Electronic Auctions with Different Types of Constraints

Submitted By
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C E R T I F I C A T E

This is to certify that the thesis work entitled **'Electronic Auctions with Different Types of Constraints'** being submitted by **Mr. A. R. Dani** (Reg. No. **2KMCPC11)** in partial fulfillment of the requirement for the award of degree of **Doctor of Philosophy (Computer Science)** of the University of Hyderabad, is a record of *bona fide* work carried out by him under our supervision.

The matter embodied in this thesis has not been submitted for the award of any other research degree.

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To my parents, family
members and teachers without
whose support this work
would not be possible.

D E C L A R A T I O N

I, **A. R. Dani**, hereby declare that the work presented in this thesis has been carried out by me under the supervision of **Prof. Arun K. Pujari**, Department of Computer and Information Sciences, University of Hyderabad, Hyderabad, India and **Dr. V. P. Gulati**, as per the PhD ordinances of the University. I declare, to the best of my knowledge, that no part of this thesis has been submitted for the award of a research degree of any other University.

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A. R. Dani

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Abstract

An auction is an economic mechanism that is widely used to sell different commodities such as food grains, flowers, real estates, cars and air tickets etc. It is specified by set of winner determination and payment obligation rules. Auctions are commonly used when sellers do not want to decide the price and allow the market to determine the price. Auctions are used in different types of transactions to sell variety of goods. Governments of many countries use different types of auctions for selling Treasury Bills, Bonds etc.. Private companies use auction mechanism to sell different types of products like flowers, food grains, diamonds etc. In every day life, auction accounts for enormous volume of trade transactions. Different auction mechanisms differ, basically with respect to winner determination and payment rules. There are different types of auctions like English, Dutch, Sealed bid, which are used in various economic transactions in different types of markets. Auctions have been widely studied in economic literature. In any auction based mechanism, two main problems are

(i) determining the winning bids/asks
(ii) determining how much each participant has to pay/receive.

Auction theory also specifies the set of desirable properties of any auction mechanism. Auction based protocols are widely used in electronic commerce. In many electronic commerce applications different types of bidding mechanisms are commonly used for implementing electronic negotiation. In last few years there has been growing interest in Internet based market places. Ebay, uBid etc. are some of the successful Internet sites, which use auction mechanisms in their operations. Electronic markets leverage on information technology to perform different market functions efficiently and reduce cost of transactions. Electronic auctions in these markets are used in many cases for providing e-negotiation functionalities. Hence large number of auctions are carried out over the Internet. Apart from traditional auctions, many different types of auction systems based on multi attribute auctions, combinatorial auctions etc. are implemented in electronic environment. Double auction is an important auction mechanism, which is widely used in electronic markets. Most of electronic markets like multicommodity exchanges or financial markets for stock are built up around the mechanism of double auctions that involves multiple buyers, sellers, the continuous double auction and a clearinghouse. Another important application of electronic auctions in last few years is privatization of government owned enterprises. However most of these models cannot be directly applied in electronic auctions. This is due to the following reasons.

(1) In an electronic exchange, the object being sold may not be exactly identical (e.g. cotton with different grades).
(2) In an electronic procurement system, the object to be procured is required in different specification or sizes (e.g. paper of different widths).
(3) In case of privatization of government owned enterprises, there may be certain policies of government, which may constrain the ownership beyond certain percentage for group of buyers.

All these requirements give rise to auctions with constraints which have not been studied widely. These requirements means that demand and supply cannot be aggregated across all the bids and asks. Earlier there were few studies of auction

models where buyers have budget constraints [BCIMS2005]. Apart from this auctions with constraints have not been studied widely.

In our work, buyers do not have budget constraints, but constraints arise due to the very nature of electronic auctions. Auctions with different types of constraints are studied in this work. We have addressed two basic problems in the context of electronic auctions with different types of constraints. The problems which have been addressed in the context of electronic auctions are (i) finding optimum assignment of asks and bids in case of different types of constraints and (ii) determining how much each participant has to pay/receive. In this work we have developed set of algorithms to obtain optimum assignment of bids/asks. Payment mechanisms with desirable properties have been developed, so that auction mechanism satisfies desirable properties. The thesis is organized into seven chapters as follows.

(1) In first chapter, the problem of electronic auction is introduced.
(2) In second chapter, earlier work is reviewed in the context of electronic auction.
(3) Third chapter deals with the algorithms for determining optimum assignments and payment mechanism in case of Double Auctions with different types of constraints.
(4) In fourth chapter, a mechanism which is budget balanced strategy proof and individually rational is presented.
(5) In fifth chapter, algorithms for determining optimum assignments and payment mechanism in case of Single Object Multi Unit Auctions with different types of constraints are presented. A financial application of auctions with constraints is presented in chapter 6.
(6) In seventh chapter the future research directions are presented.

We study Double sided auctions under different types of constraints, which provide an efficient mechanism to implement bidding-based many to many negotiations. In these auctions, sellers and buyers submit asks and bids respectively, which are matched and cleared periodically. Efficient algorithms exist to match these asks and bids in case there is no constraint. However in many practical situations there can be different types of constraints in matching the bids and asks. These constraints can be of the form where an ask can be matched with only certain types of bids and vice-versa. In some cases the supply from different asks cannot be combined to fulfill a demand. Such constraints are known as indivisible demand bid constraints. The matching problems in case of indivisible demand have been formulated as generalized assignment or multiple knapsack problems, which require solution of NP-Hard optimization problems. In this work we investigate the problem of matching under different types of constraints. The main contributions are as follows.

(1) We formulate matching problem as nonlinear integer programming problem and develop set of new results, which form the basis of our algorithm.

(2) We develop two algorithms to obtain optimum assignments of bids and asks in case of different types of constraints like assignment and indivisibility constraints.

(3) We show that our algorithms always obtain optimum assignment with polynomial. time complexity.

(4) We then develop algorithm to compute VCG Payoff from the optimum solution without solving set of optimization problems.

(5) A payment mechanism, which is efficient, budget balanced and individually

rational is designed. We also study the effect of changing bid price by buyer and set of buyers and its effects. It has also been shown that group of buyers cannot affect our system by changing their respective bid price. The gains of the buyers are always bounded.

We also study truthful double auctions under different types of constraints. In electronic auctions, property of incentive compatibility can be very important. Incentive compatibility ensures that truthful bidding is the dominant strategy. Other important properties in electronic auctions are Budget Balanced (BB) and Individually Rational (IR). The former ensures that auction does not run in loss, whereas later ensures voluntary participation. However these can be achieved only after sacrificing efficiency. In this chapter we consider double auctions under different types of constraints, where truthful bidding is the dominant strategy. We design a mechanism which is strategy proof, individually rational and budget balanced. Here it is assumed that buyers and sellers have their private valuations independent of each other. The buyers and sellers submit their respective bids and asks depending upon their respective private valuations. In this case a part of bid and ask is public information, remaining part is private. The main contributions in chapter are as follows.
(1) We generalize multi unit double auction (MDA) mechanism to handle different types of constraints. It has been shown that mechanism is budget balanced, strategy proof and individually rational.
(2) The bounds on efficiency loss have also been established. We further show that efficiency loss tends to 0 as number of buyers or sellers become large.
(3) Then we have developed discriminatory mechanism. It has been shown that mechanism is budget balanced, strategy proof and individually rational. It has been shown that in this mechanism also efficiency loss is bounded and asymptotically goes to 0.
(4) It has been shown that this mechanism is false-name-poof. It means that buyer cannot improve his gains by submitting bids with false identification. This property is very important property in electronic auctions. It has been shown that buyer does not have incentive to reduce the demand.
(5) The conditions for the existence of desirable mechanism are also worked out. Then the case where, mechanism is strategy proof, individually rational and efficient but not budget balanced, is considered. We attempt to introduce participation fees to cover up deficit. The cases of efficiency loss are also worked out. This can be helpful in cases where demand is far excess of supply or vice versa.

Auction mechanism for privatization of state owned enterprises, a very important application of electronic auctions in many countries, in the financial domain has been studied in this thesis. Traditionally auction based mechanisms have been used for leasing of mining rights, bandwidth allocation etc. In this work, we consider different types of auctions, which can be used in financial domains for Initial Public Offering (IPO) of Government owned companies. We also consider the scenario with different types of constraints. The main contributions of this chapter are as follows.
(1) We formulate an optimization problem that can handle single object multi unit auctions with different types of constraints. It has been shown that, some of the single

object multi unit auction formulations, which have been studied earlier are particular cases of our formulation.

(2) We derive new results to obtain optimum assignment. An algorithm, which computes the optimum solution, has been developed. The algorithm generates optimum solution with polynomial time complexity.

(3) We then develop algorithm to compute VCG Payoff from the optimum solution without solving set of optimization problems.

(4) Then we generalize uniform price mechanism to handle different types of constraints. It has been shown that our mechanism is efficient, strategy proof and individually rational.

(5) Then we worked out discriminatory mechanism. It has been shown that mechanism is efficient, strategy proof and individually rational.

(4) It has been shown that this mechanism is false-name-proof. It means that buyer cannot improve his gains by submitting bids with false identification. So this property is very important in electronic auctions. It has been buyer also does not have incentive to reduce the demand.

Chapter 1

Introduction

Today's companies increasingly use Internet as a common communication medium for commercial transactions. Global connectivity and reach of Internet means that companies face increasing competition from different quarters. This also means companies must optimize, the way they do business, change their business processes and introduce new business processes. This has opened up new research issues and electronic or automated negotiation is one such area. The increased computing and networking power enabled through the Internet has provided new flexibility in designing negotiations while at the same time findings in computer science and information systems have fed back into models and procedures of negotiations. These studies have contributed to the development of negotiation support systems (NSS), software agents for negotiations and on-line electronic platforms for bidding and auctioning. A few companies have tried to introduce electronic auctions for procurement and trade negotiations. Most of the current business-to-business or business-to-consumer electronic commerce systems provide only limited support for automated negotiations. Negotiation is the process by which buyers and sellers in the marketplace arrive at a common agreement. However, electronic negotiations continue to evolve with new strategies and approaches. In this chapter we study the different types of electronic negotiations and auction mechanisms.

1.1 Negotiations and Bargaining

A commercial transaction between a buyer and a seller involves the exchange of goods or services and payment instruments/tokens. In this transaction the goods are transferred to the buyer or the buyer avails of the services. Economists have tried to build up models describing consumer buying behavior. The model of consumer behavior states that a commercial transaction consists of finite number of interactions between the business entities. These activities are classified into different phases.

Information Gathering Phase**:** In this phase, buyer obtains information about the required products. Information about vendors supplying the products and prices is also collected. In this phase the buyer tries to match his requirements with the available products. In the electronic commerce environment, this information is collected by browsing different websites.

Intention Phase: In this phase the details of demand and supply are fixed. In this phase usually the buyer informs the seller that he intends to enter into the transaction with seller.

Agreement Phase: In this phase buyers and sellers discuss terms and conditions of the transactions. This phase is also called as Negotiation Phase. This phase can terminate either in an agreement or in abandonment of the process. In the retail transactions this phase will involve bargaining for prices. In business to business transaction it may be much more complex.

Execution Phase: In this phase the transaction is executed as per the agreed terms and conditions. In this phase both the parties fulfill their respective commitments. The seller supplies the goods or services and the buyer pays for them.

In another model of consumer behavior three phases namely information phase, negotiation phase (combining intention phase and agreement phase) and execution phase are being considered [M1995]. In another model six phases namely Need Identification, Product Brokering, Merchant Brokering, Negotiation, Purchase and Delivery, Product Service and Evaluation [GMM1998] are discussed. Even though there are different models of consumer behavior there is a general agreement about what is meant by Negotiation Phase. It is generally agreed that in negotiation phase, the terms and conditions of transactions are agreed upon between the buyers and sellers. In other words, negotiation can be described as a process through which the buyer attempts to get the optimum deal (including the price and other related attributes like delivery date) for the goods and services that he is likely to purchase. The negotiations are common occurrence in any commercial transaction.

The term negotiation is almost used interchangeably with bargaining; however the term "negotiation" is also used in a broader sense. Negotiation or bargaining is also studied in Economic Theory. The term bargaining is used to refer to a situation where the following possibilities exist.

(1) Individuals have possibility of concluding a mutually beneficial agreement
(2) There is a conflict of interest about which agreement to conclude

(3) No agreement may be imposed on any individual without his consent.

The term negotiation is used in a much wider sense. It is the term used even though the parties involved in a negotiation process may not be involved in any commercial transaction. The negotiation between the union and management of a company for wages, between different countries on social and economic issues are examples of this. However, in any negotiation process the parties involved can reach a mutually beneficial agreement on the issues on which they differ.

1.2 Why Negotiations Take Place

The negotiation phase is a common occurrence in commercial transactions because this phase provides buyer and seller with price discovery mechanism. The dependence of the parties is one of the important conditions for negotiations. If one party is totally dependent on the other party, there will not be any negotiation. The dependent party can only react to the moves of other party. As an example, the relationship between the company and its employee can be considered. A company can decide that it no longer needs the services of an employee and lay him off. In such a case there cannot be any negotiations. However, if the union of employees decides to fight for the employee's job, the situation will change. This is due to the fact that the company is dependent on the union. In case of buyer seller transaction the buyer will be willing to negotiate because he may gain a better deal. In other words the negotiation takes place when both parties discover the potential for gain. This phase can help the buyer to prepare an offer tailored to individual needs. It can help him to avoid the risk of fixing price and provide the market determined price. This is especially true in the following cases.

(1) Non standard transactions

(2) New products where demand is unknown

(3) Where supply is unknown

(4) Perishable items/transactions like airline tickets

(5) Different product mix required

(6) Unknown value for goods

(7) Dynamic markets

(8) Availability of price based comparison (agents which help in shopping)

(9) Special offers with focus on long term relationships

The electronic markets have distinct features like low search costs. In the electronic markets search costs are low. Buyers are better informed about the product and prices. It is very easy for the buyer to visit a few websites and compare the prices. Such markets do not show any regional, personal or product specific preferences. They are characterized by high degree of transparency. However, these markets also require features like product differentiation, price discrimination and buyer accumulation strategies. These features can give rise to more negotiated agreements than in traditional markets. The characteristics of these markets will also have impact on the negotiation process [S1999ACM],[S1999AAAI]. Online auctions mainly benefit from the features of electronic markets. So there are many successful online Auction Sites. This means that there may be more sites supporting negotiations based on bidding. However bidding based negotiations protocols cannot support the integrative or win-win negotiations. The support to these types of negotiations is necessary in electronic commerce environment. In these cases the integrative negotiation protocols must be used. Though there are not many protocols of integrative negotiation in electronic markets at present they will be required in future to provide negotiation support in electronic markets [S1999ACM]. So both types of negotiation mechanisms will exist in these markets.

1.3 Types of Negotiations

The negotiation process can be classified according to the number of participants and attributes. The different types of negotiations are as follows.

One to One Negotiation: In this type of negotiation only two parties are involved. The simple case of this type of negotiation is where the parties involved have symmetric preferences i.e. the gain of one party is loss of the other. The negotiation for price of goods for sale between the buyer and seller represents this type of negotiation.

Many to One Negotiation: In this type of negotiation one party (say buyer) negotiates with many parties (say number of sellers) at the same time. Auctions are one example of these types of negotiations. These types of negotiations can be analyzed as a number of concurrent one to one negotiations.

Many to Many Negotiations: In this type of negotiations many parties (say many buyers) negotiate with many parties (say sellers) at the same time. In this case if there are n parties involved in the negotiation process, then there will be n(n-1)/2 simultaneous ongoing processes. These types of negotiations are hard to analyze.

Single Attribute Negotiations: In this case there is only one attribute that is involved in the negotiation process. Transactions, where buyers and sellers negotiate over the price of goods for sale, are examples for this. These types of negotiations, where one party looses while other gains, are also called as win-loose negotiations. These types of negotiations are also referred to as integrative negotiations.

Multi Attribute Negotiations: In this case, the negotiation process involves more than one attribute. The sale of car (attributes year, color, type, price etc.) is an example of this type of negotiation. In this case it is difficult to define what exactly constitutes the gain for the parties involved. In these cases a party may concede on one attribute but may gain in other attribute. Such negotiations are also referred to as distributive negotiations. In this case it is possible that no party looses in the process of negotiation. These type of negotiations are also referred to as win-all or no-win negotiations.

Combined Negotiations: These types of negotiations refer to those where the buyer negotiates for a number of interdependent items. Suppose that a customer is interested in visiting certain places. He needs to find the most optimum tour package consisting of travel tickets, hotel rooms and site tours. In this case the customer can separately negotiate with different agencies to arrive at the package. However the entities are interdependent. The customer must get the travel tickets on the days on which hotel rooms are available and site tours are available. Unless these constraints are satisfied it will not be possible for the customer to undertake the journey. These types of negotiations where all the negotiations are carried out at the same time and the interdependency is maintained are called combined negotiations [BAVK2001] and [BK2000].

In many cases, the negotiation process between two entities can give rise to a number of different negotiation processes. In a commercial transaction where terms and conditions of a large order are negotiated, it is possible that the process generates many different negotiation processes e.g., an international transaction between companies can give rise to a negotiation process with number of banks for issuing a Letter of Credit (LC), in case the LC is insisted upon by one company. This differs from the combined negotiation in the sense that the customer does not negotiate with different entities at the same time, but other entity involved in the process initiates the negotiation process.

Service Oriented Negotiations: Another class of negotiations that has been studied in the field of Distributed Artificial Intelligence (DAI) and Multi Agents System (MA) [BG1988] are called as Service Oriented Negotiations. This class of negotiation has been studied in the context of multi agents systems. An agent is a software or hardware or combination of software and hardware like robots, which carry out certain tasks for the user. The agents are autonomous and the user does not have control over how agent executes the assigned task. The user gives only the task to be carried out to the agent. In this case it is assumed that the agents have limited resources to carry out the tasks. The agent requires services or resources from others to complete its task. Negotiation provides the mechanism for managing the interaction between these agents. These agents negotiate among themselves and arrive at mutually acceptable solution. The examples of such cases are illustrated in [CML1988] and [BG1988].

Resource Allocation and Task Distributions: A set of agents share a common resource [CML1988]. The resource can be communication lines, printers, disks etc. Only one agent can use this common resource at a time. The other agents have to wait till the resource is free. The set of agents negotiate among with each other and decide how the resource is to be allocated. The agreement or allocation of resource can be a schedule that divides the usage of resources among the agents. In this problem there can be competition among the agents in the sense that each agent may seek larger share of the resource. In the task distribution problem agents are required to carry out many tasks to fulfill the goal. In this case the common goal is achieved by distributing the tasks among the agents. The agents negotiate with each other and come to a mutually beneficial agreement. The agreement will be the list of tasks and agents responsible for the tasks. The example of task distribution problem is the delivery domain [W1992]. In this domain suppose that two agents each have to deliver the message to A and B. In this case the agents can distribute the tasks in such a way that one agent delivers all the messages to A and other delivers all the messages to B. It can be seen that in this case the additional messages can be delivered at no extra cost. Both of these problems are symmetric in nature. The agents have to reach an agreement to achieve the tasks. This approach is very useful in designing the negotiation support systems in Electronic Commerce.

1.4 Negotiations in Electronic Commerce Using Different Types of Auctions

In the electronic commerce environment, auction based protocols are commonly used. It provides an easy and convenient mechanism for getting the market determined price, which is the main purpose of negotiation phase. It helps the sellers and buyers to determine the price of the transaction. In auctions the major phases are announcement of auction, opening of an auction, advertisement of offer and matching of bids to determine the winner. In auctions the negotiations occur through a sequence of bids. The process of submitting bids is continued till the auction is terminated. In this process price is the only attribute that is negotiated. There is a concept of reservation price in auctions. This price represents the minimum price in the auction.

However auctions still do not provide mechanisms for handling special and customized products, special packages or cases where more than one attributes are to be compared. In cases where more than one attribute is to be compared then it may be difficult to determine the winner. Let us suppose that a buyer sends the proposal to the seller that he is ready to buy 10 units of a product at the price of Rs. 10. The seller may send a counter proposal that he is willing to sell 20 units at that price. In such a situation it is difficult to determine the winners. In the same way auctions do not support situations like online touring plans, which include travel tickets to different destinations and hotel accommodations on different dates.

1.5 Electronic Auctions

It can be seen from the above discussion that electronic commerce systems must support both bidding type (auction based) and bargaining type of negotiations. Internet and electronic commerce have blurred the difference between auction and negotiation mechanisms. This new media provides new opportunities and mechanisms to cooperate or to compete, taking advantage of computing power and global connectivity of Internet. It has been further helped by the fact that millions of people and businesses are online simultaneously. The difference between auctions and negotiations gets blurred in the presence of two and more issues. This raises the possibility of using utility as a measure of offers and other mechanisms that have been traditionally used in negotiations (simultaneous improvements, efficiency analysis, etc.). Negotiation is a subject that has for years been thoroughly studied in the behavioral sciences. The literature concerned with negotiations does not mention auctions as a particular type of negotiations. Indeed, some economists [R1995] view bargaining as precisely the opposite of the idealized "perfect competition" that is presumed to form the basis of market models, recognizing the importance of persuasion and other human factors in determining the nature of the process and the outcomes. The Internet and electronic commerce has generated renewed interest in electronic marketplaces and auction systems, both as dynamic mechanisms to sell items to individuals and as systems for business to business transactions. A discussion on the different types of auctions in electronic commerce can be found in the book on Electronic Commerce [B2002]. If the negotiating parties can reduce the scope of negotiation to only price negotiations, then auctions provide an efficient mechanism for price discovery.

However electronic auctions (e-auctions) provide efficient mechanism to implement one to many and many to many types of negotiation systems. These auctions also provide framework for automated negotiations in retail electronic commerce settings. In last few years there has been growing interest in Internet based market places. Ebay, uBid etc. are some of the successful Internet sites, which use auction mechanism in their operations. Internet based auction companies carry out large number of auctions every day. The success of on-line auction sites like www.ebay.com, www.ubid.com. www.yahoo.com, www.freemarkets.com, www.onsale.com, www.bazee.com etc. (the list is not exhaustive) have contributed to a large extent to recent studies and development in electronic auctions. Many retailers have online consumer auctions (e.g. www.onsale.com) and auction based systems are being used for procurement in supply chain (e.g. www.freemarkets.com). The growth of private electronic market places based on EXTRANET technology (www.covisint,com) have opened up new vistas for automation of negotiation processes with the promise of higher levels of efficiency, effectiveness and quality and faster emergence of agreements in complex industrial procurement scenarios. Different types of electronic procurement systems are common examples of a scenario, where one to many or many to many type of negotiation systems are required. Different types of activities like providing services, task allocation etc. can be carried out efficiently using different types of bidding mechanisms. The process of auction and negotiation described above are almost similar. Electronic auctions have emerged as an important Internet-based tool for many business to business applications. General Electric (GE) has adopted electronic auctions for most of its procurement operations. It has conducted transactions worth 6 billion USD in the year 2000 alone [GEC2000]. This has led to the *Internet Week* magazine awarding the title "E-Business of year 2000" to GE. It has been estimated by Forrester Research that in the year 2004, online business to business trade in electronic marketplaces (i.e. auctions and exchanges) in USA alone was more than 10% of total business to business trade. In terms of values it will be over USD 1000 Billions in 2004. Many multi national corporations and government organizations have either used or in the process of using auction based protocols for procurement, selling of public assets, leasing rights of natural resources, bandwidth allocation and other operations. There are a few case studies about successful deployment of electronic auctions in procurement which can be found in [EK2003], [GIT2002],[HRNRDKLA2003] and [LOPST2002]. In an electronic commerce environment the objects to be auctioned need not always be homogenous, this can give rise to different types of assignment constraints. Hence in further discussion we restrict our scope to electronic auctions.

Auctions: Auction based protocols are widely used in electronic commerce. An auction is basically a bidding mechanism, which is specified by a set of winner determination and payment obligation rules. These rules specify how the winner is determined from the set of competing bids and how much the winner has to pay. Auctions have been in use for many years. They are used for sale of variety of objects. These objects range from bonds of public utilities to perishable items like flowers. Governments of different countries use auction mechanism to sell long-term securities, treasury bonds and treasury bills to raise funds to meet their borrowing needs. Private companies use auction mechanism to sell different types of products like flowers, food grains, diamonds etc. In every day life, auction accounts for enormous volume of trade transactions. Auctions also provide useful mechanisms for resource allocation problems with autonomous and self interested agents, respecting the autonomy and information decentralizations in open systems. Such applications range from distributed task allocation, to procurement in supply chain, to multi agents scheduling problems. The typical characteristics of such problems are local problems of agents and multiple conflicting goals. Auctions can minimize communication within a system and generate optimal or near optimal solutions that maximize the sum of value over all agents.

Different auction mechanisms differ basically with respect to winner determination and payment rules. Similar mechanisms are used by public sector utilities to sell their bonds and raise money. The process of procurement using competitive bidding is another form of auction. In this case the bidders compete for right to sell their products or services. The private and state owned enterprises use different types of bidding mechanisms for procurement of variety of products like computer stationery and this practice is fairly wide spread.

It is also being used to dispose of waste and scrap materials. The rights to use material resources from public property such as mining rights, off-shore oil leases have been sold by means of auctions in different countries. Communication companies use similar mechanisms for bandwidth allocation. In real life, large numbers of transactions are carried out using different types of auction mechanisms. Auctions are helpful to seller, as they help them to avoid the risk of determining the price of an object. Auctions provide mechanism where the price is determined by others rather than by seller himself. However the seller can decide whether to accept the bids received or not. Auctions are used mainly for the following three reasons.

- Auction helps in obtaining/revealing information about buyer's valuations
- Auctions are also helpful to avoid dishonest dealing between buyers and sellers
- They provide speed of sale

Traditionally auctions like ascending price (English), descending price (Dutch) or Sealed bid auctions were earlier used in different economic transactions. Emergence of Internet based electronic markets in the last few years has contributed significantly to the growth of different types of auction transactions. This has resulted in significant increase in number of transactions that are being carried out using different types of auction mechanisms on the Internet. Present day electronic auctions support novel applications like electronic procurement, bidding on air ticket etc. Different companies use electronic bidding to get market determined prices for their goods. Internet based auction companies have implemented many different types of auction mechanisms, apart from the traditional auctions like English, Dutch and Sealed Bid. Auction based protocols have been widely used in electronic commerce. Auction mechanisms are widely used in electronic commerce for carrying out negotiations.

One of the most important uses of this mechanism is to facilitate the transfer of assets from public to private hands. This has been a common phenomenon in different countries in the last two decades as a result of economic liberalization. Governments in countries like Britain and Scandinavia have used auctions to privatize transportation systems. In the former Soviet Union and Eastern European countries auctions have been used to sell public owned industrial enterprises. Auctions are being used for many years to acquire rights of use of natural resources. Such types of auctions, where government grants access rights to use natural resources or transfers the ownership of its enterprises can be considered as examples of forward and reverse auctions. When government is transferring public owned enterprise, it may be interested in seeing that ownership is distributed appropriately and not a single person or an enterprise gets ownership beyond certain percentage. While granting access to natural resources it may be interested to see that not a single person or a private enterprise acquires complete control. This can give rise to different types of constraints, while determining the winner.

1.6 Types of Auctions

Auctions can also be considered as a mechanism to allocate a set of goods to a set of bidders on the basis of received bids and asks. In a classical auction, the auctioneer wants to allocate a single object to a buyer among a group of bidders.

Classical Auctions: There are four classical auctions described in economic literature.

They are (1) English or British Auction (2) Dutch Auction (3) First Price Sealed Bid Auction and (4) Second Price Sealed Bid Auction. Out of these first the two are multi round auctions, whereas others are single shot mechanisms.

(1) English Auction: This auction is also called as first price or ascending price auction. In this auction each bidder submits his bid. The winning bid is the highest price bid. The winner pays that price or the highest price.

(2) Dutch Auction: This auction is also known as descending price auction. The winning bid is the lowest price bid and the winner pays the lowest price. This type of auction is initiated by seller. The seller starts auction by setting initial price. The price is lowered continuously till some buyer accepts the price. The buyer who accepts the price is the winner and pays that price.

(3) First Price Sealed Bid Auction : This auction is similar to English auction. However the bids by other buyers are not known. The buyer knows only his bid. The winning bid is the highest price bid and the same price is paid.

(4) Second Price Sealed Bid Auction : This auction is similar to First price Sealed Bid Auction. The buyer with the highest price bid is the winner. However he pays the price of the second highest bid. This auction is also known as Vickrey auction. It has been proved by Vickrey that second highest price is closer to true valuation.

The above four auction formats viz., English (ascending bid), Dutch (descending bid), simultaneous (sealed bid), Vickrey (second price) are most widely discussed and studied. Even today most of the online auctions are simple variations of these four basic auction types. There are a large number of well-known intermediaries conducting different variations of these auctions. The most common examples are Ebay, Amazon and OnSale. The most important and appealing features of auctions from a theoretical standpoint are their process efficiency and the ability to simultaneously manage large numbers of bidders. However, from a user's standpoint, the game-like aspects are often the dominant factor. Since auctions are primarily concerned with the establishment of value, most auctions focus on a single issue viz., price. The auction floors or clearing houses do not allow for the introduction and discussion about other issues than the one on the table. While the number of options and offers need not be fixed, the participants cannot add offers that are not defined by the issues (outside of the space defined by the auctioneer). Single-issue auctions are based on a fixed pie assumption and are thus distributive. Even if the participants have several objectives these objectives cannot be taken into account. Each participant may (and often does) have different objectives and explicit consideration of these, if possible, would move an auction to a series of bilateral negotiations. It has been convincingly argued in [S1992] that, "Real auctions—in contrast to theoretical models—are not exclusively or even primarily exchange processes. They are rather processes for managing the ambiguity and uncertainty of value by establishing social meanings and consensus." Auctions focus on determining the value of objects of unknown value while negotiations are about co-operating to create value. Auctions deal with known and well defined objects while negotiations may be about defining these objects and collaborating in order to obtain a common definition. Auctions are solely focused on the outcomes. The communication process is thus oriented on the achievement of an efficient outcome (compromise) through a low-cost process. However, auctions do not assure an efficient (Pareto-optimal) outcome. They are oriented towards increasing competition, with the participants not revealing their objectives and preferences. Since the outcome efficiency is defined with objectives and preferences (utility), it is possible that the result is inefficient. Auctions do not force the participants to reveal any information other than bids. If these bids fully reflect utility of the bidders, then the outcome is efficient. This is the case of single-issue negotiations with the only objective of all the parties being the negotiated issue. Multi-issue auctions cannot assure efficiency unless there are mechanisms that force the participants to reveal their utility. Single-issue auctions do not provide satisfactory mechanism for most business transactions. Therefore efforts are being made to extend the four basic auction formats.

Other Auctions: There are many variations of these classical auctions, which have been described in literature.

(1) K^{th} Price Auction: This auction works in the same way as Second Price Sealed Bid Auction. However the winner pays the price of k^{th} highest bid. This type of auction is basically generalization of Second Price Sealed Bid Auction.

(2) Multi Unit Auctions: In the classical auctions seller has a single unit of object or an item for sale. In multi unit auctions, seller has more than one item to sell. In multi unit auctions there may not be a single winner.

(3) Forward Auction: This is a variation of multi unit auction. In forward auction there is a single seller having quantity q of an item to sell and there are n buyers. The object is to maximize the selling price. There may not be a single winner in this case.

(4) Reverse Auction: This is another variation of multi unit auctions. In the reverse auction, there is one buyer who requires quantity q of certain item and there are n sellers who can supply these items. The objective is minimize the cost of purchase.

(5) Combinatorial Auctions: In these auctions a seller has bundle of items to sell instead of single item. The bids are submitted on bundle of objects. The buyers can specify the items they want. In combinatorial auctions there may not be a single winner.

(6) Multi Attribute Auctions: In these auctions, price is not the only decision variable. We need to consider other attributes as well.

(7) Double Auctions: In double auctions there are multiple sellers and buyers. Double sided auctions provide mechanism for clearing markets with multiple buyers and sellers. In double auction markets buyers submit their bids and sellers submit asks. A transaction occurs if the buyer's bid price exceeds seller's ask price. Another name for double auctions is exchanges. These are commonly used for trading securities, financial instruments and within supply chain. Two main institutions for double auctions are continuous double auction and a clearing house or continuous call double auction. A continuous double auction is one in which many individual transactions are carried out and trading does not stop. Call markets on the other hand are periodic versions of continuous double auctions, where bids from buyers and asks from sellers are collected over a specified interval of time and the market is cleared at the end of interval. The continuous call double auction is the oldest practiced type of market for exchange of stocks and few other items.

Auctions have evolved and grown far beyond the four classical auctions. A framework for classifying auctions based on requirements has been suggested in [KP2003]. These requirements fall into following six categories.

(1) Resources**:** An auction involves a set of resources over which negotiation is to be conducted. The resource could be a single item or multiple items, with single unit or multiple units of each item. Another common consideration is the type of the item. It can be standard commodity or multi attribute commodity. In second case non price attributes are required to be specified. A scoring function for trade-off is also required.

(2) Market Structure: An auction provides a mechanism for negotiation between buyers and sellers. There can be different combinations in market like one buyer many sellers, one seller many buyers and many buyers many sellers. Forward auctions, reverse auctions and double auctions characterize these scenarios.

(3) Preference Structure: The preference structure of buyers and sellers (participants - also called as agents) in an auction is an important feature. It has impact on many other factors. The preferences define the utility for different outcomes for participants. In case of multiple units, the participants may indicate a decreasing marginal utility for additional units. The preference structure is important in case of multi attribute auctions for designing scoring rules.

(4) Bid Structure: The structure of bids within the auction defines the flexibility with which buyers can express their resource requirements. In case of single unit, single commodity, the bids are simple statements of willingness to pay and accept. In case multi unit identical items, price and quantity are required to be specified. In some cases price can be a function of quantity. So volume discounts can be allowed. In case of combinatorial auctions, the bid structure may be more complicated.

(5) Matching Supply to Demand: A key aspect of auction is matching supply to demand. It is also referred to as winner determination or market clearing. The matching can be done in many different ways. There can be single sourcing, where pairs of buyers and sellers are matched or multi sourcing where multiple suppliers can be matched with single buyers and vice versa. The type of matching can influence the complexity of winner determination.

(6) Information Feedback: An auction protocol may be a direct mechanism or indirect mechanism. In direct mechanism such as sealed bid auction, the bids

are submitted without receiving feedback. In an indirect mechanism, such as descending price auction, the buyers can adjust bids in response to information feedback from auction. Feedback about the state of auction is usually characterized by a price signal and provisional allocations. It provides sufficient information about winning bids. It also helps in redefining bids. In complex settings such as multi unit auctions with bundled bids, a direct mechanism may not provide sufficient information about preferences. The focus in the design of indirect mechanism is to identify how much preference information is sufficient to achieve desired economic properties.

1.7 Approaches to Auction Based Automated Negotiations

It can be seen from above discussion that bidding or bargaining type of negotiation can be described as an iterative communication and decision making processes between two or more parties like buyers, sellers or their representatives like software agents. Each individual cannot achieve his objective alone. Due to this they engage into the process of exchanging information using offers, bids, counter offers and search for a compromise decision [BKS2003]. The process of decision making in negotiation will depend upon the attributes and information collected. It may evolve as more and more information is collected. This has lead to classification of negotiation into integrative and distributive types [KNJ2000]. In order to automate auction based negotiation, three types of approaches have been proposed.

(1) Negotiation Support System (NSS)**:** It is a software program, which is specifically oriented towards helping human negotiators make better decisions. It represents first step towards automated negotiation [BS1997] and [DHR2002]. These systems require constant human input and the final decision is left to human beings. As many of these systems are not capable of intelligent decisions, they provide only limited support to automated negotiations. The web based prototypes of negotiation support systems can be found at www.business.carleton.ca/inspire or www.business.carleton.ca/inter-neg/tools/inss/. These systems mainly support integrative negotiations.

(2) Intelligent Agents**:** Intelligent software agents which participate in electronic marketplaces typically operating on principles of economic design have been

proposed in [MS2001], [CM1996] and [ZS1996]. These software agents, each with their own agenda [T2000], electronically negotiate with each other in an environment governed by rules. The strategies for negotiation may be explicitly and completely built into an agent or agents can learn themselves. KASBAH [CM1996] is a marketplace where software agents negotiate for purchase and sale of goods is an example of an agent, where strategies are programmed. The strategies are specified by the users using a web based front end. The users also retain control over agents through out their life cycle. The agent BAZAAR [ZS1996] is a software agent, which can learn. In this agent negotiation is modeled as sequential decision making tasks. It uses Bayesian learning as underlying mechanism. The soft computing techniques like genetic algorithms have also been proposed for learning in negotiations and bargaining situations. These technologies essentially substitute the human element in negotiation with well defined ontology covering products, messages and decision rules. These agents typically address the automation requirements of bidding process, which when used in conjunction with the mechanisms within the marketplace infrastructure provide solutions for distributive negotiations.

(3) Economic Mechanism Design and Online Auctions**:** Economic mechanism design is concerned with the design of the rules of interaction, using the tools of economics and game theory, for economic transactions that will, in principle yield some desired outcome. In the context of electronic auctions, we require rules governing (1) bidding, (2) attributes and issues to be considered for winner determination, (3) winner determination and (4) payments to be made. It can be seen that classical auctions like English, Dutch, First Price Sealed Bid Auctions etc. are well understood and widely used economic mechanisms. In these cases, rules of interaction are well laid out. These mechanisms have been automated and form core constructs on which online auctions like www.ebay.com, www.freemarkets.com, www.onsale.com have been based. In order to automate different types of auction mechanisms, a number of technical issues from computational and economic perspectives are required to be understood and addressed through proper design of auction mechanism.

In our present work we have basically worked on the third approach.

Asymmetry of Information: One more important aspect of auction mechanism is the type of model being considered. The two most common models with respect to bidder's valuations are (1) Independent private values model and (2) Common values model**.** These two models actually represent two extreme cases. This aspect is also known as asymmetry of information and is a crucial element in any type of auction. In many electronic auctions, it is unreasonable to expect every bidder to possess the same amount of information. There will be differences between the valuation of item and set of items.

(1) Independent Private Values Model**:** In the independent private values model, each bidder knows precisely how he values the item. He does not know the valuation of other bidders for this item. He perceives valuation of other bidders as drawn from some probability distribution. It is assumed that the valuation of i^{th} bidder is v_i, which is drawn from a distribution with distribution function F_i. The valuation v_i, is known to the i^{th} bidder. All other bidders know only distribution function F_i. The valuations of any pair of bidders are mutually independent.

(2) Common Values Model**:** In this model, the item has single objective value. However the true value is not known. The bidders have access to different information sources and have different estimates of item's valuations. Let V be the true value of the item. Each bidder has estimated or perceived value v_i. It is assumed to be drawn from a common distribution with distribution function H. All bidders know the distribution function.

In the present work we basically work on Independent Private Values Model.

1.8 Properties Desired from an Auction

The main important properties desired from an electronic auction mechanism are solution equilibrium, efficiency, individual rationality, budget balance, incentive compatibility, solution stability, revenue maximization or cost minimization, low transaction costs, fairness and failure freeness.

(1) Solution Equilibrium: The solution of a mechanism is in equilibrium, if no buyer, seller or agent wishes to change its bid, given the information it has

about others. Many types of equilibrium can be computed given the assumptions about the preferences of agents, rationality and information availability. They include Nash equilibrium, Bayesian Nash equilibrium, and dominant strategy equilibrium. The detailed discussion about these aspects can be found in [MWG1995].

(2) Efficiency: A general criterion for evaluating a mechanism is Pareto efficiency. It means that no buyer or seller can improve his allocation without making at least one other buyer or seller worst off. Another commonly used metric of efficiency is efficiency of allocation. It is achieved when the total utility of the winners is maximized. In this case the items are allocated to the agents who value them the most.

(3) Individual Rationality: A mechanism is individually rational if its allocations do not make any agent worse off than had the agent not participated in the mechanism. In other words this property means that every agent gains non negative utility by being the participant in the mechanism.

(4) Budget Balance**:** A mechanism is said to be weakly budget balanced if the revenue to auctioneer or the exchange is nonnegative. It is said to be strongly budget balanced if this revenue is positive. Budget balance ensures that auctioneer or exchange does not make losses.

(5) Incentive Compatibility**:** A mechanism is incentive compatible, if the agents optimize their expected utilities by bidding their true valuations for the goods. This is a desirable feature because an agent's decision depends only on its local information and it gains no advantage in expending effort to model other agents' valuations. It is desirable that the truthful bidding by the agents should lead to a well defined equilibrium such as dominant strategy equilibrium. In this case the mechanism is said to be strategy proof.

(6) Solution Stability**:** The solution of a mechanism is stable, if there is no subset of agents that could have done better, coming to an agreement outside the mechanism.

(7) Revenue Maximization and Cost Minimization**:** In an auction where a seller is auctioning a set of items, the seller would like to maximize total revenue earned.

On the other hand in procurement auctions, the buyer will like to procure at minimum cost. Due to difficulties in finding equilibrium strategies, designing a cost minimizing or revenue maximizing auction is not easy.

(8) Low Transaction Cost**:** The buyers and sellers would like to minimize the costs of participating in auctions. Delay in concluding auctions is also transaction cost.

(9) Fairness and Failure Freeness**:** This influences bidder's willingness to participate in auctions. Winner algorithms, especially those based on heuristics, could lead to different sets of winners at different times. Since there could be multiple optimal solutions, different sets of winners could be produced by different algorithms used. Bidders who lose even though they could have won with a different algorithm could end up feeling unfairly treated. The property of failure freeness means that auction designs should work as intended under all but most extreme conditions. The transparency is important because (1) it simplifies bidders understanding of the situation and eases their own decision making (2) increases their trust in the auction process by improving their ability to verify that the auction rules have indeed been followed.

Auctions have been widely studied in economic literature; however most of these models cannot be directly applied in electronic auctions. This is due to the following reasons.

(4) In an electronic exchange, the object being sold may not be exactly identical (e.g. cotton with different grades).

(5) In an electronic procurement system, the object to be procured is required in different specification or sizes (e.g. paper of different widths).

(6) In case of privatization of government owned enterprises, there may be certain policies of government, which may constrain the ownership beyond certain percentage for group of buyers.

These requirements give rise to auctions with different types of constraints. Auctions with different types of constraints have not been widely studied. There are few studies of auction models where buyers have budget constraints. However in our work, constraints arise due to nature of electronic auctions. In an electronic exchange the object being sold may not be exactly identical (e.g. cotton with different grades). In an electronic procurement system, the object to be procured is required in different widths. In case of privatization of government owned enterprises, there may be certain policies of government, which may constrain the ownership beyond certain percentage for group of buyers. All these requirements give rise to auctions with constraints, which have not been studied widely. In this work the auctions with different types of constraints have been studied. We have basically addressed two problems (i) finding out optimum assignment of bids and asks, in case of auctions with different types of constraints and (ii) determining how much each participant has to pay/receive, so that our auction mechanism has desirable properties.

In this work, we have reviewed important basic literature of auction theory. We have reviewed the different types of problem formulations and how the solutions are obtained. We have also studied the desirable properties of auction mechanism.

We have studied Double Auctions under different types of constraints, which provide an efficient mechanism to implement bidding based many to many negotiations. Double auctions are widely used in electronic exchanges and financial markets. In these auctions, sellers and buyers submit asks and bids respectively, which are matched and cleared periodically. Efficient algorithms exist to match these asks and bids in case there are no assignment or any other constraints. However in many practical situations there can be different types of constraints in matching these bids and asks. These constraints can be of the form, where an ask can be matched with only certain types of bids and vice-versa. In some cases the supply from different asks cannot be combined to fulfill a demand. Such constraints are known as indivisible demand bid constraints. The matching problems in case of indivisible demand have been formulated as generalized assignment or multiple knapsack problems, which require solution of NP-Hard optimization problems. In this work we investigate the problem of matching under different types of constraints. We have formulated this problem as nonlinear integer programming problem and developed set of new results, which form the basis of our algorithm. We have developed two algorithms to obtain optimum assignments of bids and asks in case of different types of constraints like assignment and indivisibility constraints. It is shown that the time complexity of our algorithm is always polynomial.

We then develop algorithm to compute VCG Payoff from the optimum solution without solving set of optimization problems. A payment mechanism, which is efficient, budget balanced and individually rational is designed. We also study the effect of changing bid price by buyer and set of buyers and its effects. It has also been shown that group of buyers cannot affect our system by changing their respective bid price. The gains of the buyers are always bounded.

Further we study truthful Double Auctions under different types of constraints. In electronic auctions, property of incentive compatibility can be very important. Incentive compatibility ensures that truthful bidding is the dominant strategy. Other important properties in electronic auctions are Budget Balanced (BB) and Individual Rational (IR). The former ensures that auction does not run in loss, whereas the later ensures voluntary participation. However these can be achieved only after sacrificing efficiency. In this chapter we consider double auctions under different types of constraints, where truthful bidding is the dominant strategy. We design a mechanism which is strategy proof, individually rational and budget balanced. Here it is assumed that buyers and sellers have their private valuations independent of each other. The buyers and sellers submit their respective bids and asks depending upon their respective private valuations. In this case a part of bid and ask is public information, remaining part is private. In this chapter multi unit double auction (MDA) mechanism is generalized to handle different types of constraints. It has been shown that mechanism is budget balance, strategy proof and individually rational. The bounds on efficiency loss are established. It has been shown that efficiency loss tends to 0 as number of buyers or sellers become large. Then we have developed discriminatory mechanism. It has been shown that this mechanism is budget balanced, strategy proof and individually rational. It has been shown that in this mechanism also efficiency loss is bounded and asymptotically goes to 0. It has been shown that this mechanism is false-name proof. It means that buyer cannot improve his gains by submitting bids with false identification. This property is very important property in electronic auctions. It has been shown that buyer does not have incentive to reduce the demand. The conditions for the existence of desirable mechanism are also worked out. Then the case where, mechanism is strategy proof, individually rational and efficient but not budget balanced, is considered. We attempted to introduce participation fees to cover up deficit. The cases of efficiency loss are also worked out.

An auction mechanism for privatization of state owned enterprises; a very important application of electronic auctions in many countries, in the financial domain has been studied. Traditionally auction based mechanisms have been used for leasing of mining rights, bandwidth allocation etc. In this work, we consider different types of auctions, which can be used in financial domains for Initial Public Offering (IPO) of Government owned companies. We also consider the scenario with different types of constraints. We have formulated IPO problem as an optimization problem. The format proposed is single object multi unit auctions with different types of constraints. It has been shown that, some of the single object multi unit auction formulations, which have been studied earlier are particular cases of our formulation. We have derived new results to obtain optimum assignment. An algorithm, which computes the optimum solution, has been developed. The algorithm generates optimum solution with polynomial time complexity. Then an algorithm to compute VCG Payoff from the optimum solution without solving set of optimization problems has been developed. We generalize uniform price mechanism to handle different types of constraints. It has been shown that our mechanism is efficient, strategy proof and individually rational. Then we have developed discriminatory mechanism, which is efficient, strategy proof and individually rational. It has been shown that this mechanism is false-name proof. It means that buyer cannot improve his gains by submitting bids with false identification. So this property is very important in electronic auctions. It has been shown that the buyer also does not have incentive to reduce the demand.

1.9 Outline of the Thesis

The outline of rest of the thesis is as follows.

In chapter 2, basic important basic literature review is presented. In chapter 3, efficient Double Auctions under different types of constraints are studied. In chapter 4, truthful Double Auctions under different types of constraints are studied. In chapter 5, single object multiunit auctions under different types of constraints have been studied. In chapter 6, a banking application of auctions with constraints is presented. The conclusion and future work are discussed in chapter 7.

Chapter 2

Related Work

This work is basically a boundary between algorithm design, economics and game theory. In this chapter the survey of literature on auctions has been presented. This is not a complete survey but an overview. In this work literature from game theory and economics has also been reviewed. The current state of art has been briefly discussed. The process of negotiation is commonly used to arrive at consensus or agreement, whenever persons, organizations or other entities cannot arrive at it unilaterally [BKS1999]. Different types of electronic auction based systems provide efficient ways to implement different types of negotiations [KNJ2000]. Electronic auctions have rapidly proliferated on the Internet. Most of them focus on a single issue, viz., price, and support simple services (e.g., offer submission, notification and comparison). They are also usually one sided: a single seller (or buyer) considers bids from many buyers (sellers). The popularity of auctions and the requirements of e-business have led to a growing interest in the development of complex trading models. An example of a two-issue double-auction is [OPTIMARK1998], an electronic stock exchange developed for institutional traders—the issues are price and quantity. Multidimensional auctions in which bidding involves many issues [B1997], [TWW1999] as well as double auctions [WWW1998] that permit bidding by many buyers and sellers have been proposed. In this work [KNJ2000], authors have stated that even though there is difference between auctions and negotiations, computer science community involved in e-commerce have common perception that "negotiations are auctions". In order to support this argument authors have stated quotes from [S1999ACM]. This work is entirely devoted to different types of auctions, however its opening statement is that "Negotiation is a key component of e-commerce". There are other similar instances where authors write about electronic business negotiations but actually discuss auctions in [BSS1996] and [KF1998]. Authors also, cite another paper [SB1999], in which it has been stated that a new market-based negotiating paradigm, which exploits capabilities of electronic software agents on the Internet, has been designed. This agent is proposed to replace human negotiating skill with market forces. Authors also state that this is a direction, which has already gained some momentum with the use of online auctions. Authors believe that over the years, it will become more popular. However, some notable exceptions [GM1998AMET],[GM1998CIA] and [MGM1999] to this belief (i.e. equating auctions with negotiations) have also been stated. In these three papers several software agents that are engaged in multiple bilateral negotiations are proposed. These agents are, according to their developers, capable of conducting integrative negotiations. So it has been argued that there is an interest in negotiations other than auctions. It has been observed by authors that business activities in e-commerce mirror their traditional activities in many respects (e.g., contract negotiations, acquisitions, mergers). So they feel that there is a strong demand for mechanisms that allow for different types of negotiations. However this work mainly focuses on electronic auctions.

It has been observed that four traditional auction formats are not suitable for different types of business transactions. Therefore, efforts are being made to extend the auction formats to multiple issue auctions [C1992] and combinatorial auctions [RPH1998],[S1999AAAI]. In the works [C1992] and [B1997] the authors have discussed multidimensional auctions, mainly from a theoretical basis and the economics perspective. Their work involves the development of a scoring rule for the auction owner, who uses this to evaluate the bids. [TWW1999],[TWWZ1999ECBM] apply a variety of methods to web-based auction environments including the multiple issue auction, the multiple unit auction, and a multiple issue double auction (http://kvstu001.hkkk.fi/nss). They attempt to derive integrative-type benefits from auction environments by using simple preference elicitation schemes and heuristics to suggest offers that are in the neighbourhood of the efficient frontier. OptiMark [OPTIMARK1998] is a double auction stock trading system for institutional traders. The two issues involve quantity and price. They attempt to match larger traders in the two issue space in an attempt to avoid “market impact”, which is mostly made up of the price jumps once the news leaks that a large trader is re-positioning his holdings. Thus, in an integrative sense, they attempt to find price/quantity combinations that will benefit all parties in the trade as compared to the option of going out to the open market. This integrative capability is one of the main selling points they propose. However, OptiMark's objective in the trading mechanism is to maximize the volume of shares traded. Since the OptiMark system is a "black box", the traders do not know that if they were to shift to a different (probably lower quantity) price/quantity combination, they would both benefit. OptiMark justifies this seemingly irrational trait by saying that traders actually prefer, overall, higher quantities of shares traded, even though their preference scores state otherwise. This exemplifies the difficulties with a multiple issue auction system even when the participants reveal their preferences (utilities).

There are mainly three approaches to build automated negotiation support systems. These approaches are as follows.

(1) Automated Negotiation Support Systems

(2) Intelligent Agents

(3) Economic Mechanism Design

These approaches have already been discussed in the introduction section. The details can be seen in [BS1997],[CM1996],[MS2001] and [ZS2004]. The web based prototypes of negotiation support systems can be found at www.business.carleton.ca/inspire or www.business.carleton.ca/inter-neg/tools/inss/. There are online auction sites like www.ebay.com, www.freemarkets.com, www.onsale.com. Some prototypes of electronic markets or brokerage services which facilitate auctions and negotiations can be seen in [IWA1998], [BGHK1999], [CIW1998], [BS1999], MDS1998], [RP2000],[HBBPS2000], [SEMPER1998] and [SHHHLWLPLL2001]. The discussion on different aspects of electronic markets and negotiation can be seen in [B1991],[L1998],[S2000], [S2001], [MSJ1998] and [SQ2000]. However we do not discuss them then in detail here as they are only indirectly related with the present work. The models for deisgn and implementation of negotiaion support systems have been discussed in [HG1996],[HM2002],[LHSHS2002],[PC2000],[HS2002],[CLLO1999] and [S2001]. In the earlier chapter different types of auctions [MM1987],[M1989],[K1999],[K1995], requirements [KP2003], properties [MWG1995] and valuation issues [MM1987] have already been discussed. In this chapter the review of the work done in auctions is presented. It consists of five main parts (1) Survey of Auction Theory (2) Issues in

Electronic Auctions including incentive issues (3) Desirable Properties (4) Computational complexities (5) Current State of Art.

2.1 Survey of Auction Theory

A fairly detailed review of auction theory can be found in [W1996]. It begins with basic results on private value auctions, with particular emphasis on generality and limitations of revenue equivalence of different auction rules. This work is then extended to cover risk aversion, a minimum price, entry fees and other fixed cost biddings, multi unit auctions and bidder collusion. It also provides different sample applications of auction theory in economics. There are two widely used books on general auction theory [K2002] and [M2004]. The former by V. Krishna [K2002] provides detailed introduction of auction theory. It discusses the general theory of auctions as games of incomplete information. It provides an account of developments in this field in last 40 years. It discusses different models in detail and provides complete proof of propositions. The other one [M2004] mainly concentrates on application of auction theory in different fields. It provides a comprehensive introduction to modern auction theory and its important new applications, like new spectrum auction designs. It provides the analysis of traditional theories of "optimal auctions" as well as newer theories of multi-unit auctions and package auctions. It also provides examples of how these theories are used. It explores the limitations of Vickrey and other auction designs, and evaluates the practical responses to those limitations. Another book on auction theory is [K1999], which also provides survey of auction theory. This book mainly concentrates on practical applications in auction design (including many examples) and its uses in other areas of economics. In part III, it discusses practical auction design and part IV describes the examples of auctions like one-hundred-billion dollar 3G. Another book [K1995] is a comprehensive critical survey of the results and methods of laboratory experiments in economics. It provides surveys in areas of economics that have seen a concentration of experiments: public goods, coordination problems, bargaining, industrial organization, asset markets, auctions and individual decision making. The work by P. Milogram [M1989], studies different auction forms. It discusses various problems with bidding like pitfalls for bidders and compares different auction forms. It also provides a few empirical results on winner’s curse. The work

[MM1987] also discusses different types of auctions. It provides an insight into why auctions should be studied. It discusses the problem of asymmetry of information between buyers. Further it discusses the problems of commitment and uncertainty of information. A model of benchmark for comparing different auction forms is also proposed. It has further discussion on topics such as optimal auctions, royalties and incentive payments, risk averse bidders, correlated values and price discrimination. A classification framework for different types of auctions has been presented in [KP2003]. It discusses desirable economic properties of auction mechanism design and complexities arising in implementation of this design. It analyzes different types of proposed auction design to identify the future research directions. The detailed discussion on different aspects of auction theory like Nash equilibrium, Bayesian-Nash equilibrium, dominant strategy equilibrium can be found in [MWG1995]. In next section some references of combinatorial auction are presented. Since the present work does not involve combinatorial auction, only a brief description has been provided.

2.2 Combinatorial Auctions

Auctions where bidders submit bids on combinations of objects have recently received much attention. Due to increasing computing power, the Combinatorial Allocation Problem (CAP), as the problem of combinatorial auction is commonly referred to, has been studied recently. Such auctions were mainly proposed for selling of Radio Spectrum Rights [J1976]. This work discusses the method of allocating available Radio Spectrum and proposes combinatorial auction as efficient mechanism for allocation of resources. Recently Internet based/electronic auction based systems have contributed in a big way to interest in combinatorial auctions. There are a few surveys on combination auctions [VV2003],[PR2003] and [ND2003]. Many auctions involve the sale of a variety of distinct assets. Examples of such auctions are allocation of airport time slots, delivery routes and furniture. Such auctions have substitution effects between different assets. Bidders can have preferences for particular items as well as for bundle of items. Due to this, efficiency can be enhanced if bidders are allowed to bid on bundles or combinations of different assets. The work [VV2003] presents the survey on state of knowledge about design of combinatorial auctions. It proposes integer programming based tools for design of such auctions and combinatorial market in general. The work

[PR2003] discusses different issues in combinatorial auctions like computational complexity of winner determination and opportunities for cooperation among the participants. It also discusses different trade-offs between desirable properties of auction mechanism It discusses different ways of reducing computational complexities. These auctions have been proved to be extremely useful in Business to Business Exchange, e-selling and e-procurement. The work [ND2003] starts with computational foundations of auction theory. Then it discusses different business applications of auctions with particular emphasis on electronic auctions. Then current research work in respect of combinatorial auctions is discussed.

Other works on combinatorial auction are [C1996],[RPH1998],[S12002] and [FBS1999]. These papers describe different applications of combinatorial auctions in electronic businesses. There is a recent book [CSS2004], which provides comprehensive introduction to combinatorial auction. Recently FCC spectrum auction has generated a lot of debate among economists about auction mechanism for bundling problem. Spectrum licenses have non additive values in bundles because of network synergies from spatially coherent geographical regions. The final FCC auction design was a variant of simultaneous ascending price auction that allowed participants limited rights for de commitment and placed participation constraints on the participants to information exchange via prices during auction. The goal was to allow participants to find a good fit between their demand sets and the demand sets of other bidders and win coherent bundles of spectrum licenses. The details can be seen in [MM1996].

2.3 Intelligent Automated Agents

The problems of task assignments and resource allocation to self interested and automated agents have been studied in the field of Artificial Intelligence. Auctions provide useful mechanisms for handling such problems. Task assignments and distributed scheduling are typical examples of such problems. These applications are characterized with distributed information about agents' local problems and multiple conflicting goals [W1993],[W1999] and [C1996WS]. Many scenarios in electronic commerce environment are similar. Electronic commerce has generated new interest in auction based systems, both as dynamic mechanisms to sell items to individuals and as systems for business to business transactions. Intelligent software agents which participate in electronic marketplaces have been proposed in [CM1996] and [ZS1996]. In auction literature, the term agents often refers to buyers and sellers. However in the context of multi agent systems, the agent is used to refer to intelligent software agents.

Many coordination and negotiation problems in multi agent systems can be formulated as resource allocation problems, where allocation of items to agents represents a negotiated agreement between multiple self interested agents. In such problems, there is a fixed set of items, agents with private valuation problems. Here the allocation received by other agents is not of interest to any agent. Typical examples of such scenarios are as follows.

(1) Distributed Scheduling

(2) Course Registration

(3) Train Scheduling

(4) FCC Spectrum Rights

In such cooperative systems, it is necessary that agents participate truthfully and provable optimality of auction mechanism is useful in such cases. In electronic commerce environment we can consider a scenario with a single supplier with an item or items or set of items for sale and a number of buyers (also frequently referred as agents) with different values for item, items or set of items. An auction-based mechanism helps supplier to determine an allocation of items to buyers (or agents), which maximizes economic efficiency. Many ideas from multi agent systems have been used in business to business electronic commerce applications like supply chain management and other applications like flight scheduling, task allocation etc. One of the main motivating factors has been similarity of the problem domains. The work in intelligent systems development for auctions combines ideas from different fields like artificial intelligence, economics, game theory and algorithm design. A system CONTRACTNET, which handles problem of distributed task allocation among agents using bidding, has been proposed in [S1993]. In this system, agents bid on the basis of marginal values for tasks. Another system TRACONET allowed agents to bid with approximate values and continue to deliberate during the auction [DS1988]. However much of the work of intelligent agents focuses on the one to one bargaining type of negotiations. As such we cite a few relevant papers here.

