SOME COMBINATORIAL RESULTS IN TOPOLOGICAL DYNAMICS

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CERTIFICATE

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This is to certify that I, I. Subramania Pillai, have carried out the research embodied in the present thesis entitled **SOME COMBINATORIAL RESULTS IN TOPOLOGICAL DYNAMICS** for the full period prescribed under Ph.D. ordinance of the University.

I declare that, to the best of my knowledge, no part of this thesis was earlier submitted for the award of research degree of any university.

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Dedicated to my beloved grandmother

Chenbhaga Kutti Ammal

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- Subramania Pillai. I

Contents

List of Symbols

$\mathbb{R}$	The set of all real numbers
$\mathbb{Q}$	The set of all rational numbers
$\mathbb{N}$	The set of all natural numbers
$\mathbb{Q}_1$	The set of all rational numbers in [0,1)
I	A compact subinterval of $\mathbb{R}$
A^c	The complement of the set A
$P(f)$	The set of all periodic points of f
$Per(f)$	$\{n \in \mathbb{N}$ such that f has a point of period $n\ \}$
$PER(X)$	$\{A \subset X \mid A = Per(f)$ for some continuous self map f of $X\}$
$Det(A)$	The determinant of A
p_n	$\lvert Det(A^n - I)\rvert$
$\triangle$	Symmetric difference of sets.
q_n	$\lvert Trace(A^n)\rvert$
$N(f)$	The set of all nonordinary points of f
$S(f)$	The set of all special points of f
G_f	The set of all self-conjugacies of f
$G_{f\uparrow}$	The set of all increasing self-conjugacies of f
$\overleftarrow{P(f)}$	The set of preperiodic/eventually periodic points of f
$\overleftarrow{C(f)}$	The set of precritical points of f
$W^s(x,f)$ or $W^s(x)$	The basin of attraction of f at x
$\mathbb{T}^2$	The two dimensional torus
T_A	The toral automorphism induced by the matrix A
$GL(2,\mathbb{Z})$	$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z} \text{ and } ad-bc=\pm 1\right\}$

Abstract

The main results of this thesis are combinatorial in nature. We will be mainly working with the continuous automorphisms on the torus $\mathbb{T}^2$ and with the continuous self maps on $\mathbb{R}$. The thesis is conveniently divided into five chapters.

The general mathematical setting is that of an abstract dynamical system with discrete time parameter, that is, a pair (X, f) where X is a topological space and f a continuous mapping of X into itself. We are interested in the action of the iterates of f on X.

Chapter-1 is introductory in nature. We explain the basic notions of discrete dynamical systems and some important results emphasizing the role of the set of periodic points and the set of periods in chaos. We discuss briefly about the definitions of chaos due to Devaney and Li-Yorke.

It is already known [21] that for a hyperbolic (having no eigen values on the unit circle) continuous toral automorphism, the periodic points are precisely the rational points. In chapter-2, we calculate the set of periodic points for other continuous toral automorphisms; this happens to be the subgroup generated by $\mathbb{Q} \times \mathbb{Q} \cup$ (a line with rational slope). In fact, for all non-hyperbolic continuous toral automorphisms, there are uncountably many periodic points. Also, we prove that: every such subgroup is the set of periodic points for some continuous toral automorphism.

In Chapter-3, we first discuss some well known results about $Per(f)$ due to Sharkovski [17], Baker [5] and many others; our main result of this chapter is similar in spirit to these. We prove that there are exactly 8 subsets of $\mathbb{N}$ which can occur as $Per(T)$ for some continuous toral automorphism T. We solve the problem separately for hyperbolic and nonhyperbolic automorphisms. It is interesting that, for nonhyperbolic toral automorphisms, we are able to list the set $Per(T)$ in terms of the minimal polynomial

of T.

In chapter-4, we introduce the notion of *special points* and *nonordinary points* of a dynamical system. These notions are new to the literature, though they arise very naturally. By a special point we mean a point in the system which is unique by possessing some dynamical property. We call a point to be *ordinary* if it is “like” points near it. The points which are not ordinary are called *nonordinary*. It is observed that for systems with finitely many nonordinary points, the idea of nonordinary points and the idea of special points coincide.

We prove in [25] that the special points are actually inside the closure of full orbits of periodic points, critical points and possibly the limits at infinity. We call a system $(\mathbb{R}, f)$ to be *simple* if there are only finitely many kinds of orbits (upto order conjugacy). We describe completely, a class of simple systems namely, homeomorphisms with finitely many nonordinary points and give a general formula for counting. Also we prove that there are exactly 26 continuous self maps on $\mathbb{R}$ with a unique nonordinary point.

The main result of chapter-5 is obtained while making an attempt to answer the question: Exactly which maps on $\mathbb{R}$ are topologically conjugate to a polynomial?. The main theorem of this chapter, gives a necessary condition for a continuous self map on $\mathbb{R}$ to be conjugate to a polynomial.

Chapter 1

Periodic points, Periods and Conjugacy

1.1 General introduction to discrete dynamical systems

1.1.1 Definition and Examples

A *dynamical system* is simply a pair (X, f), where X is a topological space and $f : X \to X$ is a continuous self map of X. The space X can be thought of as the underlying set on which the motion takes place and f can be thought of as the rule according to which motion takes place. For $x \in X$ the point $f(x) \in X$ is thought of as the position to which x moves (in one unit of time). The composition of f with itself, denoted by $f \circ f$ is also a continuous self map of X. For $x \in X$, the element $f(f(x))$ is called the position to which x moves after two instants of time in the dynamical system (X, f).

If n is a positive integer, the element $f^n(x)$ [where f^n denotes the composition of f with itself $n-1$ times] is thought of as the position to which x moves after n instants of time. Thus in our study "time" is "discrete" and parametrised by the set $\mathbb{N}$ of natural numbers. (The space X need not be discrete.) This is the reason that it is called a " *discrete* " *dynamical system* .

Example 1.1.1. (Translation)

Let $f(x) = x + 5$ for all $x \in \mathbb{R}$. Then $(\mathbb{R}, f)$ is a dynamical system. This represents motion on a straight line with constant velocity. [The distance between $f^n(x)$ and $f^{n+1}(x)$ does not depend on n]. The element $\frac{1}{2}$ after two instants of time reaches the position $10\frac{1}{2}$. No element is fixed.

Example 1.1.2. (Rotation)

Let S^1 be the unit circle in the complex plane. i.e., $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Let $a \in S^1$ be fixed. Let $\rho_a : S^1 \to S^1$ be defined by $\rho(z) = az$ for all $z \in S^1$. Such maps ρ_a are called *rotations*. They describe motions on the circle with constant velocity. When $a \neq 1$, there are no fixed points.

Example 1.1.3. (Logistic function)

Let $r > 0$. The logistic function h_r on $[0, 1]$ is defined by the formula $h_r(x) = rx(1 - x)$. One can prove that when $0 < r \leq 4$, h_r takes $[0, 1]$ into $[0, 1]$ and hence $([0, 1], h_r)$ is a dynamical system. When $r = 4$, the point $\frac{1}{2}$ goes to 0 after two instants of time. There are two fixed points 0 and 1. But when $r > 4$, $([0, 1], h_r)$ is not a dynamical system.

Definition 1.1.4. Let (X, f) be a dynamical system. For each positive integer n, we define the function f^n now. It is recursively defined by $f^1 = f$ and $f^{n+1} = f \circ f^n$ for all $n \in \mathbb{N}$. We also use the convention $f^0 =$ the identity function. [It is easily seen that $f^m \circ f^n = f^{m+n}$ holds for all nonnegative integers m, n.]

1.1.2 Orbits, periodic points

Let (X, f) be a dynamical system and let $x \in X$. Then the sequence $x, f(x), f^2(x), ...$ is called the *trajectory* of x and the set of all points in the trajectory of x is called the *orbit* of x.

In dynamical systems, we are normally interested in studying the nature of the orbits of distinct points of the system.

Suppose for some $x \in X$, sequence $x, f(x), f^2(x), ...$ converges to some point say $x_0 \in X$, then we must have $f(x_0) = x_0$, because f is continuous. Such points we call as *fixed points*. In dynamics we say that the point x is attracted by the fixed point x_0. The set of all points in X *attracted* by x_0 is called the *stable set* or the *basin of attraction* of the fixed point x_0 and is denoted by $W^s(x_0, f)$ or simply $W^s(x_0)$. A fixed point x_0 is said to be *attracting* if its stable set is a neighbourhood of it.

Example 1.1.5. Consider the map $f(x) = x^2 : \mathbb{R} \to \mathbb{R}$. The fixed points are 0 and 1. We can easily see that, if $-1 < x < 1$, then the sequence $(f^n(x))_{n=1}^{\infty}$ converges to 0. Hence 0 is an attracting fixed point. Also, if $|x| > 1$ then $(f^n(x))_{n=1}^{\infty} \to \infty$. The point -1 is mapped to 1 and then fixed, such points we call eventually fixed.

Definition 1.1.6. The points which will reach a fixed point after finitely many iterations are called *eventually fixed* .

Definition 1.1.7. Let (X, f) be a dynamical system. A subset A of X is said to be *invariant* if $f(A) \subset A$. When A is an invariant set , $(A, f|_A)$ also becomes a dynamical system.

Example 1.1.8. 1. The set of all rational numbers in $[0, 1]$ is an invariant set in the system $([0, 1], h_r)$ when r is rational. This is because $rx(1 - x)$ is rational whenever x is rational.

2. $\mathbb{Z}$ is invariant under $(\mathbb{R}, f)$, where $f(x) = x + 5$ for all $x \in \mathbb{R}$.

3. In the system $(\mathbb{C}, f)$ where $f(z) = z^2$ for all $z \in \mathbb{C}$, the singleton set $\{1\}$ and the set $\{z \in \mathbb{C} : |z| \text{ is an integer } \}$ are examples of invariant sets.

Definition 1.1.9. Let (X, f) be a dynamical system. A point $x \in X$ is said to be *periodic* if there exists a positive integer $n \in \mathbb{N}$ such that $f^n(x) = x$. Least such n is called the *period* of x and the set $\{x, f(x), f^2(x), \ldots, f^{n-1}(x)\}$ is called an n*-cycle*. The set of all periodic points of f is denoted by $P(f)$.

The points which will reach a periodic point after finitely many iterations are called *eventually periodic* .

Let $Per(f)= \{n \in \mathbb{N} \text{ such that } f \text{ has a point of period } n \}$ and we call this as the *set of periods* of f or simply the *period set* of f.

Let X be a topological space and let $A \subset \mathbb{N}$. We write $A \in PER(X)$ if there exists a continuous map $f : X \to X$ such that $Per(f) = A$.

Example 1.1.10. 1. When f is the identity map or a constant map, all points are mapped to fixed points and hence $Per(f) = \{1\}$.

2. When f is the reflection map $x \mapsto -x$ on $\mathbb{R}$, the point 0 is fixed and all other points are periodic with period 2. Hence $Per(f) = \{1, 2\}$.

3. When f is the translation map $x \mapsto x + 1$ on $\mathbb{R}$, then $Per(f)$ is the empty set, since every orbit is strictly monotone.

4. On the unit circle, if we consider the rotation by the angle $\frac{2\pi}{3}$, all points are periodic with period 3. In this case $Per(f) = \{3\}$.

5. For the shift map(defined at the last section of this chapter) on $\sum_2 = \{0, 1\}^{\mathbb{N}}$ the periodic points are of the form $x = \overline{w} = www\ldots$ for some word w over $\{0, 1\}$ and $Per(f) = \mathbb{N}$.

We now in the following proposition state ten important simple results that are easy to prove.

Proposition 1.1.11. *Let (X, f) be a dynamical system, where X is a Hausdorff space. Then the following hold:*

1. The set of all fixed points is a closed subset of X.

2. In any trajectory, either all terms are distinct, or only finitely many terms are distinct.

3. Orbits of any two periodic points are either identical or disjoint.

4. If a trajectory converges, it converges to a fixed point.

5. An element is eventually periodic if and only if it has a finite orbit.

6. Every orbit is an invariant set; the orbits of periodic points are minimal invariant sets.

7. A subset of X is invariant if and only if it is a union of orbits.

8. The closure of an invariant set is also invariant.

9.The set of all periodic points is an invariant set.

10. For each subset A of X, the set $\cup_{n=0}^{\infty} f^n(A)$ is the smallest invariant set containing A.

1.2 The role of $Per(f)$ in Chaos

A question: *Can we find f continuous from $\mathbb{R}$ to $\mathbb{R}$ such that $Per(f) = A$, where A is $\{1, 2, 3\}$ or where A is $\{1, 2, 4\}$?*

The answer turns out to be 'NO' for the former and 'YES' for the latter. This is because of the following theorem:

Theorem 1.2.1. ***(Li and Yorke)***

Let f be continuous from $\mathbb{R}$ to $\mathbb{R}$. If $3 \in Per(f)$, then $Per(f) = \mathbb{N}$. [In other words , if there is a $3 - cycle$ then there is an $n - cycle$ for all n.]

This theorem is hard to prove. But here is an easy observation. If in the dynamical system $(\mathbb{R}, f)$, two points x and y move in the opposite directions, then there should be a fixed point between them.

More precisely: If $x < f(x)$ and if $f(y) < y$, then there exists z between x and y such that $f(z) = z$. This is proved by applying intermediate value theorem to the function $f(x) - x$. This implies that if $1 \notin Per(f)$, then the motion is uni-directional and so no point can be periodic. In other words, $Per(f) \neq \emptyset \implies 1 \in Per(f)$.

This elementary result, in combination with the result of Li and Yorke, exhibit the numbers 3 and 1 in the two extremes of an order. If a $3 - cycle$ is there, all $n - cycles$ have to be there. If no $1 - cycle$ is there then no $n - cycles$ can be there.

This leads to a search of pairs of positive integers (m, n) such that if an $m - cycle$ there, $n - cycle$ has to be there. What are all such pairs?

Sharkovskii's theorem provides a complete answer to this question. We discuss this in Chapter-3 in detail.

We now, in the next section, explain the importance of the set of periodic points in the theory of chaos.

1.2.1 Chaos

The expression " chaos" became popularized through the paper of Li and Yorke [16], "Period three implies chaos".

Chaotic systems share the property of having a high degree of sensitivity to initial conditions. In other words, a very small change in initial values will multiply in such a way that the new computed system bears no resemblance to the one predicted.

Definition 1.2.2. We say that the system (X, f) (where X is a metric space with metric d) is *sensitive to the initial conditions* if there exists $\delta > 0$ such that for any $x \in X$ and for any $\epsilon > 0$ there exists a point $y \in X$ with $d(x, y) < \delta$ and $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) > \delta$. This $\delta > 0$ is called a *sensitive constant* .

Another important property of dynamical systems called transitivity roughly says that, the iterates of any nonempty open set are well spread throughout the space. Said precisely,

Definition 1.2.3. A dynamical system (X, f) is said to be *transitive* if for any two nonempty open sets U and V in X there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. We say that f is *totally transitive* if f^n is transitive for all $n \geq 1$.

It has been recognized by Sharkovsky [17], Li and Yorke[16] and many others that there is a hidden, self-organizing order in chaotic systems. A certain degree of order in chaotic systems has led to various definitions of chaos in the literature.

1.2.2 Devaney's definition of chaos

Definition 1.2.4. According to Devaney a dynamical system (X, f) is said to be *chaotic* if

1. f is transitive.
2. f has a dense set of periodic points.
3. f is sensitive to the initial conditions.

Remark 1.2.5. In [23], Banks et al showed that, when X is infinite, the conditions (1) and (2) in the above definition imply condition (3) of sensitive dependence on initial conditions. However, no other two conditions imply the third. Examples can be seen in [18]

Theorem 1.2.6. *(Banks et al)*

Let $f : X \to X$ be a continuous map on an infinite compact metric space (X, d). If f is transitive and its set of periodic points is dense, then f possesses sensitive dependence on initial conditions, i.e., f is chaotic.

Remark 1.2.7. In fact, for continuous maps on intervals in $\mathbb{R}$, transitivity implies that the set of periodic points is dense (See [19]). Hence it follows from the above theorem that in this case *transitivity implies chaos.*

The following results describe the set $Per(f)$ for the class of transitive interval maps.

Theorem 1.2.8. *[15] A subset S of $\mathbb{N}$ occurs as $Per(f)$ for some transitive map f on $[0, 1]$ if and only if it satisfies the following two conditions:*

(1) $n \in S,\ n \neq 1 \Rightarrow\ n + 2 \in S$

(2) 1 and 2 are in S.

This theorem can be deduced by combining Sarkovskii's theorem (See chapter-3) with the following two results:

Theorem 1.2.9. *[15]*

Every transitive map on $[0, 1]$ has a periodic point of period 6.

Theorem 1.2.10. *[15], [22]*

Given any odd integer $k > 4$, there exists a transitive map f on $[0, 1]$ such that $k \in Per(f)$ but $k - 2 \notin Per(f)$.

The following two theorems are about total transitivity.

Theorem 1.2.11. *[30] A subset S of $\mathbb{N}$ arises as $Per(f)$ for a totally transitive map on $[0,1]$ if and only if $S = \mathbb{N}$ or $\mathbb{N} - \{3, 5, \dots, 2n+1\}$ for some $n \in \mathbb{N}$.*

Theorem 1.2.12. *[30] Let f be transitive on $[0,1]$. Then f is totally transitive if and only if $Per(f)$ has a finite complement in $\mathbb{N}$.*

1.2.3 Li- Yorke Chaos

In a dynamical system (X, f), let $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ be the respective orbits of two distinct points $x_0, y_0 \in X$. We ask two questions:

Q1: Do x_n and y_n come arbitrarily close to each other?

Q2: Do x_n and y_n keep a minimum positive distance from each other for infinitely many n?

Note that, by the nature of the questions, the two answers simultaneously cannot be ‘No’. So the respective answers can be (Yes, No), (No, Yes) or (Yes, Yes).

Example 1.2.13. If f is an isometry like rotation of the unit circle or if f is a contraction map like $x \mapsto \frac{x}{2}$ on $[0,1]$, then for any two points x_0, y_0 the answer pair is (No, Yes).

Example 1.2.14. If x_0 and y_0 are two distinct periodic points in a dynamical system, then the answer pair is (No, Yes).

Example 1.2.15. For the function $x \mapsto \sin(\frac{\pi x}{2})$ on $[-1,1]$, -1,0,1 are fixed points, $\sin(\frac{\pi x}{2}) > x$ for $0 < x < 1$ and $\sin(\frac{\pi x}{2}) < x$ for $-1 < x < 0$. So the orbit of x converges to 1 for $0 < x < 1$ and the orbit of x converges to -1 for $-1 < x < 0$. Therefore for two distinct points $x_0, y_0 \in [-1,1]$, the answer pair is (Yes, No) if $x_0 y_0 > 0$, and the answer pair is (No, Yes) if $x_0 y_0 \leq 0$.

Example 1.2.16. Let x_0 be a fixed point and y_0 be a point with dense orbit in an infinite dynamical system (X, f). Then the answer pair is (Yes, Yes).

Remark 1.2.17. For any two points in a dynamically simple system, the answer pair will be either (Yes, No) or (No, Yes). But an (Yes, Yes) answer pair indicates some complexity in the relative behavior of the two orbits. In the study of *scrambled sets* we are interested in pairs of points for which both the two questions have affirmative answers.

Definition 1.2.18. Let (X, f) be a dynamical system and let $S \subset X$ be a set with at least two points. Then, S is a *scrambled set* for f if for any two distinct points $x, y \in S$

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$$

and

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$$

From the famous paper of Li and Yorke [16] there ensued a definition of chaos:

Definition 1.2.19. A continuous self map of the interval is said to be Li-Yorke Chaotic if it has an uncountable scrambled set.

Li and Yorke proved that the occurrence of a period-3 point forces chaos.

Theorem 1.2.20. *If an interval map has a point of period three, then it is Li-Yorke chaotic.*

Here is a theorem connecting the set $Per(f)$ and Li-Yorke chaos.

Theorem 1.2.21. *[30] Let f be a self map of an interval I in $\mathbb{R}$. Then*

(i) If $Per(f)$ properly contains the set $\{1, 2, 2^2, 2^3, \ldots\}$, then f is Li-Yorke chaotic.

(ii) If $Per(f)$ is properly contained in $\{1, 2, 2^2, 2^3, \ldots\}$, then f is not Li-Yorke chaotic.

Corollary 1.2.22. *For any interval map f, Devaney's chaos implies Li-Yorke chaos. (we use theorem 1.2.9)*

Remark 1.2.23. We cannot say anything if $Per(f) = \{1, 2, 2^2, 2^3, \ldots\}$. In [20] it is shown that there are continuous interval maps f, g such that $Per(f) = Per(g) = \{1, 2, 2^2, 2^3, \ldots\}$, f is Li-Yorke chaotic but g is not Li-Yorke chaotic.

1.3 Topological Conjugacies

In order to classify dynamical systems we need a notion of equivalence. The notion of topological conjugacy in dynamical systems is analogous to the notion of isomorphism among groups and to "homeomorphisms" among topological spaces. i.e, We say that two dynamical systems are " having the same dynamical properties" or " dynamically same" if they are topologically conjugate.

Roughly speaking, by saying (X, f) and (Y, g) are topologically conjugate, we mean:

(1) X and Y have the same kind of topology.

(2) f and g have the same kind of dynamics.

Definition 1.3.1. Two dynamical systems (X, f) and (Y, g) are said to be *topologically conjugate* (or simply *conjugate*) if there exists a homeomorphism $h : X \to Y$(called topological conjugacy) such that $f \circ h = h \circ g$. we say simply, f is conjugate to g, and we write it as $f \sim g$. The case when h happens to be an increasing homeomorphism (For example, when $X = \mathbb{R}$ or an interval) we say that f and g are *increasingly conjugate* or *order conjugate* .

Remark 1.3.2. When $Y = X$ and $g = f$, we say that h is a self-conjugacy of f. Being conjugate(and as well as increasingly conjugate) is an equivalence relation among dynamical systems.

Let (X, f) and (Y, g) be two dynamical systems. Then a topological conjugacy from f to g carries orbits of f to "similar " g-orbits. Said precisely,

Theorem 1.3.3. *[13]*

Let (X, f) and (Y, g) be two dynamical systems and let $h : X \to Y$ be a topological conjugacy. Then

1. $h^{-1} : Y \to X$ *is a topological conjugacy.*

2. $h \circ f^n = g^n \circ h$ *for all* $n \in \mathbb{N}$.

3. $x \in X$ *is a periodic point of* f *if and only if* $h(x)$ *is a periodic point of* g.

