ADMISSIBILITY OVER FUNCTION FIELDS OF P-ADIC CURVES

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CERTIFICATE

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This is to certify that I, B. Surendranath Reddy, have carried out the research embodied in the present thesis entitled **Admissibility over function fields of p-adic curves** for the full period prescribed under Ph.D. ordinance of the University.

I declare that, to the best of my knowledge, no part of this thesis was earlier submitted for the award of research degree of any university.

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Dedicated to my Father

Preface

The aim of this thesis is to study the structure of admissible groups over finite extension of $\mathbb{Q}_p(t)$. We also study a certain local-global principle for division algebras.

It has been my great pleasure to work with my supervisor, Prof. V. Suresh. I am very thankful to him for his continuous help throughout my research work with much patience and also for spending a lot of time for so many valuable discussions. Learning and working with him has been one of the most enriching and fruitful experiences of my life.

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- B.Surendranath Reddy.

Contents

List of Symbols

$\mathbb{Q}$	The set of all rational numbers
$\mathbb{Z}$	The set of all integers
C_p	Cyclic group of order p
k	The residue field of a field K
K^*	The multiplicative group of a field K
ζ_n	A primitive root of unity of order n
K_ν	Completion of the field K with respect to a valuation ν
$char(K)$	The characteristic of a field K
$Gal(L/K)$	The Galois group of a field extension L/K
$Br(K)$	The Brauer group of the field K
${}_nBr(K)$	The n-torsion subgroup of $Br(K)$
$H^1(k, \mathbb{Z}/n\mathbb{Z})$	The first Galois cohomology group
PGL_n	projective general linear group of order n

Abstract

The thesis is conveniently divided into four chapters. Chapter 1 is a preliminary section. We explain basic definitions like central simple algebras, Brauer group, cyclic algebras, ramification, admissibility and Patching of fields and central simple algebras.

Chapter 2 is mainly devoted to the solution of a single problem. Let K be a complete discrete valued field with residue field k. Let F be a function field of a curve over K. Let n be an integer which is coprime to the characteristic of k. Assume that K contains a primitive n^{th} root of unity. We prove a certain Hasse principle for central simple algebras over F of index n.

In chapter 3, we discuss about necessary conditions for admissibility. Let K be a field and L a finite extension of K, then L is called K-*adequate* if there is a division ring D central over K containing L as a maximal commutative subfield. A finite group G is called K-*admissible* if there is a Galois extension L of K with $G = Gal(L/K)$, the Galois group of L over K, and L is K-*adequate*. We give necessary conditions for a finite group to be admissible over function fields of p-adic curves. In more general, we give necessary conditions for a finite group to be admissible over a finitely generated filed extension of a complete discretely valued filed of transcendence degree one.

In chapter 4, we discuss patching techniques over fields and prove admissibility of a certain class of groups over $\mathbb{Q}_p(t)$.

We now describe the results proved in the thesis chapter wise. In chapter 2, we prove the following local global principal for central simple algebras.

Theorem 1. (2.2.1) Let A be a complete discrete valuated ring with fraction field K and residue field k. Let X be a smooth, projective, geometrically integral curve over K. Let $F = K(X)$ be the function filed of X. Let D be a central division algebra over F of degree $n = \ell^r$ for some prime ℓ and $r \geq 1$. Assume that ℓ is a unit in A and K contains a primitive n^{th} root of unity. Then $D \bigotimes F_\nu$ is division for some discrete valuation ν of F.

In chapter 3, we give necessary conditions for a group to be admissible over certain class of fields.

We begin with the following Proposition which gives a necessary condition for a finite group to be admissible over a complete discretely valued fields.

Proposition 2. (3.1.2) Let K be a complete discretely valued field with residue filed k and G be a finite group such that char$(k) \nmid |G|$. If G is admissible over K then every sylow p-subgroup P of G has a filtration $P \supseteq P_1 \supseteq P_2 \supseteq (e)$ such that

(1) P_1 is normal subgroup of P and P_2 is a normal subgroup of P_1

(2) P/P_1 and P_2 are cyclic

(3) P_1/P_2 is admissible over some finite extension of the residue field k.

We know that over global fields, every p-group which is admissible, is meta cyclic. So using the above theorem we prove the following result as a corollary.

Corollary 3. (3.1.3) Let K be a complete discretely valued field with residue filed k and G be a finite group such that char$(k)\nmid |G|$. If G is admissible over K and the residue filed k is a global filed or a local field then every Sylow p-subgroup P of G has a filtration $P \supseteq P_1 \supseteq P_2 \supseteq (e)$ such that

(1) P_1 is normal subgroup of P and P_2 is a normal subgroup of P_1

(2) P/P_1 and P_2 are cyclic

(3) P_1/P_2 is meta cyclic.

In the following theorem, we give necessary conditions for admissibility over function fields of curves over complete discretely valued fields.

Theorem 4. (3.1.5) Let K be a complete discretely valued field with residue field k and $F = K(X)$ be the function filed of a curve X over K. Let G be a finite group of order n such that $char(k) \nmid n$. If G is admissible over F then every Sylow p-sub group P of G has a filtration $P \supseteq P_1 \supseteq P_2 \supseteq (e)$ such that

(1) P_1 is normal subgroup of P and P_2 is a normal subgroup of P_1

(2) P/P_1 and P_2 are cyclic

(3) P_1/P_2 is admissible over some finite extension of the residue field of a discrete valuation of F.

Corollary 5. (3.1.6) Let K be a local field with residue field k and $F = K(X)$ be the function filed of a curve X over K. Let G be a finite group of order n such that $char(k)$ is coprime to n. If G is admissible over F, then every Sylow p-sub group P of G has a filtration $P \supseteq P_1 \supseteq P_2 \supseteq (e)$ such that

(1) P_1 is normal subgroup of P and P_2 is a normal subgroup of P_1

(2) P/P_1 and P_2 are cyclic

(3) P_1/P_2 is meta cyclic.

We end chapter 3 with an example of a finite group of rank 4 which is $\mathbb{Q}_p(t)$-admissible and a finite group of rank 5 which is not $\mathbb{Q}_p(t)$-admissible.

We prove the following results in chapter 4.

Lemma 6. (4.2.1) Let R be a regular local ring of dimension two with residue field k and field of fraction F. Let n_1 and n_2 be natural numbers which are coprime to the char(k). Assume that F contains a primitive $n_1 n_2^{th}$ root of unity and there is an element in k^*/k^{*n_2} of order n_2. Then there is a central division algebra D over F of degree $n_1 n_2$.

Theorem 7. (4.2.2) Let K be a complete discretely valued field with residue field k and F be the function field of a curve over K. Let n be an integer which is coprime to the characteristic of k. Suppose that K contains a primitive n^{th} root of unity. Assume that for every finite extension L of k, there is an element in L^*/L^{*n} of order n. If G is a group of order n with every Sylow subgroup is isomorphic to product of at most 4 cyclic groups, then G is admissible over F.

Theorem 8. (4.2.3) Let K be a local field and k its residue field. Let F be the function field of a curve over K. Let n be an integer which is coprime to the characteristic of k. Suppose that K contains a primitive n^{th} root of unity. If G is a group of order n with every Sylow subgroup is isomorphic to a product of at most 4 cyclic groups, then G is admissible over F.

Corollary 9. (4.2.4) Let K be a local field and k its residue field. Let F be the function field of a curve over K. Let $n = p_1^{d_1} \cdots p_r^{d_r}$ with $1 \leq d_i \leq 2$ and p_i distinct primes. Assume that K contains a primitive n^{th} root of unity and n is coprime to the characteristic of k. If G is a group of order n, then G is admissible over F.

Corollary 10. (4.2.5) Let K be a local field and k its residue field. Let F be the function field of a curve over K. Let $n = p_1^{d_1} \cdots p_r^{d_r}$ with $1 \leq d_i \leq 4$ and p_i distinct primes. Assume that K contains a primitive n^{th} root of unity and n is coprime to the characteristic of k. If G is an abelian group of order n, then G is admissible over F.

Corollary 11. (4.2.6) Let K be a local field and k its residue field. Let F be the function field of a curve over K. Let G be an abelian group of order n. Assume that K contains a primitive n^{th} root of unity and n is coprime to the characteristic of k. Then G is admissible over F if and only if G is isomorphic to a product of at most four cyclic groups.

Chapter 1

Some basic definitions and results

This chapter, which is preliminary in nature, contains a rapid review of some of the basic definitions and results used in this thesis. The references for definitions and results recalled in the thesis are [GS06], [HHK11], [Pie82], [Sch68], [Sc85], [ser79].

1.1 Central Simple Algebras and the Brauer group

In this section we recall the definition of Central Simple Algebras and the Brauer group. Let R be a commutative ring with unity. By an *R-algebra* we mean a ring A which is also a unitary R module such that

$$a(xy) = (ax)y = x(ay)$$

for all a in R and x, y in A. We say that two R-algebras A and B are isomorphic if there exists an isomorphism $\phi : A \to B$ of rings which is also R-linear. Let k be a field. Let A be a k-algebra. Since k is a field, the map $k \to A$ given by $a \mapsto a \cdot 1$ is injective. Hence we identify k as a sub ring of A.