A framework for leveled commitment contracts between agents, which allows agents to cancel their respective commitments from a contract, has been proposed in [SL1995]. This framework is useful for agents having, approximate values for tasks and continue to refine their beliefs after striking initial contracts. In this way, agents can correct early mistakes as they continue to compute values for the tasks. This technique allows agents to integrate local deliberations with negotiation between many other agents. This technique is useful in case agent have to participate in multiple auctions, where there are dependencies. An example can be that of travel plan where one may be required to participate in multiple auctions for a complete package. There are dependencies like travel dates, availability of hotel accommodation on suitable date etc. In work [S1996], it has been shown that the property of strategy proof ness of an auction can break when agents have approximate values for items and options to continue computation or submit bids. An agent can make a better decision about whether or not to perform further computation about the value of an item, if it is well informed about the bids from other agents. In a way this work demonstrates the limitations of Vickrey auction in multi agent setup. The other work on agent–mediated auction can be seen in [MGM1999] and [GK1999]. An ascending price auction for distributed scheduling problems, in which agents need time slots on machines has been proposed in [WWWM2001]. Auction based protocols to coordinate agents in supply chain problem has been proposed in [WW1999W]. A fuzzy logic based bidding strategy for autonomous agents in Continuous Double Auctions (CDA) has been proposed in [HLJ2003]. In this work a set of new algorithms, which employ heuristic fuzzy rules and fuzzy reasoning mechanisms in order to determine the best bid, given the state of marketplace, have been developed. These algorithms also allow buyer and seller agents to dynamically adjust their bidding behavior to respond effectively to changes in demand and supply. Bidding strategies based on possibility based approach have been proposed in [FGA1998]. In this work, uncertainties are handled using possibilities. The bid price is decided based on possibilities derived from case base of strategies. An agent based platform for Online Auctions has been presented in [LLK2001]. This platform is basically a multi agent platform where semi autonomous agents work together to perform user's goal. The negotiation protocol is evaluated using Markov chain model. A decision theoretic framework which an autonomous agent can use to bid effectively across multiple and simultaneous auctions, has been presented in [BPJ2002]. This framework enables an agent to make rational decisions about purchasing multiple goods from a series of auctions that operate different protocols. This framework deals with different types of auctions like English, Dutch, First Price Sealed Bid and Vickery auctions. The framework is then used to characterize the optimal decision that an agent should take. A practical algorithm that provides a heuristic approximation has also been developed.

2.4 Electronic Auction Case Studies

In the last few years, growth of Internet as a low cost communication channel has resulted in many companies using Internet for innovative applications. Apart from auction based systems like e-Bay, different business have used electronic auction based systems for procurement. This has resulted in reduction in cost of procurement. There are a few case studies about successful deployment of electronic auctions (basically combinatorial auctions) in procurement like [LOPST2002], [EK2003], [GIT2002] and [HRNRDKLA2003]. The case study [LOPST2002] discusses the application of combinatorial auction in transportation domain. It refers to the company Sears Logistics Services (SLS). This type of auction was used by the company for transportation contracts on large number of lanes (a lane in this context means a path from a source to a destination) through single bid for multiple lanes. It used reverse auctions for awarding three-year transportation contracts. The work [EK2003] and [GIT2002] presents another application of combinatorial auctions in transportation domain and integrating it with Supply Chain Management Systems. These case studies discuss application of combinatorial auction techniques by Home Depot (a big retailer in USA). In order to achieve higher efficiency and effectiveness in transportation services, a flexible Internet based bidding mechanism for truckload shipments was developed. This provided detailed information about Home Depot's network and demand to transport carriers. These carriers were asked to bid for combinations of lanes as well as for individual lanes. The mechanism allowed bidders to submit information electronically in a standard format. It helped carriers to analyze better the impact of certain bidding alternatives on their own network and bid accordingly. It helped both Home Depot and bidders to reduce the cost of operations. The advantages of application of combinatorial auctions by MARS Inc. and its suppliers have been presented in [HRNRDKLA2003]. It is another application of auction theory in supply chain management. There are a few more studies on application of auctions in supply chain management and procurement like [CJRZ2002], [BW2001], [BN2002] and [BLLC2001].

2.5 Mechanism Design, Economics and Game Theory

Economic mechanism design is concerned with the design of the rules of interaction of buyers, sellers and auctioneers. It uses the tools of economics and game theory. The interaction refers to economic transactions, which yields some desirable outcome. An ascending price auction for problems when buyers or sellers (also referred to as agents) have gross substitute preferences has been proposed in [KC1982]. Gross substitutes is a condition which states that buyer who demands good j at price p(j) will continue to demand good j if the prices for other goods increase. In general it is assumed that there are no preferences. If items are gross substitutes then buyers will have sub additive valuation functions, meaning that the value for a package of all items is greater than the minimal sum of values for a partition of the package. This increases the complexity of the auction problem. Mechanism design looks for mechanisms that are self-enforcing, such that it is not in any participant's interest to manipulate the outcome by wrongly reporting their true values. The mechanism design problem is to implement a mechanism that will compute efficient allocations with self-interested buyers and sellers. In other words, it will provide incentives for buyers and sellers to report their values truthfully. A mechanism is incentive compatible if the mechanism is structured in a way that each bidder finds it "optimal" in some sense to report his valuation truthfully. An incentive compatible mechanism induces truth revelation by the bidders by designing the payoff structure in such a way that it is in the best interest of the bidders to bid truthfully. The second price sealed bid auction or Vickrey [V1961] auction for a single digit has been shown to be incentive compatible. The Generalized Vickrey Auction (GVA) [AM2002] is an example of an incentive compatible combinatorial auction mechanism. Self enforcing mechanisms are important in the systems where participants are self interested and have private information, because it prevents undetectable deviation. There is no easy way for an auctioneer to know whether a buyer or a seller is making true claims about value of an item. The desirable properties of any mechanism are Allocative efficiency, Incentive compatibility, Individual rationality and Budget Balance. These properties are discussed in detail in subsequent chapters. The problem of revenue maximization or optimal auction design problem has been addressed in [R2001]. It presents a design of generic auction which guarantees minimum half of the optimal revenue. In addition to it this auction is incentive compatible and individually rational.

2.6 Revelation Principle

The revelation principle is a central result in mechanism design, which provides directions for important theoretical contributions. The revelation principle states that the outcome of any mechanism can be implemented as a truth revealing direct mechanism. A direct revelation mechanism is any mechanism in which participants (i.e. buyers, sellers or agents) reveal their valuation function or provide direct values like price of an item instead of some other indirect information. In incentive compatible mechanism the optimal strategy is to report respective true values. There is a well known result on revelation principle [K2002]. This result is as follows.

Any mechanism (indirect or otherwise) that can be implemented with dominant strategies can be implemented as a truth revealing (or strategy proof) dominant strategy direct revelation mechanism. The revelation principle was first formulated for dominant strategy equilibrium in [G1973]. This work was later extended by [GL1977] and [M1981]. The revelation principle means that anything that can be achieved by an indirect mechanism can be achieved by a strategy proof direct mechanism, in which all buyers and sellers reveal all of their private information truthfully. The intuitive idea is that the auctioneer or some other mechanism implementer can simulate entire system – the bidding strategies of buyers and rules of an indirect mechanism with complete and perfect information about every buyer and seller. As long as the auctioneer can claim credibly to implement buyer's or seller's strategy faithfully, then it is optimal for buyer and seller to report their true valuation function. Many people [L1989] have argued that revelation principle has limited applications in real systems because of limited computation and communication. It has been stated in [L1989] that writing down one's complete preferences is considerably more difficult than reacting to a price. Another instance of breakdown of revelation mechanism can be observed in case of correlated values [MW1982]. It has been observed that ascending price auctions generate more revenue than sealed bid auctions because buyers receive information from the bids of others. It is useful to refine own valuations of buyers for items. In this case the revelation principle breaks down because auctioneer cannot faithfully predict the adjustment in bid prices in response to bids from others.

2.7 Possibilities and Impossibilities

There are many results from mechanism design theory, which state what combinations of properties are possible and what are not possible to be achieved by auction mechanism. Some such important results are [H1972], [A1979], [MS1983], [M1981], [M1992] and [V1995]. It has been shown in [H1972] that, it is impossible to achieve allocative efficiency, weak budget balance and individual rationality in a Bayesian Nash incentive compatible mechanism. It has been shown in [A1979] that allocative efficiency and strong budget balance cannot be achieved in dominant strategy equilibrium. In this work it has been shown that, in public good problems under asymmetric information, the success of voluntary bargaining is closely related to the structure of property rights. In this paper, property rights structures and mediated bargaining procedures that either lead to an efficient voluntary resolution to public good problems, or achieve the efficient outcome but slightly coerce the agents into participation are characterized. In this respect, "efficient" property rights structures are identified. It has been shown in [MS1983] that no exchange with multiple buyers and multiple sellers can be efficient, budget balanced and individual rational at the same time. This result is true with or without incentive compatibility. This result is also known as impossibility result. This result states that whenever gains from trade are possible but not certain (sometimes also referred to as values form overlapping intervals) and a trade is bilateral, then there is no mechanism which is efficient, budget balanced, individually rational and Bayes-Nash incentive compatible. This result can be summarized as follows.

(1) The auctioneer can achieve efficiency, incentive compatibility and individual rationality if it expects to make a net payment to sellers in some problems.
(2) If the valuation functions of all the buyers and sellers are known, then efficiency, incentive compatibility, individual rationality and budget balance can be achieved.
(3) If the valuation functions of all the buyers and sellers are not known, then it is impossible to implement an auction that is efficient, incentive compatible, individual rational and budget balanced.

It has been shown in [M1981] that revenue maximization, individual rationality and incentive compatibility can be achieved simultaneously. A double auction mechanism that provides dominant strategies for both buyers and sellers is analyzed in [M1992]. This mechanism satisfies (1/n) convergence to efficiency of the buyer's bid double auction. In this work it has been shown that strategy proof double auctions are possible with weak budget balance. It has also been shown that there can be loss of least significant trade in this case. The mechanism has an oral implementation using bid and asked prices. It has been shown in [V1995] that generalized Vickrey auction satisfies four properties simultaneously viz., allocative efficiency, individual rationality, weak budget balance and strategy proof ness. The impossibility theorems in [G1973] and [S1975] show that it is impossible to implement efficient outcomes with individual rational and incentive compatible dominant strategy mechanisms in a general class of problem. This impossibility theorem for allocation mechanism with agents (participants) that have general preferences was first stated in [H1972]. These theorems are not violated by Generalized Vickrey Auction (GVA), because the positive result applies only to agents (participants), with a restricted class of utility functions, quasi linear and private values, such that they do not care about the payments or allocations received by others.

The problem of allocation of goods in exchange economies with a finite number of agents who may have private information about their preferences has been studied in [BJ1995]. In such a setting, standard allocation rules such as Walrasian equilibrium or rational expectations equilibrium are not compatible with individual incentives. In this work, the set of allocation rules which are incentive compatible have been characterized. The incentive compatible allocation rules, which can be obtained from trading according to a finite number of pre specified proportions. The number of proportions which can be accommodated is proportional to the number of agents. It has also been shown that such rules are necessarily inefficient, even in the limit as economy grows.

VCG (Vickrey-Clarke-Groves) mechanisms ([V1961],[C1971],[G1973]) provide broader class of incentive compatible mechanisms. These mechanisms induce truth revelation by paying a surplus amount to each winning bidder over and above his actual bid. This surplus, which is called the Vickrey Surplus is actually the extent by which the total cost is decreased due to the presence of this bidder (marginal contribution of the bidder to the total cost). VCG mechanisms have very attractive properties. GVA mechanism is efficient, individual rational, weakly budget balanced and incentive compatible. However, these mechanisms are not commonly used for many reasons. The first reason is they are not revenue efficient because of the payment of Vickrey surplus or discounts. The second reason is the computational complexity. The computation of Vickrey discounts involves solving number of NP-hard problems (as the number of winning bidders). In our work, we develop an algorithm, which helps in obtaining Vickrey discounts without solving sequence of optimization problems. It has been established in [C1960] that the efficient outcomes will always be achieved in a market with zero transaction costs. It can be expected that this result will hold at least approximately in frictionless electronic markets, where agent-mediation and automated auctions can eliminate many traditional transactions and search costs. However, there are still costs of private information and agent's self interest. It has been assumed in [C1960] that all gains from trade can be detected, which is not true if agents can benefit from misstating information about their values for different outcomes. In an economic mechanism where resource allocation is done based on decentralized information, computations are involved at agent (buyer/seller) level and secondly at the mechanism level. In this work, the complexity at mechanism level is addressed. The complexity at agent level is not addressed. The complexity at the mechanism level will involve computational complexity of winner determination problem and communication complexity. In the context of electronic marketplaces, in which transaction costs are small, incentive compatibility i.e. truth extraction is one of the most important problems to be solved in order to make markets truly efficient. Due to this, an attempt is made in present work, to design an incentive compatible mechanism. In present work, the complexity of winner determination problem is addressed by developing a set of efficient algorithms. The problem settings in this case are Double Auctions, where valuations are not revealed iteratively. As such the issue of communication complexity is not addressed in the present work. In the same way in this work discriminative bidding or volume discount bidding [DK2001] is not permitted. In these type of auctions, suppliers provide bids as a function of the quantity that is being purchased. The winner determination problem for this type of auction mechanism is to select a set of winning bids, where for each bid we select a price and quantity so that the total quantity of demand of the buyer is satisfied at minimum cost. The winner determination for this type of auction is fairly straight forward if there are no business constraints such as maximum/minimum number of units that are to be purchased from a supplier, minimum/maximum number of winning suppliers etc.

Design issues pertaining to multiple issue (attribute) auction in WWW environment have been discussed in [TWWZ2001]. It proposes "NegotiAuction" an algorithmic Internet based auction procedure, which combines certain elements of negotiation (bargaining) and auction. It can be used in reverse or forward auction. The notion of constraints in multi object auctions has been introduced in [PT2000]. In this work, the problem of allocation of goods with possible constraints has been studied. It also deals with the computational complexity of constrained multi object auctions. Further it establishes connection between related computational problem and optimal b-matching problem. It has been shown that the time complexity of the algorithm for basic variant of constrained multi object is polynomial.

Multi criteria auctions, i.e. auctions where price is not the only decision variable, have been studied in [Y2003]. This work introduces multi criteria auctions. It has been stated that emergence of auction mechanisms which support bids characterized by several attributes is one of the most recent evolutions in auction theory. These mechanisms referred to as multi attribute, multiple issue or multi dimensional auctions have evolved from disciplines of multi criteria decision making and auction theory

[DSK2002AAMAS]. A general preference structure has been introduced. It also discusses multi attribute auctions and dominance based auctions.

Inherent complexity of clearing auctions and reverse auctions with multiple indistinguishable units for sale has been discussed in [SS2001]. The algorithm design issues have also been discussed. The settings where bidders express their preferences via price quantity curves and bids are price quantity pairs have been considered. It has been shown that the markets with piecewise linear supply/demand curves and non discriminatory pricing can be cleared in polynomial time. It has been shown that the problem becomes NP-complete in case of discriminatory pricing.

The problem of optimal clearing where buyers and sellers express their bids via supply/demand curves has been considered in [SS2002]. In this case also there are multiple indistinguishable items for sale. It has been observed that discriminatory pricing leads to greater profit for the party who runs the market than non discriminatory pricing. In this work [SS2002], it has been shown that this comes at the cost of computational complexity. A fast polynomial time algorithm for non discriminatory clearing has been presented. It has been shown that discriminatory clearing is NP complete.

The analysis of two double auction markets, the clearing house auction and continuous double auction, has been presented in [PPM2004]. Authors state that the complexity of these institutions is such that they defy analysis using traditional game theoretic techniques. Due to this reason, heuristic strategy approximation to provide an approximate game theoretic analysis has been proposed and used in the work. The heuristic strategy equilibrium for these mechanisms have been determined. These are subjected to evolutionary game theoretic analysis. Using this analysis the likelihood of occurrence of equilibrium has been quantified. Then, design objectives for each mechanism are weighted according to probability distribution over equilibrium.

A comparison of the two algorithms for multi unit k-Double Auctions has been presented in [BW2003]. In this work, two algorithms to manage bid data in flexible, multi-unit double auctions have been presented. The first algorithm is a multi-unit extension of the 4-HEAP algorithm and the second is an algorithm based on Internal Path Reduction (IPR) tree. In order to generate price quotes, IPR tree algorithm has been enhanced. It maintains the information about M^{th} and $(M+1)^{st}$ units. The experimental results have shown that IPR algorithm can outperform 4-HEAP in some cases.

The basic problems addressed by the adaptation of economic mechanisms, and auctions in particular, to the Internet have been studied in [MT2000]. An upper bound on the revenue obtained by a seller in any auction with a fixed number of participants has been derived. The cases where this upper bound is same as the least upper bound are also found out. It has been shown that the revenue obtained by standard auctions such as English auctions approaches the theoretical upper bound when the number of participants is large. These results rely basically on risk averse assumption. It has been shown that if this assumption is relaxed and the participants are sufficiently risk seeking, seller's revenue may exceed the upper bound.

In many electronic commerce applications agents have complimentary preferences for objects in the marketplace. This scenario is a case of allocation of discrete complimentary problem. It has been shown that the competitive equilibrium bundle prices that support the efficient allocation always exist [WW1999]. Families of auctions that use bundle pricing policy and new ascending k-Bundle auction have been presented.

An English auction Protocol for a procurement multi-attribute auction in which the item for sale is defined by several attributes, has been proposed in [DSK2002]. In this setup, buyer agent is the auctioneer and seller agents are bidders. Such domains include auctions on task allocation, services or compound problems. The required properties are announced at the beginning of the auction. The different bids are proposed. These bids are composed of specific configurations that match request. Each proposed bid should be better for the buyer agent than the previous bid with respect to the announced requirements. The last suggested bid would win. Authors consider two utility function models for English auction protocols and provide optimal bidding strategies.

The problem of maximizing total revenue in the simultaneous auction of a set of items, where bidders have individual budget constraints, has been considered in [GKP2001]. In this case, each bidder is permitted to bid on all the items of his choice and specifies his budget constraint to the auctioneer. He must select bids to maximize the revenue while ensuring that no budget constraint is violated. It has been shown that the problem of maximizing revenue is NP-hard and an algorithm of factor of 1.62 is presented.

The problem of procurement in English auction settings with deadlines has been studied in [DSK2003]. The protocol proposed in this work can be used for agents who try to reach agreement on an item or issue, which is characterized by several quality attributes in addition to the price. The protocol allows specification of a deadline. The deadline defined here diminishes the phenomenon of last minute bidding strategy and prevents bottlenecks.

The agents, which are able to make decisions for different objectives than currently possible, in the context of auction and negotiation, have been proposed in [PSJ1998],[MZGM2000],[CGL1999],[LCL2000],[TWWZ2001] and [W1992]. In these works the problem of building intelligent agents for high-stake single instance decision situations in the context of different applications including auctions and negotiations have been studied. This is done by generalizing the results from the game theory and operations research in three directions viz., (1) deriving optimal bidding function that maximizes the expected utility of profit (2) integrating agents into the production process and (3) using simulation of combined process.

Optimal and stable strategies for the buyer agent and seller agents participating in multi-attribute auction have been proposed in [DKS2002AAMAS]. In addition to this the expected revenue of buyers have been analyzed. An optimal scoring rule based on this analysis has been proposed. It also considers the variations of English auctions for the case of multi attribute auctions. Which variation is appropriate for which situation has also been discussed.

A service composition agent that buys components and sells services through auctions has been proposed in [PBB2003]. The agent buys services by participating in many English auctions. It sells the composite services by participating in Request-for-Quote reverse auctions. It does not hold long term inventory of component services and it must take risks. It must make offers in reverse auctions prior to purchasing all components needed. It must bid in English auction prior to having guaranteed customer for composite goods. The algorithm that can manage this risk is also presented.

In this work [KDL2001], single item multi unit double auctions have been considered. However, the problem is considered in the case of different types of assignment and other constraints. The computational challenges involved in the winner determination problem are described in [DK2001] and [EGKL2001].

The problem of winner determination in case of single item, single attribute, multi unit auctions where bidders use marginal decreasing piecewise constant functions to bid for homogenous goods has been considered in [KPS2003]. The objective is to minimize cost for buyer. It is shown that the winner determination problem is a generalization of the classical 0/1 knapsack problem, and hence NP-hard. Computing VCG payments is also addressed. The authors provide a fully polynomial time approximation scheme (FPTAS) for the generalized knapsack problem. This leads to FPTAS for allocation in the auction, which is approximately strategy proof and approximately efficient. It is also shown that VCG payments for the auctions can be computed in worst-case O(Tlogn) time, where T is the running time to compute a solution to the allocation problem. In our case these types of auctions have been presented in the context of recent problems of IPO (Initial Public Offering) for the sale of government owned enterprises.

Multi unit auctions, where bids are piece wise linear curves, have been considered in [DJ2003]. Algorithms are provided for solving the winner determination problem. In the case of multi unit single item auctions, the complexity of the clearing algorithm is $O(n(k+1)^n)$, where n is the number of bidders and k is an upper bound on the number of segments of the piecewise linear pricing functions. The clearing algorithm has exponential complexity in the number of bids.

The problem of trade determination in multi attribute exchanges has been considered in [KN2003]. Electronic exchanges are a double sided marketplace that allow multiple buyers to trade with multiple sellers, with aggregation of demand and supply across the bids and asks to maximize the revenue in the market. Two important issues in the design of exchanges are (1) trade determination – determining optimum assignment of asks and bids and (2) pricing. In this work the problem of trade determination for one shot multi attribute exchanges that trade multiple units of the same good is discussed. The bids are configurable with separable additive price functions over the attributes and each function is continuous and piecewise linear. The trade determination problem is modeled as mixed integer programming problem for different possible bid structures. It has been shown that even in two attribute exchanges, trade determination is NP-hard for certain bid structures. In our work we consider the similar problem under different types of constraints.

A model of information processing and strategy choice for participants (buyers and sellers) in a double auction has been developed in [GD1998]. In this model, sellers form beliefs that an offer will be accepted by some buyer. Buyers also have similar beliefs. The beliefs are formed on the basis of observed market data, including frequencies of asks, bids, accepted asks and accepted bids. Then traders choose actions, which maximize their own expected surplus. The trading activity resulting from these beliefs and strategies is sufficient to achieve transaction prices at competitive equilibrium and complete market efficiency after several periods of trading. The model developed in this paper demonstrates that competitive equilibrium outcomes can be reached in double auctions.

A general framework in which real time Dynamic Programming (DP) can be used, to formulate agent bidding strategies in a broad class of auctions characterized by sequential bidding and continuous clearing, has been presented in [TB2002]. This framework uses belief function approach used in [GD1998]. In next step transitional probabilities are obtained. The holdings of different agents represent the states. The belief function combined with a forecast of how it changes over time is used as an approximate state transition model in DP. The DP is solved, when the agent has opportunity to bid. The resulting algorithm then optimizes cumulative long term discounted profitability. This algorithm has been tested in a simplified model of a Continuous Double Auction (CDA). The framework of the algorithm is extensible and can accommodate many market and research aspects. The authors have claimed that the algorithm presented here may offer the best performance of any published CDA bidding strategy.

It can be seen that problems of auction under assignment constraints have not been widely studied. In our present work we study the problem of auctions with different types of constraints. In many cases the object under auction may have some differentiating features and buyers may require an object with certain features. There are many examples of such scenarios like buyer requiring paper of different widths, cotton of different grades etc. These attributes and buyers requirements can give rise to different types of assignment constraints. In many cases, these constraints are imposed by buyers. In the present work we start with double auctions with different types of assignment constraints. The literature on double auction has been presented in the relevant chapters. In the subsequent chapter the literature on mechanism design is also presented and studied. Later we study the problem of sale of state owned enterprises.

Chapter 3

Efficient Double Auctions under constraints

3.1 Introduction

Double Auction [M1992] and [GD1998] is a widely used auction mechanism. Most of the electronic markets like multi commodity exchanges or financial markets are built up around this mechanism, which involves multiple buyers, sellers, the continuous double auction and a clearinghouse. It is a mechanism for clearing markets with multiple buyers and sellers. In double auction markets, buyers submit their bids and sellers submit asks. A transaction occurs if the buyer's bid price exceeds ask price of the seller. Two main institutions for double auctions are continuous double auction and clearing house or continuous call double auctions. A continuous double auction is one in which many individual transactions are carried out and trading does not stop. Call markets, on the other hand, are periodic versions of continuous double auctions, where bids from buyers and asks from sellers are collected over a specified interval of time and the market is cleared at the end of interval. The continuous call double auction is the oldest practised type of market for exchange of stocks, where buyers and sellers post their respective bids and asks continuously. Online trading systems based on double auction mechanism have been implemented in many stock exchanges worldwide. Stock Exchanges such as New York, Tokyo [MRS1992] use this mechanism for online trading. In Arizona Stock Exchange [www.azx.com] bids and asks are transparent. In other words, all the other users know it. Most of the electronic markets like multi commodity exchanges or financial markets are built up around the mechanism of double auction that involves multiple buyers, sellers, the continuous double auction and a clearinghouse. Generally, stocks are homogenous goods and buyers do not have preferences over a designated stock. Double auction-based mechanisms have also been widely used in multi commodity stock exchanges like National Stock Exchange of India [MCS2003] and [NSE1993]. In these cases, the commodity is substitutable and buyer does not have any preference for a particular seller. So the supply from different sellers can be aggregated to satisfy the demand from different buyers. Another feature of these types of auctions is that goods (e.g. stocks, commodities in stock exchanges) do not have any differentiating features apart from price and quantity. In these cases price and quantity become only differentiating factors and matching is done based on them. However in some cases this assumption may not be valid. Even though the commodity in exchange is the same, there may be some distinct features. A commodity like cotton has different grades and the demand for a particular grade cannot be satisfied by supply of another grade. Similarly in case of commodity like paper, buyers may require paper of different widths and sellers may supply paper of different widths. In such cases, the price and quantity need not be the only differentiating factors. These requirements give rise to different types of constraints, as they restrict the assignment of asks and bids. Such constraints where a set of asks can be assigned to only certain sets of bids are called as "*assignment constraints"*. The assignment constraints can be of several types. In one type only certain subset of asks can be assigned to a set of bids. Though there can be restrictions on assigning an ask to a bid, the supply from several asks can be combined to meet the demand of a bid. There are other cases where it is not allowed to meet the demand of a buyer from multiple sources. Consider a case where buyer requires a contiguous paper roll of specified width and length. In such cases it may not be possible to combine supply from different asks to satisfy the demand of the buyer. Such constraints are called as "*indivisible demand"* bid constraint. In such "*indivisible demand bid constraints"*, a single ask is to be assigned to a bid for which demand can not be fulfilled by combining more than one asks. We consider an example from the process industry to illustrate the notion of double auctions with different types of constraints.

3.2 Different Types of Assignment Constraints and Indivisibility Constraints

In order to illustrate the notions of different types of assignment constraints, indivisible demand bid constraints with or without assignment constraints and the present work, consider the following example of sale of paper rolls. Let us assume that there are 5 sellers and 5 buyers. The paper rolls sold by different sellers have different specifications in terms of grade, width, selling price and quantity. Similarly the buyers specify their respective requirements in terms of buying price, quantity, grade and width. The details of sellers are given in Table 3.1. It can be seen from Table – 3.1 that, first seller (seller – 1) has 3 tons of paper roll of width 2000cm and grade 1. The selling price per ton is 100. The detailed requirements of buyers are shown in Table 2. It can be seen from Table 3.2 that, the first buyer requires 5 tons of paper roll with width of 2000cm. The grade of the required paper is 3 and the buying price is Rupees 175 for each ton. In this example different grades of paper are indicated by positive integers 1,2,3 and 4. The buyers and sellers indicate the grade of paper required by them. The paper of higher grade is represented by higher integer value. The width of the paper roll is specified in the units of centimeters. Both buyers and sellers specify it in their respective bids and asks.

Table. 3.1 Asks from different sellers in paper exchange

Seller	Price	Width (CM)	Supply (Quantity Tons)	Grade
1	100	2000	3	1
2	105	2000	2	2
3	110	2000	5	3
4	114	1000	5	1
5	119	1000	5	1

Table. 3.2 Bids from different buyers in paper exchange

Buyer	Price	Width	Demand	Grade	Higher Grade
1	175	2000	5	3	N
2	170	1200	5	4	N
3	165	800	3	1	Y
4	163	800	3	1	Y
5	161	1000	10	1	Y

It can be seen that there are certain assignment constraints in above example, which must be considered while matching. The assignment constraints for matching are as follows:

(1) A bid requiring paper of a particular grade cannot be matched with asks supplying paper of lower grades. In some cases it can be matched with asks supplying paper of

higher grades. It can be seen that the first bid can be matched with asks supplying paper of grade 3 only.

(2) It can be seen from the above example that the bid from the buyer requiring paper of width w can be assigned with asks supplying paper with width w or more. It can be seen that the second buyer requiring paper of width 1200 cm can be matched with only those asks, which supply paper of width 1200 cm or more. It can also be seen that the supply from any ask can be used to satisfy the demand of buyers 3 and 4 by cutting the paper into rolls of 800cm.

(3) It can be seen that, in the above example, any bid has to be matched with the set of asks supplying paper of same or higher width. This is an implicit assignment constraint. The above example illustrates a scenario of assignment constraints. Here demand of 10 tons of a bid (say bid 5) can be satisfied by combining supply of 5 tons from ask 4 and 5 tons form ask 5. In such cases it has to be ensured that width, grades and other attributes must match.

In addition to assignment constraints there are scenarios, which give rise to indivisibility constraints. This scenario is illustrated by the following example (Table 3.3). There can be indivisibility constraints, in addition to different types of assignment constraints. Indivisibility constraints mean that, the demand of bid with such constraints cannot be fulfilled by combining supply from different asks. A demand of bid with such constraint can be satisfied by only a single ask. Such a scenario is shown in Table 3.3. One can see that the last column of Table 3.3, gives rise to indivisibility requirements. This table is similar to earlier table (Table 3.2). However the buyer specifies the requirement of contiguous paper roll. This gives rise to indivisibility constraint. It can be seen from Table – 3.3, that the first buyer requires the 5 tons of contiguous paper roll and not the broken one. (In this case grade attribute is omitted for illustration). This gives rise to indivisible demand constraint in the sense that this demand cannot be satisfied by pooling supply of 5 tons from say first two asks i.e. (3 tons from first ask and 2 tons from second ask).

The indivisible demand constraint can arise implicitly due to the width of paper rolls as well as minimum quantity specified by buyer. In case of the first bid, the buyer has specified that he needs 5 paper rolls of width 2000cm. He also needs entire roll as contiguous and not broken one. So this bid cannot be matched with first two asks having total supply of 5. In the same way this bid cannot be matched with ask 4. It has to be satisfied by assigning third ask only. In the same way bid 4 is another indivisible demand bid.

Table. 3.3 - Buyers bids for paper exchange

Bids				
Buyer	Price	Width (Cm)	Demand	Minimum Contiguous Roll Quantity
1	175	2000	5	5
2	170	2000	5	-
3	165	1200	5	-
4	163	800	5	5
5	161	800	5	-

In case of bid 2, its demand of width 2000cm cannot be satisfied by combining two asks, each supplying rolls of width 1000cm. Thus the indivisible demand bid constraint may constrain one or more attributes to be indivisible.

We observe that the indivisibility restriction is imposed only on the supply and demand quantities. But in real life, it is possible to have such constraints on multiple attributes (and not only on quantity). Moreover, the indivisibility restriction may be characteristic of the attribute or of the bid. Thus there can be bids, which may or may not have indivisibility restriction on a particular attribute.

In the last few years, there has been a growing interest in study of auctions due to emergence of Internet based electronic markets. Electronic markets leverage on the information technology, to perform different market functions efficiently and reduce cost of transactions. Large numbers of auctions are carried out over the Internet and apart from traditional auctions; many different types of auction systems are implemented in an electronic environment. An auction is an economic mechanism that is widely used to sell different commodities such as food grains, flowers, real estates, cars, air tickets etc. Auctions are commonly used when the sellers do not want to decide the price and allow the market to determine it. The market is a very important component of economy and it provides legal and institutional framework for transactions. There are different types of auctions like English, Dutch, Sealed bid [W1996], which are used in various economic transactions in different types of markets. A detailed survey of auction mechanisms can be found in [W1996]. More recently there have been many novel applications of electronic auctions like electronic procurement [EGKL2001], bidding on air ticket etc. An auction-based process, which minimizes the procurement cost using auctions, is proposed in [KL2001]. Unlike the traditional English or Dutch auctions where price is the only attribute, a multi-attribute auction [BKS1999] is based on more than one attribute. In combinatorial auctions [RPH1998] [S2002], the seller wishes to sell a combination of goods and buyers bid on one or more goods. An approximate but tractable strategy proof mechanism for single-good-multi-unit allocation problem is presented in [KPS2003]. It also presents a polynomial time approximation scheme for reverse and forward auctions.

Even though auctions have been widely studied in economic theory, auctions under different types of constraints have not been studied widely. The problem of determining optimal matching with indivisible demand bid constraint is first proposed in [KDL2001]. In the absence of any assignment constraints, the matching problem can be solved using the price discovery mechanism [GD1998] and [KDL2001]. This work [KDL2001] basically investigates computational complexity of clearing markets in continuous call auction in three cases viz, (1) no constraints on assignment (unconstrained assignment), (2) constraints which restrict assignment of an ask to bid (assignment constraints) and (3) indivisible demand bid constraints where supply from different asks cannot be combined to satisfy demand (indivisible demand). An algorithm to find optimal matching of asks and bids, when there are no assignment constraints is presented in [KDL2001]. This algorithm is a multi unit extension of single unit double auction markets studied in [SW1989] and [M1992]. In case of assignment constraints, problem of finding optimal assignment is formulated as a network flow problem [AMO1993],[KDL2001],[CH1995],[CHM1996]. All bids and asks form the intermediate nodes of this network. The arcs are constructed depending upon the prices and quantities. The arcs between the source node and bids are constructed using quantity of bid and 0 (cost). In the same way, the arcs from asks to sink node are constructed. These arcs are constructed using the difference between the prices as cost and quantity as weight. This problem is then solved as a network flow optimization problem. There are many algorithms available to solve this problem [AMO1993]. The complexity of maximum flow problem i.e. matching of demand and supply is $O[(nm+ n^2 \log n)]$. In this case, n represents number of nodes and m number of edges. Three different formulations are suggested for finding optimum assignment of asks and bids in case of indivisible demand [KDL2001]. These are multiple knapsack problem, bin packing problem or generalized assignment problems [MT1989], [GJ1979] and [S2002]. Apart from this the problem of double auctions with constraints has not been studied widely. Further this work [KDL2001] only partly addresses trade determination problem. It does not address payoff determination problem. Electronic auctions are now used widely for different types of problems and in many such problems there can be different types of constraints. In many cases price and quantity need not be the only deciding attributes. It can be seen that the existing approaches are not adequate to handle different types of problems. It is therefore essential that the problem of double auctions with constraints be studied in detail.

In the present formulation the bids can be of mixed types i.e. some bids can have indivisible demand constraints, whereas other bids need not have these constraints. They can also optionally have other assignment constraints based on different attributes (e.g. width, in this case). We observe that price and quantity are not the only attributes that are to be constrained for indivisibility. The above example shows that even another attribute, say width, can also have such restriction. Thus we address the problem in much more general settings.

In the present work the algorithms for finding optimum assignment of asks and bids, under different types of constraints, are developed. Later we develop algorithms for computing VCG Payments for different buyers and sellers.

3.3 Problem Formulation

In this section the problem of optimal matching for continuous call auctions with indivisible bid constraint is formulated. It is observed that our formulation is closer to the real life applications. Further, we develop some new results that help in devising an efficient algorithm.

The object under sale in an auction is referred to as an item. The *attributes* describe different characteristics of the items. There can be number of attributes to describe each item. However, we identify three special attributes for each item namely, the price, quantity and size or width. Each attribute assumes value from a set of specified *domains*. Let there be m asks and n bids. Let AS be the set of all asks and BD be the set of all bids. The set of asks AS and bids BD can be decomposed to AS_{o1}, AS_{o2}, …,AS_{on} and BD_{o1}, BD_{o2} ,…, BD_{om}. So that we can write AS and BD as

$AS = AS_{o1} \cup AS_{o2} \cup \ldots \cup AS_{on}$ and $B = BD_{o1} \cup BD_{o2} \cup \ldots \cup BD_{om}$.

The sets BD_{oi} are disjoint i.e. $BD_{oi} \cap BD_{oj} = \phi$ for all $i \neq j$.

The set of asks AS_{oi} are not required to be mutually exclusive. This decomposition can be done based on values of set of attributes.

Definition 3.1 (Ask)

An *ask* A_i is an ordered list of k attribute values, $A_i = (v^i_1, v^i_2, \ldots, v^i_k)$, where v^i_j is the value of the j^{th} attribute of i^{th} ask.

Definition 3.2 (Bid)

A *bid* B_i is an ordered list of k attribute values, $B_i = (v^i_1, v^i_2, \ldots, v^i_k)$, where v^i_j is the value of the j^{th} attribute of i^{th} bid.

A seller describes the details of an item he wants to sell in an ask whereas the buyer describes the details of items and the price that he is willing to pay in a bid. Let bq_i and aq_j be the quantities of bid B_i and ask A_j, respectively. Also, let bp_i be the price of bid B_i and ap_j be the price of ask A_j. Let as_j be the size of the ask A_j and bs_i be the size of bid B_i specified in same units.

Once the asks and the bids are grouped in this way, the matching can be done for each subset of asks and bids separately, considering each group as an unconstrained matching. This approach can be efficient when the constraints are of equality type. The difficulty with this approach is that, in some cases it may not be possible to exactly group asks and bids. For example, in Table 3.1, bids 3 and 4 require paper roll of 800cm. It can be possibly matched with any ask where width is 800 cm or more. There can also be another potential problem. Suppose that a bid, which requires paper of width 800 cm (quantity 5 tons) is matched with an ask of 1200 cm (quantity 5 tons). Then remaining part of ask i.e. 400cm of width (quantity 5 tons) will either be wasted or should be moved to another set. In our approach we do not follow this procedure of multiple order books due to difficulties mentioned above and treat it as a single order book. We also take into account other attributes like width while matching.

Let us define variable x_{ij} for $1 \le i \le n$, and $1 \le j \le m$ as follows.

$$x_{ij} = \begin{cases} 1, & \text{if the jth ask is assigned to ith bid} \\ 0 & \text{otherwise} \end{cases}$$

The quantity q_{ij} of item that is matched between j^{th} ask and i^{th} bid is given by

$0 \le q_{ij} \le \min(bq_i, aq_j)$ for all j ($1 \le i \le n$, $1 \le j \le m$). In case $q_{ij} = bq_i$, it means that the demand of bid B_i is completely satisfied. In case $q_{ij} = aq_j$, it means that supply from the ask A_j is completely fulfilled. If it is neither, it signifies partial fulfillment of bids and asks.

Definition 3.3 (Price Spread)

The price spread between bid B_i and ask A_j is

$p_{ij} = (bp_i - ap_j)$.

Definition 3.4 (Surplus and Total Surplus)

The surplus from the matching quantity q_{ij} between i^{th} bid and j^{th} ask is $p_{ij}q_{ij}x_{ij}$.

The total surplus R of matching process is given by

$$R = \sum_i \sum_j p_{ij}q_{ij}x_{ij} \tag{3.1}$$

While matching asks and bids, an attempt is made to maximize the total surplus (i.e. to obtain the matching such that the total of difference between ask price and bid price is maximum).

Let us consider a situation where it is necessary to account for even the wastage during the matching process. The wastage could arise due to difference in sizes (width in example) of asks and bids. An ask can be matched with a bid only if its size is either same as the bid size or exceeds it. Indivisible demand bid constraints arise because asks of two different sizes cannot be combined. It can be seen from Table 3.1, if ask 4 is matched with bid 4 and ask 5 is matched with bid 3, paper roll of width 200cm and of quantity 6 tons is wasted. On the other hand if the first ask is matched with bid 2, then there will not be any wastage.

Definition 3.5 (Wastage and Wastage Penalty):

Suppose that i^{th} bid with value of width attribute bs_i is assigned to j^{th} ask with value of width attribute as_j. Then quantity $(as_j - bs_i)$ (difference between respective values of width attributes) is the wastage resulting from this assignment. The wastage is minimum when $as_j = bs_i$, and the minimum value is 0. Its values are always nonnegative.

Let w_{ij} be the per unit wastage penalty expressed in monetary units (i.e. in the same units as price) for wastage resulting from assignment of i^{th} bid and j^{th} ask.

This per unit wastage penalty is a linear function of wastage $(as_j - bs_i)$. So we have

$w_{ij} = wp(as_j - bs_i)$, which is a linear function of wastage. Its important properties are as follows.

(1) It assumes only nonnegative values. It is an one to one function; meaning that for any quantity of wastage there is unique associated wastage penalty value and vice versa.
(2) Its value is 0 when wastage is $(as_j - bs_i) = 0$. It is the minimum value for wastage penalty. Its value is minimum, when wastage is minimum.
(3) Its value increases as value of $(as_j - bs_i)$ increases.
(4) If wastage from two different assignments is same i.e. $(as_{i1} - bs_{i2}) = (as_{i3} - bs_{i4})$, then the wastage penalty is same i.e. $w_{i2i1} = w_{i4i3}$.

(5) Let B_{i1} and B_{i2} be two bids and asks A_{i3} and A_{i4} are assigned to them. Then total per unit wastage penalty, resulting these two assignments is sum of per unit wastage penalties of individual assignments. In other words
$wp(as_{i3}\text{-}bs_{i1}+ as_{i4}\text{-}bs_{i2}) = wp(as_{i3}\text{-}bs_{i1}) + wp(as_{i4}\text{-}bs_{i2})$.

The wastage penalty from assignment of quantity q_{ij} of i^{th} bid and j^{th} ask is defined is $w_{ij}q_{ij}$. The total wastage penalty W from matching is given by

$$W = \sum_i \sum_j w_{ij}q_{ij}x_{ij}$$

Our objective from matching is to maximize the surplus and minimize the wastage penalty.

So the problem can be formulated as a nonlinear integer programming problem as follows.

$$\begin{aligned}
&Max \quad \sum_i \sum_j (p_{ij} - w_{ij}) q_{ij}x_{ij} && (3.2)\\
&\sum_j q_{ij}x_{ij} = bq_i \quad \text{for every indivisible demand bid } B_i && (3.3)\\
&\sum_j x_{ij} = 1 \quad \text{for every indivisible demand bid } B_i && (3.4)\\
&\sum_j q_{ij}x_{ij} \le bq_i \quad \text{for all other bids} && (3.5)\\
&\sum_j q_{ij}x_{ij} \le aq_j \quad \text{for all asks } A_j && (3.6)\\
&p_{ij}, w_{ij} \ge 0 \text{ for all } i \text{ and } j && (3.7)
\end{aligned}$$

Let A be the set of bids and asks which cannot be assigned to each other.

$$\begin{aligned} x_{ij} &= 0 \quad \text{if } (i, j) \in A \\ &= 0, 1 \text{ otherwise} \end{aligned}$$

This set A includes all equality as well as inequality constraints on different attributes of asks and bids. If no bid has indivisibility constraints, then constraints in (3.4) can be omitted. The optimization problem Max R (as defined in 3.1) subject to constraints in (3.3) - (3.7) is a particular case of this formulation, where $w_{ij} = 0$ for $1 \le i \le m$, $1 \le j \le n$. It is an optimization problem Max R subject to constraints (3.3) - (3.7). This will be referred as formulation (1). It is a generic surplus maximization problem.

Thus the above formulation is a more general one and its special cases are no assignment constraints, assignment constraints with or without indivisible demand bid constraints. The following result is the basis of our proposed algorithms.

Definition 3.6 (Contribution)

Let quantity q of ask A_k with price ap_k be assigned to bid B_i. Let bp_i be the price attribute of this bid. Let as_k and bs_i be the size attributes of ask A_k and bid B_i respectively. Then contribution $cov_{ik}(q)$ to the value of objective function from this assignment is defined as $cov_{ik}(q) = (bp_i – ap_k)q - w_{ik}q = covs_{ik}(q) – covo_{ik}(q)$ (3.8).

$q \leq \min(aq_j,bq_i)$ (3.9).

If q = 1, we call it as unit contribution and indicate it by $ucov_{ik}$**.** The maximum possible unit contribution for any bid can be obtained, when it is assigned to an ask with minimum price and having value of size attribute same as that of bid size, if such an ask exists. It can be noted that both the terms in (3.8) are always nonnegative**.** The contribution decreases if ask size increases and the contribution increases if ask size decreases. This happens because wastage penalty decreases if ask size decreases. The contribution to the value of objective function consists of two components - contribution to the surplus and wastage penalty. The contribution to surplus is

$covs_{ik}(q) = (bp_i – ap_k)q$ and contribution to wastage penalty is $covo_{ki}(q) = w_{ki} = wp(as_k - bs_i)q$.

Let us indicate the assignment of quantity q of j^{th} ask to i^{th} bid (i = 1, 2, …, m , j =1, 2, …, n) by A_{sg} as $A_{sg} = \{(B_i, A_j, q)\}$.

The quantity q satisfies the condition $q \leq \min(bq_i,aq_j)$. Further

(i) If $q = bq_i$, it is a complete assignment of bid.

(ii) If $q = aq_j$, it is a complete assignment of ask

(iii) If $q = bq_i = aq_j$, it is a full assignment

(iv) In all other cases it is a partial assignment

In the same way, assignment of more than one asks i.e. A_1 with quantity q_1, A_2, with quantity q_2, … , A_{k1} with quantity q_{k1} for a bid B_1 is indicated as

$A_{sg} = \{(B_1, A_j, q_j), j =1, 2, …, k_1\}$. In a similar manner assignment of one ask to many bids can be expressed.

Theorem 1 - Suppose that in optimum assignment of the problem defined in (3.2), the assignment of asks in respect of bids B_i and B_j satisfies the following conditions.

(1) $A_{sg1} = \{(B_i, A_{k1}, q)\}$ - quantity q of ask A_{k1} is assigned to bid B_i

(2) $A_{sg2} = \{(B_j, A_{il}, q_{il}), 1 \le l \le k, l$ is positive integer$\}$ - quantities q_{il} of ask a_{il}, ($1 \le l \le k$, l is positive integer) respectively, are assigned to bid B_j

(3) The quantity q satisfies the relation $q = \sum_{l=1}^{k} q_{il}$ and conditions $q \le \min(bq_i, aq_{k1})$.

Then the value of objective function remains unchanged if A_{sg1} and A_{sg2} are changed as follows.

(i) $A_{sg1} = \{(B_i, A_{il}, q_{il}), 1 \le l \le k, l$ is positive integer$\}$

(ii) $A_{sg2} = \{(B_j, A_{k1}, q)\}$

(iii) There is no change in remaining assignment.

Proof : The contribution to value of objective function for assigning quantity q of A_{k1} to B_i is given by,

$cov_{ik1}(q) = (bp_i - ap_{k1})q - w_{ik1}q = covs_{ik1}(q) - covo_{ik1}(q)$, where

$covs_{ik1}(q) = (bp_i - ap_{k1})q$ and $w_{ik1}q = covo_{ik1}(q) = wp(as_{k1} - bs_i)q$

The total contribution to value of objective function $tcov_{jk}(q)$, for assigning A_{il} with quantity q_{il}, $(1 \le l \le k)$ to B_j is given by,

$$= \sum_{l=1}^{k} ((bp_j - ap_{il})q_{il} - w_{jil}q_{il}) \, (1 \le l \le k)$$

$$= tcovs_j(q) - tcovo_j(q) \text{ , where}$$

$tcovs_j(q) = \sum_{l=1}^{k} (bp_j - ap_{il})q_{il}$, $tcovo_j(q) = \sum_{l=1}^{k} w_{jil}q_{il}$ and $w_{jil} = wp(as_{il} - bs_j)$

The total contribution is

$$cov_{ik1}(q) + tcov_j(q)$$

$$= (bp_i - ap_{k1})q - w_{ik1}q + \sum_{l=1}^{k} ((bp_j - ap_{il})q_{il} - w_{jil}q_{il}) \, , (1 \le l \le k)$$

We can write $q = \sum_{l=1}^{k} q_{il}$

$$= (bp_i - ap_{k1}) \sum_{l=1}^{k} q_{il} - w_{ik1} \sum_{l=1}^{k} q_{il} + \sum_{l=1}^{k} ((bp_j - ap_{il})q_{il} - w_{jil}q_{il})$$

$$= \sum_{l=1}^{k} (bp_i - ap_{k1}) q_{il} - \sum_{l=1}^{k} w_{ik1}q_{il} + \sum_{l=1}^{k} ((bp_j - ap_{il}) q_{il} - w_{jil}q_{il})$$

$$= \sum_{l=1}^{k} ((bp_i - ap_{k1}) - w_{ik1} + (bp_j - ap_{il}) - w_{jil}) q_{il}$$

$= \sum_{l=1}^{k} ((bp_i - ap_{k1}) + (bp_j - ap_{il}) - (w_{ik1} + w_{jil})) q_{il}$ (3.10).

We can write $w_{ik1} = wp(as_{k1} - bs_i)$ and $w_{jil} = wp(as_{il} - bs_j)$.

The wastage resulting from assignment of quantity q of ask A_{k1} to bid B_i is $(as_{k1} - bs_i)q$. It can be written as $\sum_{l=1}^{k} (as_{k1} - bs_i) q_{il}$.

The total wastage from assigning A_{il} with quantity q_{il}, $(1 \leq l \leq k)$ to B_j is given by, $\sum_{l=1}^{k} (as_{il} - bs_j)q_{il}$.

The total wastage resulting from two assignments (assignment of ask A_{k1} to bid B_i and set of asks A_{il} $(1 \leq l \leq k)$ to bid B_j) is

$= \sum_{l=1}^{k} (as_{k1} - bs_i) q_{il} + \sum_{l=1}^{k} (as_{il} - bs_j)q_{il}$.

$= \sum_{l=1}^{k} ((as_{k1} - bs_i) q_{il} + (as_{il} - bs_j)q_{il})$. It can be rewritten as

$= \sum_{l=1}^{k} ((as_{il} - bs_i) q_{il} + (as_{k1} - bs_j)q_{il}) = \sum_{l=1}^{k} (as_{il} - bs_i) q_{il} + (as_{k1} - bs_j) \sum_{l=1}^{k} q_{il}$

$= \sum_{l=1}^{k} (as_{il} - bs_i) q_{il} + (as_{k1} - bs_j) q$ = Total wastage resulting from assignment of set asks A_{il} $(1 \leq l \leq k)$ to bid B_i and ask A_{k1} to bid B_j. Due to this total wastage penalty remains same by property (4). So we can write

Total wastage penalty from (assignment of quantity q of A_{k1} to B_i and quantity q_{il}, $(1 \leq l \leq k)$ of asks A_{il} to B_j) = Total wastage penalty from (assignment of quantity q of A_{k1} to B_j and quantity q_{il}, $(1 \leq l \leq k)$ of asks A_{il} to B_i). (3.11).

Using property (5) two sides of identity (3.11) can be written as

$wp(as_{k1} - bs_i) q + \sum_{l=1}^{k} wp(as_{il} - bs_j)q_{il} = \sum_{l=1}^{k} wp(as_{il} - bs_i)q_{il} + wp(as_{k1} - bs_j) q$.

This can be written as

$w_{ik1}q + \sum_{l=1}^{k} w_{jil}q_{il} = \sum_{l=1}^{k} w_{iil}q_{il} + w_{jk1}q$. Using $q = \sum_{l=1}^{k} q_{il}$, it can be further rewritten as

$\sum_{l=1}^{k} (w_{ik1} + w_{jil}) q_{il} = \sum_{l=1}^{k} (w_{iil} + w_{jk1}) q_{il}$. (3.12).

Substituting this in (3.10) we get

$= \sum_{l=1}^{k} ((bp_i - ap_{k1}) + (bp_j - ap_{il}) - (w_{iil} + w_{jk1}) q_{il}$

$= \sum_{l=1}^{k} ((bp_i - ap_{il}) - w_{iil} q_{il} + \sum_{l=1}^{k} ((bp_j - ap_{k1}) - w_{jk1}) q_{il}$

$$= \sum_{l=1}^{k} ((bp_i - ap_{il}) - w_{iil} q_{il} + (bp_j - ap_{k1}) - w_{jk1} \sum_{l=1}^{k} q_{il}$$

$$= (\sum_{l=1}^{k} ((bp_i - ap_{il}) - w_{iil}) q_{il}) + ((bp_j - ap_{k1})q - w_{jk1} q)$$

$$= tcov_i (q) + cov_{jk1}(q)$$

$$= tcovs_i(q) - tcovo_i(q) + covs_{jk1}(q) - covo_{jk1}(q)$$

$tcov_i (q)$ is the total contribution to the value of objective function from the assignment of quantity q of set of asks $a_{i1}, a_{i2}, \ldots, a_{ik}$ to B_i and $cov_{jk1}(q)$ is the contribution to the value of objective function from the assignment of quantity q of ask A_{k1} to bid B_j. This shows that an interchange of quantity q of an ask, assigned to one bid, with set of asks assigned to another bid, will not affect the value of objective function. We call this as an interchange result.

In other words, the value of objective function will not be affected by interchanging quantity q of ask A_{k1} (assigned to bid B_i) with quantity q of set of asks a_{il} ($1 \le l \le k$), (assigned to bid B_j). (In the new assignment quantity q of ask A_{k1} is assigned to bid B_j and set of asks A_{il} with quantity q_l ($l = 1, 2, \ldots, k$) is assigned to bid B_i.) In this case it is not assumed that A_{sg1} or A_{sg2} is a complete assignment in respect of bids B_i and B_j. In case it is a partial assignment of bids B_i and B_j, then the remaining assignment is not changed by this interchange. In this case, if $q = aq_{k1}$, then it is a complete interchange of ask A_{k1}. It can be easily proved, that the above result is true even if size attribute or wastage factor is omitted (i.e. only for surplus). This result shows that, if we have an optimum matching of asks and bids which maximizes the net surplus, it is possible to obtain an assignment that can minimize the wastage (as in this example), without affecting the total surplus. This shows that it is possible to consider other attributes while matching. The formulation in (3.2) helps in obtaining such matching assignment in one stage, rather than attempting it as two stage optimization problem. Further, it can be seen from this result that the surplus from the assignment does not change because of interchange. It can be seen that our objective function is of type O = (A – B), where both A and B have non negative values. So O will have maximum value when A is maximum and B is minimum. Due to this, once we get optimum value of objective function defined in (3.2), subject to constraints in (3.3) to (3.7), we will also get optimum of (3.1), subject to constraints (3.3) to (3.7). Because of this reason, objective function is defined as in (3.2). However our formulation and algorithm can be applied even in case of two stage optimization (first maximizing surplus and then minimizing wastage) process. The second attribute used here is width, as seen in the example of paper roll. However, any numeric attribute can be used in its place. We now state few particular cases of the above result.

Theorem 2: Suppose that there is full assignment of quantity q in optimum matching as follows.

(1) $A_{sg1} = \{(B_i, A_k, q)\}$

(2) $A_{sg2} = \{(B_j, A_h, q)\}$

Then the value of objective function remains unchanged if A_{sg1} and A_{sg2} are changed, without changing any other assignment of asks and bids, as

(1) $A_{sg1} = \{(B_i, A_h, q)\}$

(2) $A_{sg2} = \{(B_j, A_k, q)\}$

Proof: The contribution to value of objective function for assigning A_k to B_i is given by,

$$cov_{ik}(q) = (bp_i - ap_k)q - w_{ik}q$$

Similarly, the contribution to value of objective function for assigning A_h to B_j is given by,

$$cov_{jh}(q) = (bp_j - ap_h)q - w_{jh}q$$

In this case $bq_i = bq_j = aq_k = aq_h = q$

The total contribution is

$$cov_{ik}(q) + cov_{jh}(q)$$

$$= (bp_i - ap_k)q - w_{ik}q + (bp_j - ap_h)q - w_{jh}q.$$

Following a similar argument as in theorem (1) it can be shown that

$w_{ik}q + w_{jh}q = w_{ih}q + w_{jk}q$.

This expression can be simplified as

$$= ((bp_i - ap_h)q - w_{ih}q + (bp_j - ap_k)q - w_{jk}q)$$

$$= cov_{ih}(q) + cov_{jk}(q)$$

where $cov_{ih}(q)$ and $cov_{jk}(q)$ are contributions to the value of objective function for assigning A_h to B_i and A_k to B_j, respectively. This shows that a complete interchange of an ask, assigned to one bid to another, will not affect the value of objective function.

It can be easily shown that the value of objective function is not affected, if assignment of asks and bids are partly interchanged. It can also be seen that the above result holds good even if interchange is done partly assuming that it is allowed. The partial interchange of quantity q is not allowed for bids with indivisible demand bid constraints.

Theorem 3 – Suppose that in optimum matching assignment of asks in respect of bids B_i and B_j is as follows.

(1) $A_{sg1} = \{(B_i, A_k, aq_k)\}$

(2) $A_{sg2} = \{(B_j, A_h, aq_h)\}$

Then the value of objective function will remain unchanged if A_{sg1} and A_{sg2} are changed, without changing any other assignment of asks and bids, as

(1) $A_{sg1} = \{(B_i, A_k, aq_k\text{-}q), (B_i, A_h, q)\}$

(2) $A_{sg2} = \{(B_j, A_h, aq_h\text{-}q), (B_j, A_k, q)\}$

Proof : Let us assume that the asks A_k and A_h, are assigned to bids B_i and B_j with quantities aq_k and aq_h respectively. In the new assignment quantity q of ask A_k (out of aq_k) is assigned to bid B_j and quantity q of ask A_h (out of aq_h) is assigned to bid B_i. There is no change in other assignments. In this case the contribution to the value of objective function for assigning quantity aq_k of A_k to B_i is given by

$$cov_{ik}(aq_k) = (bp_i - ap_k)aq_k - w_{ik}aq_k.$$

Then the contribution to value of objective function for assigning quantity aq_h of A_h to B_j is given by,

$$cov_{jh}(aq_h) = (bp_j - ap_h)aq_h - w_{jh}aq_h$$

The total contribution is

$$cov_{ik}(aq_k) + cov_{jh}(aq_h)$$

$$= (bp_i - ap_k)aq_k - w_{ik}aq_k + (bp_j - ap_h)aq_h - w_{jh}aq_h$$

substitute $aq_k = (aq_k - q) + q$ and $aq_h = (aq_h - q) + q$, in above expression

$= ((bp_i - ap_k) - w_{ik})((aq_k - q) + q) +$

$((bp_j - ap_h) - w_{jh})((aq_h - q) + q)$

$= ((bp_i - ap_k) - w_{ik})(aq_k - q) + (bp_i - ap_k) - w_{ik}) q +$

$= ((bp_j - ap_h) - w_{jh})(aq_h - q) + ((bp_j - ap_h) - w_{jk})q$

Rearranging and simplifying as in theorem 1, we get

$= ((bp_i - ap_k) - w_{ik})(aq_k - q) + ((bp_j - ap_h) - w_{jh})(aq_h - q) +$

$((bp_i - ap_h) - w_{ih}) q + ((bp_j - ap_k) - w_{jk})q$

$= cov_{ik}(aq_k - q) + cov_{ih}(q) + cov_{jh}(aq_h - q) + cov_{jk}(q)$

where $cov_{ik}(aq_k-q)$ and $cov_{ih}(q)$ are the contributions, to the value of objective function, by assigning quantities $(aq_k - q)$ of ask A_k and q of ask A_h to bid B_i respectively. Similarly $cov_{jh}(aq_h-q)$ and $cov_{jk}(q)$ are the contributions, to the value of objective function, by assigning quantities $(aq_h - q)$ of ask A_h and q of ask A_k to bid B_j respectively. This shows that partial interchange of an ask, assigned to one bid to another, does not affect the value of objective function. We now consider the interchange of any quantity q, in case of two bids.