4. *If* x *is a periodic point of* f *with stable set* $W^s(x)$, *then the stable set of* $h(x)$ *is* $h(W^s(x))$.

5. *The periodic points of* f *are dense in* X *if and only if the periodic points of* g *are dense in* Y.

6. f *is topologically transitive on* X *if and only if* g *is topologically transitive on* Y.

7. f *is chaotic on* X *if and only if* g *is chaotic on* Y.

1.3.1 Dynamical properties

Definition 1.3.4. Properties preserved by topological conjugacy are called *dynamical properties* . Many kinds of examples are provided below.

Example 1.3.5. (Dynamical properties of a point)

(a) Fixed point

(b) periodic point

(c) periodic point of period n_0

(d) Eventually fixed point

(e) Eventually periodic point

(f) Point whose orbit has exactly n elements.

Example 1.3.6. (Dynamical properties of subsets)

(a) Invariant subset

(b) The property that $f(A) = A$

(c) The property that $f^{-1}(A) = A$

(d) Dense set

(e) Having a unique limit point

(f) Finite set.

Example 1.3.7. (Dynamical properties dynamical systems)

(a) Having no fixed point

(b) Having no invariant sets, except the whole set and the empty set

(c) Having exactly n periodic points

(d) Having every point periodic

(e) Surjectivity

(f) Injectivity

(g) Having dense range.

(h) Transitivity.

1.3.2 The shift map - An example

Let $\sum_2 = \{(s_0, s_1, s_2, ...)|\ s_i = 0 \text{ or } 1 \text{ for all } i\}$ be the set of all sequences of 0's and 1's. If $s = s_0, s_1, s_2, ...$ and $t = t_0, t_1, t_2, ...$, then define $d(s,t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$. Note that $(\sum_2, d)$ is a metric space and $d(s,t) \leq 2$ for all $s, t \in \sum_2$.

The following proposition which can be proved easily , asserts that, if we start with any sequence from $\sum_2$, by keep on changing the initial terms we can go arbitrarily close to any point in the space $\sum_2$.

Proposition 1.3.8. *Let $s, t \in \sum_2$. Then*

1. *If the first $n+1$ digits in s and t are identical, then $d(s,t) \leq \frac{1}{2^n}$.*
2. *If $d(s,t) \leq \frac{1}{2^n}$, then the first n digits in s and t are identical.*

Definition 1.3.9. The shift map $\sigma : \sum_2 \to \sum_2$ is defined by,

$$\sigma(s_0 s_1 s_2 ....) = s_1 s_2 s_3 \ldots .$$

In other words, the shift map forgets the first term.

It follows from the above proposition that the shift map is continuous.

Theorem 1.3.10. *[13]*

The shift map has the following properties.

1. *The set of periodic points of the shift map is dense in $\sum_2$.*
2. *The shift map has 2^n periodic points whose period divides n.*
3. *The set of eventually periodic points of the shift map that are not periodic is dense in $\sum_2$.*
4. *There is an element of $\sum_2$ whose orbit is dense in $\sum_2$.*
5. *The set of points that are neither periodic nor eventually periodic is dense in $\sum_2$.*

Proof. 1. Suppose $s = s_0 s_1 s_2 ....$ is a periodic point of σ with period k. Then $\sigma^k(s) = s$. That is $s_k s_{k+1} s_{k+3} ... = s_0 s_1 s_2 ....$ This implies that $s_{n+k} = s_n$ for all n. That is s is a periodic point with period k if and only if s is a sequence formed by repeating the k-digits $s_0 s_1 s_2 ... s_{k-1}$ infinitely often.

To prove that the periodic points of σ are dense in $\sum_2$, we must show that for all points $t \in \sum_2$ and all $\epsilon > 0$, there is a periodic point s of σ such that $d(t, s) < \epsilon$. For this , if $t = t_0t_1t_2...$ then we choose n such that $\frac{1}{2^n} < \epsilon$, then we can let $s = t_0t_1t_2...t_nt_0t_1t_2...t_nt_0t_1t_2...t_n....$. As t and s agree on the first $n + 1$ digits, by previous proposition 1.3.8, we get $d(s, t) \leq \frac{1}{2^n} < \epsilon$

Proof of 2 and 3 are similar to that of 1.

4. The sequence which begins with 0 1 00 01 10 11 and then includes all possible blocks of 0 and 1 with three digits, followed by all possible blocks of 0 and 1 with four digits, and so forth - called the *Morse sequnce* - has dense orbit.

5. Since the set of nonperiodic points includes as a subset the orbit of the Morse sequence, proof follows from (4).

□

Theorem 1.3.11. *Let s be any point in $\sum_2$ and $\epsilon > 0$. Then there is $t \in \sum_2$ and n_0 such that $d(s, t) < \epsilon$ and $d(\sigma^n(s), \sigma^n(t)) = 2$ for all $n \geq n_0$.*

Proof. Choose $n_0 \in \mathbb{N}$ such that $\frac{1}{2^{n_0}} < \epsilon$. We then choose t such that,

$$t_i = \begin{cases} s_i & \text{if } \ i \leq n_0 \\ (s_i + 1) \ mod \ 1 & \text{if } i > n_0 \end{cases}$$

That is $s_i = t_i$ if and only if $i \leq n_0$. Then for any $n \geq n_0$, $\sigma^n(s)$ and $\sigma^n(t)$ differ on every digit. Hence $d(\sigma^n(s), \sigma^n(t)) = 2$.

□

Theorem 1.3.12. *The shift map σ is chaotic on $\sum_2$.*

Remark 1.3.13. To study a particular dynamical system, we often look for a conjugacy with a better-understood model. We illustrate this with the following example.

Example 1.3.14. Let K denote the Cantor middle third set (with 0 and 1 are identified). Let $f : K \to K$ be defined by $f(x) = 3x(mod 1)$. Then f is topologically conjugate to the shift map.

For, every number $a \in K$ can be written uniquely in the form $a = \frac{a_0}{3} + \frac{a_1}{3^2} + \frac{a_2}{3^3} + \ldots + \frac{a_n}{3^{n+1}} + \ldots$ where each $a_n \in \{0, 2\}$. [This is called the ternary expansion of a.] To this element a, we associate the sequence $s_0 s_1 s_2 \ldots \in \sum_2$ where $s_n = \frac{a_n}{2}$ for all n. This defines a map T from K to $\sum_2$. One can prove that this map T is, in fact a topological conjugacy from f to the shift map σ. Hence in the view of theorem 1.3.3 all the properties of the shift map specified in theorem 1.3.10 are true for the map f and in particular, f is chaotic.

1.4 Organization of the Thesis / Synopsis

The main results of chapter 2 and 3 are about the continuous automorphisms of the torus. The torus $\mathbb{T}^2 = \frac{\mathbb{R}^2}{\mathbb{Z}^2}$ is here viewed as the topological group $[0, 1) \times [0, 1)$ with coordinate wise addition modulo 1.

For the well known systems like, tent map $f : [0, 1] \to [0, 1]$ defined by $f(x) = 1 - |1 - 2x|$, and for shift map on $\{0, 1\}^{\mathbb{N}}$ the set of periodic points is well studied (See [21]). It is already known [21] that for a hyperbolic(having no eigen values with absolute value 1) continuous toral automorphism, the periodic points are precisely the rational points.

In **chapter-2**, we calculate the set of periodic points for other continuous toral automorphisms; this happens to be the subgroup generated by $\mathbb{Q} \times \mathbb{Q} \cup$ (a line with rational slope).

In fact, for all non-hyperbolic continuous toral automorphism, there are uncountably many periodic points.

Notation 1.4.1. Let $\mathbb{Q}_1$ be the set of all rational points in $[0,1)$. Given $r \in \mathbb{Q}$, we write $S_r = \{(x,y) \in \mathbb{T}^2$ such that $rx + y$ is rational $\}$ and let $S_\infty = \mathbb{Q}_1 \times [0,1)$. Note that $S_0 = [0,1) \times \mathbb{Q}_1$.

Definition 1.4.2. For $m, n \in \mathbb{Z}$, we define

$$A_{m,n} = \begin{cases} \begin{pmatrix} m & n \\ \frac{-(m-1)^2}{n} & 2-m \end{pmatrix} & \text{if } n \neq 0 \\ \begin{pmatrix} 1 & 0 \\ m-1 & 1 \end{pmatrix} & \text{if } n = 0 \end{cases}$$

Note that $Det(A_{m,n}) = 1$ and $Trace(A_{m,n}) = 2$ for all $m, n \in \mathbb{Z}$.

Proposition 1.4.3. *If $A \in GL(2,\mathbb{Z})$ is such that $Det(A) = 1$ and $Trace(A) = 2$ then $A = A_{m,n}$ for some $m, n \in \mathbb{Z}$ such that n divides $(m-1)^2$.*

Theorem 1.4.4. *For $m, n \in \mathbb{Z}$, the set of all periodic points of $T_{A_{m,n}}$ is either the set S_r for some $r \in \mathbb{Q} \cup \{\infty\}$ or $\mathbb{T}^2$, as given in the following table:*

$A_{m,n}$	$P(T_{A_{m,n}})$
$m \neq 1$ and $n \neq 0$	$S_{\frac{m-1}{n}}$
$m = 1$ and $n = 0$	$\mathbb{T}^2$
$m \neq 1$ and $n = 0$	S_∞
$m = 1$ and $n \neq 0$	S_0

Remark 1.4.5. The set S_r can be thought of as the points on the line through the origin with slope $-r$ and its rational translates in $[0,1) \times [0,1)$. From the above proposition, $r = \frac{(m-1)}{n}$when $m \neq 1$. On the other hand given any rational $r = \frac{p}{q} \in \mathbb{Q}$, we can find $m, n \in \mathbb{Z}$ with $n|(m-1)^2$ such that $\frac{p}{q} = \frac{m-1}{n}$. Choose $m = pq + 1$, $n = q^2$. Hence every S_r arises as $P(A_{m,n})$ for some $m, n \in \mathbb{Z}$.

From the above remark it follows that,

Proposition 1.4.6. *The following are equivalent for a subset of the torus.*

(1) It is $P(T_{A_{m,n}})$ *for some* $A_{m,n} \in GL(2, \mathbb{Z})$.

(2) It is S_r *for some* $r \in \mathbb{Q} \cup \{\infty\}$.

Theorem 1.4.7. *For any continuous toral automorphism* T_A*, the set* $P(T_A)$ *of periodic points of* T_A *is one of the following:*

1. $\mathbb{Q}_1 \times \mathbb{Q}_1$.
2. S_r *for some* $r \in \mathbb{Q} \cup \{\infty\}$*; where* $S_r = \{(x, y) \in \mathbb{T}^2 \mid rx + y \text{ is rational }\}$.
3. $\mathbb{T}^2$

In **Chapter-3**, we first discuss some well known results about $Per(f)$ (the set of periods of f), due to Sharkovski [17] and Baker [5] and many others, which are similar to our main result of this chapter. We prove that there are exactly 8 subsets of $\mathbb{N}$ which can occur as $Per(T)$ for some continuous toral automorphism T. We solve the problem separately for hyperbolic and nonhyperbolic automorphisms.

Theorem 1.4.8. *For any hyperbolic toral automorphism* $T_A : \mathbb{T}^2 \to \mathbb{T}^2$*, the set of periods* $Per(T_A) \supset \mathbb{N} \setminus \{2\}$.

Theorem 1.4.9. *Let* T_A *be a toral automorphism. Then* $Per(T_A)$ *is one of the following 8 subsets of* $\mathbb{N}$.

(1) $\{1\}$

(2) $\{1, 2\}$

(3) $\{1, 3\}$

(4) $\{1, 2, 4\}$

(5) $\{1, 2, 3, 6\}$

(6) $2\mathbb{N} \cup \{1\}$

(7) $\mathbb{N} \setminus \{2\}$

(8) $\mathbb{N}$.

In **chapter-4**, we study certain simple systems on $\mathbb{R}$. Since $\mathbb{R}$ has order structure, we would like to consider the conjugacies preserving the order. Let $f : \mathbb{R} \to \mathbb{R}$ be a dynamical system and let $x, y \in \mathbb{R}$. Then, we write $x \sim y$ if there exists an order conjugacy $h : \mathbb{R} \to \mathbb{R}$ of f such that $h(x) = y$. Note that $\sim$ is an equivalence relation. Let $[x]$ to denote the equivalence class of x.

We introduce the notions of *special points* and *nonordinary points*, which are new to the literature. The properties which are preserved under topological conjugacies are called *dynamical properties*. Having every point periodic, having a point with dense orbit, having exactly n periodic points are examples of dynamical properties.

Under a topological conjugacy a point can be mapped to a point with similar dynamics. Therefore, if a point is unique upto some dynamical property, then it must be fixed by all order conjugacies. This motivates us to define,

Definition 1.4.10. A point $x \in (\mathbb{R}, f)$ is said to be "dynamically special" if $[x] = \{x\}$.

We call a point to be *ordinary* if it is like points near it. That is, $[x]$ is a neighbourhood of x. The points which are not ordinary are called *nonordinary*.

It is observed that for systems with finitely many nonordinary points, the idea of nonordinary points and the idea of special points coincide.

Definition 1.4.11. Let (X, f) be a dynamical system. By the *full orbit* of a point $x \in X$ we mean the set

$$\widetilde{O}(x) = \{y \in X | f^n(x) = f^m(y) \text{ for some } m, n \in \mathbb{N}\}.$$

Definition 1.4.12. A point x in a dynamical system (X, f) is said to be a *critical point* if f fails to be one-one in every neighbourhood of x. The set of all critical points of f is denoted by $C(f)$.

We proved in [25], the following characterization theorem for the set of special points $S(f)$.

Theorem 1.4.13. *For continuous self-maps of the real line $\mathbb{R}$, the set of all nonordinary points (and hence the set of all special points) is contained in the closure of the union of full orbits of critical points , periodic points and the limits at infinity (if they exist and are finite).*

We prove that: If S_P denote the set of all points having the dynamical property P then the points of ∂S_P (the boundary of S_P) are nonordinary. In particular, being a point in a particular equivalence class is a dynamical property of the point. Hence by the very nature of the order conjugacies, it follows that when there are finitely many nonordinary points(therefore special points) there are only finitely many equivalence classes. These are the simple systems we study in this chapter.

We describe completely, the homeomorphisms on $\mathbb{R}$, having finitely many nonordinary points and give a general formula for counting.

Notation 1.4.14.

a_n = The number of increasing bijections, with exactly n nonordinary points, on $\mathbb{R}$, upto order conjugacy.

t_n = The number of increasing bijections, with exactly n nonordinary points, on $\mathbb{R}$, upto topological conjugacy.

s_n = The number of decreasing bijections, with exactly n nonordinary points, on $\mathbb{R}$, upto order conjugacy.

k_n = The number of decreasing bijections, with exactly n nonordinary points, on $\mathbb{R}$, upto topological conjugacy.

Theorem 1.4.15. *For every positive integer n, we have*

$a_n = c_1(1+\sqrt{3})^n + c_2(1-\sqrt{3})^n$ *where* $c_1 = \frac{5+3\sqrt{3}}{2\sqrt{3}}$ *and* $c_2 = \frac{3\sqrt{3}-5}{2\sqrt{3}}$.

$$s_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ a_{\frac{n-1}{2}} & n \text{ is odd} \end{cases}$$

$$t_n = \begin{cases} \frac{a_n + 2a_{\frac{n-4}{2}}}{2} & \text{if } n \text{ is even} \\ \frac{a_n + 2a_{\frac{n-3}{2}}}{2} & n \text{ is odd} \end{cases}$$

$$k_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ t_{\frac{n-1}{2}} & n \text{ is odd} \end{cases}$$

Where $t_0 = 2$, $t_1 = 5$ *and* $t_2 = 12$ *by direct computation.*

Also we prove that,

Theorem 1.4.16. *Upto order conjugacy, there are exactly 26 self maps on $\mathbb{R}$ with a unique nonordinary point.*

The main result of **chapter-5**, is obtained while making an attempt to answer the question: *Exactly which maps on $\mathbb{R}$ are topologically conjugate to a polynomial?.* The main theorem of this chapter, gives a necessary condition for a continuous self map of $\mathbb{R}$ to be conjugate to a polynomial.

Points which will reach a critical point (res. periodic point) after finitely many instants of time are called precritical points (res. preperiodic points).

Theorem 1.4.17. *Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial such that $f^2 \neq id$(the identity map on $\mathbb{R}$). Then $\overleftarrow{C(f)} \Delta \overleftarrow{P(f)}$ is a discrete set in the relative topology.*

Where $\overleftarrow{C(f)}$ denotes the set of all precritical points of f and $\overleftarrow{P(f)}$ denotes the set of all preperiodic points of f.

Chapter 2

Periodic points of toral automorphisms

2.1 General introduction

The automorphisms of the two-dimensional torus are rich mathematical objects possessing interesting geometric, algebraic, topological and measure theoretic properties.

The torus $\mathbb{T}^2 = \frac{\mathbb{R}^2}{\mathbb{Z}^2}$ *is here viewed as the topological group* $[0,1) \times [0,1)$ *with coordinate-wise addition modulo 1.*

Let $GL(2,\mathbb{Z})$ be the set of all 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbb{Z}$ and $Det(A) = ad - bc = \pm 1$.

Each such matrix A gives a linear map on $\mathbb{R}^2$ by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

We define an automorphism on the torus $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ by $T_A(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2)(mod 1)$.

Now we have,

Proposition 2.1.1. *[29] Every automorphism T_A (as defined above) on the torus is a homeomorphism.*

Proof. The map T is clearly continuous, since if $|x_1-y_1|, |x_2-y_2| < \epsilon$ then $|(T_A(x_1,x_2))_1-(T_A(y_1,y_2))_1| < (|a|+|b|)\epsilon$ and $|(T_A(x_1,x_2))_2-(T_A(y_1,y_2))_2| < (|c|+|d|)\epsilon$.(Here suffix 1 refer to the first coordinate and 2 refers to the second coordinate.)

To see that T_A is invertible we note that if we write the inverse matrix $A^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$ then since $ad-bc = \pm 1$ we see that $A^{-1} \in GL(2,\mathbb{Z})$. The inverse to T_A is then the toral automorphism associated to A^{-1}, i.e. $T_{A^{-1}}$.

□

On the other hand, in the following proposition we prove that every continuous automorphism on the torus is induced by a matrix from $GL(2,\mathbb{Z})$. Let $Aut(\mathbb{T}^2)$ to denote the set of all continuous automorphisms on the torus.

Proposition 2.1.2. *The above map $A \mapsto T_A$ from $GL(2,\mathbb{Z})$ to $Aut(\mathbb{T}^2)$ is surjective.*

Proof. let $\phi : \mathbb{T}^2 \to \mathbb{T}^2$ be any continuous toral automorphism. Since ϕ is continuous at $(0,0)$ there exist $\delta > 0$ such that $\phi(([0,\delta) \times [0,\delta))) \subset [0,\frac{1}{2}) \times [0,\frac{1}{2})$ and such that $\phi(X+Y) = \phi(X)+\phi(Y)$ for all $X, Y \in [0,\delta) \times [0,\delta)$, where $+$ denotes the usual addition in $\mathbb{R}^2$.

Now, observe that $\phi(\lambda X) = \lambda\phi(X)$ for all $\lambda \in (0,1)$ and for all $X, Y \in [0,\delta)\times[0,\delta)$, because the set $\{\lambda \in (0,1) | \ \phi(\lambda X) = \lambda\phi(X)\}$ is a closed set contains all dyadic rationals (Since the set contains $\lambda = \frac{1}{2}$, using the additivity of ϕ, it contains all numbers of form $\lambda = \frac{m}{2^n}$). The set of all dyadic rationals is dense in $[0,1]$ and then by continuity the set contains all $\lambda \in (0,1)$.

Hence $\phi|_{[0,\delta)\times[0,\delta)} = L|_{[0,\delta)\times[0,\delta)}$ for some linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$. This linear transformation induces an integer matrix A with determinant ± 1 such that

$Ax = \phi(x)$ for all $x \in \mathbb{T}^2$ [The kernel of an endomorphism (different from the zero map), on a connected topological group cannot have nonempty interior]. Hence the proof.

$\square$

Remark 2.1.3. In fact the 1-1 correspondence in the above proposition is a group isomorphism.

For each $A \in GL(2, \mathbb{Z})$, let $P(T_A)$ denote the set of all periodic points of T_A.

Proposition 2.1.4. *[21]*

For any $A \in GL(2, \mathbb{Z})$ *the set* $P(T_A)$ *is dense in* $[0, 1) \times [0, 1)$.

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$.

We prove that $P(T_A) \supset \mathbb{Q}_1 \times \mathbb{Q}_1$, where $\mathbb{Q}_1 = \mathbb{Q} \cap [0, 1)$. A general element in $\mathbb{Q}_1 \times \mathbb{Q}_1$ is of the form $x = (\frac{p_1}{q}, \frac{p_2}{q})$ where $p_1, p_2, q \in \mathbb{Z}$ with $0 \leq p_1, p_2 < q$. We note that $T_A(X) =$ (fractional part of $\frac{ap_1}{q} + \frac{bp_2}{q}$, fractional part of $\frac{cp_1}{q} + \frac{dp_2}{q}$) $=$ an element of the form $(\frac{m}{q}, \frac{n}{q})$ where $0 \leq m, n < q$. Note that, for a fixed $q \in \mathbb{N}$, the set $\{(\frac{m}{q}, \frac{n}{q})/0 \leq m, n < q; m, n \in \mathbb{N}\}$ is invariant and finite. Hence the orbit of x is finite and therefore eventually periodic. Now, the result follows from the fact that for invertible maps the eventually periodic points are periodic points.

$\square$

Note that, for a toral automorphism T_A, the periodic points with period n are solutions of the congruent equation $A^n x = x(mod 1)$. The following proposition is in this direction.

Lemma 2.1.5. [28] If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an isomorphism then for every Riemann measurable set (having Jordan content) $S \subset \mathbb{R}^2$, $T(S)$ is Riemann measurable and

$$Area(T(S)) = |Det(T)|Area(S)$$

Proposition 2.1.6. *[21]*

Let $A \in GL(2,\mathbb{Z})$. Then

(1) The number of solutions of $A^n x = x$ in $[0,1) \times [0,1)$, is $|Det(A^n - I)|$, provided $Det(A^n - I) \neq 0$.

(2) If $Det(A^n - I) = 0$ then $A^n x = x$ has infinitely many solutions in $[0,1) \times [0,1)$.

