FOCAL BOUNDARY VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

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DECLARATION

This is to certify that I, Modali Venkata Rama, have carried out the research work embodied in the present thesis under the guidance of Dr.S. Umamaheswaram for the period prescribed under the Ph.D. Ordinances of the University.

I declare to the best of my knowledge that no part of this thesis was earlier submitted for the award of research degree of any University.

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TO MY PARENTS

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CONTENTS

CHAPTER I

INTRODUCTION

Consider the nth order differential equation

$$y^{(n)} = f(x, y, \ldots, y^{(n-1)}) \qquad (1.1)$$

along with the '(n(1), ..., n(k)) – right focal' boundary conditions (BC's) denoted by

$$y^{(i)}(x_r) = y_{r\,i}\,, \quad i = s(r-1), \ldots, s(r) - 1 \qquad (1.2R)$$
$$r = 1, \ldots, k$$

where n is fixed, k $(1 < k \leq n)$, $n(1), \ldots, n(k)$ are arbitrary but fixed integers ; $s(0) = 0$, $s(r) = n(1) + \ldots + n(r)$, $r = 1, \ldots, k$; $s(k) = n$, $x_1 < \ldots < x_k$ are arbitrary real numbers in I, $y_{r\,i} \in \mathbb{R}$ are arbitrary and f is continuous on $I \times \mathbb{R}^n$.

The above boundary value problem (BVP) and its special cases for particular values of n and k have been studied by several authors in several different contexts. For instance for $n = k = 2$ and $I = (a,b)$, Klaasen [21] gave sufficient conditions in terms of lower and upper solutions for the existence of a solution for the BVP (1.1), (1.2R). For arbitrary n and k with $I = (a,b)$, Henderson in [12] proved that under the hypotheses (1) to (4) stated below the BVP (1.1), (1.2R) has a solution.

(1) Solutions of initial value problems (IVP's) for (1.1) are unique.

(2) Solutions of IVP's for (1.1) extend to (a,b).

(3) Each $(1, 1, \ldots, 1)$ - right focal BVP for (1.1) on (a,b) has at most one solution.

(4) If $\{y_k(x)\}$ is a sequence of solutions of (1.1) and $[c,d]$ is a compact subinterval of (a,b) such that $\{y_k(x)\}$ is uniformly bounded on $[c,d]$, then there exists a subsequence $\{y_{k_j}(x)\}$ such that $\{y_{k_j}^{(i)}(x)\}$ converges uniformly on $[c,d]$, $0 \leq i \leq n-1$.

Again for arbitrary n, k and $I = [a,b]$ Eloe and Handerson in [7] obtained local and global existence theorems for the BVP (1.1), (1.2R) using the topological transversality method. Further in [28] considering the differential equation $y^{(n)} = \pm\, p(x)y$ along with the BC's (1.2R) with $k = 2$ under the assumptions that $p(x) \geq 0$ and is continuous on $[x_1,x_2]$, Peterson showed that $(-1)^{n - n(1)}\, G^{(i)}(x,t)$ for $i = 0,\ldots,n(1)-1$ is positive semidefinite on $(x_1,x_2) \times (x_1,x_2)$ where $G(x,t)$ is the Green's function associated with the above BVP.

Besides the above mentioned results for focal BVP's there are also some results concerning the ' k - point conjugate ' BVP (1.1) and

$$y^{(i)}(x_r) = y_{r\,i}\,, \quad i = 0, \ldots, n(r) - 1 \qquad (1.2C)$$
$$r = 1, \ldots, k$$

($n(1), \ldots, n(k)$ as in (1.2R))

which are of interest to us. In particular, we recall the papers by Gustafson [8] and Das and Vatsala [6] wherein, among other things they have explicitly computed the Green's function associated with the BVP (1.1), (1.2C).

Further in [29] assuming the linear differential operator L defined by

$$Ly \equiv y^{(n)} + p_1(x)\, y^{(n-1)} + \ldots + p_n(x)y$$

($p_i(x)$, $i = 1, \ldots, n$ are continuous on I) is disconjugate on I, Schmitt has given sufficient conditions for the existence of a solution for the BVP $Ly = f(x,y)$ and (1.2R), with f continuous on $I \times \mathbb{R}$ in terms of some algebraic and differential inequalities to be satisfied by a pair of auxiliary functions $u(x)$ and $v(x)$.

Sufficient conditions for the existence of a solution to the BVP $y^{(n)} = f(x,y)$ and (1.2C) and sufficient conditions for the existence as well as uniqueness of solution of the same BVP for $k = 2$ entirely in terms of f have been given by Schrader and Umamaheswaram in [32].

There is also a result concerning differential inequalities that is of immediate interest to us, namely the one due to Schrader [30] giving necessary and sufficient conditions for lower solutions of $y'' = f(x,y,y')$ to be subfunctions with respect to solutions of the above differential equation.

This thesis is motivated mainly by the above mentioned results for focal and conjugate BVP's. For other related results concerning these BVP's reference can be made to [1, 9, 10, 13, 14, 15, 16, 17, 18, 19, 20, 23, 24, 27, 31] and to other references contained therein. We now mention the results obtained in this thesis chapterwise.

In Chapter II we consider the differential equation

$$y'' = f(x,y,y') \tag{2.1}$$

along with the 'right focal', 'left focal' and conjugate BC's denoted respectively by

$$y(x_1) = y_1 \quad , \quad y'(x_2) = y_2 \tag{2.2R}$$

$$y'(x_1) = y_1 \quad , \quad y(x_2) = y_2 \tag{2.2L}$$

$$y(x_1) = y_1 \quad , \quad y(x_2) = y_2 \tag{2.2C}$$

where $x_1 < x_2$, $x_1, x_2 \in I$, an interval in $\mathbb{R}$ and $y_1, y_2 \in \mathbb{R}$ are arbitrary and along with some of the following hypotheses which, for the sake of convenience we label as follows.

A. f is continuous on $I \times \mathbb{R}^2$.

UR. Solutions of right focal boundary value problems (BVP's) of (2.1) if they exist are unique on I (that is ' $y(x)$, $z(x)$ are solutions of the BVP (2.1), (2.2R) for arbitrary x_1, x_2 in I, $x_1 < x_2$ and $y_1, y_2 \in \mathbb{R}'$ implies $y(x) \equiv z(x)$ on $[x_1,x_2]$).

UL. S olutions of left focal BVP's of (2.1) if they exist are unique on I (that is ' $y(x)$, $z(x)$ are solutions of the BVP (2.1), (2.2L) for arbitrary x_1 , x_2 in I , $x_1 < x_2$ and y_1 , $y_2 \in \mathbb{R}'$ implies $y(x) \equiv z(x)$ on $[x_1,x_2]$).

UC. Solutions of conjugate BVP's of (2.1) if they exist are unique on I (that is ' $y(x)$, $z(x)$ are solutions of the BVP (2.1), (2.2C) for arbitrary x_1 , x_2 in I, $x_1 < x_2$ and y_1 , $y_2 \in \mathbb{R}'$ implies $y(x) \equiv z(x)$ on $[x_1,x_2]$).

E. All solutions of (2.1) exist on I.

We also need the following well-known definitions which we state here for the sake of reference.

<u>Definition 1.1</u>. $u(x) \in C^2(I)$ is a lower solution of (2.1) if $u''(x) \geq f(x, u(x), u'(x))$ for all x in I and $v(x) \in C^2(I)$ is an upper solution of (2.1) if $v''(x) \leq f(x, v(x), v'(x))$ for all x in I.

<u>Definition 1.2</u>. $u(x) \in C(I)$ is a ' conjugate subfunction with respect to solutions of (2.1) on I ' if the inequality $u(x) \leq y(x)$ holds on $[x_1, x_2]$ whenever $u(x_1) \leq y_1$, $u(x_2) \leq y_2$ holds and $y(x)$ is a solution of the BVP (2.1), (2.2C) for arbitrary $x_1, x_2 \in I$, $x_1 < x_2$ and $y_1, y_2 \in \mathbb{R}$.

In order to motivate further discussion we recall the following two theorems which are due to Jackson and Schrader respectively.

<u>Theorem J1 (Theorem 3.2, [15])</u>. If $u \in C^2(I)$ is a conjugate subfunction with respect to solutions of (2.1) on I, then it is a lower solution of (2.1) on I.

It may be noted that in [15], u is assumed to be in $C(I) \cap C^1(I^o)$. Thus Theorem J1 is a special version of Theorem 3.2 of [15].

<u>Theorem S1 (Theorem 1, [30])</u>. Assume hypotheses A, UC and E hold on I. Then lower solutions of (2.1) on I are conjugate subfunctions on I.

We also need the following two definitions for further discussion.

<u>Definition 1.3</u>. $u \in C^1(I)$ is a ' right focal subfunction with respect to solutions of (2.1) on I ' if the inequality $u(x) \leq y($ holds on $[x_1,x_2]$ whenever $u(x_1) \leq y_1$, $u'(x_2) \leq y_2$ holds and $y(x)$ is a solution of the BVP (2.1), (2.2R) for arbitrary $x_1, x_2 \in I$, $x_1 < x_2$ and $y_1, y_2 \in \mathbb{R}$.

<u>Definition 1.4</u>. $u \in C^1(I)$ is a 'left focal subfunction with respect to solutions of (2.1) on I ' if the inequality $u(x) \leq y(x,$ holds on $[x_1,x_2]$ whenever $u'(x_1) \geq y_1$, $u(x_2) \leq y_2$ holds and $y(x)$ is a solution of the BVP (2.1), (2.2L) for arbitrary $x_1, x_2 \in I$, $x_1 < x_2$ and $y_1, y_2 \in \mathbb{R}$.

Hereafter for the sake of brevity we shall simply use the term lower solution (right focal subfunction) of equation (2.1) omitting the words 'with respect to (with respect to solutions of) equation (2.1)'.

It follows (Lemma 2.2) from the above definition that a right focal subfunction on an interval I is necessarily a conjugate subfunction on I and hence a C^2 - right focal subfunction on I is a lower solution on I by Theorem J1. Furthe it is shown on page 17 by means of an example that even in the case of a linear differential equation satisfying hypotheses A, UC and E, a lower solution need not be a right focal subfunction.

So now one can raise the question whetner under the stronger hypotheses A, UR (For the result UR implies UC refer to Lemma 2.6) and E lower solutions of (2.1) on an interval I, are right focal subfunctions on I. We answer this question in the affirmative in Theorem 2.12.

We further show in Theorem 2.18, that if I is an interval which is open at the left end point then under hypotheses A, UR and E the BVP (2.1), (2.2R) has a solution for x_1 , $x_2 \in$ I , $x_1 < x_2$ and y_1, $y_2 \in \mathbb{R}$ arbitrary. This theorem removes the restrictions of I being open at the ri..ht end point and the uniqueness of solutions of IVP's but yields the same conclusion as that of Lasota and Lucszynski in [23] and Henderson (Theorem 3 with n = 2, [12]). We also show on page 34 by means of an example that the above stated theorem is not true if I is a closed interval. However it remains an open question whether the theorem is true or not if I is closed at the left end point, but open at the right end point.

In Chapter III, we consider the differential equation

$$y^{(n)} = 0 \tag{3.1}$$

along with the ' k - point conjugate homogeneous ' and ' k - point right focal homogeneous ' BC's denoted respectively by

$$y^{(i)}(x_r) = 0 \quad , \quad i = 0, \ldots, n(r) - 1 \qquad r = 1, \ldots, k \tag{3.2C}$$

$$y^{(i)}(x_r) = 0 \quad , \quad i = s(r-1), \ldots, s(r) - 1 \qquad r = 1, \ldots, k \tag{3.2R}$$

where k, $n(r)$, $s(r)$ and x_r, $r = 1, \ldots, k$ are as in (1.2R).

Concerning the BVP (3.1) , (3.2C), the Green's function for the case $k = n$ has been given by Beesack [4] by means of a formula involving induction on n. Further Gustafson [8] has proved that the Green's function for arbitrary k can be obtained from that for $k = n$ by a suitable limiting process. Also it has been shown by Beesack [4] that the Green's function in the case $k = n$ satisfies an inequality commonly referred to thereafter as Beesack's inequality (The same inequality was also obtained later by Nehari [26] by a different method). In addition it has been shown (Proposition 13, Ch.3 of [5]) that a function $u(x) \in C^n [x_1,x_k]$ satisfying $u^{(n)}(x) > 0$ on $[x_1,x_k]$ and the BC's (3.2C) also satisfies certain inequalities commonly known as Caplygin's inequalities.

However there does not appear to be much literature available on the Green's function or Beesack and Caplygin type inequalities associated with the focal BVP (3.1), (3.2R) for arbitrary k $(1 < k \leq n)$.

In Theorem 3.1 we obtain explicitly the Green's function $G(x,t)$ for the BVP (3.1),(3.2R) and show that $G(x,t)$ and its partial derivatives $G^{(i)}(x,t)$ with respect to x satisfy Beesack type inequalities. As illustrations, we give $G(x,t)$ and Beesack type inequalities for all focal BVP's in the cases $n = 2$ and $n = 3$.

We also determine (Theorem 3.2) for each $r = 1,\ldots,k - 1$ and $i = s(r - 1), \ldots, n - 1$ the sign of $G^{(i)}(x,t)$ on the $x - t$ strip $[x_1,x_k] \times [x_r,x_{r+1}]$ and show (Corollary 3.4) that for $i = s(r - 1), \ldots, s(r) - 1$, $(-1)^{n - s(r)} G^{(i)}(x,t)$ is positive semidefinite on the

$x - t$ strip $[x_r, x_{r+1}] \times [x_1, x_k]$, $r = 1, \ldots, k-1$. However if $i > s(r) - 1$ or $i < s(r-1)$ the last mentioned statement need not hold. This is illustrated (Remark following Corollary 3.4) by showing that in the case $n = k = 3$ and $r = 2$ if $x_2 - x_1 < x_3 - x_2$ the signs of $G(x_3, t)$, $x_1 < t < x_2$ and $G(x_3, x_3)$ are 1 and -1 respectively. We also give right focal analogue of Caplygin's inequalities (Corollary 3.5), namely if $u(x) \in C^n[x_1, x_k]$ satisfies $u^{(n)}(x) > 0$ on $[x_1, x_k]$ and the BC's (3.2R) then $(-1)^{n - s(r)} u^{(i)}(x) > 0$ for $x_r < x < x_{r+1}$, $i = s(r-1), \ldots, s(r) - 1$ and $r = 1, \ldots, k-1$.

In Chapter IV we consider the BVP's (1.1), (1.2R); (1.1) and

$$\begin{aligned} y^{(i)}(x_1) &= y_{1\,i}, \quad i = 0, \ldots, m-1 \\ y^{(i)}(x_2) &= y_{2\,i}, \quad i = m, \ldots, n-1 \end{aligned} \qquad (4.1R)$$

where f is continuous on $[a,b] \times \mathbb{R}^n$, $1 \leq m < n$ is an arbitrary integer and x_1 , $x_2 \in [a,b]$ $(x_1 < x_2)$, $y_{1\,i}$, $y_{2\,i}$ are arbitrary real numbers.

Using the Beesack type inequalities for the Green's function and its derivatives obtained in Chapter III, we obtain a local existence theorem (Theorem 4.1) for the BVP (1.1), (1.2R). This leads to the global existence theorem (Corollary 4.2) for the BVP (1.1), (1.2R).

In Theorem 4.4, we give sufficient conditions for the existence of a solution to the BVP (1.1), (4.1R) in terms of f and two auxiliary functions $u(x)$, $v(x)$ satisfying certain algebraic and differential inequalities. This theorem is a 2 - point right focal analogue of Theorem 3.1 of [29]. However it remains unknown whether an analogous theorem for k - point right focal BVP's ($k \geq 3$) is true or not. On the other hand in Theorem 4.6 we give sufficient conditions for the existence of a solution to the BVP (1.1), (4.1R) entirely in terms of f. This theorem is a 2 - point right focal analogue of Theorem 2.2 of [32]. Whether the k - point right focal analogue ($k \geq 3$) of this theorem is true or not is also an open question.

In Chapter V, we consider a special type of differential equation

$$y^{(n)} = f(x, y, \ldots, y^{(m)}) \qquad (5.1)_m$$

along with the BC's

$$\begin{aligned} y^{(i)}(x_1) &= c_i , \quad i = 0, \ldots, n-2 \\ y^{(r)}(x_2) &= d \end{aligned} \qquad (5.2)_r$$

or

$$y^{(r)}(x_1) = d$$

$$y^{(i)}(x_2) = c_i\ , \quad i = 0, \ldots, n-2 \tag{5.3$_r$}$$

where f is continuous on $[a,b] \times \mathbb{R}^{m+1}$, m and r are arbitrary but fixed positive integers satisfying $0 \leq m \leq r \leq n-1$, $a \leq x_1 < x_2 \leq b$ and $c_0, \ldots, c_{n-2}, d \in \mathbb{R}$ are arbitrary.