Theorem 4: Suppose that we have an assignment $A_{sg} = \{(B_j, A_i, q), i = 1, 2, 3, j = 1, 2, 3\}$, in optimum matching in respect of bids B_1, B_2 and B_3. Suppose that A_{sg} is changed as

$A_{sg} = \{(B_1, A_2, q), (B_2, A_3, q), (B_3, A_1, q)\}$ without changing any other assignment, then the value of objective function remains unchanged.

Proof: This result immediately follows from theorem 1. The interchange is equivalent to the following.

(1) Interchange quantity q of ask A_1 from bid B_1 to bid B_2 and quantity q of ask A_2 from bid B_2 to bid B_1. This interchange does not affect the value of objective function (from theorem 1).

(2) Interchange quantity q of ask A_1 from bid B_2 to bid B_3 and quantity q of ask A_3 from bid B_3 to bid B_2. This interchange does not affect the value of objective function (from theorem 1).

Theorem – 5 - Suppose that in optimum matching set of k and m asks are assigned to bids B_i and B_j respectively, as follows.

(1) A_{sg1} = {$B_{i,}$ $A_{ic,}$ q_c) c=1, 2, 3, … , k}.

(2) A_{sg2}= {(B_j, A_{jh} q_h) h=1, 2, 3, … , m}

(3) The quantities assigned satisfy the relation $q = \sum_{c=1}^{k} q_c = \sum_{h=1}^{m} q_h$.

Then the value of objective function does not change if A_{sg1} and A_{sg2} are changed as follows without changing any other assignment.

(a) A_{sg1}= {(B_i, A_{jh} , q_h) h=1, 2, 3, … , m}

(b) A_{sg2} = {($B_{j,}$ A_{ic} , q_c) c=1, 2, 3, … , k}.

Proof: It may be noted that the above interchange is equivalent to following set of successive interchanges, each of which does not change the value of objective function. If $q_{i1} < q_{j1}$, then the above interchange is equivalent to the following.

(1) Interchange ask A_{i1} of quantity q_{i1} with ask $A_{j1.}$ Then interchange quantity ($q_{j1} - q_{i1}$), between asks A_{j1} and A_{i2}. These interchanges do not change the value of objective functions. If ($q_{j1} - q_{i1}$)>q_{j2} then continue the assignment, till all quantity of ask A_{j1} is completely assigned. If ($q_{j1} - q_{i1}$)< $q_{i2,}$ then interchange quantity ($q_{j1} - q_{i1}$) between ask A_{j1} and ask A_{i2}. The next interchange is between ask A_{i2} with remaining quantity and ask A_{j2}. This procedure can be continued till interchange of quantity q is completed. At each stage equal quantity is interchanged. These successive interchanges do not change the value of objective function.

It can be seen that the interchange of asks as in (a) and (b) is equivalent to set of successive interchanges. At each stage there is no change in the value of objective function. Hence by the interchange in (a) and (b), the value of objective function remains unchanged.

It can be easily seen that the above result holds good for k = m = 2. Let us assume that the quantities q_1 and q_2 of asks A_{i1} and A_{i2} respectively, are assigned to bid B_i. Further suppose that the quantities q_3 and q_4 of asks A_{i3} and A_{i4} respectively, are assigned to bid B_j. The quantities assigned satisfy the following relation

$q = q_1 + q_2 = q_3 + q_{4.}$

Then interchanging of asks A_{i1}, $A_{i2,}$ $A_{i3,}$ A_{i4} by assigning quantities q_1 and q_2, currently assigned to bid $B_{i,}$ to bid B_j and assigning quantities q_3 and q_4 of asks A_{i3} and A_{i4}, currently assigned to bid B_j to bid B_i, will not change the value of the optimal matching. In this case

$q_1 \leq aq_{i1}$, $q_2 \leq aq_{i2}$ and $(q_1 + q_2) \leq bq_j$ where bq_j is the quantity of bid $B_{j.}$ If equality condition is satisfied then it is complete interchange of asks. The same relationship also holds good for q_3 and q_4.

Combining asks to fulfill demand – Obtaining Maximum Improvement: In this section, we determine how maximum improvement in the value of objective function as defined in (3.1) can be obtained for a single bid B_i. Let ap_1, aq_1 and as_1 be the price, quantity and size attributes of ask A_1 respectively. Let $ucov_{i1}$ be the contribution to the value of objective function by assignment of quantity 1 (unit quantity) of this ask to bid B_i. It is

$ucov_{i1} = (bp_i - ap_1) - w_{i1.}$.

Suppose that $ucov_{i1}$ is the maximum value.

Let $ucov_{ij}$ be the contribution to the value of objective function by assignment of unit quantity of ask A_j to bid B_i. Then

$ucov_{ij} = (bp_i - ap_j) - w_{ij.}$

Then we have

$ucov_{i1} > ucov_{ij}$ for j = 2, 3, …, m.

Let q_1 be the maximum quantity at price ap_1 and size attribute as_1. Let cov_{i1} be the contribution from this assignment. Then

$$cov_{i1} = (q_1) \times (ucov_{i1}) \quad (3.13)$$

This is the maximum improvement. It can be seen that this term is a product of quantity assigned and contribution per unit. This can be improved if and only if, the term $ucov_{i1}$ (i.e. unit contribution) is improved or q_1 (assigned quantity) is improved. In this case neither is possible – as there is no ask with higher unit contribution and the quantity assigned is the maximum available. Since neither is possible, (3.13) represents, the maximum improvement from a single assignment. (If $q_1 = bq_i$, then it is the maximum possible improvement from assignment to bid B_i.).

Suppose that A_2 is another ask with ask price ap_2 and size as_2 having highest contribution to value of objective function by assignment of unit quantity of ask A_2 except ask A_1. Let $ucov_{i2}$ be the contribution to the value of objective function by assignment of unit quantity of asks A_2 to bid B_i, then

$$ucov_{i2} = (bp_i - ap_2) - w_{i2} \quad (3.14).$$

Also we have

$$ucov_{i2} > ucov_{ij} \text{ for } j = 3, \ldots, n. \text{ and } ucov_{i2} < ucov_{i1} \quad (3.15).$$

Let q_2 be the maximum quantity that can be assigned at price ap_2 and size as_2. In that case these two assignments represent the maximum improvement that can be obtained by assignment of quantity $q_1 + q_2$ and the maximum contribution to the value of objective function which is

$$cov_{i1} + cov_{i2} = (q_1) \text{ x } (ucov_{i1}) + (q_2) \text{ x } (ucov_{i2}) \quad (3.16).$$

It can be seen that the contribution can be improved if any of the two terms on right hand side can be improved. As already seen the first term cannot be improved. The same argument holds good for the second term, as A_1 is the only ask which can improve unit contribution of ask A_2, and it is already assigned. So no other ask can improve unit contribution. The quantity assigned is also the maximum possible value. The second term can not be improved. Therefore it is the highest possible improvement at this stage.

Theorem 6: Let $AS_k = \{A_1, A_2, \ldots, A_{k-1}\}$ be the set of asks which can be assigned to bid B_i. Suppose asks in AS_k satisfy the following conditions.

1. For any ask $A_j \in AS_k$ $ucov_{ij} > ucov_{ij+1}$, $j = 1, 2, \ldots, k-2$. Let $q_1, q_2, .., q_{k-1}$ be the quantities assigned respectively.
2. For any ask $A_j \notin AS_k$ $ucov_{ij} < ucov_{il}$, where $l = 1, 2, .., k-1$ (i.e. set of asks belonging to AS_k

3. Let A_k be an ask such that $ucov_{ik} > ucov_{ij}$, where j = k+1, .., m (set of asks not in AS_k). The assignment of quantity q_k of this ask completely satisfies the demand of bid B_i.

Then we can obtain maximum improvement by assigning all asks in AS_k, and A_k to bid B_i. This is the maximum possible improvement by assigning set of asks or an ask to bid B_i.

Proof: This result can be easily shown by induction. As already shown the result is true for k =1 and k = 2. We assume that the result is true for (k-1) and show that it holds for k.

The total contribution $tcov_{ik}$ to the value of objective function after assignments of k asks is

$$tcovik = \sum_{l=1}^{l=k-1} (q_l) \times (ucov_{il}) + (q_k) \times (ucov_{ik}) \qquad (3.17).$$

The contribution can be improved, if and only if the second term can be improved. Since first term cannot be improved (by assumption), only second term can be improved. In this case, it is not possible to improve second term, since its unit contribution $ucov_{ik}$ is the highest among the remaining asks (and hence cannot be improved). In the same way q_k is the maximum quantity that can be assigned. So the second term cannot be improved. Therefore $tcov_{ik}$ is the maximum improvement in the value of objective function by assignment to a bid B_i. So the result follows by induction.

In case bid B_i has indivisible demand bid constraint, then only a single ask can be assigned to it. In this case, the ask with highest unit contribution and satisfying demand of bid B_i completely, gives the highest improvement.

Application of theorem 6 to a bid with indivisible demand constraint**:** Let B_i be the bid with demand bq_i and price bp_i. The net surplus generated from matching is maximum when the bid is matched with the ask having minimum price, which can satisfy demand completely, if such an ask exists. This ask generates maximum net surplus for a bid with indivisible demand. Let ap_{min} be the price of such ask. The maximum net surplus from this assignment is

$Rb_{imax} = (pb_i - ap_{min})qb_i.$ (3.18).

Suppose that there is another ask A_j with price ap_j, higher than ap_{min}, which satisfies the demand of bid completely. The surplus from this assignment will be

$Rb_s = (pb_i - ap_j) qb_i$

Also ap_{min} being the minimum price we have inequality

$ap_{min} < ap_i$

So $(pb_i - ap_{min}) > (pb_i - ap_j)$

Since qb_i. are positive quantities for all i

$(pb_i - ap_{min}) qb_i. > (pb_i - ap_j) qb_i.$

So $Rb_{imax} > Rb_s$

Suppose that there is another ask A_j with price ap_j having same price as ap_{min}, however its supply (i e. quantity) does not satisfy the demand of bid completely. Let aq_i be its quantity. This does not satisfy the demand fully i.e. $aq_j < bq_i$. The surplus from this assignment is

$Rb_s = (pb_i - ap_j)aq_j$, since $ap_j = ap_{min}$ it can be written as

$= (pb_i - ap_j)aq_j$

Here $qb_i > aq_j$

So $(pb_i - ap_j)aq_j < (pb_i - ap_{min})qb_i.$

All other assignments with higher price or with smaller quantity generates smaller net surplus. This assignment results in maximum improvement in objective function if the sizes of ask and bid are same. So Rb_{imax} is the maximum improvement that can be done in any assignment in objective function.

Improvements with two attributes price and size: Let cov_{ik} be the contribution to value of objective function for assigning quantity bq_i of A_k to B_i is given by,

$$cov_{ik} = (bp_i - ap_k)bq_i - w_{ik}bq_i \quad (3.19).$$

In this case both terms are always nonnegative. Since $bp_i \geq ap_k$ and wastage penalty is non-negative. The second term is subtracted from first term.

So, the maximum contribution occurs when the second term is minimum i.e. 0. So, the assignment that gives the maximum contribution must have minimum ask price and same value for size attributes. Let A_l be the ask with minimum ask price ap_l (also called as ap_{min}) and its size be same as bs_l. The contribution to the value of objective function is

$$cov_{il} = (bp_i - ap_{min})bq_i - 0 = (bp_i - ap_{min})bq_i \qquad (3.20).$$

This will be the maximum contribution, since for any other ask A_k,

$(bp_i - ap_{min})bq_i \geq (bp_i - ap_k)bq_i$ and $w_{ik}bq_i \geq 0$

It can be seen that if price remains same and ask size increases then contribution decreases. If price remains same and ask size decreases, then contribution improves. Let cov_{ik} be the contribution to value of objective function for assigning quantity bq_i of A_k to B_i. In the same way let cov_{ij} be the contribution for assigning quantity bq_i of ask A_j to Bi. Then they are given by

$$cov_{ik} = (bp_i - ap_k)bq_i - w_{ik}bq_i \qquad (3.21).$$

$$cov_{ij} = (bp_i - ap_j)bq_i - w_{ij}bq_i \qquad (3.22).$$

Then assignment of j^{th} ask is an improvement, if and only if ,

$cov_{ij} - cov_{ik} > 0$

$(bp_i - ap_j)bq_i - w_{ij}bq_i - ((bp_i - ap_k)bq_i - w_{ik}bq_i) > 0$

$(bp_i - ap_j) - w_{ij} - (bp_i - ap_k) + w_{ik} > 0$

$bp_i - ap_j - w_{ij} - bp_i + ap_k + w_{ik} > 0$

$ap_k - ap_j + w_{ik} - w_{ij} > 0$

$$(ap_k - ap_j) > (w_{ij} - w_{ik}) \qquad (3.23).$$

So the assignment of j^{th} ask improves the value of objective function, if condition (3.23) is satisfied. This condition is used in algorithm to find out whether any assignment improves the objective function.

3.4 Optimum Assignment for Set of Bids

In this analysis it is assumed that, all the assignments are feasible. Let AS be the set of all asks. Let B_1 and B_2 be two bids. Let AS_1 be the set of asks which are assigned to bid B_1. Let the contribution to the value of objective function cov_1, from this assignment be the maximum. So there is no ask A_i or set of asks $AS_{i,}$ ϵ (AS-AS_1), such that its contribution $cov_i > cov_1$. The contribution cannot be improved by changing the quantity assigned. Let bq_1 be the quantity assigned at this stage.

In the same way AS_2 be the set of asks assigned to bid B_2. This set AS_2 may contain some asks from AS_1, whose supply is not completely satisfied. Let the contribution to the value of objective function cov_2, from this assignment be the maximum. This assignment satisfies the property that, there is no ask A_i or set of asks $AS_{i,}$ ϵ (AS-AS_1-AS_2), such that its contribution (when assigned to bid B_2) $cov_i > cov_2$. The contribution cannot be improved by changing the quantity assigned. Let $bq_1 + bq_2$ be the quantity assigned at this stage.

In this case cov_1 is the optimum contribution, which cannot be improved. On the other assignment to bid B_2 is optimum among the currently unassigned asks (i.e. not assigned to bid B_1 or assigned to B_1 with unfulfilled supply). The contribution cov_2 may be improved by asks from AS_1. In this case improvement means assignment of asks with higher unit contribution or assignment of higher quantity from asks assigned to bid B_1. So when we refer to improvement or higher contribution, both the cases are considered. Then the assignment of all asks in AS_1 to B_1 and AS_2 to B_2 with value of objective function $cov_1 + cov_2$ is optimum in following two cases.

(a) If some or all the asks which are assigned to $B_{1,}$ can be assigned to B_2 and vice versa and this change does not involve any ask which is not in ($AS_1 \cup AS_2$), then assignment A_1 to B_1 and A_2 to B_2 is the optimum for both bids together. The quantity replaced is the same. In this case there is no change in total quantity assigned (bq_1+bq_2). This follows from interchange theorem.

(b) If no ask can be assigned to both bids B_1 and $B_{2,}$ (there is no common set of asks which can be assigned to both the bids) then above assignment is optimum. As there is no assignment, which improves the value of objective function. In this case ($AS_1 \cap AS_2$) = ϕ.

In case (b), both cov_1 and cov_2 cannot be improved so $cov_1 + cov_2$ cannot be improved. In the same way in case (a), cov_1+cov_2 cannot be improved.

We now consider the case where, an ask or set of asks, which improve contribution cov_2 to the value of objective function are assigned to bid B_1. (This includes the case where AS_2 can be an empty set.) We do not consider the case of bid B_1 as cov_1 is globally optimum. So it cannot be improved. Let AS_{2g} be the set of asks currently assigned to bid B_1 and have higher contribution than asks in AS_2, to the value of objective function from assignment to bid B_2. This set AS_{2g} contains, set of asks assigned to B_1 with higher unit contribution, as well as asks currently partly assigned to both bids B_1 and B_2, where higher quantity can be assigned to bid B_2 by decreasing quantity assigned to bid B_1, if such an assignment improves contribution from bid B_2. Let AS_{2n} be the set of asks, with the highest contribution cov_{2n} to the value of objective function from assignment to bid B_2. In other words $cov_{2n} > cov_2$ and there is no ask A_i or set of asks AS_i in $(AS - AS_{2n})$, which improves cov_{2n}. Let AS_4 be the set of asks from AS_2, replaced in new assignment. So AS_{2n} can be written as $AS_{2n} = (AS_2 - AS_4) \cup AS_{2g}$.

Let AS_{1n} be the set of asks from currently unassigned and unfulfilled asks, with contribution to the value of objective function cov_{1n} from assignment to bid B_1. Suppose that apart from sets AS_1, this is the next highest contribution. In other words $cov_1 > cov_{1n}$, however, there is no ask or set of asks in $(AS - AS_{2n})$, which improves it. Further AS_{1n} contains some asks from set $(AS - (AS_1 \cup AS_2))$.

Then the optimum assignment can be determined as follows.

(c) If $cov_1 + cov_2 \geq cov_{1n} + cov_{2n}$, then the first assignment (AS_1 to bid B_1 and AS_2 to bid B_2) is optimum.

(d) If $cov_1 + cov_2 < (cov_{1n}+cov_{2n})$ then the second assignment (AS_{1n} to bid B_1 and AS_{2n} to bid B_2) is optimum. The second assignment is obtained by repeated replacement.

This can be easily generalized for set of k bids. Suppose that AS_{k-1} be the set of asks assigned to set $BD_{k-1} = \{B_1, B_2, \ldots, B_{k-1}\}$ of (k-1) bids. Let cov_{k-1} be the optimum contribution. Let AS_k be the set of asks assigned to bid B_k. This assignment is currently optimum for bid B_k. Let cov_k be the contribution to the value of objective function from this assignment. Then optimum assignment for set k bids can be obtained as follows.

(1) Determine set of asks which can be assigned to bid B_k (currently assigned to set k-1 bids) and satisfy (i) have higher unit contribution (ii) assignment of quantity to bid B_k can be increased.

(2) Then find out which condition (a) or (b) is satisfied. In these cases, we have the optimum assignment.

(3) In other cases, we work optimum assignment as in case (c) and (d). Determine the set of asks $AS_{kg,}$ which is currently assigned to set of bids $BD_{k-1,}$ which can improve contribution (higher unit contribution as well as quantity) to the value of objective function by assignment to bid B_k. Then determine set of asks AS_{kn} with the highest possible contribution cov_{kn} to the value of objective function from assignment to bid B_k. So there is no ask or asks in ($AS - AS_{kn}$), which can improve cov_{kn}. Then determine set of asks with $AS_{k-1n,}$ which can be assigned to first (k-1) bids and have the next highest contribution.

(4) Then determine the optimum assignment for set $BD_k = \{ B_1, B_2, \ldots, B_k\}$ by applying conditions (c) and (d). So if ($cov_{k-1}+cov_k \geq cov_{k-1n}+cov_{kn}$), then assignment AS_{k-1} to bid BD_{k-1} and AS_k to bid B_k is optimum. Otherwise assignment AS_{k-1n} to bid BD_{k-1} and AS_{kn} to B_K is optimum for BD_k.

Stage-wise improvement: The value of objective function can be improved either by increasing surplus from an assignment without increasing wastage factor or by an assignment which brings down wastage without increasing surplus by corresponding amount. The optimum assignment is obtained in stages. In first stage an optimum assignment for a single bid. Then an optimum assignment for set of two bids is determined by carrying out steps described earlier in this section. This procedure is repeated till all assignments are completed. Thus at each stage we obtain the optimum assignment. We present two algorithms for generating optimum assignment in case of assignment and indivisibility constraints. First, we present algorithm to obtain optimum solution where there are only assignment constraints. In the second case, we present an algorithm, which can handle bids with indivisible demand constraints. In the second algorithm, there is an assignment tree construction, which helps in determining how the quantity can be reassigned without changing the value of objective function.

3.5 Algorithm for Obtaining Optimum Assignment

Let the terms bp_i, ap_j and $w_{ij,}$ be as defined earlier. If i^{th} bid can be assigned to j^{th} ask and $bp_i \geq ap_j$, then it is called as feasible assignment. In our algorithm, in first stage, a table of feasible assignments is created. This table consists of three elements. If i^{th} bid can be assigned to j^{th} ask and $bp_i \geq ap_j$, then three elements of the table are i (called as bid indicator), j (called as ask indicator) and unit contribution resulting from assignment of i^{th} bid and j^{th} ask. The last element is

$$ucov_{ij} = (bp_i - ap_j) - w_{ij.}$$

In case $i_1{}^{th}$ bid cannot be assigned $i_2{}^{th}$ ask or $bp_{i1} \geq ap_{i2,}$ then entry i_1,i_2, $ucov_{i1i2,}$ is absent from the table. So all feasible assignments can be determined by scanning this table.

We start assignment from the bid with the highest contribution/the highest unit contribution. Then, an ask or a set of asks, which can be assigned to current bid and have the highest possible unit contribution for this bid is determined. In the next step the maximum quantity that can be assigned is obtained. This assignment maximizes the value of objective function i.e. net surplus and minimum wastage by theorem 6. If ask with maximum unit contribution has enough supply to satisfy the demand of highest price bid, then bid and ask are marked as assigned and assignment continues from next bid. If ask quantity is less than bid quantity, then the ask is marked as temporarily assigned, while bid is marked as partly assigned. In case bid quantity is more, the assignment is continued till demand is satisfied. In the next step, assignment continues from the bid with next highest contribution/the highest unit contribution. First an ask or set of asks from unassigned asks with the highest contribution is determined. Then the optimum assignment is determined as discussed in section 3.4. The process is continued till complete either demand is fulfilled or supply is exhausted. At every stage we have an optimum assignment for set of bids and an assignment, which is optimum for current bid. The optimum assignment for set of bids and current bid is determined. This is the maximum possible improvement at any stage. This follows from the results of theorems

1,2,3,4 and 6 depending upon the assignment. The bid and all asks are marked as assigned. Then assignment is continued in decreasing order of unit contribution. If ask has more quantity, then bid is marked as assigned and ask is marked as partly assigned.

Initially a table indicating demand for different width is constructed. If the ask size is multiple of bid size (e.g. bid width 800 cm and ask width 1600 cm) then this table is used to see whether there is demand for remaining width (i.e. 800 cm). If demand is there then wastage is set to 0. This is helpful in situations to decide, whether bid is to be matched with ask 1600 cm or 1000 cm. Here matching of current bid with an ask with width of 1000 cm will show lesser wastage than that of 1600 cm. However, if there is a demand for 800 cm then it can be matched with the ask of 1600 cm width so that wastage is minimized.

The assignment is continued till one of the three conditions holds good:

(i) no ask is left

(ii) no bid is left

(iii) either total supply is exhausted or total demand is fulfilled.

In our solution assignment is carried out if bid price is either more than ask price or both are same. This assumption is reasonable in the sense that in most exchanges asks are cleared with bids of same or higher price. It can also be seen in [KDL2001], that equilibrium price is first obtained. This price is used to determine the asks and bids which can be cleared (asks below this price and bids above it). Let AS be the list of asks and BD be the list of bids. The main algorithm can be seen in figure 3.1.

```
Algorithm  findopasg(AS,BD)
  Call Create_Unit_Contr_Table
  Call Create_ Size_Demand_Table(bids)  ;
 While ( there is unassigned  bid in B  ){
      Call get_next_unassigned_bid(bids) ;
     Call get_ opt_ask(bid_quantity,bid_size,) ;
     Call assignment(bid,ask,bid_quantity) }
      return ; }
```

Figure. 3.1 Main algorithm

This algorithm calls function “Create_Size_Demand_Table”. This function creates size demand table, which indicates the demand for different values of size attribute. The usage of this table has already been explained. This function can be seen in figure 3.2.

It then calls function "get_next_unassigned_bid()". This function returns the next unassigned bid. This function can be seen in figure 3.3.

The function "get_opt_ask()" returns the set of asks which bring the maximum improvement in the value of objective function. This function in turn calls "get_next_unassigned_ask()", which returns the next unassigned ask that can be assigned to current bid. Then it determines optimum assignment by finding out whether there are assigned asks with higher contribution, which can be assigned to current bid. It then calls function "cal_obv()" to find out improvement in the value of objective function.

Then it selects assignment with maximum value. This function also calls function "search_table()", which searches the size demand table, for a particular value of width attribute, which is passed on as a parameter and returns true, if demand for that value exists. If it returns "true", then wastage is set to 0. These functions can be seen in figure 3.4.The function "assignment()" carries out the assignment of asks and bids and marks them appropriately. This function can be seen in figure 3.5.

```
Create_Size_Demand_Table(bids) {
  while ( there is next bid) {     get bid_size, bid_quantity ;
    call search_table(bid_size,table_size) ;
    if ( .not.. found  )  then  {  increase current_index by  1 ;
          store bid_size to   search_table(current_index,1) ;
         store  bid_quantity to search_table(current_index,2) ; set  table_size to current_index ;}
    else {
         add bid_quantity to   search_table(current_index,2)} read next_bid ; }    return ; }

Create_Unit_Contr_Table(bids) {
  while ( there is next bid) {  while (there is next ask) {  get bid_size, bid_price ;
   get ask_price,ask_size ;     if ( bid_prce ≥ ask_price  )  then  {  increase current_index by  1 ;
          unit_contr =(  bid_price – ask_price ) – wastage_penalty(ask_size,bid_size) ;
          store bid_index,ask_index, unit_contr  to    contr_table(current_index,1) } }; return ; }
```

Figure. 3.2 Function - Creates Size Demand Table

```
get_next_unassigned _bid(bids) {
while ( there is unassigned bid) {   if (bid_price > max_price) then {
       set max_price to bid_price;    set bid to current_bid ; } }  return ; }
```

Figure. 3.3 Function - Gets next unassigned bid

In this algorithm for any bid, we always determine the set of asks, which bring the maximum improvement in the value of objective function. Then it determines the maximum quantity that can be assigned. Then assignment is carried out. This process is repeated for each ask and bid.

Example 3.1 **:** The working of the algorithm is illustrated in the following example. There are five asks and five bids. The assignment constraints are basically size constraints i.e. a bid can be matched with an ask of same or higher size. The wastage penalty is 8 here. The output can be seen in Table – 4.4.

The algorithm has been implemented in C++. It is tested with randomly generated data sets of different sizes. Each data set consisted of ask price, ask quantity, ask size, bid size, bid price and bid quantity. The data set consisted of number of asks with ask price, quantity, ask size, bid size, bid price and bid quantity. Size of data sets varied from 5 to 100.

```
-This function returns set of asks which bring maximum improvement to the value of objective function-
get_opt_ask(bid_quantity,bid_size){while(there is unassigned ask and asg_quantity<bid_quantity))
 call get_next_unassigned_ask(bid_index) ; if ( ask_quantity >= bid_quantity ) asg_quantity =
asg_quantity + bid_quantity ;
 if ( bid_quantity > ask _quantity ) then   qty_asg = asg_quanttiy+ ask_quantity ; }
if ( optmum return ) ;
copy assignment to old_assignment ;
while {( assigned_ask with higher contribution )
  creat new_assignment
 call cal_obv(bid,ask,qty_asg) ;  }
  if (  ov > max_imp )  then {  return new_assignmet  ;}else return  old_assignment ;   }

-This function determines the next unassigned ask that can be assigned to current bid –

get_next_unassigned ask(bid_index) { {ucontr = high_neg_value;
while (  i <= table_size ) ; if ( ask unassigned  and  bid_index = bid ) and (unit_contr > ucontr )
 {  ucontr = unit_contr ; current_ask  = ask_index; qty_sup = qty_sup + ask_qty } ;
if ( ask assigned and bid index = bid ) { qty_inv = qty_inv + qty_assigned;}
if ( qty_inv = 0 ) or ( qty_inv >= ask_qty) {optimum = true} ; return
}

- This function finds the value of objective function -

cal_obv(bid,ask,qty_asg) {       net_surplus =    ( bid_price – ask_prce ) * bid_quantity ;
      wastage =    ( ask_size – bid_size ) * bid_quantity  ;
      call search_table(wastage,table_size);
      if found then set wastage = 0 ;      ov = net_surplus + wastage ;    return }

-This table searches the Size Table -
Search_Table(bid_size,table_size) {
while ( i <= table_size ) {
     If (search_table(I,1) = bid_size) then   {set  current_index to i ;    return .true ; }
    else          {increase i by 1 ;}   return false; }
```

Figure. 3.4 Functions used in main algorithm findopasg

```
assignment(bid,ask,qty_asg) {
assign ask to bid ;  quantity_assigned = qty_asg ;
if (  qty_asg = bid_quantity ) then            mark bid as assigned ;
if ( qty_asg = ask_quantity    and ( ask_size – bid_size = 0 ) ) then      mark ask as assigned ;
if ( qty_asg < bid_quantity ) then   bid_quantity = bid_quantity – qty_asg ;
if ( qty_asg < ask_quantity ) then    ask_quantity = ask_quantity – qty_asg ;
if (  ( ask_size – bid_size) > 0 ) then   ask_size = ask_size-bid_size; return ; }
```

Figure. 3.5 Function Assignment

The results were compared with unconditional optimum solution and some solutions obtained with the help of the MATLAB package. It has been observed that our algorithm always generated optimum solution. It has also been seen that if the size of ask is constant and the bid sizes are variable but take few values (as in most of the practical cases), then we can ignore the wastage factor. The approach can be, to obtain the maximum surplus and then readjust assignment without affecting value of objective function. It can also be seen that time complexity of our algorithm is always polynomial. In this algorithm a matching ask which generates the maximum improvement can be obtained by scanning unassigned asks at any point of time. In the first instance there are n unassigned asks, in the next instance there are (n-1) unassigned asks and so on. So, in all, the solution will require to scan n(n-1)/2 asks. So, the time complexity is polynomial and of the order $O(n^2)$ or O(mn). The unit contribution table created with time complexity is O(nm). So the time complexity is always polynomial. Apart from this, the demand size table and minimum or maximum price asks/bids can be obtained with linear time complexity. So, overall, the time complexity is always polynomial.

Table. 3.4 Bids and Asks Optimum Solution

Bid	Price	Quantity	Size	Ask	Price	Quantity	Size	Bid	Ask	Spread	Quantity	Net Surplus	Wastage
1	187	11	8	1	101	23	8	1	1	86	11	946	0
2	181	12	4	2	109	8	12	2	1	80	12	960	4
3	173	23	8	3	121	6	12	3	4	38	4	152	0
4	161	10	12	4	135	4	8	3	5	22	19	418	0
5	157	8	8	5	151	23	8	4	2	52	8	416	0
		64				64		4	3	40	2	80	0
								5	3	36	4	144	4
								5	5	6	4	24	0
											64	3140	8

This compares quite favorably with the complexity O[(nm+ n^2 log n)] of algorithm presented in [KDL2001], apart from its simplicity for large instance of n and m. The comparison can be seen in the figure – 3.6.

Algorithm for assignment in case of indivisible demand constraints: Now the algorithm to obtain the optimum assignment in case of indivisibility constraints is presented. This algorithm can also handle other types of assignment constraints. This algorithm differs from above algorithm, since it needs to keep track of quantity assigned in case of bids with indivisible demand constraints. It uses assignment tree construction to determine a set of bids to which assignment should be carried out without changing the value of objective function.

Assignment Tree

The algorithm uses assignment tree data structure to determine the assignment of asks and bids, which do not change the value of objective function. Assignment tree helps in determining distribution of ask quantity among different bids, so that the value of objective functions remains unchanged from unconstrained assignment and bids with indivisible demand are satisfied.

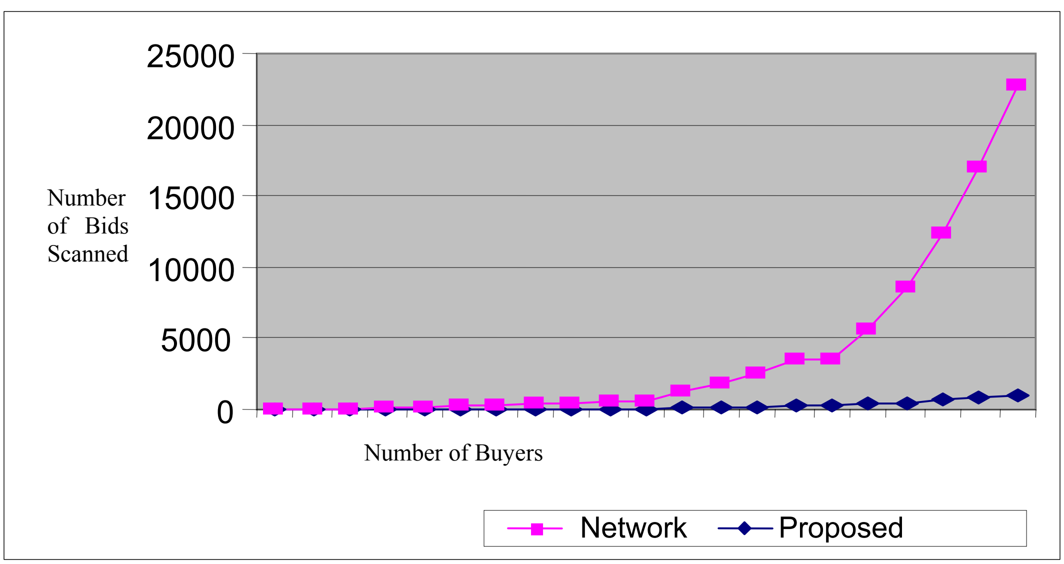
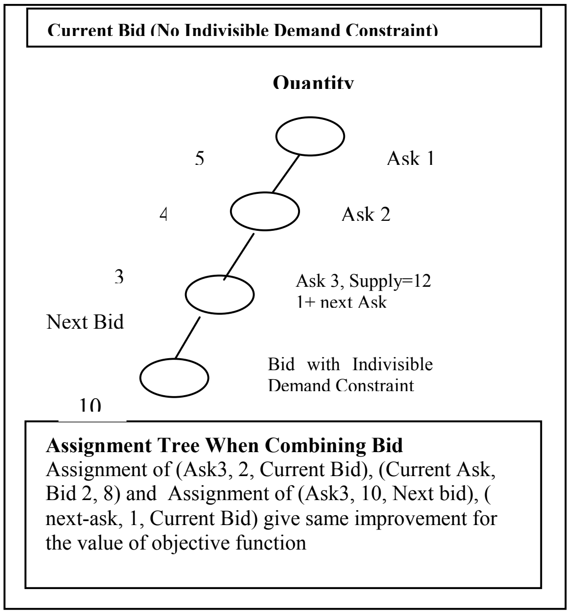


Figure. 3.6 Comparative Performance of the Algorithm (assignment constraints)

This tree is constructed when an ask has more supply than the current bid or to obtain adjustment quantity for combining different bids. The left child of assignment tree always represents bids without indivisible demand bid constraints and right child represents the bids with indivisible demand bid constraints, which can be satisfied by current ask. The nodes at first level are the ones which bring highest improvement, nodes at second level are the next highest and so on. All the paths represent different assignments of the current ask. Any one of the assignments can be selected, as they do not change the value of objective function.

The algorithm selects one assignment (application of theorem 1 to 5 as the case may be). This is helpful in cases where only one ask can be assigned to bids with indivisible demand bid constraint. It can be seen, by applying results from theorem 1 and 2, that the different assignments obtained from assignment tree, will not change the value of objective function. So, these paths can be used to determine the assignment. In the same way while assigning asks to bids, which require combining of different bids, the assignment tree is constructed to obtain adjustment quantity, especially when the next bid has indivisible demand bid constraint. Using this adjustment quantity the asks are distributed among these bids. In this case, it can be seen, by applying results from theorems 1, 2, 3, 4 and 5 that, redistribution of quantity among different bids, will not change the value of objective function. An assignment tree is shown in figure 3.7, which indicates two different situations of assignment tree.

These constructs ensure that it is possible to redistribute quantity among different bids without affecting objective function value. The adjustment scenario is shown in Figure 3.7(a). The current bid has demand of 3 and the next bid has indivisible demand constraint with demand of 10. The current ask has supply of 12. The assignment of 10 from current ask to next bid, 2 to current bid and assignment of 1 from next ask do not change the value of objective function. Its value remains the same as unconstrained assignment. Figure 3.7(b) shows where supply is more than demand. In this case assignment of current ask to bids 1 and 3, or 1 and 4, do not change the value of objective function. This follows from application of theorem 1,2,3,4 and 5.



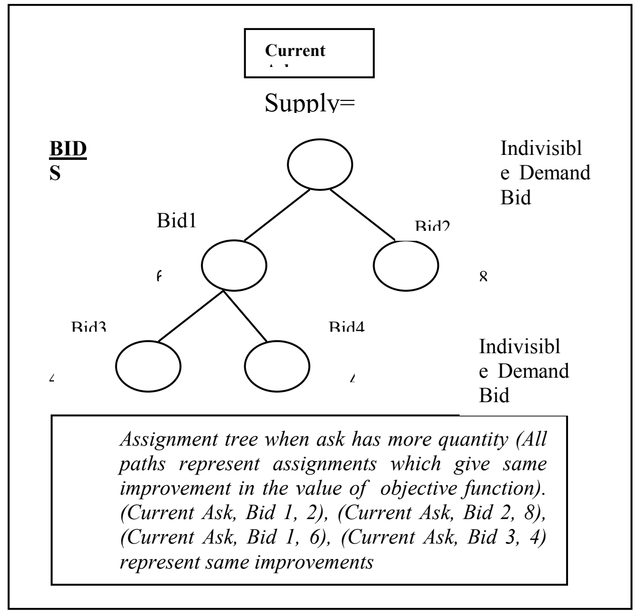


Figure. 3.7 (a) Assignment Tree adjustment scenario   Figure – 3.7(b) Assignment Tree excess supply scenario

We give brief description of working of this algorithm. The assignment starts from the bid with the highest contribution/the highest unit contribution. This bid is assigned to ask which maximizes the value of objective function i.e. net surplus and minimum wastage penalty. If this bid has indivisible demand bid constraint then an ask of same or more quantity which brings the maximum improvement in the value of objective function is selected. If the ask quantity is less than the bid quantity and bid does not have indivisible demand bid constraint, then the ask is marked as assigned, while bid is marked as partly assigned and will continue the assignment from next ask onwards.

If both quantities are same then asks and bids are marked as assigned and the assignment continues. The assignment is continued in decreasing order of unit contribution for bids. At each stage the bid is assigned to currently available ask or set of asks which gives the maximum improvement in value of objective function (theorem 6). In case of bids with indivisible demand bid constraint, we obtain an ask of equal or more quantity that brings the maximum improvement (theorem 6). When ask has more quantity, then an assignment tree is constructed. Using this assignment tree, an assignment is worked out. The assignment tree is also constructed while combining different bids for finding out adjustment quantity. In this case we apply theorems 4 and 5. After determining this assignment, optimum assignment is worked as shown in section 3.4. A table indicating demand for different width is constructed. Its usage is already explained. The assignment is continued till one of the following three conditions hold good (i) no ask is left (ii) no bid is left (iii) ask price exceeds that of bid price.

In this solution, assignment is carried out if bid price is either more than ask price or if both are same. This assumption is reasonable in the sense that in most exchanges asks are cleared with bids of same or higher price. It can also be seen in [KDL2001], that equilibrium price is first obtained. This price is used to determine the asks and bids which can be cleared (asks below this price and bids above it). Let AS be the list of asks and BD be the list of bids.

Algorithm findopasgic(AS,BD) presented here generates the optimum assignment of asks and bids which maximizes surplus and minimizes the wastage, while satisfying indivisible demand and assignment constraints. As in earlier algorithm the table of unit contributions where assignments are feasible is created. Then it creates the demand size table, which contains demand for different sizes. It uses number of functions for finding ask which generates the maximum improvement in the value of objective function, assigning asks to bids and for assignment tree construction. The main algorithm can be seen in figure 3.8.

```
Algorithm  findopasgic(AS,BD)
  Call Create_Unit_Contr_Table
   call create_ size_demand_table(bids) ;
   while ( there is unassigned  bid in B  )  { if ( nobid = false) then get  next unassigned bid ;
    call get_ opt_ask(bid_quantity,bid_size,bid_type); if(bid_size=ask_size) then
    call assignment(bid,ask,bid_quantity); if  ( ask_size > bid_size  ) then
    call   find_no_change_assignment(bid,ask);   if (bid_size > ask_size  and bid_divisible = True ) then
call    more_assignment(bid,ask);   check_optim(assignment)   return ;  }    }    }
```

Figure. 3.8 Main algorithms

It calls function "get_opt_ask()", which finds out the ask which brings out maximum improvement in the value of objective function from current assignment. This function can be seen in figure 3.9.

```
get_opt_ask(bid_quantity,bid_size,bid_type)
do while ( there is unassigned ask)  {     read next unassigned_ask ;
     if ( bid_type = indivisible)   then   qty_asg = bid_quantity ;
     if  ( ask = unassigned and   ask_quantity ≥ bid_quantity  and   ask_size ≥ bid_size) then
      qty_asg = bid_quantity ;    call cal_obv(bid,ask,qty_asg) ;
      if ( bid_type = divisible ) and   ( ask_size ≥ bid_size ) ) then {
         {  if  ( bid_quantity < ask _quantity )  then  {  qty_asg = bid_quantity ; }
           else  { qty_asg = ask_quantity ;  } }       call cal_obv(bid,ask,qty_asg) ;}
      if ( ov > max_imp )  then {    ask_ret = current_ask ;      max_imp = ov ;  }  return  ;   }
```

Figure. 3.9   Function to get optimum ask

It also calls function "create_size_demand_table()", which creates the size demand table. This function stores the total demand for different values of width attribute. The function "get_opt_ask()", calls another function "cal_obv()", which finds value of objective function after current assignment and returns ask/set of asks which bring maximum improvement. While finding value of objective function it uses "search_table()", which searches the size demand table for a particular value of width (which is passed as parameter) and returns true, if demand for that value exists. If "true" is returned then wastage is set to 0. These functions are shown in figure 3.10. The function "no_change_assignment()", constructs assignment tree as described earlier for the current ask. It calls function "get_next_bid()", to find out bids, which can become nodes of assignment tree.

These are the bids, to which assignment can be made without changing value of objective function. The function "more_assignment()" , is used to set "nobid" parameter to true, when bid quantity is more than ask quantity. When value of this parameter is true, algorithm continues assignment of current bid. Main algorithm and other functions use function "assignment()", to assign asks to bids. If the quantity of assignment is equal, then asks and bids are marked as fully assigned otherwise they are marked as partly assigned depending upon the quantity. It also adjusts width parameter.

The function check_optim() finds out whether the assignment is optimum or not. It is done by finding out whether any assignment of assigned ask or set of asks with higher unit contributions improves current value of objective function. In case there is improvement new assignment is selected as optimum assignment. These functions are shown in figure 3.11.This algorithm first locates ask/set of asks that brings out maximum improvement in value of objective function, effects assignment and repeats the process. The working of algorithm is illustrated with following simple example.

```
-This function creates demand size table –

create_ size_demand_table(bids) {
while ( there is next bid) {  get bid_size, bid_quantity ;   call search_table(bid_size,table_size) ;
 if (.not. found ) then  {  increase current_index by 1 ; store  bid_size to  search_table(current_index,1) ;
store  bid_quantity to  search_table(current_index,2) ;
set  table_size to current_index  ;}
else {  add bid_quantity to search_table(current_index,2)};  read next_bid ; }    return ; }


Create_Unit_Contr_Table(bids) {
  while ( there is next bid) {  while (there is next ask) {  get bid_size, bid_price ;
   get ask_price,ask_size ;    if ( bid_prce ≥ ask_price  )  then  {  increase current_index by  1 ;
        unit_contr =(  bid_price – ask_price ) – wastage_penalty(ask_size,bid_size) ;
        store bid_index,ask_index, unit_contr  to   contr_table(current_index,1) } }; return ; }

- This function finds ask with maximum improvement -
get_opt_ask(bid_quantity,bid_size) {
while ( there is unassigned ask) {   { {ucontr = high_neg_value;
while (  i <= table_size ) ; if ( ask unassigned  and  bid_index = bid ) and (unit_contr > ucontr )
 {  ucontr = unit_contr ; current_ask  = ask_index; qty_sup = qty_sup + ask_qty } ;
      if  ( ask_quantity >= bid_quantity    and ask_size >= bid_size) then
          qty_asg = bid_quantity       end if ;
      if  ( bid_quantity < ask _quantity ) then   qty_asg = qty_asg_+ bid_quantity ;
     mark ask temp_assigned ;      call cal_obv(bid,ask,qty_asg) ;
     if (  ov > max_imp )  then {   ask_ret = current_ask ;      max_imp = ov ;  }return  ;   }


- This function finds the value of objective function -
cal_obv(bid,ask,qty_asg) {
      net_surplus =    ( bid_price – ask_prce ) * bid_quantity ;
      wastage =    ( ask_size – bid_size ) * bid_quantity  ;
      call search_table(wastage,table_size);
      if found then set wastage = 0 ;   ov = net_surplus + wastage ;     return }       -

- This function searches the table -
Search_Table(bid_size,table_size) {
while ( i <= table_size ) {    If (search_table(I,1) = bid_size) then    {set  current_index to i ;
         return .true ; }
    else         {increase i by 1 ;}    return false; }
```

Figure. 3.10 – Functions used in main algorithm

Example 3.2**:** In Table 3.5 a simple example with 5 bids and 4 asks is shown. Out of these five bids, first three bids have indivisible demand bid constraints. The table is divided into three parts which show bids submitted by buyers, asks submitted by sellers

```
- This function finds asks which do not change objective function values. -
no_change_assignment(bid,ask) {
start_bid = bid ;
while (  qty_asg <=  ask_quantity ) {
   qty_asg = qty_asg + bid_quantity ;
    if ( bid_type = divisible ) then  add bid as left_node of tree ;
    if ( bid_type = indivisible ) and  ( bid_quantity <= ask_quantity ) then {
      add bid as right node of tree ;   ind_qty = ind_qty + bid_quantity ; }
    call get_next_bid(qty_asg) ;  } last_bid = current_ bid ;

if ( ask_quantity = ind_qty  or ind_qty > 0 ) then
  { while ( ask_quantity > 0 ) { while ( ind_qty > 0 ) {
      search  right subtree ;   qty_asg = bid_quantity ;
      call  assignment(bid,ask,qty_asg) ;    ask_quantity = ask_quantity – qty_asg ;
   ind_qty = ind_qty – aty_asg ;   if ( ask_quantity <= 0)  then return ;  }
  search left  subtree  ;   qty_asg = bid_quantity ;          call  assignment(bid,ask,qty_asg) ;
     ask_quantity = ask_quantity – qty_asg ;
    if ( ask_quantity <= 0)  then return ;  }   }
else { while ( ask_quantity > 0 ) {   search left subtree ;  qty_asg = bid_quantity ;
         do  assignment(bid,ask,qty_asg) ;
ask_quantity = ask_quantity – qty_asg ;   }   return ; } }

- This function obtains next bid -

get_next_bid(qty_asg) {
while ( there is unassigned bid) {  if ( bid_type = indivisible and  bid_quantity <= qty_asg and bid_size
>= ask_size ) then  {    qty_chk = bid_quantity ;  call cal_obv(bid,ask,qty_asg) ;}
else  if  ( bid_size >= ask_size )  then {
    if ( bid_quantity > rty_asg ) then            qty_chk = bid_quantity ;  }
    else {  qty_asg = ask_quantity  ; }          call cal_obv(bid,ask,qty_asg) ;
   if ( ov > max_imp )  then {  bidk_ret = current_bid ;  max_imp = ov ; } } return ; }

- This function sets no bid parameter to true -

more_assignment(bid,ask)   {
qty_asg = ask_quantity ;     do assignment(bid,ask,qty_asg) ;
 bid_quantity = bid_quantity – ask_quantity ; nobid = true ;  return ; }
-This function carries oot assignment –
assignment(bid,ask,qty_asg) {_assign ask to bid ;  quantity_assigned = qty_asg ;
if ( (  qty_asg = bid_quantity ) and  ( ask_size – bid_size = 0 )  )  then  mark bid as assigned ;
if ( qty_asg = ask_quantity )  then  mark ask as assigned ;
if ( qty_asg < bid_quantity )  then   bid_quantity = bid_quantity – qty_asg ;
if ( qty_asg < ask_quantity )  then   ask_quantity = ask_quantity – qty_asg ;
if (  ( ask_size – bid_size) > 0 )  then ask_size = ask_size-bid_size;  return ; }
check_optimum() { if no common ask or ask_inerchange t return optimum else save current assignment ;
while ( assigned ask > unit_contr than current ask) get_opt_ask ; if imp > oval then oval – imp ;
 if  imp > oval
     change_assignment ;
      return ; }
```

Figure. 3.11 Functions used in algorithm findopasgic

and the optimum assignment generated by our algorithm. It generates the net surplus of 1024 without any wastage, which is same as optimum solution.

3.6 Experimental Results and Conclusion

The algorithm is implemented in C++. It is tested with randomly generated data sets of different sizes. Each data set consisted of ask price, ask quantity, ask size, bid size, bid price and bid quantity. The sizes of data sets varied from 5 to 100. The results were compared with unconditional optimum solution and some solutions obtained with the help of the MATLAB package. The performance of our algorithm against other algorithms [KDL2001] for number of different bids is plotted in figure 3.12. . It shows

Table. 3.5 - Illustrative example of solution generated by algorithm

Bids				Asks				Assignment obtained by Algorithm					
Bid	Price	Quantity	Size	Ask	Price	Quantity	Size	Bid	Ask	Spread	Quantity	Net Surplus	Wastage
1	171	5	8	1	121	6	12	1	4	34	5	170	
2	167	8	8	2	127	6	12	2	3	35	8	280	
3	161	6	12	3	132	8	8	3	1	40	6	240	
4	154	6	4	4	137	8	8	4	2	27	6	162	
5	151	8	8					5	2	24	6	144	
		33						5	4	14	2	28	
												1024	

the number of asks and bids required to be scanned by our algorithm against the other

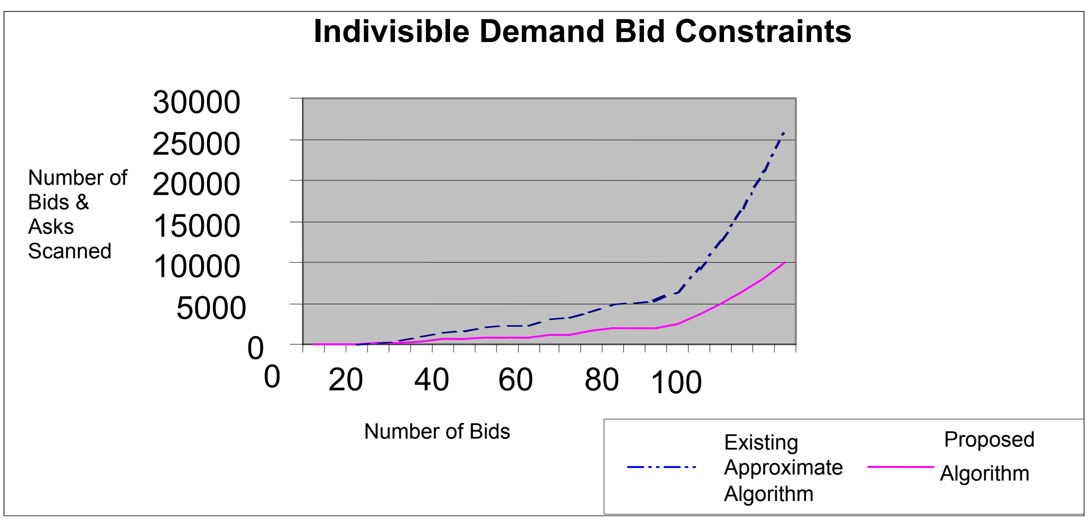


Figure. 3.12 Comparative performance of algorithm

algorithms. In the graph, x-axis represents number of bids and y-axis represents number of asks and bids required to be scanned to obtain the optimum solution. The time complexity of our algorithm is polynomial. An ask/set of asks which can give maximum improvement can be obtained by scanning at most n asks.

Theorem 7: It can be seen that the algorithm will generate optimum solution in both cases. (Indivisibility constraints with or without assignment constraints and only assignment constraints).

Proof: Let AS be the set of all asks. Let AS_1 be the set of ask/asks, which when assigned to bid B_1, brings the maximum improvement in the value of objective function. Let cov_{a1} be the contribution value of objective function after assignment of these asks/ask to bid B_1. If bid has indivisible bid constraint then there will be only one element in AS_1. There is no ask A_i or set of asks $AS_i \epsilon (AS - AS_1)$, such that cov_{a1} can be improved. The contribution cannot be improved by changing quantity assigned in respect of any ask in AS_1. At this stage bq_1 be the quantity assigned.

Let AS_2 be the set of ask/asks, which brings maximum improvement in the value of objective function after assignment to second bid B_2. This set will not contain set of asks in AS_1, whose supply is completely satisfied but may contain some asks in AS_1, which are partly assigned to first bid. Let cov_{a2} be the contribution to the value of objective function from this assignment. The value of cov_{a2} is 0, when AS_2 is empty set. There is no ask A_i or set of asks $AS_i \epsilon (AS- AS_1-AS_2)$, such that cov_{a2} can be improved. The quantity assigned is maximum possible. Let OV_2 be the value of objective function after second assignment, then clearly that is the maximum possible value. The contribution cannot be improved by changing quantity assigned in respect of any ask in AS_2. At this stage $(bq_1 +bq_2)$ be the quantity assigned.

It can be seen that $OV_2 = cov_{a1} + cov_{a2.}$ (3.24).

Let cov_{a1i} be the highest contribution to the value of objective function from assignment of asks $\epsilon (AS – AS_1)$ to bid B_1. Then we have $cov_{a1} > cov_{a1i.}$

Let cov_{a2i} be the highest contribution to the value of objective function from assignment of asks $\epsilon (AS – AS_1- AS_2)$ to bid B_2. Then we have $cov_{a2}> cov_{a2i.}$

So we have $OV_2 = cov_{a1} + cov_{a2} > cov_{a1i} + cov_{a2i.}$ (3.25)

It is possible that assignment of asks from set AS_1 may improve the contribution from second bid B_2. Three possible cases here are as follows.

(1) No common set of asks between bids B_1 and B_2.

(2) It is possible that assignment of asks from set AS_1 may improve the contribution from second bid B_2. However these asks can be completely replaced by asks in AS_2 and can be assigned to bid B_1. In this case only asks from ($AS_1 \cup AS_2$) are assigned to bids B_1 and B_2. The total quantity assigned is (bq_1+bq_2).

(3) An ask or set of asks in AS_1 (currently assigned to bid B_1) can improve contribution to the value of objective function, if they are assigned to bid B_2 (both cases i.e. asks assigned to bid B_1 with higher unit contribution or assignment of higher quantity than currently assigned to bid B_2 in case of asks assigned to both bids are considered). However this requires assignment of asks, currently not in ($AS_1 \cup AS_2$) to bid B_1 (in order to satisfy demand of B_1 and B_2). In other words it requires assignments of asks not in ($AS_1 \cup AS_2$). In some cases the total quantity assigned can be smaller than (bq_1+bq_2).

In case (1), it can be seen that both terms of expression (3.24) i.e. cov_{a1} and cov_{a2} cannot be improved from asks ϵ ($AS - AS_1 - AS_2$). Further no ask from AS_1 can be assigned to bid B_2 and no ask from AS_2 can be assigned to bid B_1. So the value of objective function cannot be improved. Thus expression (3.25) is the optimum value.

In case (2), the current value of objective function is $OV_2 = cov_{a1} + cov_{a2}$. Here cov_{a1} cannot be improved. The term cov_{a2} can be improved by asks currently assigned to bid B_1. However it is possible to assign asks from AS_2 to B_1, to replace the asks now assigned to bid B_2. This will not change the value of objective function by interchange theorem. So (3.25) is the optimum value, which cannot be improved.

In case (3) assignment of asks in AS_1 to bid B_2 decreases the contribution to the value of objective function from B_1. Let AS_3 be the set of asks (currently not in AS_1 or AS_2) which replaces the asks in AS_1 with higher unit contribution (or assignment of higher quantity) to bid B_2. (The asks from AS_3 are now assigned to bid B_1). Suppose that this assignment results in the next highest contribution cov_{n1} to bid B_1. In other words $cov_{a1} \geq cov_{n1}$. The quantity assigned to bid B_1 is the maximum possible at this stage. There is no unassigned ask, which can produce higher contribution than cov_{n1}.

Let cov_{n2} be the highest contribution to the value of objective function from new assignment to bid B_2. So the value of objective function is
$OV_{n2} = cov_{n1}+cov_{n2}$.

In this case second value cannot be improved. If $cov_{a1} + cov_{a2} > cov_{n1}+cov_{n2}$, then first assignment is optimum. This follows because no other assignment of asks to bids B_1 and B_2 improves the value of objective function. There is no ask or set of asks which can improve cov_{a1}. The quantity assigned is the maximum possible. Only ask or set of asks assigned to bid B_1, improves the contribution cov_{a2}. However this change does not improve the value of objective function as earlier value of objective function is higher.

In case $cov_{a1} + cov_{a2} < cov_{n1}+cov_{n2}$, the second assignment is optimum. This follows because contribution ($cov_{n1}+cov_{n2}$), cannot be improved.
Here $cov_{a1} > cov_{n1}$ and $cova_2 < cov_{n2}$, such that $cov_{a1} + cov_{a2} < cov_{n1}+cov_{n2}$.

The contribution cov_{n2} cannot be improved. The first term cov_{a1} can be improved, but this results in decrease in value of second term so that total value is not optimum i.e. $cov_{a1} + cov_{a2} < cov_{n1}+cov_{n2}$. Any other assignment decreases the value of objective function. So the second assignment is optimum.

So for k = 1 and k = 2 the result is optimum. Suppose that after assignment to k bids B_1, B_2, … , B_k the value of objective function is OV_k. Let this be the optimum value. Let AS_k be the set of asks which are assigned to all these k bids. Let AS_{k+1} be the set of unassigned asks (remaining ask or set of asks from AS_k, which are not completely assigned) which has the highest contribution to the value of objective function after assignment to bid B_{k+1}. Let cov_{ak+1} be this contribution. Then the value of objective function is $O_{k+1} = O_k + cov_{ak+1}$.

This value can be improved only if any two terms can be improved. The first term cannot be improved by assumption. In case of second term the contribution cannot be improved from remaining asks. As earlier there are three possible cases here.
In case (1), there are no common set of asks which can be assigned to set of bids

$Bd_k = \{B_1, B_2, \ldots, B_k\}$ and B_{k+1}. Each of this term is individually optimal. Hence following similar argument as in case of k = 2, it can be shown that the assignment is optimum.

In case (2), cov_{ak+1} can be improved by ask or set of asks currently assigned to bids $BD_K=\{B_1, B_2, \ldots, B_k\}$. If these asks are assigned to bid B_{k+1}, the ask or set of asks assigned to bid B_{k+1} can replace them without affecting any other assignment. There is no change in quantity replaced. So it is an interchange of asks hence will not change the value of objective function. It can be further proved that the second term cannot be improved following a similar procedure as done for k = 2.

In case (3), cov_{ak+1} can be improved by ask or set of asks currently assigned to bids in BD_k. However this requires assignment of currently unassigned asks to bids in BD_k. As earlier O_{nk} be the next highest contribution from assignment to bids in BD_k and cov_{nak+1}, is the maximum contribution from assignment to bid B_{k+1}. This cannot be improved by any other ask. As earlier in this case if

$O_k + cov_{ak+1} > O_{nk} + cov_{nak+1}$, the first assignment is optimum. In other cases second assignment is optimum. This follows by extending same argument as in earlier case, when k = 2. Hence the result follows by induction. So the algorithm generates optimum solution.