Proof. (1) Suppose $Det(A^n - I) \neq 0$. Then note that the number of solutions of the equation, $A^n x = x$ in $[0,1) \times [0,1)$ is equal to the number of integer points in the image of $[0,1) \times [0,1)$ under $A^n - I$, treated as a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Note also that the image of $[0,1) \times [0,1)$ under $A^n - I$ is a parallelogram and hence the number of integer points in it, is equal to its area, which is equal to $|Det(A^n - I)|$, by previous lemma.

(2) Note that, when $Det(A^n - I) = 0$, the system $(A^n - I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ itself, has infinitely many solutions in $[0,1) \times [0,1)$. □

Observe that for any continuous toral automorphism T_A the set $P(T_A)$ is a subgroup of the torus. We now ask:

Which subgroups of $[0,1) \times [0,1)$ arise in this way?

2.2 Automorphisms with determinant 1 and trace 2

We start by listing all the matrices from $GL(2,\mathbb{Z})$ with determinant 1 and trace 2.

Definition 2.2.1. For $m, n \in \mathbb{Z}$ we define,

$$A_{m,n} = \begin{cases} \begin{pmatrix} m & n \\ \frac{-(m-1)^2}{n} & 2-m \end{pmatrix} & \text{if } n \neq 0 \\ \begin{pmatrix} 1 & 0 \\ m-1 & 1 \end{pmatrix} & \text{if } n = 0 \end{cases}$$

Note that $Det(A_{m,n}) = 1$ and $Tr(A_{m,n}) = 2$ for all $m, n \in \mathbb{Z}$.

Proposition 2.2.2. *If $A \in GL(2, \mathbb{Z})$ is such that $Det(A) = 1$ and $Trace(A) = 2$ then $A = A_{m,n}$ for some $m, n \in \mathbb{Z}$ such that n divides $(m-1)^2$.*

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$ be such that $Det(A) = 1$ and $Tr(A) = 2$. Then we have $a + d = 2$ and $ad - bc = 1$. Hence $bc = -(a-1)^2$

If $b \neq 0$ then $c = -\frac{(a-1)^2}{b}$ an integer and therefore $A = A_{a,b}$. If $b = 0$ then $a = d = 1$ and c can be any integer. Hence $A = A_{c+1,0}$.

□

Remark 2.2.3. Note that the characteristic polynomial of any matrix of type $A_{m,n}$ is $(x-1)^2$ and hence 1 is an eigen value. Therefore no matrix of type $A_{m,n}$ can be hyperbolic.

We now calculate the set of all periodic points of the continuous toral automorphisms induced by the matrices of form $A_{m,n}$. By induction we can prove that, for any $k \in \mathbb{N}$, $A^k_{m,n} = A_{km-k+1,kn}$ for all $m, n \in \mathbb{Z}$.

Let $A \in GL(2, \mathbb{Z})$. Then $X \in \mathbb{T}^2$ is a periodic point of T_A if and only if it is a solution of $A^k X = X$ for some $k \in \mathbb{N}$. Thus,

$P(T_A) = \cup_{k=1}^{\infty} \{X \in \mathbb{T}^2 : \ A^k X = X\}$.

Notation 2.2.4. Let $\mathbb{Q}_1$ be the set of all rational points in $[0, 1)$. Given $r \in \mathbb{Q}$, we write $S_r = \{(x, y) \in \mathbb{T}^2$ such that $rx + y$ is rational $\}$ and let $S_\infty = \mathbb{Q}_1 \times [0, 1)$. Note that $S_0 = [0, 1) \times \mathbb{Q}_1$.

Theorem 2.2.5. *For $m, n \in \mathbb{Z}$, the set of all periodic points of the continuous toral automorphism $T_{A_{m,n}}$ is either the set S_r for some $r \in \mathbb{Q} \cup \{\infty\}$ or $\mathbb{T}^2$.*

Proof. **Case: 1**

When $m \neq 1$ and $n \neq 0$.

The periodic points with period k can be obtained by solving the equation $A^k_{m,n}X = X(mod 1)$, which is equivalent to the system of linear equations,

$$(km - k + 1)x_1 + knx_2 = x_1 + m_1$$

$$-\frac{k(m-1)^2}{n}x_1 + (1 - km + k)x_2 = x_2 + m_2$$

for some $m_1, m_2 \in \mathbb{Z}$

That is, $(x_1, x_2) \in \mathbb{T}^2$ satisfies the equation $A^n X = X$ if and only if

$$(km - k + 1)x_1 + knx_2 = m_1 \in \mathbb{Z}$$

$$-\frac{k(m-1)^2}{n}x_1 + (1 - km + k)x_2 = m_2 \in \mathbb{Z}$$

if and only if

$$\left.\begin{array}{r} (km - k + 1)x_1 + knx_2 = m_1 \in \mathbb{Z} \\ -\frac{k(m-1)^2}{n}x_1 + (1 - km + k)x_2 = m_2 \in \mathbb{Z} \end{array}\right\} \tag{2.1}$$

which implies that,

$$\left.\begin{array}{r} (m - 1)x_1 + nx_2 \in \mathbb{Q} \\ -\frac{(m-1)^2}{n}x_1 + (1 - m)x_2 \in \mathbb{Q} \end{array}\right\} \tag{2.2}$$

which reduces to solving the single equation,

$$\frac{(m-1)}{n}x_1 + x_2 \in \mathbb{Q}.$$

Conversely, if

$$\frac{(m-1)}{n}x_1 + x_2 \in \mathbb{Q}$$

holds. Then (x_1, x_2) satisfies the equation (2.2). Then we can prove that (x_1, x_2) satisfies the equation (2.1) for some suitable k. $P(T_{A_{m,n}}) = \{(x_1, x_2) | \frac{(m-1)}{n}x_1 + x_2 \in \mathbb{Q}\} = S_{\frac{(m-1)}{n}}$ as desired.

[From the equations (2.1) and (2.2), it is observed that:

(x_1, x_2) is a fixed point of $T_{A_{m,n}}$ if and only if $(\frac{x_1}{k}, \frac{x_2}{k})$ is a fixed point of $T^k_{A_{m,n}}$.]

Case: 2

When $m = 1$ and $n = 0$. We get $A_{1,0}$ is the identity matrix and hence $P(T_{A_{m,n}}) = Fix(T_{A_{m,n}}) = \mathbb{T}^2$.

Case: 3 When $m \neq 1$ and $n = 0$.

We have $A_{m,0} = \begin{pmatrix} 1 & 0 \\ m-1 & 1 \end{pmatrix}$. Note that, for any $k \in \mathbb{N}$ we have $A^k_{m,0} = A_{km-k+1,0}$

Now, solving the congruence equation $A^k_{m,0}X = X(mod1)$ is equivalent to solving the single condition $k(m-1)x_1 \in \mathbb{Z}$. This is implies that $x_1 \in \mathbb{Q}_1$.

Conversely, as before if $x_1 \in \mathbb{Q}_1$ then we can find $k \in \mathbb{Z}$ such that $k(m-1)x_1 \in \mathbb{Z}$.

Hene $P(T_A) = \mathbb{Q}_1 \times [0, 1) = S_\infty$.

Case: 4 When $m = 1$ and $n \neq 0$. In this case $A_{1,n} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

Then for any $k \in \mathbb{N}$ we have $A^k_{1,n} = A_{1,kn}$ Now, solving the congruent equation $A^k_{1,n}X = X(mod1)$ is equivalent to solving the single condition $knx_2 \in \mathbb{Z}$. This implies that $x_2 \in \mathbb{Q}_1$.

Conversely, if $x_2 \in \mathbb{Q}_1$ then we can find some $k \in \mathbb{Z}$ such that $k(m-1)x_2 \in \mathbb{Z}$. Hence $P(T_A) = [0,1) \times \mathbb{Q}_1 = S_0$.

□

Remark 2.2.6. Let $A \in GL(2,\mathbb{Z})$ be of $A_{m,n}$ type. Then from the relation, $A^k_{m,n} = A_{km-k+1,kn}$(for all $k \in \mathbb{N}$), it is clear that A and its all powers pertain to the same case among the four cases discussed in the previous proposition. Hence they have the same set of periodic points.

Remark 2.2.7. The set S_r can be thought of as the points on the line through the origin with slope $-r$ and its rational translates in $[0,1)\times[0,1)$. From above proposition, $r = \frac{(m-1)}{n}$when $m \neq 1$. On the other hand given any rational $r = \frac{p}{q} \in \mathbb{Q}$, we can find $m, n \in \mathbb{Z}$ with $n|(m-1)^2$ such that $\frac{p}{q} = \frac{m-1}{n}$. For, choose $m = pq+1, n = q^2$. Hence every S_r arises as $P(A_{m,n})$ for some $m, n \in \mathbb{Z}$.

Proposition 2.2.8. *The following are equivalent for a subset of the torus.*

(1) It is $P(T_{A_{m,n}})$ for some $A_{m,n} \in GL(2,\mathbb{Z})$.

(2) It is S_r for some $r \in \mathbb{Q} \cup \{\infty\}$.

Proof. Follows from the above remark. □

2.3 Main theorem

Definition 2.3.1. A continuous toral automorphism $T_A \in GL(2,\mathbb{Z})$ is said to be *hyperbolic* if A has no eigen values with absolute value 1.

Example 2.3.2. The matrices of the type $A_{m,n}$ are not hyperbolic, because 1 is an eigen value.

It is already known [21] that for a hyperbolic continuous toral automorphism, the periodic points are precisely the rational points. In this chapter, we calculate the set of periodic points for other continuous toral automorphisms; this happens to be the subgroup of $\mathbb{T}^2$ generated by $\mathbb{Q}_1 \times \mathbb{Q}_1 \cup$ (a line with rational slope). In fact, for all non-hyperbolic continuous toral automorphism , there are uncountably many periodic points.

Lemma 2.3.3. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$ is hyperbolic then $Det(A - I) = (a - 1)(d - 1) - bc \neq 0$.

Proof. Let A be hyperbolic. Now, suppose $(a-1)(d-1)-bc = 0 = (ad-bc)-(a+d)+1$

Case: 1 If $ad - bc = 1$, then $a + d = 2$. Which implies by proposition 2.2.2 that $A = A_{m,n}$ for some m, n. Which is a contradiction, by remark 2.2.3.

Case: 2 If $ad - bc = -1$, then $a + d = 0$. Therefore the characteristic polynomial is $x^2 - 1$. Hence the eigen values are ± 1. Which contradicts hypothesis. $\square$

Remark 2.3.4. If A is hyperbolic then so is A^n for all $n \in \mathbb{N}$. (If λ is an eigen value of A,then λ^n is eigen value of A^n). Then $Det(A^n - I) \neq 0$ for all $n \in \mathbb{N}$. Hence the above lemma will apply to all the positive powers of A.

Proposition 2.3.5. *If T_A is hyperbolic then $P(T_A) = \mathbb{Q}_1 \times \mathbb{Q}_1$.*

Proof. The fixed points of A^n are given by the congruence equation $A^n X \equiv X(mod 1)$.

If we let , $A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ then we get, $(a_n - 1)x_1 - b_n x_2 = k_1 \in \mathbb{Z}$ and $c_n x_1 - (d_n - 1)x_2 = k_2 \in \mathbb{Z}$.

Now the condition $c_n b_n - (a_n - 1)(d_n - 1) \neq 0$ guarantees that the above system of linear equations is consistent and its solutions are having rational coordinates (by Cramer's rule). Hence $P(\widetilde{A}) = \mathbb{Q}_1 \times \mathbb{Q}_1$.

□

Even though we assumed A to be hyperbolic in the above proposition, we have not used its full strength. What we needed only is that the determinant $Det(A^n - I) = c_n b_n - (a_n - 1)(d_n - 1)$ is nonzero for all $n \in \mathbb{N}$. Thus, in view of proposition 2.1.6 the previous proposition 2.3.5 can be improved/restated as

Proposition 2.3.6. *If T_A is a toral automorphism such that for each $n \in \mathbb{N}$ there are only finitely many periodic points with period n, then $P(T_A) = \mathbb{Q}_1 \times \mathbb{Q}_1$.*

We are now in a position to prove the main theorem.

Theorem 2.3.7. *For any continuous toral automorphism T_A, the set $P(T_A)$ of periodic points of T_A is one of the following:*

1. $\mathbb{Q}_1 \times \mathbb{Q}_1$.
2. S_r *for some* $r \in \mathbb{Q} \cup \{\infty\}$; *where* $S_r = \{(x, y) \in \mathbb{T}^2 \mid rx + y \text{ is rational }\}$.
3. $\mathbb{T}^2$

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$.

For any $n \in \mathbb{N}$ we write $A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ where a_n, b_n, c_n, d_n are integers.

Case: 1

If $Det(A^n - I) = c_n b_n - (a_n - 1)(d_n - 1) \neq 0$ for all $n \in \mathbb{N}$ then proof follows from Proposition 2.3.6.

Case: 2

Suppose $Det(A^n - I) = c_n b_n - (a_n - 1)(d_n - 1) = 0$ for some $n \in \mathbb{N}$.

Let $S = \{k \mid Det(A^k - I) = c_k b_k - (a_k - 1)(d_k - 1) = 0\}$.

Subcase: (2a)

If $Det(A^k) = -1$ for some $k \in S$, then $Tr(A^k) = 0$. Therefore $A^k = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$

Note that $A^{2k} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Hence $P(A) = \mathbb{T}^2 = S_0$ since $P(A) \supset P(A^k)\ \forall\ k \in \mathbb{N}$.

Subcase: (2b) If $Det(A^k) = 1$ for all $k \in S$ then $Tr(A^k) = 2$. Therefore $A^k = A_{m,n}$ for some $m, n \in \mathbb{Z}$

From remark 2.2.6, it follows that A^k and its powers namely, $A^{2k}, A^{3k}, A^{3k}, ...$ *share the same set of periodic points.* Note that, for any $j \in \mathbb{N}$ the periodic points of T_A with period j are contained in $P(A^{jk})$. Hence, from theorem 2.2.5, $P(T_A) = S_r$ for some $r \in \mathbb{Q} \cup \{\infty\}$.

□

We conclude this chapter with the following remark.

Remark 2.3.8. Even though there are apparently four kinds of subsets which can appear as the set of periodic points for some continuous toral automorphism, there are only three upto homeomorphism.

Chapter 3

The set of periods of toral automorphisms

3.1 Examples and Motivation

Let (X, f) be a dynamical system. Let $x \in X$. Recall that x is a periodic point if $f^n(x) = x$ for some $n \in \mathbb{N}$ and that least such n is called the period of x.

We denote by $Per(f)$, the set of all periods of periodic points in the dynamical system (X, f). We now ask:

Given X, which subsets of $\mathbb{N}$ arise as $Per(f)$ for some continuous self map f from X to X ?

We start with a simple example. For the class of homeomorphisms on $\mathbb{R}$ the set of periods is one among the following subsets of $\mathbb{N}$.

1. The empty set.
2. $\{1\}$.
3. $\{1, 2\}$.

This follows from the following facts:

1. For increasing maps on $\mathbb{R}$ all periodic points are fixed points.

2. For any homeomorphism f on $\mathbb{R}$, the map f^2 is increasing.

For the class of continuous maps on $\mathbb{R}$, we have:

3.1.1 Sharkovskii's Theorem

Among the uncountably many subsets of $\mathbb{N}$, there are only countably many that can arise as $Per(f)$ for some continuous self map of $[0, 1]$. These sets form a chain (any two members are comparable) and can be described through a total order on $\mathbb{N}$ different from the usual. This is a celebrated theorem of Sharkovskii.

Definition 3.1.1. The following total order of $\mathbb{N}$ is called the Sharkovskii's ordering:

$$
\begin{gathered}
3 \succ 5 \succ 7 \succ 9 \succ ... \\
\succ 2 \times 3 \succ 2 \times 5 \succ 2 \times 7 \succ ... \\
\vdots \\
\succ 2^n \times 3 \succ 2^n \times 5 \succ 2^n \times 7 \succ ... \\
\vdots \\
.............2^n \succ .... \succ 2^2 \succ 1
\end{gathered}
$$

Theorem 3.1.2. ***(Sharkovskii)*** *[17]*

Let $m \succ n$ in the Sharkovskii's ordering. For every continuous self-map of $\mathbb{R}$, if there is an $m-cycle$, then there is an $n-cycle$.

A converse of Sharkovskii's theorem : [18]

Let m and n be distinct positive integers. Let m not precede n in the above ordering. Then there is a continuous map f from $\mathbb{R}$ to $\mathbb{R}$, where there is an $m - cycle$ but no $n - cycles$.

A combined Statement:

$m \succeq n$ in the Sharkovskii's ordering if and only if for every continuous self-map of $\mathbb{R}$, the existence of an $m - cycle$ forces that of an $n - cycle$.

This can be restated as,

Theorem 3.1.3. *Let $A \subset \mathbb{N}$. Then $A = Per(f)$ for some continuous map $f : \mathbb{R} \to \mathbb{R}$ if and only if*

$m \in A$ and $m \succ n$ imply $n \in A$.

Remark 3.1.4. This theorem is special to some spaces like the real line. Its analogues for other spaces are not valid. For example, on the unit circle S^1, the rotation by 120 degrees admits a $3 - cycle$, but admits cycles of no other length.

It seems that the sharkovskii-ordering is forced by nature. There are spaces other than $\mathbb{R}$ where an analogous theorem is true.

3.1.2 Baker's Theorem

We reformulate a theorem of Baker as follows:

Theorem 3.1.5. *[5] Let p be a complex polynomial. Then the set of periods of p has to be one of the following five kinds of subsets of* $\mathbb{N}$:

1. *The whole set* $\mathbb{N}$
2. $\mathbb{N} \setminus \{2\}$
3. $\{1, n\}$, *where* $n \in \mathbb{N} \setminus \{1\}$

4. $\{1\}$

5. Empty set.

Moreover, the following hold:

(a) *Any polynomial* p *such that* $Per(p) = \mathbb{N} \setminus \{2\}$ *has to be conjugate to* $z^2 - z$.

(b) *For all polynomials* p *of degree* ≥ 2, $Per(p) \supset \mathbb{N} \setminus \{2\}$.

The following table gives the number of complex polynomials upto topological conjugacy, having the given subset of $\mathbb{N}$ as its period set.

Subset of $\mathbb{N}$	**Polynomials for which it is the period set**	**Number of conjugacy classes**
Empty set	Nontrivial translations $z + c$ where $c \neq 0$	1
$\{1\}$	Polynomials $cz + d$ satisfying $d \neq 0$; also identity z; also polynomials cz where c is not a root of unity.	Uncountably many
$\{1, n\}$, where $n \in \mathbb{N} \setminus \{1\}$.	Polynomials cz where c is a nontrivial nth root of unity.	$\phi(n)$
$\mathbb{N} \setminus \{2\}$	Polynomials of the form $az^2 + (2ab - 1)z + b^2$, where $a \neq 0$	1
$\mathbb{N}$	All other polynomials. (not in the previous rows)	Uncountably many
All other subsets not listed in the previous rows	No polynomial.	0

Incidentally, we obtain a dynamical characterization of the polynomial $z^2 - z$. Upto topological conjugacy, it is the only polynomial whose set of periods is $\mathbb{N} \setminus \{2\}$.

The chart below helps us to contrast four situations:

Complex polynomials	Exactly 5 subsets of $\mathbb{N}$
Real continuous maps	A countably infinite family $\mathfrak{F}$ of subsets of $\mathbb{N}$
Complex continuous maps	All subsets of $\mathbb{N}$
Real polynomials	An infinite proper subfamily of $\mathfrak{F}$

Here $\mathfrak{F}$ is as described in theorem 3.1.3.

The third row means: Given any subset A of $\mathbb{N}$, there is a continuous map f from $\mathbb{C}$ to $\mathbb{C}$ such that $\mathrm{Per}(f) = \mathbb{C}$.

The fourth row implies: There is a subset of $\mathbb{N}$, occurring as $\mathrm{Per}(f)$ for a continuous self map of $\mathbb{R}$, but not as $\mathrm{Per}(p)$ for a real polynomial. Explicitly, $\{2^k : k \in \mathbb{N}_0\}$ is one such set.

3.1.3 Period sets of Unit circle S^1 in the plane

The theorem of Sharkovskii specifies, for continuous maps of an interval, which sets of positive integers may occur as the sets of periods. Results along these lines have also been obtained for maps of the circle.

In 1980, L.S Block [8] proved the following interesting results on continuous self maps of the unit circle S^1. By $f \in C(S^1, S^1)$ we mean f is a continuous self map from S^1 to S^1.

Theorem 3.1.6. *Let $f \in C(S^1, S^1)$. Suppose $1 \in Per(f)$ and $n \in Per(f)$ for some odd integer $n > 1$, then for every integer $m > n$, $m \in Per(f)$.*

Theorem 3.1.7. *Let $f \in C(S^1, S^1)$ and suppose that $Per(f)$ is finite. Then there are integers m and n (with $m \geq 1$ and $n \geq 0$) such that*

$Per(f) = \{m, 2.m, 2^2.m, ..., 2^n.m\}$

Theorem 3.1.8. *Let $f \in C(S^1, S^1)$. If $\{1, 2, 3\} \subset Per(f)$ then $Per(f) = \mathbb{N}$. Conversely, if $S \subset \mathbb{N}$ with the property that for any $f \in C(S^1, S^1)$, $S \subset Per(f) \Rightarrow Per(f) = \mathbb{N}$ then $\{1, 2, 3\} \subset S$.*

Description of $Per(f)$ when f has fixed point:

Again in 1981, Block [7] proved the following main results about $Per(f)$ for $f \in C(S^1, S^1)$ when f has a fixed point.

Theorem 3.1.9. *Let $f \in C(S^1, S^1)$. Suppose $1 \in Per(f)$ and $n \in Per(f)$ for some integer $n > 1$. Then (atleast) one of the following holds:*

(i) For every integer m with $n < m$, $m \in Per(f)$

(ii) For every integer m with $n \succ m$, $m \in Per(f)$

(here $<$ denotes the usual order on $\mathbb{N}$).

He has also proved the converse of the above theorem. i.e,

Theorem 3.1.10. *Let $S \subset \mathbb{N}$ with $1 \in S$. Suppose that for each $n \in S$ with $n \neq 1$, atleast one of the following holds.*

(a) for every integer m with $n < m$, $m \in S$.

(b) for every integer m with $n \succ m$, $m \in S$

Then there exists a continuous map $f \in C(S^1, S^1)$ such that the set of periods of periodic points of f is exactly S.

These two theorems of Block characterize the sets of periods which can occur for a continuous map of the circle to itself having a fixed point. But not every self map of the circle has fixed point. One can see that if the $deg(f) \neq 1$ (where $f \in C(S^1, S^1)$)

then f has a fixed point. Partly for this reason, degree-one maps of the circle require special attention to study the periodic orbits. See [6] for more details.