Assuming that solutions to IVP's of $(5.1)_m$ are unique and f satisfies a monotonicity condition with respect to $y, y', \ldots, y^{(m)}$, we prove (Theorems (5.1) and (5.2)) that for fixed $x_1, x_2, c_0, \ldots, c_{n-2}$ the BVP $(5.1)_m$, $(5.2)_r$ is either solvable for all d or unsolvable for any d. Thus this result generalizes Theorem 3.2 of [32] for 2 - point conjugate BC's to 2 - point BC's of the form $(5.2)_r$. Further this theorem is different from the uniqueness - existence result (Corollary 4.13) of [2] for the same BVP $(5.1)_m$, $(5.2)_r$ in the sense that the monotonicity properties of f assumed in [2] are opposite to those of f assumed here.

Lastly we show by means of two examples with $n = 2$ and $n = 3$ that the conclusion of Theorem 5.2 need not hold if f does not satisfy the monotonicity property with respect to even one of the variables.

CHAPTER II

FOCAL SUBFUNCTIONS AND SECOND ORDER DIFFERENTIAL INEQUALITIES

In this chapter we are interested in the differential equation

$$y'' = f(x,y,y') \qquad (2.1)$$

along with the ri_ht focal, left focal and conjugate BC's denoted respectively by

$$y(x_1) = y_1 \quad , \quad y'(x_2) = y_2 \qquad (2.2R)$$

$$y'(x_1) = y_1 \quad , \quad y(x_2) = y_2 \qquad (2.2L)$$

$$y(x_1) = y_1 \quad , \quad y(x_2) = y_2 \qquad (2.2C)$$

where $x_1 < x_2$, x_1 , $x_2 \in I$, an interval in $\mathbb{R}$ and y_1 , $y_2 \in \mathbb{R}$ are arbitrary.

In one of the two main theorems (Theorem 2.12) of this chapter we shall prove that under the hypotheses A, UR and E lower solutions of (2.1) on an interval I are right focal subfunctions on I. This theorem is proved by means of a 'Local existence' theorem (Theorem 2.10) for the BVP (2.1), (2.2R), the result (Corollary 2.11) that lower solutions of (2.1) under the hypothesis UR are right focal subfunctions in the 'small' and an induction argument similar to that used in the proof of Theorem S1 but using hypothesis UR rather than UC .

In the other main theorem (Theorem 2.12) we shall prove that if I is an interval open at the left end point then under the hypotheses A, UR and E the BVP (2.1), (2.2R) has a solution for all $x_1, x_2 \in I$, $x_1 < x_2$ and $y_1, y_2 \in \mathbb{R}$. This theorem is proved by means of shooting method technique.

Now we shall prove some preliminary results which are useful in proving the main theorems of this chapter.

It will be assumed that hypothesis A holds throughout this chapter.

Lemma 2.1. Suppose $u \in C^1[c,d]$, $u(c) \geq 0$, $u(d) \geq 0$ and $u(x) < 0$ for some x, $c < x < d$. Then there exists an interval $[x_1,x_2] \subset [c,d)$ such that $u(x_1) = 0$, $u'(x_2) = 0$ and $u(x) < 0$ on (x_1,x_2).

Proof. By hypotheses $u(x)$ attains its negative minimum, at some point, say $x_2 \in (c,d)$. Let $x_1 = \text{Sup}\{c < x < x_2 : u(x) = 0\}$. Then $[x_1,x_2] \subset [c,d)$ and $u(x)$ satisfies $u(x_1) = 0$, $u'(x_2) = 0$ and $u(x) < 0$ on (x_1,x_2).

Lemma 2.2. If $u \in C^1(I)$ is a right focal subfunction on I, then u is a conjugate subfunction on I.

Proof. Suppose $u(x)$ satisfies the condition $u(x_1) \leq y_1$, $u(x_2) \leq y_2$ for some $x_1, x_2 \in I$, $x_1 < x_2$ and $y_1, y_2 \in \mathbb{R}$ and $y(x)$ is a solution of the BVP (2.1), (2.2C). If

$u(x) \leq y(x)$ does not hold on $[x_1, x_2]$ then there exists x', $x_1 < x' < x_2$ such that $u(x') > y(x')$ and consequently by Lemma 2.1 an interval $[x_3, x_4] \subset [x_1, x_2)$ such that $u(x_3) = y(x_3)$, $u'(x_4) = y'(x_4)$ and $u(x) > y(x)$ on (x_3, x_4). This contradicts the hypothesis that $u(x)$ is a right focal subfunction on I.

Corollary 2.3. If $u \in C^2(I)$ is a right focal subfunction on I, then u is a lower solution on I.

Proof. This is a consequence of Lemma 2.2 above and Theorem J1 of Chapter I.

However the converse of Corollary 2.3 need not be true even in the case of linear differential equations satisfying hypotheses A, UC and E as shown by the following example.

Example. Consider the equation $y'' + y = 0$, $0 \leq x \leq 3\pi/4$. Then $y(x) \equiv \text{Sin } x$ and $z(x) \equiv 0$ are both solutions satisfying the same right focal BC's $y(0) = 0$, $y'(\pi/2) = 0$ and hence not all lower solutions are right focal subfunctions on $[0, 3\pi/4]$.

Lemma 2.4. If $u \in C^1[x_1, x_2]$ and attains a minimum at a point x_0, $x_1 < x_0 \leq x_2$ then (i) $u'(x_0) = 0$ if $x_1 < x_0 < x_2$ and (ii) $u'(x_0) \leq 0$ if $x_0 = x_2$.

Lemma 2.5. Suppose $u \in C^1[c,d]$, $u(c) \geq 0$ and $u(x) < 0$ for some x in $(c,d]$. Then there exists a subinterval $[c_1,d_1] \subset [c,d]$ such that $u(c_1) = 0$, $u'(d_1) \leq 0$ and $u(x) < 0$ on (c_1,d_1).

Proof. To see this, let $c < d_1 \leq d$ be such that $u(x)$ attains its negative minimum on $[c,d]$ at d_1 and $c_1 = \text{Sup}\{c \leq x < d_1 : u(x) = 0\}$. Now the conclusion is obvious by virtue of Lemma 2.4.

Lemma 2.6. UR implies UC.

Proof. Let $y(x)$, $z(x)$ be two solutions of the BVP (2.1) (2.2C) for some $x_1, x_2 \in I$, $x_1 < x_2$ and $y_1, y_2 \in \mathbb{R}$. If $y(x) \not\equiv z(x)$ on $[x_1,x_2]$ then by Rolle's theorem we can suppose without loss of generality that there exists a largest x_3 , $x_1 < x_3 < x_2$ such that $y'(x_3) = z'(x_3)$ and consequently $y(x) \equiv z(x)$ on $[x_1,x_3]$. Another application of Rolle's theorem yields x_4 , $x_3 < x_4 < x_2$ such that $y'(x_4) = z'(x_4)$ and consequently $y(x) \equiv z(x)$ on $[x_1,x_4]$, a contradiction to the maximality of x_3. Hence $y(x) \equiv z(x)$ on $[x_1,x_2]$.

Lemma 2.7. Suppose $y(x)$, $z(x)$ are solutions of (2.1) satisfying the hypotheses UR and E. If $y(x_1) = z(x_1)$, $y'(x_2) > z'(x_2)$ for some x_1 , x_2 in I and $x_1 \leq x_2$ then $y'(x) > z'(x)$ for all $x \geq x_2$, $x \in I$.

Proof. If $y'(x') \le z'(x')$ for some $x' > x_2$ then there exists x_3, $x_2 < x_3 \le x'$ such that $y'(x_3) = z'(x_3)$. Hence we have $y(x) \equiv z(x)$ on $[x_1, x_3]$, a contradiction to hypothesis at x_2.

<u>Lemma 2.8</u>. Suppose $y(x)$, $z(x)$ are solutions of (2.1) satisfying the hypotheses UR and E. If $y(x_1) = z(x_1)$ and $y(x_2) > z(x_2)$ for some x_1, x_2 in I and $x_1 < x_2$ then $y'(x) > z'(x)$ for all $x \ge x_2$, $x \in I$.

Proof. By our hypotheses and Mean Value Theorem, there exists x_3, $x_1 < x_3 < x_2$ such that $y'(x_3) > z'(x_3)$. Now the conclusion follows from Lemma 2.7.

<u>Lemma 2.9</u>. Suppose $y(x)$, $z(x)$ are solutions of (2.1) satisfying the hypotheses UR and E. If $y(x_1) = z(x_1)$, $y(x_2) \ge z(x_2)$ and $y'(x_2) \le z'(x_2)$ for some x_1, x_2 in I and $x_1 < x_2$ then $y(x_2) = z(x_2)$ holds and hence $y(x) \equiv z(x)$ on $[x_1, x_2]$.

Proof. Suppose $y(x_2) > z(x_2)$ holds. Then by Lemma 2.8 and hypotheses we must have $y'(x_2) > z'(x_2) \ge y'(x_2)$, a contradiction. Hence $y(x_2) = z(x_2)$ holds and then the conclusion follows from Lemma 2.6.

<u>Theorem 2.10 (Local existence theorem)</u>.

Let $M > 0$, $N > 0$ be given. Let q be the maximum of $|f(x,y,y')|$ on the compact set $\{(x,y,y') : a \leq x \leq b,$ $|y| \leq 2M, |y'| \leq 2N\}$. Assume $q > 0$ and $\delta = \text{Min}\{\sqrt{(2M/q)}, N/q\}$. Then

(i) The BVP (2.1), (2.2R) with $[x_1,x_2] \subset [a,b]$, $x_2 - x_1 \leq \delta$, $|y_1| \leq M$, $|y_2| \leq N$, $|y_1 + y_2(x_2 - x_1)| \leq M$ has a solution $y(x)$.

(ii) If $\varepsilon > 0$ is given, $\delta^* = \text{Min}\{\delta, \varepsilon/q, \sqrt{(2\varepsilon/q)}\}$, $[x_1,x_2] \subset [a,b]$, $x_2 - x_1 < \delta^*$ and $\omega(x)$ is the unique linear function satisfying $\omega(x_1) = y_1$, $\omega'(x_2) = y_2$ then the BVP (2.1), (2.2R) has a solution $y(x)$ satisfying $|y(x) - \omega(x)| < \varepsilon$ and $|y'(x) - \omega'(x)| < \varepsilon$ on $[x_1,x_2]$.

Proof. The proof of this theorem is a standard application of Schauder's fixed point theorem and hence is not given here. However we remark that the estimate for δ, given in the theorem is arrived at by using the following estimates of Green's function and integrals involving the Green's function, namely

$$|G(x,t)| \leq (x_2 - x_1), \quad \int_{x_1}^{x_2} |G(x,t)|\, dt \leq (x_2 - x_1)^2/2 \text{ and}$$

$$\int_{x_1}^{x_2} |G_x(x,t)|\, dt \leq (x_2 - x_1).$$

<u>Remark</u>. The above theorem remains valid if the BC's (2.2R) are replaced by (2.2L) , the inequalities in (i) are replaced by $|y_1| \leq N$, $|y_2| \leq M$, $|y_2 - y_1(x_2 - x_1)| \leq M$ and $\omega(x)$ in (ii) is assumed to be the unique linear function satisfying $\omega'(x_1) = y_1$, $\omega(x_2) = y_2$.

<u>Note</u>. If in the above theorem $q = 0$, that is $f(x,y,y') \equiv 0$ for $a \leq x \leq b$, $|y| \leq 2M$, $|y'| \leq 2N$ then $y(x)=\omega(x)$ is the solution of the BVP (2.1) and (2.2R) ((2.2L)) for arbitrary x_1 , $x_2 \in [a,b]$, y_1, $y_2 \in \mathbb{R}$.

The following corollary which implies that lower solutions under hypothesis UR are right focal subfunctions 'in the small' is used frequently in the proof of Theorem 2.12.

<u>Corollary 2.11</u>. Let $u \in C^1[a,b]$ be given. Then

(i) there exists a $\delta > 0$ such that for $[x_1,x_2] \subset [a,b]$ and $x_2 - x_1 \leq \delta$, there exists a solution $y(x)$ of the BVP (2.1) and $y(x_1) = u(x_1)$, $y'(x_2) = u'(x_2)$.

Further

(ii) given $\varepsilon > 0$, $[x_1,x_2] \subset [a,b]$, and $\omega(x)$ the unique linear function satisfying $\omega(x_1) = u(x_1)$, $\omega'(x_2) = u'(x_2)$ there exists δ^* , $0 < \delta^* \leq \delta$ such that $x_2 - x_1 \leq \delta^*$ implies the BVP (2.1) and $y(x_1) = u(x_1)$ $y'(x_2) = u'(x_2)$ has a solution $y(x)$ satisfying $|y^{(i)}(x) - \omega^{(i)}(x)| < \varepsilon$ on $[x_1,x_2]$ for $i = 0,1$.

(iii) If in addition $u(x)$ is a lower solution of (2.1) on $[a,b]$ then there exists a δ^{**}, $0 < \delta^{**} \leq \delta$ such that the BVP in (i) has a solution $y(x)$ satisfying $u(x) \leq y(x)$ on $[x_1,x_2]$ provided $x_2 - x_1 \leq \delta^{**}$.

Proof. (i) and (ii) follow from Theorem 2.10, if we choose

$$M = \mathrm{Max}\{|u(x)| : a \leq x \leq b\} + (b-a)\,\mathrm{Max}\{|u'(x)| : a \leq x \leq b\},$$
$$N = \mathrm{Max}\{|u'(x)| : a \leq x \leq b\},$$
$$u(x_1) = y_1 \quad \text{and} \quad u'(x_2) = y_2 .$$

(iii) If we define

$$F(x,y,y') = \begin{cases} f(x,y,y') & y \geq u(x) \\ & a \leq x \leq b \\ f(x,u(x),y') - (u(x) - y) & y \leq u(x) \\ & a \leq x \leq b \end{cases}$$

then by (i) there exists δ' (depending on F), $0 < \delta' \leq b-a$ such that $[x_1,x_2] \subset [a,b]$ and $x_2 - x_1 \leq \delta'$ implies that the BVP

$$y'' = F(x,y,y')$$
$$y(x_1) = u(x_1)\ ,\ y'(x_2) = u'(x_2)$$

has a solution $y(x)$.

Suppose $u(x) > y(x)$ for some $x \in (x_1,x_2]$. Then there exists a point x_0, $x_1 < x_0 \leq x_2$ such that $u(x) - y(x)$

will have a positive maximum at x_o with $u'(x_o) = y'(x_o)$ and $u''(x_o) \leq y''(x_o)$. Consequently

$$\begin{aligned} y''(x_o) &= F(x_o\,,\, y(x_o)\,,\, y'(x_o)) \\ &= f(x_o\,, u(x_o)\,, u'(x_o)) - (u(x_o) - y(x_o)) \\ &< f(x_o\,, u(x_o)\,, u'(x_o)) \\ &\leq u''(x_o)\ , \quad \text{a contradiction.} \end{aligned}$$

Hence $u(x) \leq y(x)$ on $[x_1,x_2]$ implying that $y(x)$ is a solution of the equation (3.1) on $[x_1,x_2]$. Now choose $\delta^{**} = \min(\delta\,,\delta')$ where δ is as in (i). This completes the proof of (iii).

<u>Theorem 2.12</u>. Assume the hypotheses A , UR and E hold. Then lower solutions of (2.1) on I are right focal subfunctions on I.

Proof. Suppose $u(x)$ is a lower solution on I , but not a right focal subfunction on I. Then by Lemmas 2.4, 2.5, and definition of right focal subfunction, there exists an interval $[c,d] \subset I$ and a solution $y_1(x)$ of (2.1) such that $y_1(c) = u(c)$, $y_1'(d) = u'(d)$ and $y_1(x) < u(x)$ on (c,d).

Now since $u \in C^1[c,d]$ is a lower solution on $[c,d]$, by Corollary 2.11 (iii) there exists a δ , $0 < \delta < d - c$ such that for $[x_1,x_2] \subset [c,d]$ and $x_2 - x_1 \leq \delta$ the BVP (2.1) and $y(x_1) = u(x_1)$, $y'(x_2) = u'(x_2)$ has a solution $y(x)$

satisfying $u(x) \leq y(x)$ on $[x_1,x_2]$ (that is, $u(x)$ is a right focal subfunction ' in the small ').