3.7 Payoff Determination Problem

Double Auctions can be used to implement efficient many to many electronic negotiations. There are two main problems in double auctions as follows.

(1) Finding out optimal assignment of asks and bids. Algorithms for finding optimum assignments of asks and bids in case of assignment constraints and indivisibility constraints have been developed.

(2) Designing a payment scheme for buyers and sellers such that auction mechanism satisfies properties like Incentive Compatible, Individual Rational, Budget Balance and Efficient. At this stage net payment by each buyer and net payment to each seller is to be computed. This will be referred to as Mechanism Design under assignment constraints.

Payment Mechanism: In the next step, the payment mechanism, which computes payoff of different buyers and sellers, is to be designed. Essential properties of any payment mechanism are:

(1) Strategy Proof (IC): - The payment mechanism should be strategy proof. It means that truthful bidding should be the dominant strategy. This property is also known as Incentive Compatibility.

(2) Individual Rationality (IR)**:** The payment mechanism should be individual rational. It means each buyer and seller should gain some amount by participating in auction. Otherwise there will not be any incentive for participating in auction.

(3) Budget Balance (BB)**:** The payment mechanism should be budget balanced. It means that difference between total payments to the sellers and total receipts from buyers should be nonnegative. It means that the auction should not be run in loss.

(4) Efficient (EF)**:** It means that the total profit obtained through the auction mechanism should be the maximum. The objects are assigned to those who value them the most.

However there is a well known impossibility result of Myerson and Satterthwaite [MS1983], which states that no mechanism can be efficient, budget balanced, incentive compatible and individual rational in respect of double auctions, at the same time. There can be many approaches to this problem.

Approaches for Mechanism Design

The payment mechanism, which is strategy proof, weakly budget balanced and individually rational is presented in [HWS2002]. This mechanism is based on single price for buyers and sellers. Other approach (known as VCG Mechanism) is to compute VCG Payments [V1961],[C1971],[G1973]. However VCG Payment mechanism is not budget balanced. This happens if amount to be paid to sellers exceeds the total receipts from buyers. In this case the auction has to run in loss. There can be number of approaches to achieve budget balance property.

(1) One approach can be to design the payment scheme to minimize the distances from VCG Payments. This approach is followed in [PKE2001],[PKE2002].

(2) The participation fees from the buyer can be introduced and budget balance can be achieved. However in this case some buyers and sellers may not choose to participate. This can make the scheme inefficient. In our analysis we follow different payment schemes and compare them.

In VCG mechanism in the first step optimum assignment of asks and bids is obtained. Then amount payable by each buyer and receivable by each seller is computed. Let amp_{bj} be the amount payable by j^{th} buyer. Let amp_{si} be the amount payable by i^{th} seller. These amounts are computed as

$$amp_{bj} = bp_j - (V_o - V_{bj}) \qquad (3.26)$$

where V_o is the optimum solution of (3.1) or the total surplus from the optimum assignment of (3.1) and where V_{bj} is the optimum solution of (3.1) or the total surplus from the optimum assignment of (3.1) after deleting j^{th} buyer and keeping all other conditions same. In the same way the amount receivable by i^{th} seller will be

$$amr_{si} = ap_i + V_o - V_{si} \qquad (3.27)$$

where V_{si} is the optimum solution of (3.1) or the total surplus from the optimum assignment of (3.1) after deleting i^{th} seller, keeping all other conditions same. So computation of VCG Payments requires that number of optimization problems should be solved.

3.8 VCG Payment Computation

In this section we derive some important results which can be used to VCG Payments without requiring solution of new optimization problems.

Cyclic interchange of asks and bids : Suppose that there is a set of bids $BD_k = \{B_i, i=1, 2, \ldots, k\}$. Let bq_i (i = 1, 2, …, k) be the demand of i^{th} bid. Let the total demand of these k bids be $q = \sum_{i=1}^{k} bq_i$. Let $AS_{k1} = \{A_i, i = 1, 2, .., k_1\}$ be the set of asks which is assigned to bid set BD_k. Let aq_i ($i=1,2..,k_1$) be the assigned quantity. Then we have $q = \sum_{i=1}^{k1} aq_i$. The assignment is defined by set $A_{sg} = \{(B_i, A_j, aq_j), i = 1, 2, \ldots, k, j = 1, 2, .., k_1\}$.

So that quantity aq_1 of ask A_1 is assigned to bid B_1 and quantity aq_{k1} of ask Ak_1 is assigned to bid B_k. Then we define cyclic interchange of quantity q_c ($q_c \leq aq_1$) as a change, which changes the assignment set A_{sg} to $\{(B_i, A_j, (aq_j - q_c)), i = 1, 2, \ldots, k\text{-}1, j = 2, \ldots, k_1, (B_k, A_1, q_c)\}$. A cyclic interchange of quantity q_c, in any assignment of asks to bids, will be called feasible if it is allowed as per the constraints. We call a feasible cyclic interchange as partial cyclic interchange, in case the assignment set gets changed as

$\{(B_i, A_j, (aq_j - q_c)), i = 1, 2, \ldots, k\text{-}1, j = 2, \ldots, k_1\}$

In other words a cyclic interchange is an interchange where quantity q_c of ask A_1 is assigned to bid B_k and quantity q_c of asks A_i ($i = 2, \ldots, k_1$) are assigned to bid B_i ($i=2, 3, \ldots, k\text{-}1$). In case some asks A_i ($i = 2, 3, \ldots, k_1$), have assigned quantity $aq_i < q_c$, then before interchanging asks are combined fully or partly such that assigned supply equals q_c. In partial cyclic interchange quantity q_c of ask A_1 is not assigned to bid B_k and the demand of quantity q_c for bid B_k remains unfulfilled.

Suppose that there are three bids, with bid prices bp_1, bp_2 and bp_3 respectively. Let bq_1, bq_2 and bq_3 be their respective quantities and bs_1, bs_2 and bs_3 be respective sizes. Suppose that optimum assignment of asks and bids, is as shown in Table 3.6.

.

Table. 3.6 Optimum assignment

Bid	Bid Price	Ask	Ask Price	Ask Size	Bid Size	Quantity Assigned
b_1	bp_1	a_1	ap_1	as_1	bs_1	q_1
		a_2	ap_2	as_2		q_2
b_2	bp_2	a_3	ap_3	as_3	bs_2	q_3
		a_4	ap_4	as_4		q_4
		a_5	ap_5	as_5		q_5
b_3	bp_3	a_6	ap_6	as_6	bs_3	q_6

An example of cyclic interchange and partial cyclic interchange is shown in Table 3.7 and 3.8.

We state the following two results in case of bids. However they are symmetric and can be applied to asks as well.

Theorem 8 : A feasible cyclic interchange of quantity q_c in an optimum assignment will not change the value of objective function.

Proof : Suppose that in optimum assignment we have set of k bids B_i with demand bq_i (i=1, 2, ... , k). Suppose that set of k_1 asks A_i (i=1, 2, 3, …, k_1) with supply aq_i, be assigned to these set of k bids. So we have assignment set

A_{sg} = {(B_i, A_j, aq_j), i =1, 2,…, k, j = 1, 2, ... , k_1}.

Suppose that a feasible cyclic interchange of quantity q_c of ask A_1 is carried out. Then the assignment set will change to

{(B_i, A_j, ($aq_j - q_c$)), i =1, 2, …, k-1, j = 2, ... , k_1, (B_k, A_1, q_c)}.

In this interchange quantity q_c of ask A_1 is assigned to bid B_k. In the same way quantity q_c of asks A_i (i=2, 3, …, k_1) are assigned to bids B_1, B_2, ... , B_{k-1}. The quantity q_c of ask A_1 is assigned to bid B_k. All these assignments are feasible. It can be seen that this interchange is equivalent to following set of interchanges.

(1) Interchange quantity q_c of ask A_1 assigned to bid B_1 with quantity q_c of ask A_2 assigned to bid B_2. This interchange does not change the value of objective function as well as surplus (By theorem 1).

(2) In next step interchange quantity q_c of ask A_1 assigned to bid B_2 with quantity q_c of ask A_3. This interchange does not change the value of objective function as well as surplus.

Table. 3.7 Cyclic interchange

Bid	Bid Price	Ask	Ask Price	Ask Size	Bid Size	Quantity Assigned
b_1	bp_1	a_2	ap_2	as_2	bs_1	q_2
		a_4	ap_4	as_4		q_1
b_2	bp_2	a_3	ap_3	as_3	bs_2	q_3
		a_4	ap_4	as_4		$(q_4 - q_1)$
		a_5	ap_5	as_5		q_5
		a_6	ap_6	as_6		q_1
b_3	bp_3	a_6	ap_6	as_6	bs_3	$q_6 - q_1$
b_3	bp_3	a_1	ap_1	as_6	bs_1	q_1

Table. 3.8 Partial cyclic interchange

Bid	Bid Price	Ask	Ask Price	Ask Size	Bid Size	Quantity Assigned
b_1	bp_1	a_2	ap_2	as_2	bs_1	q_2
		a_4	ap_4	as_4		q_1
b_2	bp_2	a_3	ap_3	as_3	bs_2	q_3
		a_4	ap_4	as_4		$(q_4 - q_1)$
		a_5	ap_5	as_5		q_5
		a_6	ap_6	as_6		q_1
b_3	bp_3	a_6	ap_6	as_6	bs_3	$q_6 - q_1$

The above steps are repeated till bid B_k is reached and quantity q_c is assigned to bid B_k. It may be noted that k is finite. As each interchange does not change the value of objective function and surplus the result follows.

Theorem 9: In case of partial cyclic interchange of quantity q_c of ask A_1 with ask price ap_1 is not assigned to bid B_k with bid price bp_k with q_c being unfulfilled demand, then the value of objective function changes by $cov_{1k}(q_c) = (bp_k - ap_1)q_c - (as_1 - bs_k)q_c$.

Proof : This result follows directly by application of earlier theorem. Let ov be the value of objective function in the optimum assignment.

Suppose that in optimum assignment we have set of k bids B_i with demand bq_i (i=1, 2, … , k). Suppose that set of k_1 asks A_i (i=1, 2, 3…,k_1) with supply aq_i, be assigned to these set of k bids. So we have assignment set

$A_{sg} = \{(B_i, A_j, aq_j), i = 1, 2, …, k, j = 1, 2, …, k_1\}$

Suppose that a partial cyclic interchange of quantity q_c of ask A_1 is carried out. Then the assignment set will change to

$\{(B_i, A_j, (aq_j - q_c)), i = 1, 2, …, k-1, j = 2, ..., k_1\}$.

In case complete interchange of quantity q_c is applied the value of objective function does not change. In this case the value of objective function is

$ov = ov_{k-1} + cov_{1k}(q_c)$, ov_{k-1} is the contribution to the value of objective function, from assignment of bids excluding bid B_k and ask A_1. Let $cov_{1k}(q_c)$, be the contribution to the value of objective function by assigning ask A_1 to bid B_k. So the difference when complete interchange is carried out and partial interchange is carried out is

$cov_{1k}(q_c) = (bp_k - ap_1)q_c - (as_1 - bs_k)q_c$.

Further the change in surplus is ($bp_k - ap_1$)$q_{c.}$.

These two results help in obtaining VCG payment without actually solving set of optimization problems. We start with an optimum assignment of (3.1). Suppose that there are n buyers. In order to obtain VCG payment of k^{th} buyer, we remove k^{th} buyer. After deleting k^{th} buyer we have an optimization problem with (n-1) buyers and m sellers. Let $BD_{k-1} = \{B_1, B_2, \ldots, B_{k-1}\}$, be the set bids of buyers 1, 2, …, k-1. Similarly let BD_2, be set of bids of buyers (k+1) to n. Let AS_1 and AS_2 be the set of asks assigned to bid sets $BD_{k-1,}$ prior to assignment to bid B_k, and BD_2. Let AS_k be the set of asks assigned to k^{th} buyer's bid.

```
Algorithm  VCG(BD,AS,k) {
  while ( new buyer) {
     for ( k + 1 to n ) do {
       add current  ask  to unassigned ask set
       for current bid determine the ask with the highest unit contribution ;
       assign ask }
    change in surplus =  (current_bid_price – unassigned_ask_price ) current bid }
    while ( new seller) {
     for ( k + 1 to m ) do {
        add current bid to unassigned bid set
        for current ask  determine the bid with the highest unit contribution ;
       assign bid}
    change in surplus =  (current_bid_price – unassigned_ask_price ) current bid }}
```

Figure. 3.13 – Algorithm for computation of VCG Payment

In order to determine VCG payment of k^{th} buyer, we first determine the set of bids, to which the asks in AS_k (assigned to the bid of k^{th} buyer) can be assigned. In this case it is not necessary to verify the bids in set BD_{k-1}. This is due to the fact our optimization algorithm assigns asks with the highest unit contribution to any bid at any point of time. So all the asks in set AS_1, which are assigned to bids in set BD_{k-1} have higher unit contribution than the asks in set AS_k. It is not necessary to carry out assignment again, as we have optimum assignment for BD_{K-1}. So we start assignment from $(k+1)^{th}$ bid and continue verifying bids in set BD_2. We add asks in set AS_k to set AS_2. Let AS_3 be this set.

Then the assignment between BD_{k-1}, BD_2 and set of asks in AS_3 is carried out by applying our optimization algorithm. This process is continued till all bids in set BD_2 are assigned. As the supply now is more than demand, quantity equal to demand of k^{th} buyer from some ask will remain unfulfilled. So the surplus from this ask and k^{th} buyer ask will give the change in the value of surplus and then using (3.26) and (3.27) VCG payment can be obtained. This process is repeated for all buyers and sellers. Then we obtain VCG payment fro each buyer and seller in polynomial time.

Let BD be the set of bids and AS be the set of asks. So the main algorithm to compute VCG payment is shown in figure 3.13.

Theorem 10: The solution obtained by above algorithm is optimum for new problem of (n-1) buyers and m sellers and vice-versa.

Proof : Suppose that there are n buyers and m sellers in original optimization problem. In order to obtain VCG payment of k^{th} buyer, we remove k^{th} buyer. After deleting k^{th} buyer we have an optimization problem with (n-1) buyers and m sellers. Let $BD_{k-1} = \{B_1, B_2, ... , B_{k-1}\}$, be the set bids of buyers 1, 2, ... , k-1. Similarly let BD_2, be set of bids of buyers (k+1) to n. Let AS_k and AS_2 be the set of asks assigned to bid sets BD_{k-1} and BD_2 respectively. Let AS_k be the set of asks assigned to k^{th} buyer's bid. Let AS_3 be the set of asks such that $AS_3 = AS_2 \cup AS_k$.

The set of asks in AS_1 are assigned to the set of bids in set BD_{k-1}. Let ov_1 be the contribution to value of objective function from this assignment. In our algorithm the assignment is the set of asks in AS_3 is carried out with set of bids in BD_{k-1} and BD_2. After this assignment we get an assignment which optimum for bids in set BD_{k-1} and B_{k+1}. This process is carried out till assignment of all remaining bids is completed.

At each stage we get optimum assignment. It may be noted that number of buyers is finite. So the assignment is the optimum and the surplus generated is also the optimum surplus. Since there is no assumption about value of k, the result is true for all buyers.

This argument can be easily extended for group of asks or in case of sellers.

It can be seen that the time complexity of the algorithm is always polynomial. It can be seen that an ask or bid with highest unit contribution can be determined by scanning (n-1) asks or (m-1) bids in worst case for a single bid or ask. There are (n-1) asks and m bids or vice versa. So an optimum solution can be obtained with worst case time complexity of m(n-1) or n(m-1). In worst case scenario all VCG payment for all buyers and sellers can be obtained with worst time complexity of $O(n^3)$. It can be seen that in case of optimization formulation (1), the performance of this algorithm will be linear, as the change in surplus can be determined by subtraction of last bid and first ask in the set. If a table maintaining unit contribution of different asks and bids is created then the performance of the algorithm can be speeded up considerably.

3.9 Achieving Budget Balance

VCG Payment mechanism is not budget balanced. This means that the total amount payable to all sellers exceeds the total amount receivable from all buyers. In such a scenario auction has to run in loss.

One approach to achieve budget balance has been presented in [PKE2001]. In this the total surplus is allocated to individual buyers in such a way that the distance to Vickrey discount of each individual buyers and sellers is minimized. The problem of distance minimization is formulated as constrained optimization problem treating budget balance (BB) and individual rationality (IR) as hard constraints. However, this approach is not truthful i.e. truthful bidding is not the dominant strategy.

Let V be the total surplus obtained after optimum assignment. Suppose that VCG Discount for each participant has been obtained. A participant in the transaction can be a buyer or a seller. This discount is obtained for individual buyers and sellers. Let there be n buyers and m sellers. In all there are (n + m) participants. Let bv_i and sv_j be the Vickrey discounts of i^{th} buyer and j^{th} seller respectively. Let bd_i and sd_j be the discounts of i^{th} buyer and j^{th} seller, so that the scheme becomes budget balanced. These discounts are computed in such a way, they minimize the distance from VCG discounts.

Let $D(bd_i,sd_j,bv_i,sv_j)$ be distance function. Then the problem is formulated as

$$\min D(bd_i,sd_j,bv_i,sv_j) \qquad (3.28)$$

$$\text{s.t.} \quad \Sigma\, bd_i + \Sigma\, sd_j \leq V \qquad (3.29)$$

$$bd_i \leq bv_i \quad \text{for } i = 1,2,\ldots,n \qquad (3.30)$$

$$sd_j \leq sv_j \quad \text{for } j = 1,2,\ldots,m \qquad (3.31)$$

$$bd_i \geq 0 \quad \text{for } i = 1,2,\ldots,n \qquad (3.32)$$

$sd_j \geq 0 \quad \text{for } j = 1,2,\ldots,m$ (3.33)

In this case constraint in (3.29) ensures budget balance. The constraints in (3.30) and (3.31) ensure that the individual discount of each buyer and seller does not exceed their respective Vickrey discount. The constraints (3.32) and (3.33) ensure individual rationality. So all the participating buyers and sellers make positive gains. The different types of distance function considered are

(1) square distance i.e. $\Sigma (bv_i - bd_i)^2 + \Sigma (sv_j - sd_j)^2$,

(2) maximum distance

(3) relative error function

(4) squared relative error

(5) weighted error function.

Then the problem is not directly solved, an analytic function for the solutions that correspond to each distance function is obtained. Each family of solution is a parameterized payment rule. The buyers and sellers (participants) are arranged in the order of Vickrey discounts. The different payment rules those are implemented are as follows:

Threshold: If threshold parameter falls into interval k, then first k participants receive discount, while others do not receive any discount.

Small: If parameter falls into interval k, then the first k participants do not receive any discount, while others receive.

Fractional: In this scheme each participant receive discount, which is equal to proportional share of Vickrey discount of the participant.

Large: This is reverse of small.

Reverse: This is reverse of threshold.

Pay what you bid: In this scheme the buyer pays the amount, which is equal to product of his bid price and the quantity assigned. The seller gets the amount equal to the product of his ask price and the quantity assigned.

VCG Discount**:** In this scheme, VCG discounts are computed and each buyer and seller gets VCG Discount. So the amount payable by the buyer will be the difference between product of bid price and quantity assigned, and VCG Discount of the buyer. Each seller will get an amount equal to the sum of the ask price and the quantity assigned, and VCG Discount of the seller.

It can be seen that in above rules, except in case of fractional rule, there is possibility of some buyers and sellers not getting any discount. If VCG scheme is not budget balanced then our scheme computes the pay off so that budget balance is achieved.

In this section, we work out the payment scheme which is budget balanced as follows : The payment by each buyer and to be paid to each seller has to be determined after the optimum assignment has been found out. The main problem is how to distribute the total surplus obtained. The budget balance constraint means that total payment from all the buyers and the total amount to be paid to all the sellers must be the same. It is also desirable that electronic exchange remains neutral between buyers and sellers. In double auctions with assignment constraints, different buyers and sellers trade multiple units of goods, on multiple attributes, with aggregation over demand and supply. In such a scenario it may not be possible to decide the criteria on distribution of surplus. We attempt to determine some criteria for distribution of surplus.

3.10 Payment Scheme Based on Contribution

Let BD_i be the set of bids selected from i^{th} buyer in the optimum assignment. It consists of three elements viz., bid price (bp_i), matching ask price (ap_k) and quantity matched q_i. Then we define the contribution of the i^{th} buyer to the total surplus as

$$covs_{bi}(q) = \Sigma_{BDi} (bp_i - ap_j)q_j \qquad (3.34).$$

The summation is over all elements of set BD_i.

If $\sum_{BDi} q_i = q$, it means that the demand of i^{th} buyer is completely fulfilled, otherwise it will be partially fulfilled. In case the demand of i^{th} buyer has a single bid with indivisibility constraint, there will be only one element in set BD_i. The buyer's contribution is sum of products of differences between matched bids and asks and quantity matched in between different asks and bids. It can be easily seen that the total surplus from matching will be sum of all the contributions of all buyers whose bids are matched. So we get

$$V=\Sigma_{BD}covs_{bi}(q) \tag{3.35}$$

where BY is the set of buyers, whose bids are partly or fully matched. Let AS_j be the set of asks selected from j^{th} seller in the optimum assignment. It consists of three elements ask price (ap_j), matching bid price (bp_i) and quantity matched q_j. Then we define the contribution of the i^{th} seller to the total surplus as

$$covs_{Asj}(q)=\Sigma_{ASj}(bp_i - ap_j)q_j \tag{3.36}$$

The summation is over all elements of set AS_j. (Its definition is symmetric to that of buyer's contribution).

If $\sum_{ASj} q_j = q$, it means that the supply of j^{th} seller is completely utilized, otherwise it will be partially utilized. In case the ask of j^{th} seller is completely assigned to a bid with indivisibility constraints, there will be only one element in set AS_j. The seller's contribution is the sum of products of differences between matched bids and asks and quantity matched in between different asks and bids. It can be easily seen that the total surplus from matching will be sum of all the contributions of all sellers whose bids are matched. So we get

$$V=\Sigma_{SL}\, covs_{sj}(q) \tag{3.37}$$

where SL is the set of sellers, whose bids are partly or fully matched.

The contribution is used as the value that each buyer and seller brings to the assignment problem. This concept is similar to the concept of added value concept in game theory [BS1996]. Added value measures how much each player contributes to the game. It is computed as the difference between the surplus when all participants are there and the surplus by removing one participant each time. It is computed for each participant. However instead of added value, we use buyer's and seller's contribution The reason for using this is that in case of double auctions with assignment constraints, it may not be possible to get added value, as removing a seller means it may not be possible to match certain bids. This will make computation of added value difficult. So we use each participant's contribution in the same way as added value. We can define certain fairness criteria for surplus distribution. The surplus can then be distributed using these criteria as follows.

(1) A participant gets higher proportion of surplus if his contribution is higher. If i^{th} buyer has higher contribution than $(i+1)^{th}$ buyer, then i^{th} buyer gets higher proportion of surplus. The fairness of the scheme also means that a participant with no contribution, does not get any proportion of surplus and the one with the same contribution gets the same proportion of surplus.
(2) A proportion of surplus should be linear function of contribution and should vary at constant rate with change in contribution. This makes that payments worked out are fair to all the participants, and there is no favour to any particular buyer or seller.

The budget balance constraint means that the sum of proportions of surplus for both the buyers and the sellers should be 1. In this case we allocate exactly half of the available surplus to buyers and remaining half to the sellers. Let sp_{bi} and sp_{sj} be the proportion of surplus allocated to i^{th} buyer and j^{th} seller respectively. The proportion for the i^{th} buyer can be calculated as follows

$$sp_{bi} = \frac{2covs_{bi}(q)}{V} \tag{3.38}$$

It can be seen that the payment scheme worked out satisfies the requirements stated in (1) and (2).

$$sp_{bi} = sl(covs_{bi}(q)) + inc$$, where sl and inc are constants.

This follows from the linearity assumption and the requirement that if $covs_{bi}(q)$ is same then the proportion should be same. Since if contribution is 0 the proportion is also 0. This means that inc = 0 and we get

$$sp_{bi} = sl(covs_{bi}(q)) \tag{3.39}$$

Adding it for all buyers we get

$$\Sigma\, sp_{bi} = \Sigma\, sl(covsbi(q)) \tag{3.40}$$

Since $\Sigma\, sp_{bi} = 1$, we get

$$sl\, \Sigma\, covsbi(q) = 1 \quad sl = \frac{1}{V} \tag{3.41}$$

This means $sp_{bi} = \frac{sp_{bi}}{V}$.

Apart from payment based on surplus the payment schemes based on quantity, price-quantity product and VCG discount are also proposed.

Payment Scheme based on Quantity

We can also use quantity purchased for distribution of surplus, instead of contribution. This can also be considered as a fair scheme, since the surplus is distributed as per quantity purchased. Extending the above argument to quantity purchased means that:

(1) A participant gets higher proportion of surplus if quantity purchased is higher. If i^{th} buyer purchases higher quantity than $(i+1)^{th}$ buyer, then i^{th} buyer gets higher proportion of surplus. The fairness of the scheme also means that a participant with no quantity sold or purchased does not get any proportion of surplus and the one with same quantity gets the same proportion of surplus.

(2) A proportion of surplus should be linear function of quantity purchased/sold and should vary at constant rate with change in contribution. This makes that payments worked out is fair to all the participants, and there is no favour to any particular buyer or seller.

In the same way as above we can work out the formula for proportion as

$sp_{bi} = \frac{q_{bi}}{q}$, where q is the total quantity and q_{bi} is the quantity purchased by i^{th} buyer. In this scheme some buyers or sellers may not gain much by bidding untruthfully. It can be seen that the buyer or seller purchasing smaller amount of quantity can make only limited gain, though the scheme may not prevent them from bidding untruthfully. We do not consider the case of misreporting of other attributes by buyers here. The requirements of buyers change (e.g. different widths), if they misreport other attributes.

Payment Scheme based on Price Quantity Factor

It is expected that buyer who bids with higher price and higher quantity need not have higher contribution than a buyer with lower price. This can happen in case of constrained assignment, where buyer's bid can be matched with an ask of higher price (though lower than matching bid price). So the combination of price and quantity (i.e. pay what you bid can also be used for surplus) is used. So the proportion can also be calculated as

$$sp_{bi} = \frac{p_{bi}}{\sum p_{bi}} \qquad (3.42).$$

where pb_i is the amount payable by i^{th} buyer in case of no discount.

Payment Scheme based on Discount

Another method that can be used, for calculating discount can be based on Vickrey discount for individual participant. We can use following two functions for calculating the Vickrey discount.

$$df = \frac{V}{\sum_i bdi + \sum_j sdj} .$$

It will be a constant factor for all buyers and sellers.

The other formula can be to multiply individual discount by $\frac{1}{\sum_i bdi + \sum_j sdj}$ to arrive at , df_{bi} the individual discount factor for i^{th} buyer. Let bd_i be the discount allocated to the i^{th} buyer by any of the above schemes. Then the amount payable of i^{th} buyer is

$$bpa_i = bp_i\, q_i - bd_i \qquad (3.43)$$

Where bp_i is the per unit bid price of the bid submitted by i^{th} buyer and q_i is the quantity allocated. Since $bd_i > 0$, each buyer pays less than what he bids, and hence makes positive gain. So the scheme is individual rational (IR). The scheme is also budget balanced (BB), since the total amount payable to the sellers is equal to the total amount receivable from the buyers. So the auction does not run in loss. The scheme is efficient as the objects are assigned to those who value them the most. Further it can be seen that $bd_i \leq V$ for all i, so the maximum gain by any buyer is always bounded. The same holds for the seller. In the next section the effect of changing prices by different buyers is studied. It may be noted that that all the schemes are efficient. The analysis has been carried out in respect of payment scheme by contribution. The analysis in case of other payment schemes is similar and hence it is omitted. We carry out the experimental analysis for all the above schemes.

3.11 Effect of Changing Bid Price by a Single Buyer

The above schemes are not strategy proof. So truthful bidding is not the dominant strategy. So we attempt to find the gain that buyer can make by bidding untruthfully and how the effect can be minimized. Suppose that each buyer and seller have their private valuation and they use it while bidding. Initially it is assumed that only one buyer bids untruthfully, while others bid truthfully. Let bpv_i be the private valuation of the i^{th} buyer. Let bpa_i be the amount payable by i^{th} buyer when auction clears. Then gain (also called as utility of the buyer) of the buyer bg_i is

$$bg_i = bpa_i - bpv_i \qquad (3.44).$$

It can be seen that the buyer can improve his gain by bidding for a lower amount, by decreasing the quantity required and keeping price unchanged or by decreasing both. However without loss of generality it can be assumed that the buyer prefers to lower price rather than lowering the quantity required. Also price and quantity are generally related e.g. buyer is willing to pay lower price for higher quantity and so on. Hence it is assumed that the buyer is willing to lower price rather than quantity, while bidding. In bidding truthfully the i^{th} buyer submits a bid with bid price based on his true valuation of the payable amount. Let bp_i be the bid price submitted by the buyer based on his true valuation. However if truthful bidding is not the dominant strategy, the buyer sets his bid price as

$$butp_i = bp_i - x \qquad (3.45)$$

where x is a positive real number.

If the i^{th} buyer bids the above amount, whereas all other bid truthfully the surplus will change by amount xq_i, where q_i is the quantity allocated. The surplus decreases because the buyer submits lower price. So the new surplus will be

$$V_n = V - xq_i \qquad (3.46)$$

The second term is positive and hence the $V_n < V$.

The i^{th} buyer's contribution decreases by xq_i. There will also be similar decrease in price quantity multiplication factor.

Theorem 11: If only one buyer decreases the amount

(1) the proportion of surplus allocated to that buyer also decreases under contribution payment scheme

(2) the scheme allocates entire decrease in surplus to the buyer. The utilities of all others remain unchanged and the gain of the buyer (who decreases his price) is always bounded.

Proof: It can be seen that if i^{th} buyer decreases his price by x, the contribution and the surplus decreases by xq_i.

Let V be the total surplus. Let $covsb_i(q)$ be the contribution of i^{th} buyer to the surplus. Let ocovs be the contribution of other buyers. Then we can write

$$V = covsb_i(q) + ocovs \tag{3.47}$$

If the i^{th} buyer decrease his bid price by x, the contribution will change to

$$covsn_{bi}(q) = covs_{bi}(q) - xq_i \tag{3.48}$$

The surplus will change to

$$V_n = V - xq_i = covsb_i(q) + ocovs - xq_i$$
$$= (covsb_i(q) - xq_i) + ocovs \tag{3.49}$$

So the proportion of surplus that is allocated changes in two schemes as

in contribution based scheme

$$spn_{bi} = \frac{2\,cov\,sn_{bi}(q)}{V_n}$$

$$= \frac{2covsb_i(q) - 2xq_i}{(covsb_i(q) - xq_i) + ocovs} \tag{3.50}$$

If the buyer bids truthfully then the proportion will be

$$sp_{bi} = \frac{2covsb_i(q)}{V}$$

$$= \frac{2covsb_i(q)}{covsb_i(q) + ocovs} \tag{3.51}$$

Substituting $a = covsb_i(q)$, $b =$ ocovs , $y = xq_i$, in (3.50) and (3.51) these expressions can be written as

$$spn_{bi} = \frac{2a - 2y}{(a - y + b)} \tag{3.52}$$

$$sp_{bi} = \frac{2a}{(a+b)} \tag{3.53}$$

Subtracting (3.52) from (3.53) we get

$$sp_{bi} - spn_{bi} = \frac{2a}{(a+b)} - \frac{2a - 2y}{(a - y + b)}$$

$$= \frac{2a(a - y + b) - (a+b)(2a - 2y)}{(a+b)(a - y + b)}$$

$$= \frac{2a\,a - 2ay + 2ab - 2aa + 2ay - 2ab + 2by}{(a+b)(a - y + b)}$$

$$= \frac{2by}{(a+b)(a - y + b)} \tag{3.54}$$

Since y > 0 and y < a, the above term is always positive. So

$$sp_{bi} - spn_{bi} > 0$$

So the proportion of surplus allocated to the i^{th} buyer decreases. The discount also decreases (As there is decrease in surplus).

The discount of i^{th} buyer db_i, when he bids truthfully is

$$db_i = sp_{bi} V = \left(\frac{2covsb_i(q)}{V}\right) V = 2covsb_i(q) .$$

The discount of i^{th} buyer dbn_i, when he bids untruthfully is

$$dbn_i = spn_{bi} V = \left(\frac{2covsnb_i(q)}{V_n}\right) V_n = 2covsnb_i(q) = 2(covsb_i(q) - xq_i) .$$

So change in discount is $db_i - dbn_i = 2xq_i > 0$. So entire decrease in surplus is allocated to i^{th} buyer.

So the difference in amount payable is

$bp_i\, q_i - sp_{bi} V - (bp_i\, q_i - xq_i - spn_{bi} V_n)$

$= bp_i\, q_i - 2\, covsb_i(q) - (bp_i\, q_i - xq_i - 2(covsb_i(q) - xq_i))$

$= bp_i\, q_i - 2\, covsb_i(q) - (bp_i\, q_i - xq_i - 2covsb_i(q) + 2\, xq_i)$

$= bp_i\, q_i - 2\, covsb_i(q) - (bp_i\, q_i + xq_i - 2covsb_i(q))$

$= bp_i\, q_i - 2\, covsb_i(q) - bp_i\, q_i - xq_i + 2covsb_i(q) = - xq_i.$

So i^{th} buyer can improve utility only by xq_i. The scheme does not allocate any additional discount. This can be similarly proved for price quantity scheme.

Further if only i^{th} buyer decrease bid price and prices of others remain the same, then the proportion of discount of j^{th} buyer ($j \neq i$) increases. If all buyers bid truthfully, then the proportion of surplus to be allocated to j^{th} buyer is

$sp_{bj} = \frac{2covsb_j(q)}{V}$, where $covsb_j(q)$ be the contribution of j^{th} buyer to the surplus.

If i^{th} buyer decreases, his bid price and no other buyer change their respective bid prices, then the proportion of surplus to be allocated to j^{th} buyer is

$spn_{bj} = \frac{2covsb_j(q)}{V_n}$,

as $V > V_n$, we get $sp_{bj} < spn_{bj}$. So discount of all other buyers improve. So the new discount of j^{th} buyer is

$$dbn_j = spn_{bj} V_n = \left(\frac{2covsb_j(q)}{V_n}\right)V_n = covsb_j(q) = db_j \qquad (3.55).$$

As there is no change in bid price and discount, the utilities of others do not change. So if any buyer decreases his bid price, his gain is always be bounded and the utilities or the payable amount of other buyers remain unchanged. The gain is bounded. This is due to the fact that if the buyer continues to decrease his price, he may be out of auction after some stage. The scheme is not strategy proof, however it ensures that the gain of buyer will be bounded and that of others who bid truthfully remains does not change. This analysis can similarly be extended to seller.

Extending this result to the scenario where more than one buyer or seller bid untruthfully may not be straightforward. If two or more buyers bid untruthfully, the proportion of surplus allocated to all may not decrease. This can be seen from the following.

Suppose that two buyers i and j bid untruthfully. As earlier let $covs_{bi}(q)$ and $covs_{bj}(q)$ be their respective contributions to the surplus V when bidding truthfully. Let ocovs be the contribution of other buyers. Suppose that by bidding untruthfully the contribution decrease by x and y respectively. The proportion of surplus that is allocated to these two buyers is

$sp_{bj} = \frac{a}{(a+b+c)}$, where $a = covs_{bi}(q)$, $b = covs_{bj}(q)$ and $c = ocovs$.

$$sp_{bj} = \frac{b}{(a + b + c)}$$

The proportion of discount, when they bid untruthfully is

$$spn_{bj} = \frac{(a - x)}{((a - x) + (b - y) + c)} \quad (3.56)$$

$$spn_{bj} = \frac{(b - y)}{((a - x) + (b - y) + c)} \quad (3.57)$$

The proportion that is allocated to buyer B_i will change by

$$\frac{a}{(a + b + c)} - \frac{(a - x)}{((a - x) + (b - y) + c)}$$

$$= \frac{a((a - x) + (b - y) + c) - (a - x)(a + b + c)}{(a + b + c)((a - x) + (b - y) + c)}$$

$$= \frac{(a\mathit{a} - ax + ab - ay + ac) - aa - ab - ac + ax + xb + xc}{(a + b + c)((a - x) + (b - y) + c)}$$

$$= \frac{bx + cx - ay}{(a + b + c)((a - x) + (b - y) + c)}$$

This expression is positive if (bx+cx-ay) > 0.

In this case the proportion increases. It decreases if (bx+cx-ay) < 0, and does not change if (bx+cx-ay) = 0.

In the same way the difference in proportion of the second buyer is

$$\frac{b}{(a + b + c)} - \frac{(b - y)}{((a - x) + (b - y) + c)}$$

$$= \frac{b((a - x) + (b - y) + c)}{(a + b + c)((a - x) + (b - y) + c)}$$

$$= \frac{(ab - bx + bb - by + bc) - ab - bb - bc + ay + yb + yc}{(a + b + c)((a - x) + (b - y) + c)}$$

$$= \frac{ay + cy - bx}{(a + b + c)((a - x) + (b - y) + c)}$$

This expression is positive if (ay+cy-bx)>0. In this case proportion increases.

It decreases if (ay+cy-bx)<0, and does not change if (ay+cy-bx) = 0. However this does not guarantee the discount increase as the overall surplus also goes down. We can generalize this as follows.

Suppose that three buyers i, j and k bid untruthfully. As earlier let $covs_{bi}(q)$, $covs_{bj}(q)$ and $covs_{bk}(q)$ be their respective contributions to the surplus V when bidding truthfully. Let ocovs be the contribution of other buyers. Suppose that by bidding untruthfully the contribution decrease by x, y and z respectively. The proportion of surplus that is allocated to these three buyers is

$$sp_{bi} = \frac{a}{(a+b+c+d)},$$

where a= $covs_{bi}(q)$, b = $covs_{bj}(q)$ and c = $covs_{bk}(q)$, d = ocovs.

$$sp_{bj} = \frac{b}{(a+b+c+d)}$$

$$sp_{bk} = \frac{c}{(a+b+c+d)}$$

The proportion of discount, when they bid untruthfully is

$$spn_{bi} = \frac{a-x}{((a-x)+(b-y)+(c-z)+d)} \qquad (3.58)$$

$$spn_{bj} = \frac{b-y}{((a-x)+(b-y)+(c-z)+d)} \qquad (3.59)$$

$$spn_{bk} = \frac{c-z}{((a-x)+(b-y)+(c-z)+d)} \qquad (3.60)$$

The proportion that is allocated to buyer i will change by

$$\frac{a}{(a+b+c+d)} - \frac{a-x}{((a-x)+(b-y)+(c-z)+d)}$$

$$= \frac{a((a-x)+(b-y)+(c-z)+d)-(a-x)(a+b+c+d)}{(a+b+c+d)((a-x)+(b-y)+(c-z)+d)}$$

$$= \frac{(aa-ax+ab-ay+ac-az+ad)-aa-ab-ac-ad+ax+xb+xc+xd}{(a+b+cd)((a-x)+(b-y)+(c-z)+d)}$$

$$= \frac{bx+cx+dx-ay-az}{(a+b+cd)((a-x)+(b-y)+(c-z)+d)}$$

This expression is positive if bx+cx+dx-ay-az>0. In this case the proportion increases. It decreases if (bx+cx+dx-ay-az)<0, and does not change if (bx+cx+dx-ay-az) = 0. In the same way the difference in proportion of the second buyer j is

$$\frac{b}{(a+b+c+d)} - \frac{b - y}{((a - x)+(b - y)+(c - z)+d)},$$

which simplifies to

$$= \frac{ay + cy + dy - bx - bz}{(a+b+c+d)((a - x)+(b - y)+(c - z)+d)}$$

This is positive if (ay+cy+dy-bx-bz)>0. In this case the proportion increases. It decreases if (ay+cy+dy-bx-bz)<0, and does not change if (ay+cy+dy-bx-bz)= 0. However this does not guarantee that the discount increases as overall surplus goes down.

In the same way the difference in proportion of the third buyer k is

$$\frac{c}{(a+b+c+d)} - \frac{c - z}{((a - x)+(b - y)+(c - z)+d)}$$

Which can be simplified as

$$= \frac{ay + by + dy - cx - cy}{(a+b+c+d)((a - x)+(b - y)+(c - z)+d)}$$

This expression is positive if (ay+by+dy-cx-cy)>0. In this case proportion increases. It decreases if (ay+by+dy-cx-cy)<0, and does not change if (ay+by+dy-cx-cy)= 0. However this does not guarantee that the discount increases as overall surplus goes down.

Generalizing this for k buyers, we obtain the general formula for change in the proportion of surplus that can be allocated. The change will be given as under.

Let $ccov_{bi}$ be the change in contribution of i^{th} buyer contribution by bidding untruthfully. Then change in proportion is positive, negative or 0 depending upon

$$ccov_{bi}\, \Sigma_{j\pm i} covs_{bj}(q) - covs_{bi}(q)\, \Sigma_{j\pm i} ccov_{bi}$$

The new discount is

$$df_{bi} = \left(\frac{\mathrm{cov}_{bi}(q) - c\,\mathrm{cov}_{bi}}{V - \sum \mathrm{cov}_{bi}}\right)(V - \Sigma ccov_{bi})$$

This gives idea about the effect of untruthful bidding by any buyer. However our contribution payment scheme ensures that even if any k of n buyers decrease their respective bid prices, the utilities of others remain does not change. The decrease in surplus is allocated among the k buyers only irrespective of changes in proportion. We state this as

Theorem 12: If any k buyers decreases the amount

(3) the decrease in surplus is allocated to these k buyers under contribution payment scheme.

(4) the utilities of all others remain unchanged and the gain of the k buyers (who decrease their prices) is always bounded.

Proof: Let V be the total surplus. Let $covsb_i(q)$ be the contribution of i^{th} buyer to the surplus. It is assumed without any loss of generality that buyers 1, 2, …, k decrease their respective bid prices by x_i. Let q_i be the quantities purchased by them. The buyers k+1, k+2, …, n do not change their respective bid prices. Let V_n be the new surplus. Let sp_{bi} be the proportion of surplus allocated, when all the buyers bid truthfully. Let spn_{bi} be the new proportion of surplus allocated, when k buyers bid decrease their respective bid prices. It can be seen that

$V_n = V - \sum_{i=1}^{k} x_i q_i$

For buyers i = 1, 2 , … , k the new contribution will be

$covsnb_i(q) = covsb_i(q) - x_i q_i$. The contributions of others remain unchanged.

The discount of i^{th} buyer db_i, when he bids truthfully is (i = 1, 2, …, k)

$$db_i = sp_{bi} V = \left(\frac{2covsbi(q)}{V}\right) V = 2covsb_i(q) .$$

The discount of i^{th} buyer dbn_i, when he bids untruthfully is

$$dbn_i = spn_{bi} V = \left(\frac{2covsnbi(q)}{V_n}\right) V_n = 2covsnb_i(q) = 2(covsb_i(q) – xq_i) .$$

So change in discount is $db_i - dbn_i = 2xq_i > 0$. In the same way as earlier it can be shown that the change in amount payable is $x_i q_i$. So i^{th} buyer can improve utility only by xq_i. The scheme does not allocate any additional discount.

Further if buyers (i = 1, 2, …, k) decrease bid their respective prices and prices of others remain the same, then the proportions of discount to remaining buyers increase. If all buyers bid truthfully, then the proportion of surplus to be allocated to j^{th} (j > k) buyer is

$sp_{bj} = \frac{2covsb_i(q)}{V}$, where $covsb_j(q)$ be the contribution of j^{th} buyer to the surplus.

If buyers (i = 1, 2, …,k) decrease their respective bid prices and no other buyers change their respective bid prices, then the proportion of surplus to be allocated to j^{th} (j > k) buyer is

$$spn_{bj} = \frac{2covsb_j(q)}{V_n},$$

as $V > V_n$, we get $sp_{bj} < spn_{bj}$. So discount of all other buyers improve. So the new discount of j^{th} (j>k) buyer is

$$dbn_j = spn_{bj} V_n = (\frac{2covsb_j(q)}{V_n})V_n = covsb_j(q) = db_j \qquad (3.61).$$

As there is no change in bid price and discount, the utilities of others do not change. So even if any k buyers decrease their respective bid prices, their gains will always be bounded and the utilities or the payable amount of other buyers remain unchanged. The gain is always bounded. This is due to the fact that if the buyers continue to decrease his price, they may be out of auction after some stage.

In experimental analysis payoff using contribution and VCG schemes were obtained for randomly generated data sets. The results have been presented in following six graphs. It can be seen that:

(1) As expected, VCG discount does not change with price (Incentive Compatibility). However the payment scheme based on contribution is not strategy proof. It can be seen from graphs, that as buyer's price decreases, his payoff also decreases (reverse for sellers). This ensures that buyer cannot make only bounded gains.

(2) The buyer's gains are always be bounded irrespective of number of buyers and sellers changing their respective pries. The scheme always budget balanced.

(3) It can be further seen that even if more than two buyers decrease bid prices, the discount of the buyer decreases. This is due to the fact that surplus also decreases. So a buyer or set of buyers can manipulate our scheme only to limited extent, by bidding untruthfully. In other words our scheme is efficient, budget balanced and individually rational.

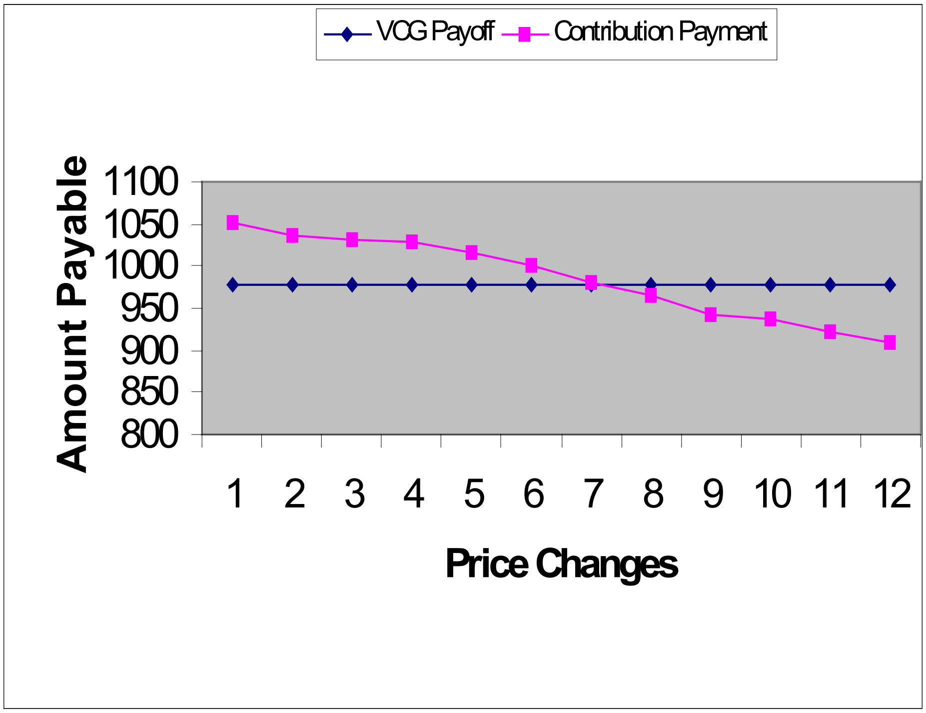


Figure. 3.14 Effect of changes in prices on VCG Payoff and Contribution by a single buyer. (Horizontal line is VCG Payoff)

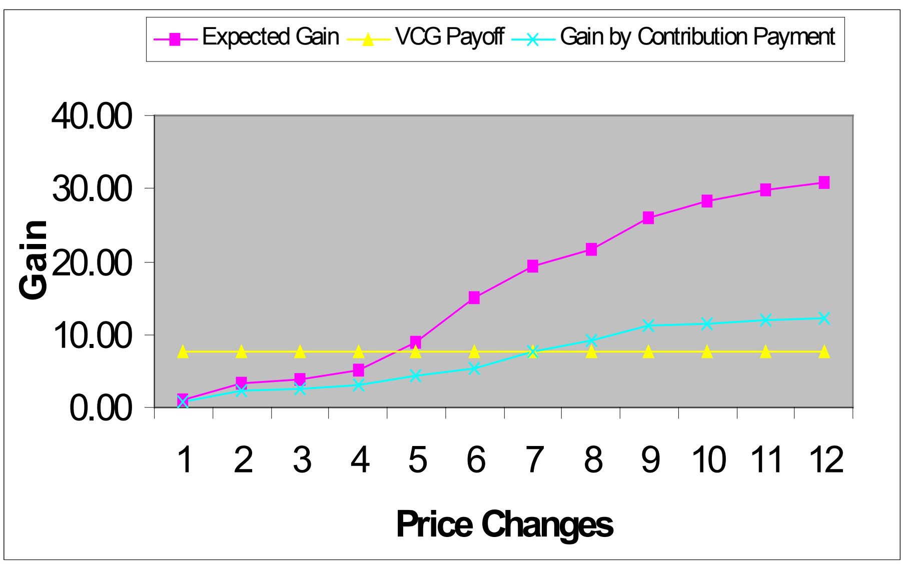


Figure. 3.15 Buyer's expected gain and changes in bid prices (Single Buyer)

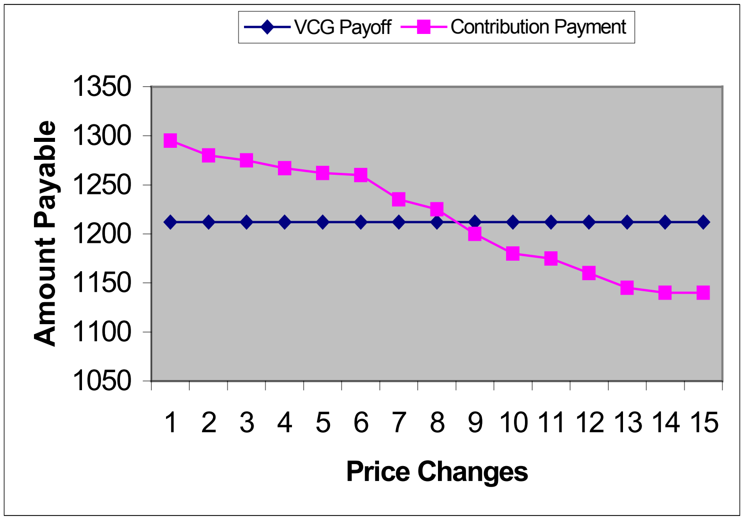


Figure. 3.16 Effect of changes in VCG Payoff and Contribution Payment, when two participants change their respective prices. The payoff of one participant has been shown here.

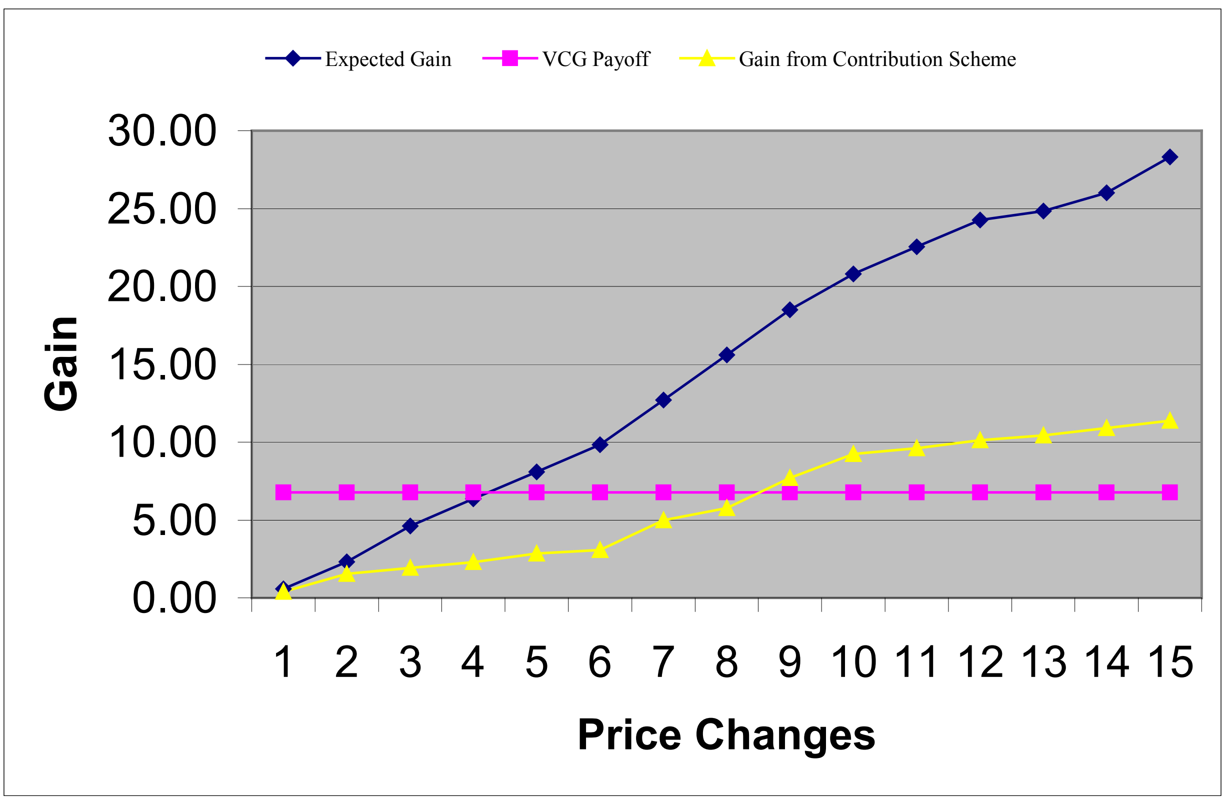


Figure. 3.17 Change in gain when two participants change their respective prices. The payoff of one participant is shown here.

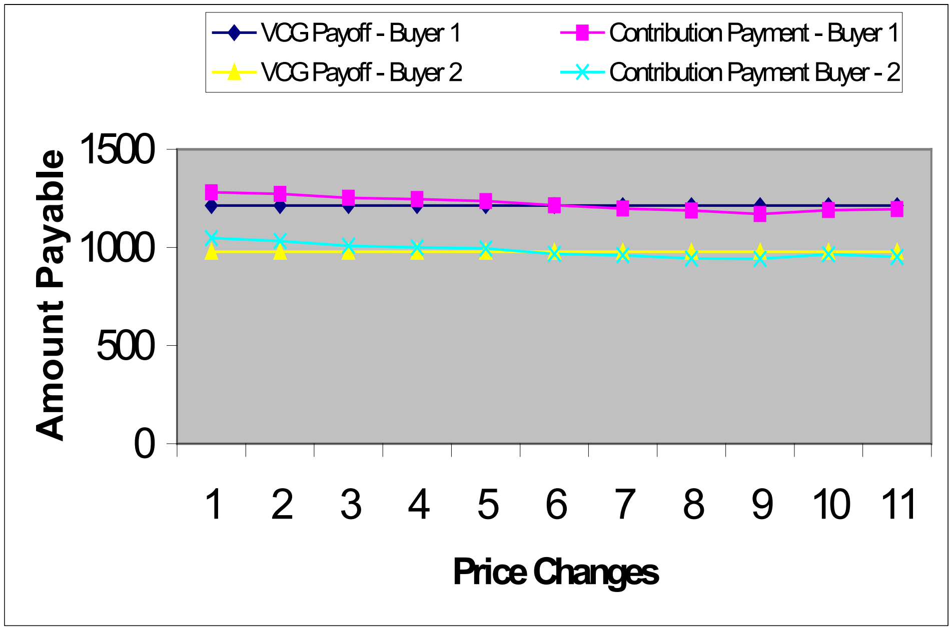


Figure. 3.18 Effect of price changes by more than two buyers on VCG Payoff and Contribution Payoff of two buyers.

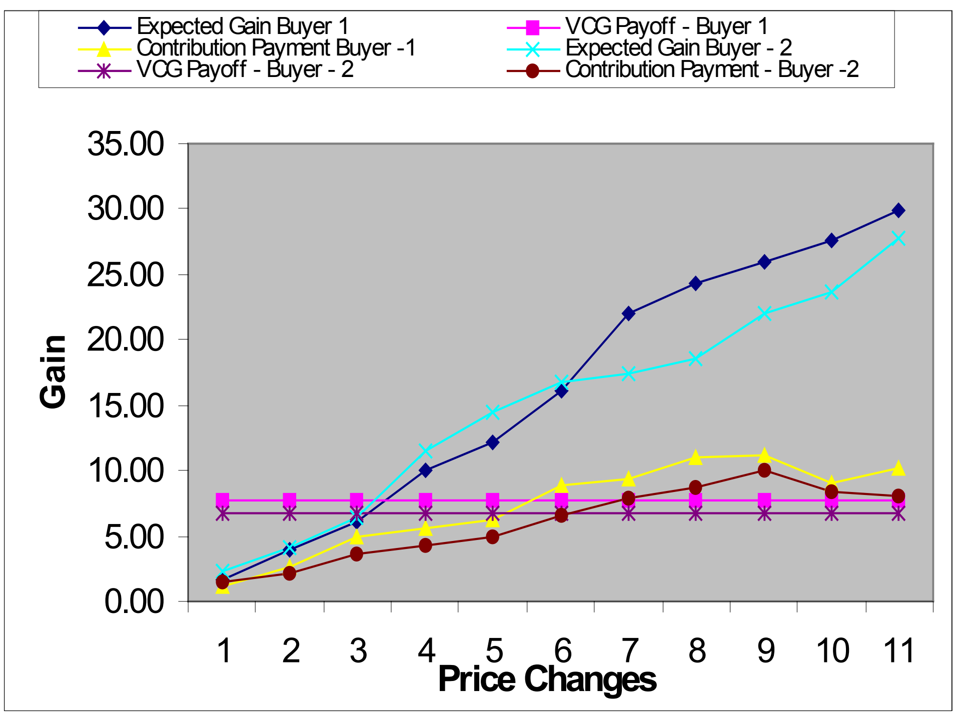


Figure. 3.19 Change in gain of two buyers when more than two buyers change their prices.

Chapter 4

Strategy Proof Double Auctions under assignment and indivisibility constraints

4.1 Introduction

In double auctions there are two main problems (1) finding optimum assignment of bids and asks – which is also known as Trade Determination Problem (TDP) and (2) determining how much each participant buyer has to pay and how much each participant seller should receive – which is also known as Pay off Determination Problem (PDP). Another phrase commonly used for these two problems is Mechanism Design (MD). The desirable properties of any mechanism are (1) Budget Balanced (BB), (2) Strategy Proof or Incentive Compatibility (IC), (3) Individual Rationality (IR) and (4) Efficiency (EF). These properties have already been described in detail in the previous chapters.

However there is a well known impossibility result of Myerson and Satterthwaite [MS1983], which states that no mechanism can be strategy proof, efficient, budget balanced and individual rational at the same time in case of double auctions.

In this study we consider double auctions where truthful bidding is the dominant strategy in order to ensure that buyers and sellers submit their true valuations. It is assumed that buyers and sellers have their private valuations independent of each other. The buyers and sellers submit their respective bids and asks depending upon their respective private valuations. It has been observed that VCG Mechanism is efficient and strategy proof (IC). However it is not budget balanced (BB). In electronic auctions, property of incentive compatibility (IC) is very important. Incentive compatibility property ensures that truthful bidding is the dominant strategy. This property is essential in electronic auctions, since bids and asks are submitted remotely. Other important desirable properties in electronic auctions are (BB) and (IR). The former ensures that auction does not run in loss, whereas later ensures voluntary participation. However these properties can be achieved only after sacrificing efficiency.

In this chapter we consider double auctions under assignment and indivisibility constraints, which are truthful. We start with Vickrey like Auctions [V1961] and VCG Mechanism [V1961], [C1971], [G1973].