<u>$Per(f)$ **when degree of** $f \neq 1$**:**</u>

In 1982, M. Misiurewicz [9] proved the result, which describes the possible sets of periods of the periodic points of a continuous degree one map (See [6]) of the circle. For any two real numbers a and b, let $M(a,b) = \{n \in \mathbb{N} : a < \frac{t}{n} < b$ for some integer $t\}$.

Definition 3.1.11. If $a \in \mathbb{R}$ and $l \in \mathbb{N} \cup \{2^\infty\}$, (Think 2^∞ as a symbol) we define a subset $S(a,l) \subset \mathbb{N}$ as follows:

If a is irrational then $S(a,l) = \emptyset$

If a is rational and if $a = \frac{t}{n}, n \in \mathbb{N}, t \in \mathbb{Z}, (t,n) = 1$ and if $l \in \mathbb{N}$ then $S(a,l)$ denotes the set of positive integers of the form ns, where $l \succ s$(in Sarkovskii ordering).

If $l = 2^\infty$ then $S(a,l)$ denotes the set of all postive integers of the form ns, where s is a power of 2.

Now we state Misiurewicz's result about continuous maps of circle.

Theorem 3.1.12. *Let f be a continuous map of the circle to itself of degree one. Then there exist $a, b \in \mathbb{R}$ with $a \leq b$ and $l, r \in \mathbb{N} \cup \{2^\infty\}$ such that $Per(f) = M(a,b) \cup S(a,l) \cup S(b,r)$.*

Conversely, for every subset A of $\mathbb{N}$ of the form $A = M(a,b) \cup S(a,l) \cup S(b,r)$ there is a continuous map of the circle to itself of degree one such that $Per(f) = A$

Remark 3.1.13. Hence we have a complete answer for describing the sets of periods for continuous self maps on circle, because

$PER(S^1) = \{Per(f) | f$ is a continuous self map on S^1 with deg 1$\} \bigcup \{Per(g) | g$ is a continuous self map on S^1 with a fixed point $\}$.

3.1.4 Period sets of the Y-space

In 1989, L.Alseda, J. Llibre and M. Misiurewicz [1] made a generalization of Sarkovskii's theorem to characterize the possible sets of periods for continuous maps f of the space $Y = \{z \in \mathbb{C} |\ z^3 \in [0, 1]\}$ into itself for which $f(0) = 0$.

Hereafter $C(Y)$ denote the set of all continuous self maps on Y.

The following theorem was proved by Mumbru in [12] for $f \in C(Y)$.

Theorem 3.1.14. *(a) If $f \in C(Y)$ and $\{2, 3, 4, 5, 7\} \subset Per(f)$ then $Per(f) = \mathbb{N}$.*

(b) If $W \subset \mathbb{N}$ is a set such that for every $f \in C(Y)$, $W \subset Per(f)$ implies $Per(f) = \mathbb{N}$, then $W \subset \{2, 3, 4, 5, 7\}$.

To describe the result of Alseda et al in [1], we need to introduce the following notations and two new orderings.

$S(k) = \{n \in \mathbb{N} : k \succ n\} \cup \{k\}$ for all $k \in \mathbb{N}$.

(here $\succ$ is the Sharkovskii order.)

$S(2^\infty) = \{2^i |\ i = 0, 1, 2, \ldots\}$

Definition 3.1.15. [1]

Green ordering is the ordering of $\mathbb{N} \setminus \{2\}$, denoted by $<_g$, and defined as follows:

$5 <_g 8 <_g 4 <_g 11 <_g 14 <_g 7 <_g 17 <_g 20 <_g 10 <_g 23 <_g 26 <_g 13 <_g \ldots <_g 3.3 <_g 3.5 <_g 3.7 <_g \ldots 3.2.3 <_g 3.2.5 <_g 3.2.7 <_g \ldots <_g 3.2^2.3 <_g 3.2^2.5 <_g 3.2^2.7 <_g \ldots <_g 3.2^3 <_g 3.2^2 <_g 3.2 <_g 3.1 <_g 1.$

The first part of the ordering can be understood as

6-1,6+2, 3+1, 2.6-1, 2.6+2, 2.3+1, 3.6-1, 3.6+2, 3.3+1,. . . .

Definition 3.1.16. [1]

Red ordering is the ordering of $\mathbb{N} \setminus \{2, 4\}$, denoted by $<_r$ and defined as :

$7 <_r 10 <_r 5 <_r 13 <_r 16 <_r 8 <_r 19 <_r 22 <_r 11 <_r 25 <_r 28 <_r 14 <_r \ldots <_r 3.3 <_r 3.5 <_r 3.7 <_r \ldots\ldots <_r 3.2.3 <_r 3.2.5 <_r 3.2.7 <_r \ldots <_r 3.2^2.3 <_r 3.2^2.5 <_r 3.2^2.7 <_r \ldots <_r \ldots 3.2^3 <_r 3.2^2 <_r 3.2 <_r 3.1 <_r 1.$

Here the first part can be viewed as,

6-1, 6+4, 3+2, 2.6+1, 2.6+4, 2.3+2, 3.6+1, 3.6+4, 3.3+2,....

Notation 3.1.17. We denote $G(n) = \{n\} \cup \{k : n <_g k\}$ for all $n \in \mathbb{N} \setminus \{2\}$

$R(n) = \{n\} \cup \{k : n <_r k\}$ for all $n \in \mathbb{N} \setminus \{2, 4\}$ and

$G(3.2^\infty) = R(3.2^\infty) = \{3.i : i \in S(2^\infty)\}$.

We also denote

$$N_s = \mathbb{N} \cup \{2^\infty\}$$

$$N_g = (\mathbb{N} \setminus \{2\}) \cup \{3.2^\infty\}$$

$$N_r = (\mathbb{N} \setminus \{2, 4\}) \cup \{3.2^\infty\}$$

Now we are ready to state the main result of Alseda, Llibre and Misiurewicz on sets of periods of Y.

Theorem 3.1.18. *(a) If $f \in C_0(Y)$ (i.e., f is a continuous self map on Y with $f(0) = 0$) then $Per(f) = S(n_s) \cup G(n_g) \cup R(n_r)$ for some $n_s \in N_s$, $n_g \in N_g$ and $n_r \in N_r$.*

If $n_s \in N_s$, $n_g \in N_g$ and $n_r \in N_r$ then there exists a continuous self map f on $C_0(Y)$ for which

$$Per(f) = S(n_s) \cup G(n_g) \cup R(n_r).$$

In [1] at the end they posed an open question to describe possible sets of periods of continuous self maps f on the space $X_k = \{z \in \mathbb{C} |\ z^k \in [0, 1]\}$ for $k > 3$, for which zero is a fixed point.

This question was taken up by S. Baldwin, and he generalized the theorem of [1]. He described the complete solution for the sets of periods of self maps on n-od with 0 as fixed point.

3.1.5 Sets of periods of n-od

The n-od, denoted by X_n is the subspace of the complex plane, which is described as $X_n = \{z \in \mathbb{C} |\ z^n \in [0, 1]\}$ [Note that 1-od and 2-od are homeomorphic]. This can be viewed as the set obtained by attaching n copies of unit interval to the central point(i.e, at 0.)

Some definitions and notations:

In order to study the structure of sets of periods of continuous maps $f : X_n \to X_n$ we need to define partial ordering $\leq_t$ for all positive integers t.

The ordering $\leq_1$ is defined by

$$2^i \leq_1 2^{i+1} \leq_1 2^{j+1}(2m + 3) \leq_1 2^{j+1}(2m + 1) \leq_1 2^j(2k + 3) \leq_1 2^j(2k + 1)$$

for all integers $i, j \geq 0$ and $k, m \geq 0$.

In other words $\leq_1$ is the usual Sharkovskii ordering.

If $n > 1$ then the ordering $\leq_n$ is defined as follows:

Let m, k be positive integers.

Case: 1

$k = 1$ then $m \leq_n k$ if and only if $m = 1$.

Case: 2

k is divisible by n then $m \leq_n k$ if and only if either $m = 1$ or m is divisible by n and $(\frac{m}{n} \leq_n \frac{k}{n})$.

Case: 3

$k > 1$, k is not divisible by n. Then $m \leq_n k$ if and only if either $m = 1, m = k$ or $m = ik + jn$ for some integers $i \geq 0,\ j \geq 1$.

In [3], some diagrams illuminating these partial orderings are given. From the definition we can see that $\leq_1$ and $\leq_2$ coincide with the Sharkovskii ordering. If $n > 2$, then $\leq_n$ is not a linear ordering. Define a set $S \subset \mathbb{N}$ to be an initial segment of $\leq_n$ if whenever k is an element of S and $m \leq_n k$ then m is also an element of S, i.e, S is closed under $\leq_n$ predecessors. Now we state the theorem of Baldwin in [3].

Theorem 3.1.19. *Let X be an n-od.*

1. Let $f : X_n \to X_n$ be a continuous map. Then $Per(f)$ is a nonempty finite union of initial segments of $\{\leq_p : 1 \leq p \leq n\}$.

2. Conversely, if S is a nonempty finite union of initial segments of $\{\leq_p : 1 \leq p \leq n\}$ then there is a continuous map $f : X_n \to X_n$ such that $f(0) = 0$ and $Per(f) = S$.

Remark 3.1.20. The n-od result is the same, regardless of whether the branching point is required to be fixed or not.

3.1.6 Period sets for Tree maps

By a tree in the plane we mean a connected graph without any cycles. A tree with its relative topology (treating as a subset of $\mathbb{R}^2$) in $\mathbb{R}^2$ is called a tree space.

The main result of [4] is the extension of n-od theorem to every continuous self map on a tree T having all branching points fixed. It is of interest to ask what $Per(f)$ can be if all branching points of T are fixed. The result on n-od has been extended to all trees(without assumptions on the branching points) by Baldwin, but these results do not specify which sets are possible if the branching points are remaining fixed. For similar results on graphs which characterize sets of periods without specifying which sets of periods correspond to which graphs, See [10]. Now we state the main result of [4].

Given a tree T, let $e(T)$ and $b(T)$ be the number of end points and branching points respectively.

Theorem 3.1.21. *Let T be a tree.*

(a) Let $f : T \to T$ be a continuous map with all branching points fixed. Then $Per(f)$ is a nonempty finite union of initial segments of $\{\leq_p : 1 \leq p \leq e(T)\}$.

(b) Conversely, if S is a nonempty finite union of initial segments of $\{\leq_p : 1 \leq p \leq e(T)\}$ then there is a continuous map $f : T \to T$ with all branching points fixed such that $Per(f) = S$.

This theorem solves a problem which was originally posed by Alseda et al in [1].

3.1.7 Saradhi's result

Theorem 3.1.22. *[27] Let X be a noncompact convex subset of $\mathbb{R}^2$ with nonempty interior. Then $PER(X) = \mathcal{P}(\mathbb{N})$, the power set of $\mathbb{N}$.*

Theorem 3.1.23. *[27]*

Let S be a closed disc. Then $PER(S) = \{$ all subsets of $\mathbb{N}$ containing 1 $\}$.

We conclude our survey here and we now prove our main theorem of this chapter.

3.2 Period sets of hyperbolic toral automorphisms

Proposition 3.2.1. *[21]*

For $2-$dimensional hyperbolic toral automorphisms, the eigen values are real and irrational.

Proof. Let A be the matrix of a toral automorphism and $(x, y)(\neq (0,0))$ be an eigenvector of A. If λ is the corresponding eigenvalue, we have

$$ax + by = \lambda x \quad .....(1)$$

$$cx + dy = \lambda y \quad .....(2)$$

First, $x \neq 0$; for if $x = 0$, then $b = 0$ from (1). $Det(A) = \pm 1$ and $b = 0$ implies $ad = \pm 1$. Therefore $d = \pm 1$ and hence from (2) we obtain $\lambda = \pm 1$ (since $y \neq 0$), contradiction to the hyperbolicity.

Next, letting $t = \frac{y}{x}$ and using it to eliminate x and y from (1) and (2), we get $a + bt = \lambda$ and $c + dt = \lambda$.

Now eliminating λ, we have a quadratic in t, namely $bt^2 + (a - d)t - c = 0$. The discriminant of this quadratic equation is $(a + d)^2 \pm 4$, never a perfect square unless $a+d = 0$. This is because the Diophantine equation $X^2 - Y^2 = 4$ has no integer solution except when $Y = 0$. But $a + d = 0$ implies $Ch(A) = x^2 \pm 1$. This not possible since A is hyperbolic. Hence from equation (2) it follows that eigen values are irrational.

□

Let p_n denote the number of solutions(in $\mathbb{T}^2$) of the equation $A^n X = X$ and let $q_n = |Trace(A^n)|$. Then it follows from the proposition 2.1.6 that $p_n = |Det(A^n - I)|$. It is observed in the following lemma that this sequence (q_n) follows a nice pattern for hyperbolic automorphisms.

Proposition 3.2.2. *Let T_A be a hyperbolic toral automorphism induced by the matrix $A \in GL(2, \mathbb{Z})$. Let t denote the trace of A and let $q_n = |Tr(A^n)|$. Then*

$q_{n+1} + Det(A)q_{n-1} = |t|q_n$ *for all* $n \geq 2$.

Proof. By Cayley Hamilton Theorem we have $A^2 - tA + (Det(A))I = 0$.

Multiplying by A^{n-1} we get

$A^{n+1} + (Det(A))A^{n-1} = tA^n$

The result follows immediately by taking the trace both sides.

Here we use the following facts :

(i) If the $Trace(A) \geq 0$ then $Trace(A^n) \geq 0$ for all $n \in \mathbb{N}$.

(i) If the $Trace(A) < 0$ then $Trace(A^n) < 0$ for all odd numbers $n \in \mathbb{N}$ and $Trace(A^n) > 0$ for all even numbers $n \in \mathbb{N}$.

□

Lemma 3.2.3. For $n \geq 3$, $n^2 - 2 > 2n$.

Proof. Proof follows by induction on n. □

Theorem 3.2.4. *For any hyperbolic toral automorphism $T_A : \mathbb{T}^2 \to \mathbb{T}^2$, $Per(T_A) \supset \mathbb{N} \setminus \{2\}$.*

Proof. Let $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ be hyperbolic. Note that, to prove $n \in Per(T_A)$ it is enough to prove that $p_{n+1} > p_1 + p_2 + ... + p_n$, where p_k = the number of solutions of the equation $A^k X = X$ which is equal to $|Det(A^k - I)|$, by proposition 2.1.6.

In fact, it is enough to prove: $p_{n+1} > p_1 + p_2 + ... + p_{n-1}$, because if x is a point of period n for T_A and $T_A^m(x) = x$ then n must divide m.

Let α be an eigen value of A with $|\alpha| > 1$. By proposition 3.2.1, $\alpha \in \mathbb{R}$ and hence either $\alpha < -1$ or $\alpha > 1$.

Case(1) : $\alpha > 1$

Subcase(1) : $Det(A) = 1$

Then the eigen values are $\alpha, \frac{1}{\alpha}$. Then $t = \alpha + \frac{1}{\alpha} \geq 3$ as t is an integer and $(\sqrt{\alpha} - \frac{1}{\sqrt{\alpha}})^2 > 0$. By proposition 2.1.6, $p_n = q_n - 2 \; \forall \; n \in \mathbb{N}$. From proposition 3.2.2, we have $q_{n+1} + q_{n-1} = tq_n \; \forall \; n \geq 2$.

Adding these equations for $n = 2, 3, \ldots, k$ we get,

$$q_1 + q_2 + 2(q_3 + q_4 + \ldots + q_{k-1}) + q_k + q_{k+1} = t(q_2 + q_3 + \ldots + q_k)$$

i.e.,

$$q_{k+1} = -q_1 + (t-1)q_2 + (t-2)[q_3 + q_4 + \ldots + q_{k-1}] + (t-1)q_k$$

$$\geq -q_1 + 2q_2 + q_3 + \ldots + q_{k-1} + 2q_k, \text{ since } t \geq 3.$$

$$= (-q_1 + 2q_2) + q_3 + \ldots + q_{k-1} + 2q_k$$

$> q_1 + q_2 + \ldots + q_{k-1} + q_k$ (since $t \geq 3$, by lemma-3.2.3 we have $t^2 - 2 > 2t$. Hence $q_2 = t^2 - 2 > 2t = 2q_1$ and therefore $q_2 - q_1 > q_1$.)

Therefore, $q_{k+1} - 2 > (q_1 - 2) + (q_2 - 2) + \ldots + (q_{k-1} - 2) + (q_k - 2)$

i.e, $p_{k+1} > p_1 + p_2 + ... + p_k$ for all $k \geq 2$.

Hence $k \in Per(T_A)$ for all $k \geq 3$ and hence $Per(T_A) \supset \mathbb{N} \setminus \{2\}$.

Subcase 2 : $Det(A) = -1$

Then eigen values are $\alpha, \frac{1}{\alpha}$. Then $t = \alpha - \frac{1}{\alpha} > 0$ and hence $t \geq 1$, as t is an integer. Using proposition 2.1.6,

$$p_n = \begin{cases} q_n - 2 & \text{if } n \textit{ is even} \\ q_n & \text{if } n \textit{ is odd} \end{cases}$$

From proposition 3.2.2, we have $q_{n+1} - q_{n-1} = tq_n$ for all $n \geq 2$.

Adding these relations for $n = 2, 3, ..., k$, we get,

$q_{k+1} = q_1 + (t+1)q_2 + t(q_3 + q_4 + ... + q_{k-1}) + (t-1)q_k$

$> q_1 + q_2 + ... + q_{k-1}$ (since $t \geq 1$.)

Hence $p_{k+1} > p_1 + p_2 + ... + p_{k-1}$. for all $k \geq 2$

Which implies that $Per(T_A) \supset \mathbb{N} \setminus \{2\}$.

<u>**Case (2)**</u> : $\alpha < -1$

Subcase 1 : $Det(A) = 1$

Therefore the eigen values are $\alpha, \frac{1}{\alpha}$. Then $-t = -\alpha - \frac{1}{\alpha} \geq 3$ as $(\sqrt{-\alpha} - \frac{1}{\sqrt{-\alpha}})^2 > 0$ and t is an integer. Using proposition 2.1.6, we get,

$$p_n = \begin{cases} q_n - 2 & \text{if } \textit{n is even} \\ q_n + 2 & \text{if } \textit{n is odd} \end{cases}$$

From proposition 3.2.2, we have $q_{n+1} + q_{n-1} = -tq_n \ \forall n \geq 2$. Adding these relation for $n = 2, 3, ..., k$, we get,

$$q_1 + q_2 + 2(q_3 + q_4 + ... + q_{k-1}) + q_k + q_{k+1} = -t(q_2 + q_3 + ... + q_k).$$

Hence

$$\begin{aligned} q_{k+1} &= -q_1 - (t+1)q_2 - (t+2)(q_3 + q_4 + ... + q_{k-1}) - (t+1)q_k. \\ &\geq -q_1 + 2q_2 + q_3 + q_4 + ... + q_{k-1} + 2q_k, \text{ as } -t \geq 3. \\ &> q_1 + q_2 + ... + q_k. \end{aligned}$$

Therefore $p_{k+1} > p_1 + p_2 + ... + p_{k-1} \ \forall k \geq 2$. Hence $Per(T_A) \supset \mathbb{N} \setminus \{2\}$.

Subcase 2 : $Det(A) = -1$

Here the eigen values are $\alpha, \frac{-1}{\alpha}$. Then $t = \lambda - \frac{1}{\lambda} < 0$ and hence $-t \geq 1$, since t is an integer. Using proposition 2.1.6,

$$p_n = \begin{cases} q_n - 2 & \text{if } \textit{n is even} \\ q_n & \text{if } \textit{n is odd} \end{cases}$$

From proposition 3.2.2, we have $q_{n-1} - q_{n+1} = tq_n$, for all $n \geq 2$.

Adding these relations for $n = 2, 3, ...k$, we get,

$$\begin{aligned} q_{k+1} &= q_1 + (1-t)q_2 - t(q_3 + q_4 + ... + q_{k-1}) - (t+1)q_k \\ &> q_1 + q_2 + \ldots + q_{k-1} \text{ (Since } -t \geq 1, \text{ we have } 1 - t > 1.) \end{aligned}$$

Which implies that $p_{k+1} > p_1 + p_2 + ... + p_{k-1}$ for all $k \geq 2$.

Hence $Per(T_A) \supset \mathbb{N} \setminus \{2\}$.

$\square$

Remark 3.2.5. From the above proposition , it is clear that for a hyperbolic automorphism T_A, the period set $Per(T_A)$ is either $\mathbb{N}$ or $\mathbb{N} \setminus \{2\}$. In proposition 3.3.5 and proposition 3.3.6 we prove that for certain class of hyperbolic automorphisms $Per(T_A)$ is $\mathbb{N}$ and for some other class of automorphisms, it is $\mathbb{N} \setminus \{2\}$.

3.3 The nonhyperbolic case

Note that $1 \in Per(T_A), \forall A \in GL(2, \mathbb{Z})$. Suppose that $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ is a non-hyperbolic toral automorphism. Then $t = |\alpha + \beta| \leq |\alpha| + |\beta| \leq 2$. That is $t \in \{-2, -1, 0, 1, 2\}$ and $Det(A) = \pm 1$. Thus, any $A \in GL(2, \mathbb{Z})$ which is nonhyperbolic will fall under one of these 10 cases . For $A \in GL(2, \mathbb{Z})$, let $ch(A)$ denote the characteristic polynomial of A .

Proposition 3.3.1. *Let $A \in GL(2, \mathbb{Z})$ be such that $Det(A) = -1$, $Trace(A) = 0$. Then $Per(T_A) = \{1, 2\}$.*

Proof. The $ch(A) = x^2 - 1$. Then by Cayley-Hamilton theorem $A^2 = I$ and $A \neq I$. Hence $Per(T_A) = \{1, 2\}$. □

Proposition 3.3.2. *Let $A \in GL(2, \mathbb{Z})$ be such that $Det(A) = 1$, $Trace(A) = 0$. Then $Per(T_A) = \{1, 2, 4\}$.*

Proof. The $ch(A) = x^2 + 1$. Then $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ with $a^2 + bc = -1$.

