CHARACTERS APPEARING IN THE WEIL REPRESENTATION OF A REGULAR CHARACTER

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by

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To Acchan and his struggles

To Amma and her love

To all my teachers and their patience

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Vishnu

DECLARATION

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I, **VISHNU NAMBOOTHIRI K** hereby declare that the work embodied in the present thesis entitled **CHARACTERS APPEARING IN THE WEIL REPRESENTATION OF A REGULAR CHARACTER** has been carried out by me under the supervision of Prof. Rajat Tandon, Department of Mathematics and Statistics, University of Hyderabad, Hyderabad - 500 046, India, as per the Ph.D ordinance of the university.

I declare that, to the best of my knowledge and belief, no part of this thesis was earlier submitted for the award of research degree of any university or institution.

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This is to certify that Mr. **VISHNU NAMBOOTHIRI K** has carried out the research work embodied in the present thesis entitled **CHARACTERS APPEARING IN THE WEIL REPRESENTATION OF A REGULAR CHARACTER** for the full period prescribed under the University rules. No part of this thesis was earlier submitted for the award of research degree of any University/Institute.

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Contents

Chapter 1

Introduction

Let F be a nonarchimedean local field of characteristic not two and K a separable quadratic extension. Then if $K = F(x_0)$ with x_0 an element of K^* whose trace to F is 0 we have an embedding of K^* into GL(2,F) given by

$$a + bx_0 \mapsto \begin{bmatrix} a & bx_0^2 \\ b & a \end{bmatrix}.$$

It was Tunnell [Tu] who first gave an answer to the following question: if π is an irreducible, admissible representation of GL(2,F) what are the characters χ of K^* that occur in the restriction of π to K^*? It is immediate that if a character χ occurs in such a restriction then $\chi|_{F^*}$ must be the central character of π. Assuming that the residue characteristic is odd he showed that the multiplicity of such a character in $\pi|_{K^*}$ must be $\frac{\epsilon(\Pi\otimes\chi^{-1},\psi_0)+1}{2}$, where Π is a base change lift of π to K and ψ_0 is nontrivial additive character of K whose restriction to F is trivial. Tunnell also showed that under the given circumstances the value of the epsilon factor must be ± 1. Hence the multiplicity is either 0 or 1. Tunnell did this, case by case, by computing 'characters'; H. Saito [S] gave a residue characteristic free proof of the same result. D. Prasad [P1] considered a slightly more general problem. He looked at representations π of GL(2,F) whose restriction to GL(2,F)$^+$ (here GL(2,F)$^+$ denotes the subgroup of index 2 in GL(2,F) consisting of those matrices whose determinant is in $N_{K/F}(K^*)$ where $N_{K/F}$ is the usual norm map from K to F) breaks up as a sum of two irreducible representations $\pi_+ + \pi_-$ (this excluded, for instance, the exceptional representations whose restriction to GL(2,F)$^+$ remain irreducible). If π is supercuspidal then $\pi = r_\theta$, the Weil representation of GL(2,F) attached to a character θ of K^* whose restriction to F^* does not factor through the norm map from K to F. Using a lemma of Langlands [L], Prasad showed that χ occurs in $r_{\theta+}$ if and only if $\epsilon(\theta\chi^{-1}, \psi_0) = \epsilon(\overline{\theta}\chi^{-1}, \psi_0) = 1$ and in $r_{\theta-}$ if and only if both the epsilon factors are -1 ($\overline{\theta}$ is the Galois conjugate of θ). His proof was based on the following identity which is in fact equivalent to his extension theorem:

$$\epsilon(\omega, \psi) \frac{\omega(\frac{x-\overline{x}}{x_0-\overline{x}_0})}{\left|\frac{(x-\overline{x})^2}{x\overline{x}}\right|_{F^*}^{\frac{1}{2}}} = \sum_{\chi \in S} \chi(x) \tag{1.1}$$

where S is the set of characters χ of K^* whose restriction to F^* is $\omega_{K/F}$ and such that $\epsilon(\chi, \psi_0) = 1$. Prasad's [P1] proof of the identity (1.1) was valid only when the residue characteristic is odd. The residue characteristic even case was open till Prasad [P2] provided a proof. Using a theorem of Waldspurger [W], his own local identity in the odd residue characteristic case, and a local-global technique he was able to conclude that the identity is also true for even residue characteristic (ch F=0). Concurrently, Saito [P2] gave a purely local proof of the above identity which was residue characteristic free. He defined an involutive intertwining operator $T : \pi(1_{F^*}, \omega_{K/F}) \longrightarrow \pi(1_{F^*}, \omega_{K/F})$ which commuted with the GL(2,F)$^+$ action. The eigenspaces corresponding to the eigenvalues 1 and -1 turned out to be the spaces for π_+ and π_- and Saito defined explicit eigenfunctions in each space and proved that for such eigenfunctions f we have $Tf = \epsilon(\chi, \psi_0)f$. His proof involves integral operators and their convergence was the only difficulty in the proof.

We give a local proof of the extension theorem here which depends on the definition of the epsilon factor and nothing else. The proof is transparent and, we hope, better explains the difference between the odd and even residue characteristic cases. It spans the third and fourth chapters of this thesis. We show that in considering $\sum_{\chi \in S} \chi(x)$ enough cancelations take place to yield the left hand side of the identity. In fact, the right hand side after cancelations reduces to a sum over the characters of a single conductor. Note that the main identity (1.1) does not make sense if ch F=2 for then $x_0 \in F$.

In chapter 3 we complete the proof of the identity (1.1). We explain how this identity could be used together with a lemma of Langlands ([L], 7.19 *the lemma with an embarrassing proof*) to prove the extension of Tunnell's theorem. We state the extension theorem precisely:

Theorem 1.0.1. *Let r_θ be an irreducible admissible representation of $GL(2,F)$ associated to a regular character θ of K^*. Fix embeddings of K^* in $GL(2,F)^+$ and in $D^{*+}{}_F$ (there are two conjugacy classes of such embeddings in general), and choose a nontrivial additive character ψ of F, and an element x_0 of K^* with $tr(x_0) = 0$. Then the representation r_θ of $GL(2,F)$ decomposes as $r_\theta = r_{\theta+} \oplus r_{\theta-}$when restricted to $GL(2,F)^+$ and the representation $r_\theta{}'$ of $D^*{}_F$ decomposes as $r_\theta{}' = r_\theta{}'_+ \oplus r_\theta{}'_-$ when restricted to $D^{*+}{}_F$, such that for a character χ of K^* with $(\chi.\theta^{-1})|_{F^*} = \omega_{K/F}$, χ appears in $r_{\theta+}$ if and only if $\epsilon(\theta\chi^{-1}, \psi_0) = \epsilon(\overline{\theta}\chi^{-1}, \psi_0) = 1$, χ appears in $r_{\theta-}$ if and only if $\epsilon(\theta\chi^{-1}, \psi_0) = \epsilon(\overline{\theta}\chi^{-1}, \psi_0) = -1$, χ appears in $r_\theta{}'_+$ if and only if $\epsilon(\theta\chi^{-1}, \psi_0) = 1$ and $\epsilon(\overline{\theta}\chi^{-1}, \psi_0) = -1$, and χ appears in $r_\theta{}'_-$ if and only if $\epsilon(\theta\chi^{-1}, \psi_0) = -1$ and $\epsilon(\overline{\theta}\chi^{-1}, \psi_0) = 1$.*

In the second part of this thesis we use the above theorem to look at the following problem:

Let r_θ be as above. We have, by definition of central character, $\theta|_{F^*} = \omega_{r_\theta}\omega$ where ω_{r_θ} is the central character of r_θ and $\omega = \omega_{K/F}$. As we have already noted, since $\theta|_{F^*} \neq \omega_{r_\theta}$ the character θ cannot occur in $r_\theta|_{K^*}$. A necessary condition for a character λ of K^* to occur in $r_\theta|_{K^*}$ is that its restriction to F^* should be equal to the central character ω_{r_θ}. The question we would like to ask at this point is whether θ twisted by some character λ of K^* can occur in r_θ where $\lambda|_{F^*} = \omega$ since it satisfies the said necessary condition. That is, whether there exist some λ such that $\lambda\theta$ occurs in $r_\theta|_{K^*}$. We will prove some results which give an affirmative answer to this question. In fact, we will try to count at each conductor level precisely how many characters are there occuring in $r_{\theta+}$ and in $r_{\theta-}$.

1.1 Organization of the thesis

We explain the basic terminologies that we use in this work in the second chapter. We also state some known theorems and facts that we frequently use. Our basic references on these will be [Se], [Ne], [Bu], [Ta], [J-L], and [L]. Those terms we have not defined anywhere but used somewhere in this thesis could be found in one of these references. In the third chapter, we lay down the foundation required for proving identity (1.1). Most of the results proved in chapter three will be used in the final chapter also. After proving these basic results, we will turn in chapter four towards proving the identity and then the extension theorem. We in fact imitate [P1] in proving the extension theorem after proving the main identity using our own techniques. We can do this this because the lemma of Langlands used in proving the extension theorem in [P1] is residue characteristic free and so true in all residue characteristics.

In chapter five, which is the last chapter of this work, we count the twists of θ that occur in the restriction of the Weil representation r_θ attached to θ, a regular character. We make use of the the above cited theorem for this counting process. We do use an interesting result given at the end of chapter 4 heavily for our computations. The last chapter is divided into two parts, when the extension K is unramified over F, and when it is ramified. Unramified case is quite simple where as the ramified situation consists of some what lengthy, but straight forward computations.

We collect our main results of this thesis for quick reference in the next section to conclude this introductory chapter.

1.2 Main results

The main theorem in the first part of this thesis (that is, chapters 3 and 4) is the following. This is in fact equivalent to the extension of Tunnell's theorem. All most all the notations used here are explained in chapter 2.

Theorem 1.2.1. *Let K be a separable quadratic extension of a local field F of characteristic not two. Let ψ be a nontrivial additive character of F, and $x_0 \in K^*$ such that $tr(x_0) = 0$. Define an additive character ψ_0 of K by $\psi_0(x) = \psi(tr[-xx_0/2])$. Then*

$$\epsilon(\omega, \psi) \frac{\omega\left(\frac{x-\overline{x}}{x_0-\overline{x}_0}\right)}{\left|\frac{(x-\overline{x})^2}{x\overline{x}}\right|_{F^*}^{\frac{1}{2}}} = \sum_{\chi \in S} \chi(x) \tag{1.2}$$

$x \in K^* - F^*$ where as is usual, the summation on the right is by partial sums over all characters of K^* of conductor $\leq n$.

The following lemma is simple, but plays a pivotal role in our computations. Note that the summation is over χ's of a fixed conductor, not over all

χ's such that $\chi|_{F^*} = \omega$.

Lemma 1.2.2.

$$\sum_{\chi \in S(2f) \cup S'(2f)} \chi(y) = 0 \textit{ if } y \notin F^* U_K^{2f-1}.$$

Theorem 1.2.3. *$|S(l)| = |S'(l)|$ for each feasible l, that is when $l = 2d-1$ or $l = 2f$ with $f \geq d$.*

The above theorem states that with a given conductor l, out of all $\chi \in \widehat{K^*}$ with restriction to F^* equal to ω, half will have epsilon factor $+1$ and the remaining will have epsilon factor -1.

Theorem 1.2.4. *Let $x = 1 + \pi_F^{r-1}\pi_K x'$ where $x' \in U_F$, then*

$$\sum_{\chi \in S(2r+2m)} \chi(x) = \begin{cases} -q^{r-1} & \textit{if } m = 0 \\ 0 & m = 1, 2, \dots \textit{ and } m \neq d-1 \\ \epsilon(\omega, \psi)\dfrac{\omega(\frac{x-\overline{x}}{x_0-\overline{x}_0})}{\left|\frac{(x-\overline{x})^2}{x\overline{x}}\right|_{F^*}^{\frac{1}{2}}} & \textit{if } m = d-1 \end{cases}$$

and

$$\sum_{\chi' \in S'(2r+2m)} \chi'(x) = \begin{cases} -q^{r-1} & \textit{if } m = 0 \\ 0 & m = 1, 2, \dots \textit{ and } m \neq d-1 \\ -\epsilon(\omega, \psi)\dfrac{\omega(\frac{x-\overline{x}}{x_0-\overline{x}_0})}{\left|\frac{(x-\overline{x})^2}{x\overline{x}}\right|_{F^*}^{\frac{1}{2}}} & \textit{if } m = d-1 \end{cases}$$

The above is in fact stronger than theorem (1.2.1). In chapter 5, we have the following results:
If K/F ramified, then

Lemma 1.2.5. *If $\frac{\theta}{\overline{\theta}} = (-1)^{v_K}$ then no $\tilde{\omega}\theta$ can occur in r_θ where $\tilde{\omega} \in S_{2d-1}$.*

Theorem 1.2.6. *Let $0 \neq a(\frac{\theta}{\overline{\theta}}) < a(\tilde{\omega})$. Then among all $\tilde{\omega} \in S(2d-1)$ half and only half will be such that $\tilde{\omega}\theta$ occur in r_θ^+ and among all $\tilde{\omega} \in S'(2d-1)$ half and only half will be such that $\tilde{\omega}\theta$ occur in r_θ^-.*

Theorem 1.2.7. *Let* $\lambda \in S(2f+2d)$, $f \geq 0 \quad a(\frac{\theta}{\bar{\theta}}) \leq a(\lambda) - 2d = 2f$. *Then all the elements in* $\{\lambda\theta : \lambda \in S(2f+2d)\}$ *will occur in* ${r_\theta}^+$. *Similarly if* $\lambda' \in S'(2f+2d)$, *then all the elements in* $\{\lambda'\theta : \lambda' \in S'(2f+2d)\}$ *will occur in* ${r_\theta}^-$. *Therefore the number of* $\lambda\theta$ *where* $\lambda \in S_{2f+2d}$ *occurring in* r_θ *is* $|S_{2f+2d}|$.

Corollary 1.2.8. *If* $\frac{\theta}{\bar{\theta}} = (-1)^{v_K}$, *then all* $\lambda \in S(2f+2d)$ *will be such that* $\lambda\theta$ *will occur in* ${r_\theta}^+$. *Similarly all* $\lambda' \in S'(2f+2d)$ *will be such that* $\lambda'\theta$ *will occur in* ${r_\theta}^-$.

Theorem 1.2.9. *Let* $\lambda \in S(2f+2d)$, $2f+2 < a(\frac{\theta}{\bar{\theta}}) < a(\lambda)$. *Then among all* $\lambda\theta$ *where* $\lambda \in S(2f+2d)$ *exactly half will occur in* $r_{\theta+}$. *Similarly, let* $\lambda' \in S'(2f+2d)$, $2f+2 < a(\frac{\theta}{\bar{\theta}}) < a(\lambda')$. *Then among all* $\lambda'\theta$ *where* $\lambda' \in S'(2f+2d)$ *exactly half will occur in* $r_{\theta-}$. *Therefore the number of* $\lambda\theta$ *where* $\lambda \in S_{2f+2d}$ *occurring in* r_θ *is* $|S_{2f+2d}|/2$.

Theorem 1.2.10. *Let* $\lambda \in S(2f+2d)$, $a(\frac{\theta}{\bar{\theta}}) = 2f+2 < a(\lambda)$. *Then number of* $\lambda\theta$ *appearing in* $r_{\theta+}$ *where* $\lambda \in S(2f+2d)$ *is* $\frac{q-2}{2}q^{f+d-1}$. *Similarly, let* $\lambda' \in S'(2f+2d)$, $a(\frac{\theta}{\bar{\theta}}) = 2f+2 < a(\lambda')$. *Then number of* $\lambda'\theta$ *appearing in* $r_{\theta-}$ *where* $\lambda' \in S'(2f+2d)$ *is* $\frac{q-2}{2}q^{f+d-1}$. *The number of* $\lambda\theta$ *where* $\lambda \in S_{2f+2d}$ *occurring in* r_θ *is therefore* $(q-2)q^{f+d-1}$.

Theorem 1.2.11. *Let* $a(\lambda) = 2f+2d < a(\frac{\theta}{\bar{\theta}}) = 2m < a(\lambda)+2d$. *Then the number of* $\lambda\theta$ *with* $\lambda \in S_{2f+2d}$ *appearing in* r_θ *is* $|S(2f+2d)| = |S_{2f+2d}|/2$.

When $a(\lambda)$ is too small compared to $a(\frac{\theta}{\bar{\theta}})$ the occurrence of $\lambda\theta$ in $r_{\theta+}$ or $r_{\theta-}$ depends only on θ.

Theorem 1.2.12. *Suppose* $\lambda \in S(2m)$, $m \geq d$ *and* $a(\frac{\theta}{\bar{\theta}}) = 2n \geq a(\lambda)+2d$. *Then either all the elements in* $\{\lambda\theta : \lambda \in S(2m)\}$ *will occur in* $r_{\theta+}$ *or all the elements in* $\{\lambda'\theta : \lambda' \in S'(2m)\}$ *will occur in* $r_{\theta-}$ *not both. Therefore the number of* $\lambda\theta$ *where* $\lambda \in S_{2m}$ *occurring in* r_θ *is* $|S_{2m}|/2$.

The only one theorem in which we didn't give an exact count, but only a lower bound is the following:

Theorem 1.2.13. *If* $a(\frac{\theta}{\bar{\theta}}) = a(\lambda) = 2f + 2d$, $\lambda|_{F^*} = \omega$ *then the number of* $\lambda\theta$ *appearing in* r_θ *is greater than or equal to* q^{f+d-1}.

And we conclude the thesis in the fifth chapter by describing the possibilities of occurrence of twists when K/F is unramified.

Chapter 2

Preliminaries

2.1 Basic ideas and notations

Let F be a non Archimedean local field, that is, a non discrete locally compact totally disconnected topological field. If the characteristic of F is 0 then it is a finite extension of $\mathbb{Q}_p$. Otherwise it is isomorphic to $\mathbb{F}_q((X))$ where $\mathbb{F}_q$ is the finite field with q elements. Throughout this work, by a local field, we will mean a non archimedean local field.

For the local field F, we denote by O_F its ring of integers and $P_F = \pi_F O_F$ the unique prime ideal in O_F, where π_F is a uniformizer (a uniformizing element or prime element) which is an element in F such that its F-valuation is 1. That is, $v_F(\pi_F) = 1$.

The residue field O_F/P_F of F is denoted by $\mathbb{F}_q$. Here $q = |O_F/P_F|$. For a finite set X, by $|X|$ we denote its cardinality. $ch\,\mathbb{F}_q$ is called the residue characteristic of F.

By U_F or O_F^* we denote $O_F - P_F$, that is, the group of units in O_F. Let $P_F^i = \{x \in F : v_F(x) \geq i\}$ and for $i \geq 0$ define $U_F^i = 1 + P_F^i$ (with the provisio that $U_F^0 = U_F$).

Note that if F^* denote the multiplicative group of F, then $U_F \supset U_F^1 \supset U_F^2 \supset \ldots \supset U_F^i \supset \ldots$ form a fundamental system of neighborhoods at identity for F^*.