Suppose that there are M buyers and N sellers. Let BD be the set of bids submitted by these M buyers. Let AS be the set of asks submitted by these N sellers. The concepts of bids and asks have already been introduced in chapter 3. The same concepts are used here.

In this chapter we consider the case where there are different types of constraints like assignment constraints and indivisibility constraints. The mechanism design in the absence of assignment and other types of constraints has been studied in case of single unit (SDA)[SW1989] as well as multi unit (MDA)[HWS2002] double auctions. However in the present chapter we consider a scenario with different types of constraints.

In many cases there can be situations where there are different types of assignment and indivisibility constraints while matching. These constraints are already introduced in chapter 3. The constraints can be of the following types.

(1) A bid can be matched only with subset of asks – Assignment Constraints
(2) A bid may have indivisibility constraints – Its demand cannot be satisfied by combining supply from different asks.

We consider the following example of a cotton exchange to illustrate the present work of mechanism design under different types of constraints.

Suppose there are different grades of cotton. Let these grades be denoted by 1, 2 and 3 with grade 3 being the highest and 1 being the lowest.

Table. 4.1 : Bids submitted by the buyers

Serial Number	Price	Quantity	Grade	Higher Grade Acceptable	Cumulative Demand
1	155	5	3	N	5
2	151	10	3	N	15
3	143	10	2	Y	25
4	138	5	2	Y	30
5	136	8	3	Y	38
6	131	7	1	Y	45
7	127	8	1	Y	53
8	124	5	2	N	58
9	111	10	1	Y	68
10	103	10	1	Y	78

The bids submitted by different buyers are shown in Table 4.1. In this case buyers 6, 7, 9 and 10 require cotton of grade 1, however the cotton of higher grade can also satisfy their demand. The buyers 3, 4 and 8 require cotton of grade 2. The buyer 8 requires cotton of grade 2 only.

All other buyers require cotton of grade 3 only and cotton of no other grade is acceptable to them. In this example there are assignment constraints on attribute *grade*. The bids with demand for cotton of grade 3 can only be matched with asks which supply cotton of grade 3. These bids represent the case where assignment constraint is of equality type. On the other hand, in case of other bids, the demand can be matched with any asks supplying cotton of same or higher grade. These bids represent bids where assignment constraints are not of equality type. Their demand can be satisfied by any asks which supply cotton of same or higher grade. However in all cases the demand cannot be satisfied by any ask supplying cotton of lower grade. In this case, apart from price and quantity, grade is also another matching attribute. The asks submitted by different sellers are shown in Table 4.2. They supply cotton of three required grades (1, 2, 3).

Table. 4.2 : Asks submitted by the sellers

Serial Number	Price	Quantity	Grade	Cumulative Supply
1	91	5	1	5
2	95	5	1	10
3	105	25	2	35
4	108	5	1	40
5	126	5	2	45
6	128	5	1	50
7	139	20	3	70
8	150	6	3	76
9	153	10	2	86
10	159	10	3	96
11	163	10	3	106

It can be seen that due to assignment constraints asks 1, 2, 4 and 6 can be matched with bids 6, 7, 9 and 10 only. On the other hand, asks 3, 5 and 9 can be matched with bids 3, 4 and 8 as well as bids 6, 7, 9 and 10. This is possible as the demand of bids 6, 7, 9 and 10 can be satisfied by cotton of higher grade. The demand of bids 1, 2 and 5 can be satisfied by asks 7, 8, 10 and 11 only. The asks from 9 to 11, can be assigned to bid of any grade.

In the above example there are assignment constraints of different types. However, the supply from different asks can be pooled to satisfy demand of a bid. In some cases the supply from different asks cannot be pooled together to satisfy the demand of a bid. This is illustrated in the example of paper exchange considered in chapter 3. We refer to the same example here.

One effect of assignment constraints can be that in many cases it may not be possible to obtain VCG pay off. This problem can arise as removing an ask may result in no solution for a particular problem, as there may not be an ask which can be matched with a bid with assignment or indivisibility constraint. Further in case of constraints, the price may be dependent on other attribute value. It can be seen in the above example that the price of higher grade of cotton is expected to be higher than that of lower grade cotton. An ask can be used to satisfy the demand of different types of bids. An ask of higher grade can be matched with a bid requiring cotton of lower grade but it is not the other way. It can be seen that assignment and other types of constraints can arise due to different attributes and hence may have an impact on prices. So, obtaining a single market clearing price itself may be difficult. It is also expected that the seller, selling cotton of higher grade should get higher price than the one selling cotton of a lower grade. In the same way the buyer purchasing cotton of higher grade should pay more than the one who is purchasing cotton of lower grade. As such, having a single market clearing price may not be desirable in such cases. If we arrange bid according to prices from the highest to the lowest and then attempt to select bids as in Vickrey like auctions, it is possible that we may select bids only of a particular type. It is quite possible that the buyers willing to buy cotton of higher grade submit bids, where prices are higher. The buyers willing to purchase cotton of lower grade may submit bids with lower prices. So, when we select say k highest price bids we may end up selecting bids requiring higher grade of cotton. In the same way when we select say j least price asks, we may end up selecting asks which supply only lower grade cotton. As a result of this, we cannot directly apply the existing mechanisms to the scenario with different types of constraints. The assignment and other types of constraints can arise in many cases in electronic commerce. In electronic commodity exchanges there can be assignment constraints based on the grade or type of commodity. In electronic procurement of computer stationery, paper of different widths are required giving rise to different types of assignment constraints. In this work we generalize the Multi Unit Double Auction mechanism (MDA) proposed in [HWS2002]. In case there are no assignment and other constraints, our mechanism reduces to the mechanism proposed in [HWS2002]. It is to be noted that the constraints of different types arise due to requirements of buyers. In some cases, the constraints can be implicit. As such it is assumed here that different types of constraints are not public information. In this setup prices and volumes of the buyers (as in MDA) is public information. Our mechanism that is referred to as Generalized Multi Unit Double Auction (GMDA) for double auctions with different types of constraints is described in next section.

4.2 Generalized Multi Unit Double Auction Mechanism (GMDA)

Let BD be the set of all bids and AS be the set of all asks.

(1) *Arrange bids in decreasing order of price.* After arranging bids in decreasing order of price we have, $bp_{(M)} \geq bp_{(M-1)} \geq \ldots \geq bp_{(1)}$.

In the same way arrange asks in increasing order of price. After arranging asks in increasing order of prices we have, $ap_{(1)} \leq ap_{(2)} \leq ap_{(3)} \ldots \leq ap_{(N)}$.

(2) *Divide set B into k mutually exclusive sets $Bo_{a1av1c2}$, $Bo_{a1av1c3}$, …, $Bo_{aiavici}$, …, $Bo_{akavkck}$, Bo_u and Bo_{in}, satisfying following*

(a) $B = Bo_{a1av1c1} \cup Bo_{a1av1c2} \cup Bo_{a1av1c3} \cup \ldots \cup Bo_{aiavici.} \cup Bo_{akavkck} \cup Bo_u \cup Bo_{in}$

and $Bo_{aiavici} \cap Bo_{ajavjcj} = \phi$ for all $i \neq j$.

(b) Each set $Bo_{aiavici}$ is a set of bids having constraints of type c_i on attribute a_i with value av_i. The attribute value and constraint type uniquely identifies each set. The bids having constraints on attributes A_1 with value v_1 and constraint type c_1 are grouped together in Boa_{1v1c1}. If bid has constraint on more than one attribute, then we group all bids having the same constraints on common attributes together. The constraints can be like

$a_i \leq v_i$ or $a_i < v_i$

$a_i \geq v_i$ or $a_i > v_i$

$a_i = v_i$ or $a_i \neq v_i$.

In addition to it there can be indivisibility type constraints. Each of these sets represents the smallest level of possible grouping. In order to obtain such bid sets, we continue to divide bids, till further division is not possible. In example in Table – 1, there are five order books namely (Grade = 3, Grade ≥ 2, Grade = 2, Grade = 1, Grade ≥ 1). However for all practical purposes the bid sets for equality constraints can be merged with corresponding inequality constraints.

(c) There are two set Bo_u - set of bids without any type of constraints and Bo_{in} - set of bids with only indivisibility constraints on quantity attribute. All bids without assignment constraints are grouped in set Bo_u.

All the bids with same constraints are grouped together. This grouping can eliminate the effect on price of dependent attribute. It is easy to verify that these sets are mutually exclusive. Hereafter, without loss of generality a single subscript index will be used to refer to bid sets. Let set BD, of bids be divided into k mutually exclusive sets BD_1, BD_2, BD_3, ..., BD_k. In each set bids are arranged in decreasing order of price.

(3) In the next step we select m_i highest price bids from each set BD_i, i = 1, 2, …, k. The number m_i of bids to be selected from each set BD_i depends on the supply available for each set. The set of m_i bids, from set of bids BD will be referred to as order book Bo_i. An ask can satisfy the demands of different sets. Due to this we do not divide asks into specific sets of asks. The matching of asks is done for each set BD_i. However supply of an ask may get distributed across different order books. We select n_i least price asks (from currently unassigned asks) for matching against bids in order book Bo_i (i=1, 2, ..., k). The number of asks to be selected depends upon the supply available for each order book. In case an ask can be assigned to more than one bid in different bid sets the assignment starts from higher price bid. First we determine set of asks from AS, which can be assigned to bids in different order books Bo_i, i = 1, 2, ..., k. Let AS_i, be the set of asks, which are matched against bids in order book Bo_i. These sets AS_i are not mutually exclusive in general. Each set is determined by identifying the asks which can be assigned to bids in different bid sets and adjusting supply across different sets. Once an ask is assigned to a bid, its supply becomes $aq_i = aq_i – bq$ (where bq is the quantity assigned). If the remaining quantity is positive, then this quantity can be assigned to other bids. The matching starts from the highest price bid and the

least price ask. This process is continued till either demand is satisfied or supply is exhausted. Then the clearing price for each order book is determined. In all order books except minimum price order book, buyer and seller pay the same price.

(4) Select m_{bc} highest price bids from each order book Bo_i for all $i = 1, 2, ..., k$. The m_i is determined as follows

Identify the asks from set AS which can be assigned to bids in order book (i.e. asks satisfying constraints of order book Bo_i). Let AS_i be this set of asks.

Then determine m_{bc} and n_{sc}. These are determined by identifying price cross-over point i.e.

$bp_i > ap_j$ *for* $i < bc$ *and* $j < sc$, $bp_{bc} \geq ap_{sc}$

and either one of the following relations is satisfied

either $bp_{bc} \geq ap_{sc} \geq bp_{bc+1}$ *or*

$ap_{sc} \leq bp_{bc} \leq ap_{sc+1}$

Determine the total demand and supply available at $i = bc$ and $j = sc$. Let the total demand at $i = bc$ be bq_{bc} and supply at $j = sc$ be aq_{sc}.

Determine $bq = min(bq_{bc}, aq_{sc})$. Then determine m_i and n_j as

$m_i = i$, where i is minimum index at which total demand exceeds or equals bq.

$n_j = j$, where i is minimum index at which total supply exceeds or equals bq.

Let $bpi_{(mi)}$ be the m_i^{th} highest bid price from bid set Bo_i.

Let $api_{(nj)}$ be the j^{th} least ask price. The m_i^{th} bid price and ask price $api_{(nj)}$ are selected such that $bpi_{(mi)} > api_{(nj)}$ and $bpi_{(mi+1)} < api_{(nj+1)}$. Let bqi_{mi} be the demand at crossover point and aqi_{nj} be the supply available. The possible volume of trade in order book Bo_i is $bq_{Bi} = min(bqi_{mi}, aqi_{nj})$.

(5) This step (4) is repeated for all remaining order book Bo_i. Each order book has m_i bids. In the same way AS_i be set of asks assigned to bids in order book Bo_i. Each of this set AS_i has n_j asks.

(6) Determine $bp_{min} = min_{mi}(bpi_{(mi)})$ i.e. minimum cross over price and determine an order book Bo_i, where $bpi_{(mi)} = bp_{min}$. If there are more than one such order books, the tie can be broken in any predetermined manner.

(7) Let Bo_k be the order book with minimum cross over price. *We call this order book as minimum price order book. In our mechanism a possible trade from this order book is sacrificed, as is done in MDA [HWS2002]. The trade that is sacrificed is the last possible feasible trade. In the remaining order books all the trades take place. As discussed in MDA [HWS2002] we sacrifice the least significant trade. In other words, the number of buyers who trade in minimum price order book Bo_k are m_k-1 and the number of sellers are n_k-1.*

(8) In all other order books no trade is sacrificed, all the selected m_i buyers and n_j sellers participate. It can be seen that in minimum price order book Bo_k our mechanism works in same way as in [HWS2002], when the order book Bo_k contains bids without any types of constraints.

It is first determined whether there is an excess demand or supply. It is determined whether inequality $\sum bqk_i \leq \sum aqk_j$ is satisfied or not. If inequality is satisfied, there is over supply otherwise there is excess demand. This excess demand or under supply is averaged over m_k-1 buyers or n_j-1 sellers in a way that the trades are feasible. If it is not feasible then assignment takes place as per constraints.

(9) In all order books except Bo_k, all the selected n_j sellers sell their volumes aq_j to m_i buyers in respective order books.

(10) *The clearing price per unit is set at $pc_{Bi} = (bpi_{(mi+1)+} api_{(nj+1)})/2$,*

if $bp_i \geq pc_{Bi}$ for all bids in i^{th} order book Bo_i and

$ap_j \leq pc_{Bi}$ for all asks in AS_i.

In all other cases including the case where there is a single bid in order book Bo_i and ask AS_i, market clearing price is set as

$pc_{Bi} = (bpi_{(mi)+} api_{(nj)})/2$,

(It may be noted that the clearing price can also be set as

pc_{Bi} = any price between [api_{ni} , bpi_{mi}]. The subsequent analysis does not change).

This ensures that total payments and receipts in all order book are equal. It can be seen that i^{th} buyer in the i^{th} order book pays ($pc_{Bi}bqi_i$) and j^{th} seller receives

($pc_{Bi}aqi_j$). The quantity purchased by the i^{th} buyer is bqi_i and volume sold by j^h seller in i^{th} order book is aqi_j.

(11) *In the order book Bo_k, the total traded volume is $min(\sum bqk_i, \sum aq_{kj})$.*

The sellers 1, 2, ..., n_k-1 sell their volumes at price $apk_{(nj)}$ and buyers 1, 2, ..., m_k-1 purchase at price $bpk_{(mi)}$. (If $\sum bqk_i < \sum aq_{kj}$, then there is excess supply, otherwise there is excess demand).

Market Clearing Price: In GMDA, in all order books, except minimum price order book buyer and seller pay same price. In i^{th} order book m_i bids and n_j asks are selected. In case i^{th} order book is not minimum price order book, the price pc_{Bi} is given by

pc_{Bi}	$= (bpi_{(mi+1)} + api_{(nj.+1)})/2$	--- If $pc_{Bi} \geq ap_j$ for all ask A_j Є AS_i , $pc_{Bi} \leq bp_i$ for all bid B_i Є Bo_i and such asks and bids exist
	$= (bpi_{(mi\ )} + api_{(nj\ )})/2$	or any price between $bpi_{(mi\ )}$ and $api_{(nj)}$

In case of minimum price order book the clearing price is

Buyer pays $pc_{Bk} = bpk_{(mk)}$

Seller receives $spc_{Bk} = apk_{(nk)}$

In this way our mechanism is generalization of basic MDA mechanism discussed in [HWS2002]. If there are no assignment or other types of constraints there will be only one order book and it reduces to MDA mechanism. Our mechanism can be applied to example in earlier section (Table – 4.1 and Table – 4.2). In this case the received bids are divided in three order books B_{G1}, B_{G2}, B_{G3} (one for each grade). The order book B_{G3} is minimum price order book. The 10 received bids are grouped in three order books and asks which can be matched are shown in Table 4.3. It can be seen that in first order book, two highest price bids (bids 1, 2) are selected. Only one least price ask is selected, since there is demand for only 15 units. However, remaining asks cannot be used to satisfy demands for other order books due to higher price. The unfulfilled supply from second order book can be used to satisfy demand from third order book. It can be seen that the third order book is minimum price order book and a trade is sacrificed in this order book. There is uniform average market clearing price for first two order books. The unfulfilled supply from the asks 3 and 5 can be used for satisfying demand of 3rd order book. In this order book 3 buyers and 4 sellers can trade, however one trade is sacrificed. It relates to last feasible trade related to bid 9, with bid price 111. In this case demand equals supply, hence seller 1, 2, 3 sell all their volumes to buyer 5 and 6. The last feasible trade related to buyer 9 and seller 4 of quantity 5 is sacrificed. This results in trading surplus of 15 (111–108) = 45. There is no further potential loss. The assignment obtained and clearing prices are shown in Table – 4.4. It can be seen that prices are generally dependent on attributes like grade. In general if k highest price bids are to be selected as in MDA mechanism, the price crossover occurs at bid 5 or price 136. So no bid requiring cotton of grade 1 is selected. On the other hand no least price ask satisfying demand of grade 3 cotton is selected. It can be seen that only possible feasible trades are in respect of bids 3 and 4. In our mechanism, since the bids are divided according to order book, it is possible to satisfy demands of different types of bids. It can be seen that it is possible to satisfy demand of two bids of grade 3, three bids of grade 2 and two of grade 1. The asks are distributed across different order books. It can be seen that supply of ask 3 has been distributed across two order books i.e. order book 2 and 3. This happens because trades are feasible. In the same way a seller also gets higher amount for the part of volume sold in different order books. In this example seller 3 is paid 108 for the part sold in order book 3 and 125 for the part sold in order book 2. The asks selected are always the least price asks.

Payoffs of Buyer and Sellers

In GMDA mechanism the amount payable by i^{th} buyer bap_i is

$bap_i = pc_{Bi}\ bqi_i$ -- If i^{th} buyer's bid is selected and is member of i^{th} order book, where pc_{Bi} is per unit clearing price of i^{th} order book and bqi_i is quantity purchased and i^{th} order book is not minimum price order book (4.1)

$= pc_{Bk}\ bqk_i$ -- If i^{th} buyer's bid is selected and is member of minimum price order book, bqk_i is quantity purchased by the buyer, where B_k is minimum price order book. (4.2)

$= 0$ -- If bid is not part of any order book. (4.3)

In the same way the amount receivable by j^{th} seller in i^{th} order book is

$sar_j = pc_{Bi}\ aqi_j$ – If j^{th} seller's ask is selected. The clearing price of i^{th} order book is pc_{Bi} and aqi_j is the quantity sold by j^{th} seller in i^{th} order book and i^{th} order book is not minimum price order book. (4.4)

$= spc_{Bk}\ aqk_j$ -- If j^{th} seller sells quantity aqk_j in minimum price order book $Bo_{k.}$ (4.5)

$= 0$ -- otherwise (4.6)

It can be seen that if j^{th} seller sells his volumes $aqi_{j,}$ in i^{th} order book , i = 1, 2, ..., k and pc_{Bi} is the clearing price of i^{th} order book, then the total amount receivable by the seller is given by $pc_{Bi}\ aqi_{j.}$

In the same way the amount receivable by j^{th} seller in i^{th} order book is

$$sar_j = \sum pc_{Bi}\, aqi_j \qquad (4.7).$$

The total number of buyers who actually traded is $\sum_{j=1}^{k} n_{ij} = n_b$. It is assumed that there are k order books. In the same way total number of sellers who sell is $m_s = \sum_{j=1}^{k} m_{ij}$.

Table. 4.3 - The bids divided into three order books and corresponding asks

Order Book	Bids	Price	Quantity	Grade		Ask	Price	Quantity	Grade
1	1	155	5	3		7	139	20	3
	2	151	10	3		8	150	6	3
	5	136	8	3		10	159	10	3
						11	163	10	3
2	3	143	10	2		3	105	25	2
	4	138	5	2		5	126	5	2
	8	124	5	2		9	153	10	2
3	6	131	7	1		1	91	5	1
	7	127	8	1		2	95	5	1
	9	111	10	1		4	108	5	1
	10	103	10	1		6	128	5	1

Table. 4.4 – GMDA Results

Order Book	Trade Bid	Ask	Quantity	Market Clearing Price	
1	1	7	5	147.50	
	2	7	10		
2	3	3	10	125	
	4	3	5		
	8	3	5		
				Buyer	Seller
3	5	1	5	111	108
	6	2	5		
		3	5		

In our mechanism in minimum price order book a trade is sacrificed and there is potential trade loss, as parts of volumes are sacrificed. Apart from feasibility of trades, there are two important differences between GMDA and MDA in respect of minimum price order book. The main differences are as follows.

(1) Suppose that all bids in minimum price order book have indivisibility constraints. Suppose that in this order book there are k_l bids, selected after price crossover. Let the demand at that point be q_{bl}. Let the demand of last selected bid be qbk_{mk}. Let aqk_{nl} be the total supply available. Let the supply of last selected ask be aqk_{nk}. Then this is the last feasible trade and is sacrificed. The potential loss is $\min(bqk_{mk,}aqk_{nk})$. Due to indivisibility constraints, the actual demand that can be satisfied may be less than the minimum of demand and supply available in that order book. This difference is not treated as a potential loss. In the worst case scenario, there is a possibility that in every trade, certain quantity is wasted or remains unassigned. This happens due to feasibility of trades and it is not sacrificed trade. Hence it is not considered as potential loss. It is an infeasible trade. In this case, only trade and not volume (as in MDA) is sacrificed. So efficiency loss is restricted to the sacrificed trade only.

(2) There is also a possibility that there may be only one feasible trade in minimum price order book. This is in fact last feasible trade. In GMDA this trade is sacrificed. So there is no trade in minimum price order book and efficiency loss is restricted to last feasible trade. In this case, there is no surplus left with the market maker. There is surplus in GMDA (weak budget balance), only if one trade takes in minimum price order book. In all other cases the mechanism works in the same way as in MDA.

In this mechanism the price paid by the buyer is dependent upon order book. Similarly seller receives the amount depending upon the order book/books in which the volumes are sold. It can be seen that in GMDA there are multiple market clearing prices. The clearing price is same for buyers and sellers in any order book, except minimum price order book. However in case of minimum price order book, buyers and sellers pay different prices. This is done to restrict efficiency loss. It can be seen that bids in all other order books have higher price than the last bid, which we select. Since we sacrifice only last feasible trade, we employ same price in other order books. The division of bids in different order books is dependent on other attributes. So we handle the problem in more general settings, where other different attributes are also taken into account.

In GMDA it is assumed that all buyers and sellers have reservation prices. It is private information and is static. It is assumed that bv_i is per unit reservation price or private valuation of i^{th} buyer and sv_j is reservation price of j^{th} seller. We define the utility for i^{th}

buyer as the difference between product of reservation price and quantity purchased and amount payable. If i^{th} buyer purchases quantity bq_i, at price bp_i per unit then his utility bu_i is

$$bu_i = (bv_i - bap_i)\, bq_i \qquad (4.8).$$

In the same way, if j^{th} seller sells quantity sq_j at price sar_j per unit, then utility su_j of j^{th} seller is

$$su_j = (sar_j - sv_j)\, aq_j \qquad (4.9).$$

As in MDA [HWS2002], it is assumed that trading prices and quantities of all buyers and sellers are separable in its utility or prices do not depend upon quantities. Since GMDA is generalization of MDA, as in MDA, discriminatory bidding, where seller may offer discount for large volume, is not allowed here. All the prices refer to per unit price or price of single unit of quantity. In this setup prices and quantities of all buyers and sellers are public information. The assignment constraints are setup by buyers or may be implicit, however it is not public information. In this case, sellers submit only asks and are not allowed to set up constraints, even though they may be selling cotton of different grades or paper rolls of different widths. There is no separate order book for asks. The separation of bids into different order books is done by market maker and buyers and sellers do not have any control over it. A buyer or a seller is not aware about order books and in which order book a buyer's bid is placed, or with which bid, an ask is matched. The order books are dependent upon the bids received, at the time of matching. It may change depending upon bids and is not static. It can be seen in the above example that if no bid, for cotton of grade 3 is received, there is no order book for grade 3. In other words even though buyers and sellers are aware that cotton of different grades are purchased and sold, they do not have complete information about internal processing and working of call market.

4.3 Properties of GMDA

In this section we show that GMDA satisfies the following

(1) Budget Balance
(2) Individual Rationality
(3) Incentive Compatibility
(4) Efficiency loss is always bounded

Theorem 1: GMDA mechanism is budget balanced.

Proof: Suppose that there are k order books in GMDA. Let k^{th} order book be the minimum price order book. In (k-1) order books the price to be paid by the buyers and sellers is the same. The budget balance is the difference between the total amount received from the buyers and total amount to be paid to the sellers.

Total amount to be received from all buyers = $\sum_{i=1}^{k}\sum_{j=1}^{mi} pc_{Bi} bqi_j = \sum_{i=1}^{k} pc_{Bi}\sum_{j=1}^{mi} bqi_j = \sum_{i=1}^{k} pc_{Bi} qb_i$,

where qb_i is the quantity purchased in i^{th} order book. Let bpc_{Bk} be the per unit price to be paid by all buyers in minimum price order book. Then the above expression can be written as

= $\sum_{i=1}^{k-1} pc_{Bi} qb_i$ + $bpc_{Bk}qb_k$. In the same way we can obtain the total amount to be paid to all the sellers.

Total amount to be paid to all the sellers = $\sum_{i=1}^{k-1}\sum_{j=1}^{ni} pc_{Bi} aqi_j$ + $spc_{Bk}qb_k$= $\sum_{i=1}^{k-1} pc_{Bi}\sum_{j=1}^{ni} aqi_j$ + $spc_{Bk}qb_k$ = $\sum_{i=1}^{k-1} pc_{Bi} qb_i$ + $spc_{Bk}qb_k$, where spc_{Bk} is the per unit price to be paid by the seller in minimum price order book.

The difference between these two amounts is the budget balance. The difference can be obtained as follows

$$= \sum_{i=1}^{k-1} pc_{Bi} qbi + bpc_{Bk}qb_k - \sum_{i=1}^{k-1} pc_{Bi} qb_i - spc_{Bk}qb_k$$

$$= \sum_{i=1}^{k-1} (pc_{Bi} qb_i - pc_{Bi} qb_i) + (bpc_{Bk} - spc_{Bk})\, qb_k$$

$$= (bpc_{Bk} - spc_{Bk})\, qb_k$$

The first term in above expression is always 0, as the total amount payable to all the selected sellers is exactly same as total amount receivable from all the buyers. So the difference is always 0. As such in (k-1) order books, total amount receivable is the same as total amount payable. In k^{th} order book (second term in above expression) there are two possibilities

(1) There is no feasible trade – In this case, there is no trade in minimum price order book, therefore no amount is paid or received. So the GMDA is budget balanced in this case, as the above expression is 0.

(2) There are feasible trades in minimum price order book. In case of k^{th} order book, the per unit price paid by the buyers is at least equal to the one to be paid to the seller i.e. $bpc_{Bk} \geq spc_{Bk}$. This results in positive surplus. The term qb_k is positive. Hence the above expression is always non negative.

So GMDA is always budget balanced.

Theorem 2: GMDA satisfies IR property.

Proof: It can be seen that in GMDA, if buyers and sellers are not selected then they do not pay and receive any amount. So they have 0 utility. In case buyers or sellers participate in auction then there are following cases.

(1) The per unit price to be paid by i^{th} buyer is pc_{Bi} or pc_{Bk}. These prices satisfy the relations, $pc_{Bi} \leq bap_i$ (if buyer's bid is not in the minimum price order book) and $pc_{Bk} \leq bap_i$ for bid in the minimum price order book.

(2) In the same way, j^{th} seller never receives the per unit amount, which is lesser than the per unit amount of ask submitted by him. The per unit price to be received by a seller is pc_{Bi} or spc_{Bk}. These prices satisfy the relations $pc_{Bi} \geq ap_j$ (if seller's ask is not in the minimum price order book) and $spc_{Bk} \geq ap_j$ for bid in the minimum price order book.

The above two statements are true for any buyer or seller. It may be noted that the per unit price paid by each buyer in auction does not exceed per unit bid amount. In the same way per unit price to be received by each seller in auction is not lesser than per price received. So, all the participants have positive utility or have non negative gains after auction clears. Due to this, GMDA is always IR.

Theorem 3: If only prices and volumes of buyers and sellers are public information then GMDA mechanism is strategy proof.

Proof: Suppose that the selected bids are distributed in k order books $Bo_1, Bo_2, \ldots, Bo_k$. Let pc_{Bi} be the clearing price of i^{th} order book. Let Bo_k be the minimum price order book. Let bv_i be the private valuation of the i^{th} buyer and bp_i be the bid price submitted. Let bq_i be the quantity purchased by the i^{th} buyer. Let aqi_j be the quantity sold by j^{th} seller in i^{th} order book. Let sv_j be the private valuation of j^{th} seller. We prove that GMDA is strategy proof in two cases (i) case 1 - i^{th} buyer is not in the minimum price order book (ii) case 2 - i^{th} buyer is in the minimum price order book.

Case (i) : Suppose that i^{th} buyer's bid is placed in i^{th} order book with pc_{Bi} as clearing price. The utility bu_i of i^{th} buyer is

$$bu_i = (bv_i - pc_{Bi})\, qb_i \, .$$

(a) Bidding higher than true valuation: Suppose that buyer's valuation $bv_i \geq pc_{Bi}$ and buyer bids bp_i. The buyer's utility is

$$\begin{aligned} bu_i &= (bv_i - pc_{Bi})\, qb_i \,, && \text{if } bp_i \geq pc_{Bi} \\ &= 0 && \text{otherwise} \end{aligned}$$

It can be seen that buyer cannot improve his utility by bidding $bp_i > bv_i$. In this case the buyer's bid is selected but his utility does not improve. This happens as the amount payable does not depend upon buyer's bid. In other words his utility by bidding higher remains same as utility, he gets if he bids bv_i.

(b) Bidding lower than true valuation: Suppose that buyer's valuation $bv_i \geq pc_{Bi}$ and buyer bids bp_i which satisfies $bp_i < bv_i$. The buyer's utility is

$$\begin{aligned} bu_i &= (bv_i - pc_{Bi})\, qb_i \,, && \text{if } bv_i > bp_i \geq pc_{Bi} \\ &= 0 && \text{if } bv_i \geq pc_{Bi} > bp_i \end{aligned}$$

It buyer bids lower than his true valuation, buyer's bid is selected only if $bp_i \geq pc_{Bi}$. In case $bp_i < pc_{Bi}$ then buyer's bid is not selected. In the first case buyer's utility does not improve by bidding lower amount. In the second case buyer's bid is not selected, even though his valuation is higher than the clearing price. The utility of buyer is 0 in this case. So by bidding lower than his true valuation buyer does not improve his utility and in some case may have 0 utility.

(c) Suppose that $bv_i < pc_{Bi}$, i.e. buyer's valuation is smaller than clearing price. Suppose that buyer bids bp_i. The utility of the buyer is

$$\begin{aligned} bu_i &= (bv_i - pc_{Bi})\, qb_i \,, && \text{if } bp_i \geq pc_{Bi} \\ &= 0 && \text{otherwise} \end{aligned}$$

In case buyer bids $bp_i > bv_i$, buyer's bid can get selected if $bp_i \geq pc_{Bi}$. However in this case buyer has negative utility. In case the buyer bids $bp_i < bv_i$, buyer's bid is not selected and the buyer has 0 utility.

Combining arguments in (a), (b) and (c) above, it can be seen that the buyer does not improve his utility by bidding untruthfully.

(d) Further, it can be seen that by changing his bid price, buyer's bid will not get selected in different order book. This is due to the fact that a bid can be in only one order book and it is already included in the order book Bo_i. So buyer cannot improve his utility by being included in an order book, where clearing price is less than pc_{Bi}. The buyer's bid is included in the order book, depending upon the constraints. In order to get included in different order book, buyer needs to change his requirements. It is independent of the buyer's price. By bidding lower amount, buyer may not be included in an order book with lower clearing price then pc_{Bi}. So lower bidding does not improve his utility.

We can extend this argument to seller's case to show that the seller's utility does not improve by reporting higher or lower price than his valuation. The utility su_j of j^{th} seller is

$$su_j = \sum (pc_{Bi} - sv_j)\, aqi_i \qquad (4.10).$$

In the same way, a seller cannot improve his utility by bidding higher or lower amount. The seller's ask is matched with bids of one or more order books, depending upon the constraints of order books and supply available in that ask. Suppose that $sv_j \geq pc_{Bi.}$ Let ap_j be the price submitted by j^{th} seller. As in earlier case, if $ap_j > pc_{Bi,}$ the seller's ask is not included in the trade for order book $B_{i.}$ However seller's ask may still be included in some other order book Bo_j with higher clearing price, if it satisfies constraints of order book Bo_j and it is among the least price asks of order book Bo_j. The seller will always get this utility, if he bids sv_j and his ask is matched against order book Bo_j. If he increases ask price, his ask may not be included in Bo_j or other order books, as his ask may not be in least price asks for these order books. So bidding higher amount does not improve his utility. (It can be seen that the seller's utility can be improved if seller's ask is matched with bids in an order book with higher clearing price. If seller's ask satisfies the constraints of any order book, it will be included in that order book, in case it is the least price ask. So utility does not change). If he submits the ask with lower price, his ask may be included in order book Bo_i, but does not improve his utility. It can be seen that seller cannot misreport his price to improve his utility. By submitting an ask of lower or higher price, it is not necessary that ask will be matched with an order book of lower or higher clearing price. By submitting an ask of lower or higher price, it is not necessary that his ask will be included in the trade. We can extend similar argument in case $sv_j < pc_{Bi.}$ In this case also the seller can not improve his utility by submitting ask of lower or higher price, as his utility does not improve.

Case 2: Suppose that buyer is included in the minimum price order book. We can show in the same way as in earlier case, that the buyer's utility does not improve by bidding higher or lesser amount.

However, in some cases there is volume decrease. This decrease is the same for all the buyers and sellers in the minimum price order book. However, this decrease is independent of reservation prices and cannot reduce the decrease by misreporting reservation prices. Even if supply from seller's ask is distributed in other order books, volume decrease takes place if there is excess supply in minimum price order book.

This proves that GMDA is strategy proof.

It can be seen that buyer and seller may not be able to improve their utilities by misreporting prices. A buyer does not have incentive to misreport the quantity to be purchased. However in GMDA set up a seller does have incentive to misreport the quantity to be sold. It can be seen that a seller can misreport the quantity to be sold and get included in an order book with higher clearing price. If seller quotes higher quantity than he has, it is possible that his ask may be matched with bids in more than one order books. If an ask is matched with order book with higher clearing price, seller improves his utility. So he does have incentive to misreport his volume. However it is possible only if seller knows completely the bids and asks submitted by all other buyers and sellers. A seller should also know which bids are placed in which order books, and against what bids his ask can be matched. In GMDA, prices and quantities of different bids and asks are only public information. A seller is not aware about the assignment and indivisibility constraints, submitted by buyers and types of asks submitted by other sellers. A seller has only partial information about bids and asks submitted by others. If seller has complete information then it will always be possible for the seller to misreport his volumes to improve his utility. One way this problem can be overcome is, by having single clearing price. However this will not be fair to sellers, also it will be unrealistic to expect that cotton of different grades will be sold at the same price. Due to this reason, in GMDA, the public information is restricted to prices and quantities. This is not an unreasonable assumption, since different buyers have their own different requirements, these requirements need not be known to others. A seller has complete information about features of his products and so can submit his asks accordingly. In most of real life transactions, the transaction takes place between a buyer and a seller. Only certain information about the transaction is known to others. Individual buyer and seller may only know the complete details of the transaction.

In this problem formulation there are different types of assignment and indivisibility constraints. The bid and ask prices may depend upon different attributes apart from quantity. One implication of assignment and indivisibility constraints is that the valuations of buyers or sellers need not be identically distributed. In usual auction theory, it is always assumed that the bid and ask prices (or valuations of individual buyers and sellers) are drawn from independent and identically distributed (iid) random variables. This assumption may not be completely valid in case of assignment and indivisibility constraints. In GMDA, bids of buyers are distributed in different order books. The bids of similar types are placed in same order books. The bids have similar types of constraints. It represents an identical set of data. It is not unreasonable to assume that valuation of these buyers is identical. So it is assumed that the valuations of buyers in one order book are identical e.g. valuations of buyers requiring cotton of grade 3. Hence the valuations of buyers in one order book, are drawn from independent and identical random variables. However valuations of buyers, whose bids are in different order books is not in general identical. The implication of this assumption is that valuations of buyers requiring cotton of grade 3 are from iid random variables. There is similar situation for bids, which require cotton of grade 2. However these two are independent and not identical. So valuations of buyers requiring cotton of different grades are not identical. In this case it is assumed that the valuations are not identically distributed, however they are still independent. So valuations of buyers in order book Bo_i are drawn from independent and identically distributed (iid) random variables. On the other hand, valuations of buyers in different order books are independent. In the same way, it can be assumed that the sellers whose asks are of similar type have identical valuations. The asks are matched with bids in different order books. Let AS_j be the set of asks, such that all asks in AS_j are matched with bids in single order book Bo_i only. Then set of asks in AS_j are identical, so valuations of sellers who have submitted these asks can be identical. In the same way, let AS_i be the set of asks, i.e. asks which are matched against bids in different order books Bo_1, Bo_2, ..., Bo_l, then asks in AS_i are similar. In other words, basic assumption here is that valuations of buyers in the same order book are drawn from independent and identically distributed random variables, whereas valuations of buyers in different order books are independent and are drawn from random variables, which are independent. In the same way, sellers whose asks are matched with bids of the same order book, or same sets of order books, have their valuations drawn from independent and identically distributed (iid) random variables. The valuations of others are drawn from random variables, which are independent. In other words, set of similar bids (i.e. having same set of assignment constraints) are drawn iid random variables.

Let the valuations of all the m_i buyers in i^{th} order book be drawn from iid random variables with distribution function F_i and probability density function f_i which is defined over closed interval $[b_l , b_u]$. This interval remains same for all order books. Let AS_j be the set of asks such that all the asks in AS_j are matched with same set of order books. It is assumed that the valuations of sellers in AS_j are from independent and identically distributed (iid) random variables with distribution function G_j and density function g_j, which is defined over closed interval $[s_l , s_u]$. This interval remains the same for all the sellers. In both cases it is assumed that the intervals are compact and density functions are continuous. Let bv_i and sv_j be the private valuations of i^{th} buyer and j^{th} seller respectively. It is also assumed that demands of the buyers are identically and independently distributed random variables. Similarly supply from the sellers is from independent and identically distributed random variables. Since volume discounts are not permitted, demand and valuations as well as supply and valuations are independent. We derive certain results using which it is proved that efficiency loss is always bounded. It is also necessary that all density functions f_i and g_j have non zero minimum values i.e.

$$\alpha_i = \min \{f_i(x): b_l \leq x \leq b_u, \; i = 1, 2, \ldots, n\} > 0 \qquad (4.11)$$

$$\beta_j = \min \{g_j(x): s_l \leq x \leq s_u, \; i = 1, 2, \ldots, m\} > 0 \qquad (4.12).$$

We derive certain results using which it can be shown that efficiency loss is bounded.

Efficiency Loss

We derive two results in respect of order statistics when they are independently distributed but not identically distributed.

Theorem 4 : Let $x_1, x_2, \ldots, x_n$ be the sample observations. Let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ be the corresponding order statistics. Let x_{avg} be the mean of sample observations respectively. Let us define

$s = \sum_{i=1}^{n} (x_{(i)}-x_{avg})^2$ (sum of squares of deviation from sample mean).Then $|x_{(i)}-x_{(j)}|$ is always bounded by $\sqrt{2}$ s.

Proof : The result is derived using Cauchy's inequality. By this inequality we have

$$\left| \sum_{i=1}^{2} c_i (x_{(i)}-x_{avg}) \right| \leq [\sum_{i=1}^{2} (c_i - c_{avg})^2 \sum_{i=1}^{2} (x_{(i)}-x_{avg})^2]^{1/2} \quad (4.13).$$

Where c_i's are constant and c_{avg} is the mean of c_i.

In present case we have $c_1 = 1$ and $c_2 = -1$ $c_{avg} = (1 - 1) /2$

So we get

$$|x_{(i)}-x_{(j)}| = |(x_{(i)} - x_{avg}) - (x_{(j)} - x_{avg})| \quad (4.14).$$

.Using inequality (4.13) we get

$$|x_{(i)}-x_{(j)}| = |(x_{(i)} - x_{avg}) - (x_{(j)} - x_{avg})| \leq [((1-0)^2+(-1-0)^2) \Sigma (x_{(i)}-x_{avg})^2]^{1/2}$$

$$\leq [2 \sum_{i=1}^{n} (x_{(i)}-x_{avg})^2]^{1/2}$$

$$\leq \sqrt{2} .s$$

This result states that the difference between any two order statistics is always bounded. It can be seen that these order statistics refer to reservation prices or valuations of any two buyers or sellers. These buyers or sellers need not be consecutive buyers or sellers.

Theorem 5 : Let $X_1, X_2, \ldots, X_n$ be the independent random variables with $E(X_i) = \mu_i$ and $var(X_i)= \sigma_i$. Let μ_{avg} be mean of μ_i. Let $x_1, x_2, \ldots, x_n$ be the sample observations. Let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ be the corresponding order statistics. Then

$$|E(x_{(i)}-x_{(j)})| \leq [2(\mu_i^2 + \mu_j^2 - 2((\mu_i + \mu_j) \mu_{avg} + \mu_{avg}^2) + ½ (\sigma_i^2 + \sigma_j^2))]^{1/2}$$

Proof : Using Cauchy inequality and noting that variables X_i are independent, we have the following basic result

$$\left| E(\sum_{i=1}^{n} c_i(x_{(i)}-\mu_{avg})) \leq [\sum_{i=1}^{n} (c_i - c_{avg})^2 \sum_{i=1}^{n} ((\mu_i - \mu_{avg})^2 + ((n-1) /n) \sigma i^2)]^{1/2} \right. \quad (4.15).$$

As earlier c_i for all i are constant and c_{avg} is the mean of c_i.

In this case we have $c_1 = 1$ and $c_2 = -1$, $c_{avg} = (1-1)/2 = 0$,

So we get

$$\left| E(x_{(i)}-x_{(j)}) \right| = \left| E((x_{(i)} - \mu_{avg})-(x_{(j)}- \mu_{avg})) \right| \quad (4.16).$$

Using inequality (4.15) we get

$$\left| E(x_{(i)}-x_{(j)}) \right| \le [((1-0)^2+(-1-0)^2)\ ((\mu_i - \mu_{avg})^2 + (\mu_j - \mu_{avg})^2 +(2-1)/2(\sigma_i^2 + \sigma_j^2))]^{1/2}$$

$$\le [2((\mu_i - \mu_{avg})^2 - (\mu_j - \mu_{avg})^2 + ½(\sigma_i^2 + \sigma_j^2)]^{1/2}$$

$$\le [2(\mu_i^2 + \mu_j^2 - 2((\mu_i - \mu_j)\ \mu_{avg} + \mu_{avg}^{\ 2}) + ½\ (\sigma_i^2 + \sigma_j^2))]^{1/2} \quad (4.17).$$

It can be seen that the square root of the right hand side usually results in lower value of right hand side term.

In case $E(X_i) = \mu$ and $var(X_i) = \sigma$ (i.e. IID case), then the above inequality reduces to

$$\left| E(x_{(i)}-x_{(j)}) \right| \le \sigma \quad (4.18).$$

The above expression defines the bound on absolute values. Let bo_{max} and bo_{min} be the upper and lower bounds on the expected values. So that

$bo_{min} \le E(x_{(i)}-x_{(j)}) \le bo_{max}$, where

$bo_{max} = [2(\mu_i^2 + \mu_j^2 - 2((\mu_i + \mu_j)\ \mu_{avg} + \mu_{avg}^{\ 2}) + ½\ (\sigma_i^2 + \sigma_j^2))]^{1/2}$ and

$bo_{min} = -bo_{max.}$

Bounds on Expectations of Order Statistics: There are further bounds on expectations of order statistics, which we use. The basic results can be seen in [OS2003].

Let $X_1, X_2, \ldots, X_n$ be the independent random variables with $E(X_i) = \mu_i$ and $var(X_i)= \sigma_i$. Let μ_{avg} be mean of $\mu_{i.}$ Let $x_1, x_2, \ldots, x_n$ be the sample observations. Let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ be the corresponding order statistics. Then let $\mu_r = E(X_{(r)})$, expected value of r^{th} order statistics. Let the variance of i^{th} order statistics be σ_i^2 i.e. $V(X_{(i)}) = \sigma_i^2$. Then bounds on μ_r are

$$\frac{1}{r}\sum_{i=1}^{r}\mu_i \le \mu_r \le \frac{1}{(n-r+1)}\sum_{i=r}^{n}\mu_i \quad (4.19).$$

If $\mu_i = \mu$ and $\sigma_i^2 = \sigma$ for all i = 1, 2, …, n then

$$\mu - \sigma \left(\frac{n-r}{r}\right)^{1/2} \le \mu_r \le \mu + \sigma \left(\frac{r-1}{n-r+1}\right)^{1/2} \quad (4.20).$$

The proof of this result can be seen in [OS2003] and hence omitted.

Efficiency Loss: In this section terms efficiency loss, market value and inefficiency ratio are extended to GMDA mechanism. In GMDA mechanism efficiency loss occurs because there is a trade that is sacrificed in minimum price order book. This is the last possible trade. Further as in [HWS2002] the part of volume is sacrificed and market maker retains a part of surplus. The volume sacrificed depends upon demand and supply in minimum price order book. Let m_b and n_s be the number of buyers and sellers who actually trade (in all order books). Let $el(m_b, n_s)$ be the efficiency loss in the market. It is defined in the same way as in MDA [HWS2002]. It has got three components already described. This efficiency loss denoted by $el(m_b, n_s)$ can be written as

$el(m_b, n_s)$ = Trading surplus retained by the market maker + Sacrificed trade between m_k^{th} buyer and n_k^{th} seller in minimum price order book + Potential trade sacrificed (4.21). Let qm_k be the point where aggregate demand and supply meet in the k^{th} order book. Then the first two terms of above expression (4.21) can be written as

$= (bv_{mk} - sv_{nk})qm_k$.

In case of excess demand the potential sacrificed trade sacrificed is bounded by

$(bp_{k(1)} - bp_{k(mk)})\, aqk_{nk}$. (Condition I in [HWS2002]).

In case of excess supply the potential trade sacrificed is bounded by

$(ap_{k(nk)} - ap_{k(1)})bqk_{mk}$. (Condition II in [HWS2002]).

So we can write

$$el(m_b,n_s) \leq (bv_{mk} - sv_{nk})qm_k + (bp_{k(1)} - bp_{k(mk)})\, aq_{knk} \quad \text{or} \tag{4.22}$$

$$el(m_b,n_s) \leq (bv_{mk} - sv_{nk})qm_k + (ap_{k(nk)} - ap_{k(1)})bq_{kmk} \tag{4.23}$$

Market Value**:** There are k order books $B_1, B_2, \ldots, B_k$. Let B_k be the minimum price order book. Let pc_{Bi} be the per unit price paid by all the buyers in i^{th} order book. Let q_i be the quantity traded in each order book. Let there be m_i buyers in each order book. In the same number of sellers in each order book are n_j. So the market value $mv(m_b,n_s)$ with m_b buyers and n_s sellers is

$$mv(m_b,n_s) = \sum_{i=1}^{k} mvb_i + \sum_{i=1}^{k} mvs_i + (bv_{mk} - sv_{nk})qm_k \tag{4.24}$$

where mvb_i is buyer's market value of i^{th} order book, mvs_i is the seller's market value of the i^{th} order book. The third term in above expression represents the value of the sacrificed trade. The buyer's and seller's market values for order books, which are not minimum price order book can be written as

$$mvb_i = \sum_{j=1}^{mi} (bv_{ij} - \frac{1}{2}(bv_{imi+1} + sv_{ini+1}))bqi_j \qquad (4.25).$$

where bv_{ij} is the private valuation of j^{th} buyer in i^{th} order book, bq_{ij} is the quantity purchased by j^{th} buyer in i^{th} order book. The terms bv_{imi+1} and sv_{ini+1} represent private valuations of $(m_i+1)^{th}$ buyer and $(n_i+1)^{th}$ seller in i^{th} order book.

In the same way seller's market value can be written as

$$mvs_i = \sum_{j=1}^{ni} (\frac{1}{2}(sv_{ini+1} + bv_{imi+1}) - sv_{ij})aqi_j \qquad (4.26).$$

The explanation of terms is symmetric with respect to buyer i.e. seller's private valuation, market clearing price and quantity sold.

In case of minimum price order book the above values are

$$mvb_k = \sum_{j=1}^{mk-1} (bv_{kj} - bv_{kmk})bqk_j \text{ and } mvs_k = \sum_{j=1}^{nk-1} (sv_{knk} - sv_{kj})aqk_j$$

The value of the sacrificed trade is given by

$$(bv_{mk} - sv_{nk})qm_k \qquad (4.27).$$

In case of expression (4.27), $bp_{mk} \geq sv_{nk,}$ further quantity $qm_k \geq 0$.

So expression (4.27) is always nonnegative as $bv_{mk} \geq sv_{nk}$ and quantity qm_k are both nonnegative. So we can write the following inequality

$$mv(n_b, m_s) \geq \sum_{i=1}^{k} mvb_i + \sum_{i=1}^{k} mvs_i \qquad (4.28).$$

Inefficiency Ratio: The inefficiency ratio $ir(n_b, m_s)$ is defined as the ratio of efficiency loss when m_b buyers and n_s sellers trade in [HWS2002]. The same concept is used here. This ratio can be written as

$$ir(m_b, n_s) = \frac{E(EfficiencyLoss)}{E(TotalMarketValue)} \qquad (4.29).$$

The expectations are taken because the valuation of buyers and sellers are random variables.

We state the following two theorems about efficiency loss.

Theorem 6: Efficiency loss in GMDA is always bounded**.**

Proof: The upper bound on efficiency loss can be obtained by using theorem 4.

As already stated one upper bound on efficiency loss is (from exp 4.22)

$el(m_b,n_s) \leq (bv_{mk} - sv_{nk})qm_k + (bp_{k(1)} - bp_{k(mk)})\, aqk_{nk.}$

Let μ_{bs} be the average of combined prices of selected m_k bids and n_k asks in minimum price order book. Let $S_{bs} = \sum_{i=1}^{mk} (bp_i - \mu_{bp})^2 + \sum_{i=1}^{nk} (ap_i - \mu_{bs})^2$

Let μ_b be the average of selected m_k bid prices in minimum price order book. Let μ_s be the average of selected m_k bids and n_k ask prices in minimum price order book. Let S_b and S_s be the corresponding sum of squares of deviations.

Then applying theorem 5, the bound on efficiency loss can be worked out as

Efficiency Loss $\leq \sqrt{2}\,(S_{bs}q_k)$,

Since the above quantity is constant, it has been shown that efficiency loss is always bounded. In the same way other upper bound can be worked out.

Theorem 7: Inefficiency ratio is always bounded by constant term and as number of buyers or sellers becomes large it tends to 0.

Proof: We now compute the bounds on expected market value and efficiency loss. These bounds are used to obtain bounds on inefficiency ratio.

E(Total Market Value) = $E(mv(m_b, n_s))$. Using inequality (4.28) we can write the following expression.

$$E(mv(m_b,n_s)) \geq \sum_{i=1}^{k} E(mvb_i) + \sum_{i=1}^{k} E(mvs_i)$$

$$E(mv(m_b,n_s)) \geq A + B \text{ , where}$$

$$A = \sum_{i=1}^{k} E(mvb_i) \quad \text{and } B = \sum_{i=1}^{k} E(mvs_i)$$

The above expression is obtained after simplification and using property E(x+y) = E(x)+E(y). Both the terms of above expression can further be simplified. It can be seen that

$$E(mvb_1)=E(\sum_{j=1}^{mi}(bv_{1j} - \frac{1}{2}(bv_{1m1+1} + sv_{1n1+1}))bq_{1j}) \qquad (4.30).$$

In our scheme discriminative bidding is not permitted. Due to this it is reasonable to assume the independence of demand and valuations as well as supply and valuations. Let the demand of all the buyers be distributed as random variable D. Similarly the supply of all the sellers be distributed as random variable S. We derive the lower bound for one order book for the buyer. It can be then extended to other order books. In order book 1, there are m_1 buyers. The expression (4.30) can be rewritten as

$$E(mvb_1)=\sum_{j=1}^{m1}(E(bv_{1j})-\frac{1}{2}E(bv_{1m1+1}+sv_{1n1+1}))E(D) \qquad (4.31).$$

There are m_1 buyers in this order book. The terms bv_{1j} are m_1 highest price bids from set Bo_1. These values correspond to the order statistics of the valuations from set Bo_1 as well as the original valuations of buyers i.e. set BD. However in case of bid set BD, these are not consecutive order statistics. In this order book (i.e. Bo_1) the valuations are drawn from independent and identically distributed random variables with mean μ_{b1} and variance σ^2_{b1}. Then using the second part of result (4.19 and 4.20) we get the lower bound on expected value of j^{th} buyer's valuation in first order book.

$$E(bv_{1j}) \geq \mu_{b1} - \sigma_{b1}\left(\frac{m_1-j}{j}\right)^{\frac{1}{2}} \quad \text{and}$$

$$E(bv_{1m1+1}) \geq \mu_{b1} - \sigma_{b1}\left(\frac{m_{b1}-m_1-1}{m_1+1}\right)^{\frac{1}{2}} \qquad \text{and}$$

$$E(sv_{1n1+1}) \geq \mu_{s1} - \sigma_{s1}\left(\frac{n_{s1}-n_1-1}{n_1+1}\right)^{\frac{1}{2}} \qquad (4.32).$$

Where mb_1 is the number of buyers in order book Bo_1 and ns_1 is number of corresponding sellers.

Using this expression for the term of (4.30) can be written as

$E(mvb_1)$

$$\geq\sum_{j=1}^{m1}\left(\frac{1}{2}\mu_{b1}+\sigma_{b1}\left(\frac{1}{2}\left(\frac{m_{b1}-m_1-1}{m_1+1}\right)^{\frac{1}{2}}-\left(\frac{m_1-j}{j}\right)^{\frac{1}{2}}\right)-\frac{1}{2}\mu_{s1}+\frac{1}{2}\sigma_{s1}\left(\frac{n_{s1}-n_1-1}{n_1+1}\right)^{\frac{1}{2}}\right)E(D)$$

$$\geq\left(\frac{1}{2}m_1\left(\mu_{b1}+\sigma_{b1}\left(\frac{m_{b1}-m_1-1}{m_1+1}\right)^{\frac{1}{2}}-\mu_{s1}+\sigma_{s1}\left(\frac{ns_1-n_1-1}{n_1+1}\right)^{\frac{1}{2}}\right)E(D)-\sigma_{b1}\sum_{j=1}^{m1}\left(\frac{m_1-j}{j}\right)^{\frac{1}{2}}E(D)\right)$$

This expression is general one and using it we can write the expression for $E(mvb_i)$. We can write

$$E(mvb_i) = \sum_{j=1}^{mi} (E(bv_{ij}) - \frac{1}{2} E(bv_{imi+1} + sv_{ini+1})) E(D)$$

There are m_i buyers in i^{th} order book. The terms bv_{ij} are m_i highest price bids from set Bo_i. These values correspond to the order statistics of the valuations from set Bo_i as well as the original valuations of buyers i.e. set BD. However in case of bid set BD, these are not consecutive order statistics. In this order book (i.e. Bo_i) the valuations are drawn from independent and identically distributed random variables with mean μ_{bi} and variance σ^2_{bi}. Then using the second part of result (4.19 and 4.20) and the argument as above the bound can be worked out as

$$E(mvb_i) \geq \sum_{j=1}^{mi} (\frac{1}{2} \mu_{bi} + \sigma_{bi}(\frac{1}{2} (\frac{m_{bi} - m_i - 1}{m_i + 1})^{\frac{1}{2}} - (\frac{m_i - j}{j})^{\frac{1}{2}}) - \frac{1}{2} \mu_{si} + \frac{1}{2} \sigma_{si} (\frac{n_{si} - n_i - 1}{n_i + 1})^{\frac{1}{2}}) E(D)$$

$$\geq (\frac{1}{2} m_i(\mu_{bi} + \sigma_{bi} (\frac{m_{bi} - m_i - 1}{m_i + 1})^{\frac{1}{2}} - \mu_{si} + \sigma_{si} (\frac{n_{si} - n_i - 1}{n_i + 1})^{\frac{1}{2}}) E(D) - \sigma_{bi} \sum_{j=1}^{mi} (\frac{m_i - j}{j})^{\frac{1}{2}} E(D)).$$

for i =1, 2, …, k-1.

In case of k^{th} order book, which is minimum price order book, the above expression can be written as

$$E(mvb_k) = \sum_{j=1}^{m_k - 1} E(bv_{kj} - bv_{kmk}) E(D).$$

Then using the second part of result (4.19 and 4.20) we can write

$$E(bv_{kj}) \geq \mu_{bk} - \sigma_{bk} (\frac{m_k - j}{j})^{\frac{1}{2}} \quad \text{and} \quad E(bv_{kmk}) \geq \mu_{bk} - \sigma_{bk} (\frac{m_{bk} - m_k}{m_k})^{\frac{1}{2}}.$$

Using these two inequalities we get

$$E(mvb_k) = \sum_{j=1}^{mk-1} (E(bv_{kj}) - E(bv_{kmk})) E(D)) \geq$$

$$(\sum_{j=1}^{m_k - 1} \mu_{bk} - \sigma_{bk} \sum_{j=1}^{m_k - 1} (\frac{m_k - j}{j})^{\frac{1}{2}} - (\sum_{j=1}^{m_k - 1} \mu_{bk} - \sum_{j=1}^{m_k - 1} \sigma_{bk} (\frac{m_{bk} - m_k}{m_k})^{\frac{1}{2}})) E(D)$$

$$= ((m_k - 1)\mu_{bk} - \sigma_{bk} \sum_{j=1}^{m_k - 1} (\frac{m_k - j}{j})^{\frac{1}{2}} - (m_k - 1)(\mu_{bk} - \sigma_{bk} (\frac{m_{bk} - m_k}{m_k})^{\frac{1}{2}})) E(D)$$

$$= \sigma_{bk}\ \Big((m_k\text{-}1)\Big(\frac{m_{bk}-m_k}{m_k}\Big)^{\frac{1}{2}} - \sum_{j=1}^{m_{k-1}} \Big(\frac{m_k-j}{j}\Big)^{\frac{1}{2}}\Big)\ E(D).$$

So we can write term A as

$$A = \sum_{i=1}^{k} E(mvb_i) \geq$$

$$\sum_{i=1}^{k-1} \Big(\frac{1}{2} m_i(\mu_{bi}+\sigma_{bi}\Big(\frac{m_{bi}-m_{i-1}}{m_{i+1}}\Big)^{\frac{1}{2}} - \mu_{si}+\sigma_{si}\Big(\frac{n_{si}-n_{i-1}}{n_{i+1}}\Big)^{\frac{1}{2}}\Big)\ E(D) - \sigma_{bi}\sum_{j=1}^{mi}\Big(\frac{m_i-j}{j}\Big)^{\frac{1}{2}} E(D)) +$$

$$\sigma_{bk}((m_{k-1})\Big(\frac{mb_k-m_k}{m_k}\Big)^{\frac{1}{2}} - \sum_{j=1}^{m_{k-1}} \Big(\frac{m_k-j}{j}\Big)^{\frac{1}{2}}\Big)E(D)) \qquad (4.33).$$

Let $\mu_{bmin} = \min(\mu_{bi})$ (minimum average of different order book averages for buyers) and $\mu_{smin} = \min(\mu_{si})$ (minimum average of different order book averages for sellers).
In the same way let us define $\sigma_{bmin} = \min(\sigma_{bi})$ (minimum standard deviation of different order book averages for buyers) and $\sigma_{smin} = \min(\sigma_{si})$ (minimum standard deviation of different order book averages for sellers). Then noting that $\sigma_{bi} \geq \min(\sigma_{bi})$ and similar ineqaulity for other terms we can write (4.33) as

$$A \geq \Big(\frac{1}{2}\mu_{bmin}m_b+\frac{1}{2}\sigma_{bmin}\sum_{i=1}^{k-1} m_i\Big(\frac{m_{bi}-m_{i-1}}{m_i+1}\Big)^{\frac{1}{2}} - \frac{1}{2}\mu_{smin}m_b+\frac{1}{2}\sigma_{smin}\sum_{i=1}^{k-1} m_i\Big(\frac{n_{si}-n_{i-1}}{n_i+1}\Big)^{\frac{1}{2}} -$$

$$\sigma_{bmin}\sum_{i=1}^{k-1}\ \sum_{j=1}^{mi}\Big(\frac{m_i-j}{j}\Big)^{\frac{1}{2}} + (m_{k-1})\sigma_{bmin}\Big(\frac{m_{bk}-m_k}{m_k}\Big)^{\frac{1}{2}} - \sigma_{bmin}\sum_{j=1}^{m_{k-1}}\Big(\frac{m_k-j}{j}\Big)^{\frac{1}{2}}\Big)E(D) - \frac{1}{2}$$

$m_k(\mu_{bmin} + \mu_{smin})E(D)$.

