 3.2.15 for sofic shifts.

3.2.3 Set of periods of sofic shifts

The notion of labeled digraph is well known (see [37], [18]). For every sofic shift there is a labeled digraph and vice versa (see Proposition 3.2.22). For a labeled digraph Γ, we denote X_Γ for the subshift induced by Γ. As for digraphs, we can define simple labeled digraph and strongly connected labeled digraph. First we have to define it for corresponding digraphs. Then consider the corresponding labeled digraphs. As for digraphs, for every labeled digraph Γ there exists another labeled digraph Γ' such that $\mathrm{Per}(X_\Gamma) = \mathrm{Per}(X_{\Gamma'})$ and Γ' is a finite union of strongly connected simple labeled digraphs.

Let Γ be a finite labeled digraph, the edges of Γ are labeled by an alphabet $\mathcal{A}_m = \{1, 2, ..., m\}$. Note that we do not assume that the different edges of Γ are labeled differently. Let $E(\Gamma)$ denotes the set of all edges of Γ. The subset $X_\Gamma \subset \mathcal{A}_m^{\mathbb{Z}}$ consisting of all infinite directed paths in Γ is closed and shift invariant (see the proof of Remark 3.2.3). If a subshift X is topologically conjugate to X_Γ for some labeled digraph Γ, then we say that Γ is a presentation of X.

Proposition 3.2.22. *[18] A subshift $X \subset \mathcal{A}^{\mathbb{Z}}$ is sofic if and only if it admits a presentation by a finite labeled digraph.*

Theorem 3.2.23. *The following are equivalent for a subset S of $\mathbb{N}$.*

(1) $S = Per(X_\Gamma)$ for some strongly connected simple labeled digraph Γ.

(2) Either S is a singleton subset of $\mathbb{N}$ or $S = k\mathbb{N} \setminus F$ for some subset $F \subset\subset \mathbb{N}$ and for some $k \in \mathbb{N}$.

Proof. $1 \implies 2$

Assume (1) and S is not singleton. Let $k = \gcd(S)$.

Claim: $k\mathbb{N} \setminus S$ is finite.

As in the proof for strongly connected digraphs we have two cases (see the proof of Theorem 3.2.8 for details)

Case 1. When $k \in S$.

If Γ is strongly connected labeled digraph then for all $m \in S$ there exists $l \in S$ such that $al + bm \in S$ for all $a, b \in \mathbb{N}$. Hence $k\mathbb{N} \setminus S$ is finite.

Case 2. When k is not in S.

We can give a proof similar to the Case 2 of the proof of Theorem 3.2.8.

$2 \implies 1$

All strongly connected simple digraphs give rise to vertex shifts. Every vertex shift is conjugate to an edge shift (see [18]). Every edge shift is given by a finite labeled digraph. Hence the proof follows from Theorem 3.2.8. □

Corollary 3.2.24. *The following are equivalent for a subset S of $\mathbb{N}$.*

(1) $S = Per(X_\Gamma)$ *for some simple labeled digraph* Γ.

(2) $S = \bigcup_{i=1}^{n}(k_i\mathbb{N} \setminus F_{k_i}) \cup F$ *for some* $k_i, n \in \mathbb{N}$ *and for some* $F_{k_i}, F \subset\subset \mathbb{N}$.

Theorem 3.2.25. *The period set of a transitive sofic shift is either a singleton subset of* $\mathbb{N}$ *or a set of the form* $k\mathbb{N} \setminus F$ *for some* $k \in \mathbb{N}$ *and for some* $F \subset\subset \mathbb{N}$.

Proof. Note that the set of all symmetric cylinders form a basis for the product topology on $\mathcal{A}^{\mathbb{Z}}$. Therefore for every non-empty open set U in X_Γ we can choose $(i_n)_{n\in\mathbb{Z}} \in U$ for $M > 0$ sufficiently large such that

$U \supset \{(x_n)_{n\in\mathbb{Z}} : x_k = i_k, -M \leq k \leq M\}$......(1).

Therefore, if X_Γ is transitive then for every $i, j \in E(\Gamma)$, there exists a direct path of length n from i to j for some $n \in \mathbb{N}$. Conversely, assume that for every $i, j \in E(\Gamma)$, there exists a directed path of length n from i to j for some $n \in \mathbb{N}$; and consider two non-empty open sets U and V. Then by (1), we can prove that $\sigma^{2M+n}(U) \cap V$ is non-empty (see the proof of Lemma 3.3.8). Therefore X_Γ is transitive if and only if Γ is a strongly connected labeled digraph. Hence the proof follows from Theorem 3.2.23. □

3.3 Transitivity, weak mixing, mixing

The following are equivalent for a topological graph map $f : G \to G$ (See [15], [20] for interval maps, See [4] for topological graph maps).

(i) f is transitive and $\mathrm{Per}(f)$ is cofinite (ie., $\mathbb{N} \setminus \mathrm{Per}(f)$ is finite).

(ii) f is weak mixing.

(iii) f is mixing.

It is natural to ask on which classes of dynamical systems a similar result will be true. We find that the same result is true in the class of non-singleton SFTs but not for sofic shifts. In the case of sofic shifts, only (ii) and (iii) are equivalent. Note that

SFTs and topological graph maps are different kinds of dynamical systems, and we cannot hope to have a similarity of proofs.

In Chapter 4, we observe that the above equivalence is true in the class of continuous 2-dimensional toral automorphism (see Theorem 4.2.12). In this section, we consider SFTs and sofic shifts.

Definition 3.3.1. Let A be a $k \times k$ adjacency matrix. We call the matrix primitive if there exists $N \in \mathbb{N}$ such that $A^N > 0$.

Proposition 3.3.2. *[18] An SFT induced by a matrix A with non-zero rows and columns is mixing if and only if A is primitive.*

Theorem 3.3.3. *The following are equivalent for a subset S of $\mathbb{N}$.*

(1) *$S = Per(X_G)$ for some strongly connected simple digraph G containing cycles of lengths $m_1, ..., m_k$ such that $gcd(m_1, m_2, ..., m_n) = 1$.*

(2) *Either $S = \{1\}$ or $S = \mathbb{N} \setminus F$ for some $F \subset\subset \mathbb{N}$.*

Proof. We can take $k = 1$ in Theorem 3.2.8. But proof of the existence of finite subset S' of S such that $\gcd(S') = k$ is not required since it is already given. □

Theorem 3.3.4. *A strongly connected simple digraph G induced by an adjacency matrix A with non-zero rows and columns contains cycles of lengths $m_1, m_2, ..., m_k$ such that $gcd(m_1, m_2, ..., m_k) = 1$ if and only if A is primitive.*

Proof. Suppose that G is a strongly connected simple digraph. First assume that G contains cycles of lengths $m_1, m_2, ..., m_k$ such that $\gcd(m_1, m_2, ..., m_k) = 1$. Without loss generality we can assume that these are simple cycles of G. Let C_i be a simple cycle having length m_i and $x_i \in C_i$ for $i = 1, 2, ..., r$. There exist a directed path from $x_{i (mod\ r)}$ to $x_{i+1 (mod\ r)}$ which doesn't contain any cycle. Since the digraph is strongly connected we can bring the remaining vertices of G that are not in the above simple cycles to the above paths. Let C be a simple cycle obtained from the cycles C_is and the above paths. Then by Lemma 3.2.5, there exists $n_0 \in \mathbb{N}$, $F \subset\subset \mathbb{N}$ such that for all $n \geq n_0$, $n \in \mathbb{N} \setminus F$ there exist $a_1, a_2, ..., a_k \in \mathbb{N}$ such that $n = a_1 m_1 + a_2 m_2 + ... + a_k m_k$ as in the Case 2 of Theorem 3.2.8. Therefore, given any vertex x there exists a cycle of length n for all $n \geq n_0$. Let $m = diam(G) = Max\{l(x, y) : x, y \in V(G)\}$ where $l(x, y)$ denotes the minimum length of directed paths from x to y. Let $N = n_0 + m$. Hence $A^N > 0$ since for every $x, y \in V(G)$ there exists a directed path of length p for all $p \leq m$ and for every $x \in V(G)$ there exists a cycle of length n for all $n \geq n_0$. Write $N = n_0 + m - p + p$. Hence A is primitive.

Conversely, suppose that A is primitive and $\gcd(m_1, m_2, ..., m_l) = p > 1$ for all cycles of length m_i, $1 \leq i \leq p$, $p \in \mathbb{N}$. Let $k =$ gcd of lengths of all cycles of G. Then there exist cycles of length $m_1, m_2, ..., m_l$ such that $\gcd(m_1, m_2, ..., m_l) = k$. Then k divides the lengths of all cycles. Also, there exists $s \in \mathbb{N}$ such that $A^s > 0$, and for every $x, y \in V(G)$ there exists a directed path of length s from x to y since A is primitive and G is strongly connected. Therefore k divides s and $s + 1$, which implies $k = 1$. A contradiction. Hence the proof. □

Corollary 3.3.5. *A strongly connected simple digraph G contains cycles of length $m_1, ..., m_k$ such that $gcd(m_1, m_2, ..., m_n) = 1$ if and only if X_G is mixing.*

Proof. This follows from Theorems 3.3.3 and 3.3.4. □

Definition 3.3.6. A tournament is a digraph whose adjacency matrix $A = (a_{ij})$ has the following property.

(a) $a_{ii} = 0$ for all i.

(b) $a_{ij} + a_{ji} = 1$ for all $i \neq j$.

Remark 3.3.7. The following results are true for any tournament having a cycle of length greater than 4, and it was separately proved in [17].

(1) Its matrix is primitive if it is irreducible.

(2) There exist a 3-cycle and a 4-cycle on it.

From Theorem 3.3.4 and the result (2), we can conclude that a strongly connected simple digraph having cycles of length greater than 4, its matrix is primitive if it is irreducible. The result proved in [17] for tournaments and at present it has been improved for a very large class of digraphs.

A slight modification of a known proof of the equivalence of (i) and (iii) in Remark 3.2.10, we can prove the following. We give the proof for self containment.

Lemma 3.3.8. An SFT X_A is weak mixing if and only if for every $1 \leq i_1, j_1, i_2, j_2 \leq k$ there exists $n \in \mathbb{N}$ such that $A^n(i_1, j_1) > 0$ and $A^n(i_2, j_2) > 0$ where A denotes a $k \times k$ adjacency matrix with non-zero rows and columns.

Proof. Assume that X_A is weak mixing. Let $U_1 = \{(x_n)_{n\in\mathbb{Z}} \in X_A : x_0 = i_1\}$, $V_1 = \{(x_n)_{n\in\mathbb{Z}} \in X_A : x_0 = j_1\}$, $U_2 = \{(y_n)_{n\in\mathbb{Z}} \in X_A : y_0 = i_2\}$ and $V_2 = \{(y_n)_{n\in\mathbb{Z}} \in X_A : y_0 = j_2\}$ where $1 \leq i_1, j_1, i_2, j_2 \leq k$. These sets are non-empty and open. Then there exists $n \in \mathbb{N}$ such that $\sigma^n(U_i) \cap V_i \neq \emptyset$, $i = 1, 2$. Hence there exist $x = (x_n)_{n\in\mathbb{Z}}, y = (y_n)_{n\in\mathbb{Z}} \in X_A$ such that $x_0 = i_1$, $y_0 = i_2$, $x_n = j_1$ and $y_n = j_2$. Note that $A^N(i,j) = \sum_{r_1=1}^{k} ... \sum_{r_{N-1}=1}^{k} A(i, r_1)A(r_1, r_2)...A(r_{N-2}, r_{N-1})A(r_{N-1}, j)$ for all $N \in \mathbb{N}$. But $A(i_1, x_1) = A(x_1, x_2) = A(x_2, x_3) = ... = A(x_{n-1}, j_1) = 1$ and $A(i_2, y_1) = A(y_1, y_2) = A(y_2, y_3) = ... = A(y_{n-1}, j_2) = 1$. Therefore $A^n(i_1, j_1) > 0$ and $A^n(i_2, j_2) > 0$.

Conversely, assume that for every $1 \leq i_1, j_1, i_2, j_2 \leq k$ there exists $N \in \mathbb{N}$ such that $A^N(i_1, j_1) > 0$ and $A^N(i_1, j_2 > 0$. Given non-empty open sets U_1, V_1, U_2, V_2 we can choose $(i_n^{(l)})_{n\in\mathbb{Z}} \in U_l$ and $(j_n^{(l)})_{n\in\mathbb{Z}} \in V_l$ such that for $M > 0$ sufficiently large; and

$U_l \supset \{(x_n)_{n\in\mathbb{Z}} \in X_A : x_k = i_k^{(l)}, -M \leq k \leq M\}$, $V_l \supset \{(x_n)_{n\in\mathbb{Z}} \in X_A : x_k = j_k^{(l)}, -M \leq k \leq M\}$ for $l = 1, 2$ (It is possible since the set of symmetric cylinders form a base for the topology on $\mathcal{A}^{\mathbb{Z}}$).

By hypothesis, there exists $N > 0$ such that $A^N(i_M^{(l)}, j_{-M}^{(l)}) > 0$ for $l = 1, 2$. This means we can find a word $x_1^l ... x_{N-1}^l$ such that $A(i_M^{(l)}, x_1^l) = A(x_1^l, x_2^l) = ... = A(x_{N-1}^l, j_{-M}^{(l)}) = 1$.

Define $x_n^{(l)} = \begin{cases} i_n^{(l)} & \text{if } n \leq M \\ x_{n-M}^l & \text{if } M+1 \leq n \leq M+N-1 \\ j_{n-(2M+N)}^{(l)} & \text{if } M+N \leq n \end{cases}$

Then $\sigma^{2M+N}(U_l) \cap V_l \neq \emptyset$ for $l = 1, 2$. Hence X_A is weak mixing. □

Theorem 3.3.9. *An SFT is weak mixing if and only if it is mixing.*

Proof. Let A be an adjacency matrix of order k. Assume that X_A is weak mixing. We have to prove that there exist cycles of lengths $m_1, m_2, ..., m_p$ such that $\gcd(m_1, m_2, ..., m_p) = 1$. Suppose not. Then there exists $s > 1$ such that s divides the lengths of all cycles (let s = gcd of lengths all cycles). Let $1 \leq v_1, w_1, v_2 \leq k$ be such that $v_1 w_1$ is a 2-block in x for some $x \in X_A^v$. Then there exist a cycle of length n through v_2 and a directed path of length n from w_1 to v_1, which implies s divides n and $n+1$. Hence $s = 1$. A contradiction. Hence X_A is mixing. Then by Proposition 3.2.4, any SFT is mixing. Converse part is easy. □

Note: Theorem 3.3.9 was independently proved by T.K.S. Moothathu using some other ideas. See [52] for his proof.

Theorem 3.3.10. *The following are equivalent for a non-singleton SFT X.*

(i) X is transitive and $Per(X)$ is cofinite.

(ii) X is weak mixing.

(iii) X is mixing.

Proof. This theorem follows from Theorems 3.3.3, 3.3.9 and 3.2.8, and by the following observations.

The period set $\mathrm{Per}(X)$ of a finite SFT X is finite. So $\mathrm{Per}(X)$ is not cofinite. Except singleton SFTs all other finite SFTs are not weak mixing and hence not mixing. □

Let $\{1, 2, ...\}$ be a countable set. With the discrete topology it is a non-compact metrizable space. Let $\sum = \{1, 2, ...\}^{\mathbb{Z}}$. With product topology $\sum$ is a totally disconnected, perfect and non-compact metric space. As in the finite case, the cylinder sets form a countable basis of clopen sets. The shift, σ, is a homeomorphism of the space to itself. The dynamical system $(\sum, \sigma)$ is the full shift on the symbols. If A is a countable, zero-one matrix then as in the finite case we use transition rules to define a shift-invariant subset of the full shift on countably many symbols, denoted by $\sum_A$. Then the subspace $\sum_A$ of $\sum$ is a non-compact, metrizable and $\sigma : \sum_A \to \sum_A$ is the countable state Markov shift defined by A.

Let A be an adjacency matrix with non-zero rows and columns.

Proposition 3.3.11. *[32] A countable state Markov shift $\sum_A$ is topologically transitive if and only if A is irreducible.*

Proposition 3.3.12. *[32] A countable state Markov shift $\sum_A$ is topologically mixing if and only if A is primitive.*

Hence we have the following remark

Remark 3.3.13. Because Propositions 3.3.11 and 3.3.12, we can extends Theorem 3.3.10 for countable Markov shifts.

Next we try to generalize Theorem 3.3.10 for sofic shifts.

Theorem 3.3.14. *The following are equivalent for a subset S of $\mathbb{N}$.*

(1) *$S = Per(X_\Gamma)$ for some strongly connected labeled digraph Γ containing cycles of length $m_1, m_2, ..., m_k$ such that gcd$(m_1, m_2, ..., m_k) = 1$.*

(2) *Either $S = \{1\}$ or $S = \mathbb{N} \setminus F$ for some $F \subset\subset \mathbb{N}$.*

Proof. We can take $k = 1$ in Theorem 3.2.23. But proof of the existence of finite subset S' of S such that $\gcd(S') = k$ is not required since it is already given. □

From the definition of transitivity, mixing, weak mixing and by using some ideas from Lemma 3.3.8, we can prove the following lemma for any directed labeled graph Γ. Recall that for every non-empty open set U in X_Γ we can choose $(i_n)_{n\in\mathbb{Z}} \in U$ such that for $M > 0$ sufficiently large; $U \supset \{(x_n)_{n\in\mathbb{Z}} : x_k = i_k, -M \leq k \leq M\}$.

Lemma 3.3.15. Let Γ be a labeled digraph. Then the following are true.

(1) X_Γ is transitive if and only if for every $i, j \in E(\Gamma)$ there exists a directed path of length n from i to j for some $n \in \mathbb{N}$.

(2) X_Γ is weak mixing if and only if for every $i_1, j_1, i_2, j_2 \in E(\Gamma)$ there exist directed paths of length n from i_1 to j_1 and from i_2 to j_2 for some $n \in \mathbb{N}$.

(3) X_Γ is mixing if and only if for every $i, j \in E(\Gamma)$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ there is a directed path of length n from i to j.

Theorem 3.3.16. *Let Γ be a strongly connected labeled digraph. Then Γ contains cycles of lengths $m_1, m_2, ..., m_k$ such that gcd$(m_1, m_2, ..., m_k) = 1$ if and only if X_Γ is mixing.*

Proof. We can give a proof similar to that of Theorem 3.3.4. Note that without loss of generality we cannot assume that the cycles are simple. Still the theorem is true. □

Corollary 3.3.17. *The period set of a mixing SFT is either* $\{1\}$ *or* $\mathbb{N} \setminus F$ *for some* $F \subset\subset \mathbb{N}$.