Now for each positive integer n, let $P(n)$ be the proposition that there exists an interval $[c_n,d_n] \subset [c,d]$ with $0 < d_n - c_n \leq d - c - (n-1)\delta$ and a solution $y_n(x)$ with $y_n(c_n) = u(c_n)$, $y_n'(d_n) = u'(d_n)$ and $y_n(x) < u(x)$ on (c_n,d_n). Obviously $P(n)$ cannot be true for all $n \geq 1$. However assuming that $u(x)$ is not a right focal subfunction on I, we will show by an induction argument that $P(n)$ is true for all n, thereby proving that $u(x)$ must be a right focal subfunction on I.

$P(1)$ is true since we can choose $[c_1,d_1] = [c,d]$ with $y_1(x)$ same as above.

Now assume $P(k)$ is true, that is, there exist an interval $[c_k,d_k] \subset [c,d]$ with $0 < d_k - c_k \leq d - c - (k-1)\delta$ and a solution $y_k(x)$ with $y_k(c_k) = u(c_k)$, $y_k'(d_k) = u'(d_k)$ and $y_k(x) < u(x)$ on (c_k,d_k). Now $d_k - c_k > \delta$ as otherwise by Corollary 2.11 (iii), there exists a solution distinct from $y_k(x)$ for the BVP (2.1) and $y(c_k) = u(c_k)$, $y'(d_k) = u'(d_k)$.

Let $z_1(x)$ be the solution of the BVP (2.1) and $y(c_k) = u(c_k)$, $y'(c_k + \delta) = u'(c_k + \delta)$ so that by Corollary 2.11 (iii)

$u(x) \leq z_1(x)$ on $[c_k , c_k + \delta]$ and hence $y_k(x) < z_1(x)$ on $(c_k , c_k + \delta]$. By Lemma 2.9, hypotheses UR and E, we have $z_1'(x) > y_k'(x)$ for all $x > c_k$ and hence $z_1'(d_k) > y_k'(d_k) = u'(d_k)$.

Now assuming $P(k + 1)$ not true, we first prove

Claim (i). $z_1(x) \geq u(x)$ on $[c_k + \delta , d_k]$.

For, if not there exists x', $c_k + \delta < x' \leq d_k$ such that $\operatorname{Min}\{z_1(x) - u(x) : c_k + \delta \leq x \leq x \leq d_k\} = z_1(x') - u(x') < 0$. However if $x' = d_k$ then $z_1'(d_k) - u'(d_k) \leq 0$ by (ii) of Lemma 2.4, a contradiction. Hence $c_k + \delta < x' < d_k$ which in turn implies by (i) of Lemma 2.4 that $P(k + 1)$ is true with $y_{k+1}(x) \equiv z_1(x)$ and $[c_{k+1} , d_{k+1}] \subsetneqq [c_k + \delta , d_k]$, an interval of length $d_k - c_k - \delta \leq d - c - k\delta$. Hence the claim is true.

Claim (ii). $d_k - (c_k + \delta) > \delta$.

For if otherwise the focal BVP (2.1) and $y(c_k + \delta) = u(c_k + \delta)$, $y'(d_k) = u'(d_k)$, by Corollary 2.11 (iii) will have a solution $z_2(x)$ with $u(x) \leq z_2(x)$ on $[c_k + \delta , d_k]$ and consequently $u'(c_k + \delta) \leq z_2'(c_k + \delta)$. By hypothesis E, $z_2(x)$ exists on I and hence either (a) there exists x', $c_k \leq x' < c_k + \delta$ such that $z_2(x') = y_k(x')$ or (b) there exists a largest x'', $c_k < x'' \leq c_k + \delta$ such that $z_2(x'') = z_1(x'')$.

If case (a) occurs, then by Lemmas 2.7 and 2.8, we must have $z_2'(d_k) > y_k'(d_k) = u'(d_k)$, a contradiction.

On the other hand if case (b) occurs again there are two possibilities, namely $x'' < c_k + \delta$ and $x'' = c_k + \delta$. If $x'' < c_k + \delta$ then since $z_1(c_k + \delta) \geq z_2(c_k + \delta)$ and $z_1'(c_k + \delta) = u'(c_k + \delta) \leq z_2'(c_k + \delta)$, we have by Lemma 2.9 that

$$w_o(x) \equiv \begin{cases} z_1(x) & c_k \leq x \leq c_k + \delta \\ z_2(x) & c_k + \delta \leq x \leq d_k \end{cases}$$

and $y_k(x)$ are two distinct solutions of the same right focal BVP, a contradiction to UR. If $x'' = c_k + \delta$ then $z_1'(c_k + \delta) = u'(c_k + \delta) \leq z_2'(c_k + \delta)$. However the strict inequality in the above statement is ruled out by virtue of the fact that $z_1'(d_k) > z_2'(d_k)$ and by Lemma 2.7. Hence $z_1'(c_k + \delta) = z_2'(c_k + \delta)$ and consequently

$$z(x) \equiv \begin{cases} z_1(x) & c_k \leq x \leq c_k + \delta \\ z_2(x) & c_k + \delta \leq x \leq d_k \end{cases}$$

and $y_k(x)$ are two distinct (distinct since $y_k(x) < u(x) \leq z_1(x)$ on $(c_k, c_k + \delta)$) solutions of the same right focal BVP, a contradiction to UR. Hence claim (ii) is true.

Now there exists by Corollary 2.11 (iii) a solution $z_2(x)$ of (2.1) such that $z_2(c_k + \delta) = u(c_k + \delta)$, $z_2'(c_k + 2\delta) = u'(c_k + 2\delta)$ and $u(x) \le z_2(x)$ on $[c_k + \delta, c_k + 2\delta]$. Consequently $z_2'(c_k + \delta) \ge u'(c_k + \delta) = z_1'(c_k + \delta)$. By hypothesis E, $z_2(x)$ exists on I and hence either (a) there exists an x', $c_k \le x' \le c_k + \delta$ such that $z_1(x') = z_2(x')$ or (b) there exists a largest x'', $c_k < x'' < c_k + \delta$ such that $z_2(x'') = y_k(x'')$.

If case (a) occurs let

$$w_1(x) = \begin{cases} z_1(x) & x \le c_k + \delta \\ z_2(x) & x \ge c_k + \delta \end{cases}$$

so that by Lemma 2.9 $w_1(x)$ is a solution of (2.1) satisfying $w_1(c_k) = u(c_k)$ and $w_1'(c_k + 2\delta) = u'(c_k + 2\delta)$. If case (b) occurs let $w_1(x) \equiv z_2(x)$, for $x \ge x''$.

Claim (iii). $d_k - (c_k + 2\delta) > \delta$.

Suppose $d_k - (c_k + 2\delta) \le \delta$. Then by Corollary 2.11(iii) there exists a solution $z_3(x)$ of equation (2.1) satisfying $z_3(c_k + 2\delta) = u(c_k + 2\delta)$, $z_3'(d_k) = u'(d_k)$ and $z_3(x) \ge u(x)$ on $[c_k + 2\delta, d_k]$ and consequently $z_3'(c_k + 2\delta) \ge u'(c_k + 2\delta) = w_1'(c_k + 2\delta)$ and $z_3(c_k + 2\delta) > y_k(c_k + 2\delta)$. By hypothesis E,

$z_3(x)$ exists on I and hence either (a) there exists an x', $c_k \leq x' \leq c_k + 2\delta$ such that $z_3(x') = w_1(x')$ or (b) there exists a largest x''', $c_k < x''' < c_k + 2\delta$ such that $z_3(x''') = y_k(x''')$.

If (a) occurs, we can consider as in the proof of claim (ii) the two possibilities $x' < c_k + 2\delta$ and $x' = c_k + 2\delta$ and in either case by using an argument iaentical to that in claim (ii), we can arrive at a contradiction to UR.

If (b) occurs, choose $w_2(x) \equiv z_3(x)$ for $x \geq x'''$. Then by Lemma 2.8, $w_2'(d_k) > y_k'(d_k) = u'(d_k) = w_2'(d_k)$, a contradiction and hence the claim.

Let $j \geq 0$ be the unique integer such that $c_k + j\delta < d_k \leq c_k + (j+1)\delta$. Now repeating the above steps a finite number of times, we arrive at a solution $w_j(x)$ of equation (2.1) satisfying $w_j(c_k + j\delta) = u(c_k + j\delta)$, $w_j'(d_k) = u'(d_k)$ and $w_j(x) \geq u(x)$ on $[c_k + j\delta, d_k]$. Consequently $w_j'(c_k + j\delta) \geq u'(c_k + j\delta) = w_{j-1}'(c_k + j\delta)$ where $w_{j-1}(x)$ is the solution obtained in the previous step of the proof. (Note that $w_{j-1}(x)$ satisfies that $w_{j-1}(x^*) = y_k(x^*)$ for some $c_k \leq x^* < c_k + (j-1)\delta, w_{j-1}(c_k+(j-1)\delta) =$ $u(c_k + (j-1)\delta)$, $w_{j-1}'(c_k + j\delta) = u'(c_k + j\delta)$ and $w_{j-1}(x) \geq u(x)$ on $[c_k + (j-1)\delta, c_k + j\delta]$). Since $w_j(x)$ extends to I by E, either (a) there exists x',

$c_k \leq x' \leq c_k + j\delta$ such that $w_j(x') = w_{j-1}(x')$, or (b) there exists a largest x^{**}, $x^* < x^{**} < c_k + (j-1)\delta$ such that $w_j(x^{**}) = y_k(x^{**})$.

If case (a) occurs, considering the two possibilities $x' < c_k + j\delta$ and $x' = c_k + j\delta$, we can arrive at a contradiction to UR by using Lemmas 2.7 and 2.9 as in the proof of claim (ii).

If case (b) occurs then the function $w_{j+1}(x)$ defined by $w_{j+1}(x) \equiv w_j(x)$, $x \geq x^{**}$ must satisfy by Lemmas 2.7, 2.8 $w'_{j+1}(d_k) > y'_k(d_k) = u'(d_k) = w'_{j+1}(d_k)$ a contradiction. This contradiction shows that $P(k+1)$ must be true and hence $P(n)$ is true for all n. This completes the proof of the theorem.

An analogue of Theorem 2.12 for left focal subfunctions is true and is stated below. Its proof is similar to that of Theorem 2.12 and hence is omitted.

<u>Theorem 2.13</u>. Assume that hypotheses A, UL and E hold. Then lower solutions of (2.1) on I are left focal subfunctions on I.

Now we give the analogous definitions and results for the right (left) focal superfunctions of (2.1).

<u>Definition 2.14</u>. $v(x) \in C^1(I)$ is said to be a 'right focal

superfunction of (2.1) on I' if the inequality $v(x) \geq y(x)$ holds on $[x_1,x_2]$ whenever $v(x_1) \geq y_1$, $v'(x_2) \geq y_2$ holds and $y(x)$ is a solution of the BVP (2.1) , (2.2R) for arbitrary x_1 , $x_2 \in I$, $x_1 < x_2$ and y_1 , $y_2 \in \mathbb{R}$.

Definition 2.15. $v(x) \in C^1(I)$ is said to be a 'left focal superfunction of (2.1) on I' if the inequality $v(x) \geq y(x)$ holds on $[x_1,x_2]$ whenever $v'(x_1) \leq y_1$, $v(x_2) \geq y_2$ holds and $y(x)$ is a solution of the BVP (2.1) , (2.2L) for arbitrary x_1 , $x_2 \in I$, $x_1 < x_2$ and y_1 , $y_2 \in \mathbb{R}$.

Theorem 2.16. Assume hypotheses A, UR and E hold. Then upper solutions of (2.1) on I are right focal superfunctions on I.

Theorem 2.17. Assume hypotheses A , UL and E hold. Then upper solutions of (2.1) on I are left focal superfunctions on I.

Now for the sake of convenience of reference we restate the existence theorems for the BVP's (2.1), (2.2R) and (2.1), (2.2C) which are due to Klaasen and Schrader respectively. These are useful in proving the 'uniqueness implies existence' theorem (Theorem 2.18) for the BVP (2.1) , (2.2R).

Theorem K1 (Theorem 5, [21]). Assume the hypothesis A holds and solutions of IVP's for (2.1) extend to $[x_1,x_2]$ or become unbounded. Then a necessary and sufficient condition that the BVP (2.1) , (2.2R) has a solution is that there are lower and upper solutions Φ and Ψ of (2.1) satisfying $\Phi \leq \Psi$ on $[x_1, x_2]$, $\Phi(x_1) \leq y_1 \leq \Psi(x_1)$ and $\Phi'(x_2) \leq y_2 \leq \Psi'(x_2)$. In the sufficiency part, the solution y satisfies $\Phi \leq y \leq \Psi$ on $[x_1, x_2]$.

<u>Theorem S2 (Theorem 3.1,[31])</u>. Assume the hypothesis A holds and solutions of IVP's for (2.1) extend to $[x_1, x_2]$ or become unbounded. Then a necessary and sufficient condition that the BVP (2.1), (2.2C) has a solution is that there are lower and upper solutions Φ and Ψ of (2.1) satisfying $\Phi \leq \Psi$ on $[x_1, x_2]$, $\Phi(x_1) \leq y_1 \leq \Psi(x_1)$ and $\Phi(x_2) \leq y_2 \leq \Psi(x_2)$. In the sufficiency part, the solution y satisfies $\Phi \leq y \leq \Psi$ on $[x_1, x_2]$.

<u>Theorem 2.18</u>. Let I be an interval open at the left end point. Assume hypotheses A, UR, E hold on I. Then the BVP (2.1), (2.2R) has a solution where $x_1 < x_2$, $x_1, x_2 \in I$ and $y_1, y_2 \in \mathbb{R}$ are arbitrary.

Proof. Let $x_1, x_2 \in I$ and $y_1 \in \mathbb{R}$ be arbitrary but fixed.

Let $S = \{\gamma \in \mathbb{R} : y(x_1) = y_1,\ y'(x_2) = \gamma$ and $y(x)$ is a solution of (2.1)$\}$.

Clearly S is non empty by hypothesis E. Now to prove the theorem it suffices to show $S = \mathbb{R}$. We do this through the following claims.

Claim 1. S is connected.

For suppose γ_1, $\gamma_2 \in S$, $\gamma_1 < \gamma_2$ and $\gamma_1 < \gamma' < \gamma_2$ is arbitrary. Let $z_i(x)$ be the solution of (2.1), (2.2R) with $y_2 = \gamma_i$, $i = 1,2$. Then by Lemma 2.8 we have $z_2(x) \geq z_1(x)$ for all $x > x_1$, $x \in I$. Applying Theorem K1 with $\Phi = z_1$ and $\Psi = z_2$ we obtain that there exists a solution of the BVP (2.1), (2.2R) with $y_2 = \gamma'$, so $\gamma' \in S$ and hence the claim.

Let $\beta_o = \text{Sup } S$ and $\gamma_o = \text{Inf } S$. Now to show $S = \mathbb{R}$ it suffices to show $\beta_o = +\infty$ and $\gamma_o = -\infty$. We will only prove $\beta_o = +\infty$, since the other proof is similar.

Claim 2. $\beta_o \notin S$.

For if not, suppose $y_o(x)$ is the solution of (2.1) and (2.2R) with $y_2 = \beta_o$. Then the solution $z(x)$ of the IVP (2.1) and

$$y(x_1) = y_1 \text{ , } y'(x_1) = y_o'(x_1) + 1$$

satisfies by Lemma 2.7 that $z'(x_2) > y_o'(x_2) = \beta_o$, a contradiction and hence the claim.

Now let $y_o(x)$ be the solution of the IVP (2.1) and

$$y(x_1) = y_1 \text{ , } y'(x_1) = 1$$

and $z_o(x)$ be the solution of the IVP (2.1) and

$$y(x_2) = y_o(x_2) + 1 \text{ , } y'(x_2) = \beta_o$$

Now $y_o(x)$ and $z_o(x)$ exist on I, $z_o(x_1) \neq y_1$ and $y_o'(x_2) < \beta_o$.

Claim 3. $z_o(x_1) < y_1$.

Suppose if possible $z_o(x_1) > y_1$. This implies $z_o(x) \geq y_o(x)$ for $x \geq x_1$, for if otherwise there exists x' , $x_1 < x' < x_2$ such that $z_o(x') < y_o(x')$, and then $y_o(x) - z_o(x)$ will have at least one zero on each of the intervals (x_1,x') and (x',x_2) , a contradiction to the conclusion of Lemma 2.6. Now $y_o(x) \leq z_o(x)$ on $[x_1,x_2]$ with $y_o(x_1) = y_1 < z_o(x_1)$ and $y_o'(x_2) < \beta_o = z_o'(x_2)$. So choosing $\Phi = y_o$ and $\Psi = z_o$ in Theorem K1 , we obtain that there exists a solution $w(x)$ of the BVP (2.1), (2.2R) with $y_2 = \beta_o$ satisfying $y_o(x) \leq w(x) \leq z_o(x)$ on $[x_1,x_2]$. This implies $\beta_o \in S$, a contradiction to claim 2. Hence claim 3 is true.