By $|\,|_F$ we mean the absolute value on F. It is chosen in such a way that $|\pi_F|_F = q^{-1}$. Note that for $u \in U_F$, $|u|_F = 1$.

If we fix a complete set of coset representatives for P_F in O_F as $\{a_1, a_2, \ldots, a_q\}$, then any $x \in O_F$ has a unique representation of the form

$$x = \sum_{i=0}^{\infty} x_i \pi_F^i$$

where $x_i \in \{a_1, a_2, \ldots, a_q\}$. Unless otherwise specified, our set of coset representatives will be $\mathbb{F}_q$. If $x \in P^k - P^{k+1}$ then $x_i = 0$ for $i = 0, 1, \ldots, k-1$ and $x_k \neq 0$.

Let K be a finite extension of F. If $\mathbb{K}_{q'}$ is the residue field of K, then $q' = q^f$ for some $f \in \mathbb{N}$. If $v_F(\pi_F) = e$ then $[K : F] = ef$.

If $e = 1$, then the extension is said to be unramified and if $f = 1$, it is said to be totally ramified. If the extension is not unramified, it is ramified.

By $N_{K/F}$ or simply N we denote the usual norm map from K to F and by $tr_{K/F}$ or simply tr we mean the usual trace map from K to F.
Note that

$$\frac{U_F}{U_F^n} \cong (\frac{O_F}{P_F^n})^* \text{ and } \frac{U_F^n}{U_F^{n+1}} \cong \frac{O_F}{P_F}$$

The polynomial $X^n + a_1 X^{n-1} + \ldots + a_n \in O_F[X]$ is said to be Eisenstein if $a_i \in P_F$ for all i and $a_n \notin P_F^2$. If K is a seperable ramified extension of F then K is generated by a uniformizer π_K of K whose minimal polynomial over F is an Eisenstein polynomial.

Let K be an unramified extension of F. Then $N(1+P_K^n) = 1+P_F^n \,\forall\, n \geq 1$

The differential exponent $d_{K/F}$ or simply d of K over F is defined to be the integer d such that $tr\, P_K^{-d} \subseteq O_F$ but $tr\, P_K^{-d-1} \not\subseteq O_F$.

Suppose K is a seperable quadratic extension of the local field F. If the residue characteristic is odd then $d = e - 1$. If the residue characteristic is even, that is, two then $d = 2t+1$ where $t = v_F(2)$ if there exists a uniformizer π_K of K such that $tr\, \pi_K = 0$ and $d = 2s$ if no such uniformizer exists and K is the splitting field of the Eisenstein polynomial $X^2 - u\pi_F^t + \pi_F$, $u \neq 0$. In this case, $s \leq t$.

If G is a locally compact abelian group by $\widehat{G}$ we mean the group of characters of G. Denote by $\omega_{K/F}$, or simply ω the character of F^* associated to K by class field theory, i.e., it is the unique nontrivial quadratic character of $F^*/N(K^*)$.

Conductor of an additive character ψ of F is $n(\psi)$ if ψ is trivial on $P_F^{-n(\psi)}$, but nontrivial on $P_F^{-n(\psi)-1}$. Let $\psi_K = \psi \circ tr_{K/F}$. Then the conductor of ψ_K is $d+n(\psi)$ where d is the differential exponent of K over F. For a character χ of F^* by $a(\chi)$ we mean the conductor of χ. Therefore $a(\chi)$ is the smallest integer $n \geq 0$ such that χ is trivial on U_F^n. We say that χ is unramified if $a(\chi)$ is zero. Also, if χ_1 and χ_2 are two characters of F then $a(\chi_1\chi_2) \leq max(a(\chi_1), a(\chi_2))$. Equality holds if $a(\chi_1) \neq a(\chi_2)$. Furthermore, $a(\chi) = a(\chi^{-1})$.

The image of $x \in K$ under the nontrivial element of the Galois group of K over F is denoted by $\overline{x}$ or x^σ. If $x_0 \in K - F$ is such that $tr\, x_0 = 0$, we define $\psi_0(x) = \psi_K(\frac{-x_0 x}{2}) = \psi(tr \frac{-x_0 x}{2})$. Note that $n(\psi_0) = n(\psi_K) - v_K(2) + v_K(x_0)$. Note that ψ_0 is an additive character of K trivial on F.

A character θ of K^* is regular if it does not factor through the norm map $N_{K/F}$. By r_θ we denote the irreducible admissible Weil representation of $GL(2, F)$ associated to a character θ of K^*.

All the above definitions stated for F hold true for K also with obvious modifications.

2.2 A little bit of representation theory of $GL(2)$

Let G be a totally disconnected locally compact group and let (π, V) be a representation of G. Then V is a complex vector space (possibly infinite dimensional) on which G acts. We say π is smooth if for any $v \in V$, the stabilizer $\{g \in G : \pi(g)v = v\}$ is open. If π is smooth and if furthermore for any open subgroup U of G, the space V^U of vectors stabilized by U is finite dimensional then π is called admissible.

The center of G acts by scalar multiplication on the irreducible admissible representation (π, V). Thus if $G = GL(n, F)$, F non archimedean local field then there exists a character ω_π of F^* called the central character of π such that the center Z (of nonzero scalar matrices) of $GL(n, F)$ acts by $\pi(zI)v = \omega_\pi(z)v$ where $v \in V, z \in F^*$, and I is the $n \times n$ identity matrix.

Let F be a non Archimedean local field. We fix an additive Haar measure $\int_F dx$ on F normalized so that the volume of O_F is one.

Let us turn our attention to representations of $GL(2, F)$. We can classify them mainly into three:

a) Supercuspidal representations

b) Principal Series representations

c) Special representation

Let V^* be the dual of V. We can define a representation $\tilde{\pi}$ on V^* satisfying $< \pi(g)(v), v^* >=< v, \tilde{\pi}(g)(v^*) >$ for all $v \in V$, $v^* \in V^*$. The representation $\tilde{\pi}$

is not admissible. However, the subspace $\tilde{V}$ of V^* consisting of vectors which are stabilized by an open subgroup is invariant under $\tilde{\pi}$ and $\tilde{\pi}$ restricted to this subspace is admissible. This restriction, by abuse of notation, is also denoted by $\tilde{\pi}$ and is called the contragradient of π. Given $v \in V$ and $\tilde{v} \in \tilde{V}$, we can define a complex valued function f on $GL(2, F)$ by $f(g) =< \pi(g)(v), \tilde{v} >$. Such functions are called coefficients of π and π is said to be supercuspidal if its coefficients have compact support modulo the center Z of $GL(2, F)$. That is, given a coefficient f there exists a compact subset C of $GL(2, F)$ such that the support of F is in CZ.

Let μ_1, μ_2 be two characters of F^* and let $\mathcal{B}(\mu_1, \mu_2)$ be the space of locally constant functions ϕ on $GL(2, F)$ which satisfy

a) for all $g \in GL(2, F)$, $a_1, a_2 \in F^*$, and $x \in F$

$$\phi\left(\begin{bmatrix} a_1 & x \\ 0 & a_2 \end{bmatrix} g\right) = \mu_1(a_1)\mu_2(a_2)|\frac{a_1}{a_2}|^{1/2}\phi(g)$$

b) there is an open subgroup U of $GL(2, O_F)$ such that $\phi(ug) = \phi(g)$ for all $u \in U$.

Let ρ be the right regular representation of $GL(2, F)$ on $\mathcal{B}(\mu_1, \mu_2)$. That is, $(\rho(g)f)(h) = f(hg)$. If $\mu_1\mu_2^{-1} \neq |x|_F$ or $|x|_F^{-1}$ then ρ is irreducible and is denoted by $\pi(\mu_1, \mu_2)$. If $\mu_1\mu_2^{-1} = |x|_F$ or $|x|_F^{-1}$, then $\mathcal{B}(\mu_1, \mu_2)$ has an irreducible, invariant subspace of codimension (respectively dimension) 1 and we let $\sigma(\mu_1, \mu_2)$ denote the restriction ρ to the invariant subspace (or the infinite dimensional quotient representation).

If π is an infinite dimensional irreducible representation of $GL(2, F)$ which is not supercuspidal, then, for some choice of $\mu_1, \mu_2 \in \widehat{F^*}$, $\pi = \pi(\mu_1, \mu_2)$ or $\sigma(\mu_1, \mu_2)$. If $\pi = \pi(\mu_1, \mu_2)$ then π is called a principal series representation and if $\pi = \sigma(\mu_1, \mu_2)$ then π is called a special representation.

2.2.1 Base change lift

Let F be a local non archimedean field and F_{nr} be the maximal unramified extension of F and $\overline{F}$ be the seperable closure of F. We have the following exact sequence:

$$1 \longrightarrow Gal(F_{nr}/F) \xrightarrow{i} Gal(\overline{F}/F) \xrightarrow{r} Gal(F_{nr}/F) \longrightarrow 1$$

Let Φ be the Frobenious of $Gal(F_{nr}/F)$ then the pull back of $< \Phi >$ with respect to r is called the Weil group of F and is denoted by W_F. If K is an extension of F in $\overline{F}$ then $W_K \subseteq W_F$.

By the fundamental theorem of local classfield theory we have a canonical isomorphism between F^* and

$$W_F^{ab} = \frac{W_F}{W_F^c}$$

where W_F^c is the closure of the commutator subgroup of W_F. This gives rise to an isomorphism $\tau_F : \widehat{F^*} \to \widehat{W_F}$ (recall that for a locally compact group G, $\widehat{G}$ denotes the group of its characters) called the reciprocity map. Similarly let $\widehat{GL(2,F)}$ denote the set of all irreducible, admissible, supercuspidal representations of $GL(2,F)$ and $\widehat{W_F^2}$ the set of irreducible 2-dimensional representations of W_F. Then the Local Langlands' correspondence predicts the existence of a canonical bijection $L_F : \widehat{W_F^2} \to \widehat{GL(2,F)}$. Local class field theory tells us that the following diagram is commutative.

$$\begin{tikzcd}
\widehat{F^*} \arrow[r, "\tau_F"] \arrow[d, "BC_{K/F}"] & \widehat{W_F} \arrow[d, "r_{K/F}"] \\
\widehat{K^*} \arrow[r, "\tau_K"] & \widehat{W_K}
\end{tikzcd}$$

where $r_{K/F}$ is the restriction taking χ to $\chi_{|W_K}$ and $BC_{K/F}$ is the base change map taking χ to $\chi \circ N$ where N is the usual norm map from K to F. Likewise we have the following commutative diagram:

$$\begin{tikzcd} \widehat{GL(2,F)} \arrow[r, "L_F"] \arrow[d, "BC_{K/F}"] & \widehat{W_F^2} \arrow[d, "r_{K/F}"] \\ \widehat{GL(2,K)} \arrow[r, "L_K"] & \widehat{W_K^2} \end{tikzcd}$$

Then for $\rho \in \widehat{GL(2,F)}$, $BC_{K/F}\rho = \Pi$ is called the base change lift of ρ.

2.3 Epsilon factors

Now we will look at the most important terminology in this work - Epsilon factors.

Let F be a local field. Let $M(W_F)$ denote the set of isomorphism classes of representations of W_F and $R(W_F)$ the group of all virtual representations of W_F. For a representation V of W_F, let $[V] \in R(W_F)$ denote the virtual representation determined by V. Let $R^0(W_F)$ denote the group of all virtual representations of degree 0 of W_F, that is, those of the form $[V] - [V']$, with dim V=dim V'.

Let λ be a function which assigns to each finite seperable extension K/F and each $V \in M(W_K)$ an element $\lambda(V)$ in an abelian group X. We say λ is additive over F if for each K and each exact sequence $0 \to V' \to V \to V'' \to 0$ of representations of W_K we have $\lambda(V) = \lambda(V')\lambda(V'')$. When that is so we can define λ on virtual representations so that $\lambda : R(K) \to X$ is a homomorphism for each K. We say λ is inductive over F if it is additive over F and the diagram

$$\begin{tikzcd} R(W_K) \arrow[d, dashed, "Ind_{K/K'}"'] \arrow[dr] & \\ R(W_{K'}) \arrow[r] & X \end{tikzcd}$$

is commutative for finite separable extensions $K/K'/F$. We say λ is inductive in degree 0 over F if the same is true with R replaced by R^0. Langlands predicted and proved the existence and uniqueness of a local function - the epsilon function (theorem (3.4.1), [Ta]):

Theorem 2.3.1. *There is a unique function ϵ which associates with each choice of a local field F, a nontrivial additive character ψ of F, an additive Haar measure dx of F and a representation V of W_F a non zero complex number $\epsilon(V, \psi, dx)$ such that $\epsilon(V, \psi, dx) = \epsilon(\chi, \psi, dx)$ if V is a representation of degree 1 corresponding to a character χ and such that if F is a local field and we choose for each finite seperable extension K of F an additive Haar measure μ_K on K, then the function which associates with each such K and each $V \in M(W_K)$ the number $\epsilon(V, \psi \circ tr_{K/F}, \mu_K)$ is inductive in degree 0 over F.*

There is no explicit formula available for the epsilon function $\epsilon(V, \psi, dx)$. But in the 1-dimensional case, that is, when χ is a character, we have the following: For a character χ of F^* and a nontrivial additive character ψ of F, we have

$$\epsilon(\chi, \psi) = \chi(c) \frac{\int_{U_F} \chi^{-1}(y)\psi(y/c)dy}{|\int_{U_F} \chi^{-1}(y)\psi(y/c)dy|}$$

where the Haar measure dy is normalized such that measure of O_F is 1 and c is an element of F^* satisfying the property $v_F(c) = a(\chi) + n(\psi)$.

2.3.1 Properties of epsilon factors

We will be using many of the known properties of epsilon factors in this work. So it would be convenient to collect them together. Some theorems that are related to epsilon factors, that we will be using, but not listed here, will be explained in the due course.

1. $\epsilon(\eta\mu, \psi) = \eta(\pi_F)^{a(\mu)+n(\psi)}\epsilon(\mu, \psi)$ if η is unramified.

2. $\epsilon(\mu, \psi)\epsilon(\mu^{-1}, \psi) = \mu(-1)$

3. $\epsilon(\lambda, \psi_1) = \epsilon(\lambda^\sigma, \psi_1^\sigma)$ where $\lambda \in \widehat{K^*}$ and ψ_1 is a nontrivial additive character of K.

4. $\epsilon(\pi, \psi_a) = \omega_\pi(a)\epsilon(\pi, \psi)$ where $\psi_a(x) = \psi(ax)$.

5. $\epsilon(\chi, \psi) = 1$ if χ is an unramified character of F^* and ψ a nontrivial additive character of F with $n(\psi) = 0$.

Chapter 3

Completing an extension of Tunnell's theorem

Let F be a non archimedean local field and K a separable quadratic extension of F. Let $S = \{\chi \in \widehat{K^*} : \chi\mid_{F^*} = \omega, \epsilon(\chi, \psi_0) = 1\}$ and $S(l) = \{\chi \in S : a(\chi) = l\}$. Analogously, we define S' and $S'(l)$ with the property that $\epsilon(\chi, \psi_0) = -1$. Also, define $S_l = S(l) \cup S'(l)$. In this chapter, we will be proving certain results which will help us to prove the following *main identity*

$$\epsilon(\omega, \psi)\frac{\omega(\frac{x-\overline{x}}{x_0-\overline{x}_0})}{\left|\frac{(x-\overline{x})^2}{x\overline{x}}\right|_{F^*}^{\frac{1}{2}}} = \sum_{\chi \in S} \chi(x)$$

using basically the definition of epsilon factors and some existing results on these local factors. This identity plays a key role in this thesis work. After proving the main identity, we will show how this identity implies the extension of Tunnell's theorem. Our proof of this identity is divided into mainly two cases, viz. K/F unramified and K/F ramified covering both residue characteristic odd and residue characteristic even cases. Note that this identity firstly appeared in [P1], but there the proof was provided only in the odd residue characteristic case.

3.1 Two theorems on epsilon factors

We will be using the following two theorems from [F-Q] and [D] in our work. So it will be convenient to state them here. The following theorem was proved by Frohlich and Queyrut[F-Q] in characteristic zero and by Deligne more generally.

Theorem 3.1.1. *Let F be a local field of characteristic not two and K a quadratic extension. Let ψ be a nontrivial additive character of F and $\psi_K = \psi \circ tr$. Then for any character χ of K^* whose restriction to F^* is trivial we have $\epsilon(\chi, \psi_K) = \chi(x_0)$ where x_0 is any element of trace zero.*

Since $\psi_0(x) = \psi_K(\frac{-x_0}{2}x)$, we have $\epsilon(\chi, \psi_0) = \chi(\frac{-x_0}{2})\epsilon(\chi, \psi_K) = \chi(\frac{-x_0}{2})\chi(x_0)$. But $\frac{-x_0^2}{2} \in F^*$ and χ is trivial on F^*. So this theorem is equivalent to saying that $\epsilon(\chi, \psi_0) = 1$.

Deligne[D] described how the epsilon factor changes under twisting by a character of small conductor in the theorem:

Theorem 3.1.2. *Let α, β be two characters of a local field F such that $a(\alpha) \geq 2a(\beta)$. Let y_α be an element of F^* such that $\alpha(1+x) = \psi(y_\alpha x)$ for $v_F(x) \geq \frac{a(\alpha)}{2}$ (if $a(\alpha) = 0$ let $y_\alpha = \pi_F^{-n(\psi)}$). Then $\epsilon(\alpha\beta, \psi) = \beta^{-1}(y_\alpha)\epsilon(\alpha, \psi)$.*

We remark that $v_F(y_\alpha) = -a(\alpha) - n(\psi)$.

3.2 The Main Theorem

We state our main theorem of this part.

Theorem 3.2.1. *Let K be a separable quadratic extension of a local field F of characteristic not two. Let ψ be a nontrivial additive character of F, and $x_0 \in K^*$ such that $tr(x_0) = 0$. Define an additive character ψ_0 of K by*

$\psi_0(x) = \psi(tr[-xx_0/2])$. *Then*

$$\epsilon(\omega, \psi)\frac{\omega(\frac{x-\overline{x}}{x_0-\overline{x}_0})}{\left|\frac{(x-\overline{x})^2}{x\overline{x}}\right|_{F^*}^{\frac{1}{2}}} = \sum_{\chi \in S} \chi(x) \tag{3.1}$$

$x \in K^* - F^*$ *where as is usual, the summation on the right is by partial sums over all characters of* K^* *of conductor* $\leq n$.

Note: From now onwards, in this and next chapters by LHS and RHS we will mean the left hand side and right hand side of the above equation. We refer to the equation in this theorem as the *main identity.*
We quote a result from [P1] which is going to be quite important in our computations.

Lemma 3.2.2. *Let* $\chi \in \widehat{K^*}$ *such that* $\chi|_{F^*} = \omega$. *Then* $\epsilon(\chi, \psi_0) = \pm 1$.