Proof. This theorem follows from Theorems 3.3.14 and 3.3.16. □

Theorem 3.3.18. *A sofic shift is weak mixing if and only if it is mixing.*

Proof. Because of Lemma 3.3.15 and Theorem 3.3.16, we can give a proof similar to that of Theorem 3.3.9. □

Note: Theorem 3.3.18 was independently proved using some other ideas by Banks et al. in [12]. We noticed their result recently.

Remark 3.3.19. There exists a sofic shift X_Γ which is transitive and its period set is cofinite, but it is not mixing.

Proof. Let X_Γ be the sofic shift based on the directed graph Γ with vertices 0 and 1, arcs labeled a, b, c from 0 to 1, and arcs labeled a, b, d from 1 to 0. Then X_Γ is the image of the topologically transitive subshift of finite type, based on Γ but with distinctly labeled edges. The period set of X_Γ is $\mathbb{N}$. But X_Γ is not topologically mixing by Theorem 3.3.16. Hence the remark follows. □

If X_Γ is a transitive non-singleton sofic shift, then the set of periodic points $P(X_\Gamma)$ is dense in X_Γ. But a compact dynamical system which is totally transitive and has a dense set of periodic points is weak mixing (See [10]). Therefore X_Γ is totally transitive if and only if X_Γ is weak mixing. Hence we can add totally transitivity case in Theorems 3.3.10 and 3.3.18. In general, the conclusion of Theorem 3.3.18 need not be true. There is a subshift which is weak mixing but not mixing (Chacon shift, See [27]). See [12] for more examples.

As in previous chapter, for each self map f on a set X, we associate a subset of $\mathbb{N}$ namely, $\text{Per}(f)$. If f belongs to a certain nice class of functions then not all subsets of $\mathbb{N}$ may arise as the set of periods.

The following table contrasts our results regarding period sets in the case of subshifts.

	The class of subshifts	**Period sets**
1	Subshifts of finite type (Sofic shifts)	Sets of the form F $\mathbb{N}\Delta G$ where F and G are finite subsets of $\mathbb{N}$
2	Transitive subshifts of finite type (Transitive sofic shifts)	Sets of the form $k\mathbb{N} \setminus F$ or $\{a\}$ where $a, k \in \mathbb{N}$ and F is a finite subset of $\mathbb{N}$
3	Mixing subshifts of finite type (Mixing sofic shifts)	Sets of the form $\mathbb{N} \setminus F$ where F is a finite subset of $\mathbb{N}$
4	Chaotic subshifts of finite type (Chaotic Sofic shifts)	Sets of the form $k\mathbb{N} \setminus F$ where $k \in \mathbb{N}$ and F is a finite subset of $\mathbb{N}$
5	All subshifts	All subsets of $\mathbb{N}$

3.4 Cellular automata

The cellular automata play an important role in various contexts such as computer graphics, parallel computing and cell biology. It is natural to ask for a neat description of the sets of periodic points and the sets of periods of cellular automata, unfortunately we do not have a complete answer.

Let $\mathcal{A}$ be a finite set having at least two elements. Let $r \in \mathbb{N}_0$. A function $f : \mathcal{A}^{2r+1} \to \mathcal{A}$ is called a local rule. It induces a function $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ by the rule $(F(x))_n = f(x_{n-r}, x_{n-r-1}, ..., x_{n-1}, x_n, x_{n+1}, ..., x_{n+r-1}, x_{n+r})$ for all $n \in \mathbb{Z}$. The pair

$(A^{\mathbb{Z}}, F)$ (simply the map $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$) is called a cellular automaton (abbreviated as CA). A map $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is a cellular automaton if and only if it is continuous and commutes with the shift (see [34]). In this section, we consider empty set also an SFT.

Lemma 3.4.1. (Alphabet Lemma) [34]

Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a CA with local rule $f : \mathcal{A}^{2r+1} \to \mathcal{A}$, $r \in \mathbb{N}$. Let $k \geq r$ and let $\mathcal{B} = \mathcal{A}^k$. Define $g : \mathcal{B}^3 = \mathcal{A}^{3k} \to \mathcal{B} = \mathcal{A}^k$ by $g(w = w_1 w_2 ... w_{3k}) = a_1 a_2 ... a_k$ where $a_j = f(w_{k+j-r} ... w_{k+j} ... w_{k+j+r})$; and the cellular automaton $G : \mathcal{B}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ by $(G(y))_i = g(y_{i-1} y_i y_{i+1})$; for $y \in \mathcal{B}^{\mathbb{Z}}; i \in \mathbb{Z}$. Then, F is conjugate to G. In fact, for any $p \in \mathbb{Z}$, the map $\phi_p : \mathcal{A}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ defined as $(\phi_p(x))i = x_{ki+p, ki+p+1, ..., ki+p+k-1}$ is a homeomorphism and satisfies $\phi_p \circ F = G \circ \phi_p$.

3.4.1 A characterization for a subshift of finite type in terms of sets of periodic points of cellular automata

There have been some papers that discussed about the sets of periodic points for continuous self maps (See [9], [18], [22]). It is natural to ask: Which sets will arise as the set of all periodic points of continuous self maps? This question is too abstract. If we ask the same question in the class of some nice class of maps then we can expect a nice answer. In this section, we consider in the case of CA.

Characterization of the sets of periodic points for a continuous self map of an interval is incomplete. See the following results of J.-P. Delahaye. He gave partial results in this context. This is our first motivation for considering CA. We completely solved in the case of a continuous 2-dimensional toral automorphism in [54] (see Theorem 3.4.6). This is our second motivation for considering CA.

In this section, we give a partial answer in the case of CA. Our result is similar to the following propositions 3.4.2 and 3.4.5, and which characterizes an SFT in terms of a CA.

Proposition 3.4.2. *[22] (i) The set of fixed points of a continuous function from* $[0, 1] \to [0, 1]$ *is a closed subset of* $[0, 1]$.

(ii) For every closed subset F *of* $[0, 1]$ *there exists a continuous function* f *whose fixed point set is* F *.*

Remark 3.4.3. See the proof of Remark 5.2.9 for another proof of the above proposition.

Definition 3.4.4. A subset F of $[0, 1]$ is symmetric if for $x \in [0, 1]$, $\frac{1}{2} + x \in F \Leftrightarrow \frac{1}{2} - x \in F$.

Proposition 3.4.5. *[22]* (i) *The set of periodic points of period* 1 *or* 2 *of a continuous function from* $[0, 1]$ *to* $[0, 1]$ *is a closed subset of* $[0, 1]$.

(ii) *For every symmetric closed subset of* $[0, 1]$ *there exists a continuous function from* $[0, 1]$ *to* $[0, 1]$ *whose set of periodic points of period* 1 *or* 2 *is* $F \cup \{\frac{1}{2}\}$.

Theorem 3.4.6. *[54]*

For any continuous toral automorphism T*, the set* $P(T)$ *of periodic points of* T *is one of the following:*

1. $\mathbb{Q}_1 \times \mathbb{Q}_1$*, where* $\mathbb{Q}_1$ *denotes the set of all rational points in* $[0, 1)$.
2. S_r *for some* $r \in \mathbb{Q} \cup \{\infty\}$*; where* $S_r = \{(x, y) \in \mathbb{T}^2 \ : \ rx + y$ *is rational*$\}$.
3. $\mathbb{T}^2$.

Definition 3.4.7. A dynamical system (X, f) has the shadowing property, if for any $\epsilon > 0$ there exists $\delta > 0$ such that any finite δ-chain is ϵ-shadowed by some point. A (finite or infinite) sequence $(x_n)_{n \geq 0}$ is a δ-chain, if $d(f(x_n), x_{n+1}) < \delta$ for all n. A point $x \in X$ ϵ-shadows a finite sequence $x_0, x_1, ..., x_n$, if for all $i \leq n$, $d(f^i(x), x_i) < \epsilon$.

ie., $\forall \epsilon > 0$, $\exists \delta > 0, \forall x_0, ..., x_n$, $(\forall i, d(f(x_i), x_{i+1}) < \delta \implies \exists x, \forall i, d(f^i(x), x_i) < \epsilon)$.

Definition 3.4.8. A dynamical system (X, f) is open, if $f(U)$ is open for any open $U \subset X$.

There are two distinct topological characterizations of SFT known in literature as follows.

Theorem 3.4.9. *[34] A subset $X \subset \mathcal{A}^{\mathbb{N}}$ is an SFT if and only if (X, σ) has the shadowing property.*

Theorem 3.4.10. *[34] A subset $X \subset \mathcal{A}^{\mathbb{N}}$ is an SFT if and only if (X, σ) is open.*

Lemma 3.4.11. For every SFT X, there exists a finite set of words $\mathcal{G}$ having odd length such that $X = X_{\mathcal{G}}$.

Proof. Let X be a k-step SFT. Then there exists a finite set of words $\mathcal{F}$ having length at most k such that $X = X_{\mathcal{F}}$. If k is odd then consider $\mathcal{G} = \{x \in W_k(\mathcal{A}^{\mathbb{Z}}) : y$ is a subword of x for some $y \in \mathcal{F}\}$. If k is even then consider $\mathcal{G} = \{x \in W_{k+1}(\mathcal{A}^{\mathbb{Z}}) : y$ is a subword of x for some $y \in \mathcal{F}\}$.

Claim: $X_{\mathcal{F}} = X_{\mathcal{G}}$.

Let $x \in X_{\mathcal{F}}$. Suppose $x \notin X_{\mathcal{G}}$. Then for some $y \in \mathcal{F}$, y is a subword of x. A contradiction. Therefore $x \in X_{\mathcal{G}}$. Next, let $x \in X_{\mathcal{G}}$. Which implies y is not a subword of x for all $y \in \mathcal{F}$. Then $x \in X_{\mathcal{F}}$. Hence the claim. □

Next we give the promised characterization as follows.

Theorem 3.4.12. *Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be any CA. Then $Fix(F)$ is an SFT. Conversely given any SFT X there exists a CA F such that $Fix(F) = X$.*

Proof. Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA defined by the local rule $f : \mathcal{A}^{2r+1} \to \mathcal{A}$. Let $\mathcal{F} = \{w \in \mathcal{A}^{2r+1} : f(w) \neq \text{the middle term of } w\}$. Note that $\mathcal{F}$ is finite (it may be empty or $\mathcal{A}^{2r+1}$). First, let $x \in X_{\mathcal{F}}$. Which implies $(F(x))_i = x_i$ for all i. ie., $F(x) = x$. Next, let $x \in \mathcal{A}^{\mathbb{Z}}$ such that $F(x) = x$. Then $f(x_{i-r}x_{i-r+1}...x_0...x_{i+r-1}x_{i+r}) = x_i$ for all i. ie., $x \in X_{\mathcal{F}}$. Hence $Fix(F) = X_{\mathcal{F}}$.

Conversely, given any SFT $X_{\mathcal{F}}$ without loss of generality assume that $\mathcal{F}$ contains words of same length (odd length) because of Lemma 3.4.11.

Define $f : A^{2r+1} \to A$ such that

$$f(w) = \begin{cases} \textit{the middle term of } w & \text{if } w \textit{ is forbidden} \\ \textit{some other symbol from the alphabet} & \textit{otherwise} \end{cases}$$

Then $Fix(F) = X_{\mathcal{F}}$. □

Remark 3.4.13. In the statement of Theorem 3.4.12, we can replace $Fix(F)$ by $Fix(F^n)$.

Proof. Let $f : A^{2r+1} \to A$ be the local rule of a CA $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$. The local rule $f : \mathcal{A}^{2r+1} \to \mathcal{A}$ induces a function $\tilde{f} : \mathcal{A}^{2s+2r+1} \to \mathcal{A}^{2s+1}$ for all s. Then by inductively, define $f_n : \mathcal{A}^{2nr+1} \to \mathcal{A}$ such that $f_n(w) = f(\tilde{f}_{n-1}(w))$ where $\tilde{f}_m : \mathcal{A}^{2r+2(m-1)r+1} \to \mathcal{A}^{2r+1}$ denotes the induced function of f_m for $s = r$, and $f = f_1$. Note that the length of $\tilde{f}(w)$ is equal to the difference between the length of w and $2r$. Let $\mathcal{F}_n = \{w : f_n(w) \neq \text{the middle term of } w\}$. Then $Fix(F^n) = X_{\mathcal{F}_n}$.

Converse part follows easily. □

Now we consider the following Questions. It is worth considering because if we have a full classification of periodic points then it will be nice.

Question 1: Given two SFTs $X_1 \subset X_2 \subset \mathcal{A}^{\mathbb{Z}}$. Does there exists a CA F such that $Fix(F) = X_1$ and $Fix(F^2) = X_2$?

Let $\mathcal{A} = \{0, 1\}$. Then either $\bar{0}$ or $\bar{1}$ is F-periodic. Hence, if X_2 does not contains $\bar{0}$ and $\bar{1}$ then the question has negative answer.

Now we ask, the improved version of **Question** 1 as follows.

Question 2: Given a collection of subshifts $\{X_n : n \in \mathbb{N}\}$ such that $X_m \subset X_n$ whenever m divides n, and each X_n contains all constant sequence. Does there exists a CA F such that $X_n = \{x \in \mathcal{A}^{\mathbb{Z}} : F^n(x) = x\}$?

Definition 3.4.14. Let $X_i \subset \mathcal{A}^{\mathbb{Z}}$, $i = 1, 2$, be two subshifts. A continuous map $c : X_1 \to X_2$ is a code if it commutes with the shifts, i.e., $\sigma \circ c = c \circ \sigma$.

Definition 3.4.15. Let X be a subshift, $k, l \in \mathbb{N}_0$, $n = k+l+1$, and let α be a map from $W_n(X)$ to an alphabet $\mathcal{B}$. The (k, l) block code c_α from X to the full shift $\mathcal{B}^{\mathbb{Z}}$ assigns to a sequence $x = (x_i) \in X$ the sequence $c_\alpha(x)$ with $c_\alpha(x)_i = \alpha(x_{i-k}, ..., x_i, ..., x_{i+l})$. Any block code is a code, since it is continuous and commutes with the shift.

The following lemma help us to prove the Theorem 3.4.18.

Lemma 3.4.16. [18](Curtis-Lyndon-Hedlund) Let X_1, X_2 be subshifts of $\mathcal{A}^{\mathbb{Z}}$. Then every code $c : X_1 \to X_2$ is a block code.

Remark 3.4.17. If $X_1 = X_2 = \mathcal{A}^{\mathbb{Z}}$ then the above code $c : X_1 \to X_2$ becomes a cellular automaton (see Page 198 in [34]).

Theorem 3.4.18. *$Fix(c)$ is an SFT for every code $c : X \to X$. Conversely given any SFT X there exists a code $c : X \to X$ such that $Fix(c) = X$*

Proof. Every code is a block-code by Lemma 3.4.16. Therefore a proof similar to Theorem 3.4.12 will work. □

Remark 3.4.19. In the statement of Theorem 3.4.18, we can replace $Fix(c)$ by $Fix(c^n)$.

3.4.2 The set of periods of cellular automata

Now we take up the natural question: Which subsets of $\mathbb{N}$ arise as the set of periods of a CA?

Definition 3.4.20. A CA is said to be additive if its local rule $f : A^{2k+1} \to A$ can be expressed as $f(x_{-k}, ..., x_k) = (\Sigma_{i=-k}^{k} \lambda_i x_i) mod\ m$, where $\lambda_i \in A$ and for some $m \in \mathbb{N}$.

T.K.S Moothathu (See [52]) has given a partial answer for the set of periods of Cellular automata in the following way.

Theorem 3.4.21. *Let F be an additive CA, where the addition is done modulo a prime p. Then, Per(F) has only four possibilities: $\{1, m\}$ for some m where $1 \leq m < p$, $\mathbb{N} \setminus \{p^m : m \in \mathbb{N}\}$, $\mathbb{N} \setminus \{2p^m : m \in \mathbb{N}_0\}$ or the whole set $\mathbb{N}$.*

His method is combinatorial. For instance, he makes use of the following lemma.

Lemma 3.4.22. Let p be a prime, let $k \in \mathbb{N}$, and let $a_0, a_1, \ldots, a_k$ be integers such that a_0 and a_k are not divisible by p. Also, let $l \geq 1$ be the smallest integer such that a_l is not divisible by p. Fix $n \in \mathbb{N}$ and write $n = p^m r$, where $m \geq 0$ and $p \nmid r$. Let β_t be the coefficient of x^t in the polynomial $(a_0 + a_1 x + \cdots + a_k x^k)^n$. Then, the smallest integer $t \geq 1$ such that β_t is not divisible by p, is $t = lp^m$.

The class considered above, is a narrow one, not even exhausting all the additive CA. Hence it is good to mention the following partial result.

Theorem 3.4.23. *(see Appendix A) Let F be any additive CA. Then Per(F) has to be closed under lcm and has to contain* 1.

On one hand, this theorem states that the fact that the subsets of $\mathbb{N}$ that have been listed previously in Theorem 3.4.21 happen to be closed under lcm, is not accidental;

it has to be so for all additive CA. On the other hand, its proof does not even make use of the hypothesis that we are working with CA, and remains true in the following more general version:

If ϕ is any endomorphism of a torsion free abelian group (as every additive CA is), $\mathrm{Per}(\phi)$ has to be closed under lcm and contains 1 (see Appendix A). But none of these results becomes applicable in the case of a general CA which may not be additive. Some partial results can be seen in [52].

Question: Find which subsets of $\mathbb{N}$ arise as sets of periods of cellular Automata. It is worth-mentioning here that the answer has to be a countable family of subsets of $\mathbb{N}$.

Chapter 4

Transitive Toral Automorphisms

Transitivity is an important property in the setting of dynamical system because which is equivalent to some type of chaos and hence it is important to study the transitivity property of various systems (see [1] [7], [6], [18] and [23]). In this chapter, we consider continuous 2-dimensional toral automorphisms, and use new facts to prove our main results. We give proofs of some known propositions and lemmas for the sake of completeness and self containment using basic algebra, and do not assume so much literature. We produce examples of zero entropy dynamical systems having Lyapunov function such that every fiber is of empty interior. We denote the determinant of a matrix A by $Det(A)$, the trace of a matrix A by $Tr(A)$, and the toral automorphism induced by a matrix $A \in GL(2, \mathbb{Z})$ by T_A. In this chapter, a toral automorphism means a continuous toral automorphism.

4.1 Basic results

Lemma 4.1.1. [40] If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an isomorphism then for every Riemann measurable set $S \subset \mathbb{R}^2$, $T(S)$ is Riemann measurable and

$$Area(T(S)) = |Det(T)|Area(S)$$

Let G be the set of all toral automorphisms and $GL(2,\mathbb{Z}) = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}$ and $ad - bc = \pm 1\}$.