Then by Cayley-Hamilton theorem, $A^2 = -I$ and hence $A^4 = I$. Note that $p_1 = |Det(A - I)| = 2$ and $p_2 = |Det(A^2 - I)| = |Det(-2I)| = 4$. Hence $2 \in Per(T_A)$. Hence $Per(T_A) = \{1, 2, 4\}$. □

Proposition 3.3.3. *Let $A \in GL(2, \mathbb{Z})$ be such that $Det(A) = 1$, $Trace = -1$. Then $Per(T_A) = \{1, 3\}$.*

Proof. The $ch(A)$ is x^2+x+1. Then$A = \begin{pmatrix} a & b \\ c & -1-a \end{pmatrix}$ with $a^2+a+bc = -1$. Now $p_1 = |Det(A-I)| = 3$ and $p_2 = |Det(A_2-I)| = 3$. Hence $2 \notin Per(T_A)$. Note that $A^3 = I$. This implies $3 \in Per(T_A)$ and hence $Per(T_A) = \{1,3\}$. ☐

Proposition 3.3.4. *Let $A \in GL(2,\mathbb{Z})$ be such that $Det(A) = 1$ and $Trace(A) = 1$. Then $Per(T_A) = \{1,2,3,6\}$.*

Proof. The $ch(A)$ is x^2-x+1. Then $A = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$ with $a-a^2-bc = 1$. By Cayley-Hamilton theorem, $A^2-A+I = 0$.

Now $p_1 = |Det(A-I)| = 1$. Hence A has unique fixed point namely 0, $p_2 = |Det(A^2-I)| = 3 > p_1 = 1$ showing that $2 \in Per(T_A)$, $p_3 = |Det(A^3-I)| = 4 > p_1 = 1$ showing that $3 \in Per(T_A)$. Again, $p_4 = |Det(A^4-I)| = 3 \ngtr p_2 = 3$ showing that $4 \notin Per(T_A)$. $p_5 = |Det(A^5-I)| = 1 \ngtr p_1 = 1$ showing that $5 \notin Per(T_A)$. Now, $A^6 = I$. This implies that $6 \in Per(T_A)$ and hence $Per(T_A) = \{1,2,3,6\}$. ☐

Proposition 3.3.5. *Let $A \in GL(2,\mathbb{Z})$ be such that $Det(A) = -1$ and $Trace(A) = \pm 1$. Then $Per(T_A) = \mathbb{N} \setminus \{2\}$.*

Proof. **Case: 1** Suppose $Det(A) = -1$ and $Trace(A) = 1$. Then $ch(A)$ is x^2-x-1 and hence the eigen values are $\frac{1}{2} \pm \frac{\sqrt{5}}{2}$, showing that A is hyperbolic. Therefore by theorem 3.2.4 we have $Per(T_A) \supset \mathbb{N} \setminus \{2\}$.

We now prove that $2 \notin Per(T_A)$.

From the hypothesis, it is clear that $A = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$ for some integers a, b, c with $a^2-a+bc = 1$.

Therefore, $p_2 = |Det(A^2-I)|$

$$= |Det(A-I)||Det(A+I)|$$

$$= p_1 |Det(A+I)|$$

$$= p_1 \text{ (Since } |Det(A+I)| = |a^2+a-bc| = 1.)$$

Hence $2 \notin Per(T_A)$.

Case: 2

Suppose $Det(A) = -1$ and $Trace(A) = -1$.

Proof of this is similar to that of Case: 1

□

Proposition 3.3.6. *Let $A \in GL(2, \mathbb{Z})$ be such that $Det(A) = -1$ and $Trace(A) = \pm 2$. Then $Per(T_A) = \mathbb{N}$.*

Proof. **Case: 1** Suppose $Det(A) = -1$ and $Trace(A) = 2$. Then $ch(A)$ is $x^2 - 2x - 1$ and hence the eigen values are $1 \pm \sqrt{2}$, showing that A is hyperbolic. Therefore by theorem 3.2.4 we have $Per(T_A) \supset \mathbb{N} \setminus \{2\}$.

We now prove that $2 \in Per(T_A)$.

From the hypothesis, it is clear that $A = \begin{pmatrix} a & b \\ c & 2-a \end{pmatrix}$ for some integers a, b, c with $a^2 - 2a + bc = 1$.

Therefore, $p_2 = |Det(A^2 - I)|$

$$= |Det(A-I)||Det(A+I)|$$

$$= p_1 |Det(A+I)|$$

$$= p_1.2 \text{ (Since } |Det(A+I)| = 2.)$$

Hence $2 \in Per(T_A)$.

□

Proposition 3.3.7. *Let $Det(A) = 1$ and $Trace(A) = 2$. Then $Per(T_A) = \mathbb{N}$, provided $A \neq I$, in which case $Per(T_A) = \{1\}$.*

Proof. $Ch(A)$ is $x^2 - 2x + 1$. Then $A = \begin{pmatrix} a & b \\ c & 2-a \end{pmatrix}$ with $-a^2 + 2a - bc = 1$.

Now $Det(A - I) = 0$ and therefore A has infinitely many fixed points.

By induction on n, we have $A^n - I = n(A - I)$ for all $n \in \mathbb{N}$ and hence the system $A^nX = X$ has infinitely many solutions in $\mathbb{T}^2$.

To prove $Per(T_A) = \mathbb{N}$, it is enough to prove that for every $k(> 1) \in \mathbb{N}$ there exist a $X = (x, y) \in \mathbb{T}^2$ such that $A^kX = X$ and $A^iX \neq X \; \forall \; 1 \leq i < k$. This is evident from the fact that the equation $A^nX = X$ is equivalent to the following conditions:

$$\left.\begin{array}{c} n[(a-1)x + by] \in \mathbb{Z} \\ n[cx + (1-a)y] \in \mathbb{Z} \end{array}\right\} \tag{3.1}$$

For, Suppose $k \in \mathbb{N}$ and $k > 1$

Case: 1

Suppose $b \neq 0$ and $1 - a \neq 0$.

Then $X = (0, \frac{1}{k.gcd(b,1-a)}) \in \mathbb{T}^2$ satisfies our requirement and hence it is a point of period k.

Case: 2

Suppose $b = 0$ and $1 - a \neq 0$.

In this case, $X = (0, \frac{1}{k(1-a)}) \in \mathbb{T}^2$ is point of period k.

Case: 3

Suppose $b \neq 0$ and $1 - a = 0$.

Let

$$X = \begin{cases} (\frac{1}{kc}, 0) & \text{if } c \neq 0 \\ (0, \frac{1}{kb}) & \text{if } c = 0 \end{cases}$$

Then X is point of period k.

Case: 4

Suppose $b = 0$ and $1 - a = 0$

If $c \neq 0$ then take $X = (\frac{1}{kc}, 0)$. If $c = 0$ then $A = I$, in which case $Per(T_A) = \{1\}$.

□

Proposition 3.3.8. *Let $Det(A) = 1$ and $Tr(A) = -2$. Then $Per(T_A) = 2\mathbb{N} \cup \{1\}$, provided $A \neq -I$, in which case $Per(T_A) = \{1, 2\}$.*

Proof. The $ch(A) = x^2 + 2x + 1$. Then$A = \begin{pmatrix} a & b \\ c & -2-a \end{pmatrix}$ with $a^2 + 2a + bc = -1$.

It can be shown by induction, with the use of Cayley-Hamilton's theorem that

$$A^n = \begin{cases} \begin{pmatrix} -na-n+1 & -nb \\ -nc & na+n+1 \end{pmatrix} & \text{if } n \textit{ is even} \\ \begin{pmatrix} na+n-1 & nb \\ nc & -na-n-2 \end{pmatrix} & \text{if } n \textit{ is odd} \end{cases}$$

Now ,

$$|Det(A^n - I)| = \begin{cases} 4 & \text{if } n \textit{ is odd} \\ 0 & \text{if } n \textit{ is even} \end{cases}$$

It is immediate that, if $n \neq 1$ is odd then $n \notin Per(T_A)$. Also, when n is even, the equation $A^n X = X$ has infinitely many solutions. Now, the equation $A_n X = X$ is equivalent to

$$\left.\begin{aligned} n[(a+1)x + by] &\in \mathbb{Z} \\ n[cx - (a+1)y] &\in \mathbb{Z} \end{aligned}\right\} \tag{3.2}$$

Now, we argue as in the previous proposition to show that $2\mathbb{N} \subset Per(T_A)$, except the case when $a + 1 = b = c = 0$, in which case $A = -I$, for which $Per(T_A) = \{1, 2\}$. Thus $Per(T_A) = 2\mathbb{N} \cup \{1\}$. □

Thus we have proved,

Theorem 3.3.9. *Let T_A be a nonhyperbolic toral automorphism. Then $Per(T_A)$ is one of the following 7 subsets of $\mathbb{N}$.*

(1) $\{1\}$

(2) $\{1, 2\}$

(3) $\{1, 3\}$

(4) $\{1, 2, 4\}$

(5) $\{1, 2, 3, 6\}$

(6) $2\mathbb{N} \cup \{1\}$

(7) $\mathbb{N}$

In the following table the set $Per(T_A)$ is listed in terms of the minimal polynomial of the induced matrix A for nonhyperbolic automorphisms.

Minimal polynomial of A	$Per(T_A)$
$x^2 - 1, x + 1$	{1,2}
$x^2 + 1$	{1,2,4}
$x^2 + x + 1$	{1,3}
$x^2 - x + 1$	{1,2,3,6}
$x^2 - 2x + 1$	$\mathbb{N}$
$x^2 + 2x + 1$	$2\mathbb{N} \cup \{1\}$
$x - 1$	{1}

Main Theorem

Theorem 3.3.10. *Let T_A be a toral automorphism. Then $Per(T_A)$ is one of the following 8 subset of $\mathbb{N}$.*

(1) $\{1\}$

(2) $\{1, 2\}$

(3) $\{1, 3\}$

(4) $\{1, 2, 4\}$

(5) $\{1, 2, 3, 6\}$

(6) $2\mathbb{N} \cup \{1\}$

(7) $\mathbb{N} \setminus \{2\}$

(8) $\mathbb{N}$

Proof. Proof follows from theorems 3.2.4 and 3.3.9. $\square$

The following table summarizes the known results, similar to our main result of this chapter.

A class of dynamical systems	**Family of period sets**
Homeomorphisms on $\mathbb{R}$	Empty set, $\{1\}$, $\{1, 2\}$
Continuous maps on $\mathbb{R}$	As in theorem 3.1.3
Continuous self maps of the upper half plane in $\mathbb{R}^2$	All subsets of $\mathbb{N}$.
Continuous maps on the closed unit disc.	All subsets of $\mathbb{N}$ containing 1.
Circle maps with degree 1	As in theorem 3.1.12
Complex polynomials.	Five subsets of $\mathbb{N}$ as in theorem 3.1.5 .
Automorphism on abelian, torsion free groups	Subsets of $\mathbb{N}$, containing 1 and closed under l.c.m.
Continuous toral automorphisms.	*8 subsets of* $\mathbb{N}$ as in theorem 3.3.10.

Chapter 4

A counting problem

4.1 Dynamically Special points

The properties of dynamical systems which are preserved by topological conjugacies are called dynamical properties. The points which are unique upto some dynamical property are called *dynamically special points.* Said differently, a special point has a dynamical property which no other point has. The idea of special points is new to the literature.

Throughout this chapter we will be working with continuous self maps of the real line. Since $\mathbb{R}$ has order structure, we would like to consider the conjugacies preserving the order. Hence the conjugacies which we consider in this chapter are order preserving conjugacies (increasing conjugacies).

When we are working with a single system, any self conjugacy can utmost shuffle points with same dynamical behavior. Therefore a point which is unique upto its behavior must be fixed by every self conjugacy . On the other hand if a point is fixed by all self conjugacies then it must have a special property (some times it may not be known explicitly). These things motivated us to call the set of all points fixed by all self-conjugacies as set of *special points*.

For $x, y \in \mathbb{R}$, we write $x \sim y$ if x and y have the same dynamical properties in the dynamical system $(\mathbb{R}, f)$. Said precisely, $x \sim y$ if there exists an increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that $h \circ f = f \circ h$ and $h(x) = y$. It is easy to see that $\sim$ is an equivalence relation. Let $[x]$ to denote the equivalence class of $x \in \mathbb{R}$.

In a dynamical system (X, f), we say that a point x is ordinary if, it is "like" points near it. That is,

Definition 4.1.1. An element $x \in \mathbb{R}$ is *ordinary* in $(\mathbb{R}, f)$ if its equivalence class $[x]$ is a neighbourhood of it. i.e, the equivalence class of x contains an open interval around x. A point which is not ordinary is called *non-ordinary* . Let $N(f)$ denote the set of all non-ordinary points of f.

We call a point to be *special* if $[x] = \{x\}$. Let $S(f)$ denote the set of all special points of f.

Remark 4.1.2. A point x in a topological space X is said to be *rigid* if it is fixed by every self homeomorphism of X. For example, the point 1 is rigid in $(0, 1]$. According to the above definition all rigid points are special, even though there is no role for the map f. we make this as a convention.

4.1.1 Examples and some characterization theorems

Definition 4.1.3. Let (X, f) be a dynamical system. By the *full orbit* of a point $x \in X$ we mean the set

$\widetilde{O}(x) = \{y \in X | f^n(x) = f^m(y) \text{ for some } m, n \in \mathbb{N}\}$.

For any subset $A \subset \mathbb{R}$, let

$$\tilde{O}(A) = \bigcup_{x \in A} \widetilde{O}(x) = \bigcup_{x \in A} \{y \in \mathbb{R} : f^n(y) = f^m(x) \ for\ some\ m,\ n \in \mathbb{N}\}.$$

Definition 4.1.4. A point x in a dynamical system (X, f) is said to be a *critical point* if f fails to be one-one in every neighbourhood of x. The set of all critical points of f is denoted by $C(f)$.

We prove in [25], the following characterization theorem for the set $N(f)$ (and hence for $S(f)$).

Theorem 4.1.5. *For continuous self-maps of the real line $\mathbb{R}$, the set of all nonordinary points is contained in the closure of the union of full orbits of critical points , periodic points and the limits at infinity (if they exist and finite).*

Remark 4.1.6. In the above theorem the inclusion can be strict.
Consider the map $f(x) = x + sinx$ for all $x \in \mathbb{R}$. All integral multiples of π are fixed points for this map but the increasing bijection $x \mapsto x + 2\pi$ commutes with f and fixes none of them .

Theorem 4.1.7. *[25] For polynomials of even degree the equality is true in the previous theorem.*

Let $D = \tilde{O}(C(f) \cup P(f) \cup \{f(\infty), f(-\infty)\})$. Where $f(\infty)$ and $f(-\infty)$ are the limits of f at ∞ and $-\infty$ respectively, provided they are finite, $C(f)$ denotes the set of all critical points and $P(f)$ denotes the set of all periodic points of f.

Theorem 4.1.8. *[25] For polynomial maps f of $\mathbb{R}$, $S(f)$ has to be either empty or a singleton or the whole $\overline{D}$*

From the definition, it is clear that the set of special points $S(f)$ is always closed. The following theorem is about the converse and it is proved in [25].

Theorem 4.1.9. *Given any closed subset F of $\mathbb{R}$, there exists a continuous map f : $\mathbb{R} \to \mathbb{R}$ such that $S(f) = F$.*

Examples

For any $f : \mathbb{R} \to \mathbb{R}$, let G_f denote the set of all topological conjugacies of f and let $G_{f\uparrow}$ denote the set of all increasing conjugacies of f.

Proposition 4.1.10. *If x is an ordinary point of f and if h is self topological conjugacy of f, then $h(x)$ is ordinary.*

Proof. Since x is ordinary there exists an open interval V contained in $[x]$. We prove that the open interval(since h is a homeomorphism) $h(V)$ is contained in $[h(x)]$.

Take $s \in h(V)$. Then $s = h(t)$ for some $t \in V$. Since $V \subset [x]$, there exists $\varphi \in G_{f\uparrow}$ such that $\varphi(t) = x$. Then the increasing homeomorphism $\psi = h\phi h^{-1}$ carries s to $h(x)$ and commutes with f.

□

Proposition 4.1.11. *If x is a nonordinary point of f and if h is a self topological conjugacy of f, then $h(x)$ is nonordinary.*

Proof. Note that, if h is a self conjugacy of f then h^{-1} is also a self conjugacy of f. Now, the proof follows from the previous proposition.

□

Example 4.1.12. (i) If $f : \mathbb{R} \to \mathbb{R}$ has a unique fixed point then it is nonordinary.

(ii) If $f : \mathbb{R} \to \mathbb{R}$ has a unique nonordinary point then it must be a fixed point.

Proof. (i) Since the topological conjugacies carry fixed points to fixed points, the unique fixed point must be fixed by every self conjugacy and hence special.

(ii) Suppose $x_0 \in \mathbb{R}$ is the unique nonordinary point of f. Then $h(x_0) = x_0$ for all $h \in G_{f\uparrow}$. Now, for any $h \in G_{f\uparrow}$ we have $h(f(x_0)) = f(h(x_0)) = f(x_0)$. That is, the point $f(x_0)$ is special. since x_0 is the only special point, we have $f(x_0) = x_0$.

□

Example 4.1.13. If $f : \mathbb{R} \to \mathbb{R}$ has finitely many fixed points(critical points) then all fixed(critical) points are special and hence nonordinary.

Proof. Follows from the fact that under a topological conjugacy fixed points will be mapped to fixed points and critical points will be mapped to critical points and the fact that it takes the finite set F (of fixed points) to F bijectively, preserving the order.

□

Example 4.1.14. If there are only finitely many periodic cycles then all periodic points are special.

Remark 4.1.15. It is immediate from the definition that every special point is nonordinary. But every nonordinary point may not be special. For example, consider the map $x \mapsto x + \sin x$ on $\mathbb{R}$ which has countably many fixed points namely $n\pi$ where $n \in \mathbb{Z}$. Among them, the fixed points $2k\pi, (k \in \mathbb{Z})$ are repelling and the fixed points $(2k+1)\pi, (k \in \mathbb{Z})$ are attracting. Recall that repelling fixed points cannot be conjugate to attracting fixed points. Note that all these fixed points are nonordinary but these fixed points constitute two conjugacy classes namely, $[0] = \{2k\pi : k \in \mathbb{Z}\}$ and $[\pi] = \{(2k+1)\pi : k \in \mathbb{Z}\}$,(Conjugacies of the form $x \mapsto x + 2k\pi$ serve the purpose.) hence they are not special.

The following proposition is about the converse.

Proposition 4.1.16. *If $f : \mathbb{R} \to \mathbb{R}$ has only finitely many nonordinary points then every nonordinary point is special.*

Proof. Let $N(f)$ to denote the set of all nonordinary points of f. Since $N(f)$ is finite, it follows from the previous proposition that $h(N(f)) = N(f)$ for all $h \in G_{f\uparrow}$. Then we must have $h(x) = x$ for all $x \in N(f)$, because of the order preserving nature of h. Hence all points of $N(f)$ are special.

$\square$

Thus, for the class of maps with finitely many nonordinary points the idea of special points and the idea of nonordinary point, coincide.

Proposition 4.1.17. *For maps with finitely many nonordinary points, $f(x)$ is nonordinary whenever x is nonordinary.*

Proof. Since x is nonordinary and since there are only finitely many nonordinary points, we have $h(x) = x$ for all $h \in G_{f\uparrow}$.

Now for any $h \in G_{f\uparrow}$, we have $h(f(x)) = f(h(x)) = f(x)$. Hence $f(x)$ is nonordinary.

$\square$

Definition 4.1.18. For any subset A of $\mathbb{R}$, we write $\partial A = \overline{A} \cap \overline{(X - A)}$ and call the boundary of A . Here $\overline{A}$ denotes the closure of A in $\mathbb{R}$.

Recall that the properties which are preserved under topological conjugacies are called dynamical properties. Hence, if two points x, y in the dynamical system (X, f), differ by a dynamical property, then no conjugacy can map one to the other, from which it follows that,

Proposition 4.1.19. *For any dynamical property P, the points of ∂S_P are non-ordinary. Here S_P denotes the set of all points in (X, f) having the dynamical property P.*

Corollary 4.1.20. *Let $f : \mathbb{R} \to \mathbb{R}$ be constant in a neighbourhood of a point x_0. Then the end points of the maximal interval around x on which f is constant, are non-ordinary.*

Lemma 4.1.21. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $a < b$ and $(a, b) \cap N(f) = \emptyset$. Then $x \sim y$ for all $x, y \in (a, b)$.

Proof. Assume WLOG that $x < y$. Suppose $x \nsim y$, so $z = \sup([x] \cap (-\infty, y])$ exists. Clearly $z \in \overline{[x]}$. If $z = y$ then $z \in \overline{[y]} \subset \overline{\mathbb{R} \setminus [x]}$. Otherwise $z < y$ and $[z, y) \cap (\mathbb{R} \setminus [x]) \neq \emptyset$ for every $y - x > \epsilon > 0$ which again shows $z \in \overline{\mathbb{R} \setminus [x]}$. Hence $z \in \partial([x])$, so $z \in N(f)$ by proposition 4.1.19. But $a < x \leq z \leq y < b$ so $z \in (a, b) \cap N(f)$ contradicting our hypothesis.

□

Theorem 4.1.22. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If $|N(f)| = n$ then $|\{[x] : x \in \mathbb{R}\}| = 2n + 1$.*

Proof. Let $N(f) = \{x_1, x_2, \cdots, x_n\}$ where $x_1 < x_2 < \cdots < x_n$. By proposition 4.1.16, each $\{x_i\}$ is an equivalence class. By remark 4.2.1, each of these intervals $(-\infty, x_1), (x_1, x_2), \cdots, (x_{n-1}, x_n), (x_n, \infty)$ is invariant under every element of $G_{f\uparrow}$, so all remaining equivalence classes are contained in one of these intervals. Lemma 4.1.21 above now shows that each of these interval is an equivalence class, giving $|\{[x] : x \in \mathbb{R}\}| = 2n + 1$. □

Remark 4.1.23. If the definition of $\sim$ is weakened to allow decreasing conjugacies (i.e, $x \sim y$ if there exists $h \in G_f$ such that $h(x) = y$), similar reasoning yields the inequality

$$n + 1 \leq |\{[x] : x \in \mathbb{R}\}| \leq 2n + 1$$

and both bounds can be obtained. This is illustrated by the maps $f(x) = 2x$ for which $|N(f)| = 1$ and $|\{[x] : x \in \mathbb{R}\}| = 2$ and $f(x) = |x|$ for which $|N(f)| = 1$ and and $|\{[x] : x \in \mathbb{R}\}| = 3$.

Remark 4.1.24. Note that, being a point in a particular equivalence class $[x]$ is a dynamical property.