Proof. Note that ω is trivial on $N(K^*)$. Now $\chi\overline{\chi}(x) = \chi(x\overline{x}) = \omega(x\overline{x}) = 1$. So $\chi^{-1} = \overline{\chi}$. Therefore,

$$\begin{aligned}
\epsilon(\chi^{-1}, \psi_0) &= \epsilon(\overline{\chi}, \psi_0) \\
&= \overline{\chi}(\frac{-x_0}{2})\epsilon(\overline{\chi}, \psi_K) \\
&= \overline{\chi}(\frac{-x_0}{2})\epsilon(\chi, \overline{\psi_K}) \text{ (by Galois invariance of epsilon factors)} \\
&= \overline{\chi}(\frac{-x_0}{2})\epsilon(\chi, \psi_K) \\
&= \omega(-1)\chi(\frac{-x_0}{2})\epsilon(\chi, \psi_K) \\
&= \omega(-1)\epsilon(\chi, \psi_0)
\end{aligned}$$

Therefore, $\omega(-1)\epsilon(\chi, \psi_0)^2 = \epsilon(\chi^{-1}, \psi_0)\epsilon(\chi, \psi_0) = \chi(-1) = \omega(-1) \Longrightarrow \epsilon(\chi, \psi_0)^2 = 1$. This proves the lemma. $\square$

For an element r of F^*, and $\chi \in S$, we have $\epsilon(\chi, (\psi_0)_r) = \chi(r)\epsilon(\chi, \psi_0) = \omega(r)\epsilon(\chi, \psi_0)$ where $(\psi_0)_r(x) = \psi_0(rx)$. So if $\omega(r) = 1$, then $\epsilon(\chi, (\psi_0)_r) =$

$\epsilon(\chi, \psi_0)$ and so
$$\sum_{\chi \in S} \chi(x) = \sum_{\epsilon(\chi,(\psi_0)_r)=1,\, \chi|_{F^*}=\omega} \chi(x).$$
On the other hand, if $\omega(r) = -1$, since $\sum_{\chi \in S \cup S'} \chi(x) = 0$, we have
$$\sum_{\chi \in S} \chi(x) = -\sum_{\chi \in S'} \chi(x) = \omega(r) \sum_{\epsilon(\chi,(\psi_0)_r)=1,\, \chi|_{F^*}=\omega} \chi(x).$$
But $\omega(\frac{x-\overline{x}}{rx_0-\overline{r}x_0}) = \omega(r)\omega(\frac{x-\overline{x}}{x_0-\overline{x}_0})$ for $r \in F^*$. Now if we fix a nontrivial additive character ψ of F, then every other nontrivial additive character ψ' of F is of the form $\psi'(x) = \psi(ax)$ for some $a \in F^*$ and if x_0 is an element of trace 0 in K^*, then any other trace 0 element in K^* will be of the form rx_0 for some r in F^*. From this it follows easily that once the theorem is proved for one choice of x_0 and ψ, it is true for any other choice of these parameters. Therefore we will choose an additive character ψ of F with conductor 0. Let x_0 a unit if K is unramified. If K is ramified and d is odd we take x_0 to be a uniformizer π_K of K and when d even it will be a unit in K. It is also clear that once the theorem is true for an $x \in K^*$, then it is so for any rx for $r \in F^*$. Hence it suffices to prove theorem (3.2.1) either when x is a unit or when x has valuation 1.

Let $\tilde{\omega}_{K/F} \in \widehat{K^*}$, or simply $\tilde{\omega}$, be an extension of ω of conductor $2d-1$ if $d \geq 1$ and if $d = 0$(that is, when K/F unramified) we assume $\tilde{\omega}$ to be trivial on U_K and -1 on any uniformizer of K. Then if $a(\chi) \geq 2a(\tilde{\omega})$ we have

$$\begin{aligned}
\epsilon(\chi, \psi_0) &= \chi(-x_0/2)\epsilon(\chi, \psi_K) \\
&= \chi(-x_0/2)\epsilon(\chi.\tilde{\omega}^{-1}.\tilde{\omega}, \psi_K) \\
&= \chi(-x_0/2)\epsilon(\chi.\tilde{\omega}^{-1}, \psi_K)\tilde{\omega}^{-1}(y_\chi) \\
&= \chi(-x_0/2).(\chi.\tilde{\omega}^{-1})(x_0)\tilde{\omega}^{-1}(y_\chi) \\
&= \chi(-{x_0}^2/2)\tilde{\omega}^{-1}(x_0)\tilde{\omega}^{-1}(y_\chi) \\
&= \tilde{\omega}(-x_0/2)\tilde{\omega}^{-1}(y_\chi) \qquad (3.2)
\end{aligned}$$

where $\chi.\tilde{\omega}^{-1}(1+x) = \psi_K(y_{\chi\tilde{\omega}^{-1}}x)$. Since $a(\chi) \geq 2a(\tilde{\omega})$, $y_\chi = y_{\chi\tilde{\omega}^{-1}}$. Here y_χ is as in theorem (3.1.2). Also, $v_k(y_\chi) = -a(\chi) - n(\psi_K)$.

3.3 Proof of the main identity in the unramified case

Consider first the relatively simple unramified case, i.e., K/F is unramified. The proof is as in the odd residue characteristic case as given in [P1]. However since no details are given in [P1], we give a detailed proof.

We have $\pi_K = \pi_F \in F$. Take $y_\chi = \pi_F^{-a(\chi)-n(\psi_K)}$ in equation (3.2). Hence $\epsilon(\chi, \psi_0) = (-1)^{a(\chi)+n(\psi_K)}$. But $n(\psi_K) = 0$ and $\tilde{\omega}^{-1} = (-1)^{v_K}$. So $\epsilon(\chi, \psi_0) = (-1)^{a(\chi)+t}$. We need only consider the case when $x \in K^* - F^*$ is a unit. Since ω unramified and $n(\psi) = 0$, we have $\epsilon(\omega, \psi) = 1$. Now if $x = a_0 + \ldots + a_r\pi_F^r + \ldots$ where $a_i \in O_K$ and r is the largest positive integer such that $a_i \in O_F$ for all $i < r$ then our LHS becomes

$$\begin{aligned} \frac{\omega(\frac{x-\overline{x}}{x_0-\overline{x}_0})}{\left|\frac{(x-\overline{x})^2}{x\overline{x}}\right|_{F^*}^{\frac{1}{2}}} &= \frac{\omega(\frac{\pi_F^r}{2})}{|\pi_F^{2r}|_{F^*}^{\frac{1}{2}}} \\ &= (-1)^{r+t}q^r \end{aligned}$$

Case 1: t even

Then $\epsilon(\chi, \psi_0) = 1$ if and only if $a(\chi)$ is even, say $2m$, for some $m \geq 0$. Let r be even. To prove the theorem we have to show that

$$q^r = \sum_{\chi \in S(2m),\, m \geq 0} \chi(x)$$

With conductor 0, note that there is only one nontrivial character. For χ to be an extension of ω it has to be nontrivial. We have $\epsilon(\chi, \psi_0) = \chi(\frac{-x_0}{2})\epsilon(\chi, \psi_K)$. But $a(\chi) = 0$ and $n(\psi_K) = 0$ gives $\epsilon(\chi, \psi_K) = 1$. Therefore $\epsilon(\chi, \psi_0) =$

$\chi(\frac{-x_0}{2}) = (-1)^{v_K(2)} = (-1)^t = 1$. Since this χ is trivial on units, we have

$$\sum_{\chi \in S(0)} \chi(x) = 1$$

If $0 < m$ and $2m \leq r$,

$$\begin{aligned}
\sum_{\chi \in S(2m)} \chi(x) &= |S(2m)| \\
&= |\frac{U_K}{U_F U_K^{2m}}| - |\frac{U_K}{U_F U_K^{2m-1}}| \\
&= (q+1)q^{2m-1} - (q+1)q^{2m-2} \\
&= q^{2m} - q^{2m-2}
\end{aligned}$$

If $a(\chi) = 2m \geq r+2$, let $\mu \in \widehat{\frac{U_K}{U_F U_K^{r+1}}}$. Then

$$\begin{aligned}
\sum_{\chi \in S(2m)} \chi(x) &= \sum_{\chi_i} \sum_{\mu} (\chi_i \mu)(x) \\
&= \sum_{\chi_i} \chi_i(x) \sum_{\mu} \mu(x) \\
&= \sum_{\chi_i} \chi_i(x).0 \\
&= 0
\end{aligned}$$

where χ_i's are distinct nontrivial characters of $\frac{U_K^{2m-1}}{U_K^{2m}}$ extended arbitrarily to K^*. On adding all these sums for all values of $m \geq 0$ we get q^r which is what is required.

Suppose r odd. Then our LHS is $-q^r$. Again

$$\sum_{\chi \in S(0)} \chi(x) = 1$$

For $0 < 2m \leq r-1$,

$$\sum_{\chi \in S(2m)} \chi(x) = q^{2m} - q^{2m-2}$$

as in the previous case. If $a(\chi) = 2m \geq r+3$, let $\mu \in \widehat{\frac{U_K}{U_F U_K^{r+2}}}$. Then

$$\begin{aligned}\sum_{\chi \in S(2m)} \chi(x) &= \sum_{\chi_i} \sum_{\mu} (\chi_i \mu)(x) \\ &= 0.\end{aligned}$$

Note that here $r+1$ is even. So $a(\chi) = r+1$ is possible.

$$\begin{aligned}\sum_{\chi \in S(r+1)} \chi(x) &= \sum_{\mu \in \widehat{\frac{U_K}{U_F U_K^r}}} \sum_{\chi_i} \chi_i \mu(x) \\ &= \sum_{\mu \in \widehat{\frac{U_K}{U_F U_K^r}}} \mu(x) \sum_{\chi_i} \chi_i(x) \\ &= |\frac{U_K}{U_F U_K^r}| \times -1 \\ &= (q+1)q^{r-1}. -1 \\ &= -q^r - q^{r-1}\end{aligned}$$

where χ_i's are distinct nontrivial characters of $\frac{U_K^r}{U_K^{r+1}}$ extended arbitrarily to K^*. So the total sum becomes $-q^r$ which is equal to the LHS.

Case 2: t odd

Then $\epsilon(\chi, \psi_0) = 1$ if and only if $a(\chi)$ is odd. Let r be odd. Then our LHS is q^r.

If $a(\chi) = 1$ then

$$\sum_{\chi \in S(1)} \chi(x) = |\frac{U_K}{U_F U_K}| - |\frac{U_K}{U_F U_K^1}| = (q+1) - 1 = q.$$

Now if $1 \neq a(\chi) = 2m+1 \leq r$ then

$$\sum_{\chi \in S(2m+1)} \chi(x) = |S(2m+1)| = |\frac{U_K}{U_F U_K^{2m+1}}| - |\frac{U_K}{U_F U_K^{2m}}| = (q+1)(q^{2m} - q^{2m-1}).$$

When $a(\chi) = 2m+1 > r+1$, we have $\sum_{\chi \in S(2m+1)} \chi(x) = 0$ as in the t even case. Hence

$$\sum_{m=0}^{\infty} \sum_{\chi \in S(2m+1)} \chi(x) = q^r = \text{ LHS}.$$

If r is even, our LHS is $-q^r$. As above, we have

$$\sum_{\chi\in S(1)} \chi(x) = 1,$$

if $1 \neq a(\chi) = 2m+1 < r$ then

$$\sum_{\chi\in S(2m+1)} \chi(x) = (q+1)(q^{2m} - q^{2m-1})$$

$$\sum_{\chi\in S(r+1)} \chi(x) = -q^r - q^{r-1} \text{ as in t even case.}$$

If $a(\chi) = 2m+1 > r+1$ then $\sum_{S(2m+1)} \chi(x) = 0$. So the total sum becomes $-q^r$ which is the LHS. Thus the identity is true when K/F is unramified, residue characteristic even or odd.

3.4 The ramified case

We will now assume that K is ramified over F. It is known (see, for instance, [K-T], section 3) that if d is odd then $d = 2t+1$ and there exists a uniformizer, denoted by π_K such that $tr\,\pi_K = 0$. Let $x_0 = \pi_K$. In this case π_K^2 is a uniformizer of F which we denote by π_F and $N\pi_K = -\pi_F$. If d is even (which can only happen if the residue characteristic is 2) then $O_K = O_F[\pi_K]$ where π_K is a uniformizer of K which satisfies the Eisenstein polynomial $X^2 - u'\pi_F^s X - \pi_F$ with $s \leq t$ and $u' \in U_F$. Again $N\pi_K = -\pi_F$. In this case $d = 2s$ and $\pi_K = \frac{\pi_F^s u'}{2}(1 + x_0)$ where x_0 is a unit of trace 0. We note that $n(\psi_0)$ is equal to 2 if d is odd and $2(s-t)$ if d is even. So $n(\psi_0)$ is always even.

Lemma 3.4.1. *If $\chi|_{F^*} = \omega$ and $a(\chi) \neq 2d-1$ then $a(\chi) = 2f \geq 2d$.*

Proof. Suppose $a(\chi) \leq 2d-2$ then $\chi|_{\frac{U_F^{d-1}}{U_F^d}} = 1$ so that χ cannot be an extension of ω. On the other hand, suppose that $a(\chi) = 2f+1 > 2d$. Then $\chi|_{\frac{U_K^{2f}}{U_K^{2f+1}}}$ has to be nontrivial. But if $\chi|_{F^*} = \omega$ then $\chi(1+\pi_F^f a) = \omega((+\pi_F^f a) = 1$ where $a \in \mathbb{F}_q$ since $a(\omega) = d < f$. So $a(\chi)$ cannot be $2f+1 > 2d$. $\square$

Recall that For a character χ of F^* and a nontrivial additive character ψ of F we have

$$\epsilon(\chi,\psi) = \chi(c)\frac{\int_{U_F}\chi^{-1}(y)\psi(y/c)dy}{|\int_{U_F}\chi^{-1}(y)\psi(y/c)dy|}$$

where the Haar measure dy is normalized such that measure of O_F is 1 and c is an element of F^* satisfying the property $v_F(c) = a(\chi)+n(\psi)$. We quote a result from [G-G-S], section (2.2.6).

Lemma 3.4.2. *If $a(\chi) \neq 0$ then*

$$|\int_{U_F}\chi^{-1}(y)\psi(\pi_F^{-n}y)dy)| = \begin{cases} q^{-a(\chi)/2} \text{ if } n = a(\chi)+n(\psi) \\ 0 \quad \text{otherwise.} \end{cases}$$

where $\chi \in \widehat{F^*}$ and ψ is a nontrivial character of $(F,+)$.

Hence the formula above reduces to

$$\epsilon(\chi,\psi) = \chi(c)q^{a(\chi)/2}\sum_{y\in\frac{U_F}{U_F^{a(\chi)}}}\chi^{-1}(y)\psi(y/c)m'({U_F}^{a(\chi)})$$

The Haar measure m' is normalized so that $m'(O_F) = 1$. But $O_F = \bigcup_{a_i\in\mathbb{F}_q}(a_i+\pi_F^n U_F : n = 1,2,\ldots)$. Since $|\mathbb{F}_q| = q$, each element of this union has measure $\frac{1}{q}$. So $m'(U_F) = 1-\frac{1}{q} = \frac{q-1}{q}$. Since $U_F = \frac{U_F}{U_F^{a(\chi)}}U_F^{a(\chi)}$, it follows that $m'({U_F}^{a(\chi)}) = \frac{m'(U_F)}{|\frac{U_F}{U_F^{a(\chi)}}|}$. Now $|\frac{U_F}{U_F^{a(\chi)}}| = (q-1)q^{a(\chi)-1}$. So we can modify the formula as

$$\epsilon(\chi,\psi) = \chi(c)q^{-a(\chi)/2}\sum_{y\in\frac{U_F}{U_F^{a(\chi)}}}\chi^{-1}(y)\psi(y/c)) \tag{3.3}$$

where $c = \pi_F^{a(\chi)+n(\psi)}$. In particular since $a(\omega) = d$ and we have chosen ψ such that $n(\psi) = 0$ we have

$$\epsilon(\omega, \psi) = \omega(\pi_F^d) q^{-d/2} \sum_{y \in \frac{U_F}{U_F^d}} \omega(y)\psi(\pi_F^{-d} y) \tag{3.4}$$

The results following help us very much in our computations.

Lemma 3.4.3. *Let $\chi \in \widehat{F^*}$ and ψ a nontrivial character of $(F, +)$.*

1. *If $n < a(\chi) + n(\psi)$ then*

$$\sum_{u \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(u)\psi(\pi_F^{-n} u) = 0$$

2. *If $n > a(\chi) + n(\psi)$ then*

$$\sum_{u \in \frac{U_F}{U_F^n}} \chi^{-1}(u)\psi(\pi_F^{-n} u) = 0$$

Proof. 1. Let $n < a(\chi) + n(\psi)$. Every $y \in U_F$ is of the form $y_1(1 + \pi_F^m x)$ for some $x \in U_F$, $y_1 \in \frac{U_F}{U_F^{a(\chi)}}$, $m \geq a(\chi)$. Then $1 + \pi_F^m x \in U_F^{a(\chi)}$. Now $\psi(\pi_F^{-n} y) = \psi(\pi_F^{-n} y_1)\psi(\pi_F^{m-n} y_1 x)$. But $n < a(\chi) + n(\psi) \implies -n(\psi) < m - n \implies \psi(\pi_F^{m-n} yx) = 1$. Therefore

$$\begin{aligned} 0 &= \int_{U_F} \chi^{-1}(y)\psi(\pi_F^{-n} y) dy \\ &= \sum_{u \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(u)\psi(\pi_F^{-n} u) m'({U_F}^{a(\chi)}) \text{ since } \psi(\pi_F^{-n}) \text{ and } \chi \text{ trivial on } U_F^{a(\chi)} \end{aligned}$$

$\implies \sum_{u \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(u)\psi(\pi_F^{-n} u) = 0$ since $m'({U_F}^{a(\chi)}) \neq 0$

2. If $n > a(\chi) + n(\psi)$ then

$$\begin{aligned} \sum_{u \in \frac{U_F}{U_F^n}} \chi^{-1}(u)\psi(\pi_F^{-n} u) &= \sum_{y_1 \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(y_1) \sum_{y_2 \in \frac{U_F^{a(\chi)}}{U_F^n}} \psi(\pi_F^{-n} y_1 y_2) \\ &= \sum_{y_1 \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(y_1) \times 0 = 0 \end{aligned}$$

since $-n + a(\chi) < -n(\psi)$.