Theorem 4.1.2. *There is an isomorphism from $GL(2,\mathbb{Z})$ to G.*

Proof. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,\mathbb{Z})$ then A induces a toral automorphism $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ by the rule $(x_1, x_2) \mapsto ($ fractional part of (ax_1+bx_2), fractional part of (cx_1+dx_2) $)$. The map T_A is continuous since if $|x_1 - y_1|$, $|x_2 - y_2| < \epsilon$ then $|(T(x_1, x_2))_1 - (T(y_1, y_2))_1| \leq (|a|+|b|)\epsilon$ and $|(T(x_1, x_2))_2-(T(y_1, y_2))_2| \leq (|c|+|d|)\epsilon$. We can easily verify that $A^{-1} \in GL(2,\mathbb{Z})$. The inverse to $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ is then the toral automorphism associated to A^{-1}. Hence the toral automorphism T_A is a homeomorphism.

Conversely, let $\phi : \mathbb{T}^2 \to \mathbb{T}^2$ be any continuous toral automorphism. Since ϕ is continuous at $(0, 0)$ there exists $0 < \delta < \frac{1}{2}$ such that $\phi(([0, \delta) \times [0, \delta))) \subset [0, \frac{1}{2}) \times [0, \frac{1}{2})$ and such that $\phi(X + Y) = \phi(X) + \phi(Y)$ for all $X, Y \in [0, \delta) \times [0, \delta)$, where $+$ denotes the usual addition in $\mathbb{R}^2$. Note that, if $X \in [0, \delta) \times [0, \delta)$ then $\frac{1}{2^n}X \in [0, \delta) \times [0, \delta)$ for all $n \in \mathbb{N}$. For any $X \in [0, \delta) \times [0, \delta)$, $\phi(x) = \phi(\frac{X}{2} + \frac{X}{2}) = \phi(\frac{X}{2}) + \phi(\frac{X}{2}) = 2\phi(\frac{X}{2})$ and hence $\phi(\frac{X}{2}) = \frac{1}{2}\phi(X)$. By induction on n, we can prove that $\phi(\frac{1}{2^n}) = \frac{1}{2^n}\phi(X)$ for all $n \in \mathbb{N}$ and then by using the additivity, we can show that $\phi(\frac{m}{2^n}) = \frac{m}{2^n}\phi(X)$ for all $m \in \{1, 2, \cdots, 2^n - 1\}$. Since the set of all dyadic rationals is dense in $[0, 1]$, by the continuity of ϕ we have $\phi(\lambda X) = \lambda\phi(X)$ for all $\lambda \in (0, 1)$. Hence $\phi|_{[0,\delta)\times[0,\delta)} = L|_{[0,\delta)\times[0,\delta)}$

for some linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$. This linear transformation induces an integer matrix A with determinant ± 1 such that $Ax = \phi(x)$ for all $x \in \mathbb{T}^2$ [The kernel of an endomorphism (different from the zero map), on a connected topological group cannot have non-empty interior]. Hence the proof. □

Remark 4.1.3. The toral automorphism induced by a matrix from $GL(2, \mathbb{Z})$ is an area preserving map.

Proof. This remark follows from Lemma 4.1.1 and Theorem 4.1.2. □

Remark 4.1.4. See [16] for a generalization of Theorem 4.1.2.

Lemma 4.1.5. For any $A \in GL(2, \mathbb{Z})$, the set $P(T_A)$ is dense in $\mathbb{T}^2$.

Proof. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{Z})$. We prove that $P(T_A) \supset \mathbb{Q}_1 \times \mathbb{Q}_1$ where $\mathbb{Q}_1$ denote the set of all rational numbers in $[0, 1)$. A general element in $\mathbb{Q}_1 \times \mathbb{Q}_1$ is of the form $x = (\frac{p_1}{q}, \frac{p_2}{q})$ where $p_1, p_2, q \in \mathbb{Z}$ with $0 \leq p_1, p_2 < q$. We note that $T_A X =$ (fractional part of $(\frac{ap_1}{q} + \frac{bp_2}{q})$, fractional part of $(\frac{cp_1}{q} + \frac{dp_2}{q})$) = an element of the form $(\frac{m}{q}, \frac{n}{q})$, where $0 \leq m, n < q$. Note that for a fixed $q \in \mathbb{N}$, the set $\{(\frac{m}{q}, \frac{n}{q}) | 0 \leq m, n < q; m, n \in \mathbb{N}\}$ is invariant and finite. Hence the orbit of x is finite and therefore eventually periodic. Now, the result follows from the fact that for invertible maps the eventually periodic points are periodic points. □

Definition 4.1.6. For $m, n \in \mathbb{Z}$,

$$\text{Define } A_{m,n} = \begin{cases} \begin{bmatrix} m & n \\ \frac{-(m-1)^2}{n} & 2-m \end{bmatrix} & \text{if } n \neq 0, n \textit{ divides } m-1 \\ \begin{bmatrix} 1 & 0 \\ m-1 & 1 \end{bmatrix} & \text{if } n = 0 \end{cases}$$

Note that $Det(A_{m,n}) = 1$ and $Tr(A_{m,n}) = 2$ for all $m, n \in \mathbb{Z}$.

Definition 4.1.7. For $m, n \in \mathbb{Z}$,

$$\text{Define } B_{m,n} = \begin{cases} \begin{bmatrix} m & n \\ \frac{-(m+1)^2}{n} & -2-m \end{bmatrix} & \text{if } n \neq 0, n \text{ divides } m+1 \\ \begin{bmatrix} -1 & 0 \\ m+1 & -1 \end{bmatrix} & \text{if } n = 0 \end{cases}$$

Note that $Det(B_{m,n}) = 1$, $Tr(B_{m,n}) = -2$ and $B_{m,n} = -A_{-m,-n}$ for all $m, n \in \mathbb{Z}$.

By induction we can prove that, for any $k \in \mathbb{N}$,

$A^k_{m,n} = A_{km-k+1,kn}$ and $B^k_{m,n} = (-1)^{k-1} B_{km+k-1,kn}$ for all $m, n \in \mathbb{Z}$.

Lemma 4.1.8. (i) Let $A \in GL(2, \mathbb{Z})$ be such that $Det(A) = 1$ and $Tr(A) = 2$. Then $A = A_{m,n}$ for some $m, n \in \mathbb{Z}$.

(ii) Let $A \in GL(2, \mathbb{Z})$ be such that $Det(A) = 1$ and $Tr(A) = -2$. Then $A = B_{m,n}$ for some $m, n \in \mathbb{Z}$.

Proof. (i) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{Z})$ be such that $Det(A) = 1$ and $Tr(A) = 2$. Then we have $a + d = 2$ and $ad - bc = 1$. Hence $bc = -(a-1)^2$.

If $b \neq 0$ then $c = -\frac{(a-1)^2}{b}$ an integer, and therefore $A = A_{a,b}$. If $b = 0$ then $a = d = 1$ and c can be any integer. Hence $A = A_{c+1,0}$.

(ii) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{Z})$ be such that $Det(A) = 1$ and $Tr(A) = -2$. Then we have $a + d = -2$ and $ad - bc = 1$. Hence $bc = -(a+1)^2$.

If $b \neq 0$ then $c = -\frac{(a+1)^2}{b}$ an integer, and therefore $A = B_{a,b}$. If $b = 0$ then $a = d = -1$ and c can be any integer. Hence $A = B_{c-1,0}$. □

4.2 Main results

Let (X, d) be a compact metric space, and $f : X \to X$ a continuous map. For each $n \in \mathbb{N}$, define $d_n(x, y) = Max_{0 \leq k \leq n-1} d(f^k(x), f^k(y))$ for $x, y \in X$. Each d_n is a metric on X and induces same topology on X. Note that $d_n \geq d_{n-1}$ and $d_1 = d$. Fix $\epsilon > 0$.

Definition 4.2.1. A subset $A \subset X$ is (n, ϵ)-spanning if for every $x \in X$ there is $y \in A$ such that $d_n(x, y) < \epsilon$. By compactness, there is a finite (n, ϵ)-spanning set. Let $span(n, \epsilon, f)$ be the minimum cardinality of an (n, ϵ)-spanning set.

Definition 4.2.2. A subset $A \subset X$ is (n, ϵ)-separated if any two distinct points in A are at least ϵ apart in the metric d_n. Any (n, ϵ)-separated set is finite. Let $sep(n, \epsilon, f)$ be the maximal cardinality of an (n, ϵ)-separated set.

Definition 4.2.3. Topological entropy (simply we call entropy) of f is defined as

$$h_{top}(f) = h(f) = lim_{\epsilon \to 0} lim_{n \to \infty} \frac{1}{n} log(span(n, \epsilon, f)).$$

4.2.1 Toral automorphisms

Definition 4.2.4. An automorphism $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ is said to be *hyperbolic* if the matrix A has no eigen value with absolute value one. An automorphism $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ which is not hyperbolic is said to be *non-hyperbolic*.

Theorem 4.2.5. *[18] Any hyperbolic toral automorphism is mixing.*

Let $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ be a non-hyperbolic toral automorphism induced by the matrix $A \in GL(2, \mathbb{Z})$. Then $Tr(A) = |\alpha + \beta| \leq |\alpha| + |\beta| = 2$, where α and β are eigen values of A. That is $Tr(A) \in \{-2, -1, 0, 1, 2\}$ and $Det(A) = \pm 1$. If $Det(A) = -1$ and $Tr(A) = \pm 1$ or ± 2 then $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ is hyperbolic. Thus there are only six cases for a non-hyperbolic toral automorphism in terms of trace and determinant. For $A \in GL(2, \mathbb{Z})$, let $p_A(x)$ denotes the characteristic polynomial of A.

Proposition 4.2.6. *Let $A \in GL(2,\mathbb{Z})$ be a matrix of one among the following type.*

(i) $Det(A) = -1$, $Tr(A) = 0$.

(ii) $Det(A) = 1$, $Tr(A) = 0$.

(iii) $Det(A) = 1$, $Trace = -1$.

(iv) $Det(A) = 1$, $Tr(A) = 1$.

Then $A^n = I$ for some $n \in \mathbb{N}$.

Proof. (i) The $p_A(x) = x^2 - 1$. Then by Cayley-Hamilton theorem $A^2 = I$.

(ii) The $p_A(x) = x^2 + 1$. Then by Cayley-Hamilton theorem, $A^2 = -I$ and hence $A^4 = I$.

(iii) The $p_A(x)$ is $x^2 + x + 1$. Then by Cayley-Hamilton theorem, $A^3 = I$.

(iv) The $p_A(x)$ is $x^2 - x + 1$. Then by Cayley-Hamilton theorem, $A^2 - A + I = 0$ and hence $A^6 = I$. □

Two matrices $A, B \in GL(2,\mathbb{Z})$ are said to be conjugate if there exists an invertible matrix $P \in GL(2,\mathbb{Z})$ such that $A = P^{-1}BP$ say, $A \sim B$. Then $\sim$ is an equivalence relation on $GL(2,\mathbb{Z})$.

Proposition 4.2.7. *[45] The set $\{A_{1,j} : j \in \mathbb{Z}\backslash\{0\}\}$ contains exactly one representative from each conjugacy class of $A_{m,n} \in GL(2,\mathbb{Z})$ for $(m,n) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$.*

Proposition 4.2.8. *The set $\{B_{-1,j} : j \in \mathbb{Z} \setminus \{0\}\}$ contains exactly one representative from each conjugacy class of $B_{m,n}$ for $(m,n) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$.*

Proof. $A_{m,n}$ is similar to $A_{m',n'}$ if and only if $-A_{m,n}$ is similar to $-A_{m',n'}$ for $m, n, m', n' \in \mathbb{Z}$ since $(-P)^{-1} = -P^{-1}$ for every invertible matrix P. Hence the result follows from Proposition 4.2.7. □

Theorem 4.2.9. *For every non-hyperbolic toral automorphism T, there exist uncountably many non-empty open connected T-invariant sets.*

Proof. Let A denotes the matrix induced by T. By Lemma 4.1.8 and Proposition 4.2.6, A should satisfy one among the following three conditions.

(1) $A^m = I$ for some $m \in \mathbb{N}$.

(2) $A = A_{m,n}$ for some $m, n \in \mathbb{Z}$.

(3) $A = B_{m,n}$ for some $m, n \in \mathbb{Z}$.

Case 1. If $A^m = I$ for some $m \in \mathbb{N}$.

First we have to prove that there exists a non-empty open connected T-invariant set. Consider a non-trivial proper open subset U of $\mathbb{T}^2$ such that $\bigcup_{i=0}^{m-1} A^iU$ has area less than one. If this union is a non-empty open connected T-invariant then the proof is over. Otherwise join U and AU by an open thin rectangular strip R such a way that $V = \bigcup_{i=0}^{m-1}(A^iU \cup A^iR)$ has area less than one. By continuity, the rectangular strips A^kR will join A^kU and $A^{(k+1)(mod\ m)}U$ for $k = 0, ..., m-1$. Then V is open connected T-invariant set since $A^m = I$.

From our discussion, it is clear that for every $0 < r_1 < r_2 < 1$, we can choose U and R such a way that either $\bigcup_{i=0}^{m-1} A^iU$ or V has area lies between r_1 and r_2. By intermediate value theorem to the area function on the torus $\mathbb{T}^2$ we will get a non-empty, open, connected, and T-invariant set V_r having area r for $r_1 < r < r_2$. Hence we done in this case.

Case 2. Either $A = A_{m,n}$ or $A = B_{m,n}$ for some $m, n \in \mathbb{Z}$.

Subcase 1. When $n \neq 0$.

First suppose that $A = A_{m,n}$ for some $m, n \in \mathbb{Z}$ and $A_{m,n}$ conjugate to $A_{1,j}$ for some j. Let $V_r = \{(x, y) \in \mathbb{T}^2 : 0 < y < r < 1\}$. Then V_r is open, connected and $T_{A_{1,j}}$-invariant having area r for each $0 < r < 1$. Let U_r denotes the $T_{A_{m,n}}$-invariant set corresponding to V_r obtained from conjugacy.

Next suppose that $A = B_{m,n}$ for some $m, n \in \mathbb{Z}$. Let $V_r' = \{(x, y) \in \mathbb{T}^2 : \frac{1-r}{2} < y <$

$\frac{1+r}{2}$, $0 < r < 1\}$. Then V'_r is non-empty, open, connected and $T_{B_{-1,j}}$- invariant having area r for each j, $0 < r < 1$. By a similar argument as in the case of $T_{A_{m,n}}$, we can obtain uncountably many $T_{B_{m,n}}$-invariant sets.

Subcase 2. When $n = 0$.

If $A = A_{m,0}$ for some $m \in \mathbb{Z}$ then take $V_r = \{(x, y) \in \mathbb{T}^2 : 0 < x < r < 1\}$. If $A = B_{m,0}$ for some $m \in \mathbb{Z}$ then take $V'_r = \{(x, y) \in \mathbb{T}^2 : \frac{1-r}{2} < x < \frac{1+r}{2}, 0 < r < 1\}$. These are similar kind of rectangular strips as in previous subcase, but parallel to vertical axis. By a similar argument as in previous subcase, we will get uncountably many non-empty, open, connected, T-invariant sets. □

Remark 4.2.10. In general, the conclusion of theorem 4.2.9 may not be true for interval maps.

Proof. Define $f(x) = \begin{cases} f_1(x) & \text{if } 0 \le x \le \frac{1}{2} \\ f_2(x) & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$ where $f_1(x) = \begin{cases} \frac{1}{2} - 2x & \text{if } 0 \le x \le \frac{1}{4} \\ 2x - \frac{1}{2} & \text{if } \frac{1}{4} \le x \le \frac{1}{2} \end{cases}$

and $f_2(x) = \begin{cases} 2x - \frac{1}{2} & \text{if } \frac{1}{2} \le x \le \frac{3}{4} \\ \frac{5}{2} - 2x & \text{if } \frac{3}{4} \le x \le 1 \end{cases}$

Then f has only three non-empty open connected f-invariant sets $(0, \frac{1}{2}), (\frac{1}{2}, 1)$ and $(0, 1)$ since f_1, f_2 are transitive on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ respectively. □

An immediate corollary to the Theorem 4.2.9

Corollary 4.2.11. *A toral automorphism is transitive if and only if it is hyperboplic.*

Now we have:

Theorem 4.2.12. *The following are equivalent for every toral automorphism $T : \mathbb{T}^2 \to \mathbb{T}^2$.*

(1) *T is transitive and $Per(T)$ is cofinite.*

(2) *T is weak mixing.*

(3) *T is mixing.*

Proof. This theorem follows from Theorem 2.1.14, Remark 2.1.15 and Corollary 4.2.11.

□

Proposition 4.2.13. *[18]*

For any $\epsilon > 0$ *and* $n \geq 1$; $span(n, \epsilon, f) \leq sep(n, \epsilon, f)$ *and* $sep(n, 2\epsilon, f) \leq span(n, \epsilon, f)$ *and hence*

$$h(f) = lim_{\epsilon \to 0} lim_{n \to \infty} \frac{1}{n} log(span(n, \epsilon, f)) = lim_{\epsilon \to 0} lim_{n \to \infty} \frac{1}{n} log(sep(n, \epsilon, f)).$$

The ideas involved in the following theorem may not be new. But it is worth mentioning because we do not find any reference for the non-hyperbolic case.

Theorem 4.2.14. *Let* $T : \mathbb{T}^2 \to \mathbb{T}^2$ *be a toral automorphism. Then the entropy* $h(T) = log|\lambda|$ *where* λ *is an eigen value of the matrix induced by* T *having modulus* ≥ 1.

Proof. If T is hyperbolic then see [18]. Next suppose that T is a non-hyperbolic toral automorphism. Fix $\epsilon > 0$. Since $\mathbb{T}^2$ is totally bounded, we can cover $\mathbb{T}^2$ by ϵ-ball centered at a finite set of points $\{X_1, X_2, ..., X_k\}$, say $B_\epsilon(X_i) = \{Z \in \mathbb{T}^2 : |Z - X_i| < \epsilon\}$ and let $Box(X_i) = \{X_i + \alpha v_1 + \beta v_2 : -\epsilon \leq \alpha, \beta \leq \epsilon\}$ where v_1, v_2 are the two distinct eigen vector corresponding to distinct eigen values with $|v_1| = |v_2| = 1$. Choose $Box(X_i)$ such a way that $\mathbb{T}^2 = \bigcup_{i=1}^{k} Box(X_i)$, it is possible since $B_\epsilon(X_i)$ covers $\mathbb{T}^2$. Let $S_p(X_i) = \{X_i - \epsilon v_1, X_i, X_i + \epsilon v_2\}$. Note that $S_p(X_i) \subset\subset Box(X_i)$. Let $S_p = \bigcup_{i=1}^{k} S_p(X_i)$.