By claim 3, the fact that $z_o(x_2) > y_o(x_2)$ and hypothesis UC , it follows that there exists x' , $x_1 < x' < x_2$ such that $z_o(x') = y_o(x')$, $z_o(x) \leq y_o(x)$ for $x < x'$, $x \in I$ with $z_o(x) < y_o(x)$ for $x_1 - \varepsilon \leq x \leq x_1$ where $\varepsilon > 0$ is sufficiently small. Now by Theorem S2 and Lemma 2.6 the BVP (2.1) and

$$\begin{aligned} . \quad y(x_1 - \varepsilon) &= z_o(x_1 - \varepsilon) \\ y(x_1) &= y_1 \end{aligned}$$

has a solution $w(x)$ satisfying $z_o(x) \leq w(x) \leq y_o(x)$ on $[x_1 - \varepsilon, x_1]$ and hence by Lemma 2.8 $w'(x) > z_o'(x)$ for all $x \geq x_1$, $x \in I$. In particular $w'(x_2) > z_o'(x_2) = \beta_o$. This contradicts the definition of β_o since $w'(x_2) \in S$. This completes the proof of the theorem.

The conclusion of Theorem 2.18 need not hold if I is a closed interval, as shown by the following example. We now state a lemma which is used in the example.

<u>Lemma 2.19</u>. The equation $y'' = -y$ is right disfocal on the interval $[0, \pi/2)$ ($(0, \pi/2]$) (that is, $y(x_1) = 0$, $y'(x_2) = 0$, $0 \leq x_1 < x_2 < \pi/2$, ($0 < x_1 < x_2 \leq \pi/2$) and $y(x)$ is a solution of the above equation implies $y(x) \equiv 0$) and hence UR holds for the above equation on $[0, \pi/2)$ ($(0, \pi/2]$).

<u>Example</u>. Consider the differential equation

$$y'' = -y + \arctan y \tag{2.3}$$

with $-\pi/2 < \arctan y < \pi/2$ and $I = [0, \pi/2]$.

The hypotheses A, UC and E hold for equation (2.3) on $[0, \pi]$ as shown on page 347 of [15].

We first claim that UR holds for (2.3) on $[0, \pi/2)$ and $(0, \pi/2]$. If it does not hold on $[0, \pi/2)$ suppose

$y(x)$, $z(x)$ are two distinct solutions of (2.3) satisfying $y(x_1) = z(x_1)$, $y'(x_2) = z'(x_2)$ for some x_1 , x_2 , $0 \le x_1 < x_2 < \pi/2$. Since UC holds for (2.3) on $[0,\pi/2]$, we can assume without loss of generality that $y(x) > z(x)$ on $(x_1,\pi/2]$. Now let $w(x) = y(x) - z(x)$ so that we have $w(x_1) = 0$, $w'(x_2) = 0$, $w(x) > 0$ and $w''(x) > -w(x)$ on $(x_1,\pi/2]$ as shown on page 347 of [15]. Consequently by Theorem 2.12 and Lemma 2.19 , $w(x)$ is a right focal subfunction with respect to solutions of the equation

$$y'' = -y \tag{2.4}$$

on $[x_1, \pi/2]$ and hence $w(x) \le 0$ on $[x_1,x_2]$, a contradiction. This proves that UR holds for (2.3) on $[0,\pi/2)$. The proof for $(0 , \pi/2]$ is analogous.

Now to show that UR holds for (2.3) on $[0,\pi/2]$ we only need to show that if $y(x)$, $z(x)$ are solutions of (2.3) satisfying $y(0) = z(0)$, $y'(\pi/2) = z'(\pi/2)$ then $y(x) \equiv z(x)$ on $[0,\pi/2]$. If the above assertion is not true then since UR holds on $[0,\pi/2)$ we can suppose without loss of generality that $y'(x) > z'(x)$ for $0 < x < \pi/2$ and hence $w(x) \equiv y(x) - z(x) > 0$ on $(0 , \pi/2]$. Thus $w(x)$ attains its positive maximum on $[0, \pi/2]$ only at $x = \pi/2$, yielding $w(\pi/2) > 0$ and $w'(\pi/2) = 0$. Further $w''(x) > -w(x)$ on $(0 , \pi/2]$.

Now let $u(x)$ be the solution of the IVP (2.4) and $y(\pi/2) = w(\pi/2)$, $y'(\pi/2) = w'(\pi/2) = 0$. Since $w''(\pi/2) > -w(\pi/2) = -u(\pi/2) = u''(\pi/2)$, it follows that $(w - u)$ has a relative minimum at $x = \pi/2$, that is, $(w - u)(x) > 0$ for $0 < \pi/2 - x < \pi/2$, sufficiently small. Since by Theorem S1 of Chapter I, $w(x)$ is a conjugate subfunction with respect to solutions of (2.4) on $[0, \pi/2]$, we must have $(w - u)(x) \neq 0$, for $0 \leq x < \pi/2$ and hence $(w - u)(x) > 0$ for $0 \leq x < \pi/2$. In particular $u(0) < w(0) = 0$ whereas $u(\pi/2) = w(\pi/2) > 0$. Therefore $u(x') = 0$ for some x', $0 < x' < \pi/2$. This together with $u'(\pi/2) = 0$ implies by Lemma 2.19 that $u(x) \equiv 0$ on $[x', \pi/2]$. In particular $u(\pi/2) = 0$, a contradiction. This shows that UR holds on $[0, \pi/2]$.

Claim. The BVP (2.3) and $y(0) = 0$, $y'(\pi/2) = 3\pi$ has no solution.

Suppose on the contrary the above BVP has a solution $y(x)$ and $y'(0) = m$. Let $v(x)$ be the solution of the IVP

$$v'' = -v + \pi$$
$$v(0) = 0, \quad v'(0) = m + 1 .$$

As shown on page 347 of [15], $v(x)$ is a lower solution of equation (2.3) on $[0, \pi/2]$, satisfying

$v(0) = y(0)$, $v(x) > y(x)$ for $0 < x$ sufficiently small. Consequently by Theorem 2.12, $v(x)$ is a right focal subfunction of (2.3) on $[0, \pi/2]$. This together with the fact $v(0) = y(0)$ must imply $v'(x) > y'(x)$ on $(0, \pi/2]$ and in particular $v'(\pi/2) > y'(\pi/2) = 3\pi$.

However an easy computation yields $v(x) = (m+1)\sin x - \pi\cos x + \pi$, whereby we get $3\pi < v'(\pi/2) = \pi$, a contradiction. Hence the claim.

An analogue of Theorem 2.18 for left focal BVP's is stated below as Theorem 2.20. The proof of this theorem is similar to that of Theorem 2.18 except that in this case we have to use, instead of Theorem K1 its analogue to left focal BVP's. Since this is not explicitly stated in [21], we first state it here for the sake of completeness.

Theorem K2. Assume the hypothesis A holds and solutions of IVP's for (2.1) extend to $[x_1, x_2]$ or become unbounded. Then a necessary and sufficient condition that the BVP (2.1), (2.2L) has a solution is that there are lower and upper solutions Φ and Ψ satisfying $\Phi \le \Psi$ on $[x_1, x_2]$, $\Psi'(x_1) \le y_1 \le \Phi'(x_1)$ and $\Phi(x_2) \le y_2 \le \Psi(x_2)$. In the sufficiency part, the solution y satisfies $\Phi \le y \le \Psi$ on $[x_1, x_2]$.

<u>Theorem 2.20</u>. Let I be an interval open at the right end point. Assume the hypotheses A, UL and E hold on I. Then the BVP (2.1), (2.2L) has a solution where $x_1 < x_2$, $x_1, x_2 \in I$, $y_1, y_2 \in \mathbb{R}$ are arbitrary.

<u>Remark</u>. The conclusion of the above theorem need not hold if I is a closed interval and whether the theorem is true or not if I is closed at the right end point and open at the left end point remains an open question.

CHAPTER III

GREEN'S FUNCTIONS FOR k - POINT FOCAL BOUNDARY VALUE PROBLEMS

In this chapter we consider the k-point right focal BVP

$$y^{(n)} = 0 \tag{3.1}$$

$$y^{(i)}(x_r) = 0\ ,\quad i = s(r-1)\ ,\ \ldots,\ s(r) - 1 \qquad r = 1,\ \ldots,\ k \tag{3.2R}$$

where k, $n(r)$, $s(r)$ and x_r, $r = 1, \ldots, k$ are as in (1.2R).

We obtain explicitly the Green's function $G(x,t)$ for the BVP (3.1), (3.2R) and prove that $G(x,t)$ and its partial derivatives $G^{(i)}(x,t)$ with respect to x satisfy Beesack type inequalities. We also determine the sign of $G^{(i)}(x,t)$, $i = 0, \ldots, n-1$ on appropriate subsets of $[x_1,x_k] \times [x_1,x_k]$ and give right focal analogue of Caplygin's inequalities.

We state at the outset that the characterization of the Green's function for the BVP (3.1), (3.2R) is same as given by Coppel (page 115, [5]) for conjugate BVP's except that in the present case the conjugate BC's of [5] have to be replaced by focal BC's (3.2R). Further it follows as in [5] that if $\{y_1(x), \ldots, y_n(x)\}$ is a basis of solutions of (3.1) then $G(x,t)$ can be uniquely represented in the form

$$G(x,t) = \begin{cases} \sum_{i=1}^{n} \alpha_i(t) y_i(x) & x \leq t \\ \sum_{i=1}^{n} (\alpha_i(t) + \gamma_i(t)) y_i(x) & t \leq x \end{cases} \qquad (3.3)$$

where $G(x,t)$ is defined by the following two properties.

(i) For each fixed t, $x_1 < t < x_k$, $G(x,t)$ is a solution of equation (3.1) on each of the intervals $[x_1, t)$ and $(t, x_k]$ satisfying the conditions

$$G^{(i)}(x_r, t) = 0 \ , \quad i = s(r-1), \ldots, s(r) - 1$$
$$r = 1, \ldots, k.$$

(ii) For each fixed t, $G(x,t)$ and its first $n-2$ derivatives with respect to t are continuous at $x = t$, while

$$G^{(n-1)}(t+0, t) - G^{(n-1)}(t-0, t) = 1.$$

In (3.3), $(\gamma_1(t), \ldots, \gamma_n(t))$ is the unique solution of the linear system of equations

$$\left.\begin{aligned}&\sum_{i=1}^{n} \gamma_i(t) y_i^{(j)}(t) = 0\ , \quad j = 0, \ldots, n-2\\&\sum_{i=1}^{n} \gamma_i(t) y_i^{(n-1)}(t) = 1\end{aligned}\right\} \tag{3.4}$$

and $(\alpha_1(t), \ldots, \alpha_n(t))$ is the solution of the system of equations that can be obtained by using property (i) of $G(x,t)$.

For the sake of convenience in further discussions we use the following notation. S denotes the $x - t$ square $[x_1,x_k] \times [x_1,x_k]$; for each r, $1 \leq r \leq k - 1$, S_r denotes the $x - t$ strip $[x_1,x_k] \times [x_r, x_{r+1}]$ so that $S = \bigcup \{ S_r : 1 \leq r \leq k - 1\}$; T_r denotes the $x - t$ strip $[x_r,x_{r+1}] \times [x_1,x_k]$; for each r, $1 \leq r \leq k - 1$, let

$$y_i(x;r) = (x - x_r)^{i-1} / (i-1)!, \quad i = 1, \ldots, n, \tag{3.5}$$

$$\gamma_i(t;r) = (-1)^{n-i}(t - x_r)^{n-i} / (n-i)!, \ i = 1, \ldots, n, \tag{3.6}$$

$\alpha_i(t;r)$ be defined recursively as follows;

$$(t;r) = \begin{cases} -\gamma_i(t;r) & s(r)+1 \le i \le s(k) = n \\ 0 & s(r-1)+1 \le i \le s(r) \\ -\sum_{j=i+1}^{n} \alpha_j(t;r)(x_{r-1}-x_r)^{j-i}/(j-i)! & s(r-2)+1 \le i \le s(r-1) \\ \cdot & \\ \cdot & \\ \cdot & \\ -\sum_{j=i+1}^{n} \alpha_j(t;r)(x_1-x_r)^{j-i}/(j-i)! & 1 \le i \le s(1) \end{cases} \quad (3.7)$$

$$(x,t;r) = \begin{cases} \sum_{i=1}^{n} \alpha_i(t;r) y_i(x,r) & x \le t \\ \sum_{i=1}^{n} (\alpha_i(t;r) + \gamma_i(t;r)) y_i(x;r) & t \le x \end{cases} \quad (3.8)$$

In (3.7), those equations which give the values of α_i for $i \le s(m)$ where $m \le 0$ are to be treated as nonexistent.

<u>Theorem 3.1</u>. If $G(x,t)$ is the Green's function for the BVP (3.1), (3.2R) then $G(x,t) = G(x,t;r)$, $(x,t) \in S$, $r = 1, \ldots, k-1$.

Proof. Keep $1 \leq r \leq k-1$ fixed. Note that $\{y_i(x;r),\ i = 1,\ldots,n\}$ is a basis of solutions for the equation (3.1) and hence can be substituted for $\{y_1(x),\ldots,y_n(x)\}$ in (3.3). This substitution makes the coefficient matrix of the system (3.4) upper triangular since the equations (3.4) reduce to the following.

$$\begin{aligned}
t) + \gamma_2(t)(t-x_r) + \cdots \qquad & + \gamma_n(t)(t-x_r)^{n-1} / (n-1)! = 0 \\
\gamma_2(t) + \gamma_3(t)(t-x_r) + \cdots \qquad & + \gamma_n(t)(t-x_r)^{n-2} / (n-2)! = 0 \\
\cdots \qquad \cdots \qquad & \cdots \\
\cdots \qquad & \cdots \\
& \cdots \\
& \gamma_n(t) = 1 .
\end{aligned}$$

Solving the above system we obtain $\gamma_i(t) = \gamma_i(t;r)$, $i = 1, \ldots, n$. Now the property (i) yields a linear system of equations for $(\alpha_1(t), \ldots, \alpha_n(t))$ whose coefficient matrix is also upper triangular and which are as shown below.

$$0 = G(x_1,t) = \alpha_1(t) + \alpha_2(t)(x_1-x_r) + \cdots + \frac{\alpha_n(t)(x_1-x_r)}{(n-1)!}$$

$$0 = G'(x_1,t) = 0 + \alpha_2(t) + \alpha_3(t)(x_1-x_r) + \cdots + \frac{\alpha_n(t)(x_1-x_r)^{n-2}}{(n-2)!}$$

$$\vdots$$

$$0 = G^{(s(r-1)-1)}(x_{r-1},t) = 0 + \cdots + \alpha_{s(r-1)}(t) + \cdots + \frac{\alpha_n(t)(x_{r-1}-x_r)^{n-s(r-1)}}{(n-s(r-1))!}$$

$$0 = G^{(s(r-1))}(x_r,t) = 0 + \cdots + \alpha_{s(r-1)+1}(t) + \cdots + 0$$

$$\vdots$$

$$0 = G^{(s(r)-1)}(x_r,t) = 0 + \cdots \cdots 0 + \cdots + \alpha_{s(r)}(t) + 0 + \cdots + 0$$

$$0 = G^{(s(r))}(x_{r+1},t) = 0 + \cdots \cdots \cdots + (\alpha_{s(r)+1}(t) + \gamma_{s(r)+1}(t)) + \cdots + \frac{(\alpha_n(t)+\gamma_n(t)(x_{r+1}-x_r)^{(n-s(r)-1)}}{(n-s(r)-1)!}$$

$$\vdots$$

$$0 = G^{(n-1)}(x_k,t) = 0 + \cdots \cdots \cdots \cdots \cdots + (\alpha_n(t) + \gamma_n(t))$$

Hence we have

$$\alpha_i(t) = \begin{cases} -\gamma_i(t) & s(r) + 1 \leq i \leq s(k) = n \\ 0 & s(r-1) + 1 \leq i \leq s(r) \end{cases}$$

Now for $s(r-2)+1 \leq i \leq s(r-1)$, $\alpha_i(t)$ is computed as follows.

$$\alpha_{s(r-1)}(t) = -\alpha_{s(r-1)+1}(t)(x_{r-1}-x_r) - \cdots \quad \cdots$$

$$\cdots - \alpha_n(t)(x_{r-1}-x_r)^{n-s(r-1)} / (n-s(r-1))!$$

$$= -\alpha_{s(r)+1}(t)(x_{r-1}-x_r)^{n(r)+1} / (n(r)+1)! - \cdots \quad \cdots$$

$$\cdots - \alpha_n(t)(x_{r-1}-x_r)^{n-s(r-1)} / (n-s(r-1))!$$

$$\alpha_{s(r-1)-1}(t) = -\alpha_{s(r-1)}(t)(x_{r-1}-x_r) - \alpha_{s(r)+1}(x_{r-1}-x_r)^{n(r)+2} / (n(r)+2)! - \cdots$$

$$\cdots - \sigma_n(t)(x_{r-1}-x_r)^{n-s(r-1)+1} / (n-s(r-1)+1)!$$

.
.
.

$$\alpha_{s(r-2)+1}(t) = -\alpha_{s(r-2)+2}(t)(x_{r-1}-x_r) - \cdots \quad \cdots$$

$$\cdots - \alpha_{s(r-1)}(t)(x_{r-1}-x_r)^{n(r-1)-1} / (n(r-1)-1)! -$$

$$\alpha_{s(r)+1}(t)(x_{r-1}-x_r)^{n(r-1)+n(r)} / (n(r-1)+n(r))!$$

$$- \cdots - \alpha_n(t)(x_{r-1}-x_r)^{n-s(r-2)-1} / (n-s(r-2)-1)!$$

Hence $\alpha_i(t) = -\sum_{j=i+1}^{n} \alpha_j(t)(x_{r-1}-x_r)^{j-i} / (j-i)!$, $s(r-2)+1 \leq i \leq s(r-1)$

Similarly

$$\alpha_i(t) = -\sum_{j=i+1}^{n} \alpha_j(t)(x_{r-2} - x_r)^{j-i} \,/\, (j-i)! \quad , \quad s(r-3)+1 \le i \le s(r-2)$$

$$\vdots$$

$$\alpha_i(t) = -\sum_{j=i+1}^{n} \alpha_j(t)(x_1 - x_r)^{j-i} \,/\, (j-i)! \quad , \quad 1 \le i \le s(1)$$

Thus we have $\alpha_i(t) = \alpha_i(t;r)$, $i = 1, \ldots, n$ and hence the theorem.