$\square$

Lemma 3.4.4.

$$\sum_{\chi \in S(2f) \cup S'(2f)} \chi(y) = 0 \ \text{if } y \notin F^* U_K^{2f-1}.$$

Proof. If $v_K(y) \neq 0, y \notin F^* U_K^{2f-1}$, then we can take y to be $\pi_K^{2m+1} y_1$ where $y_1 \in U_K$ and m an integer. If $\chi \in S(2f) \cup S'(2f)$ then $\chi\mu$ is also in $S(2f) \cup S'(2f)$ where $\mu = (-1)^{v_K}$ and $\chi(y) + \chi\mu(y) = 0$. If $y = \pi_F^m y_1$ where $y_1 \in U_K - U_K^{2f-1}$ and $y_1 \notin U_F$, then

$$\sum_{\chi \in S(2f) \cup S'(2f)} \chi(y) = \sum_{\mu} \mu(y) \sum_{\chi_i} \tilde{\omega}(y) \chi_i(y) = 0$$

since the first sum is 0. Here χ_i's are distinct characters of $\frac{U_K^{2f-1}}{U_K^{2f}}$ extended arbitrarily to $\frac{K^*}{F^* U_K^{2f}}$, $\tilde{\omega} \in S(2d-1)$ and $\mu \in \widehat{\frac{U_K}{U_F U_K^{2f-1}}}$ extended to K^* by taking $\mu(\pi_K) = 1$. $\square$

We again fix an additive character ψ of F with $n(\psi) = 0$.

Theorem 3.4.5. $|S(l)| = |S'(l)|$ *for each feasible l, that is when $l = 2d-1$ or $l = 2f$ with $f \geq d$.*

Proof. If l is odd, i.e. $2d-1$, and μ is the unramified character such that $\mu(\pi_K) = -1$ then $\epsilon(\chi\mu, \psi_0) = \mu(\pi_K)^{a(\chi)+n(\psi_0)} \epsilon(\chi, \psi_0) = -\epsilon(\chi, \psi_0)$. Hence clearly $|S(l)| = |S'(l)|$. Now if the conductor is even, say $2f$, we can take $c = \pi_F^{f + \frac{n(\psi_0)}{2}}$ in equation (3.3) so that

$$\epsilon(\chi, \psi_0) = q^{-f} \omega(\pi_F^{f + \frac{n(\psi_0)}{2}}) \sum_{y \in \frac{U_K}{U_K^{2f}}} \chi^{-1}(y) \psi_0(\pi_F^{-f - \frac{n(\psi_0)}{2}} y).$$

Let $e = -1$ if d is odd and $e = -\frac{u'}{u}x_0^2$ if d is even. Note that (identifying $\frac{U_F U_K^{2f}}{U_K^{2f}}$ with $\frac{U_F}{U_F^f}$)

$$\sum_{y \in \frac{U_F U_K^{2f-1}}{U_K^{2f}}} \chi^{-1}(y)\psi_0(\pi_F^{-f-\frac{n(\psi_0)}{2}} y) = \sum_{y' \in \frac{U_F}{U_F^f}} \sum_{a \in \mathbb{F}_q} \chi^{-1}(y'(1+\pi_F^{f-1}\pi_K a)) \times$$
$$\psi_0(\pi_F^{-f-\frac{n(\psi_0)}{2}} y'(1+\pi_F^{f-1}\pi_K a))$$
$$= \sum_{a \in \mathbb{F}_q^*} \chi^{-1}(1+\pi_F^{f-1}\pi_K a) S_a$$

where $S_a = \sum_{y_1 \in \frac{U_F}{U_F^f}} \omega(y_1)\psi(\pi_F^{-1} a y_1 e)$. In view of lemma (3.4.3), S_a is in fact 0 when $d \neq 1$. Since

$$\sum_{\chi \in S(2f)\cup S'(2f)} \chi(y) = 0 \text{ if } y \notin F^* U_K^{2f-1}.$$

by lemma (3.4.4) we now have

$$\sum_{\chi \in S(2f)\cup S'(2f)} \epsilon(\chi,\psi_0) = q^{-f}\omega(\pi_F^{f+\frac{n(\psi_0)}{2}}) \sum_{a \in F_q^*} \sum_{\chi \in S(2f)\cup S'(2f)} \chi^{-1}(1+\pi_F^{f-1}\pi_K a) S_a.$$

As before $S(2f)\cup S'(2f) = \bigcup_i^{q-1} \tilde{\omega}\chi_i \widehat{\frac{K^*}{F^* U_K^{2f-1}}}$. Here χ_i's are distinct characters of $\frac{U_K^{2f-1}}{U_K^{2f}}$ extended arbitrarily to $\frac{K^*}{F^* U_K^{2f}}$ and $\tilde{\omega} \in S(2d-1)$. Hence

$$\sum_{\chi \in S(2f)\cup S'(2f)} \chi^{-1}(1+a\pi_F^{f-1}\pi_K) = \sum_i \sum_{\mu \in \widehat{\frac{K^*}{F^* U_K^{2f-1}}}} \tilde{\omega}^{-1}\chi_i^{-1}\mu^{-1}(1+a\pi_F^{f-1}\pi_K)$$
$$= |\frac{K^*}{F^* U_K^{2f-1}}| \sum_i \tilde{\omega}^{-1}\chi_i^{-1}(1+a\pi_F^{f-1}\pi_K)$$
$$= |\frac{K^*}{F^* U_K^{2f-1}}| \sum_i \chi_i^{-1}(1+a\pi_F^{f-1}\pi_K) \text{ since } a(\tilde{\omega}) < 2f$$
$$= -|\frac{K^*}{F^* U_K^{2f-1}}|$$

Hence

$$\begin{aligned}
\sum_{\chi \in S(2f)\cup S'(2f)} \epsilon(\chi,\psi_0) &= -|\frac{K^*}{F^*U_K^{2f-1}}|q^{-f}\omega(\pi_F^{f+\frac{n(\psi_0)}{2}})\sum_{a\in F_q^*}\sum_{y'\in\frac{U_F}{U_F^f}}\omega(y')\psi(-\pi_F^{-1}aey') \\
&= -|\frac{K^*}{F^*U_K^{2f-1}}|q^{-f}\omega(\pi_F^{f+\frac{n(\psi_0)}{2}})\sum_{y'\in\frac{U_F}{U_F^f}}\omega(y')\sum_{a\in F_q^*}\psi(-\pi_F^{-1}aey') \\
&= |\frac{K^*}{F^*U_K^{2f-1}}|q^{-f}\omega(\pi_F^{f+\frac{n(\psi_0)}{2}})\sum_{y'\in\frac{U_F}{U_F^f}}\omega(y') \\
&= 0
\end{aligned}$$

This completes our theorem. □

Remark:The beginning of the above proof shows that if $\chi \in S(2d-1)$ and if $\mu \in \widehat{\frac{U_K}{U_F U_K^{2d-2}}}$ then μ may be extended to K^* such that $\chi\mu \in S(2d-1)$.

We can in fact make a stronger statement which is a corollary to the proof of our previous theorem.

Corollary 3.4.6. *If $\chi \in S(2f)\cup S'(2f)$ and $d \neq 1$ then*

$$\sum_{\mu\in\widehat{\frac{K^*}{F^*U_K^{2f-1}}}} \epsilon(\chi\mu,\psi_0) = 0.$$

Proof.

$$\begin{aligned}
\sum_{\mu\in\widehat{\frac{K^*}{F^*U_K^{2f-1}}}} \epsilon(\chi\mu,\psi_0) &= q^{-f}\omega(\pi_F^{f+\frac{n(\psi_0)}{2}})\sum_{y\in\frac{U_K}{U_K^{2f}}}\sum_{\mu\in\widehat{\frac{K^*}{F^*U_K^{2f-1}}}}\mu^{-1}(y)\chi^{-1}(y)\psi_0(\pi_F^{-f-\frac{n(\psi_0)}{2}}y) \\
&= q^{-f}\omega(\pi_F^{f+\frac{n(\psi_0)}{2}})|\frac{K^*}{F^*U_K^{2f-1}}|\sum_{y\in\frac{U_F U_K^{2f-1}}{U_K^{2f}}}\chi^{-1}(y)\psi_0(\pi_F^{-f-\frac{n(\psi_0)}{2}}y) \\
&= q^{-f}\omega(\pi_F^{f+\frac{n(\psi_0)}{2}})|\frac{K^*}{F^*U_K^{2f-1}}|\sum_{a\in F_q^*}\chi^{-1}(1+\pi_F^{f-1}\pi_K a)S_a \qquad = 0
\end{aligned}$$

by the proof of the last theorem. □

Remark: When $d = 1$ and $a(\chi) = 2f \geq 2$ then the calculation in section (3.2) tells us that $\epsilon(\chi, \psi_0) = \omega((-1)^f a_0(\chi)/2)$ where $y_\chi = \pi_F^{-f}\pi_K^{-1}a_0(\chi)(1 + a_1(\chi)\pi_K)(1 + a_2(\chi)\pi_F)$.... so that the epsilon factor depends only on $\chi|_{U_K^{2f-1}}$. This fact together with the above corollary shows the essential difference in the $d \neq 1$ (residue characteristic even) and the $d = 1$ (residue characteristic odd) cases. When $d \neq 1$ then any character χ of $\frac{U_K^{2f-1}}{U_K^{2f}}$ $(f \geq d)$ will extend to K^* such that $\epsilon(\chi, \psi_0) = 1$ and in fact of all the possible extensions exactly half will have epsilon factor 1. But if $d = 1$ then of the $q - 1$ nontrivial characters χ of $\frac{U_K^{2f-1}}{U_K^{2f}}$ exactly half will extend to K^* such that $\epsilon(\chi, \psi_0) = 1$ and if we take one such character then all its possible extensions will have epsilon factor 1.

Chapter 4

Proof of the extension theorem

In this chapter, we will be proving our main identity. The idea behind the proof is simple: we have a formula for finding epsilon factor of a character (equation (3.3)), we know that if $\chi \in S$, then the left hand side of this formula is 1, multiply left and right hand sides by $\chi(x)$ and find the sum over S, using the results obtained in the previous chapter, try to deduce that the right hand side of our summation is actually equal to the LHS of our main identity. We will also prove the extension theorem using this identity.

4.1 Proof of the main identity when x has valuation 1

Every element of valuation 1 in K can be written in the form $u_1 u'' \pi_K$ where u_1 is a unit in F, u'' is of the form $1 + x'\pi_F^{r-1}\pi_K$, $r \geq 1$, $x' \in U_F$ or $x' = 0$. In view of the fact that identity (3.1) is true for x if and only if it is true for ax for any $a \in F$ we may without loss of generality assume that $x = u''\pi_K$ where u'' is as above. We have either (1) $d = 2t + 1$, $x_0 = \pi_K$ and $\pi_F = \pi_K^2$

in which case

$$\mathrm{LHS} = \epsilon(\omega, \psi) q^t \tag{4.1}$$

or (2) $d = 2s$, $x_0 =$ a unit, and $\pi_K = \frac{\pi_F^s u'}{2}(1 + x_0)$ so that

$$\mathrm{LHS} = \epsilon(\omega, \psi) q^{s-\frac{1}{2}} \omega(\pi_F^{s-t} u u'(1 + x' u' \pi_F^{s+r-1})) \tag{4.2}$$

To compute the sum in the RHS, we first consider the case when $\chi \in S(2d-1)$. Taking $c = \pi_F^{d+\frac{n(\psi_0)}{2}}(u'')^{-1}\pi_K^{-1}$ in equation (3.3), multiplying both sides of the equation by $\chi(u''\pi_K)$, and by applying the transformation $y \longrightarrow u''y$, we get

$$\chi(u''\pi_K) = \omega(\pi_F^{d+\frac{n(\psi_0)}{2}}) q^{-(d-\frac{1}{2})} \sum_{y \in \frac{U_K}{U_K^{2d-1}}} \chi^{-1}(y) \psi_0(\pi_F^{-d-\frac{n(\psi_0}{2}} \pi_K \, u'' y)$$

Now if $y \notin U_F U_K^{2d-1} = U_F U_K^{2d-2}$ then

$$\sum_{\chi \in S(2d-1)} \chi(y) = \sum_{\chi_i} \sum_{\mu \in \widehat{\frac{U_K}{U_F U_K^{2d-2}}}} (\chi_i \mu)(y) = 0$$

where the χ_i's are some characters in $S(2d-1)$ such that $\chi_i \mu$ form all distinct characters in $S(2d-1)$. Note that if $\chi \in S(2d-1)$ and $\chi\mu \notin S(2d-1)$, then we can extend μ to K^* taking $\mu(\pi_K) = -1$ so that $\chi\mu \in S(2d-1)$. Otherwise, take $\mu(\pi_K) = 1$. So in the summation, we need to consider only those y's which are in $U_F U_K^{2d-2}$. But on these y's, χ is just ω. So we get

$$\begin{aligned} \sum_{\chi \in S(2d-1)} \chi(u''\pi_K) &= |S(2d-1)| \omega(\pi_F^{d+\frac{n(\psi_0)}{2}}) q^{-d+\frac{1}{2}} \sum_{y \in \frac{U_F}{U_F^d}} \omega(y) \psi_0(\pi_F^{-d-\frac{n(\psi_0)}{2}} \pi_K \, u'' y) \\ &= q^{-\frac{1}{2}} \omega(\pi_F^{d+\frac{n(\psi_0)}{2}}) \sum_{y \in \frac{U_F}{U_F^d}} \omega(y) \psi(-\frac{\pi_F^{-d-\frac{n(\psi_0)}{2}}}{2} y \, \mathrm{tr}(\pi_K \, u'' x_0)) \end{aligned}$$

since $|S(2d-1)| = q^{d-1}$.

Case 1: $d = 2t+1$

Here $n(\psi_0) = 2$, $\pi_K^2 = \pi_F$, and $x_0 = \pi_K$ has trace 0. So $tr(\pi_K u'' x_0) = 2\pi_F$. Then

$$\sum_{\chi \in S(2d-1)} \chi(u''\pi_K) = q^{-\frac{1}{2}} \omega(\pi_F^{d+\frac{n(\psi_0)}{2}}) \sum_{y \in \frac{U_F}{U_F^d}} \omega(y)\psi(-\pi_F^{-d} y)$$

which is equal to LHS by equation (3.4) since $\omega(\pi_F) = \omega(-1)$.

Case 2: $d = 2s \leq 2t$

Here $n(\psi_0) = 2s - 2t$ and $tr(\pi_K u'' x_0) = \pi_F^s u' x_0^2 (1 + x' u' \pi_F^{r+s-1})$. Then

$$\begin{aligned} \sum_{\chi \in S(2d-1)} \chi(u''\pi_K) &= q^{-\frac{1}{2}} \omega(\pi_F^{s-t}) \sum_{y \in \frac{U_F}{U_F^d}} \omega(y)\psi(-\pi_F^{-d} u^{-1} u' x_0^2 y(1 + x' u' \pi_F^{r+s-1})) \\ &= q^{-\frac{1}{2}} \omega(\pi_F^{s-t} u u' (1 + x' u' \pi_F^{r+s-1})) \sum_{y \in \frac{U_F}{U_F^d}} \omega(y)\psi(\pi_F^{-d} y) \end{aligned}$$

since $\omega(-x_0^2) = \omega(N(x_0)) = 1$ this is equal to LHS again by equation (3.4).
If $\chi \in S(2f)$, then as already noted, $\chi\mu$ is also in $S(2f)$ where $\mu = (-1)^{v_K}$. So $\sum_{\chi \in S(2f)} \chi(u''\pi_K) = 0$. Therefore the RHS is just $\sum_{\chi \in S(2d-1)} \chi(u''\pi_K)$. This completes the proof when x has valuation 1.

4.2 Proof of the main identity when x is a unit

If $x \in U_K - U_F$ then x is of the form $a_0 + a_1\pi_K + a_2\pi_F\pi_K + \ldots$ where $a_i \in \mathbb{F}_q$, $a_0 \neq 0$ and not all a_{2i-1} equal to 0, $i = 1, 2, \ldots$. So $x = (a_0 + a_1\pi_F + \ldots)(1 + a_1'\pi_K + a_3'\pi_F\pi_K + \ldots) = u(1 + \pi_F^{r-1}\pi_K x')$ where $u, x' \in U_F$. So without loss of generality we can take $x = 1 + \pi_F^{r-1}\pi_K x'$ where $x' \in U_F$.

When $d = 2t + 1$,

$$\text{LHS} = \epsilon(\omega, \psi)\omega(\pi_F^{r-1}x')q^{r+t-\frac{1}{2}} \tag{4.3}$$

and when $d = 2s \leq 2t$, we have

$$\text{LHS} = \epsilon(\omega, \psi)\omega(\pi_F^{r+s-t-1}uu'x')q^{r+s-1} \tag{4.4}$$

We have

$$\begin{aligned}\sum_{\chi\in S}\chi(x) = \sum_{\chi\in S(2d-1)}\chi(x) + \sum_{\chi\in S(2f),\, d\leq f\leq r-1}\chi(x) + \\ \sum_{\chi\in S(2r)}\chi(x) + \sum_{\chi\in S(2f),\, r+1\leq f\leq r+d-2}\chi(x) + \\ \sum_{\chi\in S(2r+2d-2)}\chi(x) + \sum_{\chi\in S(2f),\, f\geq r+d}\chi(x).\end{aligned} \tag{4.5}$$

keeping in mind that if $d = 1$ the fourth term does not exist and the third and fifth terms coincide. If $r < d$ the first three sums do not exist. Now $\sum_{\chi\in S(2d-1)}\chi(x) + \sum_{\chi\in S(2f),\, d\leq f\leq r-1}\chi(x)$ is equal to the number of such χ's which is q^{r-1}. If $r \geq d > 1$ every non trivial character of $\frac{U_K^{2r-1}}{U_K^{2r}}$ can be extended in $2|\frac{U_K}{U_F U_K^{2r-1}}|$ different ways to get a character of K^* whose restriction to F^* is ω. Of these $|\frac{U_K}{U_F U_K^{2r-1}}|$ will have $\epsilon(\chi, \psi_0) = 1$ in view of theorem (3.4.5). Hence

$$\begin{aligned}\sum_{\chi\in S(2r)}\chi(x) &= |\frac{U_K}{U_F U_K^{2r-1}}| \sum_{\chi\in\widehat{\frac{U_K^{2r-1}}{U_K^{2r}}},\,\chi\,\text{nontrivial}}\chi(x) \\ &= -|\frac{U_K}{U_F U_K^{2r-1}}| \text{ since } \sum_{\chi\in\widehat{\frac{U_K^{2r-1}}{U_K^{2r}}}}\chi(x) = 0 \\ &= -q^{r-1}\end{aligned}$$

making the first three sums in equation (4.5) vanish.