For any $Z \in \mathbb{T}^2$, choose $1 \leq i \leq k$ such that $Z \in Box(X_i)$. Then $Z = X_i + \alpha v_1 + \beta v_2$ for some $-\epsilon \leq \alpha, \beta \leq \epsilon$. Choose $-1 \leq j \leq 1$ with $|\alpha - j\epsilon| \leq \frac{\epsilon}{2}$. Then $W = X_i + j\epsilon V_1 \in S_p(X_i)$. For every $n \in \mathbb{N}$, $|T^n Z - T^n W| \leq |(\alpha - j\epsilon)| + |\beta| < 2\epsilon$. Hence

S_p is an $(n, 2\epsilon)$-spanning set. Note that the cardinality of S_p is at most $3k$. Therefore $h(T) = lim_{\epsilon\to 0} lim_{n\to\infty} \frac{1}{n} log(span(n, \epsilon, T)) \leq 0$ and hence $h(T) = 0$. □

By Theorem 4.2.5, a toral automorphism is transitive if and only if it is hyperbolic. But by Lemma 4.1.5, every toral automorphism has a dense set of periodic points. So by Theorem 1.2.4, a toral automorphism is chaotic if and only if it is hyperbolic. This gives a characterization for toral automorphisms. By Theorem 4.2.14, we can say that a toral automorphism T is chaotic if and only if $h(T) > 0$. This gives another characterization for toral automorphisms. In short, we can say that hyperbolic and non-hyperbolic toral automorphisms are in two extremes.

4.2.2 Existence of a Lyapunov function

Let (X, f) be a dynamical system. A continuous map $\phi : X \to \mathbb{R}^+ \cup \{0\}$ is said to be a Lyapunov function on (X, f) if $\phi(f(x)) \leq \phi(x)$ for all $x \in X$. For each $t \in \mathbb{R}^+ \cup \{0\}$, the set $\phi^{-1}(t) = \{x \in X : \phi(x) = t\}$ is called a *fiber* in X. Note that having Lyapunov function is a dynamical property.

This section is mainly motivated by the following question:

Can the entropy of a dyanamical system is zero whenever there is a Lyapunov function on it such that every fiber is of empty interior?

Unfortunately we do not have complete answer. But we have so many affirmative examples. For this purpose, we define three properties as follows.

A dynamical system (X, f) is said to satisfy property 'C_1' if there exists a continuous function $\phi : X \to \mathbb{R}^+ \cup \{0\}$ such that every fiber has empty interior, and every set of the form $\phi^{-1}([0, \alpha))$ is f-invariant for all $\alpha \in \mathbb{R}^+$. It is said to satisfy property 'C_2' if there exists a compact subset K of X having non-empty interior for which there is a neibourhood base such that every member of which is f-invariant. ie., if V is open and contains K then there exists f-invariant open W such that $K \subset W \subset V$. It is said to satisfy property 'C_3' if there exists a Lyapunov function such that every fiber has an empty interior.

Now, let $x \in \phi^{-1}([0, \alpha))$ for some $\alpha > 0$. This implies $\phi(x) < \alpha$. Then $f(x) \in \phi^{-1}([0, \alpha))$ if $\phi^{-1}([0, \alpha))$ is f-invariant. Which implies $\phi(f(x)) < \alpha$. This is true for all α. Hence $\phi(f(x)) \leq \phi(x)$. Conversely, assume that $\phi(f(x)) \leq \phi(x)$ for all $x \in X$. This implies $\phi^{-1}([0, \alpha))$ is f-invariant for all $\alpha > 0$. Hence the property 'C_1' is equivalent to the property 'C_3'.

Definition 4.2.15. Let X be a metric space with metric d. A map $f : X \to X$ is said to be a contraction if there exists $0 < c < 1$ such that $d(f(x), f(y)) \leq cd(x, y)$ for all $x, y \in X$.

Proposition 4.2.16. *Every contraction map on a non-isolated complete metric space satisfies both the properties 'C_1' and 'C_2'.*

Proof. Let X be a complete metric space with metric d, and $T : X \to X$ be a contraction map. Let x_0 be the unique fixed point of T.

Define $\phi : X \to [0, \infty)$ by $\phi(x) = d(x, x_0)$. Then for all $\alpha > 0$, $\phi^{-1}([0, \alpha)) = B(x_0, \alpha)$ is open and T-invariant. Then the property 'C_1' holds, if each $\phi^{-1}(\alpha)$ has empty interior. This is true whenever X is non-isolated. Hence X satisfies property 'C_1'.

Next consider, $K = \{x_0\}$ and note that each ball open ball with center x_0 and radius $r > 0$ is T-invariant. Therefore K has a local base of T-invariant sets. Then property 'C_2' holds if $\{x_0\}$ is not open ; ie., true whenever X is non-isolated. □

Proposition 4.2.17. *Every isometry with a periodic point satisfies the properties 'C_1' and 'C_2'.*

Proof. Let $f : X \to X$ be an isometry, and K be an f-periodic orbit with $|K| = n$.

Define $\phi(x) = \text{Min}\{d(x,y) : y \in K\} = \phi_1 \wedge \phi_2 \wedge ... \wedge \phi_n$ where $\phi_i(x) = d(x, f^i(x_0))$ and $\wedge$ denotes the minimum of functions. Then ϕ is continuous and hence $\phi^{-1}([0,\alpha))$ is open for all $\alpha > 0$. Let $y \in \phi^{-1}([0,\alpha))$. Then $\phi(y) < \alpha$. Therefore $d(y,v) < \alpha$ for some $v \in K$. Then $d(f(y), f(v)) < \alpha$ for some $v \in K$. This implies $\phi(f(y)) < \alpha$. Hence $f(y) \in \phi^{-1}([0,\alpha))$. Hence $f : X \to X$ satisfies property property 'C_1'.

Next consider, $x_0 \in K$. For all $r > 0$, let $V_r = \bigcup_{i=1}^{n} B(f^i(x_0), r)$. Then V_r is open and f-invariant. Let $y \in V_r$. Then $d(y, f^i(x_0)) < r$ for some i. This implies $d(f(y), f^{i+1}(x_0)) < r$. Hence $f(y) \in V_r$. Given any open W contains K, we can find $r > 0$ such that $W \supset V_r$, $f^i(x_0) \in K \subset W$, and there exists $B(f^i(x_0), r_i) \subset W$ for $i = 1, 2, ..., n$. Take $\text{Min}\{r_i : i = 1, 2, ..., n\} = r$. Hence $V_r = \bigcup_{i=1}^{i=r} B(f^i(x_0), r) \subset W$.

Hence the proof. □

Recall, for $A \subset X$, $\overline{A}$ denotes the closure of A.

Lemma 4.2.18. Let V be an attractor of a dynamical system (X, f). Then $\bigcap_{n=0}^{\infty} f^n(V) = \bigcap_{n=0}^{\infty} f^n(\overline{V}) = \overline{\bigcap_{n=0}^{\infty} f^n(V)}$.

Proof. From the definition of an attractor, we have $f^{m+1}(V) \subset f^{m+1}(\overline{V}) \subset f^m(V)$ for all m. Therefore $\bigcap_{n=0}^{\infty} f^n(V) = \bigcap_{n=0}^{\infty} f^n(\overline{V})$. By continuity $f(\overline{V})$ is compact. Then $\overline{f^{m+1}(V)} \subset f^{m+1}(\overline{V}) \subset \overline{f^m(V)}$ for all m. Therefore $\bigcap_{n=0}^{\infty} f^n(\overline{V}) = \overline{\bigcap_{n=0}^{\infty} f^n(V)}$. Hence the proof. □

Proposition 4.2.19. *If (X, f) is an open dynamical system having an attractor then it satisfies property 'C_2'.*

Proof. Let V be an attractor of an open dynamical system (X, f).

Claim: If W is an open set containing V then $f^n(V) \subset W$ for all but a finitely many n.

Suppose not. ie., $f^n(V)$ does not contained in W infinitely many n. Choose $(x_n) \in f^n(V) \setminus W$. Then (x_n) has a convergent subsequence (x_{n_i}). Say (x_{n_i}) converges to x. Then $x \in \overline{f^{m+1}(V)}$ for all m since $x_{m+1} \in f^{m+1}(V)$ for all m. By the definition of attractor, $f^{m+1}(V) \subset f^m(f(V)) \subset f^m(V)$. Hence $x \in K$. A contradiction. But by Lemma 4.2.18, $\bigcap_{n=0}^{\infty} f^n(V) = \bigcap_{n=0}^{\infty} f^n(\overline{V}) = \overline{\bigcap_{n=0}^{\infty} f^n(V)}$. Therefore $f^n(V) \subset W$ for all but a finitely many n. Hence $\{f^n(V)\}$ form a local base. □

Theorem 4.2.20. *Every non-hyperbolic toral automorphism satisfies the properties 'C_2' and 'C_3'.*

Proof. By looking the proof Theorem 4.2.9, we can prove this theorem. □

Theorem 4.2.21. *Let T be a non-hyperbolic toral automorphism. Then*

(1) there exist uncountably many non-empty open connected T-invariant sets,

(2) there exists a Lyapunov function for T, and

(3) the topological entropy, $h(T) = 0$.

Proof. Proof follows from Theorems 4.2.9, 4.2.14 and 4.2.20. □

We can prove that isometries and contractions on a complete metric space are having zero entropy. Now we ask the following question that is open to us.

Question: Does there exists a dynamical system which satisfies either the property 'C_2' or the property 'C_3' having positive entropy?

Chapter 5

Some Simple Dynamical Systems

In this chapter, we study the class of some simple systems on $\mathbb{R}$ induced by continuous maps having finitely many non-ordinary points. We characterizes this class using labeled digraphs and maximal dynamically independent sets. In particular, we discuss the class of continuous maps having unique non-ordinary point, and the class of continuous maps having exactly two non-ordinary points separately.

5.1 Introduction

The properties of dynamical systems which are preserved by topological conjugacies are called dynamical properties. The points which are unique upto some dynamical property are called *dynamically special points*. Said differently, a special point has a dynamical property which no other point has. The idea of special points and non-ordinary points are relatively new to the literature (see [3] [53], [45]). Recently, we studied the class of simple dynamical systems induced by homeomorphisms. Reader may refer [3] to get an idea of simple systems induced by homeomorphisms having finitely many non-ordinary points.

Throughout this chapter we will be working with continuous self maps of the real line. Since $\mathbb{R}$ has order structure, we would like to consider the conjugacies preserving the order. Hence the conjugacies which we consider in this chapter are order preserving conjugacies (increasing conjugacies).

When we are working with a single system, any self conjugacy can utmost shuffle points with same dynamical behavior. Therefore a point which is unique upto its behavior must be fixed by every self conjugacy. On the other hand if a point is fixed by all self conjugacies then it must have a special property (some times it may not be known explicitly). These things motivated to call the set of all points fixed by all self conjugacies as set of *special points*. For $x, y \in \mathbb{R}$, we write $x \sim y$ if x and y have the same dynamical properties in the dynamical system $(\mathbb{R}, f)$. Said precisely, $x \sim y$ if there exists an increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that $h \circ f = f \circ h$ and $h(x) = y$. It is easy to see that $\sim$ is an equivalence relation. Since the equivalence relation is coming from self conjugacy it is important in the field of topological dynamics. Let $[x]$ to denote the equivalence class of $x \in \mathbb{R}$. Let I, J be two sub intervals of $\mathbb{R}$. We say that $I < J$ if $x < y$ for all $x \in I$ and $y \in J$.

In a dynamical system (X, f), we say that a point x is ordinary if points like it form a neighbourhood of it. That is, an element $x \in \mathbb{R}$ is *ordinary* in $(\mathbb{R}, f)$ if its equivalence class $[x]$ is a neighbourhood of it. i.e, the equivalence class of x contains an open interval around x. A point which is not ordinary is called *non-ordinary*. Let $N(f)$ to denote the set of all non-ordinary points of f. Note that a point is *special* if $[x] = \{x\}$. Let $S(f)$ to denote the set of all special points of f.

5.2 Some basic results

Definition 5.2.1. Let (X, f) be a dynamical system. The *full orbit* of a point $x \in X$ we mean the set

$O(x) = \{y \in X : f^n(x) = f^m(y) \text{ for some } m, n \in \mathbb{N}\}$.

For any subset $A \subset \mathbb{R}$, let

$$O(A) = \bigcup_{x \in A} O(x) = \bigcup_{x \in A} \{y \in \mathbb{R} : f^n(y) = f^m(x) \ for \ some \ m, \ n \in \mathbb{N}\}.$$

Definition 5.2.2. A point x in a dynamical system (X, f) is said to be a *critical point* if f fails to be one-one in every neighbourhood of x. The set of all critical points of f is denoted by $C(f)$.

Definition 5.2.3. For any subset A of $\mathbb{R}$, we write $\partial A = \overline{A} \cap (\overline{X - A})$ and call the boundary of A where $\overline{A}$ denotes the closure of A in $\mathbb{R}$.

See [45], for the following characterization theorem for the set $N(f)$ (and hence for $S(f)$).

Theorem 5.2.4. *For continuous self maps of the real line $\mathbb{R}$, the set of all non-ordinary points is contained in the closure of the union of full orbits of critical points, periodic points and the limits at infinity (if they exist and finite).*

Remark 5.2.5. In the above theorem the inclusion can be strict.

Proof. Consider the map $f(x) = x + sinx$ for all $x \in \mathbb{R}$. All integral multiples of π are fixed points for this map but the increasing bijection $x \mapsto x + 2\pi$ commutes with f and fixes none of them. $\square$

Remark 5.2.6. [45] For polynomials of even degree the equality is true in Theorem 5.2.5.

For a dynamical system (X, f), let $D = O(C(f) \cup P(f) \cup \{f(\infty), f(-\infty)\})$ where $f(\infty)$ and $f(-\infty)$ are the limits of f at ∞ and $-\infty$ respectively, provided they are finite.

Theorem 5.2.7. *[45] For polynomial maps f of $\mathbb{R}$, $S(f)$ has to be either empty or a singleton or the whole $\overline{D}$.*

From the definition, it is clear that the set of special points $S(f)$ is always closed. The following theorem is about the converse and it is proved in [45].

Theorem 5.2.8. *Given any closed subset F of $\mathbb{R}$, there exists a continuous map $f : \mathbb{R} \to \mathbb{R}$ such that $S(f) = F$.*

Remark 5.2.9. For every closed subset F of $\mathbb{R}$ there exists continuous maps $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that $S(f) = Fix(g) = F$. Conversely, for every closed subset F of $\mathbb{R}$ there exist continuous maps $f, g : \mathbb{R} \to \mathbb{R}$ such that $S(f) = Fix(g) = F$.

Proof. For every closed subset F of $\mathbb{R}$ there exists strictly increasing continuous bijection $f : \mathbb{R} \to \mathbb{R}$ such that $Fix(f) = F$. This is because, if we define $f(x) = x + \frac{1}{2}d(x, F)$ for all x, then f is as required. Now the remark follows from Theorem 5.2.8. □

For any continuous $f : \mathbb{R} \to \mathbb{R}$, let G_f denote the set of all topological conjugacies of f and let $G_{f\uparrow}$ denote the set of all order conjugacies of f.

We prove the Propositions 5.2.10 to 5.2.26 in [3]. Reader may refer [3] for more detailed proof.

Proposition 5.2.10. *If x is an ordinary point of f and if h is self topological conjugacy of f, then $h(x)$ is ordinary.*

Proof. Since x is ordinary there exists an open interval V contained in $[x]$. We prove that the open interval (since h is a homeomorphism) $h(V)$ is contained in $[h(x)]$.

Take $s \in h(V)$. Then $s = h(t)$ for some $t \in V$. Since $V \subset [x]$, there exists $\varphi \in G_{f\uparrow}$ such that $\varphi(t) = x$. Then the increasing homeomorphism $\psi = h\phi h^{-1}$ carries s to $h(x)$ and commutes with f. □

Proposition 5.2.11. *If x is a non-ordinary point of f and if h is a self topological conjugacy of f, then $h(x)$ is non-ordinary.*

Proof. Note that, if h is a self conjugacy of f then h^{-1} is also a self conjugacy of f. Now, the proof follows from the Proposition 5.2.10. □

Now we ask: For a continuous map $f : \mathbb{R} \to \mathbb{R}$, how the equivalence classes looks like?

The following lemma answer this question.

Lemma 5.2.12. [3] Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $a < b$ and $(a, b) \cap N(f) = \emptyset$. Then $x \sim y$ for all $x, y \in (a, b)$.

Proof. Assume without loss of generality that $x < y$. Suppose $x \nsim y$, so $z = \sup([x] \cap (-\infty, y])$ exists. Clearly $z \in \overline{[x]}$. If $z = y$ then $z \in \overline{[y]} \subset \overline{\mathbb{R} \setminus [x]}$. Otherwise $z < y$ and $[z, y) \cap (\mathbb{R} \setminus [x]) \neq \emptyset$ for every $y - x > \epsilon > 0$ which again shows $z \in \overline{\mathbb{R} \setminus [x]}$. Hence $z \in \partial([x])$, so $z \in N(f)$ by Proposition 5.2.19. But $a < x \leq z \leq y < b$ so $z \in (a, b) \cap N(f)$ contradicting our hypothesis. □

Theorem 5.2.13. *[3] Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If $|N(f)| = n$ then $|\{[x] : x \in \mathbb{R}\}| = 2n + 1$.*

Proof. Let $N(f) = \{x_1, x_2, \cdots, x_n\}$ where $x_1 < x_2 < \cdots < x_n$. By Proposition 5.2.17, each $\{x_i\}$ is an equivalence class. By Remark 5.3.15, each of these intervals $(-\infty, x_1), (x_1, x_2), \cdots, (x_{n-1}, x_n), (x_n, \infty)$ is invariant under every element of $G_{f\uparrow}$, so all remaining equivalence classes are contained in one of these intervals. Lemma

5.2.12 above now shows that each of these interval is an equivalence class, giving $|\{[x] : x \in \mathbb{R}\}| = 2n + 1$. □

Remark 5.2.14. Note that, being a point in a particular equivalence class $[x]$ is a dynamical property.

Remark 5.2.15. There are continuous maps $f : \mathbb{R} \to \mathbb{R}$ having finitely many equivalence classes, but infinitely many non-ordinary points. For example, consider the map $f(x) = x + \sin x$ on $\mathbb{R}$. There are two classes of fixed points. Since increasing orbits must map to increasing orbits under increasing conjugacies, points like $\frac{\pi}{2}$ (increasing orbit) and $\frac{3\pi}{2}$ (decreasing orbit) cannot be equivalent. Hence there must be at least four equivalence classes. To see that there are exactly four equivalence classes, let $I_k = (2k\pi, (2k + 1)\pi), D_k = ((2k + 1)\pi, 2(k + 1)\pi)$ and observe that $I_k \cap N(f) = \emptyset = D_k \cap N(f)$ for each $k \in \mathbb{Z}$ by Proposition 5.2.25. Hence by Lemma 5.2.12, each I_k and D_k is contained in a single equivalence class. Conjugacies of the form $x \mapsto x + 2k\pi$ complete the argument.