We say that an R-algebra A is *simple* if the ring A is simple i.e., A has no two sided ideals other than (0) and A. A finite dimensional simple k-algebra with center a field k is called a *central simple algebra* over k. Let D be a division ring and k be the center of D. If D is finite dimensional over k, then D is a central simple k-algebra and we call D a *central division* k-algebra.

Let k be a field. By the classical theorem of Wedderburn, we know that every central simple k-algebra is isomorphic to a matrix algebra $M_n(D)$ for some central division k-algebra D. We also know that if $M_n(D)$ is isomorphic to $M_{n'}(D')$ for some central divsion k-algebras D and D', then $n = n'$ and $D \simeq D'$. Let A be a central simple k-algebra. Then $A \simeq M_n(D)$ for some n and a central division k-algebra D. Let $L \subset D$ be a maximal subfield. Then we have $D \otimes_k L \simeq M_d(L)$. In particular, the dimension of a central simple k-algebra is a square. The square root of the dimension of a central simple k-algebra is called the *degree* and denoted by $deg(A)$. Let A be a cental simple k-algebra. Then $A \simeq M_n(D)$ for some central division k-algebra. The degree of D is called the *index* of A and denoted by $ind(A)$.

Two central simple k-algebras A and B are called *similar*, denoted by $A \sim B$, if $M_m(A)$ and $M_n(B)$ are isomorphic for some m, n. It is easy to see that this is an equivalence relation on the set of isomorphic classes of central simple k-algebras. The set of equivalence classes of central simple k-algebras is denoted by $Br(k)$. By the Wedderburn theorem, $Br(k)$ can be identified with the set of isomorphism classes of central division k-algebras. For a central simple k-algebra A, let $[A]$ denote the equivalence class containing A. Let A and B be two central simple k-algebras of same dimension. Then $[A] = [B]$ if and only if $A \simeq B$.

Let A and B be two central simple k-algebras. Then $A \otimes_k B$ is a central simple algebra over k. The tensor product of central simple algebras induces a group structure on $Br(k)$. This group $Br(k)$ is called the *Brauer group* of k. The equivalence class $[M_n(k)]$ is the identity element of $Br(k)$. For a central simple k-algebra A, the class $[A^o]$ is the inverse of the class $[A]$, where for any ring B, B^o denotes the opposite ring. Since $A \otimes_k B \simeq B \otimes_k A$, the Brauer group $Br(k)$ is abelian. Let A be a central simple k-algebra. It is known that $deg(A) \cdot [A] = 0$. In particular every element of $Br(k)$ is a torsion element. The order of the class $[A]$ in $Br(k)$ is called the *exponent* of A.

If k is a algebraically closed field, then $Br(k)$ is trivial.

Let L/k be an extension of fields. Let A be a central simple k-algebra. Then $A \otimes_k L$ is a central simple L-algebra and this induces a homomorphism $Br(k) \to Br(L)$.

The n-torsion subgroup of $Br(K)$, denoted by ${}_nBr(K)$, is defined as

$$ {}_nBr(K) = \{[A] | [A]^{\otimes n} = [K]\} $$

. Suppose that ν is a discrete valuation of K with residue field κ. Let n be a natural number which is coprime to the characteristic of κ. Then we have a *residue homomorphism* $\partial_\nu : {}_nBr(K) \to H^1(\kappa, \mathbb{Z}/n\mathbb{Z})$, where for any field $H^1(k, \mathbb{Z}/n\mathbb{Z})$ denotes the first Galois cohomology group.

1.2 The cyclic algebras

Let L/K be a Galois extension of degree n such that the Galois group $G = Gal(L/K)$ is cyclic. Let σ be a generator of G. Let $a \in K^* = K - \{0\}$. Now we construct an algebra A which is denoted by $(L/K, \sigma, a)$ as follows : Let A be a L-vector space of dimension n. Choose a basis of A containing 1 and denote it by $1, e, ..., e^{n-1}$. We have

$$ A = L.1 \oplus Le \oplus ... \oplus Le^{n-1}. $$

Define the multiplication on A as follows:

$$ e^n = a.1,\ \lambda e^i \mu e^j = \lambda\mu e^{i+j} \text{ and } e(\lambda\,.1) = \sigma(\lambda)\, e \qquad \text{for } \lambda, \mu \in L. $$

We denote this algebra A by $(L/K, \sigma, a)$. It is well know that $(L/K, \sigma, a)$ is a central simple algebra over K and L is a maximal subfield of $(L/K, \sigma, a)$. We call this algebra a *cyclic algebra*.

For a finite extension L/K, let $N_{L/K} : L \to K$ be the norm map. Let L/K be a Cyclic extension, σ be a generator of $Gal(L/K)$ and $a, b \in K^*$. We have the following:

Theorem 1.2.1. $(L/K, \sigma, a) \simeq (L/K, \sigma, b)$ if and only if $ba^{-1} \in N_{L/K}(L^*)$.

Corollary 1.2.2. If the degree of L/K is a prime number, then $(L/K, \sigma, a)$ is a division algebra if and only if $a \notin N_{L/K}(L^*)$.

Let K be a field and n be a natural number not equal to the characteristic of k. Suppose that K contains a primitive n^{th} root of unity ζ. Let $a \in K^*$. If a is not an n^{th} power in K, then $L = K(\sqrt[n]{a})$ is a cyclic extension and the automorphism of L given by $\sigma(\sqrt[n]{a}) = \zeta\sqrt[n]{a}$ is a generator of the Galois group $Gal(L/K)$. For any $b \in K^*$, the cyclic algebra $(L/K, \sigma, b)$ is denoted by $(a, b)_n$ and called an *n-symbol algebra*. Hence $(a, b)_n$ denote the cyclic algebra generated by x, y with $x^n = a$, $y^n = b$ and $xy = \zeta yx$. Then $(a, b)_n$ represents an element in ${}_nBr(K)$. Suppose κ contains a primitive n^{th} root of unity. Then, by fixing a primitive n^{th} root of unity, we identify $H^1(\kappa, \mathbb{Z}/n\mathbb{Z})$ with κ^*/κ^{*n}. With this identification we have $\partial_\nu((a, b)_n) = \overline{\frac{a^{\nu(b)}}{b^{\nu(a)}}} \in \kappa^*/\kappa^{*n}$, where for any $c \in K^*$ which is a unit at ν, $\overline{c}$ denotes its image in κ^*.

Let k be a field and n an integer coprime to the char(k). Then $H^1(k, \mathbb{Z}/n\mathbb{Z})$ classifies pairs (E, σ) with natural equivalences, where E is a cyclic Galois field extension of k of degree a factor of n and σ a generator of the Galois group $Gal(E/k)$ of E/k.

In fact we have

$$H^1(k, \mathbb{Z}/n\mathbb{Z}) = Hom(Gal(k^{sep}/k, \mathbb{Z}/n\mathbb{Z})$$

where k^{sep} is the separable closure of k. For a given homomorphism $\phi : G \to \mathbb{Z}/n\mathbb{Z}$, let $H = ker(\phi)$ and $E = k^{sep^H}$, fixed field of H. Then E/k is Galois extension with Galois group $Gal(k^{sep}/k)/H$. Since $Gal(k^{sep}/k)/H$ is isomorphic to a subgroup of $\mathbb{Z}/n\mathbb{Z}$, it is cyclic and the degree of E/k divides n. For any $m \geq 1$, let $H_m = ker(\phi^m)$ and $E^m = k^{sep^{H_m}}$. Since $H \subset H_m$, we have $E^m \subset E$. Since $H^1(k, \mathbb{Z}/n\mathbb{Z})$ is a group, we have $E^m = k$ if and only if m divides the order of ϕ in $H^1(k, \mathbb{Z}/n\mathbb{Z})$.

Let K be a complete discretely valued field with residue field k. Let n be a natural number which is coprime to the characteristic of k. Let E_0/k be a cyclic extension of degree n. Then there exists a unique (up to isomorphism) cyclic extension E of K with residue field E_0 and a natural isomorphism $Gal(E/K) \to Gal(E_0/k)$. Let $(E_0, \sigma_0) \in H^1(k, \mathbb{Z}/n\mathbb{Z})$. Then we have a unique $(E, \sigma) \in H^1(K, \mathbb{Z}/n\mathbb{Z})$ and it is called the lift of (E_0, σ_0).

1.3 Ramification

Let $\mathcal{X}$ be a regular integral scheme with function field F. Let n be an integer which is a unit on $\mathcal{X}$. Let $f \in F$ and $P \in \mathcal{X}$ be a point. If f is regular at P, then we denote its image in the residue field $\kappa(P)$ at P by $f(P)$. Let $\mathcal{X}^1$ denote the set of codimension one points of $\mathcal{X}$. For each codimension one point x of $\mathcal{X}$, we have discrete valuation ν_x on F. Let $\kappa(x)$ denote the residue field at x. Since n is a unit on $\mathcal{X}$, n is coprime to the char$(\kappa(x))$ and we have the residue homomorphism $\partial_x : {}_nBr(F) \to H^1(\kappa(x), \mathbb{Z}/n\mathbb{Z})$. Let $\alpha \in {}_nBr(F)$. We say that α is *unramified* at x if $\partial_x(\alpha) = 0$. We say that α is *unramified on* $\mathcal{X}$ if it is unramified at every codimension point of $\mathcal{X}$. Let A be a central simple algebra over F. We say tat A is *unramified* if its class in $Br(F)$ is unramified. If $\mathcal{X} = Spec(B)$ for a ring, then we say that α is unramified on B if it is unramified in $\mathcal{X}$.