Lemma 4.2.1. *Let $f \geq r + d$. Then for any $\nu \in \widehat{\frac{K^*}{F^*U_K^{2r}}}$ we have $\chi\nu \in S(2f)$ whenever $\chi \in S(2f)$.*

Proof. **Case 1:**($r \geq d-1$). Then $a(\chi) = 2f \geq 2(r+d) \geq 2(2d-1)$. Therefore $\epsilon(\chi, \psi_0) = \tilde{\omega}(-x_0/2)\tilde{\omega}^{-1}(y_\chi)$. We write

$$y_\chi = \begin{cases} \pi_F^{-f-\frac{d-1}{2}}\pi_K a_0(\chi)(1 + a_1(\chi)\pi_K)(1 + a_2(\chi)\pi_F)... & \text{if } d \text{ is odd} \\ \pi_F^{-f-\frac{d}{2}}x_0 a_0(\chi)(1 + a_1(\chi)\pi_K)(1 + a_2(\chi)\pi_F)... & \text{if } d \text{ is even} \end{cases}$$

If $\nu \in \widehat{\frac{K^*}{F^*U_K^{2r}}}$ then $a(\chi\nu) = 2f$ so $\epsilon(\chi, \psi_0)$ is given by the same formula with y_χ replaced by $y_{\chi\nu}$. We have $\chi\nu = \chi$ on $U_K^{2r} \supseteq U_K^{2f-(2d-1)}$ and $a_i(\chi)$ is completely determined by $\chi|_{U_K^{2f-(i+1)}}$. Hence $a_i(\chi) = a_i(\chi\nu)$ for $i = 0, 1, 2..., 2d-2$. Since

$\epsilon(\chi, \psi_0) = \tilde{\omega}(-x_0/2)\tilde{\omega}^{-1}(\pi_F^{-f-\frac{d-1}{2}}\pi_K a_0(\chi)(1 + a_1(\chi)\pi_K)...(1 + a_{2d-2}(\chi)\pi_F^{d-1}))$ if d is odd with a similar expression if d is even we have $\epsilon(\chi\nu, \psi_0) = \epsilon(\chi, \psi_0)$ whenever $\nu \in \widehat{\frac{K^*}{F^*U_K^{2r}}}$. Hence if $\chi \in S(2f)$ then $\chi\nu \in S(2f)$.

Case 2:($r < d-1$). Again if $\nu \in \widehat{\frac{K^*}{F^*U_K^{2r}}}$ with $\nu(\pi_K) = 1$ since $a(\chi) = 2f \geq 2(r+d) > 2.2r \geq a(\nu)$ we have by theorem (3.1.2)

$$\begin{aligned} \epsilon(\chi\nu, \psi_0) &= \chi\nu(-x_0/2)\epsilon(\chi\nu, \psi_K) \\ &= \chi\nu(-x_0/2)\nu^{-1}(y_\chi)\epsilon(\chi, \psi_K) \\ &= \nu(x_0)\nu^{-1}(y_\chi)\epsilon(\chi, \psi_0) \end{aligned}$$

$$\begin{aligned} &= \begin{cases} \nu(x_0)\nu^{-1}(\pi_F^{-f-\frac{d-1}{2}}\pi_K a_0(\chi)(1 + a_1(\chi)\pi_K)\ldots)\epsilon(\chi, \psi_0) \text{ if } d \text{ is odd} \\ \nu(x_0)\nu^{-1}(\pi_F^{-f-\frac{d}{2}}x_0 a_0(\chi)(1 + a_1(\chi)\pi_K)\ldots)\epsilon(\chi, \psi_0) \text{ if } d \text{ is even} \end{cases} \\ &= \nu^{-1}((1 + a_1(\chi)\pi_K)(1 + a_3(\chi)\pi_F\pi_K)...(1 + a_{2r-1}(\chi)\pi_F^{r-1}\pi_K))\epsilon(\chi, \psi_0) \end{aligned}$$

since ν is trivial on F^*. If d is odd, then note that $x_0 = \pi_K$ and that $\pi_K^2 = \pi_F$. So, since $f > r + d$ and $a(\omega) = d$, we have $1 = \omega(1 + a\pi_F^{f-1}) = \chi(1+a\pi_F^{f-1}) = \psi_K(\pi_F^{f-1}\pi_F^{-f-\frac{d-1}{2}}\pi_K^2 a a_0(\chi)a_1(\chi)) = \psi(\pi_F^{-1}a a_0(\chi)a_1(\chi))$ where $a \in \mathbb{F}_q$. But ψ is nontrivial on $\pi_F^{-1}U_F$. So $a_1(\chi)$ has to be 0. If, on the other hand, d is even and $a \in F_q^*$ then $1 = \omega(1 + a\pi_F^{f-1}) = \chi(1 + a\pi_F^{f-1}) =$

$\psi_K(a\pi_F^{f-1}\pi_F^{-f-\frac{d}{2}}x_0a_0(\chi)(1+a_1(\chi)\pi_K)) = \psi(\pi_F^{-1}aa_0(\chi)a_1(\chi)u'x_0^2)$. This implies that $a_1(\chi)=0$ in this case also. Similarly $a_i(\chi)=0$ for $i=1,3,...,2r-1$ since $f-r\geq d$ implies $\omega(1+a\pi_F^{f-i})=1$ for all $1\leq i\leq r$. Therefore $\epsilon(\chi\nu,\psi_0)=\epsilon(\chi,\psi_0)$ and if $\chi\in S(2f)$ then $\chi\nu\in S(2f)$. □

Corollary 4.2.2. *Let $x=1+\pi_F^{r-1}\pi_Kx'$ where $x'\in U_F$ and let $f\geq r+d$. Then*

$$\sum_{\chi\in S(2f)}\chi(x)=0. \tag{4.6}$$

Proof. Clear. □

We now proceed further with the proof of theorem (3.2.1). By the above corollary, the right hand side of equation (4.5) is $q^{r-1}+\sum_{\chi\in S(2r)}\chi(x)$ if $d=1$ and $\sum_{\chi\in S(2f),r+1\leq f\leq r+d-1}\chi(x)$ if $d>1$.

If $\chi\in S(2r+2m)$, $1\leq m\leq d-1$ or $m=0$ and $d=1$ then $\epsilon(\chi,\psi_0)=1$. In equation (3.3), if we take $c=\pi_F^{r+m+\frac{n(\psi_0)}{2}}$, multiply both sides of the equation with $\chi(x)$ and apply the transformation $y\longrightarrow yx$, we will get

$$\chi(x)=q^{-r-m}\omega(\pi_F^{r+m+\frac{n(\psi_0)}{2}})\sum_{y\in\frac{U_K}{U_K^{2r+2m}}}\chi^{-1}(y)\psi_0(\pi_F^{-r-m-\frac{n(\psi_0)}{2}}xy)$$

Let $e_1=q^{-r-m}\omega(\pi_F^{r+m+\frac{n(\psi_0)}{2}})$. Then

$$\sum_{\chi\in S(2r+2m)}\chi(x)=e_1\sum_{\chi\in S(2r+2m)}\sum_{y\in\frac{U_FU_K^{2r+2m-1}}{U_K^{2r+2m}}}\chi^{-1}(y)\psi_0(\pi_F^{-r-m-\frac{n(\psi_0)}{2}}xy)$$

$$+e_1\sum_{\chi\in S(2r+2m)}\sum_{y\in\frac{U_K}{U_K^{2r+2m}}-\frac{U_FU_K^{2r+2m-1}}{U_K^{2r+2m}}}\chi^{-1}(y)\psi_0(\pi_F^{-r-m-\frac{n(\psi_0)}{2}}xy)$$

where as usual we identify $\frac{U_FU_K^{2f+2m}}{U_K^{2f+2m}}$ with $\frac{U_F}{U_F^{r+m}}$.

Let us call the first term on the right hand side of the above equation M_m and the second term T_m. We will show that M_m is 0 if $1 \leq m \leq d-2$ and LHS (of equation (3.1)) if $m = d-1$ (even if $d = 1$).

$$M_m = e_1 \sum_{\chi \in S(2r+2m)} \sum_{y \in \frac{U_F U_K^{2r+2m-1}}{U_K^{2r+2m}}} \chi^{-1}(y)\psi_0(\pi_F^{-r-m-\frac{n(\psi_0)}{2}} xy)$$

$$= e_1 \sum_{\chi \in S(2r+2m)} \sum_{y' \in \frac{U_F}{U_F^{r+m}}} \sum_{a \in \mathbb{F}_q} \chi^{-1}(y')\chi^{-1}(1+\pi_F^{r+m-1}\pi_K a) \times$$

$$\psi_0(\pi_F^{-r-m-\frac{n(\psi_0)}{2}}(1+\pi_F^{r-1}\pi_K x')y'(1+\pi_F^{r+m-1}\pi_K a))$$

$$= e_1 \sum_{\chi \in S(2r+2m)} \sum_{y' \in \frac{U_F}{U_F^{r+m}}} \sum_{a \in \mathbb{F}_q} \omega(y')\psi_K^{-1}(y_\chi \pi_F^{r+m-1}\pi_K a) \times$$

$$\psi_0(\pi_F^{-r-m-\frac{n(\psi_0)}{2}}(\pi_F^{r-1}\pi_K x' + \pi_F^{r+m-1}\pi_K a + \pi_F^{2r+m-2}\pi_K^2 x' a)y') \text{ (since } \psi_0 \text{ trivial on } F)$$

where y_χ is as in theorem (3.1.2). If $d = 2t+1$ then we can take $y_\chi = \pi_F^{-r-m-t-1}\pi_K a_0(\chi)(1 + a_1(\chi)\pi_K)\ldots$ where $a_i(\chi) \in \mathbb{F}_q$, $a_0(\chi) \neq 0$.

Therefore the value of M_m is

$$= e_1 \sum_{y' \in \frac{U_F}{U_F^{r+m}}} \omega(y') \sum_{\chi \in S(2r+2m)} \sum_{a \in \mathbb{F}_q} \psi(-\pi_F^{-1} u a a_0(\chi))\psi(-\pi_F^{-m-1} y'(x' + \pi_F^m a))$$

$$= e_1 \sum_{y' \in \frac{U_F}{U_F^{r+m}}} \omega(y')\psi(-\pi_F^{-m-1} y' x') \sum_{\chi \in S(2r+2m)} \sum_{a \in \mathbb{F}_q} \psi(-\pi_F^{-1} a(u a_0(\chi) + y'))$$

For a fixed $y' \in \frac{U_F}{U_F^{r+m}}$,

$$\sum_{a \in \mathbb{F}_q} \psi(-\pi_F^{-1} a(u a_0(\chi) + y')) \neq 0$$

if and only if $u a_0(\chi) + y'$ is not a unit. But $\chi|_{U_K^{2r+2m-1}}$ determines $a_0(\chi)$. Corresponding to this $a_0(\chi)$, there are $|\frac{U_K}{U_F U_K^{2r+2m-1}}| = q^{r+m-1}$ χ's in $S(2r+2m)$. For this $a_0(\chi)$,

$$\sum_{a \in \mathbb{F}_q} \psi(-\pi_F^{-1} a(u a_0(\chi) + y')) = |\mathbb{F}_q| = q.$$

So

$$\begin{aligned} M_m &= q^{-r-m+r+m}\omega(\pi_F^{r+m+1})\omega(-x') \sum_{y' \in \frac{U_F}{U_F^{r+m}}} \omega(y')\psi(\pi_F^{-m-1}y') \\ &= \omega(\pi_F^{r+m}x') \sum_{y' \in \frac{U_F}{U_F^{r+m}}} \omega(y')\psi(\pi_F^{-m-1}y') \end{aligned}$$

which is equal to LHS if $m = d-1$ (even when $d = 1$) and 0 if $m < d-1$ by lemma (3.4.3) and equation (3.4). Similarly, if $d = 2s$ take $y_\chi = \pi_F^{-r-m-s}x_0\pi_K a_0(\chi)(1 + a_1(\chi)\pi_K)\ldots)$. Note that here $n(\psi_0) = 2s - 2t$, $2 = \pi_F^t u$, and $\pi_K = \frac{\pi_F^s u'}{2}(1 + x_0)$. So $tr\, x_0\pi_K = \pi_F^s u' x_0^2$ and we get

$$\begin{aligned} M_m &= e_1 \sum_{y' \in \frac{U_F}{U_F^{r+m}}} \omega(y') \sum_{\chi \in S(2r+2m)} \sum_{a \in F_q} \psi(-\pi_F^{-1}u'aa_0(\chi)x_0^2) \\ &\quad \psi(-\pi_F^{-m-1}y'(u^{-1}u'x'x_0^2 + \pi_F^m u^{-1}au'y')) \\ &= e_1 \sum_{y' \in \frac{U_F}{U_F^{r+m}}} \omega(y')\psi(-\pi_F^{-m-1}y'(u^{-1}u'x'x_0^2)) \times \\ &\quad \sum_{\chi \in S(2r+2m)} \sum_{a \in \mathbb{F}_q} \psi(-\pi_F^{-1}u'a(a_0(\chi)x_0^2 + u^{-1}y')) \\ &= q^{-r-m+r+m}\omega(\pi_F^{r+m+s-t})\omega(-u^{-1}u'x'x_0^2)) \sum_{y' \in \frac{U_F}{U_F^{r+m}}} \omega(y')\psi(\pi_F^{-m-1}y') \end{aligned}$$

Since $\omega(-x_0^2)$=1, this gives the LHS if $m = d-1$ and 0 if $m < d-1$ in $d = 2s$ case also.

Recall $\sum_{\chi \in S(2r+2m)} \chi(x) = M_m + T_m$ where

$$T_m = e_1 \sum_{\chi \in S(2r+2m)} \sum_{y \in \frac{U_K}{U_K^{2r+2m}} - \frac{U_F U_K^{2r+2m-1}}{U_K^{2r+2m}}} \chi^{-1}(y)\psi_0(\pi_F^{-r-m-\frac{n(\psi_0)}{2}}xy).$$

Doing the same calculations for $\chi' \in S'(2r + 2m)$, we have

$$-\sum_{\chi' \in S'(2r+2m)} \chi'(x) = M_m + T'_m$$

where

$$T'_m = e_1 \sum_{\chi' \in S'(2r+2m)} \sum_{y \in \frac{U_K}{U_K^{2r+2m}} - \frac{U_F U_K^{2r+2m-1}}{U_K^{2r+2m}}} \chi'^{-1}(y)\psi_0(\pi_F^{-r-m-\frac{n(\psi_0)}{2}} xy).$$

Adding we get

$$\sum_{\chi \in S(2r+2m)} \chi(x) - \sum_{\chi' \in S'(2r+2m)} \chi'(x) = 2M_m + T_m + T'_m.$$

On the other hand by lemma (3.4.4) and the computations before lemma (4.2.1)

$$\sum_{\chi \in S(2r+2m)} \chi(x) + \sum_{\chi' \in S'(2r+2m)} \chi'(x) = \begin{cases} 0 & \text{if } m \geq 1 \\ -2q^{r-1} & \text{if } m = 0,\ d = 1 \end{cases}.$$

By the same lemma, $T_m + T'_m = 0$. Hence

$$\sum_{\chi \in S(2r+2m)} \chi(x) = \begin{cases} M_m & \text{for } m = 1, 2, ..., d-1 \\ M_0 - q^{r-1} & \text{for } m = 0. \end{cases}$$

We have already shown that $M_m = 0$ except when $m = d-1$ in which case it is LHS. This completes the proof of the theorem (3.2.1) when x is a unit. As we have already proved theorem (3.2.1) when x has valuation 1 this then completes the proof of theorem (3.2.1) in all cases.

We have an interesting theorem as a result of these tedious computations. This is in fact stronger than theorem (3.2.1) when x is a unit.

Theorem 4.2.3. *Let $x = 1 + \pi_F^{r-1}\pi_K x'$ where $x' \in U_F$, then*

$$\sum_{\chi \in S(2r+2m)} \chi(x) = \begin{cases} -q^{r-1} & \textit{if } m = 0 \\ 0 & m = 1, 2, \ldots \textit{ and } m \neq d-1 \\ \epsilon(\omega, \psi) \frac{\omega(\frac{x-\overline{x}}{x_0-\overline{x}_0})}{\left|\frac{(x-\overline{x})^2}{x\overline{x}}\right|_{F^*}^{\frac{1}{2}}} & \textit{if } m = d-1 \end{cases}$$

and

$$\sum_{\chi' \in S'(2r+2m)} \chi'(x) = \begin{cases} -q^{r-1} & \text{if } m = 0 \\ 0 & m = 1, 2, \ldots \text{ and } m \neq d-1 \\ -\epsilon(\omega, \psi) \frac{\omega\left(\frac{x-\overline{x}}{x_0-\overline{x}_0}\right)}{\left|\frac{(x-\overline{x})^2}{x\overline{x}}\right|_{F^*}^{\frac{1}{2}}} & \text{if } m = d-1 \end{cases}$$

This theorem is obtained by combining corollary (4.2.2) and the expression given above.

4.3 Proof of the extension of Tunnell's theorem

Once we use the main theorem proved in last section and a lemma of Langlands([L], lemma 7.19, *lemma with an embarrassing proof*) the proof of the extension theorem is straight forward. We will follow the lines of Prasad[P1] here. We state Langlands' lemma first:

Lemma 4.3.1. *Let r_θ (resp. r'_θ) be the representation of $GL(2,F)$ (resp. D_F^*) associated to a character θ of K^*. Then r_θ restricted to $GL(2,F)^+ = \{x \in GL(2,F) : \det(x) \in N(K^*)\}$ and r'_θ restricted to $D_F^{*+} = \{x \in D_F^* : \det(x) \in N(K^*)\}$ decomposes into two irreducible representations. If we fix a nontrivial additive character ψ of F, an element $x_0 \in K^*$ with $tr\, x_0 = 0$, and embeddings of K^* in $GL(2,F)^+$ and D_F^{*+}, then we can write the two irreducible components of r_θ as $r_{\theta+}$ and $r_{\theta-}$ with characters χ_+ and χ_-, and of r'_θ as $r'_{\theta+}$ and $r'_{\theta-}$ with characters χ'_+ and χ'_- such that on K^**

$$\chi_+ - \chi_- = \epsilon(\omega, \psi) \frac{\omega\left(\frac{x-\overline{x}}{x_0-\overline{x}_0}\right)}{\left|\frac{(x-\overline{x})^2}{x\overline{x}}\right|_{F^*}^{\frac{1}{2}}} [\theta(x) + \theta(\overline{x})],$$

and

$$\chi'_+ - \chi'_- = \epsilon(\omega, \psi) \frac{\omega(\frac{x-\overline{x}}{x_0-\overline{x}_0})}{\left|\frac{(x-\overline{x})^2}{x\overline{x}}\right|_{F^*}^{\frac{1}{2}}} [\theta(x) - \theta(\overline{x})],$$

We will firstly state the Tunnel's theorem[Tu], but in a modified form, as appeared in [P1]. For a discrete series representation π of $GL(2, F)$, we let π' denote the representation of D_F^* associated by Jacquet-Langlands to π.