Remark 1. In view of the above theorem, to compute $G(x,t)$ on S, it suffices to compute $G(x,t;r)$ on S_r for $r = 1, \ldots, k-1$.

Remark 2. In the case $k = n$, the expressions for $\alpha_i(t,r)$ in equations (3.7) reduce to

$$\alpha_i(t;r) = \begin{cases} -\gamma_i(t;r) & r+1 \le i \le n \\ 0 & i = r \\ -\sum_{j=i+1}^{n} \alpha_j(t;r)(x_i - x_r)^{j-i} \,/\, (j-i)! & 1 \le i \le r-1 \end{cases}$$

whereas $\gamma_i(t;r)$ and $y_i(x;r)$ for $i = 1, \ldots, n$ are the same as already stated and hence $G(x,t)$ for the n-point focal BVP can be computed on $[x_1,x_n] \times [x_1,x_n]$ using Remark 1.

Now we shall obtain Beesack type inequalities for $G(x,t)$ and ts derivatives with respect to x.

From the equations (3.5), (3.6) and (3.7) it can be easily seen that for each $r = 1, \ldots, k-1$, $i = 1, \ldots, n$ the following inequalities hold.

$$y_i^{(j)}(x;r)| \leq (x_k - x_1)^{i-j-1} / (i - j - 1)! \quad , x_1 \leq x \leq x_k \; ,$$
$$j = 0, \ldots, i - 1 \; ,$$

$$\gamma_i(t;r)| \leq (x_k - x_1)^{n-i} / (n - i)! \quad , x_1 \leq t \leq x_k$$

and

$$\alpha_i(t;r)| \leq C_{i\;k\;n} (x_k - x_1)^{n-i} \quad , x_1 \leq t \leq x_k$$

where $C_{i\;k\;n}$ are non-negative constants depending on i, k and n that can be computed for fixed k and n recurssively with respect to i. Consequently for $(x,t) \in S$ and $i = 1, \ldots, n - 1$, we obtain from e uation 3.8 that

$$|G^{(i-1)}(x,t)| \leq D_{i\;k\;n} (x_k - x_1)^{n-i}$$

Whereas for $(x,t) \in S$, $x \neq t$ (3.9)

$$|G^{(n-1)}(x,t)| \leq D_{i\;k\;n}$$

where $D_{i\;k\;n}$ are some non-ne_ative constants depending on k and n and can be computed in terms of $C_{i\;k\;n}$ for given k and n.

We now state $G(x,t)$ and Beesack type inequalities in the following cases.

Case (i) $n = 2$, $k = 2$, $n(1) = n(2) = 1$, $S = [x_1,x_2] \times [x_1,x_2]$.

Then for $(x,t) \in S$,

$$G(x,t) = \begin{cases} -(x - x_1) & x_1 \leq x \leq t \\ -(t - x_1) & t \leq x \leq x_2 \end{cases}$$

$|G(x,t)| \leq x_2 - x_1$ and for $(x,t) \in S$, $x \neq t$ $|G'(x,t)| \leq 1$.

Case (ii) $n = 3$, $k = 2$, $n(1) = 2$, $n(2) = 1$, S same as in case (i). Then for $(x,t) \in S$,

$$G(x,t) = \begin{cases} -(x - x_1)^2/2 & x_1 \leq x \leq t \\ (t - x_1)^2/2 - (t - x_1)(x - x_1) & t \leq x \leq x_2 \end{cases}$$

$|G(x,t)| \leq (x_2 - x_1)^2/2$; $|G'(x,t)| \leq (x_2 - x_1)$ and for $(x,t) \in S$, $x \neq t$, $|G''(x,t)| \leq 1$.

Case (iii) $n = 3$, $k = 2$, $n(1) = 1$, $n(2) = 2$, S same as in case (i).

Then for $(x,t) \in S$,

$$G(x,t) = \begin{cases} (t - x_1)(x - x_1) - (x - x_1)^2/2 & x_1 \leq x \leq t \\ (t - x_1)^2/2 & t \leq x \leq x_2 . \end{cases}$$

The bounds of G , G' , G'' are the same as in case (ii).

Case (iv) $n = 3,\ k = 3,\ n(1) = n(2) = n(3) = 1$,

$S = [x_1, x_3] \times [x_1, x_3]$. Then for $(x,t) \in S$,

$$G(x,t) = \begin{cases} (t-x_1)(x-x_1) - (x-x_1)^2/2 & x_1 \le x \le t \\ (t-x_1)^2/2 & t \le x \le x_3 \end{cases} \quad x_1 \le t \le x_2$$

$$ \begin{cases} (x_1-x_2)^2/2 - (x-x_2)^2/2 & x_1 \le x \le t \\ (x_1-x_2)^2/2 + (t-x_2)^2/2 - (t-x_2)(x-x_2) & t \le x \le x_3 \end{cases} \quad x_2 \le t \le x_3$$

$|G(x,t)| \le 3(x_3 - x_1)^2/2$; $|G'(x,t)| \le x_3 - x_1$ and for $(x,t) \in S$, $x \ne t$ $|G''(x,t)| \le 1$.

The following theorem determines the signs of $G^{(i)}(x,t)$, on the strip S_r for each fixed r, $1 \le r \le k-1$ and $i = s(r-1), \ldots, n-1$.

Theorem 3.2. Let $G(x,t)$ be as in Theorem 3.1. Then the following inequalities hold.

(a) $(-1)^{n-s(r)}\, G^{(i)}(x,t) > 0$, for $(x,t) \in S_r$, $x > x_r$,
$i = s(r-1), \ldots, s(r) - 1$, $1 \le r \le k-1$.

(b) $(-1)^{n-i}\, G^{(i)}(x,t) > 0$, for $(x,t) \in S_r$, $x < x_r$,
$i = s(r-1), \ldots, s(r) - 1$, $1 < r \le k-1$.

(c) $G^{(i)}(x,t) = 0$, for $(x,t) \in S_r$, $x \ge t$,
$i = s(r), \ldots, n-1$, $1 \le r \le k-1$, $(i,x) \ne (n-1, t)$.

(d) $(-1)^{n-i}\, G^{(i)}(x,t) > 0$, for $(x,t) \in S_r$, $x < t$,
$i = s(r), \ldots, n-1$, $1 \le r \le k-1$.

Proof. From Theorem 3.1 and equations 3.5 – 3.8 , we obtain after $s(r) - 1$ differentiations for $(x,t) \in S_r$, $x > t$,

$$G^{(s(r)-1)}(x,t) = (\alpha_{s(r)}(t) + \gamma_{s(r)}(t)) + (\alpha_{s(r)+1}(t) + \gamma_{s(r)+1}(t))(x - x_r) + \dots + (\alpha_n(t) + \gamma_n(t))(x - x_r)^{n-s(r)} / (n - s(r))!$$

$$= \gamma_{s(r)}(t)$$

$$= (-1)^{n-s(r)} (t - x_r)^{n-s(r)} / (n - s(r))!$$

and $G^{(i)}(x,t) = 0$, $i = s(r), \dots , n - 1$. Hence we have the inequality in (a) for $i = s(r) - 1$, $x \geq t$ and the inequalities (c).

Similarly for $(x,t) \in S_r$ and $x < t$, we have

$$G^{(s(r)-1)}(x,t) = \alpha_{s(r)}(t) + \alpha_{s(r)+1}(t)(x - x_r) + \dots \dots + \alpha_n(t)(x - x_r)^{n-s(r)} / (n - s(r))!$$

$$= - \gamma_{s(r)+1}(t)(x - x_r) - \dots \dots - \gamma_n(t)(x - x_r)^{n-s(r)} / (n - s(r))!$$

$$= - (-1)^{n-s(r)-1}(t - x_r)^{n-s(r)-1}(x - x_r) / (n-s(r)-1) \dots - (x - x_r)^{n-s(r)} / (n - s(r))!$$

Adding and subtracting $(-1)^{n-s(r)}(t - x_r)^{n-s(r)} / (n - s(r))!$ to the above expression on the RHS we have

$$G^{(s(r)-1)}(x,t) = (-1)^{n-s(r)+1}(t-x)^{n-s(r)} / (n-s(r))! +$$

$$(-1)^{n-s(r)}(t-x_r)^{n-s(r)} / (n-s(r))! \ .$$

Consequently we must have $(-1)^{n-s(r)} G^{(s(r)-1)}(x,t) > 0$ for $x_r < x < t$, and $(-1)^{n-s(r)+1} G^{(s(r)-1)}(x,t) > 0$ for $x_1 \leq x < x_r$. This completes the proof of inequalities in (a) and (b) for $i = s(r) - 1$. Further differentiation of $G^{(s(r)-1)}(x,t)$ for $x < t$, a suitable number of times with respect to x yields the inequalities (d).

The inequalities in (a) and (b) for $i = s(r-1),\ldots,s(r)-2$ are obtained by integrating with respect to x sufficient number of times the inequalities in (a) and (b) for $i = s(r) - 1$ respectively and using $G^{(i)}(x_r,t) = 0$, $i = s(r-1),\ldots,s(r)-2$. This completes the proof of the theorem.

In the following Theorem, we determine the signs of $G^{(i)}(x,t)$ for $0 \leq i \leq s(r-1)$ on certain subsets of S_r depending on i.

<u>Theorem 3.3</u>. Let $G(x,t)$ be as in Theorem 3.1, $1 \leq r \leq k-1$ be fixed. Then we have

$$\text{(i) } \operatorname{Sgn} G^{(i)}(x,t) = \begin{cases} (-1)^{n-s(j-1)} & \text{if } x_{j-1} < x < x_j \\ (-1)^{n-i} & \text{if } x_1 < x < x_{j-1} \end{cases}$$

for $s(j-2) \leq i \leq s(j-1) - 1$, $j = r, r-1, \ldots, 3$, $(x,t) \in S_r$ and

(ii) $\operatorname{Sgn} G^{(i)}(x,t) = (-1)^{n - s(1)}$ if $x_1 < x < x_2$ for
$0 \leq i \leq s(1) - 1$, $(x,t) \in S_1$.

Proof. From (b) of Theorem 3.2, we have

$(-1)^{n - s(r - 1)} G^{(s(r - 1))}(x,t) > 0$ for $(x,t) \in S_r$, $x < x_r$.

So it follows from

$$G^{(s(r-1)-1)}(x,t) = G^{(s(r-1)-1)}(x_{r-1},t) + \int_{x_{r-1}}^{x} G^{(s(r-1))}(s,t)ds$$

that

$$\operatorname{Sgn} G^{(s(r-1)-1)}(x,t) = \begin{cases} (-1)^{n-s(r-1)} & \text{if } x_{r-1} < x < x_r \\ (-1)^{n-s(r-1)+1} & \text{if } x_1 < x < x_{r-1} \end{cases}$$

Now $G^{(s(r-1)-2)}(x,t) = G^{(s(r-1)-2)}(x_{r-1}, t) + \int_{x_{r-1}}^{x} G^{(s(r-1)-1)}(s,t)ds$

and hence

$$\operatorname{Sgn} G^{(s(r-1)-2)}(x,t) = \begin{cases} (-1)^{n-s(r-1)} & \text{if } x_{r-1} < x < x_r \\ (-1)^{n-s(r-1)+2} & \text{if } x_1 < x < x_{r-1} \end{cases}$$

Continuing this process, we obtain after a finite number of integrations,

$$\operatorname{Sgn} G^{(s(r-2))}(x,t) = \begin{cases} (-1)^{n-s(r-1)} & \text{if } x_{r-1} < x < x_r \\ (-1)^{n-s(r-2)} & \text{if } x_1 < x < x_{r-1} . \end{cases}$$

This proves the conclusion for $j = r$. Now

$$G^{(s(r-2)-1)}(x,t) = G^{(s(r-2)-1)}(x_{r-2}, t) + \int_{x_{r-2}}^{x} G^{(s(r-2))}(s,t)ds$$

implies by virtue of the conclusion for $j = r$ and the BC's satisfied by $G(x,t)$ at $x = x_{r-2}$ that

$$\operatorname{Sgn} G^{(s(r-2)-1)}(x,t) = \begin{cases} (-1)^{n-s(r-2)} & \text{if } x_{r-2} < x < x_{r-1} \\ (-1)^{n-s(r-2)+1} & \text{if } x_1 < x < x_{r-2}. \end{cases}$$

By further integrations, it follows that

$$\operatorname{Sgn} G^{(i)}(x,t) = \begin{cases} (-1)^{n-s(r-2)} & \text{if } x_{r-2} < x < x_{r-1} \\ (-1)^{n-i} & \text{if } x_1 < x < x_{r-2} \end{cases}$$

for $s(r-3) \leq i \leq s(r-2)-1$ and hence the conclusion for $j = r-1$. The proof for other values of j is similar and hence is omitted.

The following corollary is an immediate consequence of Theorems 3.2 and 3.3.

Corollary 3.4. Let $G(x,t)$ be as in Theorem 3.1 and $1 \leq r \leq k-1$ be fixed. Then

(i) $\operatorname{Sgn} G^{(i)}(x,t) = (-1)^{n-s(r)}$, $(x,t) \in T_r$, $x_r < t$, $s(r-1) \leq i \leq s(r)-1$,

$G^{(i)}(x,t) = 0$, $(x,t) \in T_r$, $t < x_r$, $s(r-1) \leq i \leq s(r)-1$

and

$$\text{(ii) Sgn } G^{(i)}(x,t) = \begin{cases} 0 & , (x,t)\in S\ , x \geq t,\ i = s(k-1),\ldots,n-1 \\ & \quad (i,x) \neq (n-1,\ t) \\ (-1)^{n-i} & , (x,t)\in S\ ,\ x < t\ ,\ i = s(k-1),\ldots,n-1. \end{cases}$$

Remark. If $n \geq k \geq 3$ and $i < s(r-1)$ or $i > s(r) - 1$ then the conclusion (i) of the above Corollary need not hold. For instance in the BVP (3.1) , (3.2R) , let $n = k = 3$, $n(1) = n(2) = n(3) = 1$ and $x_2 - x_1 < x_3 - x_2$. Then $(x,t) = (x_3 , x_1)$ and $(x,t) = (x_3 , x_3) \in T_2$ and $0 < s(1) = 1$. However from the definition of $G(x,t)$ given on page 49 it follows that for $x_1 < t < x_2$, $G(x_3 , t) = (t - x_1)^2/2 > 0$ and $G(x_3 , x_3) = (x_3 - x_1)\ [x_2 - x_1 - (x_3 - x_2)] < 0$.