Remark 5.2.16. (1) If $f : \mathbb{R} \to \mathbb{R}$ has a unique fixed point then it is non-ordinary and vice-versa.

(2) If $f : \mathbb{R} \to \mathbb{R}$ has finitely many fixed points (critical points) then all fixed (critical) points are special and hence non-ordinary.

(3) If there are only finitely many periodic cycles then all periodic points are special.

(4) Every special point is non-ordinary. But every non-ordinary point may not be special.

Proof. (1) Since the topological conjugacies carry fixed points to fixed points, the unique fixed point must be fixed by every self conjugacy and hence special.

Suppose $x_0 \in \mathbb{R}$ is the unique non-ordinary point of f. Then $h(x_0) = x_0$ for all $h \in G_{f\uparrow}$. Now, for any $h \in G_{f\uparrow}$ we have $h(f(x_0)) = f(h(x_0)) = f(x_0)$. That is, the point $f(x_0)$ is special. Since x_0 is the only special point, we have $f(x_0) = x_0$.

(2) Follows from the fact that under a topological conjugacy fixed points will be mapped to fixed points and critical points will be mapped to critical points and the fact that it takes the finite set of fixed points (critical points) to itself bijectively, preserving the order.

Proof of (3) is easy.

(4) It is immediate from the definition that every special point is non-ordinary. For the converse, consider the map $x \mapsto x + \sin x$ on $\mathbb{R}$ which has countably many fixed points. Note that all the fixed points are non-ordinary and they form two distinct equivalence classes, hence they are not special. □

Proposition 5.2.17. *If $f : \mathbb{R} \to \mathbb{R}$ has only finitely many non-ordinary points then every non-ordinary point is special.*

Proof. Since $N(f)$ is finite, it follows from the Proposition 5.2.11 that $h(N(f)) = N(f)$ for all $h \in G_{f\uparrow}$. Then we must have $h(x) = x$ for all $x \in N(f)$, because of the order preserving nature of h. Hence all points of $N(f)$ are special. □

Thus, for the class of maps with finitely many non-ordinary points the idea of special points and the idea of non-ordinary point, coincide.

Proposition 5.2.18. *For maps with finitely many non-ordinary points, $f(x)$ is non-ordinary whenever x is non-ordinary.*

Proof. Since x is non-ordinary and since there are only finitely many non-ordinary points, we have $h(x) = x$ for all $h \in G_{f\uparrow}$.

Now for any $h \in G_{f\uparrow}$, we have $h(f(x)) = f(h(x)) = f(x)$. Hence $f(x)$ is non-ordinary. □

Recall that the properties which are preserved under topological conjugacies are called dynamical properties. Hence, if two points x, y in the dynamical system (X, f), differ by a dynamical property, then no conjugacy can map one to the other, from which it follows that,

Proposition 5.2.19. *For any dynamical property P, the points of ∂S_P are non-ordinary where S_P denotes the set of all points in (X, f) having the dynamical property P.*

Corollary 5.2.20. *Let $f : \mathbb{R} \to \mathbb{R}$ be constant in a neighbourhood of a point x_0. Then the end points of the maximal interval around x_0 on which f is constant, are non-ordinary.*

Recall that, if $f : \mathbb{R} \to \mathbb{R}$ has a unique non-ordinary point then it is a fixed point.

Proposition 5.2.21. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map. Then,*

(i) If $x \in \mathbb{R}$ is both critical and ordinary then f is locally constant at x.

(ii) If x is ordinary and f is not locally constant at x then $f(x)$ is ordinary.

Proof. (i) Let $x_0 \in \mathbb{R}$ be both critical and ordinary.

Claim: f is constant in some neighbourhood of x_0.

Since x_0 is ordinary, there exist $\eta > 0$ such that all points in $(x_0 - \eta, x_0 + \eta)$ will look alike. So it is enough to prove that f is somewhere constant in $(x_0 - \eta, x_0 + \eta)$.

Case 1:

Now , suppose some point of $(x_0 - \eta, x_0 + \eta)$ is point of local maximum. Then we can prove easily that every point of $(x_0 - \eta, x_0 + \eta)$ is a point of local maximum. That is there exist $\epsilon > 0$ such that $f(x_0) \geq f(t)\ \forall\ t \in (x_0 - \epsilon, x_0 + \epsilon)$. Next, choose $\delta < \epsilon, \eta$. Then there exist $y \in [x_0 - \delta, x_0 + \delta]$ such that $f(y) \leq f(t)\ \forall\ t \in [x_0 - \delta, x_0 + \delta]$ ....(1).

But y is a point of local maximum (since $\delta < \eta$). That is there exist $\alpha > 0$ such that $f(y) \geq f(s)\ \forall\ s \in (y - \alpha, y + \alpha)$ ....(2).

From equations (1) and (2), it follows that f is constant in some neighbourhood y and hence constant in some neighbourhood of x_0.

Case 2:

No point is a point of local maximum. That is, in every subinterval f attains its maximum at one of the end points.

If f assumes supremum always on the right end or always on the left end then f is strictly monotone.

Note that, it is enough if we prove monotone somewhere. Take a neighbourhood (α, β) of x_0 such that $(\alpha, \beta) \subset (x_0 - \eta, x_0 + \eta)$ and let $\sup f$ on (α, β) is attained at the right end point β. Suppose $\sup f$ is attained at the right end point in every subinterval of (α, β) containing x_0. Then f is increasing in (x_0, β). We are done.

Suppose there is a subinterval say $(x_0 - \epsilon_1, x_0 + \epsilon_2)$ of (α, β) on which f attains its supremum at the left end point. Then f attains its infimum on $(x_0 - \epsilon_1, \beta)$ at some interior point. We now argue as in Case.1, with infimum instead of supremum.

Proof of (ii): We make use of (i).

Assume that f is not constant on any neighbourhood of x. Because x is ordinary, there exist an open interval J around x in which all points are equivalent such that f is not constant on J. It follows that f is not constant on any non-trivial subinterval of J, because the end points of intervals of constancy are non-ordinary. From (i), it follows that J has no critical point. Therefore $f(J)$ is an open interval. We claim that any two elements of $f(J)$ are equivalent. Let $f(y)$ be a general element of $f(J)$ where $y \in J$,

$y \neq x$. By choice of J, there exists a self conjugacy h of f such that $h(y) = x$. Which implies $hf(y) = fh(y) = f(x)$. Therefore $f(y)$ is equivalent to $f(x)$. This proves $f(x)$ is ordinary. □

Remark 5.2.22. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then $\sup f(\mathbb{R})$, $\inf f(\mathbb{R})$, $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are special (in particular, non-ordinary) provided they are finite.(Note that, For maps with finitely many non-ordinary points both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ always exists in $\mathbb{R} \cup \{-\infty, \infty\}$).

Proof. For any $h \in G_{f\uparrow}$, $h(f(\mathbb{R})) = f(h(\mathbb{R})) = f(\mathbb{R})$. That is h takes the range of f to itself. Since h is increasing, $h(\sup f) = \sup f$ and $h(\inf f) = \inf f$.

To prove $\lim_{x\to\infty} f(x)$ **is special:**

First we prove that for maps with finitely many non-ordinary points, $\lim_{x\to\infty} f(x)$ always exists in $\mathbb{R} \cup \{\infty, -\infty\}$.

For, let t_0 be the largest non-ordinary point and let A be the set of all critical points $> t_0$.

Suppose A is empty. Then f is monotone on $[t_0, \infty)$ and hence $\lim_{x\to\infty} f(x)$ exists.

Suppose A is nonempty. Then ∂A is nonempty. But every element of ∂A is non-ordinary. Hence $\partial A = \{t_0\}$. Therefore $A = (t_0, \infty)$. Therefore f is constant on A (We argue as in the proof of Case-2 of (i), in the previous proposition). Hence $\lim_{x\to\infty} f(x)$ exists.

Now to prove $\lim_{x\to\infty} f(x)$ is special:

Denote $\lim_{x\to\infty} f(x)$ by l. Let $h \in G_{f\uparrow}$. Note that for any sequence $(x_n) \to \infty$ the sequence $f(x_n) \to l$ and the sequence $h(x_n) \to \infty$.

Let $(x_n) \to \infty$. Then $f(x_n) \to l$. Hence $h(f(x_n)) = f(h(x_n)) \to h(l)$. But the sequence $h(x_n) \to \infty$. Hence by the definition of l, $f(h(x_n)) = h(f(x_n)) \to l$. Hence $h(l) = l$. This completes the proof.

Similarly we can prove that $\lim_{x\to-\infty} f(x)$ is special. □

Proposition 5.2.23. *The maps $x+1$ and $x-1$ on $\mathbb{R}$ are topologically conjugate; but not order conjugate.*

Proof. The maps $x+1$ and $x-1$ are conjugate to each other through $-x+\frac{1}{2}$.

If possible, let h be an order conjugacy from $f(x)=x+1$ to $g(x)=x-1$. Then $h(x+1)=h(f(x))=g(h(x))=h(x)-1$. i.e, $h(x+1)-h(x)=-1<0$. Which is a contradiction to the assumption that h is increasing. □

Remark 5.2.24. Note that for the map $x+1$ on $\mathbb{R}$, all points are ordinary. For, if $a,b\in\mathbb{R}$, then the map $x+b-a$ is the order conjugacy of $x+1$ which maps a to b.

Recall that for any subset A of a metric space X,

$$(\partial A)^c = int(A)\cup int(A^c) \tag{5.1}$$

Proposition 5.2.25. *Let $f:\mathbb{R}\to\mathbb{R}$ be an increasing bijection and let $x\in\mathbb{R}$. Then x is non-ordinary if and only if x is in the boundary of Fix(f).*

The above proposition gives a characterization for the non-ordinary points of increasing homeomorphisms.

Proposition 5.2.26. *Let $f:\mathbb{R}\to\mathbb{R}$ be a homeomorphism without fixed points. Then*

(i) If $f(0)>0$ then f is order conjugate $x+1$.

(ii) If $f(0)<0$ then f is order conjugate $x-1$.

Proof. Define $h:\mathbb{R}\to\mathbb{R}$ as follows. Assume $f(0)>0$. Define $h(t)=\frac{t}{f(0)}$, $0\le t<f(0)$. We know that $(f^n(0))$ increases and diverges to ∞ and $(f^{-n}(0))$ decreases and diverges to $-\infty$ for all $n\in\mathbb{N}$. Moreover for $t\in\mathbb{R}$ there exists unique $n\in\mathbb{Z}$ such that,

$f^n(0) \leq t < f^{n+1}(0)$. Define $h(t) = h(f^{-n}(t)) + n$. Then $h \circ f(t) = h(t) + 1 \ \forall \ t \in \mathbb{R}$. This h gives a conjugacy from f to $x + 1$.

If $f(0) < 0$ then we can give a similar proof. □

We denote graph$(f) = \{(x, f(x)) : x \in \text{domain}\}$.

Corollary 5.2.27. *Let $f, g : (a, b) \to (a, b)$ be homeomorphisms without fixed points. Then f is order conjugate to g if and only if both graph(f) and graph(g) are on the same side of the diagonal.*

In particular,

(i) If $f(x) > x$ for all $x \in (a, b)$ then f is order conjugate to $x + 1$.

(ii) If $f(x) < x$ for all $x \in (a, b)$ then f is order conjugate to $x - 1$.

Remark 5.2.28. In fact, in the previous corollary, the interval (a, b) can be replaced by any open ray in $\mathbb{R}$.

Remark 5.2.29. For an increasing bijection $f : \mathbb{R} \to \mathbb{R}$ with finitely many non-ordinary points, all non-ordinary points are fixed points.

Proof. We know that, for maps with finitely many non-ordinary points, all non-ordinary points are fixed by every order conjugacy. Here f itself is a self conjugacy. □

5.3 Class of continuous maps

Note that, under a topological conjugacy a point can be mapped to a point with similar dynamics. By definition, the points of $[x]$ are dynamically same.

We now consider the systems for which there are only finitely many equivalence classes. This means there are only finitely many kinds of orbits upto conjugacy. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and Per(f) properly contained in $\{1, 2, 2^2, ...\}$, then f is not

Li-Yorke chaotic. Also note that, if $f : \mathbb{R} \to \mathbb{R}$ is Devaney chaotic then $6 \in \mathrm{Per}(f)$. Therefore, if $f : \mathbb{R} \to \mathbb{R}$ is a continuous map having finitely many non-ordinary points then it is neither Li-York chaotic nor Devaney chaotic because of Sharkovskii's theorem. For these reasons we call such systems as *simple systems*. These are the system in which the phase portrait can be drawn. Phase portraits (see [30]) are frequently used to graphically represent the dynamics of a system. A phase portrait consists of a diagram representing possible beginning positions in the system and arrows that indicate the change in these positions under iteration of the function. The phase portrait drawable systems are interested to physicist. So, it is better to study the class of simple dynamical systems. Recall that, if S_P denote the set of all points having the dynamical property P then the points of ∂S_P (the boundary of S_P) are non-ordinary. As in Remark 5.2.14, being a point in a particular equivalence class is a dynamical property of the point. Hence by the very nature of the order conjugacies, it follows that when there are finitely many non-ordinary points (therefore special points) there are only finitely many equivalence classes. We now study, the class of simple systems induced by continuous maps having finitely many non-ordinary points.

Throughout this section we will be working with the alphabet $\{\mathbf{A},\mathbf{B},\mathbf{O}\}$. Let $\tilde{\mathbf{A}} = \mathbf{B}$, $\tilde{\mathbf{B}} = \mathbf{A}$ and $\tilde{\mathbf{O}} = \mathbf{O}$. If $w = w_1w_2...w_n$, then the *dual* of w is defined as

$$\tilde{w} = \tilde{w}_n\tilde{w}_{n-1}...\tilde{w}_1.$$

If $\tilde{w} = w$ then w is said to be *self conjugate*. The tilde defined here is different from the tilde introduced in Chapter 2. Here **A** stands for "**a**bove the diagonal" and **B** stands for "**b**elow the diagonal" and **O** stands for "**o**n the diagonal".

Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism with finitely many non-ordinary (hence special points) say, $x_1 < x_2 < ... < x_n$ for some $n \in \mathbb{N}$. This finite set of points gives rise to an ordered partition $\{(-\infty, x_1), (x_1, x_2), ...(x_n, \infty)\}$ of $\mathbb{R} \backslash \{x_1, x_2, ..., x_n\}$.

Note that, On each component interval exactly one of the following holds, by proposition 5.2.25 (Since the only subsets of $\mathbb{R}$ with empty boundary are the empty set and $\mathbb{R}$).

(i) $f(t) > t \ \forall \ t$ (ii) $f(t) < t \ \forall \ t$ (iii) $f(t) = t \ \forall \ t$.

This gives rise to a word $w(f)$ over $\{\mathbf{A},\mathbf{B},\mathbf{O}\}$ of length $n+1$ by associating **A** to (i), **B** to (ii) and **O** to (iii).

Next, note that the sub word **OO** is forbidden. For, suppose **O** is occurring at i^{th} and $(i+1)^{th}$ place then in both(x_i, x_{i+1}) and (x_{i+1}, x_{i+2}) all points are fixed. Then x_{i+1} becomes ordinary. A contradiction to the assumption that x_{i+1} is a non-ordinary point.

Conversely,

Suppose a word w of length $(n+1)$(in which **OO** is forbidden) is given.

Then we can construct an increasing bijection on $\mathbb{R}$ such that its associated word is w, as follows:

Take the points $0, 1, 2, ..., n-1$ and consider the partition $\{(-\infty, 0), (0, 1), (1, 2), ..., (n-1, \infty)\}$ of $\mathbb{R}$. If $w = w_1 w_2 .....w_{n+1}$ then associate w_1 to $(-\infty, 0)$, w_2 to $(0, 1), ...$, and w_{n+1} to $(n-1, \infty)$. Now it is easy to construct an increasing bijection $f : \mathbb{R} \to \mathbb{R}$ such that $w(f) = w$. To be precise if $i - 1 < t < i$ then

$$f(t) = \begin{cases} i - 1 + (t - i + 1)^2 & \text{if } \ w_i = \mathbf{B} \\ i - 1 + \sqrt{t - i + 1} & \text{if } w_i = \mathbf{A} \\ t & \text{if } w_i = \mathbf{O} \end{cases}$$

Now we state the following propositions and theorems. Reader may refer [3] for a proof.

Proposition 5.3.1. *Let f, g be two increasing bijection on $\mathbb{R}$ with finitely many (same* *number of) non-ordinary points. Then f and g are order conjugate if and only if $w(f) = w(g)$.*

Proposition 5.3.2. *There is a one to one correspondence between the set of all increasing continuous bijections (upto order conjugacy) on $\mathbb{R}$ with exactly n non-ordinary points and the set of all words of length $n+1$ on three symbols* ***A,B,O*** *such that* ***OO*** *is forbidden.*

Theorem 5.3.3. *The number of all increasing continuous bijections (upto order conjugacy) on $\mathbb{R}$ with exactly n non-ordinary points is equal to a_n where $a_n = C_1(1+\sqrt{3})^n + C_2(1-\sqrt{3})^n$, $C_1 = \frac{(5+3\sqrt{3})}{2\sqrt{3}}$ and $C_2 = \frac{(3\sqrt{3}-5)}{2\sqrt{3}}$.*

Proposition 5.3.4. *Two decreasing bijections f and g are order conjugate (res. topologically conjugate) if and only if $f^2|_{[a,\infty)}$ and $g^2|_{[b,\infty)}$ are order conjugate (res. topologically conjugate) where a and b are the fixed points of f and g respectively.*

Proposition 5.3.5. *Two decreasing bijections f and g are order conjugate (res. topologically conjugate) if and only if $f^2|_{(-\infty,a]}$ and $g^2|_{(-\infty,b]}$ are order conjugate (res. topologically conjugate) where a and b are the fixed points of f and g respectively.*

Proposition 5.3.6. *If f is a decreasing bijection from $\mathbb{R}$ to $\mathbb{R}$ with fixed point a. Then f has $2n+1$ non-ordinary points if and only if $(f \circ f)|_{(a,\infty)} : (a,\infty) \to (a,\infty)$ has n non-ordinary points.*

Theorem 5.3.7. *If s_n denotes the number of decreasing homeomorphisms upto order conjugacy, then*

$$s_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ a_{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

for all n.

5.3.1 Maps with unique non-ordinary point

Proposition 5.3.8. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be such that*

(1) $f(0) = g(0) = 0$.

(2) $f|_{(0,\infty)}, g|_{(0,\infty)} : (0,\infty) \to (0,\infty)$ *are increasing bijections.*

(3) $f|_{(-\infty,0)}, g|_{(-\infty,0)} : (-\infty,0) \to (0,\infty)$ *are decreasing bijections.*

Then f is order conjugate to g if and only if $f|_{(0,\infty)}$ is order conjugate to $g|_{(0,\infty)}$.