Theorem 4.3.2. *Let π be an irreducible admissible infinite dimensional representation of $GL(2, F)$ with central character ω_π and let σ_π be the associated two-dimensional representation of the Weil-Deligne group of F. Let χ be a character of K^* such that $\chi|_{F^*} = \omega_\pi$. Let ψ be a nontrivial additive character of F and x_0 an element of K such that $tr(x_0) = 0$. Define an additive character ψ_0 of K by $\psi_0(x) = \psi(tr[-xx_0/2])$. Then the epsilon factor $\epsilon(\sigma_\pi|_K \otimes \chi^{-1}, \psi_0)$ is independent of the choice of ψ and x_0, and takes the value 1 if and only if χ appears in π and takes the value -1 if and only if χ appears in π'.*

Prasad[P1] introduced the character ψ_0 in place of the character ψ_K present in Tunnel's original theorem[Tu] for computational convenience. If the representation π of $GL(2, F)$ comes from a regular character θ of K^*, the representation σ_π of the Weil group is induced from the character θ of K^* (cf. [J-L], p.396). Therefore $\sigma_\pi|_{K^*} = \theta + \overline{\theta}$. Therefore $\epsilon(\sigma_\pi|_K \otimes \chi^{-1}, \psi_0) = \epsilon(\theta\chi^{-1}, \psi_0).\epsilon(\overline{\theta}\chi^{-1}, \psi_0)$ in Tunnell's theorem. But since $(\theta\chi^{-1})|_{F^*} = \omega$ by the condition on central characters, we have $\epsilon(\theta\chi^{-1}, \psi_0)$ and $\epsilon(\overline{\theta}\chi^{-1}, \psi_0)$ both take values in $\{\pm 1\}$. So the set of characters $\chi \in \widehat{K^*}$ with $(\theta\chi^{-1})|_{F^*} = \omega$ and $\epsilon(\sigma_\pi|_K \otimes \chi^{-1}, \psi_0) = 1$ is exactly the set of characters χ of K^* such that $\epsilon(\theta\chi^{-1}, \psi_0)$ and $\epsilon(\overline{\theta}\chi^{-1}, \psi_0)$ are either both 1 or both -1.

Now we will prove the extension theorem. We recall the statement here.

Theorem 4.3.3. *Let r_θ be an irreducible admissible representation of $GL(2,F)$ associated to a regular character θ of K^*. Fix embeddings of K^* in $GL(2,F)^+$ and in $D^{*+}{}_F$ (there are two conjugacy classes of such embeddings in general), and choose a nontrivial additive character ψ of F, and an element x_0 of K^* with $tr(x_0) = 0$. Then the representation r_θ of $GL(2,F)$ decomposes as $r_\theta = r_{\theta+} \oplus r_{\theta-}$ when restricted to $GL(2,F)^+$ and the representation $r_\theta{}'$ of $D^*{}_F$ decomposes as $r_\theta{}' = r_\theta{}'_+ \oplus r_\theta{}'_-$ when restricted to $D^{*+}{}_F$, such that for a character χ of K^* with $(\chi.\theta^{-1})|_{F^*} = \omega_{K/F}$, χ appears in $r_{\theta+}$ if and only if $\epsilon(\theta\chi^{-1},\psi_0) = \epsilon(\overline{\theta}\chi^{-1},\psi_0) = 1$, χ appears in $r_{\theta-}$ if and only if $\epsilon(\theta\chi^{-1},\psi_0) = \epsilon(\overline{\theta}\chi^{-1},\psi_0) = -1$, χ appears in $r_\theta{}'_+$ if and only if $\epsilon(\theta\chi^{-1},\psi_0) = 1$ and $\epsilon(\overline{\theta}\chi^{-1},\psi_0) = -1$, and χ appears in $r_\theta{}'_-$ if and only if $\epsilon(\theta\chi^{-1},\psi_0) = -1$ and $\epsilon(\overline{\theta}\chi^{-1},\psi_0) = 1$.*

Note: instead of writing "a character χ appears in $r_\theta|_{K^*}$" we write "χ appears in r_θ" which need not of course cause any confusion.

Proof. As the proofs for $GL(2,F)$ and D_F^* are completely analogous, we will be proving only the $GL(2,F)$ case here. By theorem (3.2.1),

$$\begin{aligned}
\epsilon(\omega,\psi)\frac{\omega\left(\frac{x-\overline{x}}{x_0-\overline{x}_0}\right)}{\left|\frac{(x-\overline{x})^2}{x\overline{x}}\right|_{F^*}^{\frac{1}{2}}}\theta(x) &= \sum_{\chi\theta^{-1}\in S}(\chi\theta^{-1})(x)\theta(x) \\
&= \sum_{\chi\theta^{-1}\in S}\chi(x)
\end{aligned}$$

So by lemma (4.3.1)

$$\begin{aligned}
\chi_+ - \chi_- &= \sum_{\chi\theta^{-1}\in S}\chi(x) + \sum_{\chi\overline{\theta}^{-1}\in S}\chi(x) \\
&= \sum_{\substack{\chi\theta^{-1}\in S\\ \chi\overline{\theta}^{-1}\in S}}\chi(x) + \sum_{\substack{\chi\theta^{-1}\in S\\ \chi\overline{\theta}^{-1}\in S'}}\chi(x) + \sum_{\substack{\chi\overline{\theta}^{-1}\in S\\ \chi\theta^{-1}\in S}}\chi(x) + \sum_{\substack{\chi\overline{\theta}^{-1}\in S\\ \chi\theta^{-1}\in S'}}\chi(x) \\
&= 2\sum_{\substack{\chi\theta^{-1}\in S\\ \chi\overline{\theta}^{-1}\in S}}\chi(x) + \sum_{\substack{\chi\theta^{-1}\in S\\ \chi\overline{\theta}^{-1}\in S'}}\chi(x) + \sum_{\substack{\chi\overline{\theta}^{-1}\in S\\ \chi\theta^{-1}\in S'}}\chi(x) + \sum_{\substack{\chi\overline{\theta}^{-1}\in S'\\ \chi\theta^{-1}\in S'}}\chi(x) - \sum_{\substack{\chi\overline{\theta}^{-1}\in S'\\ \chi\theta^{-1}\in S'}}\chi(x)
\end{aligned}$$

Note that the sum of all characters χ with $(\chi\theta^{-1})|_F^* = \omega$ is zero. So the above sum becomes

$$\chi_+ - \chi_- = \sum_{\substack{\chi\theta^{-1}\in S\\ \chi\bar{\theta}^{-1}\in S}} \chi(x) - \sum_{\substack{\chi\bar{\theta}^{-1}\in S'\\ \chi\theta^{-1}\in S'}} \chi(x)$$

By Tunnell's theorem,

$$\chi_+ + \chi_- = \sum_{\substack{\chi\theta^{-1}\in S\\ \chi\bar{\theta}^{-1}\in S}} \chi(x) + \sum_{\substack{\chi\bar{\theta}^{-1}\in S'\\ \chi\theta^{-1}\in S'}} \chi(x)$$

Adding the above two equations give

$$\chi_+ = \sum_{\substack{\chi\theta^{-1}\in S\\ \chi\bar{\theta}^{-1}\in S}} \chi(x)$$

and subtracting them gives

$$\chi_- = \sum_{\substack{\chi\bar{\theta}^{-1}\in S'\\ \chi\theta^{-1}\in S'}} \chi(x)$$

This complete the proof of our extension theorem.

□

Chapter 5

Counting twists of a character appearing in its associated Weil representation

We use theorem (4.3.3) to determine whether a $\chi \in S$ is such that $\chi\theta$ occurs in $r_{\theta+}$ or $r_{\theta-}$. By this theorem, $\chi\theta$ occurs in $r_{\theta+}$ if and only if $\epsilon(\theta(\chi\theta)^{-1}, \psi_0) = \epsilon(\chi^{-1}, \psi_0) = 1 = \epsilon(\overline{\theta}(\chi\theta)^{-1}, \psi_0) = \epsilon(\chi^{-1}\frac{\overline{\theta}}{\theta}, \psi_0)$ and $\chi\theta$ occurs in $r_{\theta-}$ if and only if $\epsilon(\chi^{-1}, \psi_0) = -1 = \epsilon(\chi^{-1}\frac{\overline{\theta}}{\theta}, \psi_0)$. Note that a character $\chi\theta$ can occur in r_θ if and only if it occurs in either $r_{\theta+}$ or in $r_{\theta-}$. Also, if $\chi \in S(l)$ for some l then $\chi\theta$ can occur in r_θ if and only if it occurs in $r_{\theta+}$. Furthermore if $\chi \in S(l)$, then $\chi\theta$ cannot occur in $r_{\theta-}$ since for that χ, $\epsilon(\chi^{-1}, \psi_0) = +1$. As in previous chapters, we fix ψ to be a nontrivial additive character of F with $n(\psi) = 0$.

5.1 Counting the twists when K/F is ramified

Theorem (4.2.3) finds an important place in our computations. But for computational convenience, we would like to slightly change our notation S from this point onwards. We define $S = \{\chi \in \widehat{K^*} : \chi\mid_{F^*} = \omega, \epsilon(\chi^{-1}, \psi_0) = 1\}$ and $S(l) = \{\chi \in S : a(\chi) = l\}$. Analogously, we redefine S', $S'(l)$ and S_l. So we are forced to create a changed version of theorem (4.2.3) for our computations. With this redefined notations, we have

Theorem 5.1.1. *Let $x = 1 + \pi_F^{r-1}\pi_K x'$ where $x' \in U_F$, then*

$$\sum_{\chi \in S(2r+2m)} \chi(x) = \begin{cases} -q^{r-1} & \text{if } m = 0 \\ 0 & m = 1, 2, \ldots \text{ and } m \neq d-1 \\ \omega(-1)\epsilon(\omega,\psi)\dfrac{\omega(\frac{x-\overline{x}}{x_0-\overline{x}_0})}{\left|\frac{(x-\overline{x})^2}{x\overline{x}}\right|_{F^*}^{\frac{1}{2}}} & \text{if } m = d-1 \end{cases}$$

and

$$\sum_{\chi' \in S'(2r+2m)} \chi'(x) = \begin{cases} -q^{r-1} & \text{if } m = 0 \\ 0 & m = 1, 2, \ldots \text{ and } m \neq d-1 \\ -\omega(-1)\epsilon(\omega,\psi)\dfrac{\omega(\frac{x-\overline{x}}{x_0-\overline{x}_0})}{\left|\frac{(x-\overline{x})^2}{x\overline{x}}\right|_{F^*}^{\frac{1}{2}}} & \text{if } m = d-1 \end{cases}$$

Note that there is an extra term $\omega(-1)$ in this new version. This is due to the fact that $\chi\overline{\chi} = 1$ since their restriction to F^* is ω and so $\epsilon(\chi^{-1}, \psi_0) = \omega(-1)\epsilon(\chi, \psi_0)$. Note also that the comments made in the first paragraph of this chapter hold true even with our new notation.

We have the following simple lemma.

Lemma 5.1.2. *For a regular character θ of K^*, $a(\frac{\theta}{\overline{\theta}})$ is always even.*

Proof. If not, suppose $a(\frac{\theta}{\overline{\theta}}) = 2r + 1$, $r \geq 0$. Then it has to be nontrivial on $\frac{U_K^{2r}}{U_K^{2r+1}}$. But $\frac{\theta}{\overline{\theta}}(1 + \pi_F^r a) = \frac{\theta(1+\pi_F^r a)}{\overline{\theta}(1+\pi_F^r a)} = 1$, where $a \in \mathbb{F}_q$ which is a contradiction. □

5.1.1 Twist by characters of odd conductor

We use the symbol $\tilde{\omega}$ and $\tilde{\omega}_{K/F}$ to denote elements of S_{2d-1} if K/F is ramified.

Lemma 5.1.3. *If $\frac{\theta}{\overline{\theta}} = (-1)^{v_K}$ then no $\tilde{\omega}\theta$ can occur in r_θ.*

Proof. By theorem (4.3.3), $\tilde{\omega}\theta$ can occur in $r_{\theta+}$ if and only if $\epsilon(\frac{\overline{\theta}}{\theta}\tilde{\omega}^{-1}, \psi_0) = \epsilon(\tilde{\omega}^{-1}, \psi_0) = 1$ and in $r_{\theta-}$ if and only if $\epsilon(\frac{\overline{\theta}}{\theta}\tilde{\omega}^{-1}, \psi_0) = \epsilon(\tilde{\omega}^{-1}, \psi_0) = -1$. Since $\frac{\overline{\theta}}{\theta}$ unramified, we have

$$\begin{aligned}
\epsilon(\frac{\overline{\theta}}{\theta}\tilde{\omega}^{-1}, \psi_0) &= \frac{\overline{\theta}}{\theta}(\pi_K)^{a(\tilde{\omega}^{-1})+n(\psi_0)}\epsilon(\tilde{\omega}^{-1}, \psi_0) \\
&= \frac{\overline{\theta}}{\theta}(\pi_K)^{2d-1}\epsilon(\tilde{\omega}^{-1}, \psi_0) \text{ (since } n(\psi_0) \text{ even)} \\
&= -\epsilon(\tilde{\omega}^{-1}, \psi_0)
\end{aligned}$$

which shows that $\epsilon(\tilde{\omega}^{-1}, \psi_0) = -\epsilon(\tilde{\omega}^{-1}\frac{\overline{\theta}}{\theta}, \psi_0) \, \forall \, \tilde{\omega} \in S(2d-1)$. Similarly $\epsilon(\tilde{\omega}^{-1}, \psi_0) = -\epsilon(\tilde{\omega}^{-1}\frac{\overline{\theta}}{\theta}, \psi_0) \, \forall \, \tilde{\omega} \in S'(2d-1)$. So $\tilde{\omega}\theta$ can occur neither in $r_{\theta+}$ nor in $r_{\theta-}$ by theorem (4.3.3) for any $\tilde{\omega} \in S_{2d-1}$. Therefore it cannot occur in r_θ. □

Theorem 5.1.4. *Let $0 \neq a(\frac{\theta}{\overline{\theta}}) < a(\tilde{\omega})$. Then among all $\tilde{\omega} \in S(2d-1)$ half and only half will be such that $\tilde{\omega}\theta$ occur in $r_{\theta+}$ and among all $\tilde{\omega} \in S'(2d-1)$ half and only half will be such that $\tilde{\omega}\theta$ occur in $r_{\theta-}$.*

Remark: When $d = 1$, $a(\tilde{\omega}) = 1$. Therefore, since $a(\frac{\theta}{\overline{\theta}}) \neq 0$, this theorem is not applicable in $d = 1$ case.

Proof. We will show that $\displaystyle\sum_{\tilde{\omega} \in S(2d-1)} \epsilon(\tilde{\omega}^{-1}\frac{\overline{\theta}}{\theta}, \psi_0) = 0$ so that half of $\tilde{\omega} \in S(2d-1)$ will be such that $\epsilon(\tilde{\omega}^{-1}\frac{\overline{\theta}}{\theta}, \psi_0) = +1$ and the other half -1. The first half will occur in $r_{\theta+}$. The remaining half will not occur either in $r_{\theta+}$ or in $r_{\theta-}$. The other part of the proof is similar.

The calculations in this proof is similar to those carried out in section (4.1).

Note that $n(\psi_0)$ is always even irrespective of d. Also, $a(\tilde{\omega}^{-1}) = a(\tilde{\omega}^{-1}\frac{\overline{\theta}}{\theta})$. Taking $c = \pi_F^{d+\frac{n(\psi_0)}{2}}\pi_K^{-1}$, in equation (3.3) we have if $\tilde{\omega} \in S(2d-1)$, then

$$\epsilon(\tilde{\omega}^{-1}\frac{\overline{\theta}}{\theta}, \psi_0) = q^{-\frac{2d-1}{2}}\tilde{\omega}^{-1}\frac{\overline{\theta}}{\theta}(\pi_F^{d+\frac{n(\psi_0)}{2}}\pi_K^{-1}) \sum_{y\in\frac{U_K}{U_K^{2d-1}}} \tilde{\omega}\frac{\theta}{\overline{\theta}}(y)\psi_0(\pi_F^{-(d+\frac{n(\psi_0)}{2})}\pi_K y)$$

Write $y \in \frac{U_K}{U_K^{2d-1}}$ as $y = y_1(1+\pi_F^{r-1}\pi_K y_2)$, $r \geq 1, y_1 \in \frac{U_F}{U_F^d}$, $y_2 = 0$ or $y_2 \in \frac{U_F}{U_F^d}$. Also note that $\frac{\theta}{\overline{\theta}}$ is trivial on F^*. Summing over $S(2d-1)$, we get

$$\sum_{\tilde{\omega}\in S(2d-1)} \epsilon(\tilde{\omega}^{-1}\frac{\overline{\theta}}{\theta}, \psi_0) = q^{(-\frac{2d-1}{2})}\omega(\pi_F^{d+\frac{n(\psi_0)}{2}})\frac{\overline{\theta}}{\theta}(\pi_K) \sum_{y_1,y_2,r,\tilde{\omega}} \tilde{\omega}(\pi_K y_1(1+\pi_F^{r-1}\pi_K y_2))$$
$$\frac{\theta}{\overline{\theta}}(1+\pi_F^{r-1}\pi_K y_2)\psi_0(\pi_F^{-(d+\frac{n(\psi_0)}{2})}\pi_K y_1(1+\pi_F^{r-1}\pi_K y_2))$$

But from the main identity (in theorem (3.2.1)) and section (4.1), it follows that

$$\sum_{\tilde{\omega}} \tilde{\omega}(\pi_K y_1(1+\pi_F^{r-1}\pi_K y_2)) = \begin{cases} \omega(-1)\epsilon(\omega,\psi)q^t\omega(y_1) \text{ if } d = 2t+1 \\ \omega(-1)\epsilon(\omega,\psi)q^{s-\frac{1}{2}}\omega(\pi_F^{s-t}uu'y_1(1+\pi_F^{s+r-1}u'y_2)) \text{ if } d = 2s \end{cases}$$

The extra $\omega(-1)$ factor is due to the changed definition of $S(2d-1)$. Also,

$$\psi_0(\pi_F^{-(d+\frac{n(\psi_0)}{2})}\pi_K y_1(1+\pi_F^{r-1}\pi_K y_2)) = \begin{cases} \psi(-\pi_F^{-d}y_1) \text{ if } d = 2t+1 \\ \psi(-\pi_F^{-d}u^{-1}u'x_0^2 y_1(1+\pi_F^{r+s-1}u'y_2)) \text{ if } d = 2s \end{cases}$$