Let B be a regular Neotherian integral domain of dimension at most 2 and F its field of fractions. Let A be a central simple algebra over F. If A is unramified on B, then there exists a unique Azumaya algebra $\mathcal{A}$ over B such that $\mathcal{A} \otimes_B F \simeq A$ ([CTS, 6.13, see also [CTPS], 4.2). Let A be a central simple algebra over F which is unramified on B. For an ideal I of B, we denote the algebra $\mathcal{A} \otimes_B B/I$ by $A(I)$.

1.4 Admissibility and Patching

We first recall the definition of *Admissibility* and some results about *admissible* groups over $\mathbb{Q}$ and $\mathbb{Q}(t)$ ([Sch68], [FS95]).

Let K be a field and L a finite extension of K, then L is called K-*adequate* if there is a division ring D central over K containing L as a maximal commutative subfield. A finite group G is called K-*admissible* if there is a Galois extension L of K with $G = Gal(L/K)$, the Galois group of L over K, and L is K-*adequate*.

For a given field K, one can ask which finite groups are admissible over K. This question was originally posed by Schacher, who gave partial results in the case $K = \mathbb{Q}$. In [sch68], Schacher gave a criteria that is necessary for admissibility of a group over the field $\mathbb{Q}$, and which he conjectured also sufficient:

Conjecture[Sch68]: Let G be a finite group. Then G is admissible over $\mathbb{Q}$ if and only if every sylow subgroup is metacyclic.

A finite group G is called *metacyclic* if G has a normal subgroup H such that H is cyclic and G/H is cyclic. Although the above conjecture is still open in general, many particular groups satisfying this criterion have been shown in fact to be *admissible* over $\mathbb{Q}$. Also Corollary 10.3 of [Sch68] shows that admissible groups over a global field of characteristic p have metacyclic Sylow subgroups at the primes other than p. Schacher proved an important result in the case of number fields in [Sch68]. The theorem he proved in that paper is the following.

Theorem 1.4.1. Let K be a number field and G be a finite group. Then G is $K-admissible$ if and only if there exists a Galois extension L/K that satisfies:

(1) $Gal(L/K) \cong G$

(2) For every rational prime $p \mid |G|$, there are two primes υ_1 and υ_2 of K such that $\mathrm{Gal}(L_{\upsilon_i}/K_{\upsilon_i})$ contains a p-Sylow subgroup of G.

Fein and Schacher gave the following criterion for admissibility over $\mathbb{Q}(t)$ in ([FS95], Theorem 4).

Theorem 1.4.2. Let t be transcendental over $\mathbb{Q}$ and G be a group of odd order. Assume that for every Sylow subgroup P of G, there exists $P_0 \lhd P$ with P/P_0 cyclic such that either

(1) P_0 is *meta-cyclic*, or

(2) P_0 can be generated by two elements and $[P : P_0] \geq |P_0|$.

Then G is $\mathbb{Q}(t)$- admissible.

They also proved that if a G is $\mathbb{Q}((t))$- admissible, then it is $\mathbb{Q}(t)$- admissible, they also exhibit a $\mathbb{Q}(t)$- admissible group but not a $\mathbb{Q}((t))$- admissible.

We now recall the method of patching over fields ([HHK08], [HHK11]). Let R be a complete discrete valuation ring with uniformizer t, fraction field K, and residue filed k. We consider a finitely generated filed extension F/K of transcendence degree one. Let $\hat{X}$ be a regular connected projective R-curve with function field F such that reduced irreducible components of its closed fiber X are regular (Given F, such an $\hat{X}$

always exists by resolution of singularities; cf. [Abh69] or [Lip75]). Let $f{:}\hat{X} \to \mathbf{P^1_R}$ be a finite morphism such that the inverse image S of $\infty \in P^1_k$ contains all the points of X at which distinct irreducible components meet. We will call $(\hat{X}, S)$ a regular *R-model* of F.

For each point $Q \in S$ as above, we let R_Q be the local ring of $\hat{X}$ at Q, and we let $\hat{R_Q}$ be its completion at the maximal ideal corresponding to the point Q. Also, for each connected component U of $X \setminus S$ we let R_U be the subring of F consisting of the rational functions that are regular at the points of U, and we let $\hat{R_U}$ denote its t-adic completion. If $Q \in S$ lies in the closure $\bar{U}$ of a component U, then there is a unique branch $\wp$ of X at Q lying on $\bar{U}$ (since $\bar{U}$ is regular). Here $\wp$ is a height one prime ideal of $\hat{R_Q}$ that contains t, and we may identify it with the pair (U, Q). We write $\hat{R_\wp}$ for the completion of the discrete valuation ring obtained by localizing $\hat{R_Q}$ at its prime ideal $\wp$. Thus $\hat{R_Q}$ is naturally contained in $\hat{R_\wp}$.

In the above situation, with $\wp{=}(U, Q)$, there is also a natural inclusion $\hat{R_U} \hookrightarrow \hat{R_\wp}$. To see this, first observe that the localizations of $\hat{R_U}$ and of $\hat{R_Q}$ at the generic point of $\bar{U}$ are the same; and this localization is naturally contained in the t-adically complete ring $\hat{R_\wp}$. Thus so is R_U and hence its t-adic completion $\hat{R_U}$. The inclusions of $\hat{R_U}$ and of $\hat{R_Q}$ into $\hat{R_\wp}$, for $\wp{=}(U, Q)$, induce inclusions of the corresponding fraction fields F_U and F_Q into the fraction filed $F_\wp$ of $\hat{R_\wp}$. Let I be the index set consisting of all U,Q,$\wp$ described above. Via the above inclusions, the collection of all F_ξ, for$\xi \in I$, forms an inverse system with respect to the ordering given by setting $U \succ \wp$ and $Q \succ \wp$ if $\wp$=(U, Q).

Under the above hypotheses, suppose that for every field extension L of F, we are given a category $\mathcal{C}(\mathcal{L})$ of algebraic structures over L (i.e. finite dimensional L-vector spaces with additional structure,e.g. associative L-algebras), along with base-change functors $\mathcal{C}(\mathcal{L}) \to \mathcal{C}(\mathcal{L}')$ when $L \subseteq L'$. An $\mathcal{C}$ - *patching problem* for $(\hat{X}, S)$ consists of an object V_ξ in $\mathcal{C}(\mathcal{F}_\xi)$ for each $\xi \in I$, together with isomorphism $\Phi_{U,\wp}$:$V_U \bigotimes_{F_U} F_\wp \to V_\wp$ and $\Phi_{Q,\wp}$:$V_Q \bigotimes_{F_Q} F_\wp \to V_\wp$ in $\mathcal{C}(\mathcal{F}_\wp)$. These patching problems form a category, denoted by $PP_{\mathcal{C}}(\hat{X}, S)$, and there is a base change functor $\mathcal{C}(\mathcal{F}) \to PP_{\mathcal{C}}(\hat{X}, S)$.

If an object $V \in \mathcal{C}(\mathcal{F})$ induces a given patching problem up to isomorphism, we will say that V is a solution to that patching problem, or that it is obtained by patching the objects V_ξ. We similarly speak of obtaining a morphism over F by patching morphisms in $PP_{\mathcal{C}}(\hat{X}, S)$.

Given a finite group G, a subgroup $H \subseteq G$, and an H-Galois field extension L/F, there is an induced G-Galois F-algebra $E = Ind_H^G L$ given by a direct sum of copies of L indexed by the left cosets of H in G;e.g. see [HH07], section 7.2. In particular if $H = 1$, then E is a split extension of F i.e., $E \cong F^{\oplus|G|}$.

We now recall the results which are useful in proving main results.

Theorem 1.4.3.([HHK08], 5.1.) Let F be as above and A be a central simple F-algebra. Then $ind(A) = lcm_{\xi \in S \cup \mathbb{U}} ind(A \otimes_F F_\xi)$, where $\mathbb{U}$ is the collection of all irreducible components of $X \setminus S$.

Corollary 1.4.4. Let F be as above and D be a division algebra over F of degree $n = \ell^r$ for some prime ℓ and $r \geq 1$ such that ℓ is not equal to $char(k)$. Then either $D \otimes F_U$ is division for some irreducible component U of $X \setminus S$ or $D \otimes F_P$ is division for some $P \in S$.

Proof. Since degree of D is a power of a prime, the degree of $D \otimes F_U$ and the degree of $D \otimes F_P$ is also a power of the same prime. Since lcm of power of a prime is the maximum, by the above theorem, either $ind(D) = ind(D \otimes F_U)$ for some irreducible component U of $X \setminus S$ or $ind(D) = ind(D \otimes F_P)$ for some $P \in S$. Since $deg(D) = ind(D)$, either $deg(D) = ind(D \otimes F_U)$ or $deg(D) = ind(D \otimes F_P)$. Hence either $D \otimes F_U$ is division for some irreducible component U of $X \setminus S$ or $D \otimes F_P$ is division for some $P \in S$. $\square$

Theorem 1.4.5.([HHK11], 4.2.) Let G be a finite group, and F and $(\hat{X}, S)$ be as in above notation. Suppose that for each $Q \in S$, we are given a subgroup $H_Q \subseteq G$ and an H_Q-Galois adequate field extension L_Q/F_Q such that $L_Q \bigotimes_{F_Q} F_\wp$ is a split extension $F_\wp^{\oplus |H_Q|}$ of $F_\wp$ for each branch $\wp$ at Q. Assume that the greatest common divisor of the indices $(G : H_Q)$ is equal to 1. Then G is admissible over F. $\square$

The following corollary is contained in the proof of the Proposition 4.3 of [HHK11].