<u>Corollary 3.5.(Analogues of Caplygin's inequalities)</u>. Let $u(x) \in C^n[x_1,x_k]$ satisfy $u^{(n)}(x) > 0$ for $x_1 \leq x \leq x_k$ and the BC's (3.2R). Then

(i) $\text{Sgn } u^{(i)}(x) = (-1)^{n-s(r)} = \text{Sgn } G^{(i)}(x,t)$ for $x_r < x < x_{r+1}$, $x_1 \leq t \leq x_k$, $s(r-1) \leq i \leq s(r) - 1$ and $1 \leq r \leq k-1$ and

(ii) $\text{Sgn } u^{(i)}(x) = (-1)^{n-i}$, for $x_1 < x < x_k$ and $s(k-1) \leq i \leq n-1$.

Proof. Define $h(x) = u^{(n)}(x)$, $x_1 \leq x \leq x_k$. Then $u(x)$ is the solution of the BVP $y^{(n)} = h(x)$ and (3.2R). Hence

$$u^{(i)}(x) = \int_{x_1}^{x_k} G^{(i)}(x,t)\ h(t)\ dt\ ,\ x_1 \leq x \leq x_k\ ,$$ which implies

$$u^{(i)}(x) = \int_{x_1}^{x_r} G^{(i)}(x,t)\, h(t)\, dt + \int_{x_r}^{x_k} G^{(i)}(x,t)\, h(t)\, dt = \int_{x_r}^{x_k} G^{(i)}(x,t) h(t)\, dt.$$

Consequently by (i) of Corollary 3.4 and the hypothesis that $h(x) > 0$ for $x_1 \le x \le x_k$ we have for $x_r < x < x_{r+1}$ and $s(r-1) \le i \le s(r) - 1$ that $\mathrm{Sgn}\, u^{(i)}(x) = \mathrm{Sgn}\, G^{(i)}(x,t) = (-1)^{n-s(r)}$ for $r = 1,\ldots,k-1$. Also by (ii) of Corollary 3.4 if $s(k-1) \le i \le n-1$ then for $x_1 < x < x_k$ we have

$$u^{(i)}(x) = \int_{x_1}^{x} G^{(i)}(x,t)\, h(t)\, dt + \int_{x}^{x_k} G^{(i)}(x,t)\, h(t)\, dt = \int_{x}^{x_k} G^{(i)}(x,t)\, h(t)\, dt.$$

Hence $\mathrm{Sgn}\, u^{(i)}(x) = \mathrm{Sgn} \int_{x}^{x_k} G^{(i)}(x,t)\, h(t)\, dt = (-1)^{n-i}$.

<u>Remark</u>. The first equation in conclusion (i) of Corollary 3.5 can be considered as the analogue for k - point right focal BVP's of Caplygin's inequalities and can also be obtained directly by integrating with respect to x both sides of the inequality $u^{(n)}(x) > 0$ and using the BC's (3.2R). Thus in the conclusion (i) of the above Corollary, the nonobvious observation is that $u^{(i)}(x)$ and $G^{(i)}(x,t)$ are of the same sign for $x_r < x < x_{r+1}$, $x_1 \le t \le x_k$, $s(r-1) \le i \le s(r) - 1$ and $r = 1, \ldots, k-1$.

<u>Remark</u>. Corollary 3.5 leads us to the following definition and two conjectures concerning the n - th order differential equation

$$y^{(n)} = f(x,y,\ldots,y^{(n-1)}) \tag{1.1}$$

where f satisfies the following hypotheses.

A'. f is continuous on $I \times \mathbb{R}^n$

$U(n(1), \dots, n(k))$ R. Solutions of $(n(1),\dots,n(k))$ right focal BVP's of (1.1) if they exist are unique, i.e. if $y_1(x)$, $y_2(x)$ are solutions of (1.1) such that $y_1(x) - y_2(x)$ satisfies the BC's (3.2R) then $y_1(x) \equiv y_2(x)$ on $[x_1, x_k]$.

E'. All solutions of (1.1) exist on I.

<u>Definition</u>. A function $u \in C^n(I)$ is a '$(n(1), \dots, n(k))$ - right focal subfunction' with respect to solutions of (1.1) on I if whenever $u(x) - y(x)$ satisfies the BC's (3.2R) where $y(x)$ is a solution of (1.1) then we have

$$(-1)^{n - s(r)} (u - y)^{(i)}(x) \geq 0 \tag{3.10}$$

on (x_r, x_{r+1}), $s(r-1) \leq i \leq s(r) - 1$, $r = 1, \dots, k-1$.

In the case $n = 2$ it is shown (Theorem 2.12) that if $u(x)$ is a lower solution on I then under the hypotheses A', U(1,1)R and E', $u(x)$ is a $(1,1)$ - right focal subfunction with respect to solutions of (1.1) on I.

<u>Conjecture 1</u>. In the case $n = 3$, $1 < k \leq 3$ arbitrary and $(n(1), \dots, n(k))$ a fixed k - tuple satisfying $n(1) + \dots + n(k) = 3$, does $u(x) \in C^3(I)$ and

$u'''(x) \geq f(x,u(x),u'(x),u''(x))$ on I imply that u is a $(n(1), \ldots, n(k))$ - right focal subfunction with respect to solutions of (1.1) on I where f is such that the hypotheses A', $U(n(1), \ldots, n(k))R$ and E' are satisfied?

<u>Conjecture 2</u>. In the case of arbitrary n and $(n(1),\ldots,n(k))$ a fixed k - tuple satisfying $n(1) + \ldots + n(k) = n$, does $u(x) \in C^n(I)$ and $u^{(n)}(x) \geq f(x, u(x), \ldots, u^{(n-1)}(x))$ on I imply that u is a $(n(1), \ldots, n(k))$ - right focal subfunction with respect to solutions of (1.1) on I where f is such that the hypotheses A', $U(n(1), \ldots, n(k))R$, E' and (4) (as stated on page 2) are satisfied?

CHAPTER IV

EXISTENCE THEOREMS FOR FOCAL BOUNDARY VALUE PROBLEMS

In this chapter we are interested in the differential equation (1.1) along with the k - point right focal BC's (1.2R) and with the '2 - point right focal' BC's

$$y^{(i)}(x_1) = y_{1\,i}, \quad i = 0, \ldots, m-1$$
$$y^{(i)}(x_2) = y_{2\,i}, \quad i = m, \ldots, n-1 \qquad (4.1R)$$

where $n > 1$ is a fixed positive integer , f is continuous on $[a,b] \times \mathbb{R}^n$, $1 \leq m < n$ is an arbitrary integer and x_1 , $x_2 \in [a,b]$ $(x_1 < x_2)$, $y_{1\,i}$, $y_{2\,i}$ are arbitrary real numbers.

Theorem 4.1 and Corollary 4.2 of this chapter are local and global existence theorems respectively for the BVP (1.1), (1.2R). In Theorem 4.4 we give sufficient conditions in terms of f and some auxiliary functions u , v satisfying certain inequalities for the existence of a solution of the BVP (1.1), (4.1R). We also give in Theorem 4.6 , sufficient conditions entirely in terms of f for the existence of a solution of the BVP (1.1) , (4.1R).

<u>Theorem 4.1 (Local existence theorem).</u> Let $f(x,y,\ldots,y^{(n-1)})$ be continuous on $[a,b] \times \mathbb{R}^n$, $N_1, \ldots, N_n$ be given positive constants, $0 < Q = \max \{ | f(x,y,\ldots,y^{(n-1)}) | : a \le x \le b ,$ $| y^{(i-1)} | \le 2N_i$ for $1 \le i \le n \}$ and $\omega(x)$ be the solution of the BVP (1.1) with $f \equiv 0$ and (1.2R). Let $\delta =$ $\min \{ (N_i | D_i \; Q)^{1|(n-i+1)} : 1 \le i \le n \}$ where $D_i = D_{i\,k\,n}$ of inequalities (3.9). Then the BVP (1.1), (1.2R) has a solution provided $a \le x_1 < \ldots < x_k \le b$, $x_k - x_1 \le \delta$ and $| \omega^{(i-1)}(x) | \le N_i$ on $[x_1,x_k]$ for $1 \le i \le n$.

We omit the proof of this theorem as it is a standard application of Schauder's fixed point theorem. However, it may be noted that the estimate for δ given in the theorem is arrived at by using the Beesack type inequalities for $| G^{(i)}(x,t) |$, $i = 0, \ldots, n-1$ given in (3.9).

<u>Corollary 4.2 (Global existence theorem).</u> Assume $f(x,y,\ldots,y^{(n-1)})$ is continuous and satisfies $| f(x,y,\ldots,y^{(n-1)}) | \le \Phi(M)$ for $a \le x \le b$, $| y^{(i)} | \le M$, $i = 0, \ldots, n-1$ where $\Phi : [0, +\infty) \longrightarrow \mathbb{R}$ is a positive nondecreasing function satisfying $M | \Phi(M) \longrightarrow \infty$ as $M \longrightarrow \infty$. Then the BVP (1.1), (1.2R) has a solution.

Proof. If we define $Q = \max \{ |f(x,y, \ldots, y^{(n-1)})| : a \le x \le b, |y^{(i)}| \le 2M, i = 0, \ldots, n-1 \}$ then by the hypothesis on f we have $Q \le \Phi(2M)$. Since by hypothesis $M / \Phi(2M) \longrightarrow \infty$ as $M \longrightarrow \infty$ we can choose M so large that $(M / D_i \Phi(2M))^{1/(n-i+1)} \ge x_2 - x_1$ for $i = 1, \ldots, n$. Now by Theorem 4.1 the BVP (1.1), (1.2R) has a solution.

<u>Remark 1</u>. In the above corollary if f is such that either

$$|f(x,y, \ldots, y^{(n-1)})| \le h + \sum_{i=1}^{n} k_i \, |y^{(i-1)}|^{\alpha_i}$$

($h > 0$, $k_i > 0$ and $0 \le \alpha_i < 1$, $i = 1, \ldots, n$ constants) or

$$|f(x,y, \ldots, y^{(n-1)})| \le h + k \prod_{i=0}^{n-1} |y^{(i)}|^{\alpha_i}$$

($h > 0$, $k > 0$, $\alpha_i \ge 0$, $i = 0, \ldots, n-1$ constants with $\sum_{i=0}^{n-1} \alpha_i < 1$) holds on $[a,b] \times \mathbb{R}^n$, then the BVP (1.1), (1.2R) has a solution.

<u>Remark 2</u>. Remark 1 with the latter inequality holding for f is same as Corollary 3.2 of [7] where it has been proved by a different method.

Lemma 4.3. Let $u(x) \in C^n[x_1, x_k]$ satisfy $u^{(n)}(x) > 0$ for $x_1 \leq x \leq x_k$ and the BC's (1.2R) with $y_{r\,i} = 0$ for all $(r, i) \neq (1, n(1) - 1), (k, n(k) - 1)$. Further suppose

$$(-1)^{n - n(1)} u^{(n(1) - 1)}(x_1) \geq 0 \quad \text{and}$$
$$u^{(n - 1)}(x_k) \leq 0 \text{ . Then}$$

(i) $\operatorname{Sgn} u^{(i)}(x) = (-1)^{n - s(r)}$ for $x_r < x < x_{r+1}$, $s(r - 1) \leq i \leq s(r) - 1$ and $1 \leq r \leq k - 1$

and

(ii) $\operatorname{Sgn} u^{(i)}(x) = (-1)^{n - i}$ for $x < x_k$ and $s(k - 1) \leq i \leq n - 1$.

Proof. The inequality $u^{(n)}(x) > 0$, the equations

$$u^{(i)}(x) = u^{(i)}(x_k) + \int_{x_k}^{x} u^{(i+1)}(t)\,dt$$

and the BC's at x_k yield

$$\operatorname{Sgn} u^{(i)}(x) = (-1)^{n - i} \quad \text{for } x < x_k, \quad s(k - 1) \leq i \leq n - 1.$$

Again the above equality for $i = s(k - 1)$, the equations

$$u^{(i)}(x) = u^{(i)}(x_{k-1}) + \int_{x_{k-1}}^{x} u^{(i+1)}(t)\,dt$$

along with the BC's at x_{k-1} imply

$$\operatorname{Sgn} u^{(i)}(x) = \begin{cases} (-1)^{n - s(k - 1)} & \text{for } x_{k-1} < x < x_k \\ (-1)^{n - i} & \text{for } x < x_{k-1} \end{cases}$$

for all i, $s(k - 2) \leq i \leq s(k - 1) - 1$.

Repeating this process for each of the intervals $(x_{k-2}, x_{k-1}), \ldots, (x_2, x_1)$ and using the BC's satisfied by u at $x_{k-2}, \ldots, x_1$ respectively, we arrive at the conclusions of the lemma.

The following theorem is a 2 - point right focal analogue of Theorem 3.1 of [29]. However in our theorem the function on the right hand side of equation (1.1) involves variables $x, y, \ldots, y^{(n-1)}$ unlike in [29] where f is a function of x and y alone.

Theorem 4.4. Let $u, v \in C^n[x_1, x_2]$ satisfy the BC's (4.1R) for all $(r, i) \neq (1, m - 1)$ and $(2, n - 1)$ and the inequalities

$$(-1)^{n-m} u^{(m-1)}(x_1) \geq (-1)^{n-m} y_{1\, m-1} \geq (-1)^{n-m} v^{(m-1)}(x_1),$$

$$u^{(n-1)}(x_2) \leq y_{2\, n-1} \leq v^{(n-1)}(x_2)$$

and

$$(-1)^{j(i)}\,[u^{(i)} - v^{(i)}]\,(x) \geq 0 \quad \text{on} \quad [x_1, x_2] \quad \text{for}$$
$$i = 0 , \ldots , n-1$$

where

$$j(i) = \begin{cases} n - m & \text{for} \quad 0 \leq i \leq m-1 \\ n - i & \text{for} \quad m \leq i \leq n-1. \end{cases}$$

Also suppose $u^{(n)}(x) \geq f(x , y , \ldots , y^{(n-1)}) \geq v^{(n)}(x)$ for all $(x , y , \ldots , y^{(n-1)}) \in W$, where W is given by

$$W = \{(x,y,\ldots,y^{(n-1)}) : x_1 \leq x \leq x_2 , \ (-1)^{j(i)} u^{(i)}(x) \geq (-1)^{j(i)} y^{(i)} \geq (-1)^{j(i)} v^{(i)}(x) , \ i = 0, \ldots, n-1\}.$$

Then the BVP (1.1) , (4.1R) has a solution $y(x)$ such that $(x, y(x) , \ldots, y^{(n-1)}(x)) \in W$ for $x_1 \leq x \leq x_2$.

Proof. Define recursively the functions $f = F_0 , F_1 , \ldots , F_n = F$ on $[x_1,x_2] \times \mathbb{R}^n$ as follows.

$$F_{i+1}(x,y,\dots,y^{(n-1)}) = \begin{cases} F_i(x\,,y\,,\dots\,,y^{(i-1)}\,, \\ \quad u^{(i)}(x)\,,y^{(i+1)}\,,\dots\,,y^{(n-1)})\,, \\ \quad (-1)^{j(i)}[y^{(i)} - u^{(i)}(x)] \geq 0 \\ F_i(x\,,y,\dots\,,y^{(n-1)})\,, \\ \quad (-1)^{j(i)}u^{(i)}(x) \geq (-1)^{j(i)}y^{(i)} \geq \\ \quad (-1)^{j(i)}v^{(i)}(x) \\ F_i(x\,,y\,,\dots,y^{(i-1)}\,, \\ \quad v^{(i)}(x), y^{(i+1)}\,,\dots, y^{(n-1)})\,, \\ \quad (-1)^{j(i)}[y^{(i)} - v^{(i)}(x)] \leq 0\,. \end{cases}$$

The function F so defined is bounded and continuous on $[x_1\,,x_2] \times \mathbb{R}^n$ and hence by Remark 1 of Corollary 4.2 for $k = 2$ there exists a solution $y(x)$ of

$$y^{(n)} = F(x\,,y\,,\dots\,,y^{(n-1)})$$

and (4.1R). Moreover by the definition of F and the inequalities satisfied by $u^{(n)}(x)$ and $v^{(n)}(x)$ in the hypothesis , we have

$$u^{(n)}(x) \geq F(x\,,y\,,\dots\,,y^{(n-1)}) \geq v^{(n)}(x)$$

on $[x_1\,,x_2] \times \mathbb{R}^n$.