Let $d = 2t+1$. Keeping y_2 fixed, we have

$$\begin{aligned} & \sum_{y_1,\tilde{\omega}} \tilde{\omega}(\pi_K y_1(1+\pi_F^{r-1}\pi_K y_2))\psi_0(\pi_F^{-(d+\frac{n(\psi_0)}{2})}\pi_K(1+\pi_F^{r-1}\pi_K y_2)) \\ = \;& \epsilon(\omega,\psi)q^t\sum_{y_1}\omega(-y_1)\psi(-\pi_F^{-d}y_1) \\ = \;& \epsilon(\omega,\psi)q^t\epsilon(\omega,\psi)\omega(\pi_F^d)q^d \end{aligned}$$

which is a multiple of $\epsilon(\omega,\psi)$ independent of y_1 and y_2. So

$$\sum_{\tilde{\omega}\in S(2d-1)} \epsilon(\tilde{\omega}^{-1}\frac{\overline{\theta}}{\theta}, \psi_0) = C\sum_{r,y_2}\frac{\theta}{\overline{\theta}}((1+\pi_F^{r-1}\pi_K y_2) = C\times 0 = 0$$

since $\frac{\theta}{\overline{\theta}}$ is a nontrivial character of $\frac{U_K}{U_F U_K^{a(\frac{\theta}{\overline{\theta}})}}$ and $a(\frac{\theta}{\overline{\theta}}) \leq 2d-2$. Here C is a constant independent of y_1 and y_2. Similarly if $d = 2s$, then keeping y_2 again fixed,

$$\sum_{y_1, \tilde{\omega}} \tilde{\omega}(\pi_K y_1(1 + \pi_F^{r-1}\pi_K y_2))\psi_0(\pi_F^{-(d+\frac{n(\psi_0)}{2})}\pi_K(1 + \pi_F^{r-1}\pi_K y_2)y_1) =$$
$$\epsilon(\omega, \psi)q^{s-\frac{1}{2}}\omega(-\pi_F^{s-t}uu')\sum_{y_1}\omega((1 + \pi_F^{s+r-1}u'y_2)y_1)\psi(-\pi_F^{-d}u^{-1}u'x_0^2 y_1(1 + \pi_F^{r+s-1}u'y_2))$$

which is again a constant multiple of $\epsilon(\omega, \psi)$ independent of y_1 and y_2. So

$$\sum_{\tilde{\omega} \in S(2d-1)} \epsilon(\tilde{\omega}^{-1}\frac{\overline{\theta}}{\theta}, \psi_0) = C' \sum_{r, y_2} \frac{\theta}{\overline{\theta}}((1 + \pi_F^{r-1}\pi_K y_2) = C' \times 0 = 0$$

where C' is a constant multiple of $\epsilon(\omega, \psi)$. This completes the proof of the theorem. □

Corollary 5.1.5. *The number of $\tilde{\omega} \in S_{2d-1}$ such that $\tilde{\omega}\theta$ occurs in r_θ is $|S_{2d-1}|/2 = |S(2d-1)| = |S'(2d-1)|$.*

Proof. This is clear since occurring in r_θ means occurring in either $r_{\theta+}$ or in $r_{\theta-}$. Equality follows from theorem (3.4.5). □

Lemma 5.1.6. *If $a(\frac{\theta}{\overline{\theta}}) > a(\tilde{\omega})$ then the number of $\tilde{\omega} \in S_{2d-1}$ such that $\tilde{\omega}\theta$ occurs in r_θ is $|S_{2d-1}|/2$.*

Proof. This is quite easy to verify. In this case, $a(\frac{\theta}{\overline{\theta}}) = a(\tilde{\omega}^{-1}\frac{\theta}{\overline{\theta}})$. Note that $a(\frac{\theta}{\overline{\theta}})$ is even. So if

$$\epsilon(\tilde{\omega}^{-1}, \psi_0) \neq \epsilon(\tilde{\omega}^{-1}\frac{\overline{\theta}}{\theta}, \psi_0), \tag{5.1}$$

consider the character $\mu = (-1)^{v_K}$ of K^* and take $\tilde{\omega}_2^{-1} = \tilde{\omega}^{-1}\mu$. If we consider the expression for epsilon factors on both the sides of (5.1), since $a(\tilde{\omega}^{-1}\frac{\overline{\theta}}{\theta})$ is even, no π_K is present but only π_F on the RHS of this equation. Therefore the twist by μ will not make any difference on the RHS. But on the

LHS, an extra $\mu(\pi_K) = -1$ will appear changing the sign of LHS. Similarly if $\epsilon(\tilde{\omega}^{-1}, \psi_0) = \epsilon(\tilde{\omega}^{-1}\frac{\overline{\theta}}{\theta}, \psi_0)$, we can make them unequal by the same sort of twisting. So for half of $\tilde{\omega} \in S_{2d-1}$, the corresponding epsilon factors are equal and for the other half they are unequal. This proves the lemma. □

5.1.2 Twist by characters of even conductor

Note that if $a(\lambda) = 2f \geq 2d$, then in the expression for $\epsilon(\lambda^{-1}, \psi_0)$ there is no π_K, but only π_F.

Theorem 5.1.7. *Let $\lambda \in S(2f + 2d)$, $f \geq 0$ $\quad a(\frac{\overline{\theta}}{\theta}) \leq a(\lambda) - 2d = 2f$. Then all the elements in $\{\lambda\theta : \lambda \in S(2f + 2d)\}$ will occur in $r_{\theta+}$. Similarly if $\lambda' \in S'(2f + 2d)$, then all the elements in $\{\lambda'\theta : \lambda' \in S'(2f + 2d)\}$ will occur in $r_{\theta-}$. Therefore the number of $\lambda\theta$ where $\lambda \in S_{2f+2d}$ occurring in r_θ is $|S_{2f+2d}|$.*

Proof. Consider the two sums $\displaystyle\sum_{\lambda\in S(2f+2d)} \epsilon(\lambda^{-1}, \psi_0)$ and $\displaystyle\sum_{\lambda\in S(2f+2d)} \epsilon(\lambda^{-1}\frac{\overline{\theta}}{\theta}, \psi_0)$, We have

$$\sum_{\lambda\in S(2f+2d)} \epsilon(\lambda^{-1}\frac{\overline{\theta}}{\theta}, \psi_0) = q^{-f-d}\omega(\pi_F^{f+d+\frac{n(\psi_0)}{2}}) \times \sum_{\lambda\in S(2f+2d)} \sum_{y\in \frac{U_K}{U_K^{2f+2d}}} \lambda(y)\frac{\theta}{\overline{\theta}}(y)\psi_0(\pi_F^{-(f+d+\frac{n(\psi_0)}{2})}y)$$

In this summation, by theorem (5.1.1), $\displaystyle\sum_{\lambda\in S(2f+2d)} \lambda(y) \neq 0$ only for two types of y's: 1). when $y = y_1(1 + \pi_F^f\pi_K y_2)$ where $y_1, y_2 \in U_F$, and 2). when $y = y_1(1 + \pi_F^{f+d-1}\pi_K y_2)$ where $y_1, y_2 \in U_F$ or $y_2 = 0$. But since $a(\frac{\theta}{\overline{\theta}}) \leq 2f$ and $\frac{\theta}{\overline{\theta}} = 1$ on F^* we have $\frac{\theta}{\overline{\theta}}$ trivial on these y's. So both the sums are independent of $\frac{\theta}{\overline{\theta}}$ and so they are the same. That is,

$$\sum_{\lambda\in S(2f+2d)} \epsilon(\lambda^{-1}, \psi_0) = \sum_{\lambda\in S(2f+2d)} \epsilon(\lambda^{-1}\frac{\overline{\theta}}{\theta}, \psi_0)$$

which means

$$\epsilon(\lambda^{-1}, \psi_0) = \epsilon(\lambda^{-1}\frac{\overline{\theta}}{\theta}, \psi_0) \quad \forall \lambda \in S(2f + 2d)$$

since $\epsilon(\lambda^{-1}, \psi_0) = 1$ for each $\lambda \in S(2f + 2d)$. The remaining part follows similarly. This proves the theorem. □

Corollary 5.1.8. *If $\frac{\theta}{\overline{\theta}} = (-1)^{v_K}$, then all $\lambda \in S(2f + 2d)$ are such that $\lambda\theta$ occur in $r_{\theta+}$. Similarly all $\lambda' \in S'(2f + 2d)$ are such that $\lambda'\theta$ occur in $r_{\theta-}$.*

Proof. It follows by taking $a(\frac{\theta}{\overline{\theta}}) = 0$ in the above theorem. □

Note: The above corollary shows the differrence between characters of even conductor and characters of odd conductor. This corollary is extremely opposite to lemma (5.1.3).
Let $\lambda \in S(2f + 2d)$, $2f < a(\frac{\theta}{\overline{\theta}}) < a(\lambda)$. Note that if $d = 1$, then no such θ exists. So we have $d \geq 2$ and so q even. By definition, we have

$$\epsilon(\lambda^{-1}\frac{\overline{\theta}}{\theta}, \psi_0) = q^{-f-d}\omega(\pi_F^{f+d+\frac{n(\psi_0)}{2}}) \sum_{\lambda \in S(2f+2d)} \sum_{y \in \frac{U_K}{U_K^{2f+2d}}} \lambda(y)\frac{\theta}{\overline{\theta}}(y)\psi_0(\pi_F^{-(f+d+\frac{n(\psi_0)}{2})}y)$$

Again, by theorem (5.1.1), the sum $\sum_{\lambda \in S(2f+2d)} \lambda(y) \neq 0$ only for two types of y's:

1. $y = y_1(1 + \pi_F^{f+d-1}\pi_K y_2)$, $y_1 \in \frac{U_F}{U_F^{f+d}}$, $y_2 \in \mathbb{F}_q$
2. $y = y_1(1 + \pi_F^{f}\pi_K y_2)$, $y_1 \in \frac{U_F}{U_F^{f+d}}$, $y_2 \in \frac{U_F}{U_F^{d}}$

Consider first type of y's. $\frac{\theta}{\overline{\theta}}$ is trivial on the these y's. Now

$$\sum_{\lambda \in S(2f+2d)} \lambda(y) = \omega(y_1) \sum_{\lambda \in S(2f+2d)} \lambda(1 + \pi_F^{f+d-1}\pi_K y_2) = -q^{f+d-1}\omega(y_1)$$

Also $\psi_0(\pi_F^{-f-d-\frac{n(\psi_0)}{2}}y)$ is $\psi(-\pi_F^{-1}y_1y_2)$ when $d=2t+1$ and is $\psi(-\pi_F^{-1}u^{-1}u'x_0^2y_1y_2)$ when $d=2s$. Now

$$\sum_{\lambda\in S(2f+2d)}\lambda(y)\psi_0(\pi_F^{-f-d-\frac{n(\psi_0)}{2}}y)$$

$$=\begin{cases}-q^{f+d-1}\sum_{y_2\in\mathbb{F}_q}\sum_{y_1\in\frac{U_F}{U_F^{f+d}}}\omega(y_1)\psi(-\pi_F^{-1}y_1y_2)) \text{ if } d \text{ odd}\\ -q^{f+d-1}\sum_{y_2\in\mathbb{F}_q}\sum_{y_1\in\frac{U_F}{U_F^{f+d}}}\omega(y_1)\psi(-\pi_F^{-1}u^{-1}u'x_0^2y_1y_2) \text{ if } d \text{ even}\end{cases}$$

$$=\begin{cases}-q^{f+d-1}\sum_{y_2\in\mathbb{F}_q}\omega(-y_2)\sum_{y_1\in\frac{U_F}{U_F^{f+d}}}\omega(y_1)\psi(\pi_F^{-1}y_1)) \text{ if } d \text{ odd}\\ -q^{f+d-1}\sum_{y_2\in\mathbb{F}_q}\omega(-y_2x_0^2uu')\sum_{y_1\in\frac{U_F}{U_F^{f+d}}}\omega(y_1)\psi(\pi_F^{-1}y_1)) \text{ if } d \text{ even}\end{cases}$$

Since $a(\omega)=d\neq 1$, by lemma (3.4.3) $\displaystyle\sum_{y_1\in\frac{U_F}{U_F^{f+d}}}\omega(y_1)\psi(\pi_F^{-1}y_1))=0$. So

$$\sum_{\lambda\in S(2f+2d)}\lambda(y)\sum_{y_1\in\frac{U_F}{U_F^{f+d}}}\lambda(y)\psi_0(\pi_F^{-f-d-\frac{n(\psi_0)}{2}}y)=0$$

Consider the second type of y's. On these, we have

$$\sum_{\lambda\in S(2f+2d)}\lambda(y_1(1+\pi_F^f\pi_Ky_2)) \;=\; \begin{cases}\omega(-1)\omega(y_1)\omega(\pi_F^fy_2)q^{f+t+\frac{1}{2}}\epsilon(\omega,\psi) \text{ if } d \text{ odd}\\ \omega(-1)\omega(y_1y_2uu')\omega(\pi_F^{f+s-t}y_2)q^{f+s}\epsilon(\omega,\psi) \text{ if } d \text{ even}\end{cases}$$

by equations (4.3), (4.4), and theorem (5.1.1)

and

$$\psi_0(\pi_F^{-f-d-\frac{n(\psi_0)}{2}}y_1(1+\pi_F^f\pi_Ky_2))=\begin{cases}\psi(-\pi_F^{-d}y_1y_2) \text{ if } d \text{ odd}\\ \psi(-\pi_F^{-d}y_1y_2u^{-1}u'x_0^2) \text{ if } d \text{ even}\end{cases}$$

Let $d = 2t+1$. Then

$$\sum_{\lambda \in S(2f+2d)} \sum_{y_1 \in \frac{U_F}{U_F^{f+d}}} \sum_{y_2 \in \frac{U_F}{U_F^{d}}} \lambda(y)\frac{\theta}{\overline{\theta}}(y)\psi_0(\pi_F^{-f-d-1}y)$$

$$= \omega(-1)\omega(\pi_F^f)q^{f+t+\frac{1}{2}}\epsilon(\omega,\psi) \sum_{y_1 \in \frac{U_F}{U_F^{f+d}}} \sum_{y_2 \in \frac{U_F}{U_F^{d}}} \omega(y_1y_2)\psi(-\pi_F^{-d}y_1y_2)\frac{\theta}{\overline{\theta}}(1+\pi_F^f\pi_K y_2)$$

$$= q^f q^{f+t+\frac{1}{2}}\omega(\pi_F^f)\epsilon(\omega,\psi) \sum_{y_2 \in \frac{U_F}{U_F^{d}}} \frac{\theta}{\overline{\theta}}(1+\pi_F^f\pi_K y_2) \sum_{y_1 \in \frac{U_F}{U_F^{d}}} \omega(y_1y_2)\psi(\pi_F^{-d}y_1y_2)$$

$$= q^{2f}q^{\frac{d}{2}}\omega(\pi_F^f)\epsilon(\omega,\psi)\epsilon(\omega,\psi)\omega(\pi_F^d)q^{\frac{d}{2}} \sum_{y_2 \in \frac{U_F}{U_F^{d}}} \frac{\theta}{\overline{\theta}}(1+\pi_F^f\pi_K y_2)$$

$$= q^{2f+d}\omega(\pi_F^{f+d})\epsilon(\omega,\psi)^2 \sum_{y_2 \in \frac{U_F}{U_F^{d}}} \frac{\theta}{\overline{\theta}}(1+\pi_F^f\pi_K y_2)$$

If $d = 2s$ we will get the same sum with an extra $\omega(-1)$ factor. Now if $a(\frac{\theta}{\overline{\theta}}) \geq 2f+4$ then

$$\sum_{y_2 \in \frac{U_F}{U_F^{d}}} \frac{\theta}{\overline{\theta}}(1+\pi_F^f\pi_K y_2) = 0 \text{ and so } \sum_{\lambda \in S(2f+2d)} \epsilon(\lambda^{-1}\frac{\overline{\theta}}{\theta}, \psi_0) = 0.$$

Therefore half of elements in $\{\lambda\theta : \lambda \in S(2f+2d)\}$ will appear in $r_{\theta+}$. Similarly, half of elements in $\{\lambda'\theta : \lambda' \in S'(2f+2d)\}$ will appear in $r_{\theta-}$.
Let $a(\frac{\theta}{\overline{\theta}}) = 2f+2$. Then $\sum_{y_2 \in \frac{U_F}{U_F^{d}}} \frac{\theta}{\overline{\theta}}(1+\pi_F^f\pi_K y_2) = q^{d-1}\sum_{a \in \mathbb{F}_q} \frac{\theta}{\overline{\theta}}(1+\pi_F^f\pi_K a) = -q^{d-1}$

Therefore if $d = 2t+1$, then

$$\sum_{\lambda \in S(2f+2d)} \sum_{y_1 \in \frac{U_F}{U_F^{f+d}}} \sum_{y_2 \in \frac{U_F}{U_F^{d}}} \lambda(y)\frac{\theta}{\overline{\theta}}(y)\psi_0(\pi_F^{-f-d-1}y) = -q^{2f+2d-1}\omega(\pi_F^{f+d})\epsilon(\omega,\psi)^2$$

So

$$\sum_{\lambda \in S(2f+2d)} \epsilon(\lambda^{-1}\frac{\overline{\theta}}{\theta}, \psi_0) = q^{-f-d}\omega(\pi_F^{f+d}) \times -q^{2f+2d-1}\omega(\pi_F^{f+d})\epsilon(\omega,\psi)^2 = -q^{f+d-1}\epsilon(\omega,\psi)^2$$

Similarly we will get $\sum_{\lambda \in S(2f+2d)} \epsilon(\lambda^{-1}, \psi_0) = (q-1)q^{f+d-1}\epsilon(\omega,\psi)^2$ (This is because in place of $\sum_{y_2 \in \frac{U_F}{U_F^d}} \frac{\theta}{\overline{\theta}}(1+\pi_F^f\pi_K y_2)$ we have $|\frac{U_F}{U_F^d}|$). But

$$\sum_{\lambda \in S(2f+2d)} \epsilon(\lambda^{-1}, \psi_0) = |S(2f+2d)| = (q-1)q^{f+d-1}.$$

So $\epsilon(\omega,\psi)^2 = 1$. (If $d = 2s$, instead of this, we have $\omega(-1)\epsilon(\omega,\psi)^2 = 1$). Therefore number of λ such that $\lambda\theta$ appear in $r_{\theta+}$ is $\sum_{\lambda \in S(2f+2d)}(\epsilon(\lambda^{-1},\psi_0) + \epsilon(\lambda^{-1}\frac{\overline{\theta}}{\theta},\psi_0)) = \frac{(q-1)-1}{2}q^{f+d-1} = \frac{q-2}{2}q^{f+d-1}$. If $d = 2s$ then also we can show that the sum is $\frac{q-2}{2}q^{f+d-1}$.

So in $r_{\theta+}$, the number of $\lambda\theta$ occurring where $\lambda \in S(2f+2d)$ is $\frac{q-2}{2}q^{f+d-1}$. Similarly in $r_{\theta-}$, the number of $\lambda'\theta$ occurring where $\lambda' \in S('2f+2d)$ is $\frac{q-2}{2}q^{f+d-1}$. We summarize the above computations in the following two theorems.

Theorem 5.1.9. *Let $\lambda \in S(2f+2d)$, $2f+2 < a(\frac{\theta}{\overline{\theta}}) < a(\lambda)$. Then among all $\lambda\theta$ where $\lambda \in S(2f+2d)$ exactly half will occur in $r_{\theta+}$. Similarly, let $\lambda' \in S'(2f+2d)$, $2f+2 < a(\frac{\theta}{\overline{\theta}}) < a(\lambda')$. Then among all $\lambda'\theta$ where $\lambda' \in S'(2f+2d)$ exactly half will occur in $r_{\theta-}$. Therefore the number of $\lambda\theta$ where $\lambda \in S_{2f+2d}$ occurring in r_θ is $|S_{2f+2d}|/2$.*

Theorem 5.1.10. *Let $\lambda \in S(2f+2d)$, $a(\frac{\theta}{\overline{\theta}}) = 2f+2 < a(\lambda)$. Then number of $\lambda\theta$ appearing in $r_{\theta+}$ where $\lambda \in S(2f+2d)$ is $\frac{q-2}{2}q^{f+d-1}$. Similarly, let $\lambda' \in S'(2f+2d)$, $a(\frac{\theta}{\overline{\theta}}) = 2f+2 < a(\lambda')$. Then number of $\lambda'\theta$ appearing in $r_{\theta-}$ where $\lambda' \in S'(2f+2d)$ is $\frac{q-2}{2}q^{f+d-1}$. The number of $\lambda\theta$ where $\lambda \in S_{2f+2d}$ occurring in r_θ is therefore $(q-2)q^{f+d-1}$.*

Note: These two theorems are not valid for $d = 1$ since no $\frac{\theta}{\overline{\theta}}$ satisfies the condition in the theorem.