Proof. Suppose $h : (0,\infty) \to (0,\infty)$ is an order conjugacy from $f|_{(0,\infty)}$ to $g|_{(0,\infty)}$.

For $x < 0$, define $h(x) = (g|_{(-\infty,0)})^{-1} h f(x)$, and $h(0) = 0$. □

Proposition 5.3.9. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be such that*

(1) $f(0) = g(0) = 0$.

(2) $f|_{(-\infty,0)}, g|_{(-\infty,0)} : (-\infty,0) \to (-\infty,0)$ *are increasing bijections.*

(3) $f|_{(0,\infty)}, g|_{(0,\infty)} : (0,\infty) \to (-\infty,0)$ *are decreasing bijections.*

Then f is order conjugate to g if and only if $f|_{(-\infty,0)}$ is order conjugate to $g|_{(-\infty,0)}$.

Proof. Suppose $h : (-\infty,0) \to (-\infty,0)$ is an order conjugacy from $f|_{(-\infty,0)}$ to $g|_{(-\infty,0)}$.

For $x > 0$, define $h(x) = (g|_{(0,\infty)})^{-1} h f(x)$, and $h(0) = 0$. □

Theorem 5.3.10. *[53] There are exactly 26 maps on $\mathbb{R}$ with a unique non-ordinary point, upto order conjugacy.*

Proof. By Corollary 5.3.14, Propositions 5.3.8, 5.3.9, 5.3.16 (1), and 5.3.16 (2) theorem follows. □

Definition 5.3.11. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map. Let $I \subset \mathbb{R}$ be an interval such that $f^k(I) = I$, and $f^m(I) \neq I$ for all $1 \leq m < k$. Then we say that $I \to f(I) \to f^2(I) \to ... \to f^{k-1}(I) \to I$ is a k-cycle through I. Cycles through I, J are said to be distinct if $f^m(I) \neq f^n(J)$ for all $m, n \in \mathbb{N}$.

We can represent each member of this class as a graph as follows.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map with a unique non-ordinary point. Now, let I_1, I_2 be non-singleton equivalence classes such that $I_1 < I_2$. Define a labeled digraph (this is different from the labeled digraphs correspond to sofic shifts) (G, V_f) with vertex set $V_f = \{I_1, I_2\}$, an edge from I_j to I_k if $f(I_j) = I_k$, and a symbol I (respectively D) on this edge whenever f is increasing (respectively decreasing) on I_j. Label the symbols $\mathbf{A}, \mathbf{B}, \mathbf{O}$ on the least vertex of each k-cycle depends on the graph of $f^k,\ k = 1, 2$ which is above the diagonal or below the diagonal or on the diagonal respectively.

Note: If f is a decreasing homeomorphism then we can consider the labeled digraph of f^2 instead of the labeled digraph f (see Proposition 5.3.5).

5.3.2 Maps with exactly two non-ordinary points

We denotes the symbol $\biguplus$ for disjoint unions.

Proposition 5.3.12. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous maps having finitely many non-ordinary points such that $N(f) = N(g)$ and let $N(f)^c = \biguplus I_n$. Let $f(\bar{I}_n) = \bar{I}_m$ and $g(\bar{I}_n) = \bar{I}_m$. Suppose there exist increasing bijections $h_n : \bar{I}_n \to \bar{I}_n$ and $h_m : \bar{I}_m \to \bar{I}_m$ such that $g|_{\bar{I}_n} \circ h_n = h_m \circ f|_{\bar{I}_n}$ for all $m, n \in \mathbb{N}$ then $f \sim g$.*

Proof. Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = h_n(x)$$

for all $x \in \bar{I}_n$ and $n \in \mathbb{N}$. Then h is an increasing bijection such that $h \circ f = g \circ h$. Hence the proposition. □

Note that if a continuous bijection on $\mathbb{R}$ has finitely many non-ordinary points then we can assume that these points arbitrarily. Because, let $a_1 < a_2 < \ldots < a_n$ be the non-ordinary points of a continuous map $f : \mathbb{R} \to \mathbb{R}$. Let $b_1 < b_2 < \ldots < b_n$ be arbitrary points in $\mathbb{R}$. Let $h : \mathbb{R} \to \mathbb{R}$ be an increasing continuous bijection such that $h([a_i, a_{i+1}]) = [b_i, b_{i+1}]$ for $i = 1, 2, ..., n-1$. Then define $g = h \circ f \circ h^{-1}$. We can easily verify $b_1, b_2, ..., b_n$ are the only non-ordinary points of g since h is an order conjugacy from f to g.

Now we give a theorem which help us to classify the class of continuous maps having finitely many non-ordinary points.

Theorem 5.3.13. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous maps having finitely many non-ordinary points such that $N(f) = N(g)$. Let $N(f)^c = \biguplus_n I_n$.*

(1) If f and g have same type of monotonicity in the closure of each non-singleton equivalence class and contains exactly n distinct cycles of length k_i through $I_{j_{k_i}}$ for some j_{k_i} for $i = 1, 2, ..., n$ and for some j_{k_i}. Then $f \sim g$ whenever graph $(g^{k_i}|_{\bar{I}_{j_{k_i}}})$ and graph $(g^{k_i}|_{\bar{I}_{j_{k_i}}})$ are same side of the diagonal.

(2) If f and g have same type of monotonicity in the closure of each non-singleton equivalence class and does not contain any cycle then $f \sim g$.

Proof. For simplicity we consider the case when f and g have exactly one cycle through its equivalence classes.

Let f and g have same type of monotonicity in the closure of each non-singleton equivalence class and contains exactly one cycle through its equivalence classes (say of length k and through I_j for some j).

Claim: $f \sim g$ whenever graph $(f^k|_{\bar{I}_j})$ and graph $(g^k|_{\bar{I}_j})$ are same side of the diagonal.

Let $I_j = J_1 \to J_2 = f(J_1) \to ... \to J_k = f^{k-1}(J_1) \to J_1$ be the k-cycle. Given that graph $(f^k|\bar{I}_j)$ and graph $(g^k|\bar{I}_j)$ are same side of the diagonal. Then there exists an increasing bijection $h : \bar{J}_1 \to \bar{J}_1$ such that $f^k \circ h = h \circ g^k$ ......(1).

Choose $h_1 = h$. Find $h_i : \bar{J}_i \to \bar{J}_i$ for $i = 2, 3, ..., k$ and $h'_1 : \bar{J}_1 \to \bar{J}_1$ such that

$f|_{\bar{J}_i} \circ h_i = h_{i+1} \circ g|_{\bar{J}_i}$ and $f \circ h_k = h'_1 \circ g$. Recursively, we can prove that $h'_1 = f^k \circ h_1 \circ g^{-k}$. This implies $h'_1 = h$ by equation (1). Define $s_m(x) = x$ for all $x \in \bar{I}_m$ whenever f, g are constant on I_m. In the closure of all other non-singleton equivalence class choose h arbitrary and define h' (or vice versa) such that $f \circ h = h' \circ g$ on it. This gives a well defined $h : \mathbb{R} \to \mathbb{R}$ such that $h(x) = s_n(x)$ where $s_n : I_n \to I_n$ is a continuous increasing bijection obtained as above such that $f \circ s_n = s_m \circ g$. This implies $f \circ h = h \circ g$. Similarly we can prove the general case and (2). □

Corollary 5.3.14. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be two increasing bijections having finitely many fixed points such that $Fix(f) = Fix(g)$ and let $Fix(f)^c = \biguplus I_n$. If $f|_{\bar{I}_n} \sim g|_{\bar{I}_n}$ for every n then $f \sim g$.*

Remark 5.3.15. Note that the complement of $Fix(f)$ is a countable union of open intervals (including rays) whose end points are fixed points. Since f is increasing and the end points are fixed, no point in a component interval can be mapped to a point in any other component interval by f.

Proposition 5.3.16. *(1) Let $f : (-\infty, 0] \to (-\infty, 0]$ be an increasing bijection (It follows that $f(0) = 0$).*

a. If $f(x) > x$ for all $x \in (-\infty, 0)$ then f is order conjugate to $\frac{x}{2}$.

b. If $f(x) < x$ for all $x \in (-\infty, 0)$ then f is order conjugate to $2x$.

(2) Let $f : [0, \infty) \to [0, \infty)$ be an increasing bijection (It follows that $f(0) = 0$).

a. If $f(x) > x$ for all $x \in (0, \infty)$ then f is order conjugate to $2x$.

b. If $f(x) < x$ for all $x \in (0, \infty)$ then f is order conjugate to $\frac{x}{2}$.

(3) Let $f : [1, \infty) \to [1, \infty)$ be an increasing bijection (It follows that $f(1) = 1$).

a. If $f(x) > x$ for all $x \in (1, \infty)$ then f is order conjugate to $2x - 1$.

b. If $f(x) < x$ for all $x \in (1, \infty)$ then f is order conjugate to $\frac{x+1}{2}$.

Proof. (1). **Proof of (a):**

Let $f : (-\infty, 0] \to (-\infty, 0]$ be an increasing bijection satisfying $f(x) > x \forall x < 0$. (It follows that $f(0) = 0$). Note that for any such map $\biguplus_{n\in\mathbb{Z}}[f^n(x), f^{n+1}(x)) = (-\infty, 0)$ for all point $x \in (-\infty, 0)$.

Then f is topologically conjugate to the map $x/2$. We construct a topological conjugacy $h : (-\infty, 0] \to (-\infty, 0]$ as follows: Take any point other than 0, say -1 in the domain. We take an arbitrary increasing homeomorphism h from $[-1, f(-1))$ *to* $[-1, -1/2)$. Then as noted above, $\biguplus_{n\in\mathbb{Z}}[f^n(-1), f^{n+1}(-1)) = (-\infty, 0)$. That is, for every $x \in (-\infty, 0)$, there exists a unique $n_0 \in \mathbb{Z}$ such that $f^{n_0}(x) \in [-1, f(-1))$. We define $h(x) = 2^{n_0}h(f^{n_0}(x))$. This is well defined. It is an increasing homeomorphism from $(-\infty, 0)$ to $(-\infty, 0)$. This h commutes with f. This h is a conjugacy from f to the map $x/2$.

Similarly, we can prove (1).b, (2) and (3). □

Next we define order conjugacy for non-self maps on the subset of $\mathbb{R}$.

Definition 5.3.17. Let $A, B \subset \mathbb{R}$, $A \neq B$, and $f, g : A \to B$ be continuous maps. We say that f is order conjugate to g if there exist increasing bijections $h_f : A \to A$ and $h_g : B \to B$ such that $f \circ h_f = h_g \circ g$.

Proposition 5.3.18. *(1) Let $f : (-\infty, 0] \to [0, 1)$ be a decreasing bijection (It follows that $f(0) = 0$). Then f is order conjugate to $\frac{x}{x-1}$.*

(2) Let $f : [1, \infty) \to [0, 1)$ be an increasing bijection (It follows that $f(1) = 0$). Then f is order conjugate to $\frac{x-1}{x}$.

(3) Let $f : [1, \infty) \to (0, 1]$ be a decreasing bijection (It follows that $f(1) = 1$). Then f is order conjugate to $\frac{1}{x}$.

(4) Let $f : [1, \infty) \to (-\infty, 0]$ be a decreasing bijection (It follows that $f(1) = 0$).

Then f is order conjugate to $1-x$.

(5) Let $f:(-\infty,0]\to(0,1]$ be an increasing bijection (It follows that $f(0)=1$). Then f is order conjugate to $\frac{1}{1-x}$.

Proof. This Proposition easily follows from the following fact.

Let $A,B\subset\mathbb{R}$ be intervals such that either both $f,g:A\to B$ are decreasing bijections or both are increasing bijections. Then there exist increasing bijections $h_f:A\to A$ and $h_g:B\to B$ such that $f\circ h_f=h_g\circ g$. Because take an arbitrary h_f and define $h_g=f\circ h_f\circ g^{-1}$. □

Proposition 5.3.19. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous map with exactly two non-ordinary points $0,1$ such that*

1. $f(0)=0$.

2. Either $f|_{[1,\infty)}:[1,\infty)\to[1,\infty)$ and is an increasing bijection or $f|_{[1,\infty)}:[1,\infty)\to(0,1]$ and is a decreasing bijection.

3. Either $f|_{(-\infty,0]}:(-\infty,0]\to(-\infty,0]$ and is an increasing bijection or $f|_{(-\infty,0]}:(-\infty,0]\to[0,1)$ and is a decreasing bijection.

Then there are only 48 such maps upto order conjugacy.

Proof. Proof follows from Theorem 5.3.13, Remark 5.2.22, and Propositions 5.2.21, 5.3.16 and 5.3.18 (see figure 5.1 (a)). □

Remark 5.3.20. There are sixty eight maps (upto order conjugacy) having exactly two non-ordinary points $0,1$ such that both are fixed.

Proof. This remark follows from Corollary 5.2.20 together with the results used for the proof of the Proposition 5.3.19 (see figure 5.1 (a)). □

Proposition 5.3.21. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous map with exacly two non-ordinary points $0,1$ such that*

1. $f(0) = g(0) = 0$.

2. Either $f|_{(-\infty,0]} : (-\infty, 0] \to (0, 1]$ *and is a decreasing bijection or* $f|_{(-\infty,0]} : (-\infty, 0] \to (-\infty, 0]$ *and is an increasing bijection.*

3. Either $f|_{[1,\infty)} : [1, \infty) \to (-\infty, 0]$ *and is a decreasing bijection or* $f|_{[1,\infty)} : [1, \infty) \to [0, 1)$ *and is an increasing bijection.*

Then there are only 8 *such maps upto order conjugacy.*

Proof. Proof follows from Theorem 5.3.13, Remark 5.2.22 and Propositions 5.3.16, 5.3.18 and 5.2.21 (see figure 5.1 (b)). □

Remark 5.3.22. There are eleven continuous maps on $\mathbb{R}$ (upto order conjugacy) having exactly two non-ordinary points $0, 1$ such that 0 is a fixed point and the image of 1 is 0.

Proof. This remark follows from Corollary 5.2.20 together with results used for the proof of the Proposition 5.3.21 (see figure 5.1 (b)). □

Proposition 5.3.23. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous map with exactly two non-ordinary points* $0, 1$ *such that*

1. $f(1) = 1$.

2. Either $f|_{[1,\infty)} : [1, \infty) \to [1, \infty)$ *and is an increasing bijection or* $f|_{[1,\infty)} : [1, \infty) \to (0, 1]$ *and is a decreasing bijection.*

3. Either $f|_{(-\infty,0]} : (-\infty, 0] \to (-\infty, 0]$ *and is a decreasing bijection or* $f|_{(-\infty,0]} : (-\infty, 0] \to (0, 1]$ *and is an increasing bijection.*

Then there are only 8 *such maps upto order conjugacy.*

Proof. Proof follows from Theorem 5.3.13, Remark 5.2.22 and Propositions 5.3.16, 5.3.18 and 5.2.21 (see figure 5.1(c)). □

Remark 5.3.24. There are eleven maps on $\mathbb{R}$ (upto order conjugacy) having exactly two non-ordinary points $0, 1$ such that 1 is a fixed point and the image of 0 is 1.

Proof. This remark follows from Corollary 5.2.20 together with the results used for the proof of the Proposition 5.3.23 (see figure 5.1(c)). □

Theorem 5.3.25. *(Main Theorem 1) There are exactly* 90 *continuous maps on* $\mathbb{R}$ *with exacly two non-ordinary points, upto order conjugacy.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map having exactly two non-ordinary points a and b such that $a < b$. Then $\{a, b\}$ is invariant under f by Proposition 5.2.18. Hence at least one of these two points is a fixed point; the other is either a fixed point or goes to a fixed point.

Case 1. Both a and b are fixed points.

Without loss of generality we can assume that $a = 0$ and $b = 1$. This is because, let $h : \mathbb{R} \to \mathbb{R}$ be an increasing bijection such that $h([a, b]) = [0, 1]$. Then consider $g = hfh^{-1}$. Then $g(0) = 0$ and $g(1) = 1$. From Remark 5.3.20, it follows that that there are only 68 continuous maps of this type upto order conjugacy.

Case 2. $f(a) = a = f(b)$.

Without loss of generality we can assume that $a = 0,\ b = 1$ (a similar proof as in Case 1 will work). From Remark 5.3.22, it follows that there are only 11 continuous maps of this type upto order conjugacy.

Case 3. $f(b) = b = f(a)$.

Without loss of generality we can assume that $a = 0$ and $b = 1$ (a similar proof as in Case 1 will work). From Remark 5.3.24, it follows that there are only 9 continuous maps of this type upto order conjugacy.

Hence the proof. □

Remark 5.3.26. From Corollary 5.2.20, Remark 5.2.22, Theorem 5.3.13 and Propositions 5.3.16, 5.3.18 and 5.2.21, it follows that there are 16 somewhere constant continuous maps upto order conjugacy such that interval of constancy is bounded, 31 somewhere constant continuous maps upto order conjugacy such that interval of constancy is unbounded, 18 nowhere constant continuous maps upto order conjugacy with unique critical point (among them 9 maps having unique critical point as a local maximum, remaining 9 maps having unique critical point as a local minimum), 3 continuous maps upto order conjugacy with exactly two critical points, and 22 continuous maps upto order conjugacy with no critical points. Hence there are exactly 90 continuous maps (upto order conjugacy) on $\mathbb{R}$ with having exactly two non-ordinary points. This gives another way of count for Theorem 5.3.25.

We can represent each member of this class as a digraph as follows.

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous map having exactly two non-ordinary points. Let I_1, I_2, I_3 be the non-singleton equivalence classes such that $I_1 < I_2 < I_3$, and define a graph (G, V_f) with vertex set $V_f = \{I_1, I_2, I_3\}$, and an edge from I_j to I_k if $f(I_j) = I_k$, and a symbol I (respectively D) on this edge whenever f is increasing (respectively decreasing) on I_j for $j = 1, 2, 3$. If there is a k-cycle, label one of the symbols $\mathbf{A}, \mathbf{B}, \mathbf{O}$ in the least vertex (say J_1) of the cycle depends on the $graph(f^k|_{J_1})$ is above the diagonal or below the diagonal or on the diagonal respectively.

Note: If f is a decreasing homeomorphism then we can consider the labeled digraph of f^2 instead of the labeled digraph of f because of Proposition 5.3.5.

5.3.3 Maps with finitely many non-ordinary points

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map having finitely many non-ordinary point. Then the set of all non-ordinary points is invariant by Proposition 5.2.18. By Corollary 5.2.20, the end points of the maximal interval around every point on which f is constant are non-ordinary. By Proposition 5.2.21, if $x \in \mathbb{R}$ is both critical and ordinary then f is locally constant at x, and if x is ordinary then so is $f(x)$ whenever f is not locally constant in a neighbourhood of x. Also recall, $lim_{x\to-\infty}f(x)$ and $lim_{x\to+\infty}f(x)$ are non-ordinary whenever f has only finitely many non-ordinary points. Hence the following informations will help us to characterizes the set of all continuous maps from $\mathbb{R}$ to $\mathbb{R}$ having finitely many non-ordinary points (see Theorem 5.3.30).