It may be noted that in RHS of above expression all terms except first term others are independent of terms m_b. We rewrite above expression as

$A \geq (MV_1 m_b + MV_2)E(D)$.

Where $MV_1 = \frac{1}{2}(\mu_{bmin} - \mu_{smin})$

and $MV_2 = \frac{1}{2}\sigma_{bmin}\sum_{i=1}^{k-1} m_i\left(\frac{m_{bi}-m_i-1}{m_i+1}\right)^{\frac{1}{2}} + \frac{1}{2}\sigma_{smin}\sum_{i=1}^{k-1} m_i\left(\frac{n_{si}-n_i-1}{n_i+1}\right)^{\frac{1}{2}}$ -

$\sigma_{bmin}\sum_{i=1}^{k-1}\sum_{j=1}^{mi}\left(\frac{m_i-j}{j}\right)^{\frac{1}{2}} + (m_k-1)\sigma_{bmin}\left(\frac{m_{bk}-m_k}{m_k}\right)^{\frac{1}{2}} - \sigma_{bmin}\sum_{i=1}^{k-1}\sum_{j=1}^{m_k}\left(\frac{m_k-j}{j}\right)^{\frac{1}{2}} -$

$$\frac{1}{2} m_k(\mu_{bmin} - \mu_{smin}) \qquad (4.34)$$

and is indepndent of m_b (actual number of buyers who trade in all order books).

In the same way the bound for the term B can be worked out.

$B = \sum_{i=1}^{k} E(mvs_i) = \sum_{i=1}^{k-1} E(mvs_i) + E(mvs_k)$

We can write for any i^th order book, where n_i sellers are selected.

$E(mvs_i) = E(\sum_{j=1}^{ni}(\frac{1}{2}(bv_{imi+1} + svi_{ni+1}) - sv_{ij})aqi_j)$. Simplifying this as earlier, we get

$$= \sum_{j=1}^{ni}(\frac{1}{2}E(bv_{imi+1}) + \frac{1}{2}E(sv_{ini+1}) - E(sv_{ij}))E(aqi_j).$$

$$= \sum_{j=1}^{ni}(\frac{1}{2}E(bv_{imi+1}) + \frac{1}{2}E(sv_{ini+1}) - E(sv_{ij}))E(S) \qquad (4.35).$$

As already stated supply is independent of valuations and is indicated by variable S. The bounds on diffrenet terms of (4.35) can be obtained as earlier. The seller's asks are matched with bids depending upon constraints. An ask can be matched in more than one order book. There may be some asks which are matched with bids of only one order book. Some asks may be matched with bids of more than one order book. The asks which are matched with same set of order books are drawn from identically distributed random variables. In general the valuations need not be from iid random variables. We can write

$$E(bv_{imi+1}) \geq \mu_{bi} - \sigma_{bi}\left(\frac{m_{bi}-m_i-1}{m_i+1}\right)^{\frac{1}{2}}. \qquad (4.36)$$

The different terms in expression (4.34) are already explained. The bounds on other two terms can be worked out as follows.

$$E(sv_{ini+1}) \geq \mu_{si} - \sigma_{si}\left(\frac{n_{si}-n_i-1}{n_i+1}\right)^{\frac{1}{2}} \qquad (4.37)$$

$$E(sv_{ij}) \geq \frac{1}{j}\sum_{l=1}^{j} \mu s_l = \mu s_{si} \tag{4.38}$$

This expression is obtained based on the bounds on expectations of order statistics stated earlier. (It may be noted that al sv_{ij} need not be identically distributed.) Then using the expressions (4.36), (4.37) and (4.38) we can work out the bond on $E(mvs_i)$ sa follows.

$$E(mvs_i) \geq \sum_{j=1}^{ni} \left(\left(\frac{1}{2}\mu_{bi} - \sigma_{bi}\left(\frac{m_{bi}-m_i-1}{m_i+1}\right)^{\frac{1}{2}}\right) + \left(\frac{1}{2}\mu_{si} - \sigma_{si}\left(\frac{n_{si}-n_i-1}{n_i+1}\right)^{\frac{1}{2}}\right) - \mu s_{si}\right)E(S) \tag{4.39}$$

This expression can be simplified as

$$E(mvs_i) \geq \left(\frac{1}{2} n_i\left(\mu_{bi} - \sigma_{bi}\left(\frac{m_{bi}-m_i-1}{m_i+1}\right)^{\frac{1}{2}} + \mu_{si} - \sigma_{si}\left(\frac{n_{si}-n_i-1}{n_i+1}\right)^{\frac{1}{2}}\right) - n_i \mu s_{si}\right)E(S) \tag{4.40}$$

In case of minimum price order book we can write

$$E(mvs_k) = \sum_{j=1}^{nk-1} (sv_{knk} - sv_{kj})E(S) \tag{4.41}$$

Then we can write using argument similar to expression 4.38

$$E(sv_{kj}) \geq \mu s_{sk} \quad \text{and} \quad E(svk_{nk}) \geq \mu sn_{sk} \tag{4.42}$$

So we can write

$$E(mvs_k) \geq \sum_{j=1}^{nk-1} (\mu s_{sk} - \mu sn_{sk})$$

This can be written as $E(mvs_k) \geq MV_{sk}E(S)$. The term MVsk is bound on seller's market value and is constant. The expressions is

$$MV_{sk} = \sum_{j=1}^{nk-1} (\mu_{sk} - \mu sn_{sk}). \tag{4.43}$$

The bound on the value of B can be worked as

$$B = \sum_{i=1}^{k} E(mvs_i) \geq \left(\sum_{i=1}^{k-1} \left(\left(\frac{1}{2} n_i\left(\mu_{bi} - \sigma_{bi}\left(\frac{m_{bi}-m_i-1}{m_i+1}\right)^{\frac{1}{2}} + \mu_{si} - \sigma_{si}\left(\frac{n_{si}-n_i-1}{n_i+1}\right)^{\frac{1}{2}}\right) - n_i \mu s_{si}\right) + MV_{sk}\right)E(S) \tag{4.44}$$

As earlier we substitute $\mu_{bmin} = \min(\mu_{bi})$ and $\mu_{smin} = \min(\mu_{si})$ for all sellers. We also substitute $\sigma_{bmin} = \min(\sigma_{bi})$ and $\sigma_{smin} = \min(\sigma_{si})$ for all sellers. Then noting that $\mu_{bmin} \geq \min(\mu_{bi})$ and similar ineqaulity for other terms we can write RHS of (4.44) as

$$(\frac{1}{2}(n_s(\mu_{bmin}+\mu_{smin}) - \sigma_{bmin}\sum_{i=1}^{k-1} n_i \left(\frac{m_{bi}-m_i-1}{m_i+1}\right)^{\frac{1}{2}} - \sigma_{smin}\sum_{i=1}^{k-1} n_i \left(\frac{n_{si}-n_i-1}{n_i+1}\right)^{\frac{1}{2}}) - \sum_{i=1}^{k-1} n_i\, \mu s_{si} + MV_{sk}$$

$$-\frac{1}{2} n_k(\mu_{bmin}+\mu_{smin}))E(S) \qquad (4.45)$$

It can be written as $(\frac{1}{2} n_b(\mu_{bmin} + \mu_{smin}) + MV_{sk-1})\ E(S)$ (4.46)

where $MV_{sk-1}=MV_{sk}-\frac{1}{2}\sigma_{bmin}\sum_{i=1}^{k-1} n_i \left(\frac{m_{bi}-m_i-1}{m_i+1}\right)^{\frac{1}{2}} - \frac{1}{2}\sigma_{smin}\sum_{i=1}^{k-1} n_i \left(\frac{n_{si}-n_i-1}{n_i+1}\right)^{\frac{1}{2}} - \sum_{i=1}^{k-1} n_i\, \mu s_{si} - \frac{1}{2} n_k(\mu_{bmin}+\mu_{smin})$.

We also write $MV_{s1} = \frac{1}{2}(\mu_{bmin} + \mu_{smin})$

The second term is constant term. Then the bound on the market value is obtained by using expression (4.45) and (4.46). The bound is

$A + B \geq (MV_1 m_b + MV_2)E(D) + (n_s MV_{s1} + MV_{sk-1})\ E(S)$ (4.47)

We obtain the bounds in four different cases. Out of these four cases the first two correspond to excess demand and excess supply case of [HWS2002]. The other two cases correspond to scenario where all bids in minimum price order book have indivisibility constraints and scenario where there is only one feasible trade in minimum price order book.

(1) Case I: In this case there is excess demand in minimum price order book. The efficiency loss can be written as

$$el(n_b,m_s) \leq (bv_{kmk} - sv_{knk})\sum_{i=1}^{mk} bq_{ki} + (bv_{k1} - bv_{kmk})aq_{knk} \qquad (4.48).$$

Taking expectations of both sides in expression (4.51) we get

$E(el(m_b, n_s)) \leq E(bv_{kmk} - bv_{knk+1})E(D) + E(bv_{k1} - bv_{kmk})E(S)$

Using theorem 6 and noting that bv_{ki}, are order statsistics and from iid variables

$E(bv_{k1} - bv_{km}) \leq \sigma_{bk}$ and

$E(bv_{kmk} - sv_{knk}) \leq [2(\mu_{bk}^2 + \mu_{sk}^2 - 2((\mu_{bk} + \mu_{sk})\ \mu_{bsavg} + \mu_{bsavg}^2) + ½\ (\sigma_{bk}^2 + \sigma_{sk}^2))]^{1/2}$

So we can write

$E(bv_{kmk} - sv_{knk}) \leq el_{bk}$, where

$el_{bk} = 2(\mu_{bk}^2 + \mu_{sk}^2 - 2((\mu_{bk} + \mu_{sk})\ \mu_{bsavg} + \mu_{bsavg}^2) + ½\ (\sigma_{bk}^2 + \sigma_{sk}^2))]^{1/2}$ and is a constant independent of n_b and m_b.

$E(el(m_b, n_s)) \leq el_{bk}\ E(D) + \sigma_{bk}\ E(S)$ (4.49).

(2) Case II: In this case there is excess supply. The bound is worked out for case of excess supply as

$$el(m_b, n_s) \leq (bv_{kmk} - sv_{knk}) \sum_{i=1}^{nk} aq_{ki} + (sv_{knk} - sv_{k1})\ bqk_{mk}$$

Taking expectations of both sides we can write

$$E(el(m_b, n_s)) \leq E(bv_{kmk} - sv_{knk})E(\sum_{i=1}^{mk} bq_{ki}) + E(sv_{knk} - sv_{k1})E(bqk_{mk}) \quad (4.50).$$

The bound on the first value is same as in above case. Using theorem 6, we get the bound on expected value as

$E(sv_{knk} - sv_{k1}) \leq \sigma_{sk}$ and

$E(bv_{kmk} - sv_{knk}) \leq [2(\mu_{bk}^2 + \mu_{sk}^2 - 2((\mu_{bk} + \mu_{sk})\ \mu_{bsavg} + \mu_{bsavg}^2) + ½\ (\sigma_{bk}^2 + \sigma_{sk}^2))]^{1/2}$

$E(bv_{kmk} - sv_{knk}) \leq el_{bk}$ as earlier.

So we get

$E(el(n_b,m_s)) \leq el_{bk}E(D) + \sigma_{sk}E(D)$ (4.50).

The standard deviation of n_k least price asks, which can be assigned to minimum price order book is σ_{sk}.

(3) Case III: In this case all the bids in minimum price order book have indivisibility constraints. There is some volume that is sacrificed, as all bids may have indivisibility constraints. This is not part of sacrificed trade and is not efficiency loss. So the bound remains same as earlier two cases depending upon demand andsupply. So we get

$E(el(n_b,m_s)) \leq el_{bk}E(D) + \sigma_{sk}E(D)$or (4.51).

$E(el(n_b,m_s)) \leq el_{bk}\ E(D) + \sigma_{bk}\ E(S)$ (4.52)

.

(4) Case IV: In this case there is no trade in minimum price order book. So the second term is absent. In this case the bound is

$E(el(n_b,m_s)) \leq el_{bk}E(D)$ (4.53).

Using these inequalities we work out bounds on inefficiency ratio $ir(m_b,n_s)$. The upper bound on numerator and the lower bound on denominator have been worked out. Using these bounds we get the bounds on inefficiency ratio in all the four cases.

Case I :

$$ir(m_b,n_s) \leq \frac{el_{bk}E(D) + \sigma_{bk}E(S)}{(MV_1 m_b + MV_2)E(D) + (n_s MV_{S1} + MV_{Sk-1})E(S)} \quad (4.54).$$

It can be seen that above term is constant as mean, standard deviations and other terms are constant. So efficiency loss is always bounded by a constant term. This can be similarly seen for all other cases.

Case II :

$$ir(m_b,n_s) \leq \frac{el_{bk}E(D) + \sigma_{Sk}E(D)}{(MV_1 m_b + MV_2)E(D) + (n_s MV_{S1} + MV_{Sk-1})E(S)} \quad (4.55).$$

Case III **:**

$$ir(m_b,n_s) \leq \frac{el_{bk}E(D) + \sigma_{bk}E(S)}{(MV_1 m_b + MV_2)E(D) + (n_s MV_{S1} + MV_{Sk-1})E(S)} \quad (4.56)$$

In case of excess demand and otherwise it is

$$ir(m_b,n_s) \leq \frac{el_{bk}E(D) + \sigma_{Sk}E(D)}{(MV_1 m_b + MV_2)E(D) + (n_s MV_{S1} + MV_{Sk-1})E(S)} \quad (4.57).$$

Case IV :

$$ir(m_b,n_s) \leq \frac{el_{bk}E(D)}{(MV_1 m_b + MV_2)E(D) + (n_s MV_{S1} + MV_{Sk-1})E(S)} \quad (4.58).$$

It can be seen that in all the above cases efficiency loss is bounded by the constant term. The numerator terms are independent of n_s and m_b. So it can also be seen that if number of buyers or sellers go to ∞, the above ratios tend to 0.

So $\lim_{m_b \to \infty} ir(m_b,n_s) = 0$ or $\lim_{n_s \to \infty} ir(m_b,n_s) = 0$, the same is case if both are large. So asymptotic efficiency loss is 0 as long as the denominator in expressions (4.54)-(4.58) is non zero.

4.4 Misreporting of Volumes

As already stated, in GMDA, a buyer or seller does not have incentive to misreport his price. A buyer does not have incentive to report higher volumes. However in some cases buyers and sellers can misreport quantities and this can affect overall market clearing price. For example, a buyer can decrease and report his volume. In some cases this can bring down the market clearing price for an order book. On the other hand seller can misreport his volumes to improve his utility. This can happen because an ask may be matched against bids in more than one order book. Consider the following scenario

Suppose that in i^{th} order book there is price crossover at $i = m_i$ and $j = n_i$. Suppose that there are three bids $(m_i - 2)$, $(m_i - 1)$, and m_i such that

$bp_i \leq ap_{(ni+1)}$ for $i = m_i - 1,\ m_i - 2,\ m_i$ and $bp_i \geq ap_{(ni)}$ for $i = m_i - 1,\ m_i - 2,\ m_i$

Further suppose that total demand at $bp_{(mi-2)}$ is bq, which is supply available at ask price $ap_{(ni)}$. In this case market clearing price is set to

$$(bp_{(mi-1)} + ap_{(ni+1)})/2 \tag{4.59}$$

However if any buyer with bid price $\geq bp_{(mi-1)}$ decreases the demand by q, the bid $bp_{(mi-1)}$ is included. So the market clearing price is

$$(bp_{(mi)} + ap_{(ni+1)})/2 \tag{4.60}$$

In other words, market clearing price has been pushed down. In some cases sellers can misreport the volumes. If the seller can misreport volume, it can push up the market clearing price. However such manipulations are quite difficult in our set up. We propose the following modification to GMDA to minimize manipulations by buyers and sellers. In order to achieve it, an attempt is made to minimize incentive to misreport volumes; however, the scheme requires that there are minimum two buyers and sellers in an order book. We use the following notations for different quantity terms.

Let q_i be the total quantity sold or purchased in i^{th} order book B_{oi}.

Let bq_{ji} be quantity purchased by j^{th} buyer in i^{th} order book B_{oi}.

Let aq_{ji} be quantity sold by j^{th} sellers to buyers in i^{th} order book B_{oi}.

It is easy to verify $\sum_{j=1}^{mi} bq_{ji} = \sum_{i=1}^{ni} aq_{ji} = q_i$ for all $i = 1, 2, ..., k$

Let bq_{oji} be the quantity purchased by all other buyers except j^{th} buyer in order book B_{oi}.

So $bqo_{ji} = q_i - bq_{ji}$.

Similarly let aq_{oji} be the quantity sold by all other sellers except j^{th} seller to buyers in order book B_{oi}.

So $aqo_{ji} = q_i - aq_{ji}$.

Let us define $bin_{oi} = \frac{1}{2}\ (bp_{i(mi)} – pc_{Bi})$ and $ain_{oi} = \frac{1}{2}(\ pc_{Bi} – ap_{i(ni)})$. for all i= 1, 2, …, k-1 (excluding minimum price order book). In case of i = k (i.e. minimum price order book), these are defined as

$bin_{ok} = bp_{k(mk-1)} – bp_{k(mk)}$ and $ain_{ok} = ap_{k(nk)} – ap_{k(nk-1).}$

It can be easily verified that $bin_{oi} \geq 0$ and $ain_{oi} \geq 0$.

This follows from the fact that $bp_{(mi)} \geq pc_{Bi}$ for all i. In this case our mechanism works in the same way as GMDA, however there is no uniform clearing price. In this case if j^{th} buyer's bid is selected in i^{th} order book the per unit payable amount for this buyer is

$$bpua_{ji}=pc_{Bi} + \left(\frac{bqo_{ji}}{q_i}\right) bin_{oi} \text{ for } i=1, , ..., k , \ j = 1, 2, ..., m_i \qquad (4.61).$$

So if bq_j is the total quantity sold by j^{th} seller, then the total amount payable is

$bpa_{ji} = bpua_{ji}\, bq_j$

Similarly per unit price to be received by j^{th} seller in i^{th} order book is

$$srua_{ji} = (\ pc_{Bi} - \left(\frac{aqo_{ji}}{q_i}\right) ain_{oi}\ )$$

Let AS_j, be the set of order books, where the asks of j^{th} sellers are assigned. $AS_j = \{i_1, i_2, ..., k\}$ (i.e. set of integers). Then the total per unit price to be received by j^{th} seller is

$\Sigma_i srua_{ji}$ for $i \in AS_j$. (4.62).

If the aq_{ij} is the quantity sold by j^{th} seller in i^{th} order book, then the total amount receivable by j^{th} seller is

$sra_j = \Sigma_i\ aq_{ij} srua_{j.}$

In this scheme each buyer and seller who participates in the auction pays and receives different amount per unit price, unlike in GMDA. In GMDA, there is a single price for all buyers and sellers in all order books except minimum price order book. In minimum price order book there are two prices (one for all buyers and other for all the sellers). This scheme is referred to as Discriminatory GMDA or DGMDA mechanism.

Theorem 8: DGMDA satisfies IR property.

Proof: It can be seen that in DGMDA as in GMDA, if buyers and sellers are not selected then they do not pay and receive any amount. So they have 0 utility. In case buyers or sellers participate in auction then

In DGMDA per unit price paid by any buyer in i^{th} order book is

$$bpua_{ji} = pc_{Bi} + \left(\frac{bqo_{j_i}}{q_i}\right) bin_{oi} \qquad (4.63).$$

It can be easily verified that $bpua_{ji} \leq bp_{(mi)}$ for i = 1, 2, ..., k-1 and

$bpua_{ji} \leq bp_{(mk)}$. This follows as

$$\left(\frac{bqo_{j_i}}{q_i}\right) < 1 \quad \text{as } q_i = bq_{oji} + bq_{ji} \text{ (by definition) and } bq_{ji} > 0 \qquad (4.64).$$

Further for any order book i ≠ k, we have

$$bp_{(mi)} - bpua_{ji} = bp_{(mi)} - pc_{Bi} - \left(\frac{bqo_{j_i}}{q_i}\right) bin_{oi}$$

$$= bp_{(mi)} - pc_{Bi} - \left(\frac{bqo_{j_i}}{q_i}\right)\left(\frac{1}{2}\,(bp_{(mi)} - pc_{Bi})\right)$$

$$= (bp_{(mi)} - pc_{Bi})\left(1 - \left(\frac{bqo_{j_i}}{q_i}\right)\frac{1}{2}\right), \quad \text{we have } \left(\frac{bqo_{j_i}}{q_i}\right) < 1.$$

So we get $(bp_{(mi)} - pc_{Bi}) \geq 0$ and $\left(1 - \left(\frac{bqo_{j_i}}{q_i}\right)\frac{1}{2}\right) > 0$

So $bp_{(mi)} - bpua_{ji} \geq 0$ for any order book i= 1, 2, …, k-1. (4.65).

It may be noted that j^{th} buyer's bid is selected in i^{th} order book if and only if $bp_j \geq bp_{(mi)}$. This, together with (4.66), proves the result that all participating buyers have non negative gain from participation.

In k^{th} order book the clearing price to be paid by each buyer is

$$bpua_{jk} = pc_{Bk} + \left(\frac{bqo_{jk_i}}{q_i}\right) bin_{ok} = pc_{Bk} + \left(\frac{bqo_{jk_i}}{q_i}\right)(bp_{(mk-1)} - bp_{(mk)})$$

$$= bp_{(mk)} + \left(\frac{bqo_{jk_i}}{q_i}\right)(bp_{(mk-1)} - bp_{(mk)})$$

So we get $bp_{(mk-1)} - bpua_{jk} = bp_{(mk-1)} - bp_{(mk)} - \left(\frac{bqo_{jk_i}}{q_i}\right)(bp_{(mk-1)} - bp_{(mk)})$

$$= (bp_{(mk-1)} - bp_{(mk)})\left(1 - \left(\frac{bqo_{jk_i}}{q_i}\right)\right)$$

It can be seen that both the terms on RHS are non negative

(1 - $\left(\frac{bqo_{jk_i}}{q_i}\right)$) > 0 and ($bp_{(mk-1)} - bp_{(mk)}$) ≥ 0.

So $bp_{(mk-1)}$ - $bpua_{jk} \geq 0$

It may be noted that j^{th} buyer's bid is selected in k^{th} order book if and only if $bp_j \geq bp_{(mk)}$. This proves the result that all participating buyers have nonnegative gain from participation.

Exactly similar argument can be extended to show that all sellers have nonnegative gains from participation.

As, per unit price paid by each buyer in auction does not exceed the bid amount and the same to be received by each seller in auction is not lower than the ask price received, all participants have positive utility or have non negative gains after auction clears. Due to this, DGMDA is always IR.

Theorem 9: DGMDA mechanism satisfies budget balanced property.

Proof: It can be seen that in all order books i = 1, 2, …, k per unit price paid by any buyer is $bpua_{ji} = pc_{Bi} + \left(\frac{bqo_{ji}}{q_i}\right) bin_{oi}$

Similarly per unit price paid by the seller is

$srua_{ji} = pc_{Bi} - \left(\frac{aqo_{ji}}{q_i}\right) ain_{oi}$

It may be noted that $\left(\frac{bqo_{ji}}{q_i}\right) bin_{oi} \geq 0$ and $\left(\frac{aqo_{ji}}{q_i}\right) ain_{oi} \geq 0$.

So per unit price paid by any buyer in any order book is more than per unit price received by the seller.

So $bpua_{ji} \geq pc_{Bi}$ and $srua_{ji} \leq pc_{Bi}$ and hence

$bpua_{ji} \geq srua_{li}$ for any j^{th} buyer and l^{th} seller in i^{th} order book.

Further we get $\max_j (srua_{li}) \leq \min_j(bpua_{ji})$.

Let $\max_j (srua_{ji}) = sruam_i$ and $\min_j(bpua_{ji}) = mbpua_i$

The quantity sold and purchased in all or any order book is same. In i^{th} order book there are m_i buyers and n_i sellers. Let q_i be the quantity purchased or sold in i^{th} order book. Let

bq_{ji} be the quantity purchased by j^{th} buyer in i^{th} order book. Similarly aq_{ji} be the quantity sold by j^{th} seller in i^{th} order book. Then we have

$(bq_{1i}+bq_{2i}+\ldots+bq_{m_i i}) = (aq_{1i}+aq_{2i}+\ldots+aq_{n_i i}) = q_i$ (4.66).

The total amount to be paid by all the buyers in i^{th} order book is

$$amp_i = (bpua_{1i}\, bq_{1i}+ bpua_{2i}\, bq_{2i}+\ldots+ bpua_{mii}\, bq_{m_i i}) = \sum_{j=1}^{mi} bpua_{ji}\, bq_{ji} \quad (4.67).$$

The total amount to be received by all the sellers in i^{th} order book is

$$amr_i = (srua_{1i}\, aq_{1i}+ srua_{2i}\, aq_{2i}+\ldots+ srua_{nii}\, aq_{n_i i}) = \sum_{j=1}^{ni} srua_{ji}\, aq_{ji} \quad (4.68).$$

Then budget balanced means that $\sum_{i=1}^{k} amp_i \geq \sum_{i=1}^{k} amr_i$

This result follows because for i^{th} order book the total quantity sold is

$(aq_{1i}+aq_{2i}+\ldots+aq_{n_i i}) = q_i$

Then we have

$sruam_i\, q_i - amr_i = sruam_i\, q_i - (srua_{1i}\, aq_{1i}+ srua_{2i}\, aq_{2i}+\ldots+ srua_{nii}\, aq_{n_i i})$

$= sruam_i\,(aq_{1i}+aq_{2i}+\ldots+aq_{n_i i}) - (srua_{1i}\, aq_{1i}+ srua_{2i}\, aq_{2i}+\ldots+ srua_{nii}\, aq_{n_i i})$

$= \sum_{j=1}^{ni} (sruam_i - srua_{ji})\, aq_{ji} \geq 0$ as $(sruam_i - srua_{ji}) \geq 0$ ($sruam_i$ is maximum and $aq_{ji} \geq 0$).

So we get $sruam_i q_i \geq amr_i$ (4.69).

In the for i^{th} order book the total quantity purchased is

$(bpua_{1i}\, bq_{1i}+ bpua_{2i}\, bq_{2i}+\ldots+ bpua_{mii}\, bq_{m_i i}) = q_i$

Then we have

$amp_i - mbpua_i q_i = (bpua_{1i}\, bq_{1i}+ bpua_{2i}\, bq_{2i}+\ldots+ bpua_{mii}\, bq_{m_i i}) - mbpua_i\, q_i$

$= (bpua_{1i}\, bq_{1i}+ bpua_{2i}\, bq_{2i}+\ldots+ bpua_{mii}\, bq_{m_i i}) - mbpua_i(bq_{1i}+bq_{2i}+\ldots+bq_{m_i i})$

$= \sum_{j=1}^{mi} (bpua_{ji} - mbpua_i) bq_{ji} \geq 0$

as $(bpua_{ji} - mbpua_i) \geq 0$ ($mbpua_i$ is minimum and $bq_{ji} \geq 0$).

So we get

$mbpua_i q_i \leq amp_i$ (4.70).

We also have $mbpua_i \geq sruam_i$ and hence $mbpua_i\, q_i \geq sruam_i\, q_i$ (4.71).

So combining, (4.70)-(4.71), we get

$amp_i \geq amr_i$

Since this is true for all order books including minimum price order book the mechanism satisfy budget balanced property. In case of minimum price order book the amount paid by the j^{th} buyer is

$$bpua_{jk} = bpc_{Bk} + \left(\frac{bqo_{jk}}{q_k}\right) bin_{ok}$$

Since $\left(\frac{bqo_{jk}}{q_k}\right) bin_{ok} > 0$, all buyers pay prices above pc_{Bk}.

Similarly j^{th} seller receives

$$srua_{ji} = spc_{Bk} - \left(\frac{aqo_{jk}}{q_k}\right) ain_{ok}, \text{ since } \left(\frac{aqo_{jk}}{q_i}\right) ain_{ok} > 0.$$

All sellers receive price, which is smaller than spc_{Bk}. In minimum price order book $bpc_{Bk} \geq spc_{BK}$. So budget balanced property is satisfied for minimum price order book.

Theorem 10 : DGMDA mechanism satisfies incentive compatibility property. Further if any buyer (seller) decreases (increases) the quantity, he pays (receives) more (less) price respectively per unit.

Proof: It can be easily verified the amount paid/received by any buyer or seller is independent of his bid/ask price. It depends upon the bid prices and ask prices submitted by others. Further as in GMDA a buyer or seller cannot improve his gain by bidding higher or lower. The utility of j^{th} buyer in i^{th} order book for a single unit of object is

$bu_{ji} = bv_{ji} - bpua_{ji}$ for $i = 1, .2, \ldots, k$, $j = 1, 2, \ldots, m_i$

If buyer's valuation, $bv_{ji} \geq pc_{Bi}$, then buyer's utility does not increase by bidding higher amount, as the amount to be paid does not change. He may loose auction by bidding lower amount. On the other hand if buyer's valuation, $bv_{ji} < pc_{Bi}$ and if buyer bids higher amount, buyer may win but will have $bu_{ji} < 0$. In case he bids lower amount, he has 0 utility. So the mechanism is incentive compatible.

Suppose that j^{th} buyer in i^{th} order book requires quantity bq_i and instead of bq_i buyer submits his demand as $bnq_j < bq_i$.

Let q_i be the total quantity sold in i^{th} order book, if j^{th} buyer submits his true demand and qn_i be the demand when buyer submits bnq_j. Then we have the following relations.

$q_i = bqo_{ji} + bq_{ji.}$

$qn_i = bqo_{ji} + bnq_j$,   since $bnq_j < bq_{ji}$, so $q_i > qn_i$,

$\frac{1}{q_i} < \frac{1}{qn_i}$, so we get

$$\left(\frac{bqo_{ji}}{q_i}\right) < \left(\frac{bqo_{ji}}{qn_i}\right)$$

This proves the result, that when buyer reduces the quantity required he pays more amount per unit of quantity. This happens even if there is no change in total quantity traded.

Conflict and false name proof bids: It can be further seen that there is a conflict between the different buyers or sellers in some cases in the sense that, if any buyer or seller decreases his volume and the volumes of others and overall volume remain unchanged, then others gain at the cost of this buyer/seller. This can be easily verified.

Let d = $bq_j - bqn_{j.}$ So for any other buyer say k, the factor $\left(\frac{bqo_{ki}}{q_i}\right)$, changes to $\left(\frac{bqo_{ji} - d}{q_i - d}\right)$. It may be noted that $\left(\frac{bqo_{ji} - d}{q_i - d}\right) \leq \left(\frac{bqo_{ki}}{q_i}\right)$.

This can happen in case there is decrease in total quantity traded. However in some cases amother bid may get selected because of volume reduction. In this case there is no change in total quantity traded. Here the utility of others remain unaffected but the buyer's utility decreases. This happens because the quantity purchased by others increases and the utility of this buyer/seller decreases. It can be seen that the amount payable by others does not change, as there is no change in numerator and denominator of $\left(\frac{bqo_{ki}}{q_i}\right)$ for others (j≠k). So utilities of others remain unaffected.

So all other participants may gain or their utilities remain unaffected. So buyer pays more by decreasing his demand and at the same time in some cases other buyers gain because they pay lesser price. In other cases their utilities remain unchanged. Due to this conflict no buyer or seller has incentive to reduce his volume.

In electronic auctions bids are submitted remotely. It is possible that buyers may submit false bids. Such false bids are bids submitted under different identification just to improve utility. Such bids are submitted under fictitious names [YSM2000]. It can be seen that DGMDA buyer does not have incentive to submit false name bids. Suppose that k^{th} buyer submits false name bids. On account of these bids, there is increase in demand say by d units.

In case these bids are not selected then there is no effect on clearing price. Suppose that these bids get selected, then k^{th} buyer pays more price. This is due to the fact that the factor $\left(\frac{bqo_{k_i}}{q_i}\right)$, changes to $\left(\frac{bqo_{j_i}+d}{q_i+d}\right)$. It may be noted that $\left(\frac{bqo_{j_i}+d}{q_i+d}\right) \geq \left(\frac{bqo_{k_i}}{q_i}\right)$.

The buyer cannot improve his utility even if there is no change in total quantity traded. This happens because buyer can improve his utility by increasing his demand. His demand does not change by false name bid. In fact by submitting false name bids a buyer may increase the per unit price to be paid by others in some cases. In the same way buyer cannot improve his utility by submitting false name bids as sellers, if all others bid truthfully. This is due to the fact that buyer cannot pushdown price crossover point by submitting a false name bid as seller. In case he submits the ask above crossover point, his ask will not be selected. If he submits an ask with ask price below cross over point, his ask will be selected. However he pays same price, as price crossover does not change. There are other possibilities like the quantity allocated to j^{th} buyer decreases, whereas there is no change in other allocation (and total quantity traded in i^{th} order book). Suppose that false name bid is selected and quantity q_f is allocated to it. However total quantity in i^{th} order book q_i remains unchanged. In other words false name bid replaces a bid selected earlier either fully or partly. In any case this does not affect total volume of other buyers but can decrease the quantity allocated to j^{th} buyer. Due to this buyer cannot improve his utility. So the mechanism is false name proof. Exactly similar argument can be followed to show that seller does not have incentive to increase his volumes. It can be show that the mechanism is incentive compatible for sellers and seller receives lesser price per unit in case he reports higher quantity. So DGMA is false name proof.

Theorem 11: In DGMDA, efficiency loss is always bounded and inefficiency ratio tends to 0.

Proof: In this proof, we use the same concepts from theorem 7, since the main difference between DGMDA and GMDA is only the market clearing price. In i^{th} (i = 1, 2, …, k-1) order book, the market clearing price paid by j^{th} buyer is

$$pc_{Bi} + \left(\frac{bq_{-ji}}{q_i}\right) bin_{oi} \quad , \quad \text{with } bin_{oi} = \frac{1}{2}(bp_{(mi)} - pc_{Bi}).$$

As rhere is no change in the sacrificed volume expected efficiency loss does not change. In DGMDA the new expected total market value can be worked out as follows.

$$E(mvd(m_b,n_s)) \geq \sum_{i=1}^{k} E(mvdb_i) + \sum_{i=1}^{k} E(mvds_i) \quad ,$$

In above expression term ‘d’ is added to indicate that it is market value for DGMDA.

$E(mvd(m_b,n_s)) \geq A + B$, where

$$A = \sum_{i=1}^{k} E(mvdb_i) \quad \text{and } B = \sum_{i=1}^{k} E(mvds_i)$$

We can write $A = \sum_{i=1}^{k} E(mvb_i) + \sum_{i=1}^{k} \frac{1}{2} E(\left(\frac{bqo_{j_i}}{q_i}\right) bp_{i(mi)} - pc_{Bi}))$.

It may be noted that $\left(\frac{bqo_{j_i}}{q_i}\right) \leq 1$. So we can write

$$E(bp_{i(mi)} - pc_{Bi}) \geq E(\left(\frac{bqo_{j_i}}{q_i}\right)(bp_{i(mi)} - pc_{Bi})).$$

The lower bound on the LHS can be obtained as in theorem 7. It may be noted tha

$$E(bp_{(mi)}) \geq \mu_{bi} - \sigma_{bi}\left(\frac{m_i - j}{j}\right)^{\frac{1}{2}} \quad \text{and}$$

$$E(bp_{i(mi+1)}) \geq \mu_{bi} - \sigma_{bi}\left(\frac{m_{bi} - m_{i} - 1}{mi + 1}\right)^{\frac{1}{2}} \quad \text{and}$$

$$E(sv_{ini+1}) \geq \mu_{si} - \sigma_{si}\left(\frac{n_{si} - ni - 1}{ni + 1}\right)^{\frac{1}{2}}$$

$$E(bp_{(mi)} - pc_{Bi}) \geq \frac{1}{2}(\mu_{bi} + \sigma_{bi}\left(\frac{m_{bi} - m_{i} - 1}{mi + 1}\right)^{\frac{1}{2}} - \sigma_{bi}\left(\frac{m_i - j}{j}\right)^{\frac{1}{2}} - \mu_{si} + \sigma_{si}\left(\frac{n_{si} - ni - 1}{ni + 1}\right)^{\frac{1}{2}}) .$$

In case of k^{th} order book the bound can be worked out as

$$E(bp_{k(mk)} - bpc_{Bk}) \geq \sigma_{bk} \left(\frac{m_{bk} - m_{k} - 1}{m_{k} + 1}\right)^{\frac{1}{2}}$$

Then lower bound on A can be obtained as

$$A \geq \sum_{i=1}^{k} E(mvb_i) + \frac{1}{2}(m_k\mu_{bmin} - n_s\mu_{smin} + \sigma_{bmin} \sum_{i=1}^{k} \left(\frac{m_{bi} - m_{i} - 1}{m_{i} + 1}\right)^{\frac{1}{2}} + \sigma_{smin}\left(\frac{n_{si} - ni - 1}{ni + 1}\right)^{\frac{1}{2}} -$$

$$\sigma_{bmin} \sum_{j=1}^{mi} \left(\frac{m_i - j}{j}\right)^{\frac{1}{2}}))$$

$$A \geq \sum_{i=1}^{k} E(mvb_i) + m_b\mu_{bmin} - n_s\mu_{smin} + MVbd$$

$$MVbd = \sigma_{bmin}\sum_{i=1}^{k}\left(\frac{m_{bi} - m_{i} - 1}{m_{i} + 1}\right)^{\frac{1}{2}} + \sigma_{smin}\left(\frac{n_{si} - ni - 1}{ni + 1}\right)^{\frac{1}{2}} - \sigma_{bmin} \sum_{j=1}^{mi} \left(\frac{m_i - j}{j}\right)^{\frac{1}{2}} - m_k$$

In the same way the bound on term B can be worked out.

We can write $B = \sum_{i=1}^{k} E(mvs_i) + \sum_{i=1}^{k} \frac{1}{2} E(\left(\frac{bq_{-ji}}{q_i}\right)(pc_{Bk} - ap_{k(nk)}))$.

Then working out the bound as earlier we get

$B \geq \sum_{i=1}^{k} E(mvs_i) + m_b\mu_{smin} + MVsd$, where

$$MVsd = \frac{1}{2}\ \sigma_{bmin}\sum_{i=1}^{k} \left(\frac{m_{bi} - m_{i} - 1}{m_{i} + 1}\right)^{\frac{1}{2}} + \sum_{i=1}^{k} n_i \mu s_{si} + \frac{1}{2} \mu s_{sk+1})$$

Then combining above two expressions we can write the bound on expected market value in DGMDA.

$E(mvd(m_b,n_s)) \geq E(mv(m_b,n_s)) + m_k\mu_{bmin} - n_s\mu_{smin} + MVbd + m_b\mu_{smin} + MVsd$.

The terms MVbd and MVsd are constant terms indepndent of n_s and m_b. As already stated the nummerator term in expression for inefficency raio remaisn same as in expression (4.29). The expression for tem E(mvd(mb,ns)) can be substituted in expression (4.54) to (4.58). Then following exactly similar argument as theorem 7, the result is proved. So the result follows.

4.5 Two Price Single Unit Double Auction Mechanism

In this section we discuss two price SDA mechanism. This discussion can be helpful in certain cases to design budget balanced and efficient mechanism. In case of single unit double auction (SDA), two price mechanism has been analyzed in [W1999]. This mechanism is strategy proof. However it is not budget balanced in general. This basic mechanism can be easily generalized to multiple two price mechanism to handle different types of assignment and other constraints.

SDA Mechanism: The basic SDA mechanism analyzed in [W1999] is as follows:

Let there be n buyers and m sellers. Each seller has an indivisible item to sell and each buyer requires an indivisible item. Let bp_i be the valuation of i^{th} buyer (prices submitted by the buyer) and ap_j be the valuation of j^{th} seller (prices submitted by j^{th} seller). The mechanism is as follows.

(1) These prices of buyers and sellers are combined and arranged in an increasing order sequence $p_{(1)} \leq p_{(2)} \leq p_{(3)} \leq \ldots \leq p_{(m)} \leq p_{(m+1)} \leq \ldots \leq p_{(m+n)}$.

(1) The buyers whose reported values are at least as $p_{(m+1)}$ obtain items from sellers whose reported values are below $p_{(m)}$. All others do not trade.

(2) A buyer pays $p_{(m)}$ and seller receives $p_{(m+1)}$ when they trade.

In other words the seller's price is the best unsuccessful offer as long as it is lower than the worst successful bid. Otherwise it is the worst successful bid. The buyer's price is set to best unsuccessful bid as long as it is higher than the worst successful offer. Otherwise it is set to the worst successful offer. The necessary and sufficient condition for the existence of desirable mechanism is that the subsidy required by two price mechanism must not exceed the individualized taxes paid by the participant. A mechanism is said to be desirable [W1999], if it is ex-ante budget balanced (pre bid) and interim individually rational (stage where buyer decides what should be his bid). Let ab_i be the amount paid by the i^{th} buyer and as_j be the amount received by the j^{th} seller. The subsidy required by two price mechanism is

$E(T(p_{(m+1)}-p_{(m)}))$, where T is expected quantity which is traded. So the condition can be written as

$$E(T(p_{(m+1)}-p_{(m)})) \leq \sum_{i=1}^{m} ab_i + \sum_{j=1}^{n} as_j \qquad (4.72).$$

The amount paid by the buyers and sellers are bounded by their respective interim utilities. The valuations of buyers are assumed to be drawn from independent and identically distributed random variables with distribution function $F(b_p)$ and continuous density function $f(b_p)$ over range $[bv_l,\ bv_u]$. In the same way, seller's values are drawn from independently and identically distributed random variables with distribution function $G(a_p)$ and continuous density function $g(a_p)$ over range $[sc_l, sc_u]$. Let U_b be the interim expected utility of the buyer at bv_l. Let U_s be the interim expected utility of the seller at sc_l. These quantities are minimized as bv_l and sc_u respectively. These utilities bound the amount paid. So equation (4.73) can be written as

$E(T(p_{(m+1)}-p_{(m)})) \leq \sum_{i=1}^{m} u_b + \sum_{j=1}^{n} u_s$ and it reduces to

$$E(T(p_{(m+1)}-p_{(m)})) \leq mu_b + nu_s \qquad (4.73).$$

Conditions for existence of desirable mechanism: The equation (4.73) provides the necessary and sufficient conditions for the existence of desirable mechanism. Using this inequality the conditions under which a desirable mechanism exists, have also been established in [W1999]. The desirable mechanism in SDA mechanism described above exists only in the following cases

(1) If $bv_l \leq sc_l$, $bv_u \geq sv_u$ and $m >> n$ (i.e. m is sufficiently larger than n).

(2) If $bv_l \geq sc_l$, $bv_u \leq sv_u$ and $n >> m$ (i.e. n is sufficiently larger than m).

(3) If $bv_l \geq sc_l$, $bv_u \geq sv_u$ in this case the mechanism exists if $n >>m$ or $m >> n$.

In no other case the desirable mechanism exists. In case the number of buyers are sufficiently larger than sellers or vice versa a desirable mechanism may exists. This result can be utilized in some cases to design a mechanism, which is strategy proof, individually rational and efficient. Such a mechanism has been proposed in the next section, in case of multi unit double auctions with different types of constraints. In certain cases such mechanism can be ex-post budget balanced.

4.6 Multi Unit Extension

Let us consider the multi unit scenario (MDA) and two price mechanism. Let there be n buyers and m sellers. Each seller has aq_j items to sell and each buyer requires bq_i items. Let bp_i be the valuation of i^{th} buyer and ap_j be the valuation of j^{th} seller. All other things including the assumption about valuations of buyers and costs of sellers are same as SDA. It is assumed that valuations of buyers and costs of sellers are variable as earlier. The demand and supply are not. The mechanism is as follows.

(1) Each bid with quantity bq_i is considered as set of bq_i bids with single item. In other words there are ties for different bid prices. There are bq_i bids with bid price

bp_i and so on. Arrange these new bid prices from the highest to the lowest i.e. $bp_{(1)} \geq bp_{(2)} \geq \ldots \geq bp_{(mq)}$. The total demand is denoted by mq.

(2) In the same way each ask is considered as set of aq_j single item asks. In other words there are ties for different ask prices. Arrange these new ask prices from the lowest to the highest i.e $ap_{(1)} \leq ap_{(2)} \leq \ldots \leq ap_{(nq)}$. The total supply is denoted by nq.

(3) Then arrange these mq + nq prices in increasing order and let them be denoted by p_i. So we have series $p_{(1)} \leq p_{(2)} \leq \ldots \leq p_{(mq+1)} \leq \ldots \leq p_{(mq+nq)}$

(3) Buyers whose valuation is at least as large as $p_{(mq+1)}$ obtain items from sellers whose cost is below $p_{(mq)}$.

(4) Then determine the total supply available at price $p_{(mq)}$ and total demand at price $p_{(mq+1)}$. Let db be the total demand and sp be the supply available. Then the quantity traded qt is min(db,sp). Each buyer with bid price above $p_{(mq+1)}$ obtains the quantity required from sellers with ask prices below $p_{(mq)}$, in such a way that the quantity qt is traded.

(5) Each buyer pays $p_{(mq)}$ per unit of quantity and each seller gets $p_{(mq+1)}$ per unit of quantity.

It can be seen that this mechanism is a direct generalization of two price mechanism for SDA. The results from equation (4.73) and (4.74) can directly be applied to this mechanism replacing m by mq and n by nq. Applying these results the necessary and sufficient condition for existence of the desirable mechanism is

$$E(T(p_{(mq+1)}-p_{(mq)})) \leq \sum_{i=1}^{mq} u_b + \sum_{j=1}^{nq} u_s \quad \text{and it reduces to}$$

$$E(T(p_{(m+1)}-p_{(m)})) \leq mqu_b + nqu_s \tag{4.74}$$

where T is the expected quantity that is traded. As earlier u_b and u_s are interim expected utilities of buyers and sellers at bv_l and sc_l respectively. These utilities bound the amounts to be paid by buyers and sellers.

Conditions for existence of desirable mechanism for MDA: Using this equation and by applying earlier basic result about existence of desirable mechanism, the desirable mechanism exists in two price multi unit mechanism for MDA, if following conditions are satisfied.

(1) If $bv_l \leq sc_l$, $bv_u \geq sv_u$ and mq >> nq (i.e. demand is sufficiently large than supply).

(2) If $bv_l \geq sc_l$, $bv_u \leq sv_u$ and nq >> mq (i.e. supply is sufficiently large than demand).

(3) If $bv_l \geq sc_l$, $bv_u \geq sv_u$ in this case the mechanism exists if nq >>mq or mq >> nq.

In no other case the desirable mechanism exists.

This mechanism can be easily extended to handle assignment constraints. A multiple two price mechanism to handle assignment constraints can be obtained by some modification to GMDA mechanism presented earlier.

4.7 Extension to GMDA

In GMDA mechanism, which handles assignment constraints, there are number of different order books. In order to extend two price mechanism to GMDA, we apply multi unit extension described in earlier section to each order book including minimum price order book. Each order book is a scenario similar to the scenario described in multi unit extension.

In this case, the following change makes our mechanism generalization of two price mechanism.

(1) In each order book determine the prices to be paid by each buyer and seller by applying multi unit extension. Let $p_{i(m+1)}$ and $p_{i(m)}$ be the per unit prices to be paid by each buyer and seller respectively in i^{th} order book.

(2) Let qt_i be the quantity traded in each order book.

(3) Let ub_i be interim expected utility of buyer in i^{th} order book at bv_l. In the same way let us_i be the interim expected utility of seller in i^{th} order book at sc_l.

Then the necessary and sufficient condition for existence of desirable mechanism can be obtained by applying equation to each order book. Applying this we get the following condition

$$\sum_{i=1}^{k} E(T_i(p_{i(m+1)} - p_{i(m)})) \leq \sum_{i=1}^{k} (mq_i u_{bi} + nq_i u_{si}) \qquad (4.75).$$

This follows directly from the fact that LHS of (4.75) represents the subsidy required in GMDA and RHS is the bound represented on the payments. If it is assumed that the range all the valuations of buyers and costs of sellers is the same then the above utilities are minimized at bv_l and sc_u respectively. Let u_b and u_s be these minimum utilities, which bound the individualized taxes. Then the equation (4.74) can be reduced to

$$\sum_{i=1}^{k} E(T_i(p_{i(m+1)} - p_{i(m)})) \leq mqU_b + nqU_s \quad (4.76)$$

where k is number of order books, mq is the total demand and nq is the total supply available (in all order books).

The above discussion can be helpful in cases where there is a large demand as compared to supply or vice versa. In such cases the above result guarantees the existence of a desirable mechanism. In order to achieve budget balance, it is proposed to introduce participation fees.

Theorem 12: The necessary and sufficient condition for double auction mechanism under assignment constraints with participation fees to be efficient is

$$\sum_{ON} M_{bp} \leq \sum_{ON} E_{br} + \sum_{ON} E_{sk}$$

where M_{bp} is the bound on subsidy required in GMDA mechanism and right hand side represents the utilities generated.

Proof : In i^{th} order book the interim expected utilities of the buyers and sellers can be obtained as follows

$U_b = \begin{cases} E(b_i - sc_u) \\ 0, \text{ otherwise} \end{cases}$ if $nq_i \geq mq_i$ and $sc_u < bv_u$, where b_i is the amount paid by the seller in i^{th} order book The terms nq_i and mq_i are the supply and demand available for i^{th} order book.

In the same way the interim expected utility of the seller is

$U_s = \begin{cases} E_T(bv_l - s_i) \\ 0, \text{ otherwise} \end{cases}$ if $mq_i \geq nq_i$ and $sc_l < bv_l$, where s_i is the amount paid by the buyer in i^{th} order book.

Where $E_T(bv_l - s_i) = \sum_{ON} E(bvi - S_j)$, where $E(bv_i\text{-}s_j)$, is the interim expected utility of the j^{th} seller in i^{th} order book.

It can also be noted that the buyers and sellers are in the auction if their valuations or costs are above or below certain values (i.e. they are some order statistics). The buyer's valuations must be as large as $pi_{(mq+1)}$ or above it and seller's cost must be below $p_{(mq)}$ or the same. Let these values be br^{th} and sk^{th} order statistics respectively. It can be seen that $pi_{(mq)}$ and $pi_{(mq+1)}$ may or may not be identically distributed.

So U_b and U_s , the minimum can be obtained by using the results from order statistics and obtaining expected values of br^{th} and sk^{th} order statistics respectively.

So we write $U_b = E_{br}$ and

$U_s = E_{sk}$

The LHS of equation (4.74) can be written as

$E(T(pi_{(mq+1)}-pi_{(mq)})$, which is bounded by

$[2D(\mu_1^2 + \mu_2^2 - 2((\mu_1 + \mu_2)\mu_{avg} + \mu_{avg}^2) + ½(\sigma_1^2 + \sigma_2^2))]^{1/2} = M_{bp}$

Where μ_1 and μ_2 are expectations and σ_1^2 , σ_2^2 are variances respectively. So we get the following results.

If $mq_i > nq_i$ then the condition will be

$$M_{bp} \leq E_{br} \tag{4.77}$$

In other case it is

$$M_{bp} \leq E_{sk} \tag{4.78}$$

These conditions are satisfied in respect of each order book. So the result follows. If the expression in (4.78 or 4.79) is satisfied, then the positive participation fees can be charged to the buyers and sellers. The conditions for all the order books can also be similarly obtained. So the condition when participation fees can be charged is

$$\sum_{ON} M_{bp} \leq \sum_{ON} E_{br} + \sum_{ON} E_{sk} \tag{4.79}$$

It can be expected that in such cases buyers and sellers do not leave the market even though participation fee is charged. The conditions in (4.78 and 4.79) can be tested if the distribution functions, averages, and variances are known. The above result can help in identifying order books, where positive participation fees can be charged. In case there is large difference between demand and supply of some order books, participation fees can be charged in that order book, since above condition ensures that buyer or seller do not leave the market. So the market maker can charge participation fees to overcome budget deficit from other order books.

Theorem 13: The expected efficiency loss is always bounded, in case probabilities of buyers and sellers leaving the market are small. Further if number of buyers and sellers are large or either number of buyers or number of sellers is large (or $n_s \to \infty$ or $m_b \to \infty$) efficiency loss tend to 0.

Proof : The efficiency loss occurs, if any one of the buyers or sellers do not decide to participate in case participation fee is charged.

Let us define the indicator variables I_{bi} and I_{sj} as follows

$I_{bi} = 1$ - If i^{th} buyer participates in the auction for i =1, 2, …, n

= 0 - Otherwise

$I_{sj} = 1$ - If j^{th} seller participates in the auction for i =1, 2, ..., m

= 0 - Otherwise

It is assumed that participation fees, is charged in a fair manner without any bias. In such cases it is reasonable to assume that the probability that any buyer leaves the market is same and is *pb*. Similarly probability that any seller leaves the market is *ps*. This is true in respect of all order books.

The efficiency loss occurs when one or more buyers or sellers withdraw. It is assumed that the buyers and sellers decide to withdraw, based on their own private valuations. The decision of any buyer or seller to withdraw is independent of one another. So the expected efficiency loss can be written as

$E_l(m_b,n_s) = E(I_{bi}) + E(I_{sj})$

$= \sum_{r=0}^{ns} r(\,prbr) + \sum_{k=0}^{mb} k\,(prsk\,)$

Where prbr = Probability that exactly r buyers participate and remaining withdraw

prsk = Probability that exactly k sellers participate remaining withdraw

When all probabilities are equal, probability that r buyers withdraw and remaining participate is

$= n_sCr\ pb^r(1\text{-}pb)^{ns\text{-}r}$

Similarly probability that k sellers withdraw and remaining participate is

$= m_bC_k\ ps^k(1\text{-}ps)^{mb\text{-}k}$

Let $E_l(m_b,n_s)$ be the expected efficiency loss. Then it can be obtained as

$E_l(m_b,n_s) = \sum_{r=0}^{ns} r(\,prbr) + \sum_{k=0}^{mb} k\,(prsk\,)$

$= \sum_{r=0}^{ns} rn_sCr\ pb^r(1\text{-}pb)^{ns\text{-}r} + \sum_{k=0}^{mb} k\ m_bC_k\ ps^k(1\text{-}ps)^{mb\text{-}k}$

$= n_spb + m_bps$

This term converges for large values of n_s or m_b, in case pb and ps are small (Poisson Distribution).

However in general probabilities may not be equal. This is due to the fact that a buyer with higher valuation can make positive gain by participating in auction. A buyer with lower valuation may have smaller gain. In such cases probabilities may not be equal.

Let pr_{bi} be the probability that i^{th} buyer withdraws from auction. Let pr_{sj} be the probability that j^{th} seller withdraws from the auction.

So probability that i^{th} buyer participates in the auction = $(1 - pr_{bi})$

In the same way probability that j^{th} seller participates in the auction = $(1 - pr_{sj})$

The efficiency loss occurs when one or more buyers or sellers withdraw. It is assumed that the buyers and sellers decide to withdraw, based on their own private valuations. The decision of any buyer or seller to withdraw is independent of one another. So the expected efficiency loss can be written as

$$E_l(m_b,n_s) = E(I_{bi}) + E(I_{sj})$$

$$= \sum_{r=0}^{ns} r(\,prbr) + \sum_{k=0}^{mb} k\,(prsk\,)$$

Where prbr = Probability that exactly r buyers participate and remaining withdraw

prsk = Probability that exactly k sellers participate remaining withdraw

The probability that r buyers participate and remaining withdraw is

$= n_sCr \prod_{i=0}^{r} pr_{bi} \prod_{i=0}^{n-r} (1\text{-}pr_{bi})$. In the same way we can obtain the probability that k sellers participate and remaining withdraw.

In order to obtain bound we define

pr_{bmax} = Maximum(pr_{b1}, pr_{b2}, …, pr_{bn}), pr_{smax} = Maximum(pr_{s1}, pr_{s2}, …, pr_{sm})

pr_{bmin} = Minimum (pr_{b1}, pr_{b2}, …, pr_{bn}), pr_{smin} = Minimum(pr_{s1}, pr_{s2}, …, pr_{sm})

$$E_l(m_b,n_s) \le \sum_{r=0}^{ns} r\,(\,(\,nCr)\,pr_{bmax}^{\,r}\,(1\text{-}pr_{bmin})^{nb-r}\,) + \sum_{k=0}^{mb} k\,(\,(\,m_sCk)pr_{smax}^{\,k}\,(1\text{-}pr_{smin})^{ms-k})$$

Let us define $PB_r = pr_{bmax}$ and $QB_r = (1\text{-}pr_{bmin})$ and $PS_k = pr_{smax}$ and $QS_k = (1\text{-}pr_{smin})$. Then the above equation can be written as

$$E_l(m_b,n_s) \le \sum_{r=0}^{nb} r\,(\,(\,n_sCr)\,PB_r^{\,r}\,QB_r^{\,ns-r}) + \sum_{k=0}^{mb} k(\,(\,mbCk)\,PS_k^{\,k}\,QS_k^{\,mb-k})\,.$$

Using binomial theorem the term gets simplified as

$$E_l(m_b,n_s) \le n_sPB_r(PB_r+QB_r)^{ns}+m_bPS_k(PS_k+QS_k)^{mb.} \qquad (4.80).$$

Then we get the following bound

$E_l(m_b,n_s) \leq n_sPB_r (PB_r+QB_r)^{ns} + m_bPS_k (PS_k+QS_k)^{mb} \leq n_sPB_r(2)^{ns} + m_bPS_k (2)^{mb}$

The term in bracket in above equation is always bounded by 2 (since they are probabilities). It can be seen that the terms in bracket are close to 1, this term is always bounded In such cases, if probability that any buyer or seller withdraws is small, such that terms in bracket are close to 1, it can be ensured that above expected loss is always bounded. It can be seen that

$PB_r+QB_r = 1 + (pr_{bmax} - pr_{bmin})$

So if difference between these two probabilities is small i.e. $(pr_{bmax} - pr_{bmin}) \leq \varepsilon$, where ε is a very small positive number, then above term is close to 1 and expression (4.80) converges. In other cases i.e. when the difference is high or probability that some buyers leave the market is high, the term though bounded, has high value.