Corollary 1.4.6. Let $p_1, ..., p_r$ be the prime numbers dividing the order of G. For each $i \in 1, ..., r$, let P_i be a Sylow p_i-sbugroup of G. Assume that $Q_1, ...Q_r \in S$ are regular on X . If for each i, there is a P_i-Galois adequate field extensions $L_i = L_{Q_i}$ of F_{Q_i} such that $L_i \bigotimes_{F_{Q_i}} F_{\wp_i}$ is a split extension $F_{\wp_i}^{\oplus|P_i|}$ of $F_{\wp_i}$, and the greatest common divisor of the indices $(G : P_i)$ is equal to 1, then G is admissible over F.

Proof. Let $P_Q = 1$ and $L_Q = F_Q$ for every point $Q \in S$ other than $Q_1, ...Q_r$. Since the indices of the subgroups P_i are relatively prime for $i = 1, ..., r$, it follows that the indices of the subgroups P_i (for $i \in S$) are relatively prime. Now corollary follows from (1.4.5). □

The following is the main theorem of [HHK11]

Theorem 1.4.7([HHK11], 4.4.) Let F be a finitely generated field extension of transcendence degree one over a complete discretely valued field K with algebraically closed residue field k, and let G be a finite group of order not divisible by $char(k)$. Then G is admissible over F if and only if each of its Sylow subgroups is abelian of rank at most 2.

The following is contained in the proof of ([HHK08],5.2,p.38)

Proposition 1.4.8 Let K be a complete discretely valued field with residue filed k. Let F be the function field of a curve over K. Let n be an integer which is coprime to $char(k)$. Let D be a central division algebra over F of degree n. Then there exists a regular proper model $(\hat{X}, S)$ of F such that for any irreducible component U of $X \setminus S$, the index of the algebra $D \otimes (L_U, \sigma_U, s)$ is equal to the index of the image of $D \otimes (L_U, \sigma_U, s)$ over $k(U)$ for some suitable L_U, σ_U and s.

Chapter 2

Hasse principle for division algebras

2.1 Introduction

In this chapter, we prove a certain local-global principle for division algebras which will be the one of the key ingredient in the proof of the theorem 3.5, which we prove in chapter 3. Let K be a complete discretely valued field with residue field k. Let F be a function field of a curve over K. Let n be an integer which is coprime to the characteristic of k. Assume that K contains a primitive n^{th} root of unity. The main theorem in this chapter is to prove a certain Hasse principle for central simple algebras over F of index n.

The following conjecture was made by Colliot-Thélène, Parimala and Suresh ([CTPS])

Conjecture. Let K be a p-adic field and F the function field of a curve over K. Let G a connected linear algebraic group over F and Y/F a projective homogeneous space of G. If Y has rational points in all completions F_ν at discrete valuation, then it has an F-rational point.

In fact they proved the above conjecture for the special orthogonal group if $p \neq 2$ using the patching methods of ([HHK08]). By using the same methods, we prove the above conjecture for PGL_n if n is coprime to p.

We begin with the following well known lemma, which we use to prove the main result of this chapter.

Lemma 2.1.1. (cf., [FS95]) Let R be a complete discrete valuated ring and K its field of fractions. Let B be a central simple algebra over K of index n. Let E be an unramified cyclic extension of K of degree m and σ a generator of the Galois group of E/K. Let π be an uniformising parameter in R. Assume that mn is invertible in R and B is unramified at R. Then the index of $B \otimes (E, \sigma, \pi)$ is equal to the product of the index of $B \otimes E$ and the degree of E over K.

2.2 Hasse principle for division algebras

Let A be a complete discrete valuated ring with field of fractions K and residue field k. Let F be the function field of a curve over K. Let Ω be the set of all discrete valuations of F given by codimension one points of regular proper two dimensional schemes $\mathcal{X}$ over K with function field F. Note that there are many such schemes $\mathcal{X}$ by the resolution of singularities.

Theorem 2.2.1. Let A be a complete discrete valuation ring with fraction field K and residue field k. Let X be a smooth, projective, geometrically integral curve over K and $F = K(X)$ the function field of X. Let D be a division algebra over F of degree $n = \ell^r$ for some prime ℓ and $r \geq 1$. Assume that ℓ is a unit in A and K contains a primitive n^{th} root of unity. Then $D \otimes F_v$ is division for some discrete valuation $v \in F$.

Proof Let D be a central division algebra over F. We choose a regular proper model $\mathcal{X}/A$ of X/K such that the support of the ramification divisor D and the components of the special fibre of $\mathcal{X}/A$ are a union of regular curves with normal crossings. Let $Y = \mathcal{X} \times_A k$ denote the special fibre.

For a generic point x_i of an irreducible component Y_i of Y, there is an affine Zariski neighbourhood $W_i = SpecR^{W_i}$ of x in $\mathcal{X}$ such that the restriction of Y_i to W_i is a principal divisor.

For each irreducible curve C in the support of ramification divisor of D, let $a_C \in \kappa(C)^*$ be such that the residue of D at C is $(a_C) \in \kappa(C)^*/\kappa(C)^{*n}$.

Let S_0 be a finite set of closed points of the special fibre containing all singular points of D, all points which lie on some Y_i but not in W_i and all those points of irreducible curves where a_C is not a unit.

Let $f : \mathcal{X} \to \mathbf{P}^1_A$ be an A-morphism such that the inverse image S of the point at infinity of the special fibre $\mathbf{P}^1_k$ containg S_0 (as in 1.4.8).

Let $U \subset Y$ be a reduced, irreducible components of the complement of S in Y. Then U is a smooth affine irreducible curve over k, U is contained in an affine subscheme SpecR^U of $\mathcal{X}$ and is a principal effective divisor in this affine subscheme.

For each $P \in S$, let F_P be the field of fractions of the completion $\hat{R}_P$ of the local ring R_P of $\mathcal{X}$ at P. For each $U \subset Y \setminus S$, let R_U be the ring of elements in F which are regular on U. It is a regular ring. The ring R_U is a localisation of R^U. Thus U is a principal effective divisor on SpecR_U. It is given by the vanishing of an element $s \in R_U$. Let $k[U] = R_U/s = \hat{R}_U/s$.

Let t denote a uniformizing parameter for A. The field F_U is the field of fractions of the t-adic completion $\hat{R}_U$ of R_U. We have $t = u.s^r$ for some integer $r \geq 1$ and a unit $u \in R_U$. Thus the t-adic completion $\hat{R}_U$ coincides with the s-adic completion of R_U.

Since D is a central division algebra over F, by (Corollary 1.4.4.) either $D \otimes F_U$

is division for some irreducible components U of $Y \setminus S$ or $D \otimes F_P$ is division for some $P \in S$.

Suppose that $D \otimes F_U$ is division for some irreducible component U of $Y \setminus S$. Let $s \in R_U$ be as above. Let ν be the discrete valuation on F given by the generic point of U. Then s is a uniformising parameter at ν and R_ν the ring of integers at ν is the localisation of R_U at the prime ideal (s). We show that $D \otimes F_\nu$ is division.

Suppose that D is unramified on R_U. Since D is a division algebra, by (1.4.8), the image of D is a division algebra over $k(U)$. Since the residue field at ν is $k(U)$, $D \otimes F_\nu$ is a division by (2.1.1).