(1) graph(f) is above the diagonal or below the diagonal or on the diagonal on each equivalence class (a, b) if $f|_{(a,b)}$ is increasing, and $f(a) = a$, $f(b) = b$. (Related to this we will assign the symbols $\mathbf{A}, \mathbf{B}, \mathbf{O}$ on the vertex (a, b) of the labeled digraph of $(\mathbb{R}, f)$ depends on the graph($f|_{(a,b)}$) is above the diagonal or below the diagonal or on the diagonal respectively).

(2) Increasing or decreasing or constant on each equivalence class. (Related to this we will assign symbols I, D on the edge from the equivalence class to itself of the labeled digraph of $(\mathbb{R}, f)$ depends on the graph (f) on corresponding equivalence class is increasing or decreasing respectively).

(3) If $f^k(I) = I$, $f^m(I) \neq I$, $m < k$ then consider $f^k|_J$ and ask (1) for least $J \in \{I, f(I), ..., f^{k-1}(I)\}$ (Related to this we will assign the symbols A, B, O on the vertex J of the labeled digraph of $(\mathbb{R}, f)$ depends on the graph $(f^k|_J)$ is above the diagonal or below the diagonal or on the diagonal respectively).

Now we introduce some labeled digraph (this is different from the labeled digraph introduced for sofic shifts in Chapter 3) for each $(\mathbb{R}, f)$ as follows:

Let $I_1, I_2, ..., I_n$ be the non-singleton equivalence classes of $(\mathbb{R}, f)$ such that $I_1 < I_2 < ... < I_n$. Define a graph (G, V_f) with vertex set $V_f = \{I_1, I_2, ..., I_n\}$, and define an edge from I_j to I_k if $f(I_j) = I_k$. But these graph would not give full information of the dynamical system $(\mathbb{R}, f)$. To achieve this, we give more labels on each edge on the graph of the map, and on the least vertex of each cycle(see (1), (2) and (3) for details). Observe that, if f, g are order conjugate then the associated labeled digraph should be isomorphic.

Note: If f is a decreasing homeomorphism having odd number of non-ordinary points then we can consider the graph of f^2 instead of f because of Proposition 5.3.5.

Image of each non-trivial equivalence Class

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map having $n-1$ non-ordinary points, and $I_1 < I_2 < ... < I_n$ be the n non-singleton equivalence of f. Then $f(I_1) = I_m$ for some $m \in \{1, 2, ..., n\}$ or a constant; and $f(I_2) = I_{m+1}$ or a constant if $m = 1$, and I_m or I_{m+1} or a constant if $m > 1$. In general for $3 \leq k \leq n-2$, $f(I_k) = I_l$ or I_{l-1} or I_{l+1} or a constant, l depends on m; and $f(I_n) = I_j$ or a constant; and $f(I_{n-1}) = I_{j-1}$ or a constant if $j = n$, and I_{j-1} or I_j or a constant if $j < n$. Note that j depends on m.

Note: This information gives the possible choices of edge sets for the assigned labeled digraph.

Definition 5.3.27. A graph isomorphism between two graphs G and H can be defined as a bijection $f : V_G \to V_H$ that such that a pair of vertices u, v is adjacent in V_G if and only if the image pair $f(u), f(v)$ is adjacent in V_H. In full generality, a graph isomorphism $f : G \to H$ is a pair of bijections $f_V : V_G \to V_H$ and $f_E : E_G \to E_H$ such that for every edge $e \in E_G$, the endpoints of e are mapped onto the endpoints of $f_E(e)$. A digraph isomorphism is an isomorphism of the underlying graphs such that the edge correspondence preserves all edge directions. A labeled digraph isomorphism is an isomorphism of the underlying digraphs such that the correspondence preserves labeling. Two graphs are isomorphic if there is an isomorphism from one to the other, or informally, if their mathematical structures are identical.

Definition 5.3.28. (see [19]) Let $(S, \leq_S)$, $(T, \leq_T)$ be two partially ordered sets. An *order isomorphism* from $(S, \leq_S)$ to $(T, \leq_T)$ is a surjective map $h : S \to T$ such that for all u and v in S, $h(u) \leq_T h(v)$ if and only if $u \leq_S v$. In this case, the posets S and T are said to be order isomorphic. We can prove that, all surjective order isomorphisms are bijective.

Definition 5.3.29. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map. A subset of $\mathbb{R}$ is said to be *dynamically independent* set if any two points of the set having disjoint orbits. A subset of $\mathbb{R}$ is said to be *maximal dynamically independent* if it is dynamically independent and no other super set is dynamically independent.

Theorem 5.3.30. *(Main Theorem 2)*

Let $\mathcal{C}$ be the class of all continuous self maps of $\mathbb{R}$, having finitely many non-ordinary points. Then

(1) *for every member of $\mathcal{C}$, there exists a maximal dynamically independent set that is a finite union of intervals.*

(2) *two members of $\mathcal{C}$ are order conjugate if and only if they have order isomorphic maximal dynamically independent set as in (1), and with isomorphic labeled digraphs.*

(3) *every order isomorphism between such maximal dynamically independent set as in* (1) *extends uniquely to an order conjugacy.*

Proof. (1) Let $z_0 = x_1 < x_2 < ... < x_n$ be the n non-ordinary points. Then $I_1 = (-\infty, x_1), I_{n+1} = (x_n, \infty)$ and $I_i = (x_i, x_{i+1})$, $i = 1, ..., n-1$ be the $n+1$ non-singleton equivalence classes by Lemma 5.2.13. Choose $y_i \in I_i$ whenever f is not a constant on I_i for $i = 1, 2, ..., n$; and let J be the set of all i such that y_i has been chosen.

Let z_1 be the least x_i not in $O(z_0)$. Then choose z_{i+1} be the least x_i not in $O(z_i)$ inductively. Let Z be the the set of all elements such that $z_{i+1} \notin O(z_i)$ (It may be empty but always finite).

Define $y_{i+1} = f^{k_i}(y_i)$ if k_i is least such that $f^{k_i}(I_i) = I_i$ for $i \in \mathrm{J}$.

Then consider $\mathcal{M} = \bigcup_{i\in \mathrm{J}}(y_i, y_{i+1}) \cup Z$. Note that this $\mathcal{M}$ is always non-empty. Observe that, $\mathcal{M}$ is a maximal dynamically independent set.

(2) First part is easy because having maximal dynamically independent set is invariant under order conjugacy. ie., if $\mathcal{M}_1$ is a maximal dynamically independent set of f. Then $h(\mathcal{M}_1)$ is a maximal dynamically independent set of g whenever h is an order conjugacy from f to g.

Conversely, let f, g have order isomorphic maximal dynamically independent set as in (1) (say $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively), and with isomorphic labeled digraphs. Consider all non-empty intersection of each equivalence classes of f with $\mathcal{M}_1$ and g with $\mathcal{M}_2$. Observe that there is a one to one correspondence between these intersections. Because of maximal dynamical independency, we can extend restriction of the order isomorphism on these sets to a homeomorphism on its corresponding equivalence class. By a similar proof as in Theorem 5.3.13 we can extend it to a unique conjugacy from f to g since f and g have isomorphic labeled digraphs and order isomorphic maximal dynamically independent sets.

(3) Easily follows from (2). □

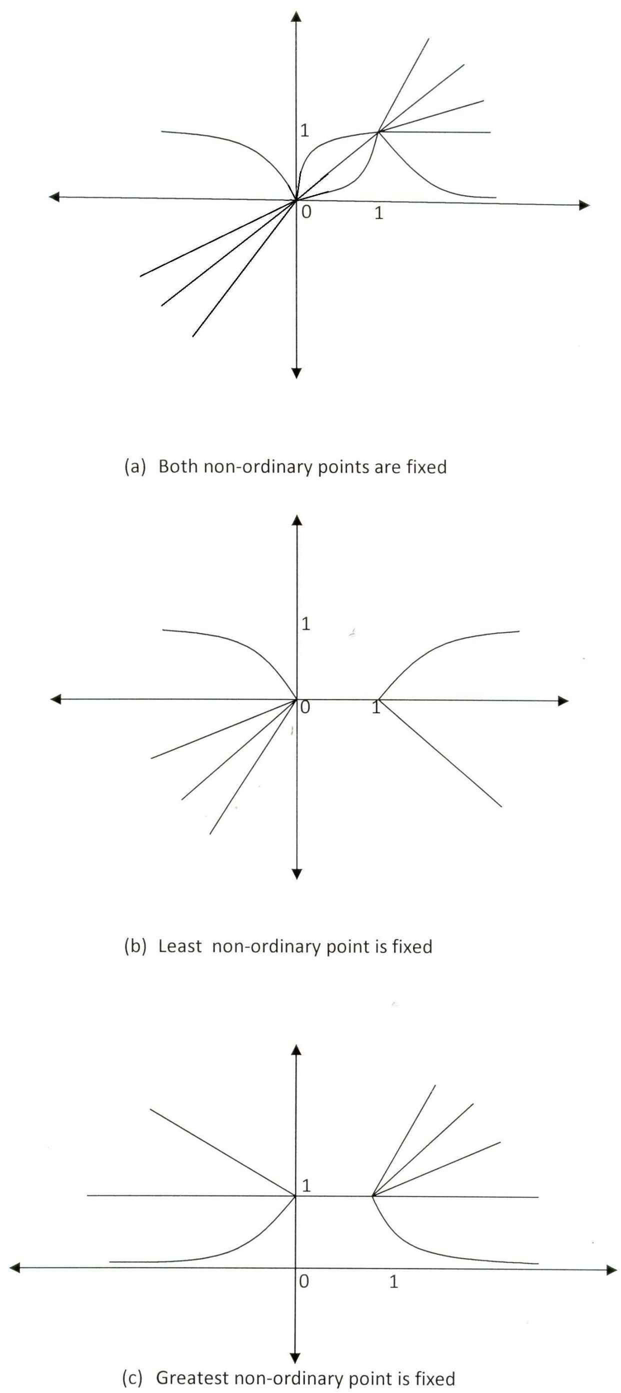

Figure 5.1: Maps with exactly two non-ordinary points

Appendix A

Periods and Orbits: Some more results

In this appendix, we discuss the period set of an endomorphism on an abelian torsion free group and different types of orbits of linear operators on $\mathbb{R}^n$. In particular, we characterize the sets of periods an endomorphism on a torsion free abelian group.

A.1 Set of periods of an endomorphism on an abelian group

An abelian group G is called a torsion group if every element of G has finite order and is called *torsion-free* if every element of G except the identity is of infinite order. A *homomorphism* of a group to itself is called an *endomorphism*; an invertible endomorphism is called an *automorphism*.

Recall the following definitions:

For $m, n \in \mathbb{N}$, $m \vee n$ denotes the lcm of m and n; for $A, B \subset \mathbb{N}$, $A \bigvee B = \{m \vee n :$

$m \in A, n \in B\}$; and a triple $(k_1, k_2, k_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ satisfies property 'P' if each number divides the lcm of the other two.

A.1.1 Torsion free abelian group

Theorem A.1.1. *Let T_i be endomorphisms on abelian groups G_i for $i = 1, 2$. Let $T = T_1 \times T_2 : G_1 \times G_2 \to G_1 \times G_2$ be defined by $T(x, y) = (T_1 x, T_2 y)$. Then $Per(T)$ is the smallest subset of $\mathbb{N}$ closed under lcm and containing both $Per(T_1)$ and $Per(T_2)$.*

Proof. Let $x \in G_1$ such that $T_1^n(x) = x$ and $T_1^m(x) \neq x$ for all $m < n$. Then $T^n(x, 0) = (x, 0)$ and $T^m(x, 0) \neq (x, 0)$ for all $m < n$. Then $Per(T_1) \subset Per(T)$. Similarly $Per(T_2) \subset Per(T)$.

Now, let $(p, q) \in G_1 \times G_2$ be T-periodic. Then $T^n(p, q) = (T_1^n(p), T_2^n(q)) = (p, q)$ for some $n \in \mathbb{N}$. It follows that p is T_1-periodic and q is T_2-periodic. Let m_1 be T_1-period and m_2 be the T_2-period. Let $l = m_1 \vee m_2$. Then $T^l(p, q) = (T_1^l(p), T_2^l(q)) = (p, q)$. It is noted that l is the T-period of (p, q). Then every element of $Per(T)$ is the lcm of some element of $Per(T_1)$ and some element of $Per(T_2)$; and the lcm of every such pair of element is in $Per(T)$. Hence the proof. □

Corollary A.1.2. *If we assume that $Per(T_1)$, $Per(T_2)$ are closed under lcm in Theorem A.1.1 then $Per(T) = Per(T_1) \bigvee Per(T_2)$.*

Proof. Let $s_1 \in Per(T_1)$ and $s_2 \in Per(T_2)$. Then $s_1 \vee s_2 \in Per(T)$ since $s_1 \vee s_2 = (m_1 \vee m_2) \vee (n_1 \vee n_2) = (m_1 \vee n_1) \vee (m_2 \vee n_2)$ for some T_1-periods m_1, n_1 and T_2-periods m_2, n_2. □

Remark A.1.3. Let T_i be endomorphisms on abelian groups G_i for $i = 1, 2, ..., n$. Define $T = T_1 \times T_2 \times ... \times T_n : G_1 \times G_2 \times .... \times G_n \to G_1 \times G_2 \times .... \times G_n$ such that $T((x_1, x_2, ..., x_n)) = (T(x_1), T(x_2), ..., T(x_n))$. Then Per$(T)$ is the smallest subset of $\mathbb{N}$

closed under lcm and containing $Per(T_i)$ for all $1 \leq i \leq n$. If we assume that Per(T_i)s are closed under lcm then Per(T) = Per$(T_1) \vee$ Per$(T_2) \vee ... \vee$ Per(T_n).

Proof. A proof similar to Theorem A.1.1 and Corollary A.1.2 will work. □

The sufficient part of the proof of the following theorem is almost similar to the proof of Theorem 2.2.1. But for self containment we give the proof.

Theorem A.1.4. *The following are equivalent for a subset A of $\mathbb{N}$.*

(1) $1 \in A$ and A is closed under lcm.

(2) There is an abelian torsion free group G and an endomorphism T of G such that $Per(T) = A$.

Proof. (1) $\implies$ (2)

Consider $(\mathbb{C}, +)$, which is an abelian torsion free group. Let $n_0 \in \mathbb{N}$. Define T_{n_0} : $\mathbb{C} \to \mathbb{C}$ by $T_{n_0}(z) = ze^{\frac{2\pi i}{n_0}}$. Then $T_{n_0}(z)$ is an endomorphism and $Per(T_{n_0}) = \{1, n_0\}$.

Let $1 \in A \subset \mathbb{N}$, and A is closed under lcm.

Suppose that A is finite, say $\{a_1, a_2, ..., a_n\}$.

Let $G_A = C_1 \times C_2 \times ... \times C_n$, $C_i = \mathbb{C}$ for all $1 \leq i \leq n$. Then G_A is an abelian torsion free group G. Define $T_A(z_1, z_2, ..., z_n) = (T_{a_1} z_1, ..., T_{a_n} z_n)$. Then T_A is an automorphism of G_A and $Per(T_A)$ is the smallest set containing $\{1, a_1, ..., a_n\} = A$ and closed under *lcm*. Hence $Per(T_A) = A$ since A is closed under lcm.

Next suppose that A is infinite.

Let $G_A = \{x \in \prod_{a \in A \setminus \{1\}} C_a : x = (x_a)_{a \in A \setminus \{1\}}, x_a = 0$ for all but finitely many $a\}$, $C_a = \mathbb{C}$ for all $a \in A$. Then G_A is an abelian torsion free group. Define $T_A : G_A \to G_A$ is such that $T_A(x) = (y_a)_{a \in A \setminus \{1\}}$ where $y_a = T_a(x_a)$ for all $a \in A \setminus \{1\}$. ie., $T_A = \prod_{a \in A \setminus \{1\}} T_a$. Then $Per(T_A)$ is the smallest set closed under lcm containing $\{1, a\}$ for all $a \in A \setminus \{1\}$; ie., containing A. Hence $Per(T_A) = A$ since A is closed under lcm.

(2) $\implies$ (1)

Note that $T(0) = 0$. Hence $1 \in Per(T)$. Next, let $x \in G$ has T-period m and $y \in G$ has T-period n. We shall find $k \in \mathbb{N}$ such that $x + ky$ has T-period $m \vee n$.

Step 1: The triple (T-period of x, T-period of y, T-period of $x+y$) satisfies property 'P'.

If $T^p(a) = a$ and $T^q(b) = b$, then $T^{p\vee q}(a+b) = T^{p\vee q}(a) + T^{p\vee q}(b) = a + b$. Next, let $c, d \in G$ be any two periodic points. Taking $a = c + d$ and $b = -d$, we obtain that the T-period of c divides the lcm of the T-period of $c+d$ and the T-period of d. Hence the result follows.

Step 2: If n is a non-zero integer, then a and na have the same T-period.

This is because, if $T^p(a) = a$, then $T^p(na) = nT^p(a) = na$. Conversely, if $T^q(na) = na$ then $n(T^q(a) - a) = 0$, but because of G is torsion free, we have $T^q(a) = a$.

Step 3: Let $k_1 = T$-period of $x+y$ and $k_2 = T$-period of $x+2y$. Then $k_1 \vee k_2 = m \vee n$.

Write $x + 2y = x + y + y$. We have that the triple (k_1, k_2, n) satisfies 'P' by Step 1 and Step 2. Therefore n divides $k_1 \vee k_2$. So, if a prime power p^r divides n, then p^r has to divide either k_1 or k_2. Now suppose that another prime power q^s divides m but not n. Then, because the triple (m, n, k_1) satisfies 'P', we have to have that q^s divides k_1. Then any prime power that divides m or n, should divide k_1 or k_2. Therefore $m \vee n$ divides $k_1 \vee k_2$. But, we already have k_1 divides $m \vee n$ and k_2 divides $m \vee n$. We conclude that $m \vee n = k_1 \vee k_2$.

Step 4: If $m, n \in \text{Per}(T)$ then $m \vee n \in \text{Per}(T)$.

Let k_t be the T-period of $x + ty$, as t varies over $\mathbb{Z}$. Then we have, $m \vee n = k_{t_1} \vee k_{t_2}$ for all $t_1 \neq t_2$ in $\mathbb{Z}$ by Step 3. This implies, every prime power divisor of m or n should divides k_t for all $t \in \mathbb{Z}$ except at most one.