Suppose that participation fees is worked out in such a way that all buyers and sellers have positive gain after participating in auction, then probability of any buyer or seller leaving the auction is very small then, expected efficiency is always bounded. In other words, while charging participation fees, auctioneer needs to ensure that probability of any buyer or seller leaving auction is very small. It can be seen that buyers or sellers with higher valuation have higher gain from participation and hence are not likely to leave in case participation fees is charged. This is due to the fact that these buyers gain by participation. If they leave market they have no gain. It is always likely that buyers with lower valuation are more likely to leave. So if all buyers have positive gain even after charging participation fees then probability that any buyer will leave the market will be smaller. The minimum participation fee required to make the mechanism budget balanced for i^{th} order book is

$pf_i=(p_{i(m)}- p_{i(m+1)}) / (ns_i + mb_i)$ (4.81).

It is easy to verify that expression in (4.79) is the minimum participation fees required so that mechanism is budget balanced.

In order to participate in auction, the valuation of buyer must be above $p_{i(m+1)}$ and seller's cost should be below $p_{i(m).}$ In case participation fees is charged as in (4.79), each buyer pays $p_{i(mq)} +pf_i$ (4.82).

and each seller receives

$p_{i(mq)}- pf_i$ (4.83).

It can be seen that term in (4.81) is lower than any participating buyer's valuation, in case $(p_{i(mq)} - p_{i(mq+1)}) \leq (ns_i + mb_i)$. (4.84).

In the same way the term in (4.82) is higher than any participating seller's cost, in case $(p_{i(mq)}-p_{i(mq+1)}) \leq (ns_i + mb_i)$. (4.85).

If these two conditions are not satisfied, it cannot be guaranteed that all buyer's have positive gain in case participation fee as given in (4.79), is charged.

In case (4.82) and (4.83) are satisfied, after charging participation fees, all buyers and sellers have positive gains and are better off than not participating. In case of non participation they have 0 utility. So the participation fees as in (4.79) ensure that probability of any buyer or seller leaving will be small. In case number of participating buyers or sellers or both are large (i.e. either n_i and m_i or both are very high)

$pf_i \rightarrow 0$ in case or $ns_i \rightarrow \infty$ or $m_{bi} \rightarrow \infty$

In case number of buyers or sellers is large, the participation fee for each participant is negligible. In this case gains are almost same as those in case no participation fee is charged. In this case buyer or seller participating in original mechanism will also participate in case of participation fee is charge. This is due to the fact that buyers pay almost same price as in original mechanism and seller receives almost same price as in original mechanism. So if they participate in original mechanism, they will also participate in the new scheme and the scheme will be efficient and budget balanced can be achieved.

In case the auctioneer knows that buyer and seller are not likely to withdraw, he can decide about suitable participation fees. It can be seen from above discussion that if demand or supply varies in different order books, two price mechanism with participation fees can be used. If probability that buyers and sellers leave auction is high, then auction is not likely to be efficient. This can be ensured in case number of buyers or sellers or both are large in at least one order book.

Efficiency loss was worked out in respect of 10 orderbooks with an average of 30, buyers and sellers in each order book and 300 buyers and sellers together overall. There were 30 buyers and sellers together in an order book on an average, however figured differed in respect of different order books. The data sets (consisting of bid prices, ask prices and values for different attibutes) were drawn randomly from different distributions. The efficiency loss as ineffiecency ratio (defined in expression 4.29) was worked out in two cases. In first case it was assumed that distributions were identical with different parameters in different order books. In second case it was assumed that data sets are from different distributions. The effect on ineffieciency ratio as number of buyers and sellers increase can be seen in figure 4.1 and 4.2. At the same time percentage gain for a buyer (as his demand decreases) was worked out from 100 randomly generated data sets. The effect on percentage gain as demand decreases can be seen in figure 4.3. At the same time effect of false name bids on his percentage gain can be seen in figure 4.4.

4.8 Conclusion

In this chapter we propose the design of strategy proof mechanism GMDA for multi unit double auctions with different types of assignment constraints. It is shown that our mechanism is strategy proof, individual rational and budget balanced. These properties ensure that truthful bidding is dominant strategy, which is a very important property in electronic auctions. It has also been shown that efficiency loss is always bounded and mechanism is asymptotically efficient. We have also generalized two price SDA mechanism in this scenario. It is helpful in cases, where either demand is far excess of supply or vice-versa. In such scenario the generalized two price mechanism with participation fees can be used.

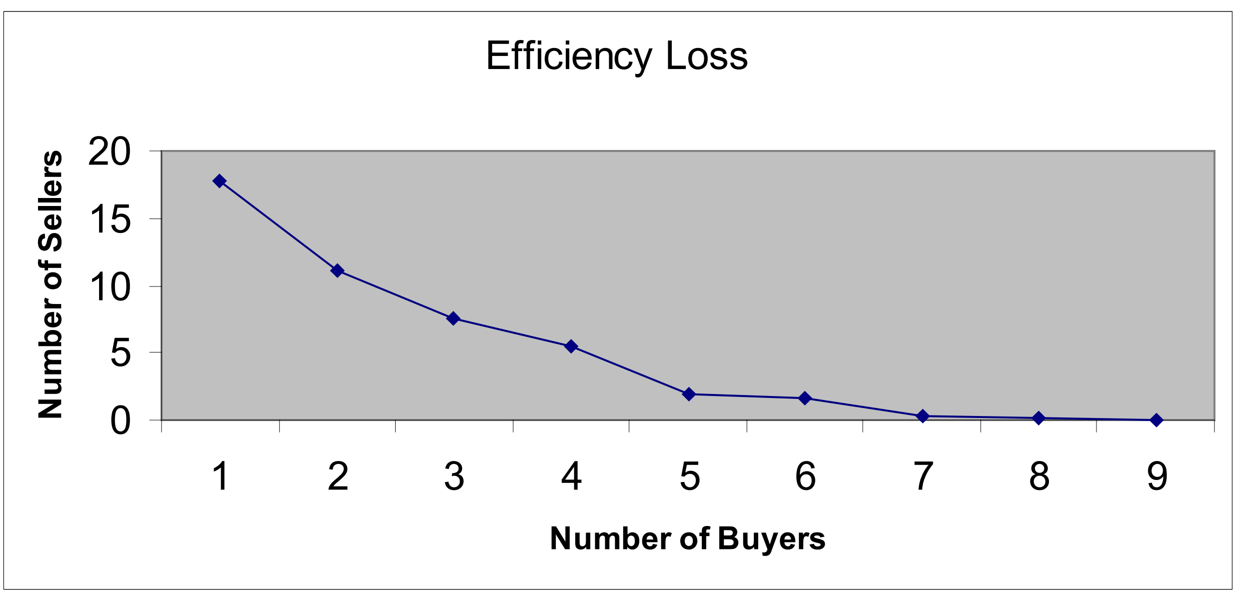


Figure. 4.1 Efficiency loss when bids in different order books have same distribution but different parameters

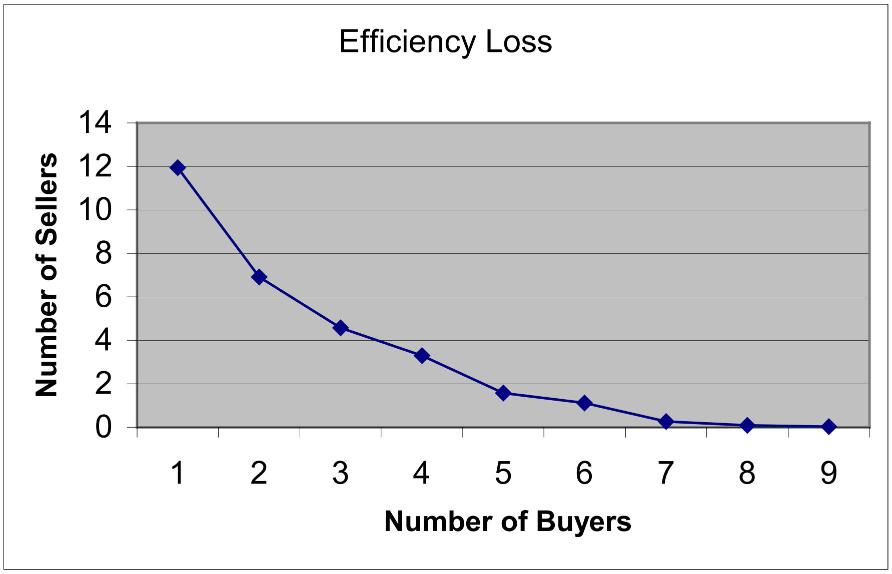


Figure. 4.2 Efficiency loss when bids in different order books have different distributions.

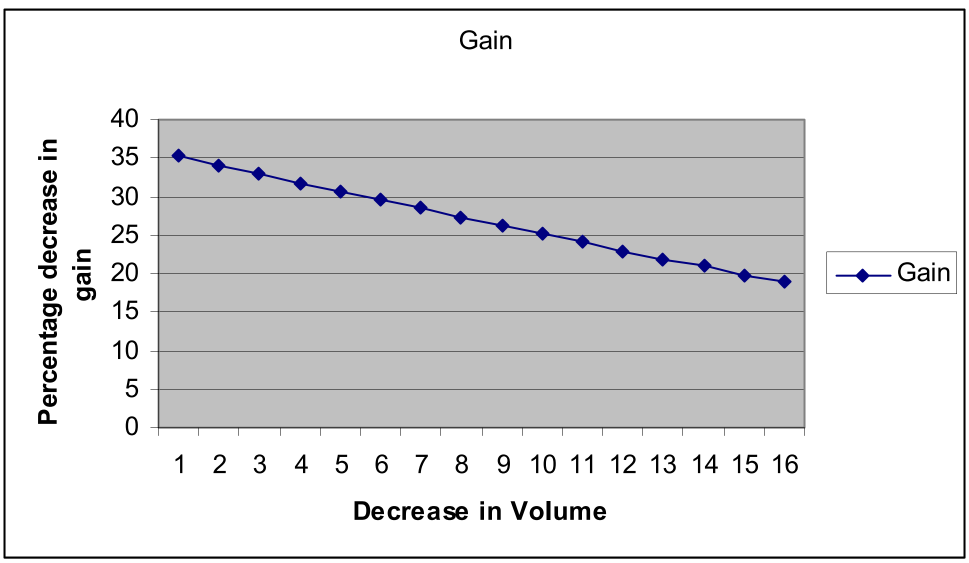


Figure. 4.3 Effect of decrease in volume on gain. It indicates how gain of buyer decreases in case he decreases his volume

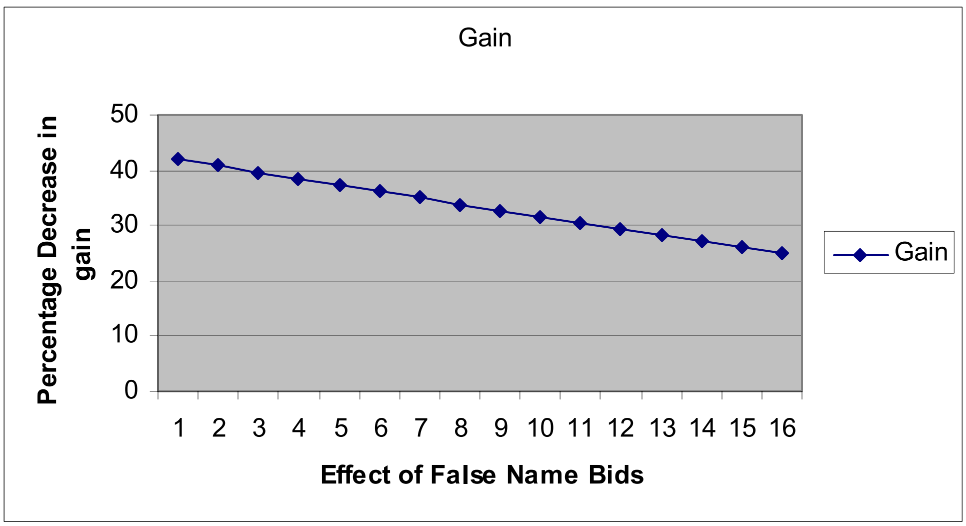


Figure. 4.4 Effect of false name – How the gain of buyer decrease by submitting false name bids

Chapter 5

Multi Unit Auctions with Constraints

5.1 Introduction

In this chapter, we analyze the problem of the sale of a government owned enterprise under different types of constraints. We present a general formulation of this problem as an optimization problem. This formulation can handle multi unit single object auctions with different types of constraints. We show that some multi unit single object auction formulations studied previously are particular cases of our formulation. Then we develop a few results and an algorithm to obtain optimum solution based on these results. It has been shown by induction that our algorithm generates an optimum solution if it exists. It has also been shown that our algorithm can solve the problem with polynomial time complexity. The Integer Programming (IP) formulations, which are typically used to obtain optimum assignment in case of multi unit single object auctions, are hard to solve when the number of buyers or sellers is large. On the other hand, our algorithm always obtains optimum solution with polynomial time complexity. The main contribution of this work is the general formulation of the problem along with an algorithm to obtain optimum solution. Then we develop strategy proof mechanisms.

Auctions have been widely studied in economic theory. A detailed survey of auction mechanisms can be found in [W1996]. An important application of auctions in the last few years has been selling of government owned enterprises in many countries across the world. As a result of economic liberalization the government is withdrawing itself from many spheres of activity. This means that governments of different countries are reducing their shares in state owned enterprises. In such cases government has to ensure that the state owned enterprises are passed over to appropriate persons/organizations. Towards this end, it needs to design an appropriate format. In some cases, in the interest of society, the government has to ensure that a single person or an organization does not monopolize state owned assets. It has to ensure that assets are distributed appropriately. This can give rise to auctions with constraints of different types. A typical constraint can be that any individual cannot acquire ownership rights beyond a certain percentage. A group of enterprises can acquire only certain percentage of the ownership. There can be similar restrictions on foreign investors. It may also be required that employees get a certain percentage of ownership. A common example of this type of situation is Initial Public Offerings (IPO) of a state owned enterprise. These constraints are different from the usual capacity constraints. Even though auctions have been widely studied, auctions under different types of constraints, in addition to usual capacity constraints, have not been studied in details.

Suppose that a government owns an enterprise fully or partially. When the State decides to privatize it either fully or partially, normally the process of competitive bidding is adopted. Different individuals and enterprises may bid for the purchase of equity. Even though individuals or private enterprises may indicate their capacity or willingness to buy certain number of shares, it may not be feasible as per policies of government to allocate them all the shares they bid. In the interest of the society, the government is interested in ensuring that the state owned assets are distributed appropriately and fairly and ensure that no individual, group of individuals or an enterprise monopolizes the public owned assets. These requirements give rise to different constraints, apart from usual capacity constraints. To illustrate this scenario, consider an example, where the government wishes to decrease its ownership in an enterprise from 100% to say 49% and decides to sell 1 million shares. In this case there can be several types of constraints. Suppose that the bids submitted by different buyers and enterprises are as given in Table 5.1. In this table, the third column indicates the types of buyers. It indicates that the second buyer is an employee, while the first one is a general individual buyer. The remaining five buyers belong to the enterprise category. The buyers of enterprise category have been divided into different groups. The third buyer belongs to group 1, fourth buyer belongs to group 2, fifth and sixth belong to group 3, while the last belongs to group 4. The usual capacity constraints indicating the minimum and maximum demand of individual buyers are shown in columns 4 and 5. The minimum and maximum demands of buyers 1 and 2 are 1000 and 2000 respectively.

The constraint indicating the maximum permissible limit on the individuals in the corresponding group is shown in the last column. It indicates that the upper limit for any individual buyer is 500 shares. On the other hand for any employee of the enterprise it is 1000 shares. In the same way, any enterprise in groups 1 to 3 can have maximum 1 Lakh shares. There is an upper limit of 10000 shares each on the enterprises belonging to group 4. These constraints can limit the minimum or maximum number of shares that can be allotted to any buyer or an enterprise. In case of buyer 1, even though he has indicated his willingness to purchase 1000 to 2000 shares, the maximum shares that can be allotted, to him are 500. The scenario in case of buyers 3 to 6 is similar. These constraints restrict the number of shares that can be sold to the buyers, irrespective of the demands specified by them.

In addition to these constraints, there can be constraints on categories. For instance the total numbers of shares to be allotted to all the individual buyers should not exceed 25% of the total 1 million shares. Other restrictions can be that the total foreign equity should not be more than say 20%. These constraints are called as group constraints, since they apply to group of buyers.

Table. 5.1: Buyer's Bids for IPO of a company

Buyer	Type	Category	Minimum Demand	Maximum Demand	Price	Upper Limit on Shares
1	Individual	Other	1000	2000	105	500
2	Individual	Employee	1000	2000	100	1000
3	Enterprise	Group 1	200000	500000	105	100000
4	Enterprise	Group 2	500000	800000	107	100000
5	Enterprise	Group 3	500000	1000000	110	100000
6	Enterprise	Group 3	300000	1000000	110	100000
7	Enterprise	Group 4	10000	100000	102	10000

An IPO with competitive bidding can be considered as an example of forward auction. In forward auction, there is a single seller having quantity q of an item to sell and there are n buyers. The objective is to maximize the selling cost or revenue. Here the government is the seller. There can be a similar problem in the reverse direction in the financial domain. Suppose that a bank or a financial institution has some fixed amount A to invest. It invites competitive bids for placement of funds from different agencies, each of which guarantees certain return and carries certain risk depending upon the funds available.

While placing the funds, the bank has to comply with the guidelines of regulatory agencies. These guidelines may prohibit the bank from investing an amount beyond certain limit in different sectors. Suppose a bank has 50 million rupees to invest and the placement offers of different types of investments in four sectors have been received. Each has certain risk associated with it and also gives certain level of return. The bank needs to determine the permissible amount to be invested in different sectors as well as in different types of investments. Then the objective is to minimize the risk. These can give rise to the constraints discussed earlier. Suppose different placements offers are as shown in Table – 5.2.

Table. 5.2: Offers for placement of funds

Agency	Sector	Minimum Amount	Maximum Amount	Estimated Rate of Risk	Upper Limit - Sector
1	Sector 1	10 Million	20 Million	5%	20 Million
2	Sector 2	10 Million	15 Million	5.5%	10 Million
3	Sector 2	10 Million	25 Million	6.5%	10 Million
4	Sector 3	15 Million	25 Million	7%	15 Million
5	Sector 3	15 Million	25 Million	6%	15 Million
6	Sector 4	1 Million	10 Million	8%	5 Million
7	Sector 1	20 Million	30 Million	6%	20 Million

It can be seen from the above table that the last column indicates the sector wise limits to be complied by the bank and hence different constraints arise. On the other hand, the problem of placement of funds is similar to the reverse auction problem, in the sense that bank is the buyer and placement agencies are sellers. In reverse auction, there is one buyer who requires quantity q of a certain item and there are n sellers, who can supply these items. The objective is to procure the items at the minimum cost. Reverse auctions are helpful in procurement. These types of constraints can arise there too. Consider a scenario where the government may ask the enterprises to procure certain quantity of items from the cooperative organizations run by groups of persons from economically weaker sections which manufacture the same items.

The present day electronic auctions support novel applications like electronic procurement, bidding on air ticket etc. Different companies use electronic bidding to set prices of their goods. A multi attribute auction system, for electronic procurement, has been studied in [BKS1999]. In multi attribute auctions, the winner determination is based on more than one attribute unlike in traditional English or Dutch auctions, where price is the only attribute. An application of auction theory in electronic procurement has also been studied in [EGKL2001], and it also gives a near optimal solution to bid evaluation problem of the buyer. A procurement process, which minimizes the cost of procurement using auctions has been proposed in [KL2001]. Another type of auction known as Combinatorial Auctions where seller wishes to sell a combination of goods and buyers bid on one or more goods, has been studied recently [RPH1998] [S2002]. A survey of combinatorial auctions problem can be found in [VV2003]. An application of auction theory in transportation services is presented in [LOPST2002]. An approximately-strategy proof and tractable multi unit auction mechanism for single good multi unit allocation problem has been presented in [KPS2003]. It presents a fully polynomial time approximation scheme for reverse and forward auction variations. We formulate this problem as a mixed integer programming problem and develop an algorithm based on branch and bound method. Later we also obtain the VCG Payoff [V1961], [C1971], [G1973]. It can also handle formulation presented in [KPS2003]. The constraints considered in this work are different from the budget constraints or liquidity constraints studied earlier [BCIMS2005]. In budget constraints buyers have constraint on available budget. These constraints restrict the total quantity that a buyer can purchase. The constraints considered in the present work are basically assignment constraints, which restrict quantity to be allocated to a buyer or group of buyers. In the present work it is assumed that buyers do not have liquidity constraints. In our formulation buyer can specify the maximum or the minimum quantity required. Our formulation can handle liquidity constraints by means of minimum and maximum quantities. We formulate IPO auction problem as single object multi unit problem with constraints. Further we develop an algorithm based on branch and bound method. Later we also obtain the VCG Payoff [V1961], [C1971], [G1973]. It can also handle formulation presented in [BVSV2002]. Our formulation allows buyers to submit bids in different formats. Apart from this, we are not aware of similar work in financial domain requiring different types of constraints.

5.2 Problem Formulation and Results

In this section the problem of optimal matching in case of IPO is formulated as multi unit auction problem with different types of constraints. This problem is formulated as non linear integer programming problem. Then a few specific cases are discussed. Our formulation can handle forward as well as reverse auctions. In IPO the State is only seller and there are a number of different buyers. This can be considered as multi unit forward auctions with different types of constraints. In forward auctions, the objective is to maximize the cost of selling whereas in reverse auction the objective is to minimize the cost of procurement. Even though it has been formulated in the context of IPO, the formulation is general and can handle any type of multi unit auctions with constraints. Our reverse auction problem for minimization of risk is symmetric and can address any similar problem. The multi unit auctions with constraints have been considered in different settings here.

The concepts of bids and asks are already introduced in the 3rd chapter, in the context of double auctions. The same concept is used here. A bid or an ask is also an ordered list of attribute names and values. A *bid* in the multi unit auction (a bid in forward auction and ask in reverse auction) describes the details of the items (shares in the context of IPO), its quantity and price that the buyer is willing to pay. Additionally a buyer (seller) can describe capacity constraints. Each B_i (Ai) is of the following type.

$B_i = (v^i_1, v^i_2, \ldots, v^i_k)$

where v^i_j is the value of the j^{th} attribute. The price and quantity are two attributes of asks and bids. The *attributes* describe different characteristics of the items. Each attribute assumes values from the set of specified *domains*. For instance, the price attribute will have values from set of positive real numbers. In this work price and quantity attributes are considered. Let there be n bids, $B_1, B_2, \ldots, B_n$. Let BD be the set of all bids. In IPO buyers can submit bids in a number of different ways. It is always assumed that the prices in bid refer to per unit price.

(1) A buyer can submit a single price-quantity pair, indicating the amount he is willing to pay for the corresponding quantity. Additionally a buyer can indicate whether it is "all or nothing bid". "All or nothing" bid means that the buyer requires the entire quantity specified and no reduction in the quantity to be purchased is acceptable. If "all or nothing" is not specified then the meaning of the bid is that the buyer has specified the maximum quantity he is willing to buy at the specified price. The price, quantity and "all or nothing" indication in the bid are of the form $\{(bp_i, bq_i)$, all-or-nothing , … } or $\{(bp_i, bq_i)$, max-qty , … }. We call these bids as bids of type 1.

(2) Alternatively a buyer can submit a set of quantity ranges and corresponding prices. A quantity range is indicated by a quantity interval. The quantity interval

is closed at the higher end. This bid can be interpreted as buyer is willing to pay the stated price for any quantity in that interval. The price and quantity intervals (but assumes integer values) in the bid is of the form

$\{((bq_1,bq_2],bp_1), ((bq_3,bq_4],bp_2), \ldots\}$.

So buyer is willing to pay bp_1 per unit for any quantity between bq_1 (excluding) and bq_2 (including) and bp_2 per unit for any quantity between bq_3 (excluding) and bq_4 (including). If $bq_1 > 0$, it means that buyer requires minimum quantity of (bq_1+1). In this case $bp_i > bp_{i+1}$ and $bq_i < bq_{i+1}$, for all i. Only one value for quantity is selected in the optimum assignment. We call these bids as bids of type 2.

(3) In another option a bid can also be submitted as a set of price, quantity pairs indicating the marginal price buyer is willing to pay for additional units. The price, quantity and all or nothing indication in the bid are of the form

$\{(bp_1,bq_1)$, all-or-nothing, $(bp_2,bq_2), (bp_3,bq_3)$, max-qty,… $\}$.

The interpretation of this bid is that buyer is willing to pay bp_1 per unit for quantity bq_1. It is all or nothing type of bids. So it is the minimum quantity required. Further, buyer is willing to pay (bp_1+bp_2) per unit for bq_2 more units. However, these units can be acquired only after getting first bq_1 units. In the subsequent price quantity pair buyer can specify only maximum quantity. After obtaining bq_1 units, buyer can get bq_2 more units. The maximum quantity that can be acquired by the buyer is $(bq_1+bq_2+bq_3)$. It further states that buyer is willing to pay $bp_1+bp_2+bp_3$ per unit for additional unit after (bq_1+bq_2). We call these bids as bids of type 3.

In our set up these different options are supported. These bids and asks are processed electronically. We have formulated IPO auction as multi unit single object auction with constraints. This formulation can handle both forward and reverse auctions. We also develop some new results that help in devising an efficient algorithm.

As already stated three different options are supported in our formulation. These can handle many practical situations. These cases are described earlier. In second case (bid of type 2) buyer can also use marginal piecewise bidding language for submitting bids [KPS2003]. If a buyer (seller) uses marginal decreasing piecewise bidding language, then

v^i_i can be a semi closed interval, which is open at one of the ends. Each buyer (seller) can specify m such intervals and price pairs. In this case we select only one point (on an interval) for each buyer In the third case buyer can submit bids as set price quantity pairs, where each pair indicates marginal price buyer is willing to pay for additional units. In case there are no assignment constraints and all buyers specify unit quantity, then it is usual single object multi unit auction. First we formulate the optimization problem and then discuss few particular cases.

Let us define

$$x_{ij} = \begin{cases} 1 \text{ if jth quantity interval is selected for ith bid} \\ 0 \text{ otherwise} \end{cases} \quad , i = 1, 2, \ldots, n \text{ and } j = 1, 2, ..., m$$

If k^{th} buyer (seller) specifies only price, quantity pair then m = 1 for i= k.

Let q_{ij} be the quantity purchased by the i^{th} buyer at price bp_{ij}. Then we define our optimization problem as follows

$$\max \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} q_{ij} bp_{ij} \tag{5.1}$$

The different constraints are defined as follows.

Quantity Constraint: It is the upper limit on the total quantity to be purchased.

$$\text{Total quantity} = \sum_{i=1}^{n} \sum_{j=1}^{m} q_{ij} \leq q \quad , i = 1, 2, .., n \quad , j = 1, 2, \ldots, m \tag{5.2}$$

Capacity Constraint: The capacity constraint indicates the minimum and maximum demand of the buyer. In some cases these constraints may be implicitly specified. Let bq_{imax} be the maximum quantity required by i^{th} buyer. Then capacity constraint of buyer i^{th} buyer is

$$\sum_{j=1}^{m} q_{ij} \leq . \; bq_{imax} \quad i = 1, 2, ..., n \tag{5.3}$$

If i^{th} buyer specifies $bq_{imin,}$ as minimum required quantity, then we have constraint

$$bq_{imin} \leq . \sum_{j=1}^{m} q_{ij} \tag{5.4}$$

In case the minimum required quantity is specified, then that minimum quantity must be allocated to that buyer, in case his bid is selected. In case i^{th} buyer specifies *all or nothing* bid, the capacity constraint for i^{th} buyer changes as

$bq_i = q_i$

Group Constraint: Let A be the set of buyers, such that there is a limitation on total quantity that can be purchased by a group of buyers. There can be more than one such set. Let q_a be the quantity, then we have

$$\sum_{i} \sum_{j=1}^{m} q_{ij} \leq q_a \text{, where } i \varepsilon A \text{ for all such groups} \qquad (5.5).$$

The other constraints are

$\sum_{j=1}^{m} x_{ij} \leq 1$ for all i ε B and all q_{ij} are non negative integers. In case any bid is of type 3, the constraint changes to

$x_{ij} = 0$ or 1 , $x_{ij} \leq x_{ij+1}$ and all q_{ij} are non negative integers (for all i and j).

$bql_{ij} < q_{ij} \leq bqu_{ij}$, where bqu_{ij} and bql_{ij} are the upper and lower limits of j^{th} price quantity interval submitted by i^{th} buyer. (5.6)

The first constraint in (5.6) ensures that only a single point in any interval is selected for buyers with bids of types 1 and 2. The second constraint ensures that quantity is an integer. The last constraint ensures that quantity allocated is within the corresponding interval. The optimization problem defined in (5.1) is basically nonlinear integer-programming problem. This happens because quantity is also decision variable. However it assumes only integer values. So both decision variables (x_{ij} and q_{ij}) assume integer values. This formulation can handle the case where a few buyers have submitted only price quantity pairs and others have submitted intervals. We state a few particular cases of our general formulation. In these cases our basic algorithm can be modified.

5.3 Particular Cases

In case of reverse auction the formulation changes as

$$\min \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} q_{ij} ap_{ij}$$

The different constraints are defined as follows

Quantity Constraint: It is the upper limit on the total quantity to be sold

Total quantity = $\sum_{i=1}^{n} \sum_{j=1}^{m} q_{ij} \leq. q$, i = 1, 2, ..., n , j = 1, 2, …, m

Capacity Constraint: The capacity constraint indicates the minimum and maximum supply by the seller. In some cases these constraints may be implicitly specified. Let

aq_{imax} be the maximum quantity available with i^{th} seller. Then capacity constraint of i^{th} seller is $\sum_{j=1}^{m} q_{ij} \leq. aq_{imax}$ $i = 1, 2, ..., n$

If i^{th} seller specifies aq_{imin}, as minimum quantity to be sold, then we have constraint

$aq_{imin} \leq. \sum_{j=1}^{m} q_{ij}$

In case minimum quantity to be sold, is specified by a seller, then that minimum quantity must be purchased from that seller, in case his ask is selected.

Group Constraint: Let A be the set of sellers, such that there is limitation on total quantity that can be purchased from them. There can be more than one such set. Let q_a be the quantity, then we have

$\sum_{i} \sum_{j=1}^{m} q_{ij} \leq q_a$, where $i \varepsilon A$ for all such groups

The other constraints are

$\sum_{j=1}^{m} x_{ij} \leq 1$ for all $i \varepsilon B$ and all q_{ij} are non negative integers. In case any bid is of type 3, the constraint changes to $x_{ij} = 0$ or 1, $x_{ij} \leq x_{ij+1}$, all q_{ij} are non negative integers (for all i and j).

$aql_{ij} < q_{ij} \leq aqu_{ij}$, where aqu_{ij} and aql_{ij} are the upper and lower limits of j^{th} price quantity interval submitted by i^{th} seller.

All bids of type 1

In case all buyers submit set of price quantity pairs then m = 1 for all the buyers. In this case second subscript j can be omitted. So the above formulation gets changed as follows

$$x_i = \begin{cases} 1 & \text{if ith bid is selected} \\ 0 & \text{otherwise} \end{cases}, \quad i = 1, 2, \ldots, n$$

if i^{th} bid is selected

Let q_i be the quantity sold to the i^{th} buyer at price bp_i. Then we define our optimization problem as follows

$$\max \sum_{i=1}^{n} x_i q_i bp_i \qquad (5.7)$$

The constraints are defined as follows

The quantity, capacity and group constraints are similarly defined. These constraints are

$$\sum_{i=1}^{n} q_i \leq q \ , i = 1, 2, ..., n, \ m = 1 \quad (5.8)$$

$$q_i \leq bq_{imax} \ i = 1, 2, ..., n \quad (5.9)$$

$$bq_{imin} \leq q_i \quad (5.10)$$

If i^{th} buyer specifies all or nothing bid then above constraint changes as

$$q_i = bq_i \quad (5.11)$$

$$\sum_{i} q_i \leq q_a \text{, where } i \in A \text{ for all such groups} \quad (5.12)$$

The other constraints are q_i are nonnegative integers for all i and $x_i = 0$ or 1 for all i.

The optimization problem defined above is a particular case of more general formulation defined in (5.1). However it is still a nonlinear integer programming problem. In case all bids are all or nothing bids, then the formulation reduces to 0-1 programming problem. In this case the variable x can be omitted and the problem becomes

$$\max \sum_{i=1}^{n} q_i bp_i \quad (5.13).$$

The constraints are as follows.

Quantity Constraint: Total quantity $\sum_{i=1}^{n} q_i \leq q \ , i = 1,2,..,n$ (5.14).

Capacity Constraint: $q_i = bq_i$ for all i = 1, 2, …, n (5.15).

These constraints capture all or nothing bids.

Group Constraint: Let A be the set of buyers, such that there is a limitation on the total quantity that can be purchased (sold) to them. There can be more than one such set. Let q_a be the quantity, then we have

$$\sum_{i} q_i \leq q_a \text{, where } i \in A \text{ for all such groups} \quad (5.16)$$

The other constraint is q_i are non negative integers for all i (5.17)

The formulation for reverse auction problem can be similarly defined and it is omitted. Apart from these two cases, the case - where all bids are marginal prices (i.e. type 3) are submitted - is considered later. The formulation presented in [BVSV2002] and usual single object multi unit auctions are particular cases of this formulation. It can be seen that our formulation is general and can handle different cases. Our algorithm is general one and can handle these different cases.

As stated earlier, the two problems addressed in this work are as follows.

(i) Determining optimum assignment of bids and asks.

(ii) Determining how much each participant has to pay or receive.

We first obtain its optimum solution and then obtain VCG Payoff. VCG Payoff is obtained first by obtaining optimum solution of (5.1) with all buyers (sellers) and then solving the optimization problem by removing a buyer (seller). Let V_o be the optimum solution to (5.1). Let V_{oi} denote the optimum solution after removing i^{th} buyer (seller). Then VCG Payoff of i^{th} buyer is

$pb_i = bp_i – (v_o\text{-}v_{oi})$, similarly for i^{th} seller $sp_i = as_i + (v_o\text{-}v_{oi})$

where as bp_i, as_i are bid and ask prices specified by i^{th} buyer or seller respectively. The LHS represents the amount payable. The VCG Payoff ensures that truthful bidding is the dominant strategy and hence buyers bid truthfully. We compute VCG Payoff without actually solving a series of optimization problems. Later on we work to generalize uniform price auction.

Assignment: An assignment of quantity qa_{ij} at price bp_{ij} from i^{th} bid B_i is denoted by (bp_{ij},qa_{ij}). In this case bp_{ij}, is the price component of j^{th} price quantity pair of bid B_i. The assigned quantity qa_{ij} satisfies the relation, $\max(0,bq_{ij-1}) \leq qa_{ij} \leq bq_{ij}$ and $bq_{i0} = 0$. The terms bq_{ij} and bq_{ij-1} are quantity components of j-1 and j^{th} price quantity pairs of bid B_i.

Contribution: We define contribution to the value of objective function at price bp_i and quantity bq as $cov_{pi}(bq)$ as follows

$$cov_{pi}(q)=bp_i bq \quad (5.18).$$

The contribution results from assignment of quantity bq at price bp_i. The value of (5.18) is the maximum (minimum) for any bp_i, when q is the maximum (minimum). In the same way for any bq the value is maximum (minimum) when bp_i is maximum (minimum). Using these we obtain the upper bounds on improvement at any stage. If there is only one bid (ask) at price bp_i, then we call it as buyer’s (seller’s) contribution and indicate it by $covb_{pi}(bq)$. If bq = 1, we call it as unit contribution and indicate it by $ucov_{pi.}$

Combination: Let there be n bids B, B_2, B_3, …, B_n. We define combination for quantity q, as set of price quantity pairs (bp_{ij},qa_{ij}) such that each pair represents assignment of quantity qa_{ij} at price bp_{ij} from bid $B_{i.}$. It is a set $\{(bp_{ij},qa_{ij})$, j = 1, 2, …, m, i = 1, 2, …, k}. In case i^{th} bid is of type 1 or 2, then m = 1 for such bids. The quantities satisfy the relation

$$\sum_{i=1}^{k}\sum_{j=1}^{m} qa_{ij} = q \;.$$

In other words a combination represents a set of assignments from bids $B_1, B_2, B_3, \ldots, B_k$. A combination is called a feasible combination, if it satisfies all the constraints of assignment. In case i^{th} bid B_i is submitted with marginal piecewise bidding language, then a feasible combination has only a single pair from bid B_i. In other words in this case m = 1. The value of the combination C_i for quantity q, $valc_i(q)$ is defined as

$$valc_i(q) = \sum_{i=1}^{k}\sum_{j=1}^{m} qa_{ij}bp_{ij} \;.$$

A feasible combination C_j is optimum combination for quantity q if $valc_j(q) \geq valc_i(q)$, (for all other feasible combinations). The value of the optimum combination is the maximum. In order to obtain optimum assignment for (5.1), an optimum combination is required to be created. We now show how such combination can be created. We begin with a combination with single assignment.

5.4 Creating an Optimum Combination

Upper bound on improvement at each stage**:** Let maximum price among the received bids be bp_1. Let q_1 be the maximum quantity that can be assigned at this price. Then it can be seen that the maximum possible improvement or upper bound on improvement at any stage is

$cov_1 = bp_1\, q_1$ (5.19).

Optimum assignment for a fixed quantity: A buyer's bid can have a single price quantity pair (bp_i,bq_i). Alternatively a bid can be submitted by using the marginal decreasing piecewise bidding language. In this case, the buyer specifies a list of quantity ranges and prices he is willing to pay $(bp_{i1},(bq_{i0},bq_{i1}])$. This can be interpreted as the buyer is willing to pay price bp_{i1}, for quantity q, where $q \; \varepsilon \; (bq_{i0},bq_{i1}]$ and assumes integer values. We call the list of prices and quantity intervals as price, quantity part of the bid. The price, quantity part of a bid (ask) denoted by B_{pi} (A_{pi}) is ordered list of price and quantity intervals as

B_{pi} = {($bp_{i1,}$(bq_{i0},bq_{i1}]), ($bp_{i2,}$(bq_{i1},bq_{i2}]), ($bp_{i3,}$(bq_{i2},bq_{i3}]), ..., ($bp_{im,}$(bq_{im-1},q_{in}])}. The prices and quantities satisfy the following relations

$bp_{ij} > bp_{ij+1}$ for all i and j and $q_{ij} < q_{ij+2}$ for all i and j .

We call bp_{ik} as price component of k^{th} element of B_{pi} and (bq_{ik-1},bq_{ik}] as the quantity interval of k^{th} element. In further discussion in this section, without loss of generality, we state further discussion for the bids. However it can be easily extended for asks. So we do not explicitly state it for ask.

Only a pair of price and quantity (bp_{ik},bq_{ik1}) , called as the point, from any one element (say k^{th} element) of the set B_{pi} can be selected for each buyer in case of an optimum assignment. In this case p_{ki} is the price of the k^{th} element and q_{k1i} is a point belonging to the corresponding quantity interval and assumes integer values.

In order to obtain optimum assignment for quantity q, we consider two cases. In the first case a buyer submits only price and quantity pair and no intervals are submitted. In this case it is assumed that all buyers submit only price quantity pair. In second case one or more quantity intervals are submitted along with corresponding price. It is also possible here that a few buyers may submit only price quantity pairs.

In order to obtain optimum solution we carry out the assignment in stages, till the assignment of quantity q is completed. In first case it is always possible to determine the maximum price and the quantity available at that price, at any stage. In the second case it is not always possible to get such an assignment. So we create different price quantity combinations and then determine the optimum assignment.

Case 1: Let B_1 be the bid with the maximum unit contribution and price bp_1. Let it be $ucov_1$. Let q be the maximum quantity that can be assigned at this price. Let $cov_1(q)$ be the contribution from this assignment. Then

$$cov_1(q) = (q) \times (ucov_1) \tag{5.20}$$

This is the maximum possible improvement from an assignment of quantity q. It can be seen that this term is a product of quantity assigned and contribution per unit. This can be improved if and only if, term $ucov_{i1}$ (i.e. unit contribution) is improved or q (assigned quantity) is improved. In this case neither is possible, as there is no bid with higher unit contribution and quantity assigned is the maximum available. So expression in (5.20) represents, the maximum possible improvement from single assignment. In case, there are more than one bid with maximum unit contribution, then we combine respective bid quantities. Due to this, we find out maximum quantity that can be assigned for that unit contribution.

Suppose that the bid B_1 with maximum unit contribution has only quantity q_1 ($< q$) available. Suppose that B_2 is another bid with bid price p_2 and having the highest contribution to value of objective function by assignment of unit quantity except ask B_1. Let $ucov_2$ be the contribution to the value of objective function by assignment of unit quantity of bid B_2. Then we have

$ucov_2 > ucov_j$ for $j = 3,\ldots,n$. and $ucov_2 < ucov_1$ (5.21).

Let q_2 be the maximum quantity that can be assigned at price bp_2 such that $q_1+q_2 = q$. Then these two assignments represent the maximum improvement that can be obtained by assignment of quantity q. This maximum contribution to the value of objective function is

$cov_1 + cov_2 = (q_1) \times (ucov_1) + (q_2) \times (ucov_2)$ (5.22).

It can be seen that contribution can be improved if any one of the two terms on right hand side can be improved. As already seen the first term cannot be improved. The same argument can be applied to the second term, as B_1, the only bid which can improve unit contribution of bid B_2, is already assigned. There is no other bid with higher unit contribution. The quantity assigned is also the maximum possible. As the second term cannot be improved so (5.22) is the highest possible improvement in this stage. In this case also we combine bids with same unit contribution.

Theorem 1: Let $BD_{k-1} = \{B_1, B_2, ..., B_{k-1}\}$ be the set of bids which can be assigned and satisfy the following condition.

1. For any bid $B_j \varepsilon BD_{k-1},$ $ucov_j > ucov_{j+1}$, $j = 1, 2 , \ldots, k-2$. Let the $q_1, q_2, ..., q_{k-1}$ be the quantities assigned respectively.

2. For any bid $B_i \not\in BD_{k-1},$ $ucov_i < ucov_l$, where $l = 1, 2, \ldots, k-1$ (i.e. set of bids belonging to BD_{k-1}).

3. Let B_k be the bid such that $ucov_k > ucov_j$, where $j = k+1, \ldots, n$ (set of bids not in BD_{k-1}). The assignment of quantity q_k of this bid completely satisfies the demand of quantity q.

Then maximum improvement can be obtained by a combination consisting of all bids in BD_{k-1}, and B_k. This is the maximum possible improvement by assigning set of bids for quantity q.

Proof: This result can be easily shown by induction. As already shown, the result is true for k = 1 and k = 2. We assume that the result is true for (k-1) and show that it holds for k.

The total contribution to the value of objective function $tcov_k$ after assignments of k bids is

$$tcov_k = \sum_{l=1}^{l=k-1} (q_l) \text{ x } (ucov_{il}) + (q_k) \text{ x } (ucov_k) \qquad (5.23)$$

The contribution can be improved, if and only if the second term can be improved. This is due to the fact that the first term cannot be improved by assumption. In this case, it is not possible to improve the second term since its unit contribution $ucov_k$ is the highest among the remaining bids (and hence cannot be improved). In the same way, q_k is the maximum quantity that can be assigned. So the second term cannot be improved. Hence $tcov_k$ is the maximum improvement in the value of objective function. So the result follows by induction.

Theorem 2: Suppose that there is a bid B with price p and the maximum quantity q and the following conditions are satisfied

(1) There is no bid with price higher than p (i.e. it is the highest price bid).

(2) No other single bid exists with quantity q.

(3) There are k bids $B_1, B_2, \ldots, B_k$, with prices p_i and quantities q_i such that $q = \Sigma q_i$ and $p_i < p$ for all i. All the remaining bids have lower price than p_k.

Then bid B has higher contribution than total contributions of bids $B_1, B_2, \ldots, B_k$.

Proof : Let cov_b be the contribution of bid B. Then

$$cov_b = pq$$

Let cov_{bi} be the contribution of bid B_i and $tcov_{Bk}$, be the contribution of bids $B_1, B_2, \ldots, B_k$.

$$cov_{bi} = p_iq_i \text{ and } tcov_{Bk} = \sum_{i=1}^{k} p_iq_i$$

$$cov_{bi} - tcov_{Bk} = pq - \sum_{i=1}^{k} p_iq_i = p\sum_{i=1}^{k} q_i - \sum_{i=1}^{k} p_iq_i$$

$= \sum_{i=1}^{k} pq_i - \sum_{i=1}^{k} p_iq_i = \sum_{i=1}^{k} (p - p_i)q_i > 0$, as $q_i > 0$ and $(p-p_i) > 0$ for all i.

Case 2: In case a buyer has submitted a set of price-quantity intervals (using marginal decreasing piecewise bidding language) it may not be possible to get a set of bids as in case 1. Here, at any stage of assignment only one point (called as price, quantity combination) can be selected for a buyer. We start with a scenario where two bids (each having set of price quantity intervals) are required to be combined to fulfill supply of quantity q. Since the case with one bid is similar to the earlier we do not consider it here.

Suppose that there are two bids B_1 and B_2 such that

$B_{p1} = \{(p_1,(q_0,q_1]), (p_3,(q_1,q_3])\}$ and

$B_{p2} = \{(p_2,(q_{02},q_2]), (p_4,(q_2,q_4])\}$

(1) The following relations are satisfied $p_1>p_3$, $p_1>p_2$, $p_2>p_4$, $p_3>p_2$

(2) The quantities satisfy $q_1+q_2 = q$ and $q_1 < q_3$, $q_3 < q$ and $q_2 < q_4$.

(3) All other bids have prices lower than p_4.

The assumptions $p_1 > p_3$, $p_2 > p_4$, $q_1 < q_3$ and $q_2 < q_4$ follow from the definition of marginal decreasing cost function and if $p_3 > p_2$ is not assumed this case reduces to earlier scenario and $p_1q_1+p_2q_2$ the optimum assignment for quantity q (from earlier results). We discuss the case where $q_3 > q$ later (combination 5 in the Table 5.3).

Table. 5.3 – Different possible combinations

Serial Number	Combination	Contribution to the Value of Objective Function	Quantity (Total =q)
1	p_1,q_1 and p_2,q_2	$p_1q_1+p_2q_2$ (=co_1)	$q = q_1+q_2$
2	$p_1,(q_1-1)$and $p_4,(q_2+1)$	$p_1q_1+p_4q_2-p_1+p_4$ (=co_2)	$q = (q_1-1)+(q_2+1)$
3	p_3,q_3 and $p_2,(q-q_3)$	$p_3q_3 + p_2q - p_2q_3$ (=co_3)	$q = q + q_3 - q_3$
4	p_3,q_3 and $p_4,(q-q_3)$	$p_3q_3 + p_4q - p_4q_3$ (=co_4)	$q = q + q_3 - q_3$
5	p_3q	p_3q (=co_5)	If $q_3 > q$

It can be easily seen that $q_1 + q_4 \geq q$ (as $q_4>q_3$). The possible combinations for quantity q with contribution to the values of objective functions are It can be seen that

$co_1 - co_2 = p_1q_1+p_2q_2 - p_1q_1 - p_4q_2 + p_1 - p_4 = (p_2-p_4)q_2+p_1-p_4 > 0$ as $p_2 > p_4$ and $p_1 > p_4$.

$co_3 - co_4 = p_3q_3 + p_2q - p_2q_3 - p_3q_3 - p_4q + p_4q_3 = (p_2-p_4)q - (p_2-p_4)q_3 = (p_2-p_4)(q-q_3) > 0$ as $p_2 > p_4$ and $q > q3$.

So combinations (2) and (4) (in Table – 5.3) cannot be optimum assignment for quantity q. It can be further seen that, by decreasing quantity assigned at higher price (p_1 or p_3) or by increasing quantity assigned at lower price, the contribution to the value of objective function cannot be improved. Out of above five combinations, combinations 1,3 or 5 can be possibly optimum.

Then the conditions under which combination (3) is optimum assignment are

$p_3q_3+p_2q - p_2q_3 > p_1q_1+p_2q_2$.

$p_3q_3+p_2(q_1+q_2)-p_2q_3 > p_1q_1+p_2q_2$

$$p_3q_3+p_2(q_1- q_3) > p_1q_1 \tag{5.24}$$

It can be seen that if (5.24) is satisfied, then combination (3) (in Table –5.3) is optimum for quantity q, otherwise one of the combinations, (1) or (5) is optimum. It can be seen that combination (5) is optimum if

$p_3q > p_1q_1+p_2q_2$

$p_3(q_{1+}q_2) > p_1q_1+p_2q_2$

$$(p_3-p_2)q_2 >(p_1-p_3)q_1 \tag{5.25}$$

If condition (5.25) is satisfied then combination (5) is optimum, otherwise combination (1) is optimum. Using this we obtain optimum assignment in different cases.

Suppose that there are n buyers. Each buyer has submitted a bid consisting of m quantity intervals. The price, quantity part of the bid (ask) of i^{th} buyer denoted by B_{pi} is an ordered list of price and quantity intervals as

$B_{pi} = \{(p_{i1,}(q_{i0},q_{i1}]), (p_{i2,}(q_{i1},q_{i2}]), (p_{i3,}(q_{i2},q_{i3}]), \ldots, (p_{im,}(q_{in-1},q_{in}])\}$.

A combination C_i is a set of price quantity pairs (p_{ij},q_{ji}), such that p_{ij} is the price component of j^{th} element of B_{pi} and q_{ji} is the point from corresponding quantity interval. A combination has property that it can have at most one point (p_{ij},q_{ij}) from B_{pi} in case if bids of type 1 and 2. The contribution of combination C_i is the value of the combination, which is already defined. These two terms mean the same thing in the context of a combination. An assignment (p_{ij+k},q_{ij+k}), can be added to the combination C_i with contribution co_i and quantity q_i if $k \neq 0$ and there is a bid with price quantity interval $(p_{ij+k,}(q_{ij+k-1},q_{ij+k}])$. Then the contribution of combination becomes $co_i + p_{ij+k}q_{ij+k}$ and quantity becomes $q_i + q_{ij+k}$. We state theorems (3) and (4) in case where there is no single highest price bid for quantity q.

Theorem 3: Let C_i, be the combination with contribution to the value of objective function co_i, which is optimum for quantity q_i. Suppose that an assignment (p_j,q) , can be added to this combination. Further suppose that this assignment has the highest possible contribution for quantity q. Then combination C_j , consisting of C_i and an assignment (p_j,q) is optimum combination for quantity q_i+q with optimum contribution to the value of objective function co_i+p_jq.

Proof: The result can be shown to be true for k = 1. It can be easily verified that a combination with a single assignment (p_i,q) has optimum contribution p_iq for quantity q, when p_i is maximum. Suppose that there is a combination C_i consisting of k assignments, which is optimum for quantity q_i with contribution co_i. Suppose that the assignment (p_i,q) can be added to this combination. The price p_i is the highest for quantity q. After adding this point to combination C_i the contribution of new combination is

$$co_i + p_iq \tag{5.26}$$

and quantity is q_i+q.

It can be easily seen that (5.26) is the optimum value for quantity q_i+q, as neither the first term nor the second term of (5.26) can be improved. So the result follows by induction.

Theorem 4: Suppose that there exist a set of k combinations $C_1, C_2, \ldots, C_k$ consisting of different price quantity assignments from bids submitted by different buyers. Let co_i be the contribution of each combination and q_i be the respective quantities. Each combination is optimum for corresponding quantity q_i. The following conditions are satisfied by these combinations

(1) Let q be the quantity such that $\sum q_i < q$ and also $q_i < q$ for all i

(2) Apart from $C_1, C_2, \ldots, C_k$ no other optimum combination can be formed for any quantity $q_k < q$ from the submitted bids.

(3) Suppose that (p_j, q_j) is an assignment with the highest price p_j, which can be added to each of above combination. After adding this assignment, quantity of each combination becomes $q_i+q_j > q$ for all i. Let us define $qtrem_i = q - q_i$. Let qtremm = $\max(qtrem_i)$. Suppose that there is no price quantity assignment having price higher than that of p_j and quantity smaller than qtremm.

Then the combination formed by adding point (p_j, $qtrem_i$) to C_i having maximum contribution is an optimum combination for quantity q.

Proof: After adding point (p_j, $qtrem_i$) to each combination, the contributions of each combination is

$co_i + p_j qtrem_{i.}$ The quantity of each combination is q. The combination with max(co_i+ $p_j qtrem_i$) has the highest contribution. So it is optimum combination for quantity q (among C_1, C_2, …, C_k). In case, there is more than one such combination, any combination can be selected.

We further show that apart from these combinations, there is no combination, which can be formed from the given bids and having optimum contribution for quantity q.

It can be seen from result (1) and (2), that we can obtain optimum value for any quantity by selecting (1) the highest price bid with quantity q, (2) selecting combination of optimum bids. In this case, scenario (1) is not possible. So we have to select a combination. If combination is one among C_1, C_2, ..., C_k, then the result is proved. If it is not one among C_1, C_2, …, $C_{k,}$ then another optimum combination can be formed from the given bids for given quantity $q_k < q$. This is a contradiction, so there is no other combination, which is optimum for q. Hence the result follows. All the above results are also applicable to ask in reverse auctions (by changing sign).

5.5 Algorithm for Assignment

In our system the bids or asks are submitted by the users till certain deadline. After the deadline the buyers, to whom allotment is to be carried out are selected. After selecting the buyers, the constraints are formulated as per IPO policy. In formulating constraints some quantity intervals submitted by buyers are deleted and in some cases its bounds are replaced. Then group constraints are formulated. The constraint formulation has been separated from the main algorithm. Then our algorithm is executed on the formulated problem. The algorithm is a general one and can be used for any forward/reverse auction system. After obtaining optimum solution our algorithm finds out the VCG Payoff. Our algorithm is based on the above results and works as follows

(1) We start with the highest (lowest) available price. Then we determine the maximum quantity that is available at that price after taking into account all the constraints. If there are two bids (asks) with same price, then we combine them. The

price, quantity combination is added to assignment list. If the quantity allocated exceeds the required quantity we stop. If we get a combination that cannot be assigned to an earlier combination of bids, we create another new combination. We save its contribution value, quantity remaining (look ahead), combination list and buyer's (seller's) contribution in separate table structure.

(2) When we select next bid, which of the conditions 1-4 (Table-5.3) hold is tested. After verifying the results we form an optimum combination for quantity q_i at any stage. If the allocated quantity exceeds the quantity required then we stop. Otherwise we add the price quantity points to appropriate combinations to obtain an assignment. We also eliminate the combinations, which are not likely to be in optimum solution.

(3) Repeat the above two steps till requirement is completely fulfilled.

(4) Then determine VCG Payoff for each buyer (seller). This is determined by subtracting the respective buyer's (seller's) contribution from the final solution.

We hereby state forward and reverse auction algorithms. The constraint formulation algorithm is not stated as it is a straight forward read and replace operation. Let A be the list of asks and B be the list of bids. The algorithms are as follows.

Algorithm (1.1) forwauct /* Main Algorithm */

1. Sort all bids on descending order of price within same price sort on descending order of quantity

2. while (there is an unassigned bid or unfulfilled demand) repeat steps 3 thru 9

3.Find out maximum quantity available for price p_i after verifying capacity constraints on buyers and other group constraints

4. Verify whether optimality condition is satisfied. If verified exit

5. Add the price, quantity combination to appropriate assignment combination. Eliminate combinations which will not be in optimum solution.

6. Calculate and save the buyer's contribution

7. Calculate the value of objective function

8 Mark the bid as assigned

9 Add quantity to demand_fulfilled

10. Select the combination with the highest value.

11. Compute VCG Payoff

Figure. 5.1. Algorithm for Forward Auction

The algorithm for reverse auction is similar and is as follows.

Algorithm (1.2) revauct /* Main Algorithm */
1. Sort all asks on descending order of price within same price sort on descending order of quantity
2.while (there is an unassigned ask or unfulfilled supply) repeat steps 3 thru 9
3.Find out maximum quantity available for price p_i after verifying capacity constraints on buyers and other group constraints
4. Verify whether optimality condition is satisfied. If verified exit
5. Add the price ,quantity combination to appropriate assignment combination. Eliminate combinations which will not be in optimum solution.
6. Calculate and save the buyer's contribution
7. Calculate the value of objective function
8 Mark the ask as assigned
9 Add quantity to supply_fulfilled
10. Select the combination with the minimum value.
11. Compute VCG Payoff

Figure. 5.2. Algorithm for Reverse Auction

The working of our algorithm is shown with the help of a simple example (Table 5.4).

Theorem 5: The algorithm always generates optimum solution for any quantity q if it exists.

Proof: At any stage our algorithm generates optimum solution for quantity q_i. If quantity exceeds the required quantity, then the optimum solution is obtained and we stop. In other case we get an optimum solution for quantity $q_i + q_j$ in next stage. Repeat this till assigned quantity exceeds quantity q. This is the optimum solution. The algorithm will terminate when quantity q is assigned or all combinations are created and no bid is left out. At this stage we will have optimum solution for quantity q_k. If $q_k < q$, the demand is more and supply is less. So no optimum solution for quantity q, from the existing bids exists.

Complexity: The complexity of our algorithm is always polynomial. There are n buyers and each buyer has submitted m points. In the best-case scenario, a combination can be created by scanning n bids. Each bid has m assignments. Scanning nm assignments in the worst case can create any combination. Then each combination can be tested for optimality in linear time. So the time complexity of our algorithm is O(nm). In each stage we scan n bids and one interval. In the worst case time complexity is O(nm) or polynomial. Once combination is obtained we need to scan them to obtain the combination with the highest value. This can be done in linear time complexity. Apart from this sorting is the hardest part. Since sorting time complexity is of the order of O(nmlognm), the time complexity of this algorithm is $O(n^2)$ + O(nmlognm) in worst case, which compares favorably with the time complexity order $O(n^3)$ presented in [KPS2003]. Additionally VCG Payoff gets computed in linear time complexity.

Example**:** Suppose that four buyers have submitted bids as follows. The output of our algorithm for different quantities is shown in column 5-9 of Table – 5.4. At each stage algorithm combination arrives at optimum solution. The algorithm has been implemented in C++. The data sets of asks and bids of different sizes were generated randomly. Each data set consisted of number of asks with ask price, quantity, ask size, bid size, bid price and bid quantity. Size of data sets varied from 5 to 1500. The results were compared with unconditional optimum solution and some solutions obtained with the help of MATLAB package. It can also be seen that time complexity of our algorithm is always polynomial. The figure – 5.1 indicates the comparative performance of proposed solution against algorithm proposed in [KPS2003]. It can be seen that the above algorithm works in case a few or all buyers submit price quantity pair instead of set of price quantity intervals.