Suppose D is ramified on R_U. Then by the choice of U, D is ramified on R_U only at the prime ideal (s) of R_U. Let C be the closure of U in Y. Then C is in the support of ramification of D. By the choice of U, $a_C \in \kappa(C)$ is a unit at every point of U. Hence $a_C \in k[U]$ is a unit. Let $u_C \in R_U$ with image $a_C \in k[U]$. Since $\hat{R}_U$ is (s)-adically complete and the image of u_C modulo (s) is a unit, u_C is a unit in $\hat{R}_U$. Let $\hat{R}_{\tilde{U}}$ be the integral closure of $\hat{R}_U$ in $F_U(\sqrt[n]{u_C})$. Since u_C is a unit in $\hat{R}_U$, the cyclic algebra (u_C, s) is ramified on $\hat{R}_U$ only at (s) and the residue of (u_C, s) at (s) is (a_C). Since D is ramified only at the prime ideal (s) of R_U, and (a_C) is the residue of D at s, we have $D \otimes F_U = D' \otimes (u_C, s)$ for some division algebra D' over F_U which is unramified on $\hat{R}_U$. In particular, index$(D \otimes F_U)$ divides index$(D' \otimes F_U(\sqrt[n]{u_C})) \cdot [F_U(\sqrt[n]{u_C}) : F_U]$. Since D' is unramified on $\tilde{R}_U$, let $\overline{D'}$ be its image over $k(U)$. Since a_C is a unit in $\hat{R}_U$ and $\hat{R}_U$ is complete, the index of $D' \otimes F_U(\sqrt[n]{u_C})$ equal to the index of $\overline{D'} \otimes k(U)(\sqrt[n]{a_C})$

(cf. [HHK08],4.5). Similarly $[F_U(\sqrt[n]{u_C}) : F_U] = [k(U)(\sqrt[n]{a_C}) : k(U)]$. Since D' is unramified on R_U, it is also unramified at ν. By (2.1.1), we have

$$\begin{aligned} index(D \otimes F_\nu) &= index(D' \otimes (u_C, s)) \\ &= index(D' \otimes F_\nu(\sqrt[n]{u_C})) \cdot [F_\nu(\sqrt[n]{u_C}) : F_U] \\ &= index(\overline{D'} \otimes k(U)(\sqrt[n]{a_C})) \cdot [k(U)(\sqrt[n]{a_C}) : k(U)]. \end{aligned}$$

The last equality is due to the completeness of F_ν. Thus index $(\overline{D'} \otimes k(U)(\sqrt[n]{a_C})) \cdot [k(U)(\sqrt[n]{a_C}) : k(U)] = index(D \otimes F_\nu) \leq index(D \otimes F_U) \leq index(\overline{D'} \otimes k(U)(\sqrt[n]{a_C})) \cdot [k(U)(\sqrt[n]{a_C}) : k(U)]$. Hence index$(D \otimes F_\nu) = index(D \otimes F_U)$. Since $D \otimes F_U$ is division, $D \otimes F_\nu$ is division.

Suppose that $D \otimes F_P$ is division for some $P \in S$. By the choice of S, the local ring R_P is a regular local ring with maximal ideal (x, y) such that D is ramified on R_P at most at x and y.

Suppose that D is unramified at P. Let ν be the discrete valuation of F given by x. Then it is easy to see that $index(D \otimes F_U) = index(D \otimes F_\nu)$.

Suppose that D is ramified on R_P only at the prime ideal (x). Let ν be the discrete valuation on F given by ν. By ([Sal97]), we have $D = D' \otimes (u, x)$ for some unit u in R_P and division algebra D' over F which is unramified on R_P. As above we can show that $index(D \otimes F_P) = index(D \otimes F_\nu)$.

Similarly the case where D is ramified on R_P only at the prime ideal (y).

Assume that D is ramified on R_P at both the primes (x) and (y). Then by

([Sal97]), either $D = D' \otimes (u_1, x) \otimes (u_2, y)$ or $D = D' \otimes (uy^r, x)$ where u_1, u_2, u are units in R_P, r coprime with n and D' unramified on R_P.

Suppose that $D = D' \otimes (u_1, x) \otimes (u_2, y)$ for some units $u_1, u_2 \in R_P$ and D' unramified on R_P. Let ν_y be the discrete valuation on F given by y and $\nu_{\hat{y}}$ the discrete valuation on F_P given by y. Let $\hat{F}_P$ be the completion of F_P with respect to the discrete valuation $\nu_{\hat{y}}$. Then by (2.1.1), we have index$D \otimes \hat{F}_P =$ index$(D' \otimes (u_1, x) \otimes \hat{F}_P(\sqrt[n]{u_2})) \cdot [\hat{F}_P(\sqrt[n]{u_2}) : \hat{F}_P]$. Since $D' \otimes (u_1, x)$ is unramified at y and u_1 is a unit, index$(D' \otimes (u_1, x) \otimes \hat{F}_P(\sqrt[n]{u_2})$ is equal to the index of its image D'' over $\kappa(\nu_{\hat{y}})(\sqrt[n]{\overline{u_2}})$. Since $\kappa(\nu_{\hat{y}})$ is the field of fractions of $\hat{R}_P/(y)$, the image of x gives a discrete valuation ν on the residue field $\kappa(\nu_{\hat{y}})$ and $\kappa(\nu_{\hat{y}})$ is complete with respect to ν. We have $\kappa(\nu) = \kappa(P)$. Since $\overline{u_2}$ is a unit at ν, the $\kappa(\nu_{\hat{y}})(\sqrt[n]{\overline{u_2}})$ is an unramified extension of $\kappa(\nu_{\hat{y}})$. Hence ν extends to a unique valuation $\tilde{\nu}$ to $\kappa(\nu_{\hat{y}})(\sqrt[n]{\overline{u_2}})$ with residue field $\kappa(\tilde{\nu}) = \kappa(P)(\sqrt[n]{\overline{u_2}})$. We have $D'' = (\overline{D'} \otimes (\overline{u}_1, \overline{x})) \otimes \kappa(\nu_{\hat{y}})(\sqrt[n]{\overline{u_2}})$, where 'bar' denotes the image in $\kappa(\nu_{\hat{y}})$. Since $\kappa(\nu_{\hat{y}})(\sqrt[n]{\overline{u_2}})$ is complete with respect to $\tilde{\nu}$ and $\overline{x}$ is a parameter at $\tilde{\nu}$, by (2.1.1), we have

$$index(D'') = index(\overline{D'} \otimes \kappa(\nu_{\hat{y}})(\sqrt[n]{\overline{u_2}}, \sqrt[n]{\overline{u_1}}) \cdot [\kappa(\nu_{\hat{y}})(\sqrt[n]{\overline{u_2}}, \sqrt[n]{\overline{u_1}}) : \kappa(\nu_{\hat{y}})(\sqrt[n]{\overline{u_2}})].$$

Since $\overline{D'}$ is unramified at $\tilde{\nu}$, u_1, u_2 are units and $\kappa(\nu_{\hat{y}})$ is complete, we have $index(\overline{D'} \otimes \kappa(\nu_{\hat{y}})(\sqrt[n]{\overline{u_2}}, \sqrt[n]{\overline{u_1}}) = index(D'(P) \otimes \kappa(P)(\sqrt[n]{u_2(P)}, \kappa(\nu_{\hat{y}})(\sqrt[n]{u_1(P)}))$ and $[\kappa(\nu_{\hat{y}})(\sqrt[n]{\overline{u_2}}, \sqrt[n]{\overline{u_1}}) : \kappa(\nu_{\hat{y}})(\sqrt[n]{\overline{u_2}})] = [\kappa(P)(\sqrt[n]{u_2(P)}, \sqrt[n]{u_1(P)}) : \kappa(P)(\sqrt[n]{u_2(P)})]$. Thus we have

$$index(D \otimes F_{\nu_{\hat{y}}}) = index(D' \otimes (u_1, x) \otimes F_{\nu_{\hat{y}}}(\sqrt[n]{u_2})) \cdot [F_{\nu_{\hat{y}}}(\sqrt[n]{u_2}) : F_{\nu_{\hat{y}}}]$$

$$= index(\overline{D'} \otimes (\overline{u_1}, \overline{x}) \otimes \kappa(\nu_{\hat{y}})(\sqrt[n]{\overline{u_2}})) \cdot [\kappa(\nu_{\hat{y}})(\sqrt[n]{\overline{u_2}}) : \kappa(\nu_{\hat{y}})] =$$

$$index(D'(P) \otimes \kappa(P)(\sqrt[n]{u_2(P)}, \sqrt[n]{u_1(P)}))\cdot$$

$$[\kappa(P)(\sqrt[n]{u_2(P)}, \sqrt[n]{u_1(P)}) : \kappa(P)(\sqrt[n]{u_2(P)})] \cdot [\kappa(P)(\sqrt[n]{u_2(P)}) : \kappa(P)]$$

$$= index(D'(P) \otimes \kappa(P)(\sqrt[n]{u_2(P)}, \sqrt[n]{u_1(P)})) \cdot [\kappa(P)(\sqrt[n]{u_2(P)}, \sqrt[n]{u_1(P)}) : \kappa(P)].$$

On the other hand we have $index(D \otimes F_P) = index(D' \otimes (u_1, x) \otimes (u_2, y))$ divides $index(D' \otimes F_P(\sqrt[n]{u_1}, \sqrt[n]{u_2})) \cdot [F_P(\sqrt[n]{u_1}, \sqrt[n]{u_2}) : F_P]$. Since F_P is the field of fractions of the two dimensional regular complete local ring $\hat{R}_P$ with u_1, u_2 units in $\hat{R}_P$, we have

$$index(D' \otimes F_P(\sqrt[n]{u_1}, \sqrt[n]{u_2})) \cdot [F_P(\sqrt[n]{u_1}, \sqrt[n]{u_2}) : F_P] =$$

$$index(D'(P) \otimes \kappa(P)(\sqrt[n]{u_2(P)}, \sqrt[n]{u_1(P)})) \cdot [\kappa(P)(\sqrt[n]{u_2(P)}, \sqrt[n]{u_1(P)}) : \kappa(P)].$$

We have

$$index(D'(P) \otimes \kappa(P)(\sqrt[n]{u_2(P)}, \sqrt[n]{u_1(P)})) \cdot [\kappa(P)(\sqrt[n]{u_2(P)}, \sqrt[n]{u_1(P)}) : \kappa(P)] =$$

$$index(D \otimes \hat{F}_P) \leq index(D \otimes F_P)$$

$$\leq D'(P) \otimes \kappa(P)(\sqrt[n]{u_2(P)}, \sqrt[n]{u_1(P)})) \cdot [\kappa(P)(\sqrt[n]{u_2(P)}, \sqrt[n]{u_1(P)}) : \kappa(P)].$$

Hence $index(D \otimes \hat{F}_P) = index(D \otimes F_P)$. Since $D \otimes F_P$ is a division algebra, it follows that $D \otimes \hat{F}_P$ is also a division algebra. Since $F_{\nu_y} \subset \hat{F}_P$, $D \otimes F_{\nu_y}$ is a division algebra.