It follows that, barring finitely many elements of $\mathbb{Z}$, for all other $t \in \mathbb{Z}$, we have

$m \vee n$ divides k_t. Thus there are infinitely many elements in G whose T-period is $m \vee n$. This is more than we claimed. □

A topological group is a topological space G with a group structure such that group multiplication $(g, h) \to gh$, and the inverse $g \to g^{-1}$ are continuous maps. Many important examples of dynamical systems arise as continuous endomorphisms of topological groups (see [18]).

Now we have:

Remark A.1.5. The following are equivalent for a subset A of $\mathbb{N}$.

(1) $1 \in A$ and A is closed under lcm.

(2) There is a torsion free abelian group G and a continuous endomorphism T of G such that $Per(T) = A$.

A.1.2 Torsion abelian group

Now we ask for a neat description of the sets of periods of an endomorphism on abelian torsion group. Unfortunately we do not have a complete answer. But we present some results in this context.

Let S be a set and let $\phi : S \to S$ be a bijection. Let G_S be the set of all functions $f : S \to \{0, 1\}$ such that $f^{-1}(1)$ is finite. Then G_S is a group under pointwise addition modulo 1. Any element is of order 2 since $f + f = \mathbf{0}$, $\mathbf{0}$ denote the zero function on S. Define $T_\phi : G_S \to G_S$ by $T_\phi(f)(s) = f(\phi(s))$ for all $s \in S$ and $f \in G_S$.

<u>T_ϕ is a homomorphism</u>

$T_\phi(f + g)(s) = (f + g)(\phi(s)) = f(\phi(s)) + g(\phi(s)) = (T_\phi(f) + T_\phi(g))(s)$ for all $s \in S$.

<u>T_ϕ is one to one</u>

$T_\phi(f) = 0 \Leftrightarrow f(\phi(s)) = 0\ \forall s \in S$. Which implies $f(t) = 0\ \forall\ t \in S$. Hence f is equal to the zero map $\mathbf{0}$.

T_ϕ is onto

$T_\phi(f \circ \phi^{-1}) = f$.

Therefore T_ϕ is an automorphism.

Proposition A.1.6. *$Per(T_\phi)$ contains the smallest subset of $\mathbb{N}$ containing $Per(\phi) \cup \{1\}$ and closed under lcm.*

Proof. Let s be a periodic point of ϕ with period n. ie., $\phi^n(s) = s$ and $\phi^m(s) \neq s$ for all $m < n$. By induction, we can prove that $T_\phi^m f = f \circ \phi^m$ for every m. Therefore $(T_\phi^n f)(s) = f(s)$ since $\phi^n(s) = s$. Take $f = \chi_{\{s\}}$. Then $T_\phi^n f(t) = \chi_{\{s\}}(t) = 1$ if $t = s$, and $\chi\phi^n(t) = 0$ if $t \neq s$ (since ϕ^n is a bijection). Then $\chi_{\{s\}}$ is T_ϕ-periodic. If $m < n$ then $T_\phi^m(f(s)) = \chi_{\{s\}}(\phi^m(s)) = 0$, and $f(s) = \chi_{\{s\}}(s) = 1$ for $s \in S$. Therefore $T_\phi^m(f) \neq f$. Hence the T_ϕ period of f is n. Therefore $\mathrm{Per}(T_\phi) \supset \mathrm{Per}(\phi)$.

Next, let s_1 be a periodic point of ϕ with period m and s_2 be a periodic point of ϕ with period n. Take $f = \chi_{\{s_1,s_2\}}$. Then $T_\phi^{m\vee n} f(t) = \chi_{\{s_1,s_2\}}(t) = 1$ if $t \in \{s_1, s_2\}$, and $\chi\phi^{m\vee n}(t) = 0$ if $t \notin \{s_1, s_2\}$. If $k < m \vee n$ and $k \notin \{m, n\}$ then $T_\phi^k(f(t)) = 0$ if $t \in \{s_1, s_2\}$, and $f(t) = 1$ if $t \in \{s_1, s_2\}$. If $k = m$ or n then $T_\phi^k f(t) = \chi_{\{s_1,s_2\}}(\phi^k(t)) = 1$ if $t = s_1$, 0 if $t = s_2$, and $f(t) = 1$ if $t \in \{s_1, s_2\}$. Therefore $m \vee n \in \mathrm{Per}(\phi)$. Note that $T_\phi(0) = 0$. So $1 \in Per(T_\phi)$.

Hence the proof. □

Now we ask:

Question: Which subsets of $\mathbb{N}$ arise as sets of periods of an endomorphism on a torsion abelian group?

A.2 Types of orbits

Definition A.2.1. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Two orbits $\{T^n x : n \in \mathbb{N}\}$ and $\{T^n y : n \in \mathbb{N}\}$ are of "Same kind", if there exists an invertible linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$ such that $L(T^n x) = T^n y$ for all $n \in \mathbb{N}_0$.

Definition A.2.2. Let (X, f) be a dynamical system. Let $x \in X$ have finite orbit. Let m be the least non-negative integer such that $f^m(x)$ is repeated in its trajectory. Let $n \in \mathbb{N}$ be the least such that $f^m(x) = f^{m+n}(x)$. Then we say that x is of type (m, n). Simply, we say that a finite orbit is of type n, whenever it of type $(0, n)$.

Let $Fiorb(f) = \{(m, n) \subset \mathbb{N}_0 \times \mathbb{N} :$ there exists $x \in X$ of type $(m, n)\}$. Call, $Fiorb(f)$ as set of all types of finite orbits. Observe that $Per(f) = \{n \in \mathbb{N} :$ there exists a point of type $(0, n)\}$. If f is one-one, then $Fiorb(f) = \{0\} \times Per(f)$ since every eventually periodic point is periodic.

Now we ask:

Describe explicitly, the family of subsets of $\mathbb{N}_0 \times \mathbb{N}$ that occur as $Fiorb(f)$, where f belonging to given family of dynamical system. We consider the following families.

1. Toral automorphisms

Let $T : \mathbb{T}^2 \to \mathbb{T}^2$ be a toral automorphism. Then $Fiorb(T)$ is $\{0\} \times C$ where C is one of the eight sets listed in Theorem 2.1.14.

2. Linear maps on all vector spaces

Identity map: $Fiorb(f) = \{(0, 1)\}$

Zero map : $Fiorb(f) = \{(0, 0), (1, 1)\}$

Reflection : $Fiorb(f) = \{(0, 1), (0, 2)\}$

There are Banach spaces (for example $\mathbb{R}$, and see Theorem A.2.3 for more general case), for which only these sets appear as $Fiorb(f)$ for linear f.

3. On $\mathbb{R}^n$ for $n \in \mathbb{N}$

First we have the following theorem. We borrowed some ideas from Sarahi's thesis ([44]) for a proof.

Theorem A.2.3. *For every sequence (a_n) in $\mathbb{N} \cup \{0, \infty\}$ there exits a continuous map $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $Fiorb(f) = \bigcup_{n=1}^{\infty} \bigcup_{k \in K_{a_n}} \{(k, n)\}$ where $K_{a_n} = \{k \in \mathbb{N}_0 : k \leq a_n\}$.*

Proof. Let (a_n) be a sequence in $\mathbb{N} \cup \{0, \infty\}$. For $n \in \mathbb{N}$, we define $X_n = \{(x, y) \in \mathbb{R}^2 : 1 \leq x \leq n + a_n\}$

$$f_n(x,y) = \begin{cases} (x+1, y+t-t^2) & \text{if } 1 \leq x \leq n-1 \\ (n^2 - n + 1 + x - nx, y + t - t^2) & \text{if } n-1 \leq x \leq n \\ (n^2 - n + 1 + x' - nx', y + t - t^2) & \text{if } n \leq x \leq n+1, x' = x - 2d(n,x) \\ (x-1, y+t-t^2) & \text{if } n+1 \leq x \leq a_n + n - 1 \\ (x-1, x+y+t-t^2-n-a_n+1) & \text{if } a_n + n - 1 \leq x \leq a_n + n \end{cases}$$

where $t = x - [x]$= fractional part of x. Then f_n is a self map on X_n. To see that f_n is continuous, we have only to note that the pieces of definition agree on the boundary lines $x = n-1$, $x = n$, $x = n+1$, and $x = a_n + n - 1$.

Observe that $t - t^2$ is zero for every integer x where $t = x - [x]$. Therefore $f_n(x, y) = (x+1, y)$ for every integer $1 \leq x \leq n-1$ and $f_n(n-1, y) = (n, y)$. Hence every point of the form (m, y) where $1 \leq m \leq n$ is a periodic point of period exactly n. If $n + 1 \leq x \leq n + a_n - 1$ is an integer then $f_n(x, y) = (x-1, y)$.

Next we consider the case when x is not an integer. If x is not an integer then the second coordinate of $f(x, y)$ is strictly greater than y. If the coordinate $f_n^k(x, y)$ is an integer for some k in $\mathbb{N}$. Then all higher values of k, the first coordinate of $f_n^k(x, y)$ remains an integer and therefore (x, y) cannot be an eventually periodic point since x is not an integer. If the first coordinate of $f_n^k(x, y)$ is an not an integer for all k in $\mathbb{N}$

then the y-coordinate of $(x, y), f_n(x, y), f_n^2(x, y), \ldots$ form a strictly increasing sequence of numbers and hence (x, y) is not eventually periodic.

Therefore $Fiorb(f_n) = \bigcup_{k \in K_{a_n}} \{(k, n)\}$ where $K_{a_n} = \{k \in \mathbb{N}_0 : k \leq a_n\}$.

Next we construct a bigger strip Y_n strictly containing X_n, and a continuous extension $\tilde{f}_n$ of f_n (as a self map of Y_n) such that $Fiorb(\tilde{f}_n) = Fiorb(f_n)$.

For this purpose we proceed as follows:

Let $Y_n = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq a_n + 1\}$.

Next we define, for $n \geq 2$,

$$\tilde{f}_n(x, y) = \begin{cases} (2x, y - x + 1) & \text{if } 0 \leq x < 1 \\ f_n(x, y) & \text{if } 1 \leq x \leq n + a_n \\ \tilde{f}'_n(x, y) & \text{if } n + a_n < x \leq n + a_n + 1 \end{cases}$$

where $\tilde{f}'_n(x, y) = \begin{cases} ((n + a_n)x - (n + a_n)^2 + n + a_n - 1, x + y + n - a_n + 1) & \text{if } a_n \neq 0 \\ (nx - n^2 + 1, x + y - n) & \text{if } a_n = 0 \end{cases}$

, and $\tilde{f}_1(x, y) = \begin{cases} (x, y - x + 1) & \text{if } 0 \leq x < 1 \\ f_1(x, y) & \text{if } 1 \leq x \leq 1 + a_1 \\ \tilde{f}'_1(x, y) & \text{if } 1 + a_1 < x \leq 2 + a_1 \end{cases}$

where $\tilde{f}'_1(x, y) = \begin{cases} (x - 1, x + y - a_1 + 1) & \text{if } a_1 \neq 0 \\ (x, x + y - 1) & \text{if } a_1 = 0 \end{cases}$.

To see that $\tilde{f}_n$ is continuous, we have only to note that the pieces of definition agree on the boundary lines $x = 1$ and $x = n$. Next, we note that if $x \in Y_n \setminus X_n$, then the y-coordinate of $f_n(x, y)$ is strictly greater than y. Then none of these points can be eventually periodic points of $\tilde{f}$.

Next define $Z_1 = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq a_1 + 2\}$, $Z_k = \{(x, y) \in \mathbb{R}^2 : \frac{k(k+1)}{2} + a_1 + \ldots + a_{k-1} - 1 \leq x \leq \frac{(k+1)(k+2)}{2} + a_1 + \ldots + a_k - 2\}$ for $k \geq 2$, and

$$g(x,y)=\begin{cases}(x,y+1) & \text{if } x\le 0\\ \tilde{f}_1(x,y) & \text{if } (x,y)\in Z_1\\ \tilde{f}_k(z_x,y)+(\frac{(k+1)(k+2)}{2}+a_1+...+a_k-2,0) & \text{if } (x,y)\in Z_k, k\ge 2\end{cases}$$

where $z_x = x-(\frac{k(k+1)}{2}+a_1+...+a_{k-1}-1)$.

To prove that g is continuous, we note that

(i) On the line $x=0$, the two pieces in the definition of $g(x,y)$ namely $(x,y+1)$ and $\tilde{f}_1(x,y)$ coincide.

(ii) On the common boundary of Z_1 and Z_2, that is, on the line $x=a_1+2$, the pieces in the definition of $g(x,y)$, namely $\tilde{f}_1(x,y)$ and $f_2(x-a_1-2,y)+(n_1+a_1+1,0)$ coincide because both are equal to (n_1+a_1+1,y) and so on. In fact, on each of the boundary lines of each of Z_ks, $g(x,y)$ is nothing but $(x,y+1)$. Therefore $g:\mathbb{R}^2\to\mathbb{R}^2$ is continuous.

Next note that each Z_k invariant under g. That is $g(Z_k)\subset Z_k$. If $\phi_k : Y_k \to Z_k$ is the translation map defined by $\phi_k(x,y)=(x+\frac{k(k+1)}{2}-1,y)$ then $\phi_k\circ\tilde{f}_k\circ\phi_k^{-1}=g|_{Z_k}$. Therefore $Fiorb(g|_{Z_k})=Fiorb(\tilde{f}_k)$, and hence $Fiorb(g)=\bigcup_{n\in\mathbb{N}}Fiorb(\tilde{f}_n)=\bigcup_{n=1}^{\infty}\bigcup_{k\in K_{a_n}}\{(k,n)\}$. □

Next we have:

Theorem A.2.4. *For every linear operator $T:\mathbb{R}^n\to\mathbb{R}^n$, the set $Fiorb(T)$ is finite.*

Proof. We prove this theorem by induction on n, as under.

Any linear operator $T:\mathbb{R}\to\mathbb{R}$ is of the form $Tx=\alpha x$ for some $\alpha\in\mathbb{R}$. For $\alpha=0$, there are two kinds, type 1 (for the point 0) and type $(1,1)$ (eg: {2,0}). For $\alpha=1$, only the type 1. For $\alpha=-1$, there are two kinds, type 1 and type 2. For any other α, the orbits are infinite.

Assume that the result is true for $n = k$, and let $T : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ be a linear operator.

Case 1. T is invertible.

Every finite orbit is a periodic orbit whenever T is invertible. This is because, let (m, n) be the kind of orbit, say $\{y, Ty, ..., T^m y = x, Tx, ..., T^{n-1}x\}$ where $T^n x = x$. Then $y = T^{-m}x$. Now, $T^n(T^{-m}x) = T^{-m}(T^n x) = T^{-m}x$. Which implies $T^{-m}x$ is periodic.

By Theorem 2.3.12, there are only finitely many periods possible for $T : \mathbb{R}^n \to \mathbb{R}^n$. Any two periodic orbits with same period (ie., the points in the orbits should have same period) are of the same kind. This is because, let $\{T^n x : n \in \mathbb{N}_0\}$ and $\{T^n y : n \in \mathbb{N}_0\}$ be two orbits such that x and y have same period. Fix two bases for $\mathbb{R}^n$, one containing x and the other containing y. Define $L : \mathbb{R}^n \to \mathbb{R}^n$ as $Lx = y$. Therefore these orbits are of same kind. So, when T is invertible, there are only finitely many kinds of finite orbits. Therefore $Fiorb(T) = \{0\} \times Per(T)$ and $Per(T) \in \mathfrak{F}_n$, and hence $Fiorb(T)$ is finite.

Case 2. T is not invertible.

Let Range of $T = X$. Therefore $dimX < k + 1$ since T is not invertible. Consider $T|_X : X \to X$. By induction hypothesis, $T|_X$ has finite number of different kinds of orbits, say (m_i, n_j), $1 \leq i \leq r$, $1 \leq j \leq s$ for some $r, s, m_i, n_j \in \mathbb{N}$.

Let $x \in \mathbb{R}^n$ be an eventually periodic point. Then $Fiorb(T) \subset Fiorb(T|_X) \cup \{(m+1, n) : (m, n) \in Fiorb(T|_X)\}$. Note that $x \in \mathbb{R}^n$ implies $T(x) \in X$.

Subcase 1. $x \notin$ X.

Then $Tx \in X$ since the range of $T = X$. Which implies the orbit of Tx is of type (m_l, n_k) for some l and k......(1).

Claim: If $y, z \in \mathbb{R}^n \setminus X$ such that $Ty = Tz = x \in X$ then the orbits of y and z are of same kind.

Let $\mathcal{B}$ be a basis of X. We can find two bases $\mathcal{B}_1$ and $\mathcal{B}_2$ for $\mathbb{R}^n$ such that $\mathcal{B}\cup\{y\} \subset \mathcal{B}_1$ and $\mathcal{B} \cup \{z\} \subset \mathcal{B}_2$. Define $L : \mathbb{R}^n \to \mathbb{R}^n$ such that $Ly = z$ and $Lw = w$ for all $w \in \mathbb{B}$. Hence the orbits of y and z are of same kind. By (1), the point $x \in X$ has the orbit of type $(m_l + 1, n_k)$. Hence the possible kinds of orbit are $(m_i + 1, n_j)$, $1 \leq i \leq r$, $1 \leq j \leq s$ whenever $x \notin X$. Therefore, for any eventually periodic point $x \in \mathbb{R}^n$, the possible kinds of orbits are (m_i, n_j) and $(1 + m_i, n_j), 1 \leq i \leq r, 1 \leq j \leq s$. ie., $T(x)$ is of type (m, n) in $Fiorb(T|_X)$ would implies x is of type $(m + 1, n)$ in $Fiorb(T)$.

Subcase 2. $x \in X$.

In this case, a point $x \in X$ is of $T|_X$-type if and only if it is of T-type.

Hence the proof follows by induction hypothesis. □

Remark A.2.5. The proof of Theorem A.2.4 may even work for the infinite orbits, the possible gap is the case of 'invertible T'.

Question: For the set of all interval maps, which subsets of $\mathbb{N}_0 \times \mathbb{N}$ will arises as $Fiorb(f)$?

It has to satisfy the two necessary conditions

(1) $(0, m) \in S, m \succ n \Rightarrow (0, n) \in S$

(2) $(m, n) \in S, m \neq 0 \Rightarrow (m - 1, n) \in S$ where $\succ$ denotes the Sharkovsii's ordering.

For each self map f on a set X, we associate a subset $\text{Per}(f)$ of $\mathbb{N}$ as in Chapter 2 and in Chapter 3; and a subset $Fiorb(f)$ of $\mathbb{N}_0 \times \mathbb{N}$. If f belongs to a certain nice class of function, then not all subsets of $\mathbb{N}$ may arise as the set of periods, and not all subsets of $\mathbb{N}_0 \times \mathbb{N}$ may arise as the set of all types of finite orbits.

It is natural to ask: Which subsets of $\mathbb{N}_0 \times \mathbb{N}$ arise as $Fiorb(f)$ for some f in that class? We hope search for $Fiorb(f)$ will leads a new research.

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