Theorem 5.1.11. *Let $a(\lambda) = 2f + 2d < a(\frac{\theta}{\overline{\theta}}) = 2m < a(\lambda) + 2d$. Then the number of $\lambda\theta$ with $\lambda \in S_{2f+2d}$ appearing in r_θ is $|S(2f + 2d)| = |S_{2f+2d}|/2$.*

Proof. Here $a(\lambda^{-1}\frac{\overline{\theta}}{\theta}) = a(\frac{\theta}{\overline{\theta}})$. Using the definition of ϵ-factors, we have

$$\sum_{\lambda\in S(2f+2d)} \epsilon(\lambda^{-1}\frac{\overline{\theta}}{\theta}, \psi_0) = q^{-m}\omega(\pi_F^{m+\frac{n(\psi_0)}{2}}) \sum_{\lambda\in S(2f+2d)} \sum_{y\in\frac{U_K}{U_K^{2m}}} \lambda(y)\frac{\theta}{\overline{\theta}}(y)\psi_0(\pi_F^{-(m+\frac{n(\psi_0)}{2})}y)$$

Recall that, from theorem (5.1.1) the sum $\sum_{\lambda\in S(2f+2d)} \lambda(y) \neq 0$ only for three types of y's:

1. $y = y_1(1 + \pi_F^f\pi_K y_2)$, $y_1 \in \frac{U_F}{U_F^m}$, $y_2 \in \frac{U_F}{U_F^{m-f}}$
2. $y = y_1(1 + \pi_F^{f+d-1}\pi_K y_2)$, $y_1 \in \frac{U_F}{U_F^m}$, $y_2 \in \frac{U_F}{U_F^{m-f-d+1}}$
3. $y = y_1(1 + \pi_F^{f+d}\pi_K y_2)$, $y_1 \in \frac{U_F}{U_F^m}$, $y_2 \in \frac{U_F}{U_F^{m-f-d}}$ or $y_2 = 0$

But on the third type of y's, λ is just ω since $a(\lambda) = 2f + 2d$. On the second type of y's,

$$\begin{aligned}\sum_{\lambda\in S(2f+2d)} \lambda(y) &= \omega(y_1) \sum_{\lambda\in S(2f+2d)} \lambda(1 + \pi_F^{f+d-1}\pi_K y_2)\\ &= \omega(y_1)(-q^{f+d-1}) \text{ (by theorem (5.1.1))}\end{aligned}$$

So $\sum\limits_{\lambda\in S(2f+2d)} \lambda(y)$ is independent of λ on these y's. Finally consider the first type of y's. Let $d = 2t + 1$.

$$\begin{aligned}\sum_{\lambda\in S(2f+2d)} \lambda(y_1(1 + \pi_F^f\pi_K y_2)) &= \omega(y_1) \sum_{\lambda\in S(2f+2d)} \lambda(1 + \pi_F^f\pi_K y_2)\\ &= \omega(-1)\omega(y_1)\epsilon(\omega, \psi)\omega(\pi_F^f y_2)q^{f+t+\frac{1}{2}}\end{aligned}$$

by equation (4.3) and theorem(5.1.1).

Therefore

$$\begin{aligned}
& \sum_{\lambda\in S(2f+2d)} \sum_{y_1\in\frac{U_F}{U_F^m}} \sum_{y_2\in\frac{U_F}{U_F^{m-f}}} \lambda(y_1(1+\pi_F^f\pi_K y_2))\frac{\theta}{\bar{\theta}}(y_1(1+\pi_F^f\pi_K y_2)) \\
& \times\psi_0(\pi_F^{-m-1}y_1(1+\pi_F^f\pi_K y_2)) \\
= \quad & \epsilon(\omega,\psi)\omega(\pi_F^f)q^{f+t+\frac{1}{2}} \sum_{y_1\in\frac{U_F}{U_F^m}} \sum_{y_2\in\frac{U_F}{U_F^{m-f}}} \omega(-y_1y_2)\psi(-\pi_F^{-m+f}y_1y_2)\frac{\theta}{\bar{\theta}}(1+\pi_F^f\pi_K y_2)
\end{aligned}$$

In this sum, we have

$$\sum_{y_1\in\frac{U_F}{U_F^m}} \omega(y_1y_2)\psi(-\pi_F^{-m+f}y_1y_2) = \sum_{y_1\in\frac{U_F}{U_F^{m-f}}} \omega(y_1y_2)\psi(-\pi_F^{-m+f}y_1y_2)|\frac{U_F^{m-f}}{U_F^m}|$$

Since $a(\omega)=d$ and $m-f>d$, by lemma (3.4.3),the above sum is zero.

Now if $d=2s$, we have $n(\psi_0)=2(s-t)$. Also, here the trace 0 element x_0 is a unit and $\pi_K=\frac{\pi_F^s u'}{2}(1+x_0)$. Considering y's first type, we have

$$\begin{aligned}
\psi_0(\pi_F^{-(m+\frac{n(\psi_0)}{2})}y) &= \psi_0(\pi_F^{-m-s+t}y_1(1+\pi_F^f\pi_K y_2)) \\
&= \psi(-\frac{\pi_F^{-m-s+t}}{2}y_1 tr\, x_0(1+\pi_F^f\pi_K y_2)) \\
&= \psi(-\frac{\pi_F^{-m-s+t}}{2}y_1 x_0^2.2\pi_F^f y_2\frac{\pi_F^s}{2}u') \\
&= \psi(-\pi_F^{-m+f}u^{-1}u'y_1y_2x_0^2)
\end{aligned}$$

and so

$$\begin{aligned}
\sum_{\lambda\in S(2f+2d)} \lambda(y_1(1+\pi_F^f\pi_K y_2)) &= \omega(-1)\epsilon(\omega,\psi)\omega(\pi_F^{f+s-t})q^{f+s}\omega(y_1y_2uu') \\
& \qquad \text{by equation (4.4) and theorem(5.1.1).}
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{\lambda\in S(2f+2d)}\sum_{y_1\in\frac{U_F}{U_F^m}}\sum_{y_2\in\frac{U_F}{U_F^{m-f}}} \lambda & \quad (y_1(1+\pi_F^f\pi_K y_2))\frac{\theta}{\overline{\theta}}(y_1(1+\pi_F^f\pi_K y_2))\times \\
& \quad \psi_0(\pi_F^{-m-s+t}y_1(1+\pi_F^f\pi_K y_2)) \\
& = \epsilon(\omega,\psi)\omega(\pi_F^{f+s-t})q^{f+s}\sum_{y_2\in\frac{U_F}{U_F^{m-f}}}\frac{\theta}{\overline{\theta}}(1+\pi_F^f\pi_K y_2) \\
& \quad \times\sum_{y_1\in\frac{U_F}{U_F^m}}\omega(y_1y_2u^{-1}u')\psi(-\pi_F^{-m+f}u^{-1}u'y_1y_2x_0^2)
\end{aligned}$$

The sum $\sum_{y_1\in\frac{U_F}{U_F^m}}\omega(y_1y_2u^{-1}u')\psi(\pi_F^{-m+f}u^{-1}u'y_1y_2)=0$ as in the $d=2t+1$ case since $m-f>d$. So the sum over the first type of y's become zero. So in both d odd and even cases, the sum $\sum_{\lambda\in S(2f+2d)}\epsilon(\lambda^{-1}\frac{\overline{\theta}}{\theta},\psi_0)$ depends only on second and third type of y's and is independent of λ. Suppose this sum is n. Using similar arguments, we have $\sum_{\lambda'\in S'(2f+2d)}\epsilon(\lambda'^{-1}\frac{\overline{\theta}}{\theta},\psi_0)=n$. So the number of $+1$'s in $\{\epsilon(\lambda^{-1}\frac{\overline{\theta}}{\theta},\psi_0):\lambda\in S(2f+2d)\}=\frac{|S(2f+2d)|+n}{2}$. Similarly, number of -1's in $\{\epsilon(\lambda'^{-1}\frac{\overline{\theta}}{\theta},\psi_0):\lambda'\in S'(2f+2d)\}=-\frac{-|S(2f+2d)|+n}{2}$. Therefore, the number of $\lambda\theta$ appearing in $r_{\theta+}$ is $\frac{|S(2f+2d)|+n}{2}$, number of $\lambda'\theta$ appearing in $r_{\theta-}$ is $\frac{|S(2f+2d)|-n}{2}$. Total number of $\lambda\theta$ appearing in r_θ is $|S(2f+2d)|$. $\square$

Theorem 5.1.12. *Suppose $\lambda\in S(2m)$, $m\geq d$ and $a(\frac{\theta}{\overline{\theta}})=2n\geq a(\lambda)+2d$. Then either all the elements in $\{\lambda\theta:\lambda\in S(2m)\}$ will occur in $r_{\theta+}$ or all the elements in $\{\lambda'\theta:\lambda'\in S'(2m)\}$ will occur in $r_{\theta-}$ not both. Therefore the number of $\lambda\theta$ where $\lambda\in S_{2m}$ occurring in r_θ is $|S_{2m}|/2$.*

Proof. We have if $\chi\in\widehat{K^*}$ with $\chi|_{K^*}=\omega$ and $a(\chi)\geq 2a(\tilde{\omega})$ then $\epsilon(\chi,\psi_0)=\tilde{\omega}(-x_0/2)\tilde{\omega}^{-1}(y_\chi)$ where

$$y_\chi=\begin{cases}\pi_F^{-f-\frac{d-1}{2}}\pi_K a_0(\chi)(1+a_1(\chi)\pi_K)(1+a_2(\chi)\pi_F)... & \text{if } d \text{ is odd}\\ \pi_F^{-f-\frac{d}{2}}x_0a_0(\chi)(1+a_1(\chi)\pi_K)(1+a_2(\chi)\pi_F)... & \text{if } d \text{ is even}\end{cases}.$$

Here $a(\lambda^{-1}\frac{\overline{\theta}}{\theta}) = a(\frac{\overline{\theta}}{\theta}) \geq 4d > 2a(\tilde{\omega})$. Therefore $\epsilon(\lambda^{-1}\frac{\overline{\theta}}{\theta}, \psi_0) = \tilde{\omega}(-x_0/2)$ $\tilde{\omega}^{-1}(\pi_F^{-f-\frac{d-1}{2}}\pi_K a_0(\lambda^{-1}\frac{\overline{\theta}}{\theta})(1 + a_1(\lambda^{-1}\frac{\overline{\theta}}{\theta})\pi_K)...(1 + a_{2d-2}(\lambda^{-1}\frac{\overline{\theta}}{\theta})\pi_F^{d-1}))$ if d odd. But note that $(\lambda^{-1}\frac{\overline{\theta}}{\theta})|_{U_K^{2n-2d+1}}$ determines $a_i(\lambda^{-1}\frac{\overline{\theta}}{\theta})$ for $i = 0, 1, \ldots 2d-2$ and on $U_K^{2n-2d+1}$, $\lambda^{-1}\frac{\overline{\theta}}{\theta} = \frac{\overline{\theta}}{\theta}$. Therefore $\epsilon(\lambda^{-1}\frac{\overline{\theta}}{\theta}, \psi_0)$ is independent of λ or $\epsilon(\lambda^{-1}\frac{\overline{\theta}}{\theta}, \psi_0) = \epsilon(\frac{\overline{\theta}}{\theta}, \psi_0)$ Now suppose that $\epsilon(\lambda^{-1}, \psi_0) \neq \epsilon(\frac{\overline{\theta}}{\theta}, \psi_0) = -1$ for one $\lambda \in S(2m)$. Then for all $\lambda' \in S'(2m)$, $\epsilon(\lambda'^{-1}, \psi_0) = -1 = \epsilon(\frac{\overline{\theta}}{\theta}, \psi_0) = \epsilon(\lambda'^{-1}\frac{\overline{\theta}}{\theta}, \psi_0)$. Therefore $\{\lambda'\theta : \lambda' \in S'(2m)\}$ will occur in $r_{\theta-}$. On the other hand if $\epsilon(\lambda^{-1}, \psi_0) = \epsilon(\frac{\overline{\theta}}{\theta}, \psi_0) = 1$ for one λ it is the same for all other $\lambda \in S(2m)$. This proves the theorem. □

Corollary 5.1.13. *Suppose* $a(\lambda) = 2f + 2d < a(\frac{\theta}{\overline{\theta}}) = 2m$. *If* $n=$ *number of* $\lambda\theta$, $\lambda \in S(2f + 2d)$, *appearing in* $r_{\theta+}$ *then number of* $\lambda'\theta$, $\lambda' \in S'(2f + 2d)$, *appearing in* $r_{\theta-}$ *is* $|S(2f + 2d)| - n = |S'(2f + 2d)| - n$. *Also, if* $2m > a(\lambda) + 2d$, *then either* $n = 0$ *or* $n = |S(2f + 2d)|$.

Proof. Follows easily from the above two theorems. □

Only one case is left now for us to handle in this exposition viz. $a(\frac{\theta}{\overline{\theta}}) = a(\lambda)$. In this case we are not giving an exact count, but still we can provide a lower bound in the next theorem.

Theorem 5.1.14. *If* $a(\frac{\theta}{\overline{\theta}}) = a(\lambda) = 2f + 2d$, $\lambda|_{F^*} = \omega$ *then the number of* $\lambda\theta$ *appearing in* r_θ *is greater than or equal to* q^{f+d-1}.

Proof. Note that $S(2f + 2d) \cup S'(2f + 2d) = \{\frac{\overline{\theta}}{\theta}\chi : \chi|_{F^*} = \omega, a(\chi) = 2d - 1, 2d, 2d + 2, \ldots, 2f + 2d - 2\} \bigcup \{\frac{\overline{\theta}}{\theta}\chi : \chi|_{F^*} = \omega, a(\chi) = 2f + 2d,\ \chi|_{U_K^{2f+2d-1}} \neq \frac{\theta}{\overline{\theta}}|_{U_K^{2f+2d-1}}\}$. Now a $\frac{\overline{\theta}}{\theta}\chi.\theta = \chi\overline{\theta}$ will appear in r_θ if and only if $\epsilon(\chi^{-1}, \psi_0) = \epsilon(\chi^{-1}\frac{\theta}{\overline{\theta}}, \psi_0)$. So the number of $\chi\overline{\theta}$ appearing in r_θ where $a(\chi) = 2d - 1, 2d, \ldots, 2f + 2d$ is greater than or equal to $|S(2d-1)| + |S(2d)| + S(2d+2)| + \ldots + |S(2f+2d-2)| = q^{d-1} + (q-1)q^{d-1} + (q-1)q^d + \ldots + (q-1)q^{f+d-2} = q^{f+d-1}$

by corollary (5.1.5), lemma (5.1.6), and theorems (5.1.11) and (5.1.12). Note that we are not considering χ's with conductor equal to $2f+2d$ and that is why we are unable to claim equality. $\square$

Remark: If $q=2$, there is no χ such that $\chi|_{U_K^{2f+2d-1}} \neq \frac{\overline{\theta}}{\theta}|_{U_K^{2f+2d-1}}$. So equality holds in the theorem.

5.2 The unramified case

Suppose K over F is unramified and let $\chi \in \widehat{F^*}$ be such that $\chi|_{K^*} = \omega$. Let $\tilde{\omega}$ be an extension of ω as in section (3.2). Note that $a(\frac{\theta}{\overline{\theta}}) \neq 0$. Otherwise, since $\pi_K = \pi_F \in F$ in this case, $\frac{\theta}{\overline{\theta}}(\pi_K) = 1$ so that $\frac{\theta}{\overline{\theta}}$ is trivial. Then $\theta = \overline{\theta}$ contradicting the regularity of θ. So $a(\frac{\theta}{\overline{\theta}}) \geq 1$.

We divide our counting into mainly 3 cases:

Case 1: $a(\frac{\theta}{\overline{\theta}}) < a(\chi)$

We have $\epsilon(\chi^{-1}, \psi_0) = \tilde{\omega}(-x_0/2)\tilde{\omega}^{-1}(y_{\chi^{-1}})$. Since $\tilde{\omega}$ is trivial on units in the unramified case, let $y_{\chi^{-1}} = \pi_F^{-a(\chi)}$. So we have $\epsilon(\chi^{-1}, \psi_0) = (-1)^{a(\chi)+t}$ where $t = v_F(2)$. Since $a(\frac{\theta}{\overline{\theta}}) < a(\chi)$ we have $\epsilon(\chi^{-1}\frac{\overline{\theta}}{\theta}, \psi_0) = (-1)^{a(\chi)+t} = \epsilon(\chi^{-1}, \psi_0)$. So all the χ's are such that all $\chi\theta$ will occur in $r_{\theta+}$ or all will occur in $r_{\theta-}$ depending on whether $a(\chi)$ is even or odd.

Case 2: $a(\chi) < a(\frac{\theta}{\overline{\theta}})$

In this case,$a(\chi^{-1}\frac{\overline{\theta}}{\theta}) = a(\frac{\theta}{\overline{\theta}})$. So $\epsilon(\chi^{-1}\frac{\overline{\theta}}{\theta}, \psi_0) = (-1)^{a(\frac{\theta}{\overline{\theta}})+t}$. Also, $\epsilon(\chi^{-1}\frac{\overline{\theta}}{\theta}, \psi_0) = (-1)^{a(\chi)+t}$. $\chi\theta$ will occur in r_θ if and only if $a(\chi) = a(\frac{\theta}{\overline{\theta}}) (mod\, 2)$.

Case 3: $a(\chi) = a(\frac{\theta}{\overline{\theta}})$

Here we have two possibilities:

1. $a(\chi^{-1}\frac{\overline{\theta}}{\theta}) < a(\chi)$ or $a(\chi) < a(\chi^{-1}\frac{\theta}{\overline{\theta}})$: In this case, if $a(\chi) = a(\frac{\theta}{\overline{\theta}}) (mod\, 2)$ then $\chi\theta$ will occur in r_θ.

2. $a(\chi^{-1}\frac{\overline{\theta}}{\theta}) = a(\chi)$: In this case $\chi\theta$ will occur in r_θ.

Remark: Since by theorem(4.3.3), $\lambda\theta$ appears in $r_{\theta+}$ (respectively $r_{\theta-}$) if and only if $\lambda\theta$ does not appear in $r'_{\theta+}$ (respectively $r'_{\theta-}$), all the theorems proved in this chapter have their obvious D_F^* analogues.

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