Now we claim that $(x, y(x), \ldots, y^{(n-1)}(x)) \in W$ for $x_1 \leq x \leq x_2$ from which it follows that $y(x)$ is a solution of (1.1). For this let $p(x)$ be the solution of

$$y^{(n)} = 1$$

satisfying the BC's (4.1R) with $y_{r\,i} = 0$ for all r and i. Then for each $\varepsilon > 0$, $\varepsilon p(x)$ satisfies all the hypotheses of Lemma 4.3 with $k = 2$ and hence we have by that lemma
$(-1)^{j(i)} \cdot \varepsilon p^{(i)}(x) > 0$ on (x_1, x_2) for $i = 0, \ldots, n-1$.
Now let

$$\begin{aligned} W_\varepsilon = \{(x, y, \ldots, y^{(n-1)}) : x_1 \leq x \leq x_2, \\ (-1)^{j(i)}[u + \varepsilon p]^{(i)}(x) \geq (-1)^{j(i)} y^{(i)} \geq \\ (-1)^{j(i)}[v - \varepsilon p]^{(i)}(x), \quad i = 0, \ldots, n-1\}. \end{aligned}$$

Clearly $W \subset W_\varepsilon$ for each $\varepsilon > 0$ and $\bigcap_{\varepsilon > 0} W_\varepsilon = W$.
So to prove our claim it suffices to show that
$(x, y(x), \ldots, y^{(n-1)}(x)) \in W_\varepsilon$ for all x, $x_1 \leq x \leq x_2$ and for each $\varepsilon > 0$. But this follows by letting

$$z(x) = -y(x) + u(x) + \varepsilon p(x)$$

$$h(x) = y(x) - v(x) + \varepsilon p(x)$$

and noting that in view of the inequalities $u^{(n)}(x) \geq F(x,y,\dots,y^{(n-1)}) \geq v^{(n)}(x)$ on $[x_1,x_2] \times \mathbb{R}^n$, $z(x)$ and $h(x)$ satisfy $z^{(n)}(x) > 0$, $h^{(n)}(x) > 0$ on $[x_1,x_2]$. Further all other hypotheses of Lemma 4.3 with $k = 2$ are satisfied by z and h and consequently the conclusions (i) and (ii) of that lemma hold, that is $(-1)^{j(i)} z^{(i)}(x) > 0$ and $(-1)^{j(i)} h^{(i)}(x) > 0$ on (x_1, x_2) for $i = 0,\dots,n-1$. Thus $(x,y(x),\dots,y^{(n-1)}(x)) \in W_\varepsilon$ for all x, $x_1 \leq x \leq x_2$. This completes the proof of the proposition.

<u>Corollary 4.5</u>. The conclusion of Theorem 4.4 also holds if inequalities $u^{(n)}(x) \geq f(x,y,\dots,y^{(n-1)}) \geq v^{(n)}(x)$ on W are replaced by the hypotheses that u, v are respectively lower and upper solutions of equation (1.1) on $[x_1, x_2]$ where f satisfies

$$f(x,y,\dots,y^{(n-1)}) \leq f(x,y,\dots,y^{(i-1)}, z^{(i)}, y^{(i+1)},\dots,y^{(n-1)})$$

for $(-1)^{j(i)}(y^{(i)} - z^{(i)}) < 0$ and fixed values of $x,y,\dots,y^{(i-1)}, y^{(i+1)},\dots,y^{(n-1)}$, $i = 0,\dots,n-1$.

($u(x) \in C^n(I)$ is said to be a 'lower solution' of (1.1) on I if $u^{(n)}(x) \geq f(x,u(x),\dots,u^{(n-1)}(x))$ for all x in I and

$v(x) \in C^n(I)$ is said to be an 'uppor solution' of (1.1) on I if $v^{(n)}(x) \leq f(x, v(x), \ldots, v^{(n-1)}(x))$ for all x in I).

Proof. Since u is a lower solution of (1.1) on $[x_1, x_2]$ we have

$$u^{(n)}(x) \geq f(x, u(x), \ldots, u^{(n-1)}(x)) \quad \text{for } x_1 \leq x \leq x_2$$

$$\geq f(x, y, u'(x), \ldots, u^{(n-1)}(x)) \quad \text{for } x_1 < x \leq x_2$$

$$(-1)^{j(0)}(y-u) < 0$$

$$\vdots$$

$$\geq f(x, y, \ldots, y^{(n-1)}) \quad \text{for } x_1 \leq x \leq x_2 \,,$$

$$(-1)^{j(i)}(y^{(i)} - u^{(i)}) < 0 \,,$$

$$i = 0, \ldots, n-1.$$

Similarly we can show that

$$v^{(n)}(x) \leq f(x, y, \ldots, y^{(n-1)}) \quad \text{for } x_1 \leq x \leq x_2 \,,$$

$$(-1)^{j(i)}(v^{(i)} - y^{(i)}) < 0 \,, \quad i = 0, \ldots, n-1.$$

Now under the present hypotheses all the hypotheses of Theorem 4.4 are satisfied and hence the conclusion.

<u>Theorem 4.6</u>. Assume f is continuous and satisfies $f(x, y, \ldots, y^{(n-1)}) \leq M$ on $[a,b] \times \mathbb{R}^n$, $f(x, y, \ldots, y^{(n-1)}) \geq K$

for $a \le x \le b$ and

$$(-1)^{j(i)} y^{(i)} \le 0 , \quad i = 0, \dots , n-1$$

where $j(i)$ is same as in Theorem 4.4. Then the BVP (1.1) , (4.1R) has a solution.

Proof. Without loss of generality assume $M > 0$ and $K < 0$. Let $u(x)$ be the solution of the BVP (1.1) with $f \equiv M$ and (4.1R). (This solution exists by Remark 1 of Corollary 4.2 with $k = 2$. Now we can write

$$u(x) = u_o(x) + \omega(x)$$

where $u_o(x)$ is the solution of the BVP (1.1) with $f \equiv M$ and (4.1R) with all $y_{r_i} = 0$ and $\omega(x)$ is the solution of the BVP (1.1) with $f \equiv 0$ and (4.1R). So by hypothesis it follows that $u^{(n)}(x) \ge f(x,y, \dots, y^{(n-1)})$ on $[a,b] \times \mathbb{R}^n$. Further by Lemma 4.3 with $k = 2$ applied to $u_o(x)$ we have for $x_1 < x < x_2$ the inequalities

$$(-1)^{j(i)}(u-\omega)^{(i)}(x) = (-1)^{j(i)} u_o^{(i)}(x) > 0 , \qquad (4.2)$$

$$i = 0 , \dots , n-1$$

Now define constants C_i by

$$C_i = \max \{ \{ \max(-1)^{j(i)} u^{(i)}(x) , a \le x \le b \} , 0 \}$$

$$\text{for } i = 0 , \dots , n-1.$$

Let $W_1 = \{ (x,y, \ldots, y^{(n-1)}) : a \le x \le b ,$ $(-1)^{j(i)} y^{(i)} \le C_i , \ i = 0, \ldots , n-1 \}$ so that by our hypothesis on f we have for all $(x, y, \ldots, y^{(n-1)}) \in W_1$ that $f(x, y, \ldots, y^{(n-1)}) \ge K_o$ for some $K_o \le K$. Now if $v(x)$ is the solution of the BVP (1.1) with $f \equiv K_o$ and (4.1R), we can show as in the case of $u(x)$ that for $x_1 < x < x_2$

$$(-1)^{j(i)}(v-\omega)^{(i)}(x) < 0 , \quad i = 0, \ldots, n-1 \tag{4.3}$$

Thus we have $u^{(n)}(x) \ge f(x, y, \ldots, y^{(n-1)}) \ge v^{(n)}(x)$ for all $(x, y, \ldots, y^{(n-1)}) \in W_1$ and hence in particular for all $(x, y, \ldots, y^{(n-1)}) \in W$ (W same as in Theorem 4.4 with the present choices of $u(x)$ and $v(x)$) since $W \subset W_1$.

Moreover combining the inequality (4.2) with (4.3) we see that $u(x)$, $v(x)$ satisfy all the hypotheses of Theorem 4.4 by which the existence of the desired solution follows.

<u>Corollary 4.7</u>. Assume f is continuous and satisfies $f(x, y, \ldots, y^{(n-1)}) \le M$ on $[a,b] \times \mathbb{R}^n$. Let f satisfy $f(x, y, \ldots, y^{(n-1)}) \ge f(x, y, \ldots, y^{(i-1)},$ $z^{(i)}, y^{(i+1)}, \ldots, y^{(n-1)})$ for $(-1)^{j(i)}(y^{(i)} - z^{(i)}) < 0$

and fixed values of $x, y, \ldots, y^{(i-1)}, y^{(i+1)}, \ldots, y^{(n-1)}$, $i = 0, \ldots, n-1$. Then the BVP (1.1), (4.1R) has a solution.

Proof. For $a \leq x \leq b$ and $(-1)^{j(i)}\, y^{(i)} < 0$, $i = 0, \ldots, n-1$ we have

$$
\begin{aligned}
f(x, y, \ldots, y^{(n-1)}) &\geq f(x, 0, y', \ldots, y^{(n-1)}) \\
&\geq f(x, 0, 0, y \;, \ldots, y^{(n-1)}) \\
&\;\;\vdots \\
&\geq f(x, 0, \ldots, 0).
\end{aligned}
$$

Let $K = \min \{ f(x, 0, \ldots, 0) : a \leq x \leq b \}$. Hence we have $f(x, y, \ldots, y^{(n-1)}) \geq K$ for $a \leq x \leq b$ and $(-1)^{j(i)}\, y^{(i)} < 0$, $i = 0, \ldots, n-1$. Now the conclusion follows from Theorem 4.6.

<u>Theorem 4.8</u>. Assume f is continuous and satisfies $f(x, y, \ldots, y^{(n-1)}) \geq M$ on $[a,b] \times \mathbb{R}^n$, $f(x, y, \ldots, y^{(n-1)}) \leq K$ for $a \leq x \leq b$ and $(-1)^{j(i)}\, y^{(i)} \geq 0$, $i = 0, \ldots, n-1$ where $j(i)$ is same as in Theorem 4.4. Then the BVP (1.1), (4.1R) has a solution.

The proof of this theorem is similar to the proof of Theorem 4.6 and hence is omitted.

Corollary 4.9. Assume f is continuous and satisfies $f(x,y,\dots,y^{(n-1)}) \geq M$ on $[a,b] \times \mathbb{R}^n$. Assume f satisfies $f(x,y,\dots,y^{(n-1)}) \leq f(x,y,\dots,y^{(i-1)}, z^{(i)}, y^{(i+1)},\dots,y^{(n-1)})$ for $(-1)^{j(i)}(y^{(i)} - z^{(i)}) \geq 0$ and fixed values of $x,y,\dots, y^{(i-1)}, y^{(i+1)}, \dots, y^{(n-1)}$, $i = 0, \dots, n-1$. Then the BVP (1.1), (4.1R) has a solution.

This Corollary follows from Theorem 4.8 in the same way as Corollary 4.7 followed from Theorem 4.6.

CHAPTER V

SOME SPECIAL TYPES OF BOUNDARY VALUE PROBLEMS

In this chapter we consider the differential equation

$$y^{(n)} = f(x, y, \ldots, y^{(m)}) \qquad (5.1)_m$$

along with the BC's

$$y^{(i)}(x_1) = c_i \ , \ i = 0, \ldots, n-2$$
$$y^{(r)}(x_2) = d \qquad (5.2)_r$$

and

$$y^{r}(x_1) = d$$
$$y^{(i)}(x_2) = c_i \ , \ i = 0, \ldots, n-2 \qquad (5.3)_r$$

where f is continuous on $[a,b] \times \mathbb{R}^{m+1}$, m and r are arbitrary but fixed positive integers satisfying $0 \leq m \leq r \leq n-1$, $a \leq x_1 < x_2 \leq b$ and $c_o, \ldots, c_{n-2}$, $d \in \mathbb{R}$ are arbitrary.

In the following two theorems we give sufficient conditions in terms of monotonicity of f which imply uniqueness and existence of solutions respectively for the BVP $(5.1)_m$, $(5.2)_r$. In fact in Theorem 5.1, we prove the uniqueness for the more general BVP $(5.1)_{n-1}$, $(5.2)_r$ where $0 \leq r \leq n-1$ is arbitrary but fixed.

<u>Theorem 5.1</u>. Consider the BVP $(5.1)_{n-1}$, $(5.2)_r$. Assume f is continuous on $[a,b] \times \mathbb{R}^n$ and is nondecreasing in $y^{(i)}$ for fixed values of $x, y, \ldots, y^{(i-1)}, y^{(i+1)}, \ldots, y^{(n-1)}$ for $i = 0, \ldots, n-1$. Also assume that solutions of IVP's of $(5.1)_{n-1}$ are unique. Then for each r, $0 \le r \le n-1$ a solution of the above BVP if it exists is unique.

Proof. We first prove the theorem for $r = n-1$. Suppose $u(x)$, $v(x)$ are solutions of $(5.1)_{n-1}$, $(5.2)_{n-1}$ and $u(x) \not\equiv v(x)$ on $[x_1, x_2]$. Since solutions of IVP's of $(5.1)_{n-1}$ are unique we have $u^{(n-1)}(x_1) \neq v^{(n-1)}(x_1)$. Without loss of generality assume that $u^{(n-1)}(x_1) > v^{(n-1)}(x_1)$. Let $h(x) = u(x) - v(x)$. Then we have that $h(x_1) = h'(x_1) = \cdots = h^{(n-2)}(x_1) = 0$, $h^{(n-1)}(x_1) > 0$ and $h^{(n-1)}(x_2) = 0$. Let $x_0 = \operatorname{Inf} \{ x : x_1 < x \le x_2$ such that $h^{(n-1)}(x) = 0 \}$. This implies $h^{(n-1)}(x_0) = 0$. Further $h^{(n-1)}(x) > 0$ for $x_1 \le x < x_0$. For if $h^{(n-1)}(x') < 0$ for some x', $x_1 < x' < x_0$ then this along with $h^{(n-1)}(x_1) > 0$ implies $h^{(n-1)}(x'') = 0$ where $x_1 < x'' < x'$. This contradicts the definition of x_0.

Now using $h^{(n-1)}(x) > 0$ for $x_1 \leq x < x_o$ together with the conditions satisfied by h at x_1 we have

$$h^{(i)}(x) = h^{(i)}(x_1) + \int_{x_1}^{x} h^{(i+1)}(t)\,dt > 0 \quad \text{for} \quad i = n-2,\ldots,0.$$

This implies $u^{(i)}(x) > v^{(i)}(x)$ for $i = 0, \ldots, n-2$, $x_1 \leq x < x_o$. Now for $x_1 < x \leq x_o$ we have

$$\begin{aligned} h^{(n)}(x) &= u^{(n)}(x) - v^{(n)}(x) \\ &= f(x,u(x),\ldots,u^{(n-1)}(x)) - f(x,v(x),\ldots,v^{(n-1)}(x)) \\ &= [f(x,u(x),\ldots,u^{(n-1)}(x)) - f(x,v(x),u'(x),\ldots,u^{(n-1)}(x))] \\ &\quad + \cdot \;\cdot \;\cdot \;\cdot \\ &\quad + [f(x,v(x),\ldots,v^{(n-2)}(x),u^{(n-1)}(x)) - f(x,v(x),\ldots,v^{(n-1)}(x))] \\ &\geq 0, \text{ by the hypothesis on } f. \end{aligned}$$

Hence $h^{(n-1)}(x) = h^{(n-1)}(x_1) + \int_{x_1}^{x} h^{(n)}(t)\,dt \geq h^{(n-1)}(x_1) > 0$ for $x_1 < x \leq x_o$ which implies $h^{(n-1)}(x_o) > 0$, a contradiction. This completes the proof for $r = n-1$.

Now let $0 \leq r < n-1$ be arbitrary but fixed and suppose u and v are two solutions $(u \neq v)$ of $(5.1)_{n-1}$, $(5.2)_r$. Let $w(x) = u(x) - v(x)$. Then we have $w^{(i)}(x_1) = 0$, $i = 0, \ldots, n-2$ $\quad (w^{(n-1)}(x_1) \neq 0$ by the uniqueness of solutions of IVP's) and $w^{(r)}(x_2) = 0$. By applying Rolle's theorem repeatedly to $w^{(r)}(x)$ on $[x_1, x_2]$ we obtain points $x_1 < t_{n-r-1} < \ldots < t_1 < x_2$ such that $w^{(r+1)}(t_1) = 0$, ..., $w^{(n-1)}(t_{n-r-1}) = 0$. This implies by the earlier part of the proof that $w(x) \equiv 0$ on $[x_1, t_{n-r-1}]$ and in particular we have $w^{(n-1)}(x_1) = 0$, a contradiction. This completes the proof of the theorem.