Assume that $D \otimes F_P = D' \otimes (uy^r, x)$ for some unit $u \in R_P$, r coprime to n and D' unramified on R_P. Let $\nu_{\hat{x}}$ be the discrete valuation on F_P given by x. Let $\hat{F}_P$

be the completion of F_P with respect to the discrete valuation $\nu_{\hat{x}}$. By (2.1.1), we have $index(D \otimes \hat{F}_P) = index(D' \otimes \hat{F}_P(\sqrt[n]{uy^r})) \cdot [\hat{F}_P(\sqrt[n]{uy^r}) : \hat{F}_P]$. As before, it can be shown that $index(D' \otimes \hat{F}_P(\sqrt[n]{uy^r})) = index(D'(P))$ and $[\hat{F}_P(\sqrt[n]{uy^r}) : \hat{F}_P] = n$. Hence $index(D \otimes \hat{F}_P) = index(D'(P)) \cdot n$. On the other hand, we have $index(D \otimes F_P)$ divides $index(D(P)) \cdot n$. Hence, as above, we have $index(D \otimes \hat{F}_P) = index(D \otimes F_P)$. Since $D \otimes F_P$ is a division algebra, $D \otimes \hat{F}_P$ is a division algebra. Let ν_x be the discrete valuation on F given by the restriction of $\nu_{\hat{x}}$. Then ν_x is given by the prime ideal (x) of R_P. Since $F_{\nu_x} \subset \hat{F}_P$, $D \otimes F_{\nu_x}$ is a division algebra. □

Remark 2.2.2 Let K, F, and Ω be as above. Then for any $\nu \in \Omega$, the residue field $\kappa(\nu)$ is either a finite extension of K or a function field of a curve over a finite extension of k.

Chapter 3

Necessary conditions for Admissibility

3.1 The main Theorem

In this section, we give a necessary condition for a group to be admissible over function fields of curves over complete discretely valued fields.

We begin with the following

Lemma 3.1.1. Let K be a complete discretely valued field with residue filed k and P a p-group with $\mathrm{char}(k)$ coprime to $|P|$. If P is admissible over K, then P has a filtration $P \supseteq P_1 \supseteq P_2 \supseteq (e)$ such that

(1) P_1 is normal subgroup of P and P_2 is a normal subgroup of P_1

(2) P/P_1 and P_2 are cyclic

(3) P_1/P_2 is admissible over some finite extension of the residue field of K.

Proof Suppose that P is admissible over K. Then there exists a Galois extension L/K and a division ring D central over K which contains L as maximal subfield such that $P = Gal(L/K)$. Let L_0 be the maximal unramified extension of K contained in L. Let l be the residue field of L. Let $\partial(D)=(E_0, \sigma_0) \in H^1(k, \mathbb{Z}/n\mathbb{Z})$ be the residue of D and $(E, \sigma) \in H^1(K, \mathbb{Z}/n\mathbb{Z})$ be the lift of (E_0, σ_0) (cf.,§1.2). Assume that L and E are in the same algebraic closure of K. Let L_{nr} be the maximal unramified extension of L over K. Since E is unramified extension of K, we have $L \cap E = L_{nr} \cap E$. Let $F = L \cap E$. Since E/K is cyclic, F/K is also cyclic.

Let P_1 be the Galois group of L/F. Since F/K is cyclic, P_1 is a normal subgroup of P and P/P_1 is cyclic.

Let P_2 be the Galois group of L/L_{nr}. Then P_2 is a subgroup of P_1. Since L/L_{nr} is a totally ramified Galois extension, P_2 is cyclic ([ser79],Corollary III.5.3). Since L_{nr}/F is a Galois extension, P_2 is a normal subgroup of P_1. The residue field F_0 of F is same as the intersection of l and E_0 (the intersection is takes in an algebraic closure of k). We now show that P_1/P_2 is admissible over F_0.

Let D' be the commutant of F in D. Then D' is a division algebra with center F and L is a maximal subfield of D'. Then D' is equal to $D \otimes F$ in $Br(F)$. Let $\pi \in K$

be a parameter. Since F/K is unramified, π is also a parameter in F. Since σ is a generator of Gal(E/K) and $F \subset E$, $\sigma_F = \sigma^{[F:K]}$ is a generator of $Gal(E/F)$. Let D'' be a central division algebra over F which is Brauer equivalent to $D' \otimes (E/F, \sigma_F, \pi)^{op}$. By the functoriality of the residue map and by the choice of (E, σ_F), it follows that D'' is unramified on F. From the following commutative diagram

$$\begin{array}{ccc} Br(F) & \longrightarrow & H^1(F_0, \mathbb{Z}/n\mathbb{Z}) \ , \\ {\scriptstyle res}\downarrow & & \downarrow{\scriptstyle e.res} \\ Br(L) & \longrightarrow & H^1(l, \mathbb{Z}/n\mathbb{Z}) \end{array}$$

where $e = [L : L_{nr}]$, we conclude that $E_0^e \subseteq l$. In particular $E^e \subseteq F$ and $[E : F] \mid [L : L_{nr}$.

We have $[L : F] = degD' = ind(D'' \otimes (E/F, \sigma_F, \pi)) = ind(D'' \otimes E)[E : F]$ (by 2.1.1). Hence $ind(D'' \otimes E)[E : F] = [L : F] = [L : L_{nr}][L_{nr} : F]$. Since $[E : F] \mid [L : L_{nr}]$, $ind(D'' \otimes E) \geq [L_{nr} : F]$.

It is easy to see that $D'' \otimes E$ splits over LE. Since $D'' \otimes E$ is unramified at the discrete valuation of E and EL/EL_{nr} is totally ramified, $D'' \otimes E$ splits over EL_{nr}. Hence $ind(D'' \otimes E) \leq [EL_{nr} : E]$. Therefore $ind(D'' \otimes E) = [EL_{nr} : E]$.

Let D''' be a division algebra with center E which is Brauer equivalent to $D'' \otimes E$. Then EL_{nr} is a maximal subfield of D'''. Since D''' is unramified at the discrete valuation of E, let $\overline{D'''}$ be its image over the residue field E_0 of E. Since E is complete, $\overline{D'''}$ is central division algebra over E_0 and lE_0 is a maximal subfield of $\overline{D'''}$.

Since $Gal(lE_0/E_0) \simeq Gal(L_{nr}E/E) \simeq Gal(L_{nr}/F) \simeq P_1/P_2$, P_1/P_2 is admissible over E_0. □

The above lemma immediately gives the following

Proposition 3.1.2. Let K be a complete discretely valued field with residue filed k and G be a finite group such that char(k) is coprime to $|G|$. If G is admissible over K then every Sylow p-subgroup P of G has a filtration $P \supseteq P_1 \supseteq P_2 \supseteq (e)$ such that

(1) P_1 is normal subgroup of P and P_2 is a normal subgroup of P_1

(2) P/P_1 and P_2 are cyclic

(3) P_1/P_2 is admissible over some finite extension of the residue field of K.

Proof. Let G be an admissible group over K. Then there is a field extension L/K and a division algebra D central over K containing L as a maximal subfield with $Gal(L/K) = G$. Let P be a Sylow p-sub group of G. Let L^P be the fixed of P. Then L^P is a complete discretely valued field. Let D' be the commutant of L^P in D. Then D' is a central division algebra over L^P and $Gal(L/L^P) = P$ is admissible over L^P. Since L^P is also a complete discrete valued field, the result follows by (3.1.1). □

Corollary 3.1.3. Let K be a complete discretely valued field with residue filed k either a local field or a global field. Let G be a finite group such that char(k) coprime to $|G|$. If G is admissible over K then every Sylow p-subgroup P of G has a filtration

$P \supseteq P_1 \supseteq P_2 \supseteq (e)$ such that

(1) P_1 is normal subgroup of P and P_2 is a normal subgroup of P_1

(2) P/P_1 and P_2 are cyclic

(3) P_1/P_2 is meta cyclic.

Proof. Every finite extension of the residue field is either a local field or a global field. The corollary follows from (3.1.2) and (1.4.6). □

Let F be a field and Ω be a set of discrete valuations of K. Let n be a natural number. We say that (F, Ω) satisfies *Hasse principle for central division algebras of degree* n over F if for every central division algebra D over F of degree n, there exists a discrete valuation $\nu \in \Omega$ such that $D \otimes F_\nu$ is division.

Theorem 3.1.4. Let F be a field and n be a natural number. Suppose that there exists a set of discrete valuations Ω of F such that (F, Ω) satisfies Hasse principle for central division algebras over F of degree n. Assume that n is coprime to the characteristic of the residue fields at all discrete valuations in Ω. Let G be a finite group of order n. If G is admissible over F, then every Sylow p-sub group P of G has a filtration $P \supseteq P_1 \supseteq P_2 \supseteq (e)$ such that

(1) P_1 is normal subgroup of P and P_2 is a normal subgroup of P_1

(2) P/P_1 and P_2 are cyclic

(3) P_1/P_2 is admissible over some finite extension of the residue field at some $\nu \in \Omega$.