Remark 4.1.25. There are maps $f : \mathbb{R} \to \mathbb{R}$ having finitely many equivalence classes, but infinitely many nonordinary points. For example, consider the map $f(x) = x + \sin x$ on $\mathbb{R}$. As noted above, (See remark 4.1.15) there are two classes of fixed points. Since increasing orbits must map to increasing orbits under increasing conjugacies, points like $\frac{\pi}{2}$(increasing orbit) and $\frac{3\pi}{2}$ (decreasing orbit) cannot be equivalent. Hence there must be at least four equivalence classes. To see that there are exactly four equivalence classes, let $I_k = (2k\pi, (2k+1)\pi), D_k = ((2k+1)\pi, 2(k+1)\pi)$ and observe that $I_k \cap N(f) = \emptyset = D_k \cap N(f)$ for each $k \in \mathbb{Z}$ by proposition 4.1.34. Hence by lemma 4.1.21, each I_k and D_k is contained in a single equivalence class. Conjugacies of the form $x \mapsto x + 2k\pi$ complete the argument.

Recall that, if $f : \mathbb{R} \to \mathbb{R}$ has a unique non-ordinary point then it is a fixed point.

We say that a function $f : \mathbb{R} \to \mathbb{R}$ is locally constant at a point $x_0 \in \mathbb{R}$ if f is constant in some open interval around x_0.

Proposition 4.1.26. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then*

(i) If $x \in \mathbb{R}$ is both critical and ordinary then f is locally constant at x.

(ii) If x is ordinary and f is not locally constant at x then $f(x)$ is ordinary.

Proof. (i) Let $x_0 \in \mathbb{R}$ be both critical and ordinary.

Claim: f is constant in some neighbourhood of x_0.

Since x_0 is ordinary, there exist $\eta > 0$ such that all points in $(x_0 - \eta, x_0 + \eta)$ will look alike. So it is enough to prove that f is somewhere constant in $(x_0 - \eta, x_0 + \eta)$.

Case: 1

Now , suppose some point of $(x_0 - \eta, x_0 + \eta)$ is point of local maximum. Then we can prove easily that every point of $(x_0 - \eta, x_0 + \eta)$ is a point of local maximum. That is there exist $\epsilon > 0$ such that $f(x_0) \geq f(t)\ \forall\ t \in (x_0 - \epsilon, x_0 + \epsilon)$. Next, choose $\delta < \epsilon, \eta$. Then there exist $y \in [x_0 - \delta, x_0 + \delta]$ such that $f(y) \leq f(t)\ \forall\ t \in [x_0 - \delta, x_0 + \delta]$ ....(1).

But y is a point of local maximum (since $\delta < \eta$). That is there exist $\alpha > 0$ such that $f(y) \geq f(s)\ \forall\ s \in (y - \alpha, y + \alpha)$ ....(2).

From equations (1) and (2), it follows that f is constant in some neighbourhood y and hence constant in some neighbourhood of x_0.

Case: 2

No point is a point of local maximum. That is, in every subinterval f attains its maximum at one of the end points.

If f assumes supremum always on the right end or always on the left end then f is strictly monotone.

Note that, it is enough if we prove monotone somewhere. Take a neighbourhood (α, β) of x_0 such that $(\alpha, \beta) \subset (x_0 - \eta, x_0 + \eta)$ and let $\sup f$ on (α, β) is attained at the right end point β. Suppose $\sup f$ is attained at the right end point in every subinterval of (α, β) containing x_0. Then f is increasing in (x_0, β). We are done.

Suppose there is a subinterval say $(x_0 - \epsilon_1, x_0 + \epsilon_2)$ of (α, β) on which f attains its supremum at the left end point. Then f attains its infimum on $(x_0 - \epsilon_1, \beta)$ at some interior point. We now argue as in Case.1, with infimum instead of supremum.

Proof of (ii): We make use of (i).

Assume that f is not constant on any neighbourhood of x. Because x is ordinary, there exist an open interval J around x in which all points are equivalent such that f is not constant on J. It follows that f is not constant on any non-trivial subinterval of J, because the end points of intervals of constancy are non-ordinary. From (i), it follows that J has no critical point. Therefore $f(J)$ is an open interval. We claim that any two elements of $f(J)$ are equivalent. Let $f(y)$ be a general element of $f(J)$ where $y \in J$, $y \neq x$. By choice of J, there exists a self conjugacy h of f such that $h(y) = x$. Which implies $hf(y) = fh(y) = f(x)$. Therefore $f(y)$ is equivalent to $f(x)$. This proves $f(x)$ is ordinary.

□

Example 4.1.27. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then $\sup f(\mathbb{R})$, $\inf f(\mathbb{R})$, $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are nonordinary (in fact, special points) provided they are finite. Note that, For maps with finitely many nonordinary points both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ always exists in $\mathbb{R} \cup \{\infty, -\infty\}$.

Proof. For any $h \in G_{f\uparrow}$, $h(f(\mathbb{R})) = f(h(\mathbb{R})) = f(\mathbb{R})$. That is h takes the range of f to itself. Since h is increasing, $h(\sup f) = \sup f$ and $h(\inf f) = \inf f$.

To prove $\lim_{x\to\infty} f(x)$ **is special:**

First we prove that for maps with finitely many nonordinary points, $\lim_{x\to\infty} f(x)$ always exists in $\mathbb{R} \cup \{\infty, -\infty\}$.

For, let t_0 be the largest nonordinary point and let A be the set of all critical points $> t_0$.

Suppose A is empty. Then f is monotone on $[t_0, \infty)$ and hence $\lim_{x\to\infty} f(x)$ exists.

Suppose A is nonempty. Then ∂A is nonempty. But every element of ∂A is nonordinary. Hence $\partial A = \{t_0\}$. Therefore $A = (t_0, \infty)$. Therefore f is constant on A (We argue as in the proof of Case-2 of (i), in the previous proposition). Hence $\lim_{x\to\infty} f(x)$ exists.

Now to prove $\lim_{x\to\infty} f(x)$ is special:

Denote $\lim_{x\to\infty} f(x)$ by l. Let $h \in G_{f\uparrow}$. Note that for any sequence $(x_n) \to \infty$ the sequence $f(x_n) \to l$ and the sequence $h(x_n) \to \infty$.

Let $(x_n) \to \infty$. Then $f(x_n) \to l$. Hence $h(f(x_n)) = f(h(x_n)) \to h(l)$. But the sequence $h(x_n) \to \infty$. Hence by the definition of l, $f(h(x_n)) = h(f(x_n)) \to l$. Hence $h(l) = l$. This completes the proof.

□

Proposition 4.1.28. *The maps $x + 1$ and $x - 1$ on $\mathbb{R}$ are topologically conjugate; but not order conjugate.*

Proof. The maps $x + 1$ and $x - 1$ are conjugate to each other through $-x + \frac{1}{2}$.

If possible, let h be an order conjugacy from $f(x) = x + 1$ to $g(x) = x - 1$. Then $h(x + 1) = h(f(x)) = g(h(x)) = h(x) - 1$. i.e, $h(x + 1) - h(x) = -1 < 0$. Which is a contradiction to the assumption that h is increasing.

□

Remark 4.1.29. Note that for the map $x + 1$ on $\mathbb{R}$, all points are ordinary. For, if $a, b \in \mathbb{R}$, then the map $x + b - a$ is the order conjugacy of $x + 1$ which maps a to b.

The following proposition is proved in [26].

Proposition 4.1.30. *Let $f : \mathbb{R} \to \mathbb{R}$ be a homeomorphism without fixed points. Then*

(i) If $f(0) > 0$ then f is order conjugate $x + 1$.

(ii) If $f(0) < 0$ then f is order conjugate $x - 1$.

Proof. Define $h : \mathbb{R} \to \mathbb{R}$ as follows. Assume $f(0) > 0$. Define $h(t) = \frac{t}{f(0)}$, $0 \leq t < f(0)$. We know that $(f^n(0))$ increases and diverges to ∞ and $(f^{-n}(0))$ decreases and diverges to $-\infty$ for all $n \in \mathbb{N}$. Moreover for $t \in \mathbb{R}$ there exist unique $n \in \mathbb{Z}$ such that, $f^n(0) \leq t < f^{n+1}(0)$. Define $h(t) = h(f^{-n}(t)) + n$. Then $h \circ f(t) = h(t) + 1 \; \forall \; t \in \mathbb{R}$. This h gives a conjugacy from f to $x + 1$.

If $f(0) < 0$ then we can give a similar proof.

□

Corollary 4.1.31. *Let $f, g : (a, b) \to (a, b)$ be homeomorphisms without fixed points. Then f is order conjugate to g if and only if both $graph(f)$ and $graph(g)$ are on the same side of the diagonal.*

In particular,

(i) If $f(x) > x$ for all $x \in (a, b)$ then f is order conjugate to $x + 1$.

(ii) If $f(x) < x$ for all $x \in (a, b)$ then f is order conjugate to $x - 1$.

Remark 4.1.32. In fact, in the previous corollary, the interval (a, b) can be replaced by any open ray in $\mathbb{R}$.

Remark 4.1.33. For an increasing bijection $f : \mathbb{R} \to \mathbb{R}$ with finitely many nonordinary points, all nonordinary points are fixed points.

Proof. We know that, for maps with finitely many nonordinary points , all nonordinary points are fixed by every order conjugacy. Here f itself is a self conjugacy.

□

For a continuous map $f : \mathbb{R} \to \mathbb{R}$, let $Fix(f)$ denote the set of all fixed points of f. It follows from the continuity of f that $Fix(f)$ is closed.

Recall that for any subset A of a topological space X,

$$(\partial A)^c = int(A) \cup int(A^c) \tag{4.1}$$

The following theorem gives a characterization for the nonordinary points of increasing homeomorphisms.

Proposition 4.1.34. *Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing bijection and let $x \in \mathbb{R}$. Then x is non-ordinary if and only if x is in the boundary of Fix(f).*

Proof. Necessary part :

Let $x \in \partial Fix(f)$. Then x is non-ordinary, since every open interval around x contains fixed and non-fixed points.

Sufficient part :

Suppose $x \notin \partial Fix(f)$. We shall prove that x is ordinary.

Now, $x \notin \partial Fix(f)$ implies $x \in (\partial Fix(f))^c = int(Fix(f)) \cup int((Fix(f))^c)$ by equation (4.1) . Hence $x \in int(Fix(f))$ or $x \in int(Fix(f)^c)$.

Case: 1

Suppose $x \in int(Fix(f))$. Then Choose $a, b \in \mathbb{R}$ such and $x \in (a, b) \subset Fix(f)$. Let $y \in (a, b)$ be such that $y \neq x$.

Then define $\phi_y : \mathbb{R} \to \mathbb{R}$ by,

$$\phi_y(t) = \begin{cases} t & \text{if } t \notin (a, b) \\ y & \text{if } t = x \\ \textit{extend linearly} & \textit{at other places} \end{cases}$$

This ϕ_y is an increasing continuous bijection on $\mathbb{R}$ which maps y to x. Both $[a, b]$ and its complement are invariant under both ϕ_y and f. This ϕ_y commutes with f since on $[a, b]$, f is identity and on the complement of $[a, b]$, ϕ_y is identity. This proves x is an ordinary point.

Case: 2

Suppose $x \in int(Fix(f)^c)$. Let (a, b) be the component interval(open) of $(Fix(f))^c$ containing x. Then $f(a) = a$ and $f(b) = b$ and since f is increasing, the map $f|_{(a,b)}$ is a fixed point free self map of (a, b). Hence by corollary 4.1.31, the map $f|_{(a,b)}$ is order conjugate to either $x + 1$ or $x - 1$, for which all points are ordinary. This completes the proof.

□

4.2 Counting homeomorphisms

Note that, under a topological conjugacy a point can be mapped to a point with similar dynamics. By definition the points of $[x]$ are dynamically same. i.e, all have the dynamics similar to that of x.

We now consider the systems for which there are only finitely many equivalence classes. This means there are only finitely many kinds of orbits upto conjugacy. For this reason we call such systems as *simple systems* .

In this chapter, we try to understand some simple systems on $\mathbb{R}$.

Recall that, if S_P denote the set of all points having the dynamical property P then the points of ∂S_P (the boundary of S_P) are nonordinary. In particular, being a point in a particular equivalence class is a dynamical property of the point. Hence by the very nature of of the order conjugacies, it follows that when there are finitely many nonordinary points(therefore special points) there are only finitely many equivalence classes. These are the simple systems we study in this chapter.

We describe completely, the homeomorphisms on $\mathbb{R}$, having finitely many nonordinary points and give a general formula for counting.

By remark 4.1.24, for systems with finitely many nonordinary points there are only finitely many equivalence classes. We now study, in the next section, the class of simple systems induced by homeomorphisms having finitely many nonordinary points.

4.2.1 Counting increasing homeomorphisms

Remark 4.2.1. Note that the complement of $Fix(f)$ is a countable union of open intervals(including rays) whose end points are fixed points. Since f is increasing and the end points are fixed, no point in a component interval can be mapped to a point in any other component interval by f.

Hence, it is observed that, for an increasing bijection f on $\mathbb{R}$, if $Fix(f)^c = \sqcup I_n$ then $f|_{I_n}$ is a self map of I_n.

Proposition 4.2.2. *Let f, g be two increasing bijections such that $Fix(f) = Fix(g)$ and let $Fix(f)^c = \sqcup I_n$. Suppose $f|_{I_n} \sim g|_{I_n}$ for every n then $f \sim g$.*

Proof. For each $n \in \mathbb{N}$ let $h_n : I_n \to I_n$ be an order conjugacy from $f|I_n$ to $g|I_n$.

Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} h_n(x) & \text{if } \; x \in I_n \\ x & \text{otherwise.} \end{cases}$$

Then h is an increasing bijection such that $hf = gh$. Hence the proposition.

□

The above proposition can be generalized as,

Proposition 4.2.3. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous with $Fix(f) = Fix(g)$, let $Fix(f)^c = \sqcup I_n$ and suppose each $\overline{I_n}$ is invariant under both f and g. Then $f \sim g$ if and only if $f|_{I_n} \sim g|_{I_n}$ for every n.*

An *alphabet* is a finite set of letters with at least two elements. A finite sequence of letters from an alphabet is often referred to as a *word*. For example, if $\Sigma = \{a, b\}$ be an alphabet then *abab*, *aaabbbab* are words over Σ. Number of letters (may not be distinct) in a word is called its *length*. Any word of consecutive characters in a word w is said to be a *subword* of w.

Throughout this section we will be working with the alphabet $\{\mathbf{A},\mathbf{B},\mathbf{O}\}$. Let $\tilde{\mathbf{A}} = \mathbf{B}$, $\tilde{\mathbf{B}} = \mathbf{A}$ and $\tilde{\mathbf{O}} = \mathbf{O}$. If $w = w_1w_2...w_n$, then the *dual* of w is defined as

$$\tilde{w} = \tilde{w}_n\tilde{w}_{n-1}...\tilde{w}_1.$$

If $\tilde{w} = w$ then w is said to be *self conjugate* . Here **A** stands for "**a**bove the diagonal" and **B** stands for "**b**elow the diagonal" and **O** stands for "**o**n the diagonal".

Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism with finitely many nonordinary (hence special points) say, $x_1 < x_2 < ... < x_n$ for some $n \in \mathbb{N}$. This finite set of points gives rise to an ordered partition $\{(-\infty, x_1), (x_1, x_2), ...(x_n, \infty)\}$ of $\mathbb{R}\backslash\{x_1, x_2, ..., x_n\}$. Note that, On each component interval exactly one of the following holds, by proposition 4.1.34 (Since the only subsets of $\mathbb{R}$ with empty boundary are the empty set and $\mathbb{R}$).

(i) $f(t) > t \ \forall \ t$ (ii) $f(t) < t \ \forall \ t$ (iii) $f(t) = t \ \forall \ t$.

This gives rise to a word $w(f)$ over $\{\mathbf{A},\mathbf{B},\mathbf{O}\}$ of length $n + 1$ by associating **A** to (i), **B** to (ii) and **O** to (iii).

Next, note that the sub word **OO** is forbidden. For, suppose **O** is occurring at i^{th} and $(i+1)^{th}$ place then in both(x_i, x_{i+1}) and (x_{i+1}, x_{i+2}) all points are fixed. Then x_{i+1} becomes ordinary. A contradiction to the assumption that x_{i+1} is a nonordinary point.

Conversely,

Suppose a word w of length $(n + 1)$(in which **OO** is forbidden) is given.

Then we can construct an increasing bijection on $\mathbb{R}$ such that its associated word is w, as follows:

Take the points $0, 1, 2, ..., n - 1$ and consider the partition $\{(-\infty, 0), (0, 1), (1, 2), ..., (n-1, \infty)\}$ of $\mathbb{R}$. If $w = w_1w_2.....w_{n+1}$ then associate w_1 to $(-\infty, 0)$, w_2 to $(0, 1)$, ..., and

w_{n+1} to $(n-1,\infty)$. Now it is easy to construct an increasing bijection $f:\mathbb{R}\to\mathbb{R}$ such that $w(f)=w$. To be precise if $i-1<t<i$ then

$$f(t)=\begin{cases} i-1+(t-i+1)^2 & \text{if } w_i=\mathbf{B} \\ i-1+\sqrt{t-i+1} & \text{if } w_i=\mathbf{A} \\ t & \text{if } w_i=\mathbf{O} \end{cases}$$

The increasing bijection so constructed is unique upto order conjugacy. This follows from the following proposition.

Proposition 4.2.4. *Let f,g be two increasing bijection on $\mathbb{R}$ with finitely many(same number of) nonordinary points. Then f and g are order conjugate if and only if $w(f)=w(g)$.*

Proof. Suppose $w(f)=w(g)=w_1w_2...w_n$. Let $x_1<x_2<...<x_n$ and $y_1<y_2<...<y_n$ be the non-ordinary points f and g respectively.

The former gives the ordered partition$\{(-\infty,x_1),(x_1,x_2),...(x_n,\infty)\}$ of $\mathbb{R}\backslash\{x_1,x_2,...,x_n\}$ and the later gives the ordered partition $\{(-\infty,y_1),(y_1,y_2),...,(y_n,\infty)\}$ of $\mathbb{R}\backslash\{y_1,y_2,...,y_n\}$.

Now, from proposition 4.1.34, it follows that, for each i, both $f|_{(x_i,x_{i+1})}$ and $g|_{(y_i,y_{i+1})}$ are fixed point free self maps(homeomorphisms) and hence by corollary 4.1.31, both are order conjugate to $x+1$ if $w_{i+1}=\mathbf{A}$, and order conjugate to $x-1$ if $w_{i+1}=\mathbf{B}$. Hence, by the proposition 4.2.2 f is order conjugate to g.

Converse follows from corollary 4.1.31. □

Thus we have proved:

Proposition 4.2.5. *There is a one to one correspondence between the set of all increasing continuous bijections (upto order conjugacy) on $\mathbb{R}$ with exactly n non-ordinary points and the set of all words of length $n+1$ on three symbols **A,B,O** such that **OO** is forbidden.*

Proposition 4.2.6. *Let a_n = the number of words of length $n+1$ over $\{\boldsymbol{A},\boldsymbol{B},\boldsymbol{O}\}$ where $\boldsymbol{OO}$ is forbidden. Then $a_n = C_1(1+\sqrt{3})^n + C_2(1-\sqrt{3})^n$ where $C_1 = \frac{(5+3\sqrt{3})}{2\sqrt{3}}$ and $C_2 = \frac{(3\sqrt{3}-5)}{2\sqrt{3}}$.*

Proof. Let A_n be the set all words of length $n+1$ over $\{\mathbf{A},\mathbf{B},\mathbf{O}\}$ in which **OO** is forbidden. A general element in A_{n+2} is of the form

(i) $\mathbf{A}w$ or $\mathbf{B}w$ for some $w \in A_{n+1}$

OR

(ii) $\mathbf{OA}v$ or $\mathbf{OB}v$ for some $v \in A_n$.

Therefore $a_{n+2} = a_{n+1} + a_{n+1} + a_n + a_n$, since A_{n+2} is the disjoint union of four types of the elements described above. Hence $a_{n+2} = 2(a_n + a_{n+1})$. This is a linear homogeneous recurrence relation with constant coefficients. The corresponding characteristic equation is $\alpha^2 - 2\alpha - 2 = 0$ which has the two distinct roots $\alpha_1 = 1+\sqrt{3}$ and $\alpha_2 = 1-\sqrt{3}$. It follows that $a_n = C_1(1+\sqrt{3})^n + C_2(1-\sqrt{3})^n$ where the constants C_1 and C_2 can be determined by using the boundary conditions $a_0 = 3$ and $a_1 = 8$. Here $C_1 = \frac{(5+3\sqrt{3})}{2\sqrt{3}}$ and $C_2 = \frac{(3\sqrt{3}-5)}{2\sqrt{3}}$. □

The following is one of the principal theorems of this chapter.

Theorem 4.2.7. *The number of all increasing continuous bijections (upto order conjugacy) on $\mathbb{R}$ with exactly n non-ordinary points is $= a_n = C_1(1+\sqrt{3})^n + C_2(1-\sqrt{3})^n$.*

Where $C_1 = \frac{(5+3\sqrt{3})}{2\sqrt{3}}$ and $C_2 = \frac{(3\sqrt{3}-5)}{2\sqrt{3}}$.

Proof. Follows from propositions 4.2.5 and 4.2.6. □

The following proposition is an analogue of proposition 4.2.4.

Proposition 4.2.8. *Let f, g are two increasing bijection on $\mathbb{R}$ with finitely many(same number of) nonordinary points. Then f and g are decreasingly conjugate if and only if $w(g) = \overline{w(f)}$.*

Let t_n to denote the number of increasing homeomorphisms upto "topological conjugacy". Then by proposition 4.2.4 and proposition 4.2.8, we get

$$t_n = \frac{a_n - Number\ of\ self\ conjugate\ words\ of\ length\ n+1}{2} +$$

$$Number\ of\ self\ conjugate\ words\ \ of\ length\ n+1.$$

<u>Case: 1</u> When n is even, say $n = 2m$. A self conjugate word w of length $2m+1$ (OO is forbidden) is of the form

$$w_1 w_2 ... w_m w_{m+1} w_{m+2} ... w_{2m+1}$$

such that $w_{m+1} = O$ and $w_m, w_{m+2} \in \{A, B\}$ such that $w_m \neq w_{m+2}$. Therefore the number of self conjugate words is $2a_{m-2}$.

Hence $t_{2m} = \frac{a_{2m}+2a_{m-2}}{2}$ for all $m \geq 2$.

<u>Case: 2</u> When n is odd, say $n = 2m + 1$.