We now state a lemma due to Kolmogorov [22], which is useful in the proof of Theorem 5.2.

Lemma KO. Given $M > 0$, $[c,d] \subset \mathbb{R}$, $y \in C^n[c,d]$ an arbitrary function with the property that $|y(x)| \leq M$ and $|y^{(n)}(x)| \leq M$ on $[c,d]$ then there exists a constant $K > 0$ depending on M and $d - c$ such that $|y^{(r)}(x)| \leq K$ on $[c,d]$ for $1 \leq r \leq n - 1$.

Theorem 5.2. Let $0 \leq m \leq n - 1$ be fixed, r $(m \leq r \leq n - 1)$ be arbitrary but fixed. Assume f in equation $(5.1)_m$ is continuous on $[a,b] \times \mathbb{R}^{m+1}$ and solutions of IVP's of $(5.1)_m$ are unique. Also assume that f is nondecreasing in $y^{(i)}$ for fixed values of x, y, ..., $y^{(i-1)}$, $y^{(i+1)}$, .., $y^{(m)}$, $i = 0, \ldots, m$. For fixed values of $x_1, x_2, c_0, \ldots, c_{n-2}$

let $S(x_2, r) = \{ \gamma \in \mathbb{R} : y(x)$ is a solution of $(5.1)_m$ and $(5.2)_r$ with $\dot{d} = \gamma \}$. Then for each x_2, $S(x_2, r) = \Phi$ or $\mathbb{R}$. If $f(x, 0, \ldots, 0) \equiv 0$, $a \leq x \leq b$ then $S(x_2, r) = \mathbb{R}$.
Also in case $S(x_2, r) = \mathbb{R}$, the solution satisfying $(5.1)_m$, $(5.2)_r$ is unique.

Proof. To show that $S(x_2, r) = \Phi$ or $\mathbb{R}$, it suffices to show that $S(x_2, r)$ is both open and closed. To show that $S(x_2, r)$ is closed, assume that $\{d_j : j \geq 1\}$ is a sequence in $S(x_2, r)$ with $d_j \longrightarrow d_o$. We show $d_o \in S(x_2, r)$.

Let $\{y_j\}$ be the sequence of solutions of $(5.1)_m$ on $[x_1, x_2]$ with $y_j^{(i)}(x_1) = c_i$, $i = 0, \ldots, n-2$ and $y_j^{(r)}(x_2) = d_j$, $j \geq 1$. Without loss of generality we can assume $\{d_j\}$ is strictly increasing to d_o, i.e. $d_1 < d_j < d_{j+1} < d_o$. Now we assert that for each $j \geq 1$,

$$y_1^{(n-1)}(x_1) < y_j^{(n-1)}(x_1) < y_{j+1}^{(n-1)}(x_1). \qquad (5.4)$$

If $y_1^{(n-1)}(x_1) = y_j^{(n-1)}(x_1)$ then by uniqueness of solutions of IVP's we have $y_1(x) \equiv y_j(x)$ on $[x_1, x_2]$ and in particular $y_1^{(r)}(x_2) = y_j^{(r)}(x_2)$, a contradiction. Similar contradiction can be obtained if $y_j^{(n-1)}(x_1) = y_{j+1}^{(n-1)}(x_1)$.

On the other hand if $y_1^{(n-1)}(x_1) > y_j^{(n-1)}(x_1)$ then there exists x', $x_1 < x' < x_2$ such that $y_1^{(n-1)}(x) > y_j^{(n-1)}(x)$ for $x_1 < x < x'$ by continuity of $y_1^{(n-1)}(x)$ and $y_j^{(n-1)}(x)$.

Now for $x_1 < x < x'$

$$y_1^{(n-2)}(x) = y_1^{(n-2)}(x_1) + \int_{x_1}^{x} y_1^{(n-1)}(t)\,dt$$

$$= y_j^{(n-2)}(x_1) + \int_{x_1}^{x} y_1^{(n-1)}(t)\,dt$$

$$> y_j^{(n-2)}(x_1) + \int_{x_1}^{x} y_j^{(n-1)}(t)\,dt = y_j^{(n-2)}(x).$$

On repetition of the above argument for lower order derivitives it follows that $y_1^{(i)}(x) > y_j^{(i)}(x)$ for $x_1 < x < x'$, $i = r, \ldots, n-3$. In particular $d_1 < d_j$ implies $y_1^{(r)}(x'') = y_j^{(r)}(x'')$ for some x'', $x' < x'' < x_2$. Hence by the uniqueness of solutions of $(5.1)_m$, $(5.2)_r$ we have $y_1(x) \equiv y_j(x)$ on $[x_1, x'']$ which implies $y_1^{(n-1)}(x_1) = y_j^{(n-1)}(x_1)$, a contradiction. A similar contradiction can be obtained if $y_j^{(n-1)}(x_1) > y_{j+1}^{(n-1)}(x_1)$. This proves the assertion (5.4).

The inequalities in (5.4) imply by the uniqueness of solutions of $(5.1)_m$, $(5.2)_{n-1}$ that $y_1^{(n-1)}(x) < y_j^{(n-1)}(x) < y_{j+1}^{(n-1)}(x)$ for $x_1 < x \leq x_2$. Also for $x_1 < x \leq x_2$ we have

$$y_1^{(n-2)}(x) = y_1^{(n-2)}(x_1) + \int_{x_1}^{x} y_1^{(n-1)}(t)\,dt$$

$$= y_j^{(n-2)}(x_1) + \int_{x_1}^{x} y_1^{(n-1)}(t)\,dt$$

$$< y_j^{(n-2)}(x_1) + \int_{x_1}^{x} y_j^{(n-1)}(t)\,dt = y_j^{(n-2)}(x)\,.$$

Similarly we can show $y_j^{(n-2)}(x) < y_{j+1}^{(n-2)}(x)$ for $x_1 < x \leq x_2$. Thus we have $y_1^{(n-2)}(x) < y_j^{(n-2)}(x) < y_{j+1}^{(n-2)}(x)$ for $x_1 < x < x_2$. By a similar reasoning for derivatives of lower order we can show that

$$y_1^{(i)}(x) < y_j^{(i)}(x) < y_{j+1}^{(i)}(x) \quad \text{for } x_1 < x \leq x_2\,, i = n-3, \ldots, 0.$$

This implies in particular $y_1^{(r+1)}(x) < y_j^{(r+1)}(x) < y_{j+1}^{(r+1)}(x)$, $x_1 < x \leq x_2$ in case $r < n-1$. In case $r = n-1$, we have for $x_1 < x \leq x_2$

$$y_1^{(n)}(x) - y_j^{(n)}(x) = f(x, y_1(x), \ldots, y_1^{(n-1)}(x)) - f(x, y_j(x), \ldots, y_j^{(n-1)}(x))$$

$$= [f(x, y_1(x), \ldots, y_1^{(n-1)}(x)) - f(x, y_j(x), y_1'(x), \ldots, y_1^{(n-1)}(x))]$$

$$+ \ldots + \ldots +$$

$$+ [f(x, y_j(x), \ldots, y_j^{(n-2)}(x), y_1^{(n-1)}(x)) - f(x, y_j(x), \ldots, y_j^{(n-1)}(x))]$$

$$\leq 0 \text{ by the monotonicity of } f.$$

Similarly we can prove that $y_j^{(n)}(x) - y_{j+1}^{(n)}(x) \leq 0$ for $x_1 < x \leq x_2$. Thus $y_1^{(n)}(x) \leq y_j^{(n)}(x) \leq y_{j+1}^{(n)}(x)$ for $x_1 < x \leq x_2$. So in any case we have $(y_j^{(r)}(x) - y_1^{(r)}(x))$ is a nondecreasing function of x for $x_1 < x \leq x_2$, i.e. $0 \leq y_j^{(r)}(x) - y_1^{(r)}(x) \leq d_j - d_1 < d_o - d_1 = M$ (say). This implies for $x_1 \leq x \leq x_2$ and $j \geq 1$

$$0 \le y_j^{(r-1)}(x) - y_1^{(r-1)}(x) = \int_{x_1}^{x} (y_j^{(r)}(t) - y_1^{(r)}(t))\, dt$$

$$\le \int_{x_1}^{x} M\, dt$$

$$\le M(x - x_1)$$

$$\le M(x_2 - x_1)\ .$$

Hence we have for $x_1 \le x \le x_2$ and $j \ge 1$

$$y_1^{(r-1)}(x) \le y_j^{(r-1)}(x) \le M(x_2 - x_1) + y_1^{(r-1)}(x)$$

By further integrations from x_1 to x of the above inequalities it can be shown that for $x_1 \le x \le x_2$ and $j \ge 1$

$$y_1^{(i)}(x) \le y_j^{(i)}(x) \le M(x_2 - x_1)^{r-i} + y_1^{(i)}(x)\ ,\ i = 0, \ldots, r.$$

Now for each i, $0 \le i \le r$ we have $\{ y_j^{(i)}(x) \}$ is uniformly bounded on $x_1 \le x \le x_2$ which in turn implies that $\{f(x, y_j(x), \ldots, y_j^{(m)}(x))\}$ and consequently $\{ y_j^{(n)}(x) \}$ are uniformly bounded on $x_1 \le x \le x_2$ since $m \le r$. Hence by Lemma KO and Kamke's convergence theorem (Theorem 3.2, p.14, [11]) there exists a solution $y(x)$ of $(5.1)_m$ such that $y^{(i)}(x_1) = c_i$, $i = 0, \ldots, n-2$, $y^{(r)}(x_2) = d_o$ which implies that $d_o \in S(x_2, r)$ and hence $S(x_2, r)$ is closed.

Now we will show $S(x_2,r)$ is open. If $S(x_2,r) = \Phi$ there is nothing to prove. So assume that $S(x_2,r) \neq \Phi$ and let $\beta \in S(x_2,r)$ so that there exists a solution $y(x)$ of $(5.1)_m$ satisfying $y^{(i)}(x_1) = c_i$, $i = 0, \ldots, n-2$; $y^{(r)}(x_2) = \beta$. Consequently there exists an $\varepsilon > 0$ such that all solutions $z(x)$ of $(5.1)_m$ which satisfy $z^{(i)}(x_1) = c_i$, $i = 0, \ldots, n-2$ and $| y^{(n-1)}(x_1) - z^{(n-1)}(x_1) | < \varepsilon$ have maximal intervals of existence (depending on $z(x)$) $\supset [x_1,x_2]$.

Now define $\Delta = \{ \gamma \in \mathbb{R} : | y^{(n-1)}(x_1) - \gamma | < \varepsilon \}$ and $T : \Delta \longrightarrow \mathbb{R}$ such that $T(\gamma) = z^{(r)}(x_2)$ where $z(x)$ is the solution of $(5.1)_m$ on $[x_1,x_2]$ satisfying $z^{(i)}(x_1) = c_i$, $i = 0, \ldots, n-2$; $z^{(n-1)}(x_1) = \gamma$.

T is well defined since solutions to IVP's are unique and T is $1-1$ by the uniqueness of solutions of $(5.1)_m$, $(5.2)_r$. Further T is continuous, for if $\{\gamma_j : j \geq 0\} \subset \Delta$ and $\gamma_j \longrightarrow \gamma_0$ then $T(\gamma_j) = z_j^{(r)}(x_2)$ where $z_j(x)$ is a solution of $(5.1)_m$ on $[x_1,x_2]$ satisfying $z_j^{(i)}(x_1) = c_i$, $i = 0, \ldots, n-2$; $z_j^{(n-1)}(x_1) = \gamma_j$, $j \geq 0$. Hence by

Kamke's convergence theorem (Theorem 3.2, P.14 [11]) and uniqueness of solutions of IVP's we have $z_j^{(r)}(x) \longrightarrow z_o^{(r)}(x)$ uniformly on $[x_1, x_2]$. In particular $z_j^{(r)}(x_2) \longrightarrow z_o^{(r)}(x_2)$. Thus we have $T(\gamma_j) \longrightarrow T(\gamma_o)$ implying T is continuous. Hence by Brouwer invariance of domain theorem it follows that T is an open mapping and $T(\Delta) = S(x_2, r)$ is open.

Finally $f(x, 0, \ldots, 0) \equiv 0$ implies $0 \in S(x_2, r)$ and hence $S(x_2, r) = \mathbb{R}$. The last conclusion of the theorem follows from Theorem 5.1.

The next theorem is the analogue of Tneorem 5.2 for the BVP $(5.1)_m$, $(5.3)_r$.

Theorem 5.3. Let $0 \leq m \leq n - 1$ be fixed, $r\,(m \leq r \leq n - 1)$ be arbitrary but fixed. Assume f in $(5.1)_m$ is continuous on $[a,b] \times \mathbb{R}^{m+1}$ and solutions of IVP's of $(5.1)_m$ are unique. Also assume that f is such that

$$f(x, y, \ldots, y^{(m)}) \leq f(x, y, \ldots, y^{(i-1)}, z^{(i)}, y^{(i+1)}, \ldots, y^{(m)})$$

for $(-1)^{n+i}(y^{(i)} - z^{(i)}) < 0$, $i = 0, \ldots, m$. For fixed values of $x_1, x_2, c_o, \ldots, c_{n-2}$ let $S(x_2, r) = \{\gamma \in \mathbb{R}: y(x)$ is a solution of $(5.1)_m$ and $(5.3)_r$ with $d = \gamma\}$. Then

$S(x_2,r) = \Phi$ or $\mathbb{R}$. If $f(x, 0, \ldots, 0) \equiv 0$, $a \leq x \leq b$ Then $S(x_2,r) = \mathbb{R}$. Also in case $S(x_2,r) = \mathbb{R}$ the solutions satisfying $(5.1)_m$, $(5.3)_r$ is unique.

The following corollary is a special case of Theorem 5.2 for $n = 2$ and $m = r = 1$. Before stating this corollary we recall the following theorem due to Lees [25] and Bebernes [3] and which can also be found in [15].

Theorem L (Corollary 4.19, [15]). Assume $f(x,y,y')$ is nondecreasing in y on $[a,b] \times \mathbb{R}^2$ for fixed x, y' and satisfies a uniform Lipschitz condition with respect to y' on $[a,b] \times \mathbb{R}^2$. Then the BVP (2.1), (2.2C) has a unique solution.

Corollary 5.4. Assume $f(x,y,y')$ is nondecreasing in y for fixed x,y' , nondecreasing in y' for fixed x,y on $[a,b] \times \mathbb{R}^2$, f satisfies a uniform Lipschitz condition with respect to y' on $[a,b] \times \mathbb{R}^2$ and solutions to IVP's of (2.1) are unique. Then the BVP $(5.1)_1$, $(5.2)_1$ for arbitrary x_1 , x_2 $(a \leq x_1 < x_2 \leq b)$ and c_0, d is uniquely solvable.

Proof. Given x_1, x_2, c_o by Theorem L the BVP (2.1) and $y(x_1) = c_o$, $y(x_2) = 0$ has a solution $y(x)$ implying $y'(x_2) \in S(x_2,1)$. Hence $S(x_2,1) = \mathbb{R}$.

The following examples show that if the hypothesis of monotonicity of f is not satisfied, the conclusion of Theorem 5.2 need not hold.

Example a. In Theorem 5.2 let $n = 2$, $m = 0$, $f(x,y) = -y$, $[a,b] = [0,\pi/2]$, $x_1 = 0$, $x_2 = \pi/2$, $c_o = 0$ and $r = 1$. This example satisfies all the hypotheses of Theorem 5.2 except the monotonicity property of f in y on $[0,\pi/2]$. However any solution of the differential equation $y'' = -y$ satisfying $y(0) = 0$ is of the form $y(x) = C \sin x$, where C is an arbitrary constant. Thus $S(\pi/2, 1) = \{0\} \neq \Phi$ or $\mathbb{R}$.

Example b. In Theorem 5.2 let $n = 3$, $m = 1$, $f(x,y,y') = -y'$ $[a,b] = [0,\pi/2]$, $x_1 = 0$, $x_2 = \pi/2$, $c_o = c_1 = 0$, $r = 2$. This example also satisfies all the hypotheses of Theorem 5.2, except the monotonicity property of f in y' on $[0,\pi|2]$. Moreover all solutions of the differential equation $y''' = -y'$ satisfying the conditions $y(0) = y'(0) = 0$ are of the form $y(x) = A(1 - \cos x)$ where A is an arbitrary constant. Hence $S(\pi/2, 2) = \{0\}$, thus violating the conclusion of the theorem.

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