Proof Since G is admissible over F, we have a field extension L/F and a division algebra D central over F containing L as a maximal sub field such that $Gal(L/F) = G$. Since (F, Ω) satisfies Hasse principle for central division algebras of degree n over F, there exists a discrete valuation $\nu \in \Omega$ such that $D \otimes F_\nu$ remains division over F_ν. Since $L \otimes F_\nu$ is a maximal subfield of $D \otimes F_\nu$ and $G = Gal(L \otimes F_\nu / F_\nu)$, the result follows from (3.1.2). □

Theorem 3.1.5. Let K be a complete discretely valued field and F the function field of a curve over K. Let G be a finite group of order n such that the characteristic of the residue field of K is coprime to n. If G is admissible over F then every Sylow p-sub group P of G has a filtration $P \supseteq P_1 \supseteq P_2 \supseteq (e)$ such that

(1) P_1 is normal subgroup of P and P_2 is a normal subgroup of P_1

(2) P/P_1 and P_2 are cyclic

(3) P_1/P_2 is admissible over some finite extension of the residue field at a discrete valuation of F.

Proof Let R be the ring of integers in K. Let Ω be the set of all discrete valuations of F given by the codimension one points of regular proper schemes $\mathcal{X}$ over R with function field F. Then, by (Lemma 2.2.1), (F, Ω) satisfies Hasse principle for central division algebras over F of degree n. Thus the result follows from (3.1.4). $\square$

Let F and Ω be as in the above corollary. Then for any $\nu \in \Omega$, the residue field $\kappa(\nu)$ is either a finite extension of K or a function field of a curve over k (cf. 2.2.2).

The following is immediate from (3.1.5) and (3.1.3).

Corollary 3.1.6. Let K be a p-adic field and F the function field of a curve over K. Let n be a natural number which is coprime to p and G a finite group of order n. If G is admissible over F then every Sylow p-sub group P of G has a filtration $P \supseteq P_1 \supseteq P_2 \supseteq (e)$ such that

(1) P_1 is normal subgroup of P and P_2 is a normal subgroup of P_1

(2) P/P_1 and P_2 are cyclic

(3) P_1/P_2 meta cyclic. $\square$

The following is proved in ([HHK11],3.5).

Corollary 3.1.7. Let K be a complete discretely valued field with residue field algebraically closed. Let F be the function field of a curve over K. Let G be a finite group of order n such that the characteristic of the residue field of K is coprime to n. If G is admissible over F then every Sylow p-sub group P of G is meta cyclic.

Proof By (3.1.5), every P-Sylow subgroup of G has a filtration $P \supseteq P_1 \supseteq P_2 \supseteq (e)$ such that

(1) P_1 is normal subgroup of P and P_2 is a normal subgroup of P_1

(2) P/P_1 and P_2 are cyclic

(3) P_1/P_2 is admissible over some finite extension of the residue field at a discrete valuation of F.

Let k be the residue field of K. By (2.2.2) and the proof of (3.1.5), the residue field at the discrete valuation given in (3) is either a finite extension of K or the function field of a curve over k. Since k is algebraically closed, there are no non-trivial division algebras over such fields. Hence $P_1 = P_2$. □

3.2 Example of a group which is not $\mathbb{Q}_p(t)$-admissible

In this section, we give an example of a finite group which is not $\mathbb{Q}_p(t)$-admissible.

Example 3.2.1. Let ℓ and p be two distinct primes. Let $P = (\mathbb{Z}/l\mathbb{Z})^5$. We claim that this group is not admissible over $\mathbb{Q}_p(t)$. Suppose that P is admissible over $\mathbb{Q}_p(t)$. Then by (3.1.6), there is a filtration $P \supseteq P_1 \supseteq P_2 \supseteq (e)$ such that

(1) P_1 is normal subgroup of P and P_2 is a normal subgroup of P_1

(2) P/P_1 and P_2 are cyclic

(3) P_1/P_2 is meta cyclic

Since P_2 and P/P_1 are cyclic their orders will be at most l. This implies that $|P_1/P_2| \geq l^3$. Since every element of P_1/P_2 has order at most l and $|P_1/P_2| \geq l^3$, P_1/P_2 can not be meta cyclic. $\square$

3.3 Example of a admissible group over $\mathbb{Q}_p(t)$

We end this chapter with an example of a finite group isomorphic to at most product of 4 cyclic groups is $\mathbb{Q}_p(t)$-admissible. In next chapter, we prove that a finite group in which every Sylow subgroup is isomorphic to at most product of 4 cyclic groups is admissible over $\mathbb{Q}_p(t)$. We use patching techniques to prove that result. However the following example is an explicit construction without using the patching techniques. We begin with the following

Lemma 3.3.1. Let R be a complete discrete valuation ring and $\pi \in R$ a parameter. Let K be the field of fractions of R and k the residue filed of R. Let $F = K(t)$ be the rational function field in one variable over K. Let n be a natural number which is coprime to the char(k). Assume that K contains a primitive nm^{th} root of unity. Suppose that $\lambda_0 \in k^*$ is such that $[k(\sqrt[n]{\lambda_0}) : k] = n$. Then $(t, \pi - \lambda t)_n \otimes (t+1, \pi)_m$ is division over F for any $\lambda \in R$ which maps to λ_0.

Proof Since π is a parameter in R, the localisation $R[t]_{(\pi)}$ of $R[t]$ at the prime ideal (π) is a discrete valuation ring. Let ν be the discrete valuation on F given the discrete valuation ring $R[t]_{(\pi)}$ and F_ν be the completion of F at ν. Let w be the extension of ν to $F_\nu(\sqrt[m]{t+1})$. To show that $(t, \pi - \lambda t)_n \otimes (t+1, \pi)_m$ is division, it is enough to show that $(t, \pi - \lambda t)_n \otimes (t+1, \pi)_m \otimes F_\nu$ is division. Since F_ν is complete and $[F_v(\sqrt[m]{t+1}) : F_v] = m$, by (2.1.1), it is enough to show that $(t, \pi - \lambda t)_n \otimes F_v(\sqrt[m]{t+1})$

is division. Since t and $\pi - \lambda t$ are units at ν, the algebra $(t, \pi - \lambda t)_n \otimes F_v(\sqrt[m]{t+1})$ is unramified at w. Thus it is enough to show that its image $(t, \lambda_0)_n$ is division over the residue filed $k(t)(\sqrt[m]{t+1})$. Let γ be the discrete valuation on $k(t)$ given by t and $\tilde{\gamma}$ be the extension of γ to $k(t)(\sqrt[m]{t+1})$. Since $t+1$ is an m^{th} power in the completion of $k(t)$ at γ, the completion of $k(t)(\sqrt[m]{t+1})$ at $\tilde{\gamma}$ is $k((t))$. It is enough show that $(t, \lambda_0)_n$ is division over $k((t))$. Since $\lambda_0 \in k*$ is an element of order n, $(t, \lambda_0)_n$ is division over $k((t))$. $\square$

Now we are in a position to give an example of a admissible group over $\mathbb{Q}_p(t)$.

Example 3.3.2. Let l_1, l_2, l_3, l_4 be natural numbers which are coprime to a prime p. Let $G = \mathbb{Z}/l_1\mathbb{Z} \times \mathbb{Z}/l_2\mathbb{Z} \times \mathbb{Z}/l_3\mathbb{Z} \times \mathbb{Z}/l_4\mathbb{Z}$. Then G is admissible over $Q_p(t)$.

Let $\lambda \in \mathbb{Z}_p$ be such that its image in $F_p^*/F_p^{*l_1 l_2}$ is of order $l_1 l_2$. let $L_1 = \mathbb{Q}_p(t)(y, z)$ where $y^{l_1} = t$, $z^{l_2} = \pi - \lambda t$. Then $D_1 = (t, \pi - \lambda t)_{l_1 l_2}$ is division and contains L_1 as maximal subfield. Let $L_2 = \mathbb{Q}_p(t)(r, s)$ where $y^{l_3} = t+1$, $z^{l_4} = \pi$. Then $D_2 = (t+1, \pi)_{l_3 l_4}$ is division and contains L_2 as maximal subfield. Then by (3.3.1), $D_1 \otimes D_2$ is division. Therefore $L_1 \otimes L_2$ is a field and contained in $D_1 \otimes D_2$ as a maximal subfield. Since $Gal(L_1 \otimes L_2/\mathbb{Q}_p(t)) = G$, G is admissible over $\mathbb{Q}_p(t)$. $\square$

Chapter 4

A class of Admissible groups over $\mathbb{Q}_p(t)$

4.1 Introduction

Let K be a complete discretely valued field with residue field k. Let F be the function field of a curve over K. Let n be an integer which is coprime to the characteristic of k. Suppose that K contains a primitive n^{th} root of unity. Then in ([HHK11]) it is proved that every finite group of order n with every sylow subgroup abelian of rank at most 2 is admissible over F. They used the patching techniques to prove this result. In this section we prove a similar result for groups with every Sylow subgroup isomorphic to product of at most 4 cyclic groups with an additional assumption on the residue field k.