In this case any self conjugate word of length $2m + 2$(OO is forbidden) is of the form

$$w_1 w_2 ... w_m w_{m+1} w_{m+2} ... w_{2m+2}$$

such that $w_{m+1}, w_{m+2} \in \{A, B\}$ and such that $w_{m+1} \neq w_{m+2}$. Hence the number of self conjugate words of length $2m + 2$ is $2a_{m-1}$.

Therefore, $t_{2m+1} = \frac{a_{2m+1}+2a_{m-1}}{2}$ for all $m \geq 1$.

Thus we have proved:

Theorem 4.2.9. *If t_n denotes the number of increasing homeomorphisms upto "topological conjugacy". Then, for $n \geq 3$ we have:*

$$t_n = \begin{cases} \frac{a_n + 2a_{\frac{n-4}{2}}}{2} & if \ \ n\ is\ even \\ \frac{a_n + 2a_{\frac{n-3}{2}}}{2} & n\ is\ odd \end{cases}$$

Where $t_0 = 2$, $t_1 = 5$ and $t_2 = 12$ by direct computation.

4.2.2 Counting decreasing homeomorphisms

We now ask:

Given a whole number n , how many decreasing bijections are there on $\mathbb{R}$ upto order conjugacy having exactly n non-ordinary points ?

Proposition 4.2.10. *Two decreasing bijections f and g are order conjugate (res. topologically conjugate) if and only if $f^2|_{[a,\infty)}$ and $g^2|_{[b,\infty)}$ are order conjugate(res. topologically conjugate).*

Here a and b are the fixed points of f and g respectively. [Note that every decreasing homeomorphism has a unique fixed point.]

Proof. Suppose f and g are order conjugate (res. topologically conjugate) then the same conjugacy between f and g when restricted, form an order conjugacy (res. topological conjugacy) between $f^2|_{[a,\infty)}$ and $g^2|_{[b,\infty)}$.

Conversely, suppose $f^2|_{[a,\infty)}$ and $g^2|_{[b,\infty)}$ are increasingly conjugate through the increasing homeomorphism h_1. Then $h_1([a,\infty)) = [b,\infty)$ and $h(a) = b$. Also note that $f((-\infty, a]) = [a,\infty)$ and $g((-\infty, b]) = [b,\infty)$. That is $f^{-1}([a,\infty)) = (-\infty, a]$ and $g^{-1}([b,\infty)) = (-\infty, b]$.

Define $h : \mathbb{R} \to \mathbb{R}$ by,

$$h(x) = \begin{cases} h_1(x) & \text{if } \ x \in [a,\infty) \\ g^{-1}h_1 f(x) & \text{if } x < a \end{cases}$$

Then h is the required increasing conjugacy from f to g. For, if $t < a$, by definition $h \circ f(t) = g \circ h(t)$.

If $t > a$ then $f(t) < a$. Therefore $h(f(t)) = g^{-1}(h_1(f(f(t)))) = g^{-1}(h_1(f^2(t))) = g^{-1}(g^2(h_1(t))) = g(h_1(t)) = g(h(t))$. Hence the proof.

□

Similarly we can prove:

Proposition 4.2.11. *Two decreasing bijections f and g are increasingly conjugate (res. topologically conjugate) if and only if $f^2|_{(-\infty,a]}$ and $g^2|_{(-\infty,b]}$ are increasingly conjugate(res. topologically conjugate).*

Here a and b are the fixed points of f and g respectively.

Definition 4.2.12. A map $f : \mathbb{R} \to \mathbb{R}$ is said to be *odd* if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

Proposition 4.2.13. *Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing bijection which is odd. Then there exists a decreasing homeomorphism f_r such that $f_r^2 = f$. (Such a decreasing homeomorphism we call as a* decreasing square root *of f).*

Proof. Note that $f(0) = 0$. Define

$$f_r(x) = \begin{cases} -f(x) & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Clearly f_r is a decreasing bijection. Then for $x \geq 0$, we have$f_r(x) \leq 0$, Therefore $f_r(f_r(x)) = -f_r(x) = f(x)$. For $x < 0$,we have $f_r(f_r(x)) = f_r(-x) = -f(-x) = f(x)$. □

Remark 4.2.14. The hypothesis of the above proposition is not true in general.

Let

$$h(x) = \begin{cases} \frac{x}{2} & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}$$

Clearly, h is an increasing bijection from $\mathbb{R}$ to $\mathbb{R}$. There is no decreasing bijection $f : \mathbb{R} \to \mathbb{R}$ such that $f \circ f = h$. Let if possible f be one such fuction. Then for all $x < 0$ we have $f(f(x) = h(x) = x$. Choose $y > 0$ such that $f(y) < 0$. Therefore

$f^2(f(y)) = f(y) = f(f^2(y))$. Since f is one-one $f^2(y) = y$. Therefore $h(y) = y$. Which is contradiction since $h(y) = \frac{y}{2}$.

Proposition 4.2.15. *Let $f : (0, \infty) \to (0, \infty)$ be an increasing bijection. Then there exists a unique decreasing bijection $f_r : \mathbb{R} \to \mathbb{R}$ upto order conjugacy such that $f_r \circ f_r|(0, \infty) = f$.*

Proof. Let $f : (0, \infty) \to (0, \infty)$ be an increasing bijection. This forces that $f(0) = 0$. Any map $f : (0, \infty) \to (0, \infty)$ can be extended uniquely to an odd function $\tilde{f} : \mathbb{R} \to \mathbb{R}$. Then by proposition 4.2.13, there exist $f_r : \mathbb{R} \to \mathbb{R}$ such that $f_r \circ f_r|(0, \infty) = f$. By proposition 4.2.11, f is unique upto order conjugacy. □

Proposition 4.2.16. *Let $f : \mathbb{R} \to \mathbb{R}$ be a decreasing bijection. Then all non-ordinary points of $f \circ f$ are non-ordinary points of f and conversely.*

Proof. Suppose x is an ordinary point for f. Then the resuts follows from the fact that if h commutes with f then it commute with $f \circ f$ also.

Conversely, suppose x is an ordinary point for $f \circ f$. Let the unique fixed point of f to be zero. i.e, $f(x) = 0$ iff $x = 0$ and let $x > 0$. Then there exist a neighbourhood $(x - \delta, x + \delta)$ such that for all y in $(x - \delta, x + \delta)$, there exists $h \in G_{f \circ f}$ such that $h(x) = y$. Then $h|_{(0,\infty)}$ is a topological conjugacy between $f \circ f|_{(0,\infty)}$ and $f \circ f|_{(0,\infty)}$. Then by proposition 4.2.11, h induces $\tilde{h}$, a conjugacy between f and f. By the way $\tilde{h}$ is defined, we have $\tilde{h}(x) = h(x) = y$. Therefore, x is an ordinary point of f.

□

Proposition 4.2.17. *If f is a decreasing bijection from $\mathbb{R}$ to $\mathbb{R}$ with fixed point a. Then f has $2n + 1$ non-ordinary points if and only if $(f \circ f)|_{(a,\infty)} : (a, \infty) \to (a, \infty)$ has n non-ordinary points.*

Proof. Suppose that f has $2n+1$ non-ordinary points. Let them be $x_1 < x_2 < ... < x_n < x_{n+1} < x_{n+2} < ... < x_{2n+1}$. Let $N = \{x_1, x_2, ...x_{2n+1}\}$. Then $f(N) \subset N$ by proposition 4.1.17. Since f is a decreasing bijection we have $f(N) = N$ and $a = x_n$. Hence $(f \circ f)|_{(a,\infty)} : (a,\infty) \to (a,\infty)$ has n non-ordinary points.

Conversely, Suppose $(f \circ f)|_{(a,\infty)} : (a,\infty) \to (a,\infty)$ has n nonordinary points. Then observe that $N(f) = N(f^2|_{(a,\infty)}) \cup f(N(f^2|_{(a,\infty)})) \cup \{a\}$ (Here we use the previous proposition). Thus, f has $2n+1$ nonordinary points. □

Remark 4.2.18. *From the above proposition it follows that there does not exist a decreasing homeomorphism with even number of nonordinary points.*

Theorem 4.2.19. *If s_n denotes the number of decreasing homeomorphisms upto order conjugacy, then*

$$s_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ a_{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

for all n.

Proof. $\underline{s_{2n} \ = 0 \ \forall \ n \in \mathbb{N}}$

Follows from proposition 4.2.17.

$\underline{s_{2n+1} \ = a_n \ \forall \ n \in \mathbb{N}}$

Let $f : \mathbb{R} \to \mathbb{R}$ be a decreasing bijection with $2n+1$ nonordinary points. Without loss of generality, we can assume that 0 is the unique fixed point. Then $g = f^2|_{(0,\infty)} : (0,\infty) \to (0,\infty)$ is an increasing bijection with n nonordinary points. Since $(0,\infty)$ is homeomorphic to $\mathbb{R}$, we get a increasing homeomorphism $g' : \mathbb{R} \to \mathbb{R}$ (unique upto order conjugacy) with n nonordinary points, which is order conjugate to g.

On the other hand, suppose $h : \mathbb{R} \to \mathbb{R}$ is an increasing bijection with n nonordinary points. Since $(0,\infty)$ is homeomorphic to $\mathbb{R}$, corresponding to each h we have a unique(upto order conjugacy) increasing bijection $h' : (0, \infty) \to (0, \infty)$ with n nonordinary points. Then by Proposition 4.2.13 there exist unique(upto order conjugacy) decreasing square root $f : \mathbb{R} \to \mathbb{R}$ for h' upto order conjugacy, such that $f \circ f|(0, \infty) = h'$. By proposition 4.2.17, f has $2n + 1$ non-ordinary points.

Thus, there is a one-one correspondence between the set of all increasing bijection with n non-ordinary points (upto order conjugacy) and the set of all decreasing bijection with $2n + 1$ non-ordinary points (upto order conjugacy). Hence $s_{2n+1} = a_n$. $\square$

Theorem 4.2.20. *If k_n denotes the number of decreasing homeomorphisms having n nonordinary points upto "topological conjugacy" then*

$$k_n = \begin{cases} 0 & \textit{if } n \textit{ is even} \\ t_{\frac{n-1}{2}} & \textit{if } n \textit{ is odd} \end{cases}$$

for all n.

Proof. Case: 1 When n is even.

We have $k_n = 0$ since $s_n = 0$.

Case: 2 When n is odd.

We will argue as in theorem 4.2.19 to prove that:

There is a one-one correspondence between the set of all increasing bijections(upto topological conjugacy) on $\mathbb{R}$ having n nonordinary points and the set of all decreasing bijections(upto topological conjugacy) on $\mathbb{R}$ with $2n + 1$ nonordinary points.

Thus $k_{2n+1} = t_n$.

$\square$

We conclude this section with the following table:

n	a_n	s_n	t_n	k_n
0	3	3	2	0
1	8	0	5	2
2	22	0	12	0
3	60	8	33	5
4	164	0	85	0
5	448	22	232	12

Where

a_n = The number of increasing bijections, with exactly n nonordinary points, on $\mathbb{R}$, upto order conjugacy.

t_n = The number of increasing bijections, with exactly n nonordinary points, on $\mathbb{R}$, upto topological conjugacy.

s_n = The number of decreasing bijections, with exactly n nonordinary points, on $\mathbb{R}$, upto order conjugacy.

k_n = The number of decreasing bijections, with exactly n nonordinary points, on $\mathbb{R}$, upto topological conjugacy.

4.3 Counting continuous maps

In this section we prove that there are exactly 26 continuous maps on $\mathbb{R}$ with a unique nonordinary point upto increasing conjugacy.

It follows from proposition 4.1.26(i), that : a continuous map on $\mathbb{R}$ with a single nonordinary point a must be either constant or injective on each of the intervals $(-\infty, a)$ and (a, ∞).

4.3.1 Some basic conjugacy results

Proposition 4.3.1. *Let $f : (-\infty, 0] \to (-\infty, 0]$ be an increasing bijection (It follows that f(0)=0). Then*

(1) If $f(x) > x$ for all $x \in (-\infty, 0)$ then f is increasingly conjugate to $\frac{x}{2}$.

(2) If $f(x) < x$ for all $x \in (-\infty, 0)$ then f is increasingly conjugate to $2x$.

Proof. **Proof of (1):**

Let $f : (-\infty, 0] \to (-\infty, 0]$ be an increasing bijection satisfying $f(x) > x \forall x < 0$. (It follows that $f(0) = 0$). Note that for any such map $\biguplus_{n\in\mathbb{Z}}[f^n(x), f^{n+1}(x)) = (-\infty, 0)$ for all point $x \in (-\infty, 0]$.

Then f is topologically conjugate to the map $x/2$. We construct a topological conjugacy $h : (-\infty, 0] \to (-\infty, 0]$ as follows: Take any point other than 0, say -1 in the domain. We take an arbitrary increasing homeomorphism h from $[-1, f(-1))$ *to* $[-1, -1/2)$. Then as noted above, $\biguplus_{n\in\mathbb{Z}}[f^n(-1), f^{n+1}(-1)) = (-\infty, 0)$. That is, for every $x \in (-\infty, 0)$, there exists a unique $n_0 \in \mathbb{Z}$ such that $f^{n_0}(x) \in [-1, f(-1))$. We define $h(x) = 2^{n_0} h(f^{n_0}(x))$. This is well defined. It is an increasing homeomorphism from $(-\infty, 0)$ to $(-\infty, 0)$. This h commutes with f. This h is a conjugacy from f to the map $x/2$.

Similarly, we can prove (2).

□

Proposition 4.3.2. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be such that*

(1) $f(0) = g(0) = 0$.

(2) $f|_{(0,\infty)}, g|_{(0,\infty)} : (0, \infty) \to (0, \infty)$ are increasing bijections.

(3) $f|_{(-\infty,0)}, g|_{(-\infty,0)} : (-\infty, 0) \to (0, \infty)$ are decreasing bijections.

Then f is order conjugate to g if and only if $f|_{(0,\infty)}$ is order conjugate to $g|_{(0,\infty)}$.

Proof. Suppose $h : (0, \infty) \to (0, \infty)$ is an order conjugacy from $f|_{(0,\infty)}$ to $g|_{(0,\infty)}$.

For $x < 0$, define $h(x) = (g|_{(-\infty,0]})^{-1} hf(x)$. □

Remark 4.3.3. The above proposition still holds (with identical proof) if the hypothesis (2) is generalized to " $[0, \infty)$ is invariant under both f and g."

Proposition 4.3.4. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be such that*

(1) $f(0) = g(0) = 0$.

(2) $f|_{(-\infty,0)}, g|_{(-\infty,0)} : (-\infty, 0) \to (-\infty, 0)$ are increasing bijections.

(3) $f|_{(0,\infty)}, g|_{(0,\infty)} : (0, \infty) \to (-\infty, 0)$ are decreasing bijections.

Then f is order conjugate to g if and only if $f|_{(-\infty,0)}$ is order conjugate to $g|_{(\infty,0)}$.

Proof. Suppose $h : (-\infty, 0) \to (-\infty, 0)$ is an order conjugacy from $f|_{(-\infty,0)}$ to $g|_{(-\infty,0)}$.

For $x > 0$, define $h(x) = (g|_{(0,\infty)})^{-1} hf(x)$. □

Remark 4.3.5. The above proposition still holds (with identical proof) if the hypothesis (2) is generalized to " $(-\infty, 0]$ is invariant under both f and g."

4.3.2 Somewhere constant maps

A map $f : \mathbb{R} \to \mathbb{R}$ is said to be *somewhere constant* if it is constant in an open interval, and we say f is *locally constant at $a \in \mathbb{R}$* if it is constant in some open interval containing a.

Theorem 4.3.6. *There are exactly 9 somewhere constant maps with a unique non-ordinary point, upto order conjugacy.*

Proof. From corollary 4.1.20 it follows that f must be constant on a ray.

Case:1

Suppose f is constant on a right ray. Without loss of generality we can assume that f is constant on $[0,\infty)$. [For, Suppose f is constant on $[a,\infty)$, then consider $g(t) = f(a+t) - a$ for $t \geq 0$. Then $g(t) = 0$ iff $t \geq 0$ and $f \sim g$ increasingly.]

From proposition 4.1.17, it follows that $f(0) = 0$.

Then by proposition 4.3.1 and by proposition 4.3.2 (means that the same formula used in the proof will work) it is clear that there are exactly five maps (including the zero map) upto increasing conjugacy whose interval of constancy is $[0,\infty)$.

They are,

$$f_1(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \qquad f_2 \equiv 0 \qquad f_3(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ \frac{x}{2} & \text{if } x < 0 \end{cases}$$

$$f_4(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases} \qquad f_5(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ 2x & \text{if } x < 0 \end{cases}$$

Case:2

Suppose f is constant on a left ray. This is dual to case:1. Hence there are four such maps.

$$f_6(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ -x & \text{if } x > 0 \end{cases} \qquad f_7(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{x}{2} & \text{if } x > 0 \end{cases}$$

$$f_8(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases} \qquad f_9(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 2x & \text{if } x > 0 \end{cases}$$

□

4.3.3 Nowhere constant maps

Proposition 4.3.7. *Let $f : \mathbb{R} \to \mathbb{R}$ be strictly monotone with a unique nonordinary point. Then f must be onto.*

Proof. Let f be increasing. Then f is an increasing homeomorphism.

Suppose $f(\mathbb{R})$ is a bounded interval then both end points are nonordinary, which is a contradiction.

Suppose $f(\mathbb{R})$ is a ray say, (a, ∞). Then a is nonordinary and since it is the only nonordinary point we have $f(a) = a$. Since f is increasing we have $f([a, \infty) = [a, \infty))$. Hence if $x < a$ then $f(x) \in [a, \infty)$. Thus f fails to be one-one.

Proof is similar, if f is decreasing.

□

Theorem 4.3.8. *There are exactly 17 nowhere constant maps with a unique nonordinary point upto order conjugacy.*

Proof. Suppose f is nowhere constant with a unique nonordinary point. Then from proposition 4.1.26 it follows that f can have at most one critical point.

Case:1 Suppose f is a map without a critical point. Then f is strictly monotone.

Case: 1(a)

Suppose f is strictly increasing. Then f must be a homeomorphism. Hence there are exactly $a_1 = 8$ such maps, by the word argument.

$$f_{10}(x) = \begin{cases} 2x & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases} \qquad f_{11}(x) = \begin{cases} \frac{x}{2} & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases}$$

$$f_{12}(x) = \begin{cases} x & \text{if } x \leq 0 \\ \frac{x}{2} & \text{if } x > 0 \end{cases} \qquad f_{13}(x) = \begin{cases} 2x & \text{if } x \leq 0 \\ \frac{x}{2} & \text{if } x > 0 \end{cases}$$

$$f_{14}(x) = \frac{x}{2} \text{ for all } x \qquad f_{15}(x) = \begin{cases} x & \text{if } x \leq 0 \\ 2x & \text{if } x > 0 \end{cases}$$

$$f_{16}(x) = 2x \text{ for all } x \qquad f_{17}(x) = \begin{cases} \frac{x}{2} & \text{if } x \leq 0 \\ 2x & \text{if } x > 0 \end{cases}$$

Case: 1(b)

Suppose f is decreasing. Since f has unique nonordinary point, f must be onto and hence it is a homeomorphism. Hence there are $s_1 = a_0 = 3$ such maps upto order conjugacy.

$f_{18}(x) = -x$ for all x

$$f_{19}(x) = \begin{cases} -2x & \text{if } x \le 0 \\ -x & \text{if } x > 0 \end{cases} \qquad f_{20}(x) = \begin{cases} \frac{-x}{2} & \text{if } x \le 0 \\ -x & \text{if } x > 0 \end{cases}$$

Case: 2

Suppose the unique nonordinary point is the critical point. Let the unique nonordinary point be zero. Now the unique nonordinary(critical) point is either a point of local maximum or a point of local minimum.

Case: 2(a)

Suppose 0 is a point of local minimum. Then f must be decreasing from $(-\infty, 0)$ on to $(0, \infty)$ and increasing from$(0, \infty)$ on to $(0, \infty)$ (since sup f and inf f are nonordinary points).

Then it follows from proposition 4.3.2 that there are exactly three such maps upto order conjugacy .

$f_{21}(x) = |x|$ for all x

$f_{22}(x) = |2x|$ for all x

$f_{21}(x) = \frac{|x|}{2}$ for all x

Case: 2(b)

Suppose 0 is a point of local maximum(infact, global maximum). Then there are exactly three such maps by a proposition 4.3.4.

$$f_{24}(x) = \begin{cases} x & \text{if } x \le 0 \\ -x & \text{if } x > 0 \end{cases} \qquad f_{25}(x) = \begin{cases} \frac{x}{2} & \text{if } x \le 0 \\ -x & \text{if } x > 0 \end{cases}$$

$$f_{26}(x) = \begin{cases} 2x & \text{if } x \le 0 \\ -x & \text{if } x > 0 \end{cases}$$

Thus there are 17 such maps.

□

4.4 Main theorem

Theorem 4.4.1. *There are exactly 26 maps on* $\mathbb{R}$ *with a unique non-ordinary point, upto order conjugacy.*

Proof. Proof follows from Theorems 4.3.6 and 4.3.8.