4.2 Admissible groups over $\mathbb{Q}_p(t)$

We begin with the following lemma.

Lemma 4.2.1. Let R be a regular local ring of dimension two with residue field k and field of fraction F. Let n_1 and n_2 be natural numbers which are coprime to the char(k). Assume that F contains a primitive $n_1n_2^{th}$ root of unity and there is an element in k^*/k^{*n_2} of order n_2. Then there is a central division algebra D over F of degree n_1n_2.

Proof. Since R is a regular local ring of dimension two, we have $m = (t, s)$. By the assumption on k, there is an element $\lambda_0 \in k^*$ such that its order in k^*/k^{*n_2} is n_2. Let $\lambda \in R$ which maps to λ_0. Let $a \in R$ be a unit with $a^{n_1} \neq 1$. Let ξ_1 be a primitive n_1^{th} root of unity and ξ_2 a primitive n_2^{th} root of unity.

Let

$$D_1 = (\frac{s}{s-t}, \frac{s-t^2}{s-a^{n_1}t^2})_{n_1}$$

and

$$D_2 = (\frac{s}{s-t^2}, \frac{s-\lambda t^2}{s-t^2})_{n_2}.$$

Let $D = D_1 \otimes_F D_2$. Then the degree of D is $n_1 n_2$. We now show that D is a division algebra.

Let $S = R[x]/(s - t^2 x)$. Then the field of fractions of S is isomorphic to F. We have

$$D_1 = (\frac{tx}{tx-1}, \frac{x-1}{x-a^{n_1}})_{n_1}$$

and

$$D_2 = (\frac{x}{x-1}, \frac{x-\lambda}{x-1})_{n_2}.$$

The ideal (t) of S is a prime ideal and gives a discrete valuation ν on F. Let $\hat{F}$ be the completion of F at ν. To show that $D_1 \otimes D_2$ is a division algebra, it is enough to show that $D_1 \otimes D_2 \otimes \hat{F}$ is a division algebra. By (2.1.1), we have

$$index(D_1 \otimes D_2 \otimes \hat{F}) = index(D_2 \otimes \hat{F}(\sqrt[n_1]{\frac{x-1}{x-a^{n_1}}}))[\hat{F}(\sqrt[n_1]{\frac{x-1}{x-a^{n_1}}}) : \hat{F}].$$

Since $\frac{x-1}{x-a^{n_1}}$ is a unit at ν and the residue field $\kappa(\nu)$ at ν is $k(x)$, we have $[\hat{F}(\sqrt[n_1]{\frac{x-1}{x-a^{n_1}}}) : \hat{F}] = n_1$. Since D_2 is unramified at ν, the index of $(D_2 \otimes \hat{F}(\sqrt[n_1]{\frac{x-1}{x-a^{n_1}}}))$ is equal to the index of its image $(\frac{x}{x-1}, \frac{x-\lambda}{x-1})_{n_2}$ over the residue field $k(\sqrt[n_1]{\frac{x-1}{x-a^{n_1}}})$. Let υ be the discrete valuation of $k(x)$ given by x. Then the residue of $(\frac{x}{x-1}, \frac{x-\lambda}{x-1})_{n_2}$ at υ is the image of $\frac{x-\lambda}{x-1}$ modulo x. Since the image of $\frac{x-\lambda}{x-1}$ modulo x is λ_0 and the order of λ_0 in k^*/k^{*n_2} is n_2, the index of D_2 is n_2. Hence the index of $(D_1 \otimes D_2 \otimes \hat{F})$ is $n_1 n_2$. Since the degree of $D_1 \otimes D_2$ is $n_1 n_2$, $D_1 \otimes D_2$ is a division algebra. □

Theorem 4.2.2. Let K be a complete discretely valued field with residue field k and F be the function field of a curve over K. Let n be an integer which is coprime to the characteristic of k. Suppose that K contains a primitive n^{th} root of unity. Assume that for every finite extension L of k, there is an element in L^*/L^{*n} of order n. If G is a group of order n with every sylow subgroup is isomorphic to product of at most 4 cyclic groups , then G is admissible over F.

Proof. Let R be the ring of integers in K. Let $\mathcal{X}$ be a regular proper two dimensional scheme over R with function field F and the reduced special fibre is a union of regular curves with normal crossings. Let $p_1, \cdots, p_r$ be the prime factors of n. Let $Q_1, \cdots, Q_r$ be regular closed points on the special fibre of $\mathcal{X}$. Let R_{Q_i} be the regular local ring at Q_i, $\hat{R}_{Q_i}$ be the completion of R_{Q_i} at the maximal ideal and F_{Q_i} the field of fractions of R_{Q_i}. Let $t_i \in R_{Q_i}$ be a prime defining the irreducible component of the special fibre of $\mathcal{X}$ containing Q_i. Let P_i be a p_i-sylow subgroup of G. By (1.4.6), it is enough to show that there exists a central division algebra D_i over F_{Q_i} and maximal subfield L_i of D_i with $Gal(L_i/F_{Q_i}) \simeq P_i$ and $L_i \otimes \hat{F}_{Q_i}$ a split algebra.

Since the residue field $\kappa(Q_i)$ of R_{Q_i} is a finite extension of k, by the assumption on k, there is an element in $\kappa(Q_i)^*/\kappa(Q_i)^{*n}$ of order n. Since P_i is isomorphic to product of at most 4 cyclic groups, $P_i \simeq C_{n_1} \times C_{n_2} \times C_{n_3} \times C_{n_4}$ with $|P_i| = n_1n_2n_3n_4$. Since $\hat{R}_{Q_i}$ is regular local ring of dimension 2 and t_i is a regular prime, we have $m_{Q_i} = (t_i, s_i)$. Let ξ_1 be a primitive ${n_1n_2}^{th}$ root of unity and ξ_2 a primitive ${n_3n_4}^{th}$ root of unity.

Let

$$D_1 = (\frac{s_i}{s_i - t_i}, \frac{s_i - t_i^2}{s_i - a_i^{n_1 n_1} t^2})_{n_1 n_2}$$

and

$$D_2 = (\frac{s_i}{s_i - t_i^2}, \frac{s_i - \lambda t_i^2}{s_i - t_i^2})_{n_3 n_4}.$$

for suitable a and λ as in (4.2.1). Let $D = D_1 \otimes_F D_2$. Then, by (4.2.1), D is a division algebra.

In particular D_1 and D_2 are division algebras. The cyclic algebra D_1 is generated by x_1 and y_1 with relations

$$x_1^{n_1 n_2} = \frac{s_i}{s_i - t_i},\ y_1^{n_1 n_2} = \frac{s_i - t_i^2}{s_i - a^{n_1 n_2} t_i^2} \text{ and } x_1 y_1 = \xi_1 y_1 x_1.$$

Similarly D_2 is generated by x_2 and y_2 with relations

$$x_2^{n_3 n_4} = \frac{s_i}{s_i - t_i^2},\ y_2^{n_3 n_4} = \frac{s_i - \lambda t_i^2}{s_i - t_i^2} \text{ and } x_2 y_2 = \xi_2 y_2 x_2.$$

Let L_1 be the sub algebra of D_1 generated by $x_1^{n_1}$ and $y_1^{n_2}$. Let L_2 be the sub algebra of D_2 generated by $x_2^{n_3}$ and $y_2^{n_4}$. Then $L = L_1 \otimes L_2$ is a maximal subfield of $D_1 \otimes D_2$, $Gal(L/F) = C_{n_1} \times C_{n_2} \times C_{n_3} \times C_{n_4}$ and $L \otimes \hat{F}$ is a split algebra. □

Theorem 4.2.3. Let K be a local field and k its residue field. Let F be the function field of a curve over K. Let n be an integer which is coprime to the characteristic of k. Suppose that K contains a primitive n^{th} root of unity. If G is a group of order n with every Sylow subgroup is abelian of rank at most 4, then G is admissible over F.

Proof. Since k is a finite field every finite extension of k is a finite field. For any finite field L and for any natural number n coprime to the characteristic of L, we have L^*/L^{*n} is cyclic group of order n. Hence the result follows by (4.2.2). $\square$.

Corollary 4.2.4. Let K be a local field and k its residue field. Let F be the function field of a curve over K. Let $n = p_1^{d_1} \cdots p_r^{d_r}$ with $1 \leq d_i \leq 2$ and p_i distinct primes. Assume that K contains a primitive n^{th} root of unity and n is coprime to the characteristic of k. If G is a group of order n, then G is admissible over F.

Corollary 4.2.5. Let K be a local field and k its residue field. Let F be the function field of a curve over K. Let $n = p_1^{d_1} \cdots p_r^{d_r}$ with $1 \leq d_i \leq 4$ and p_i distinct primes. Assume that K contains a primitive n^{th} root of unity and n is coprime to the characteristic of k. If G is an abelian group of order n, then G is admissible over F.

Corollary 4.2.6. Let K be a local field and k its residue field. Let F be the function field of a curve over K. Let G be an abelian group of order n. Assume that K contains a primitive n^{th} root of unity and n is coprime to the characteristic of k. Then G is admissible over F if and only if G is isomorphic to a product of at most four cyclic groups. $\square$

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