□

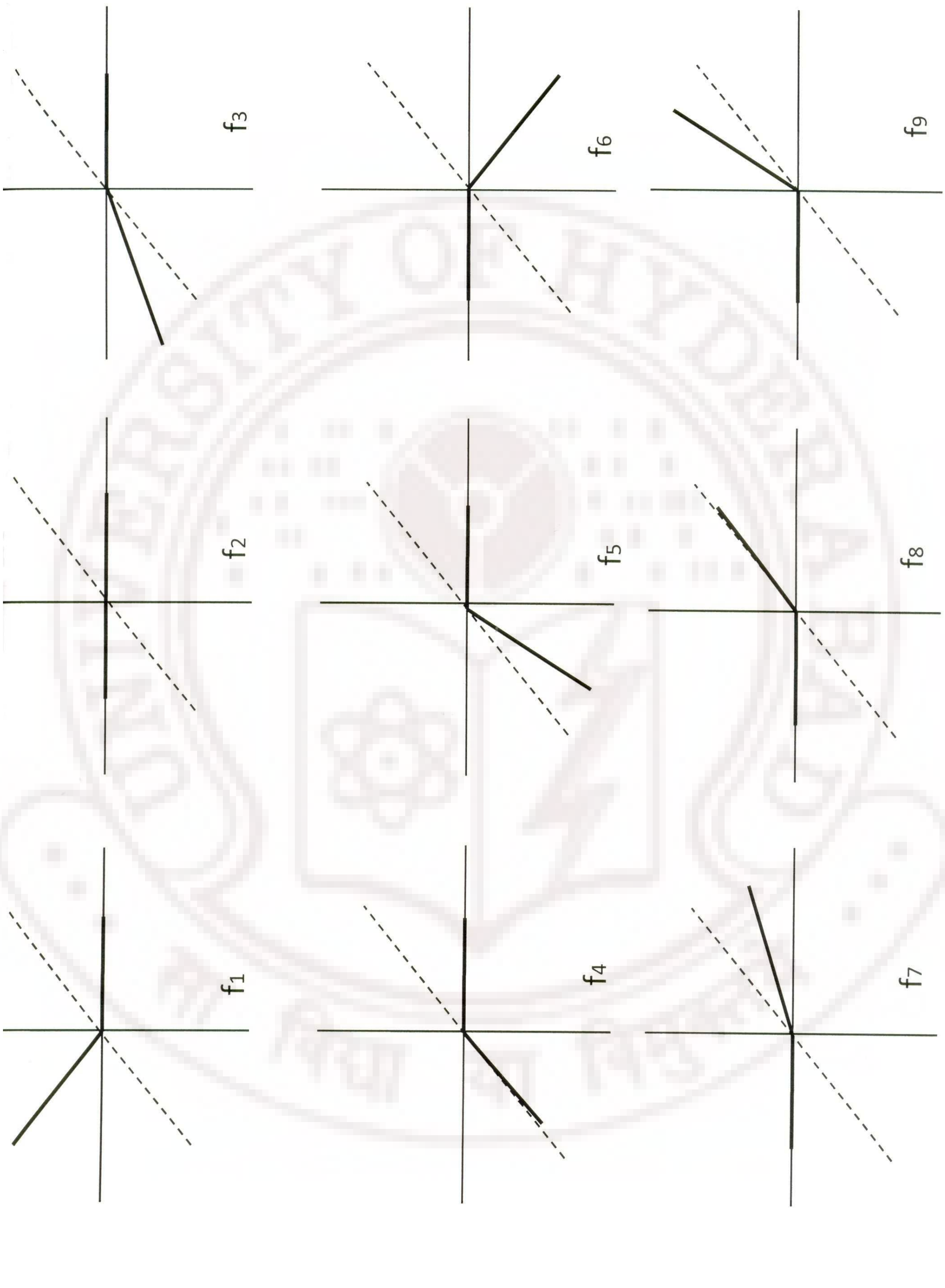

Figure 4.1: Maps with one nonordinary point

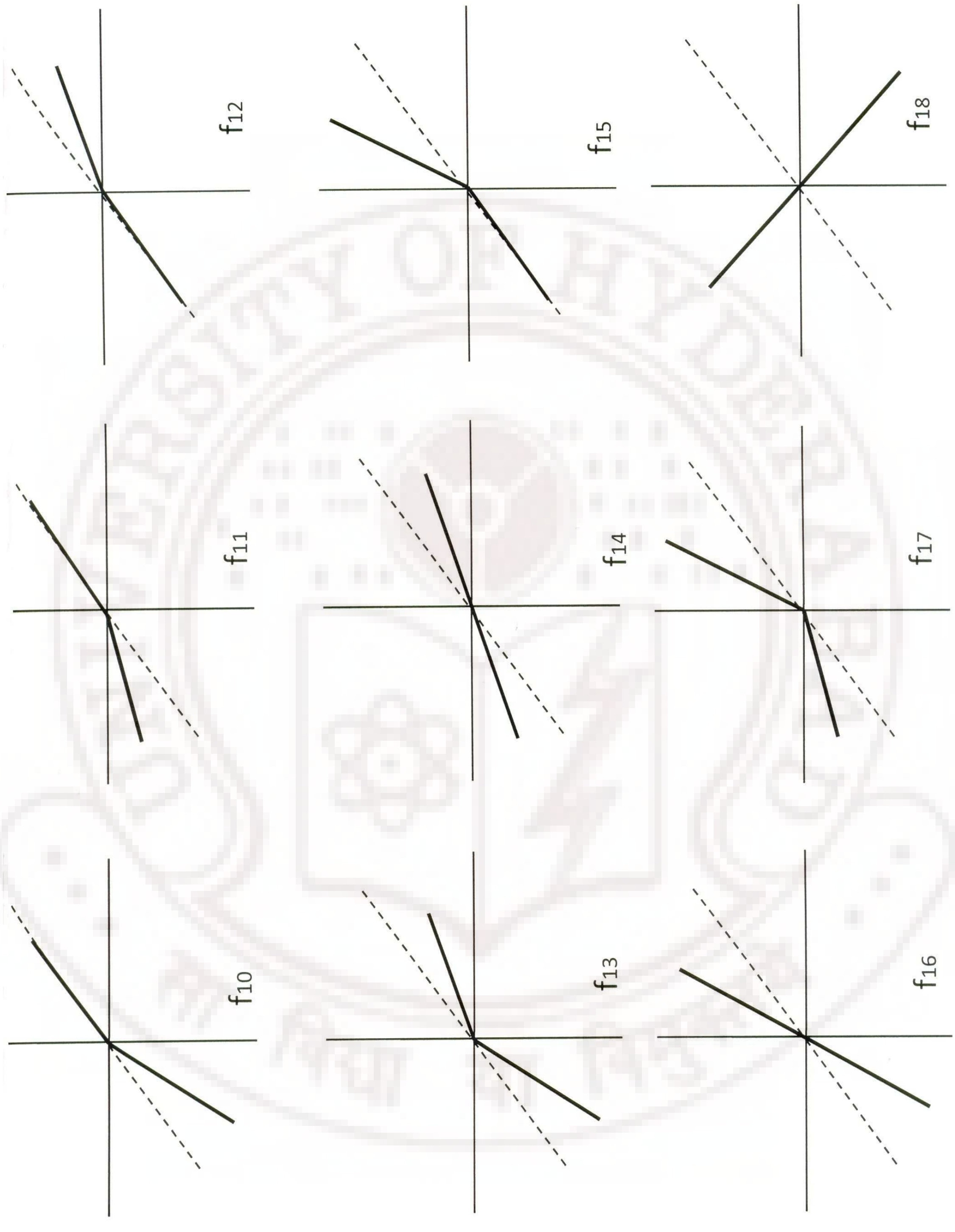

Figure 4.2: Maps with one nonordinary point

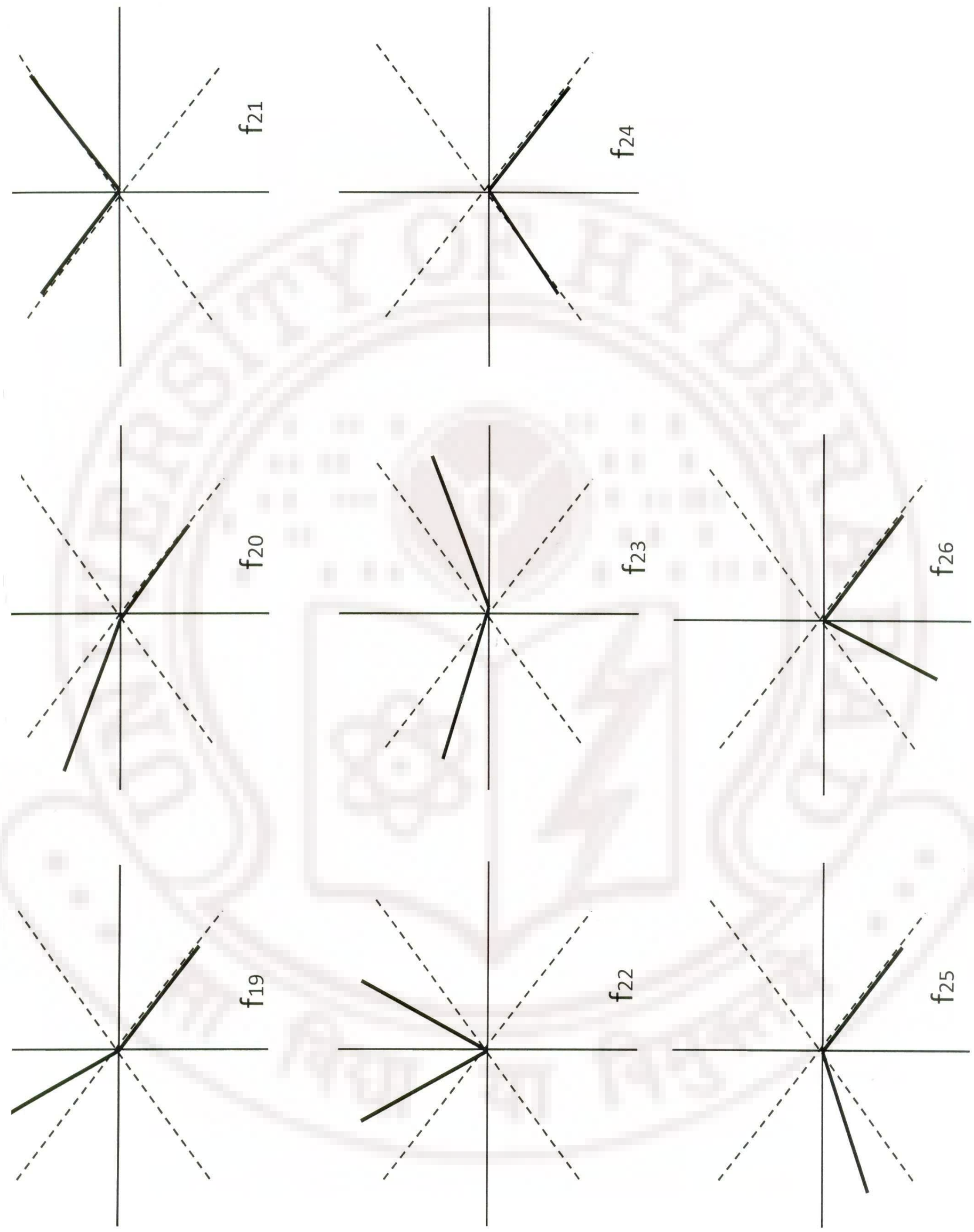

Figure 4.3: Maps with one nonordinary point

Chapter 5

Periodic points Vs Critical points

5.1 Motivation

In this chapter, first we discuss some interesting results connecting critical points and periodic points. *We prove a necessary condition for a continuous self map on $\mathbb{R}$ to be conjugate to a polynomial.*

Recall that an element $c \in X$ is said to be a critical point of the dynamical system (X, f) if f fails to be one-one on every neighbourhood of x.

Example 5.1.1. (1) When $X = \mathbb{R}$, local maxima and minima points are critical points. These are the points where the graph of f takes a turn.

(2) For complex polynomials, the roots of the derivative are critical points.

Definition 5.1.2. Let x be a fixed point of a dynamical system (X, f). We say that x is an attracting fixed point if there is a neighbourhood V of x such that for every y in V, the trajectory of y converges to x.

Example 5.1.3. Let $X = \mathbb{R}$ or $\mathbb{C}$. Let f be a continuously differentiable function. Let x be a fixed point of (X, f). If $|f'(x)| < 1$, then x is an attracting fixed point.

Definition 5.1.4. Let x be an attracting fixed point in a dynamical system (X, f). Then the set of all points whose trajectory converges to x, is called the *basin of attraction* of x.

Example 5.1.5. For the map z^2 on $\mathbb{C}$, the number 0 is an *attracting fixed point.* It is also the unique critical point. The basin of attraction is the open unit disc.

Theorem 5.1.6. ***(Fatou)*** *[13] Every attracting periodic point of a complex polynomial contains a critical point in its basin of attraction.*

Remark 5.1.7. For a complex polynomial, there are only finitely many critical points. Some of these critical trajectories may converge. Their limits may be attracting periodic points. Our search for attracting periodic points may be confined to them. We can't find them elsewhere.

Definition 5.1.8. Let x be a periodic point of a dynamical system (X, f). Let n be its period. Then x is a fixed point of (X, f^n). If x is an attracting fixed point of (X, f^n), then x is called an *attracting periodic point* of (X, f).

Definition 5.1.9. Let (X, f) be a dynamical system. For any subset $A \subset X$, define $\overleftarrow{A} = \{x \in X |\ f^n(x) \in A \text{ for some } n \in \mathbb{N} \cup \{0\}\}$.

If $C(f)$ denotes the set of all critical points of f, then the elements of $\overleftarrow{C(f)}$ are called *precritical points* . Similarly, the elements of $\overleftarrow{P(f)}$ are called the *preperiodic points* .

Definition 5.1.10. A point x in a dynamical system (X, f) is said to be a *recurrent point* if there exists an increasing sequence of natural numbers (n_k) such that $f^{n_k}(x) \to x$. We say x is nonwandering if $f^{n_k}(x_k) \to x$ for some sequence of points $x_k \to x$ and some sequence of integers $n_k \to \infty$.

The main theorem of this chapter is similar in spirit to the following known theorems:

Theorem 5.1.11. *[6] If f is any continuous map from I to I, then the set of periodic points and the set of recurrent points have the same closure.*

Theorem 5.1.12. *[6] The set of all nonwandering points of a continuous map $f : I \to I$ is contained in the closure of the set of all preperiodic points.*

Theorem 5.1.13. *[14] Let $f : I \to I$ be continuous. If the set of sensitive points is dense in I, then the set of preperiodic points is also dense in I.*

Theorem 5.1.14. *[24] Let $f : I \to I$ be a polynomial such that f^2 is not identity. Then $\overleftarrow{C(f)}$ is dense in I if and only if $\overleftarrow{P(f)}$is dense in I.*

5.2 Main Theorem

Example 5.2.1. Consider the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x(1 - x)$. For this map $\frac{1}{2}$ is the only critical point.

Let $\Lambda = \{x \in [0, 1] \mid f^n(x) \in [0, 1] \text{ for all } n \in \mathbb{N}\}$.

That is, if $J = \{x \in [0, 1] \mid f(x) \notin [0, 1]\}$ and if $E_n = [0, 1] \setminus f^{-n}(J)$, then $\Lambda = \cap_{n=1}^{\infty} E_n$.

Note that, for any $n \in \mathbb{N}$, $f^{-n}\{\frac{1}{2}\}$ is a set of 2^n points in $[0, 1]$. Also, note that $f^{-n}(J)$ is a disjoint union of 2^n open intervals such that each component interval contains exactly one element of $f^{-n}\{\frac{1}{2}\}$.

Since $f^{-n}(J) \cap f^{-m}(J) = \emptyset$ whenever $m \neq n$, we conclude that the set $\cup_{n=0}^{\infty} f^{-n}\{\frac{1}{2}\} = \overleftarrow{C(f)}$ is a discrete set.

Now, We can prove that $\overline{\overleftarrow{C(f)}} = \overleftarrow{C(f)} \cup \Lambda$.(We use the fact that Λ is perfect and totally disconnected.)

Since the map $f|_{\Lambda} : \Lambda \to \Lambda$, is topologically conjugate to the shift map (See [13]), we have $\overline{\overleftarrow{P(f)}} = \Lambda$.

Thus, $\overline{\overleftarrow{C(f)}} \Delta \overline{\overleftarrow{P(f)}}$ is discrete.

Proposition 5.2.2. *For monotone maps, all the periodic points are fixed points or period-2 points.*

Proof. Proof follows from the fact that for an increasing map the orbits are uni-directional. When the given map f is decreasing consider f^2 which is increasing.

$\square$

Proposition 5.2.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and piecewise monotone. Let A be any backward invariant subset of $\mathbb{R}$ and let J be any component interval of $\mathbb{R} \setminus \overline{A}$. Then $f(\overline{J}) \subset \overline{K}$ for some component interval K of $\mathbb{R} \setminus \overline{A}$.*

Proof. Since A is backward invariant $int f(J)$ cannot meet $\overline{A}$ and therefore $int f(J) \subset K$ for some component interval K of $\mathbb{R} \setminus \overline{A}$. Now $f(\overline{J}) \subset \overline{f(J)} = \overline{int f(J)} \subset \overline{K}$. $\square$

As noted earlier, the main theorem gives a necessary condition for a continuous self map on $\mathbb{R}$ to be conjugate to a polynomial.

Theorem 5.2.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial such that $f^2 \neq id$(the identity map on $\mathbb{R}$). Then $\overline{\overleftarrow{C(f)}} \Delta \overline{\overleftarrow{P(f)}}$ is a discrete set in the relative topology.*

Proof. ***Part: 1***

First we prove that $\overline{\overleftarrow{P(f)}} \setminus \overline{\overleftarrow{C(f)}}$ is a discrete set. For this , consider $\mathbb{R} \setminus \overline{\overleftarrow{C(f)}}$ which is open and hence a countable union of disjoint intervals(including open rays) say, $\mathbb{R} \setminus \overline{\overleftarrow{C(f)}} = \bigcup_{\alpha \in S} I_{\alpha}$. Note that for any I_{α}, the *open* intervals $I_{\alpha}, f(I_{\alpha}), f^2(I_{\alpha}), \cdots$ are disjoint from $\overline{\overleftarrow{C(f)}}$, as $\overline{\overleftarrow{C(f)}}$ is backward f invariant(that is $f^{-1}(A) \subset A$) and hence induces a map $f^* : S \to S$.

Step: 1

We claim that each I_α can contain atmost finitely many periodic points.

1. Suppose I_α has a periodic point for some $\alpha \in S$. Let $k \in \mathbb{N}$ be the least such that $f^k(I_\alpha) \subset I_\alpha$. From the choice of k it is clear that a point $x \in I_\alpha$ is f-periodic if and only if it is f^k-periodic. We know that $f^k : I_\alpha \to I_\alpha$ is strictly monotone. Hence by proposition 5.2.2, the periodic points of f^k are fixed points and period-2 points. Since f is a polynomial such that $f^2 \neq Id$, they are finite in number. This completes the proof of step-1.

Step: 2

Claim: Each I_α can have only finitely many preperiodic points. For, suppose I_α has a preperiodic point. Let k^1 be the least such that $f^{k^1}(I_\alpha)$ has a periodic point. Then $f^{k^1}(I_\alpha) \subset I_\beta$ for some $\beta \in S$. By step:1 this I_β can have atmost finitely many periodic points. Let this finite set be F. Then $f^{-k^1}(F)$ is also a finite set , as f is a polynomial. To complete the argument it is enough to prove that any preperiodic point in I_α will land in I_β as a periodic point after k^1 instants of time. For this, let $x \in I_\alpha$ be a preperiodic point. We claim that $f^{k^1}(x)$ is a periodic point. Now $f^{k^1}(x) \in I_\beta$ and I_β has a periodic point. Let p be the f^*-period of β. ($f^p : I_\beta \to I_\beta$ is strictly monotone and hence all preperiodic points are periodic.) Therefore $f^{k^1}(x)$ is f^p-periodic and hence f-periodic.

Now, Suppose $x \in I_\alpha \bigcap \overline{\overleftarrow{P(f)}}$ for some α. Then x must be a preperiodic point because I_α contains only finitely many preperiodic points. Hence $I_\alpha \bigcap \overline{\overleftarrow{P(f)}}$ is finite.

Part: 2

We now prove that $\overline{\overleftarrow{C(f)}} \setminus \overline{\overleftarrow{P(f)}}$ is a discrete set.

Write $\mathbb{R} \setminus \overline{\overleftarrow{P(f)}} = \bigcup_{\alpha \in T} J_\alpha$, a disjoint union of open intervals . Then by proposition 5.2.3, for every $\alpha \in T$ there exists $\beta \in T$ such that $f(\overline{J_\alpha}) \subset \overline{J_\beta}$. This induces a map

$f^* : T \to T$. Consider a component interval J. Fix $c \in C(f)$. It is enough to prove that $|\overleftarrow{c} \cap K| < \infty$ for every compact subinterval $K \subset J$.

Suppose $c \in f^n(J)$ for some $n \in \mathbb{N} \cup \{0\}$. Then $f^n(J) \subset \overline{J_\beta}$ for some $\beta \in T$. If this β is not f^*-periodic, then $\overleftarrow{c} \cap K = f^{-n}(c) \cap K$ which is a finite set.

If β is f^*-periodic, let p be the least such that f^p maps J_β to J_β. Then we can find $\delta > 0$ such that $\mid x - f^p(x) \mid > \delta$ on $f^n(K)$, as $f^n(K)$ cannot contain any periodic point. Thus the motion under f^p on $f^n(K)$ is unidirectional. Hence if $c_1 \in f^n(K)$ and if $f^{tp}(c_1) = c$ for some t, then this t can be chosen to be $\leq \frac{length\ of\ f^n(K)}{\delta}$. This forces that, there can be only finitely many precritical points in K which reach the critical point c. This completes the proof. $\square$

Remark 5.2.5. It is clear that the result is true for all continuous functions which are topologically conjugate to a polynomial. Still, among interval maps this can fail for nonpolynomials. This is illustrated in the following proposition.

Proposition 5.2.6. *Let F be a nonempty closed subset of I. Then there exists a strictly increasing continuous function $f : I \to I$ such that $Fix(f) = F$*

Proof. In each component interval of F^c, we keep a copy of the map $x^2 : [0, 1] \to [0, 1]$.

By a copy of $x^2 : [0, 1] \to [0, 1]$ on $[a, b]$ we mean the map $a + \frac{(x-b)^2}{b-a} : [a, b] \to [a, b]$. $\square$

Remark 5.2.7. Note that the theorem is not true in case of functions on $\mathbb{C}$. For, if $f(z) = z^2$ then $C(f) = \{0\}$, $\overleftarrow{C(f)} = \{0\}$, $\overleftarrow{P(f)} = S^1 \cup \{0\}$ and hence $\overleftarrow{C(f)} \Delta \overleftarrow{P(f)} = S^1$.

Appendix A

Some open questions

1. In chapter- 2, we answered the question : Which subsets of $\mathbb{N}$ can arise as the set of periods for some continuous automorphism on the 2-torus. The question is open for a general n- torus.

2. The above question can be asked for a general automorphism on the torus (may not be continuous).

We proved elsewhere:

Let G a torsion free abelian group. Then a subset A of $\mathbb{N}$ is the set of periods for some automorphism of G if and only if $1 \in A$ and is closed under l.c.m.

3. Given $n \in \mathbb{N}$. Find the number of continuous maps on $\mathbb{R}$ up to topological conjugacy, with exactly n non-ordinary points.

4. Find the number of bijective cubic real polynomials on $\mathbb{R}$ upto conjugacy.

Partial answer: In between 6 and 10.

5. Which continuous maps on $\mathbb{R}$ are conjugate to a polynomial?

(Answer is expected in terms of dynamical properties.)

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