Table. 5.4 Bids received from Buyers Optimum Solution for different Quantities

Buyer	Minimum Quantity	Maximum Quantity	Price	Quantity	Buyer	Quantity	Price	Cost
1	1	5	10	20	4	10	17	170
	6	10	8		3	10	18	180
	11	15	6	Total				350
	15	20	5	27	4	10	17	170
			0		3	10	18	180
2	1	7	15		2	15	7	105
	8	12	12	Total				355
	13	17	8	45	3	20	15	300
	18	22	6		2	7	15	105
	22	26	5		4	18	14	252
				Total				657
3	1	10	18	60	3	20	15	300
	11	20	15		2	7	15	105
	21	30	12		4	30	12	360
	31	40	10		1	3	10	30
								790
4	1	10	17					
	11	20	14					
	21	30	12					
	31	40	10					

This case is treated as a single point being submitted by the buyer. So algorithm creates a combination by selecting price, quantity pair in appropriate combinations. In case a bid is all or nothing type of bid, then algorithm selects this combination, in case quantity remaining is higher than quantity q. If the remaining quantity does not exceed q, then the point cannot be selected as buyer's constraint means that buyer requires entire quantity

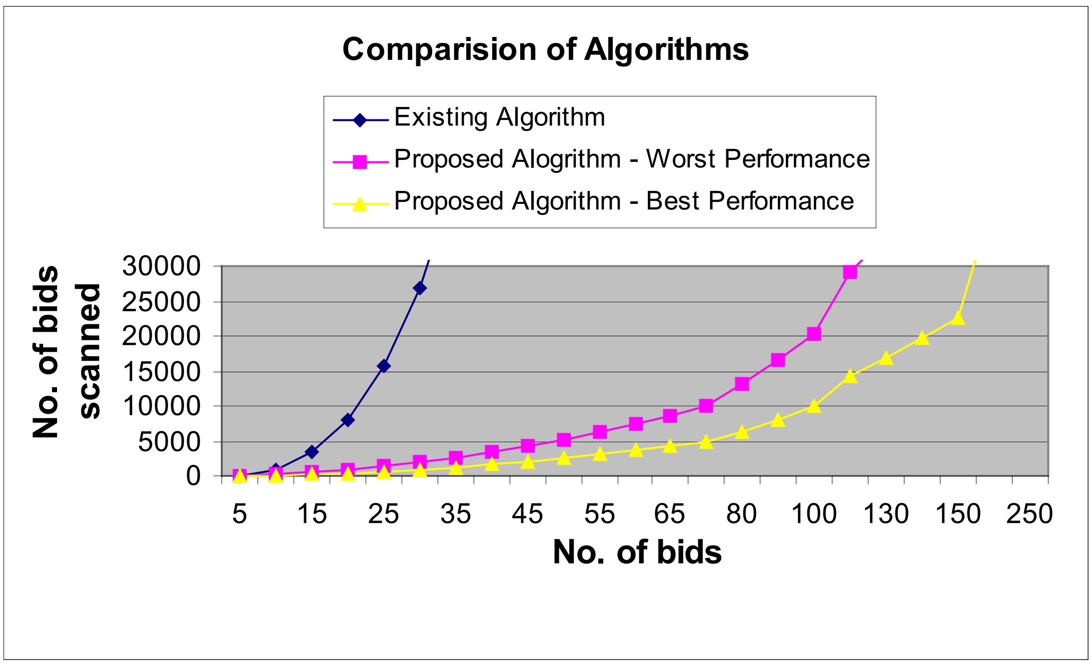


Figure. 5.3 Comparative performance of the algorithms

So in case there are bids with price higher than the bid with all or nothing constraint, then this bid (i.e. with all or nothing constraint) may not get selected. Our algorithm handles all these different cases.

5.6 Generalization of Multi Unit Auctions with Constraints

We now consider the case where all buyers submit bids as ordered pairs of prices and quantities. All bids are of type 3. In this case i^{th} buyer's bid is $[(bp_{i1},bq_1)$, v_i, (bp_{i2},bq_2), (bp_{i3},bq_3), …, and $(bp_{iK},bq_k)]$. It is interpreted as i^{th} buyer is willing to pay bp_{i1} per unit for bq_1 units, $(bp_{i1}+bp_{i2})$ per unit for (bq_1+bq_2) units, $(bp_{i1}+bp_{i2}+bp_{i3})$ per unit for

$(bq_1+bq_2+qb_3)$ units and $(\sum_{j=1}^{K} bp_{ij})$ $(\sum_{j=1}^{K} bq_{ij})$ for $\sum_{j=1}^{K} bq_{ij}$ units. The set v_i is the set of values of other attributes of the bid.

It can be seen that in this case the bid price represents the marginal cost that buyer is willing to pay for additional units. The main deviations, here from the case where buyer submit bids using marginal decreasing piecewise bidding language are as follows.

(1) A buyer submits price quantity pair instead of a price and quantity interval

(2) In optimum solution one or more price quantity pairs can be selected

(3) Unlike in marginal decreasing piecewise bidding language, where price and quantities in different pairs may not always have well defined relationships. In marginal decreasing piecewise bidding language the price decreases and quantity increases.

Further when $q_i = 1$ for all i, then this is a usual case of bids in multi unit auctions. This is the first deviation from usual multi unit auctions. In another deviation we consider capacity constraints and group constraints. In the absence of any type of constraint and when $q_i = 1$, our auction reduces to usual single object multi unit auction. Further buyer can specify whether his bid is all or nothing type. This bid is interpreted as buyer requires entire quantity $\sum_{j=1}^{K} q_{ij}$ and is not willing to accept anything lesser. The group constraints and capacity constraints are exactly same as those considered earlier. We can the rewrite above bid excluding other attribute set v_i as (it will be referred as price-quantity part of the bid).

$B_i^T = [(bp_{i1},bq_{i1}), (bp_{i2},bq_{i2}), (bp_{i2},bq_{i3}), ..., (bp_{ik},bq_{iK})]$

where $bp_{il} = \sum_{j=1}^{l} p_{ij}$ and $bq_{il} = \sum_{j=1}^{l} q_{ij}$

Suppose that n buyers (sellers) submit such bids (asks). Let us define

$$x_{ij} = \begin{cases} 1 \text{ if jth element of ith bid is selected} \\ 0 \text{ otherwise} \end{cases}$$

i = 1, 2, …, n and j = 1, 2, …, K.

$x_{ij} = 1$ if j^{th} element of B_i^T is selected for the i^{th} bid, i = 1, 2, …, n and j = 1, 2, …, K

If a buyer (seller) specifies only a single price, quantity pair then k = 1.

Let q_{ij} be the quantity sold to the i^{th} buyer at price bp_{ij} (i.e. j^{th} element). Then we define our optimization problem as follows

$$\max \sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij}q_{ij}bp_{ij}$$

The constraints are defined as follows

Quantity Constraint: Total quantity $\sum_{j=1}^{K} q_{ij} \leq q$, i = 1, 2, ..., n , j = 1, 2, ..., K (5.27).

Capacity Constraint: Let bq_{imax} be the maximum quantity required by i^{th} buyer with i^{th}. Then capacity constraint of buyer is

$$\sum_{j=1}^{K} q_{ij} \leq. bq_{imax} \quad i = 1, ..., n \qquad (5.28).$$

If the buyer specifies bq_{imin} as minimum quantity that must be purchased by i^{th} buyer

$$bq_{imin} \leq. \sum_{j=1}^{K} q_{ij} \qquad (5.29).$$

In case i^{th} buyer specifies bid as all or nothing type of bid then it can be captured by the constraint

$$\sum_{j=1}^{K} q_{ij} = \sum_{j=1}^{l} q_{ij} \qquad (5.30).$$

It may be noted that more than one buyer can submit such bids. It can be seen that if the buyer specifies minimum required quantity, then the quantity is required to be allocated to that buyer.

Group Constraint: Let A be the set of buyers, such that there is limitation on total quantity that can be purchased (sold) to them. There can be more than one such set. Let q_a be the quantity, then we have

$$\sum_{i}\sum_{j=1}^{K} q_{ij} \leq q_a, \text{ where } i \varepsilon A \qquad (5.31).$$

The other constraints are $\sum_{j=1}^{K} x_{ij} \leq K$, $x_{ij} = 0,1$, and $x_{ij-1} \leq x_{ij}$

for all i = 1, 2, ..., n and j = 1, 2, …, K and all bq_{ij} are non negative integers. Further $x_{i0} = 0$ for all i

$q_{ij} \leq bqu_{ij}$

where bqu_{ij} is the upper limit of j^{th} pair submitted by i^{th} buyer. (5.32).

It can be seen that this formulation is a particular case of formulation (5.1). The main difference is constraint (5.32). It ensures that buyer acquires q_{i1} units before acquiring further q_{i2} units. In this formulation a buyer can obtain different units at different prices, which is not possible in case of bids of type 2. Due to this, right hand side (RHS) of first constraint in (5.32) is K here. In this case we can improve our earlier algorithm. The problem for reverse auctions can also be similarly defined. The formulation presented in [BVSV2002] is a particular case of this formulation. It is a case where $bq_{ij} = 1$ for i and j, and group constraints are not present. Since quantity is not a decision variable the formulation in [BVSV2002] is linear. We use the same concept of contribution introduced earlier**.** So the contribution to the value of objective function at price bp_i and quantity q as $cov_{pi}(q)$ is

$$cov_{pi}(q) = bp_i q \quad (5.33)$$

The value of (5.33) is the maximum(minimum) for any bp_i, when q is maximum (minimum). This helps in obtaining upper bound on improvements at any stage. If there is only one bid (ask) at price bp_i, with quantity q and if there is no other bid with higher price or combination of bids of higher price, then this represents the optimum improvement. Otherwise optimum assignment is the combination of more than one bids(asks). Further such combination may be more than one price – quantity pair from any buyer.

Let the maximum price among the received bids be bp_1. Let q_1 be the maximum quantity that can be assigned at this price. Then it can be seen that the maximum possible improvement or upper bound on improvement at any stage is

$$cov_1 = bp_1 q_1 \quad (5.34).$$

This is also the optimum assignment for quantity q_1.

Optimum assignment for a fixed quantity: In the current set up of multi unit auctions, a buyer can submit set of price quantity pairs. The price quantity part of buyer's bid is of the form

$B_i^T = [(bp_{i1},bq_{i1}), (bp_{i2},bq_{i2}), (bp_{i2},bq_{i3}), ..., (bp_{ik},bq_{iK})]$ i = 1, 2, …, n

Combination: A set formed by selecting price-quantity pairs from different bids. A combination C_i is of the form $\{(bp_{i1},bq_{i1}),(bp_{i2},bq_{i2}), \ldots, (bp_{il},bq_{il})\}$, i = 1,2, ..., n.

Each pair (bp_{ik},bq_{ik}) is a part of bid Bi^T. It is k^{th} element of i^{th} bid. A combination can have more than one pair from any bid.

The total quantity (q) of the combination C_i is

$$q = \sum_i \sum_i bq_{ij}$$

The contribution to the value of objective function $cov_{Ci}(q)$ from combination C_i is

$$cov_{Ci}(q) = \sum_i \sum_i bp_{ij}\, bq_{ij}$$

A combination C_i is optimum combination for quantity q if for any other combination C_j, if and on if $cov_{Ci}(q) \geq cov_{Cj}(q)$ for $i \neq j$

In order to obtain the optimum solution we determine the maximum price and the maximum quantity available at that price. We carry out assignment in stages, till the assignment of quantity q is completed.

Suppose that we are required to determine an optimum assignment for quantity q. So first determine the highest price bp_1 amongst all the bids. Determine the maximum quantity q_1 that can be assigned as price bp_1. If there are more than one bids with the highest price bp_1, then we combine them and obtain the maximum quantity q_1 that can be assigned. Suppose that there are i_1 bids which have pair (bp_1,bq_{ij}) , where $i = 1,2,\ldots,i_1$, $j = 1,2,..,K$. So our combination C_1 is $\{(bp_1,bq_{1j}),\ldots,(bp_1,bq_{i1j})\}$. The quantity of the combination is

$$q_1 = \sum_{i=1}^{i1} \sum_{j=}^{K} bq_{ij}$$

Its contribution $cov_{c1}(q) = bp_1 q_1$ (5.35)

In the next stage determine the next highest price. Let bp_2 be the next highest price. Then determine the maximum quantity q_2 that can be assigned at this price. Suppose that there are i_2 bids which have pair (bp_2,bq_{ij}), where $i = 1,2,\ldots,i_2$, $j = 1,2,..,K$. The quantity q2 satisfies the relation

$$q_2 = \sum_{i=1}^{i2} \sum_{j=}^{K} bq_{ij.}$$

We add these points to combination C_1. So our combination C_1 is $\{(bp_1,bq_{1j}),\ldots,(bp_1,bq_{i1j}), (bp_2,bq_{1j}),..,(bp_2,bq_{2k})\}$. The quantity of the combination is

$q_1 + q_{2.}$

Its contribution $cov_{c1}(q) = bp_1 q_1 + bp_2 q_2$ (5.36).

It can be easily verified that this combination is optimum for quantity q_1+ q_2. It can be seen that contribution of the combination can be improved if any one term in (5.36) can be improved. In this case both terms cannot be improved, so the combination is optimum for q_1+q_2. The contribution can be improved only if there is a bid, which contains higher price quantity pair. In this case there is no such bid. The quantity assigned is also the maximum possible. As the second term cannot be improved, (5.36) is the highest possible improvement in this stage. So it is optimum assignment for quantity q_1 +q_2.

Theorem 6: Let C_1 = {(bp_1,bq_{1j}), …, (bp_n,bq_{i1j})} be an optimum combination for quantity q_{k-1}. Let its contribution be $cov_{C1}(q_{k-1})$. Suppose that the following condition is satisfied.

1. For any bid B_j , where i = 1, 2, ..., n , there is no price bp_m in any price quantity pair (bp_i,bq_j) ∉ C_1, such that $bp_m > bp_i$, for all i and bp_i ϵ (bp_i,bq_j) ϵ C_1. i.e. price quantity pair in C_1. (In other words all price quantity pairs, which do not belong to C_1 have lower price.)

2 Let bp_k be the price such that, $bp_k < bp_i$ for all bp_i ϵ (bp_i,bqj) ϵ C_1 and $bp_k > bp_j$ for all bp_i ϵ (bp_i,bq_j) ∉C_1.(In other words it is the highest price among the remaining price quantity pairs.) Let bq_k be the maximum quantity that can be assigned at this price.

Then the combination C_2 = {(bp_1,bq_{1j}), …, (bp_n,bq_{i1j}), (bp_k,bq_k)} is optimum for quantity q_k (= $\sum_{i=1}^{k}\sum_{j=1}^{k} bq_{ij}$) and its contribution is $cov_{c2}(q_k)$.

Proof: This result can be easily shown by induction. As already shown the result is true for k = 1 and k = 2. We assume that the result is true for (k-1) and show that it holds for k.

The total contribution to the value of objective function of combination C_2 is

$cov_{c2}(q_k) = cov_{C1}(q_{k-1})+ bp_k bq_k$

The contribution can be improved, if and only if the second term can be improved. This is due to the fact that the first term cannot be improved by assumption. In this case it is not possible to improve the second term since its contribution $bp_k bq_k$ is the highest among the remaining bids (and hence cannot be improved). In the same way, bq_k is the maximum quantity that can be assigned. So the second term cannot be improved. Hence $cov_{c2}(q_k)$ is the maximum improvement in the value of objective function. So the result follows by induction.

This result helps us in determining optimum quantity at any stage. The remaining things remain similar to earlier setup hence they are not repeated here. The algorithm is a general one and can be used for any similar multi unit auction system. After obtaining optimum solution our algorithm finds out the VCG Payoff. Our algorithm is based on the above results and works as follows

(1) We start with the highest (lowest) available price. Then we determine the maximum quantity that is available at that price after taking into account all the constraints. If there are two bids (asks) with same price, then we combine them. The price, quantity combination is added to assignment list. If the quantity allocated exceeds the required quantity we stop.

(2) Then select next highest price. Determine the maximum quantity that can be assigned at this price after taking into account all the constraints. If there are more than one bid which have this price in some price quantity pair, then combine them. If the quantity assigned exceeds the required quantity then stop. Otherwise repeat steps (1) and (2) till either all bids are assigned or the required quantity is obtained.

(3) If all bids are assigned and required demand is not fulfilled, then no optimum assignment for that quantity exists.

(4) If at any stage, in steps in (1) and (2) it is determined that entire quantity can not be assigned to all or nothing bid, then that bid is removed from optimum solution. Ties can be resolved in any predetermined ways.

We hereby state the algorithms for multi unit auction. The constraint formulation algorithm is not stated as it is a straight forward read and replace operation. Let AK be the list of asks and BD be the list of bids. The algorithm is as follows.

Algorithm (2) multiunit /* Main Algorithm */
1.Sort all bids on descending order of price within same price sort on descending order of quantity
2. while (there is an unassigned bid or unfulfilled demand) repeat steps 3 thru 9
3.Find out maximum quantity available for price p_i after verifying capacity constraints on buyers and other group constraints
4. Verify whether optimality condition is satisfied. If verified exit
5. Add the price, quantity pair to the combination.
6. Calculate and save the buyer's contribution
7. Calculate the value of objective function
8 Mark the bid as assigned
9 Add quantity to demand_fulfilled
10. Select the combination with the highest value.
11. Compute VCG Payoff

Figure. 5.4. Algorithm for Multi Unit Auction

The working of our algorithm is shown with the help of a simple example (Table 5.5).

Example: Suppose that five buyers have submitted bids as shown in the first four columns of Table – 5.5. The first two buyers belong to a group, for which there is a group constraint. The fifth buyer has submitted a bid with all or nothing constraint. The group constraint is that only 12 units can be allocated to this group. There are totally 40 units. The output of our algorithm for different quantities is shown in column 6-9 of Table – 5.5. At each stage algorithm combination arrives at optimum solution.

Theorem 7: The algorithm always generates optimum solution for any quantity q if it exists.

Proof: At any stage our algorithm generates optimum solution for quantity q_i. If quantity q_i exceeds the required quantity, then the optimum solution is obtained and we stop. In other case we get an optimum solution for quantity $q_i + q_j$ in next stage. Repeat this till assigned quantity exceeds quantity q. This is the optimum solution. The algorithm will terminate when quantity q is assigned or all combinations are created and no bid is left out. At this stage we have the optimum solution for quantity q_k. If $q_k < q$, the demand is more and supply is less. So no optimum solution for quantity q, from the existing bids exists.

Complexity**:** The complexity of our algorithm is always polynomial. There are n buyers and each buyer has submitted m points. In the best case scenario, a combination can be created by scanning n bids. Each bid has m points. So a combination can be created by scanning nm points. Then each combination can be tested optimality in linear time. So the time complexity of our algorithm will be O(nm). In each stage we scan n bids and one interval. In the worst case, time complexity will be O(nm) or polynomial. Once combination is obtained we need to scan them to obtain the combination with the highest value. This can be done in linear time complexity. Apart from this, sorting is the hardest part. Since sorting time complexity is of the order of O(nmlognm), the time complexity of this algorithm is $O(n^2)$ + O(nmlognm) in worst case, which compares favorably with the time complexity order $O(n^3)$ presented in [KPS2003].

5.7 Determining the Payable Amount

In this section, the amount payable for each buyer or seller is determined. One option is to obtain the VCG Payoff of each buyer. As already stated, VCG mechanism, has a number of desirable properties, like incentive compatibility (IC) and efficiency (EE). Such properties may be desirable in IPO auctions, by means of which government owned enterprises or public resources are privatized.

Table. 5.5 Bids received from Buyers Optimum Assignment

Buyer	Price	Quantity	Cumulative Price	Cumulative Quantity	Buyer	Quantity Assigned	Price	Cost
1	50	5	50	5	5	8	110	110
	40	5	90	10	2	8	100	210
	30	5	120	15	4	8	96	306
	25	5	145	20	3	9	90	396
					5	7	70	466
2	100	8	100	8				
	77	7	177	15				
	60	6	237	21				
	45	5	282	26				
	32	4	314	30				
3	90	9	90	9				
	54	7	144	16				
	21	3	165	19				
4	96	8	96	8				
	60	6	156	14				
	45	5	201	19				
5*	110	8	110	8				
	70	7	180	15				

* All or nothing bid

It may be desirable that these properties are transferred to persons who value them most and truthful bidding is the dominant strategy.

Let Vo be the optimum solution of the mixed integer programming problem formulated in (1). Then VCG Payoff of i^{th} buyer $vcgb_i$ is computed as follows

$$vcgb_i = V_o - V_{oi} \quad i = 1, 2, 3, \ldots, n \qquad (5.37).$$

where V_{oi} is the optimum solution of problem (5.1) obtained after removing i^{th} buyer. The term $vcgb_i$ is referred as VCG discount of the i^{th} buyer. The amount payable (pb_i) by i^{th} buyer is obtained as

$$pb_i = bp_i\, bq_i - vcgb_i \quad i = 1, 2, 3, \ldots, n \qquad (5.38).$$

where bq_i is the quantity purchased by i^{th} buyer and bp_i is the bid price. In the case of reverse auctions (i.e. sellers), VCG discount is obtained in a similar manner and is added to the product of ask amount and quantity sold. Since the nature of problem is symmetric with respect to buyer or seller, in further analysis from now we do not exactly state seller separately.

It can be seen that in order to obtain the amount payable for each buyer and seller, one is required to solve n optimization problems. The steps are

(1) Remove a buyer (seller). There are (n-1) buyers now.
(2) Obtain optimum solution of new optimization problem
(3) Obtain VCG discount using (5.37). Obtain amount payable using (5.38).
(4) Repeat steps (1) to (3) for each buyer.

Due to this computational complexity Vickrey auctions are not widely used. We now state an algorithm, which can obtain VCG Payoff of each buyer (seller) without solving set of optimization problems.

Optimum Solution of new problem: Suppose that the optimum solution of problem (5.1) has been obtained. In order to obtain VCG discount a new optimization problem excluding k^{th} buyer is formulated. It can be seen from theorem (4) that it is always possible to obtain an optimum combination for any quantity q. Suppose that there is a combination C_i. It contains an assignment (bp_k, bq_k) for k^{th} buyer. Let $valc_i(q)$ be the value of the combination. After removing k^{th} buyer the value of combination changes to

$$valc_i(q) - bp_k bq_k.$$

In order to obtain optimum combination of new problem, we go to stage before selecting k^{th} buyer's bid. Suppose that at this stage there are l optimum combinations for different quantities. If any of these combinations have bids from k^{th} buyer go to stage before that. Then optimum combination can be obtained by finding out optimum combination for remaining quantity and then selecting combination with the maximum quantity. The algorithm can be stated as

```
Algorithm  (3) vcgcal
While there is buyer with payoff not calculated  do {
  Remove lth buyer ;
  While there is no new combination {
 Remove lth buyer's bid from combination ;
 Add combination to combination list ;}
 While there is no new combination {
 Obtain optimum combination for remaining quantity };
 Find out optimum combination ;
 Find payoff of buyer ;
 Set payoff calculated to true ; }
```

Figure. 5.5. Algorithm for VCG pay off

This algorithm generates optimum solution for set of optimization problem without actually solving the optimization problem. In this algorithm, initially the optimum combination excluding l^{th} buyer is found out. This combination is then combined with remaining combinations without l^{th} buyer to obtain optimum combinations.

Theorem 8: The algorithm always generates optimum solution if it exists.

Proof: Let C_i be the optimum combination for original optimization problem. Let q be the total quantity of this combination. In this combination at some stage l^{th} buyer's bid is added. Let C_l be the optimum combination before selecting l^{th} buyer's bid. This is an optimum combination for quantity q_l. Let bq_l be the quantity of l^{th} buyer. After adding l^{th} buyer's quantity, the combination is optimum for quantity q_l+bq_l. So we consider all such optimum combinations before selecting l^{th} buyer. These combinations are optimum for respective quantities. These combinations are independent of l^{th} buyer. All these bids are selected because they have higher contribution then l^{th} buyer's bid. Hence they are optimum even for new optimization problem. In the next stage by determining the optimum combinations for remaining quantity and combining it with the combinations before selecting l^{th} buyer's bid the optimum combination for new optimization problem can be obtained. Our algorithm exactly follows this procedure and hence obtains optimum solution for new optimization problem, if it exists.

Suppose that the assignments (bp_1,bq_1), (bp_2, bq_2), …, ($bp_{k,b}q_k$) , replace the assignment (bp_l,bq_l). Then change in the value of a combination due to this replacement is

$$bp_l bq_l - \sum_{i=1}^{k} bp_i bq_i .$$

Complexity of Algorithm: The time complexity of this algorithm is always polynomial but in the best case the solution can be obtained with linear time complexity. It can be seen that in the worst case scenario an optimum solution can be obtained by scanning all created combinations. Then this procedure is repeated for all the buyers. Suppose that before l^{th} buyer is selected there are n_l combinations. The optimum combination for the remaining quantity can always be created with polynomial time complexity. However the time complexity can be improved if combinations before including l^{th} buyer and after l^{th} buyer are stored. At any stage an optimum combination can be created by scanning these combinations.

In case of single object multi unit auctions with constraints, there is a possibility that after deleting a bid or an ask, the resulting problem may not have a feasible solution. If this happens, then it is not possible to obtain an optimum assignment. In this case the amount payable by a buyer or a seller cannot be obtained. As such we generalize Uniform Price Auction mechanism for IPO and multi unit auctions with constraints scenario. As earlier, let BD be the set of all bids. It is assumed that all buyers have their private valuations. Let bv_i be the private valuations of the i^{th} buyer. Suppose that buyer pays bap_i per unit of quantity and acquires bq_i units. The utility of the i^{th} buyer is $bu_i = (bv_i – bap_i)bq_i$..

Generalized Uniform Price Mechanism (GUP): Divide set BD into k mutually exclusive sets Bg_1, Bg_2, …, Bg_k, satisfying following

(a) $BD = Bg_1 \cup Bg_2 \cup Bg_3 \cup \ldots \cup Bg_k$

and $Bg_i \cap Bg_j = \phi$ for all $i \neq j$.

Each set B_{gi}, represents the set of buyers in i^{th} group. It can be seen that a buyer can be in only one of the sets Bg_1, Bg_2, …, Bg_k. A buyer can be an employee, individual or an enterprise. Depending upon the type of buyer he is classified into one of the sets. Let bqg_i be the upper limit on the quantity that can be allocated to group Bg_i. In the next step, the optimum assignment is determined using the earlier algorithm. Let B_{oi} be the set of selected bids from i^{th} group by the algorithm. Then the clearing price is determined. In the straight forward generalization of uniform price auction there are two possibilities

(1) use uniform price for all groups, so all buyers irrespective of group pay the same price

(2) use different clearing price for different groups, but each buyer in the group pays the same price.

Let pc_i be the clearing price for i^{th} group. Let Bl be the set loosing (i.e. the set of bids which are not selected) bids from all groups. Let bp_{ij} be bid price of i^{th} selected buyer in j^{th} group B_{oj}.

Let bpc be the price such that $bp_{ij} \geq bpc$ for all i and j, and for any bid B_m in Bl, with bid price bp_m, such that $bp_m > bpc$, there are at least one i and j , such that $bp_{ij} < bp_m$.

Then in the first case, the clearing price for i^{th} group as

$pc_i = bpc$. We set the same clearing price for all groups.

In the second case, the clearing price for the i^{th} group is set at the highest loosing bids in that group. In our mechanism we set the clearing price for j^{th} group as

$pc_j = bpc$, where bpc is the price of the highest loosing bid in Bl such that,

$bpc \leq bp_{ij}$ for all i.

In other words the clearing price for i^{th} group is set to the price of the highest loosing bid, across the groups, such that it does not exceed bid price of selected bids in that group.

We call our mechanism as generalized uniform price mechanism (GUP). The mechanism for the seller can be similarly defined. Then we show that this mechanism is individually rational, efficient and strategy proof.

Theorem 9: Generalized uniform price (GUP) mechanism satisfies the IR property.

Proof: It can be seen that in GUP, if buyers or sellers are not selected then they do not pay or receive any amount. So they have 0 utility. In case buyers or sellers participate in auction then

(3) The per unit price to be paid by any buyer in i^{th} group is pc_i. This price satisfies the relation $pc_i \leq bp_{ij}$ (if j^{th} buyer's bid is selected in i^{th} group) .

(4) In the same way the seller never receives the per unit amount, which is lesser than per unit amount of ask submitted by him. The per unit price to be received by a seller is always at least as much as his ask price.

It may be noted that per unit price paid by each buyer in auction does not exceed per unit bid amount. In the same way per unit price to be received by each seller in auction is not lower than per unit price submitted. So all the participants have nonnegative gains or utility after auction clears. Due to this, the GUP mechanism is always IR.

Theorem 10: If prices and volumes of the buyers (or sellers) are public information GUP mechanism is strategy proof.

Proof: Suppose that the selected bids are distributed in k groups Bg_1, Bg_2, …, Bg_k. Let pc_i be the clearing price of i^{th} group. As this proof is stated for i^{th} group, the subscript i is omitted from this proof. Let bv_j be the private valuation of the j^{th} buyer and bp_j be the bid price submitted. Let bq_j be the quantity purchased by the j^{th} buyer. Suppose that j^{th} buyer's bid is in i^{th} group, with pc_i as clearing price. The utility bu_j of j^{th} buyer is

$$bu_j = (bv_j - pc_i)\, qb_j.$$

(a) Bidding higher than true valuation: Suppose that buyer's valuation $bv_j \geq pc_i$ and buyer bids bp_j. The buyer's utility is

$$bu_j = (bv_j - pc_i)\, qb_j, \quad \text{if } bp_j \geq pc_i$$
$$= 0 \quad \text{otherwise}$$

It can be seen that buyer cannot improve his utility by bidding $bp_j > bv_j$. In this case the buyer's bid is selected buy his utility does not improve. This happens as the amount payable does not depend upon buyer's bid. In other words his utility by bidding higher remains same as utility, he gets if he bids bv_j.

(b) Bidding lower than true valuation: Suppose that buyer's valuation $bv_j \geq pc_i$ and buyer bids bp_j which satisfies $bp_j < bv_j$. The buyer's utility is

$$bu_j = (bv_j - pc_i)\, qb_j, \quad \text{if } bv_j > bp_j \geq pc_i$$
$$= 0 \quad \text{if } bv_j \geq pc_i > bp_j$$

It buyer bids lower than his true valuation, buyer's bid is selected only if $bp_j \geq pc_i$. In case $bp_j < pc_i$ buyer's bid is not selected. In the first case buyer's utility does not improve by bidding lower amount. In the second case buyer's bid is not selected, even though his valuation is higher than the clearing price. The utility of buyer is 0 in this case. So by bidding lower than his true valuation buyer does not improve his utility and in some cases may have 0 utility.

(c) Suppose that $bv_j < pc_i$, i.e. buyer's valuation is smaller than clearing price. Suppose that buyer bids bp_j. The utility of the buyer is

$$bu_j = (bv_j - pc_i)\, qb_j, \quad \text{if } bp_j \geq pc_i$$
$$= 0 \quad \text{otherwise}$$

In case buyer bids $bp_j > bv_j$, buyer's bid can get selected if $bp_j \geq pc_i$. However in this case buyer has negative utility. In case buyer bids $bp_j < bv_j$, this bid is not selected and he has 0 utility.

Combining arguments in (a), (b) and (c), it can be seen that buyer does not improve his utility by bidding untruthfully.

(d) Further it can be seen that by changing his bid price, buyer's bid will not get selected in different group. This is due to the fact that a bid can be in only one group and he is already included in i^{th} group. So buyer cannot improve his utility by being included in a group, where clearing price is less than pc_i. The buyer's bid is included in the group, depending upon his type. In order to get included in different group, buyer needs to change his type. It is independent of the buyer's price. By bidding lower amount, buyer may not be included in a group with lower clearing price than pc_i. So lower bidding will not improve his utility.

We can extend this argument to seller's case to show that the seller's utility does not improve by reporting higher or lower price than his valuation.

This proves that GUP mechanism is strategy proof. It may be noted that at any point our algorithm selects a bid with the highest contribution for given quantity. The contribution is the highest for any fixed quantity, when the corresponding price is the highest. So in GUP mechanism, the bids of only those buyers are selected, who value them the most. So the mechanism is efficient.

5.8 Misreporting of Volumes

It can be seen that in GUP mechanism, a buyer or seller does not have incentive to misreport his price. A buyer does not have incentive to report higher volumes. However in some cases buyers or sellers can misreport quantities and this can affect the clearing price. For example a buyer can decrease and report his volume. In some cases this can bring down the clearing price for a group. On the other hand, seller can misreport his volumes to improve his utility. In some cases buyers or sellers can misreport their respective volumes to improve utility. Consider the following scenario

Suppose that in j^{th} group, the lowest winning bid is at $i = m_i$. If i^{th} buyer reduces his demand then, $(m_i - 1)^{th}$ or subsequent bid is selected either partly or fully. This happens as demand is reduced. This reduces the clearing price and buyer can improve his utility.

In other words, clearing price has been pushed down. In some cases sellers can misreport the volumes. If seller can misreport volume, it can push up market clearing price. However such manipulations are quite difficult in our set up. We propose the following modification to GUP mechanism to minimize manipulations by buyers and sellers. In order to achieve it, an attempt is made to minimize incentive to misreport volumes. However the scheme requires that there are minimum two buyers or sellers in any group.

Let there be m_i buyers in i^{th} group B_{gi}.

Let q_i be the total quantity sold or purchased in i^{th} group B_{gi}.

Let bq_{ij} be quantity purchased by j^{th} buyer in i^{th} group B_{gi}.

It is easy to verify $\sum_{j=1}^{mi} bq_{ij} = q_i$ for all $i = 1, 2, ..., k$.

Let bq_{oji} be the quantity purchased by all other buyers except j^{th} buyer in group B_{gi}.

So $bqo_{ji} = q_i - bq_{ij}$.

Let $bpi_{(min)}$ be the minimum price of the selected bid prices in the i^{th} group. We also have

$bpi_{(min)} \geq pc_i$ for $i = 1, 2, \ldots, k$.

Let us define $bin_{oi} = (bpi_{(min)} - pc_i)$ for all $i = 1, 2, \ldots, k$

It can be easily verified that $bin_{oi} \geq 0$.

In this case our mechanism works in the same way as GUP, however there is no uniform clearing price for any group. In this case if j^{th} buyer's bid is selected in i^{th} group the per unit payable amount for this buyer is

$$bpua_{ij} = pc_i + \left(\frac{bqo_{ji}}{q_i}\right) bin_{oi} \text{ for } i = 1, 2, ..., k, \ j = 1, 2, ..., m_i \qquad (5.39).$$

So if bq_{ij} is the total quantity purchased by j^{th} buyer in i^{th} group, then the total amount payable is

$bpa_{ji} = bpua_{ji} \, bq_{ij}$.

In the similar way amount receivable by different sellers can be defined.

Let there be n_i sellers in i^{th} group B_{gi}.

Let q_i be the total quantity sold in i^{th} group B_{gi}

Let aq_{ij} be quantity sold by j^{th} seller in i^{th} group B_{gi}.

It is easy to verify $\sum_{j=1}^{mi} aq_{ij}$ = q_i for all i = 1, 2, ..., k.

Let aq_{oji} be the quantity purchased by all other sellers except j^{th} seller in group B_{gi}.

So $aq_{oji} = q_i - aq_{ij.}$

Let $api_{(max)}$ be the maximum price of the selected ask prices in the i^{th} group. We also have

$api_{(max)} \leq pc_i$ for i = 1, 2, …, k

Let us define ain_{oi} = $(pc_i - api_{(max)})$ for all i = 1, 2, …, k

It can be easily verified that $ain_{oi} \geq 0$. As $api_{(max)} \geq pc_{i.}$

Similarly per unit price to be received by j^{th} seller in i^{th} group is

$$srua_{ji} = \left(pc_i - \left(\frac{aqo_{ji}}{q_i}\right) ain_{oi}\right)$$

In this scheme each buyer or seller who participates in the auction pays and receives different per unit price, unlike in GUP. This scheme is referred to as Discriminatory Multi Unit Auction mechanism (DMA).

Theorem 11: DMA mechanism satisfies the IR property.

Proof: It can be seen that in DMA as in GUP, if buyers and sellers are not selected then they do not pay and receive any amount. So they have 0 utility. In case buyers or sellers participate in auction then

In DMA per unit price paid by any buyer in i^{th} group

$$bpua_{ji} = pc_i + \left(\frac{bqo_{ji}}{q_i}\right) bin_{oi} \qquad (5.40).$$

It can be easily verified that $bpua_{ji} \leq bp_i$ for i = 1, 2, ..., m_i.

This follows as

$$\left(\frac{bqo_{ji}}{q_i}\right) < 1 \quad \text{as } q_i = bq_{-ji} + bq_{ij} \text{ (by definition) and } bq_{ij} > 0.$$

Multiplying both sides by bin_{oi} we get

$\left(\frac{bqo_{ji}}{q_i}\right)$ $bin_{oi} \leq bin_{oi}$

Adding pc_i to both sides we get

$pc_i + \left(\frac{bqo_{ji}}{q_i}\right) bin_{oi} \leq pc_i + bin_{oi}.$

Using definition of bin_{oi} we get

$$pc_i + \left(\frac{bqo_{ji}}{q_i}\right) bin_{oi} \leq bpi_{(min)}.$$

It may be noted that j^{th} buyer's bid is selected in i^{th} group if and only if $bp_j \geq bpi_{(min)}$. So the price paid by any buyer is bounded by $bpi_{(min)}$.

This proves the result that all the participating buyers have nonnegative gains from participation.

Exactly similar argument can be extended to show that all sellers have nonnegative gains from participation.

As, per unit price paid by each buyer in auction does not exceed the bid amount and the same to be received by each seller in auction is not lower than the price received, all participants have positive utility or have non negative gains after auction clears. Due to this, DMA mechanism is always IR.

Theorem 12: DMA mechanism satisfies incentive compatibility property. Further if any buyer (seller) decreases (increases) the quantity, he pays (receives) more (less) price respectively per unit. The mechanism is also false name proof and efficient.

Proof: It can be easily verified that the amount paid by any buyer or seller is independent of his bid/ask price. It depends upon the bid prices and ask prices submitted by others. Further, as in GUP mechanism, a buyer or seller cannot improve his gain by bidding higher or lower. The utility of j^{th} buyer in i^{th} group for a single unit of object is

$bu_{ij} = bv_{ij} - pc_i$ for i = 1, .2, …, k , j =1, 2, ..., m_i

If buyer's valuation $bv_{ij} \geq pc_i$, then buyer's utility does not increase by bidding higher amount, as the amount to be paid does not change. He may loose the auction by bidding lower amount. On the other hand if buyer's valuation $bv_{ij} < pc_i$, and if buyer bids higher amount, buyer may win but has $bu_{ij} < 0$. In case he bids lower amount, he has 0 utility. So the mechanism is incentive compatible.

Suppose that j^{th} buyer in i^{th} group requires quantity bq_{ij}. Suppose that instead of bq_{ij} buyer submits his demand as $bnq_{ij} < bq_{ij}$.

Let q_i be the total quantity sold in i^{th} group, if j^{th} buyer submits his true demand. Let qn_i be the demand when buyer submits bnq_{ij}. Then we have the following relations.

$q_i = bqo_{ji} + bq_{ij}$.

$qn_i = bqo_{ji} + bnq_{ij}$, since $bnq_j < bq_{ji}$, so $q_i > qn_i$,

$\frac{1}{q_i} < \frac{1}{qn_i}$, so we get

$$\left(\frac{bqo_{ji}}{q_i}\right) < \left(\frac{bqo_{ji}}{qn_i}\right)$$

This proves the result that if a buyer reduces the quantity required he pays more amount per unit of quantity. This is true even if, another bid is selected on account of demand reduction. This happens as there is no change in q_i and quantity purchased by others increases. It can be seen that in this case

$q_i = qn_i$ and $bnq_{ij} > bq_{ij}$. So the per unit payable amount increases.

Conflict and false name proof bids: It can be further seen that there is a conflict between the different buyers or sellers in the sense that, if any buyer or seller decreases his volume and the volumes of others remain unchanged, then others gain at the cost of this buyer. This conflict occurs in case there is no change in the quantity purchased. This can be easily verified.

Let $d = bq_j - bqn_j$. So for any other buyer say k, the factor $\left(\frac{bqo_{ki}}{q_i}\right)$, changes to $\left(\frac{bqo_{ji} - d}{q_i - d}\right)$. It may be noted that $\left(\frac{bqo_{ji} - d}{q_i - d}\right) \leq \left(\frac{bqo_{ki}}{q_i}\right)$.

So all other participants gain. So buyer pays more by decreasing his demand and at the same time other buyers gain because they pay lesser price. Due to this conflict no buyer or seller has incentive to reduce his volume. In some cases there may be another bid, which gets selected due to demand reduction. Suppose that there is no change in total quantity q_i. So the factor $\left(\frac{bqo_{ki}}{q_i}\right)$ remains unchanged for k^{th} buyer ($k \neq j$). Due to this utilities of others remain unaffected.

In electronic auctions bids are submitted remotely. It is possible that buyers may submit false bids. Such false bids are bids submitted under different identification just to improve utility. Such bids are submitted under fictitious names [YSM2000]. It can be seen that in DMA mechanism a buyer does not have incentive to submit false name bids. Suppose that k^{th} buyer submits false name bids, on account of these bids, there is increase in demand say by d units.

In case these bids are not selected there is no effect on clearing price. Suppose these bids get selected, then k^{th} buyer pays more price. This is due to the fact that the factor $\left(\frac{bqo_{ki}}{q_i}\right)$, changes to $\left(\frac{bq_{-ji}+d}{q_i+d}\right)$. It may be noted that $\left(\frac{bq_{-ji}+d}{q_i+d}\right) \geq \left(\frac{bq_{-ki}}{q_i}\right)$. In fact by submitting false name bids a buyer may increase the per unit price to be paid by others in some cases. So DMA does not have incentive for submission of false name bids. It can be seen that at the same time utility of some others remain unaffected. There are other possibilities like the quantity allocated to j^{th} buyer decreases, whereas there is no change in other allocation. Suppose that false name bid is selected and quantity q_f is allocated to it. However total quantity q_i remains unchanged. In other words false name bid replaces a bid selected earlier either fully or partly. In any case this does not affect total volume of other buyers but can decrease the quantity allocated to j^{th} buyer. The quantity allocated to j^{th} buyer does not increase. In some cases it may decrease. Due to this buyer cannot improve his utility. The bids selected in this mechanism are same as in GUP mechanism. So by following similar argument as in GUP mechanism it can be proved that this mechanism is efficient.

Exactly similar argument can be followed to show that this mechanism is incentive compatible for sellers and seller receives lesser price per unit in case he reports higher quantity.

The percentage gain for a buyer (as his demand decreases) was worked out from 100 randomly generated data sets. Each data set was randomly generated and it consisted of bid prices, quantity and values for other attributes. The effect on percentage gain as demand decreases can be seen in figure 5.6. At the same time effect of false name bids on his percentage gain can be seen in figure 5.7.

5.9 Conclusion

In this chapter we have formulated the problem of IPO as single object multi unit auctions with different types of constraints. Our formulation is a general one and some of the multi unit auction formulations studied earlier are particular cases of our formulation. Then we develop an algorithm to obtain optimum solution. After obtaining optimum solution, an algorithm to obtain VCG Payoff, without solving set of optimization problem, has been presented. Then we propose a design of strategy proof mechanism GUP for single object multi unit auctions with different types of constraints. It is shown that our mechanism is strategy proof, individual rational and efficient. Then we propose design of a mechanism DMA, which is efficient, strategy proof and individually rational. These properties ensure that truthful bidding is the dominant strategy, which is a very important property in electronic auctions. We further show that our mechanism is false name proof and reduces incentive for buyer to report decrease in volume.

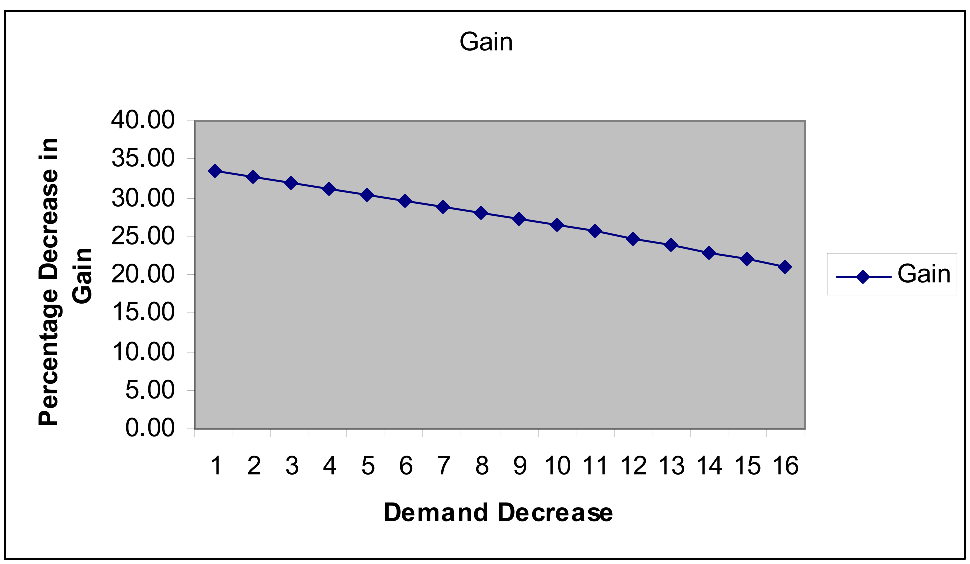


Figure. 5.6- Effect on gain due to change in demand. It indicates how gain of buyer decreases in case he decreases his volume

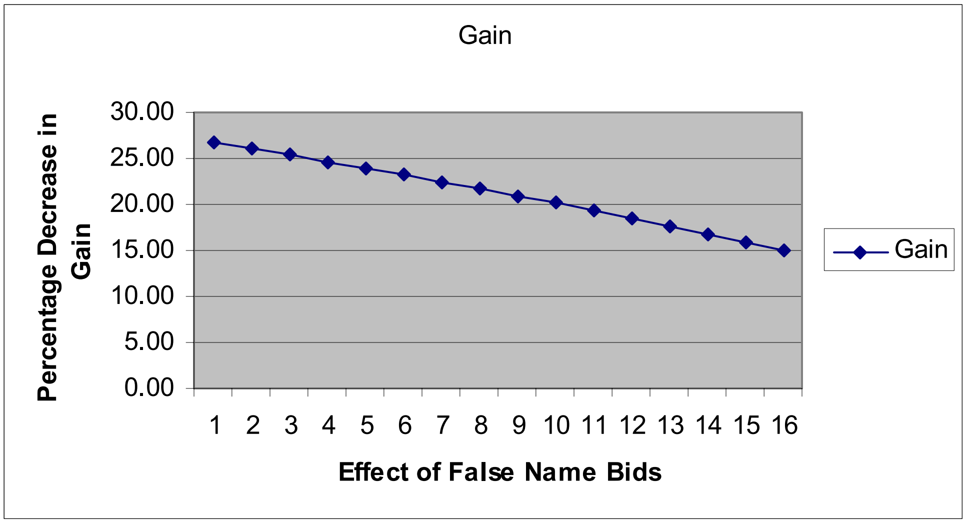


Figure. 5.7 – Effect of False Name Bids. How the gain of buyer decrease by submitting false name bids

Chapter – 6

Financial Applications of Auctions with Constraints in Banks

Introduction: In simple terms Basel II, is a term widely referred to the framework for risk management measures, which banks are required to adopt in their operations. Adoption of Basel II framework requires that banks must identify different types of risks which they are exposed to and provide adequate risk capital to cover unexpected and expected losses. It also requires that these practices should be adopted continuously. In order to adopt these practices banks must identify different types of risks, determine adequate risk capital and allocate risk capital appropriately. In earlier Basel I setup, allocation of risk capital was done on a fiat or an order. Banks were asked by the banking supervisor to provide certain percentage of capital (between 9% to 12% as risk capital). However in the new set up risk capital allocation has to be done by banks themselves. Further, the process for risk management has to be validated by the banking supervisor. In this section it has been shown that how the bidding mechanisms designed in chapters 3,4 and 5 can help banks to improve the process of estimation and allocation of risk capital.

This has given rise to two very fundamental problems (1) what should be the adequate risk capital for any banking organization and (2) how the available risk capital should be allocated. In this work the main focus is on the problem of allocation of risk capital. One of the most fundamental questions in banking organizations concerns the allocation of available risk capital within the organization in any one period. A banking organization is a multi tier structure. It has a number of different departments. These departments are divided into operational and controlling or head office departments. An operational department of the bank operates through number of branches or delivery channels.

The head office departments are engaged in planning and management or control activities. These activities are carried out centrally, keeping the entire organization in view. The work of allocation of risk capital is also carried out centrally. The set of persons carrying out these activities will be referred to as central planners.

This gives rise to the problem of ensuring that risk capital is going to its most efficient uses within the bank? This problem arises because allocation exercise is carried out centrally keeping in view entire organization. The persons carrying out this exercise may not have complete knowledge about the risks, ground realities and opportunities. The operational persons working in branches or concerned departments have hands on and practical knowledge of ground realities. However, these operational persons have little incentive to reveal this information when it is used for planning and control purposes. In fact these managers may be encouraged to understate risk capital requirements per rupee of business activity in order to improve expected risk-adjusted returns. It may be noted that the risk capital is fundamental to the financial health of banks and financial institutions. The departmental Managers know that they cannot expand their departments without at least some capital to support the department against unexpected losses, yet they are tempted to understate their total risk capital requirements per rupee of business in order to improve expected or actual returns. In case of operational departments, higher expected returns presents a higher probability of receiving the financial resources to expand the departments. As capital cannot be priced in the same way as other resources, it cannot be allocated on the basis of "who wants the most". It is a well known fact that risk capital requirements of Rs. 100 million residential mortgage portfolio and a commercial loan portfolio of the same amount are different. The former, being a portfolio with lesser risk may require less risk capital. However it may still require more risk capital than the activity of Rs. 100 million worth of interest rate swaps. The planners know that the departmental managers have incentives to misrepresent their information in order to receive a more favorable allocation of resources. The decision problem of the central planners is to decide how much capital should be allocated and the price of that allocation.

In fact if the total amount of risk capital C is known. Let X_i be the risk capital to be allocated to i^{th} department. Let R_i be the risk factor associated with i^{th} department. Then allocation problem can be formulated as a linear programming problem as follows.

$$\text{Max} \quad \sum_{i=1}^{n} R_i X_i$$

$$\text{subject to} \quad \sum_{i=1}^{n} x_i = C$$

However the main problem here is to estimate R_i.

Banks and other financial institutions determine the allocation of risk capital to different departments based on a value-at-risk approach. In economics and finance, value at risk **(VAR)** is a measure (a number) saying how the market value of an asset or of a portfolio of assets is likely to decrease over a certain time period (usually over 1 day or 10 days) under usual conditions. It is typically used by security houses or investment banks to measure the market risks of their asset portfolios **(**market value at risk**),** but is actually a very general concept that has broad application. In this approach maximum potential loss for the given activity over specific time period is estimated. This approach uses historical data from expected return on different activities. The magnitude of the worst-case losses can be estimated by this approach. So banks can compare the performance of different activities like lending, fund raising, underwriting and derivatives trading. These diverse activities can be compared, as VAR approach provides common basis for measuring the maximum potential loss. So the risk capital to be allocated to these activities can be estimated and compared. The value-at-risk approach is an objective basis for determining risk capital allocation. It does not require a specific assessment of risk by different persons, which may be subjective and a misrepresentation of what they believe or know. However value-at-risk approaches for determining risk capital depend upon historical data. So they provide an accurate assessment of risk only to the extent that historical price or earnings volatility is repeated in the planning period. These methodologies are based on statistical analysis of past prices or earnings movements. Using historical data expected ranges over which prices or earnings movements are likely to occur are worked out. In many cases historical price/earnings volatility may not be an accurate predictor of future price or earnings volatility. Due to this, resource allocation based on value at risk approach may result in a significant misallocation of capital within the organization in some cases. If historical volatility in earnings or prices proves not to be an accurate representation of actual risks, some departments will have lower risk capital than required, while others will have more than required. This can result in two potential problems as follows.

(1) In such cases it may not be always possible to transfer risk capital from department with lower risk capital allocation to department with higher allocation. In the event of unexpected loss a department may not have adequate capital exposing it to potentially high losses.

(2) The managers of departments with excess risk capital are likely to take up high risk activities to earn acceptable returns on allocated capital.

As a result the bank is exposed to higher level of risk. The main problem here is how to use the information available with the operating staff, in order to improve estimates of R_i. However the operating staff has no incentive to reveal the information to others. In order to overcome this problem, dynamic bidding based approach has been proposed. In this approach the relevant information is sought from operational people in the form of bids. The mechanism followed ensures that the operational persons have incentive to reveal the true information. This information can be useful to banks in many ways. In the first case it can be used by banks to improve the estimates obtained by VAR techniques, by following other approaches like Bayesian inference techniques (prior and posterior data). Alternatively banks can use it to allocate a part of risk capital by bidding based mechanism and remaining part by other techniques. In this work the main focus is on bidding process, as such the VAR techniques and Bayesian approaches are not discussed any more. It is assumed that the bidding mechanism is used allocate surplus or a part of total risk capital.

In a bidding based approach different departments of bank and persons attached to them can periodically compete for part of surplus risk capital. This mechanism can achieve its objectives, if persons submitting bids have incentives to truthfully reveal their correct risk assessment. The format that is employed is auctions with different types of constraints. The constraints are introduced to ensure that allocation process follows directions of supervisory bodies. We employ single object multi unit auctions as well as double auctions with constraints. It is proposed to follow uniform price auction mechanism. As already stated in this format bidders have incentive to misreport volumes. One problem with bidding mechanism approach is the subjectivity in the process due to dependence on managers. In the second stage it is proposed that double auction based format with different types of constraints whereby bids are invited from managers (buyer side) and an independent assessment based value at risk approach (seller side) may be used to allocate risk capital.

Auction mechanisms are used when sellers have uncertain information regarding the market value of the goods. In the same way banking persons involved with the exercise of allocating risk capital have uncertain information. Due to this, dynamic bidding process under which departments periodically compete for surplus risk capital has been proposed. As already stated this mechanism compliments existing approaches and helps in improving obtained estimates by incorporating the information available. The persons submitting bids must have incentives to reveal their expectations regarding volatility in prices or earnings (so that the appropriate level of risk capital can be allocated to their activities) and expected returns on allocated capital, per unit of risk.

In this bidding approach, we the have following basic assumptions.

1. All bidders have independent private values. Each bidder has his private valuation about risk associated with different activities. He uses it for bid submission.

2. The valuations of bidders are drawn from independent and identically distributed random variables. If any two bidders have the same valuations then they will submit the same bids. Bank is fair to all the bidders.

3.Banks may set the constraints and bidders are in general not aware about the constraints. Further some activities or positions may offer considerable diversification benefits to the bank and may be preferred ahead of other higher bids. These decisions are based on policies and external constraints of the organization.

4. Bidders are risk neutral. Bidders are assumed to view gains and losses symmetrically, with their only interest being to maximize expected the payout.

5. The risk capital available for allocation is fully divisible.

Operational assumptions: Further it is assumed that the bank has existing mechanism in place to verify the utilization of allocated risk capital. In other words the department, which allocates risk capital is able to assess with accuracy the actual volatility of prices or earnings for a given activity, position or departments over the predetermined time horizon. It is also able to verify whether the activities or positions are within the risk tolerance level set by the bank. It is possible to verify the net income of the department. In other words bidders cannot misreport earnings and utilization of capital. The different departments are insulated against position or risks that are beyond their control – for example, corporate lending departments are insulated from interest rate risk and funding risk but not credit risk. The system also ensures that the risk capital is linked to controllable risks within the unit. In bank there is a system in place, which insulates managers of departments or positions against different categories of risks. In case of corporate lending unit, unexpected losses derive from bad debts. In the case of under-provisioned losses, total income to the unit would be lower than expected and overall earnings volatility larger.

If the auction mechanism is to be effective it must elicit truthful revelation of risk capital requirements from managers and achieve an efficient allocation (capital is allocated to those activities that are expected to generate the highest risk-adjusted returns). The main considerations in the design of bidding mechanism are the valuation of risk capital, the term of risk capital and the minimum unit bid size.

Application of Multi Unit Auctions with Constraints: There are number of auction formats, which can be used for proposed bidding mechanism. One format that can be considered is the sealed bid discriminatory auction format. In this type of auction the highest bidders win and pay the prices they bid. This format is inappropriate for excess risk capital allocation process for two reasons. This type of auction provides incentives for participants to submit bids below their true valuations in order to try to gain the object at a lower price. There is a possibility that the participant may lose the auction. There are also incentives to learn what competitors are planning to bid. This is one of the reasons for the US Treasury experimenting with uniform-price sealed-bid auctions over the multiple-price sealed-bid auction (discriminatory auction). This format is not suitable for risk capital allocation. In this exercise, it is required to place a 'value' on risk capital and encourage truthful revelation of risk capital requirements, whereas this format does not do that exactly. Another format that can be employed is ascending bid auction. This format provides participants with information (price discovery) through the process of bidding. The sealed bid format does not provide this information to the bidders. However both formats provide truthful revelation of bidder's valuations in a private value setting.

In both formats the price paid for unit is independent of a bidder's individual bids. The sealed bid format has a slight advantage to the extent that it avoids the need to bring parties together. It has been argued that a sealed bid format may not be desirable because the incentive to reveal true value is lost if this information is relevant to subsequent transactions. This is the case in this exercise. The ascending bid format reveals only that the minimum winning price. It does not reveal other information. However this problem can be overcome in the sealed bid case if the winning bids are not made public across the bank.

The auction format that is proposed is Single Object Multi Unit Auction (considered earlier in chapter 5). The format can be used with or without constraints. A bank can impose group constraints like total risk capital allocation to all credit departments should not exceed certain fixed amount. Then the bids can be collected. Our algorithm can work out the optimum assignment. Then Vickrey payment rule can be used to determine the estimate of risk capital. This format has properties of efficiency and truthful bidding. It ensures that truthful bidding is the dominant strategy. One important characteristic of the Vickrey auction is the discount that bidder gets. The amount paid is independent of his bid. In these auctions the equilibrium strategy for participants is to bid true cost or value. This equilibrium strategy is also a dominant strategy because it is optimal to reveal the truth even if a bidder assigns a positive probability to the possibility that other bidders in the auction will deviate from their equilibrium strategies. The auction also leads to an efficient allocation because the bidders with the highest values always win. These are the desired outcomes of an auction mechanism for allocating risk capital within a banking organization. It is an essential requirement that the mechanism induce managers to truthfully reveal their beliefs or knowledge regarding the level of risk capital necessary to support their proposed activities. It is also a requirement that central planners are able to use this information to allocate part of risk capital to those activities that are able to earn the best returns per unit of risk. So it follows that this bidding process reveals expected risk capital requirements. However in case there are different types of constraints it may not be possible to obtain Vickrey payoff in some cases. In such cases it is proposed to use Uniform Price Auction. It has been shown by Vickrey that the uniform price auction is not an efficient allocation when bidders desire more than one unit of the commodity because of demand reduction incentives. However this can be overcome to some extent by introducing discriminatory mechanism discussed in chapters 4 and 5. Further double auction mechanism can also be used inviting bids from others.

Bid Format: The bid consists of department identification, required number of risk capital units and the expected return per unit of risk capital bid. A bidder can submit a single pair or multiple risk capital units and associated expected return per unit of bid capital. Let there be n bidders. Let RC be the total risk capital to be allocated. Let RC_b be the total bid capital to be allocated by bidding process and RC_a be the capital allocated by other approach. Then we have the relation $RC = RC_b + RC_a$.. Let RC_{bi} be the risk capital to be allocated to i^{th} department by bidding process. So RC_b can be written as

$$RC_b = \sum_{i=}^{n} RC_{bi}$$

Let RC_{ui} be the risk capital units required by i^{th} bidder. Let ER_{ui} be the expected returns. So the bid is of the type $(ER_{ui,}RC_{ui})$ i = 1, 2, …, n. A bidder is also allowed to submit range of risk capital units and expected returns corresponding to it. A bidder can submit m such pairs. So the bids can be of the form $\{(ER_{ui1,}RC_{ui1}), (ER_{ui2,}RC_{ui2}), \ldots, (ER_{uim,}RC_{uim})\}$. Alternatively it can be of the form $\{(ER_{ui1,} (RC_{ui0,}RC_{ui1}]), (ER_{ui2,} (RC_{ui1,} ,RC_{ui2}]), \ldots, (ER_{uim,} (RC_{uim-1,} RC_{uim}])\}$. The optimization problem in this format is as defined in chapter 5. We just state it here for reference.

Let us define

$$x_{ij} = \begin{cases} 1 \text{ if } j^{th} \text{ expected returns interval is selected for i th bid} \\ 0 \text{ otherwise} \end{cases}$$

, i = 1, 2, …, n and j = 1, 2, ..., m

If k^{th} bidder specifies only expected returns, risk capital units required, pair then m = 1 for i = k.

Let RA_{ij} be the risk capital unit allocated to i^{th} bidder at expected returns ER_{uij}. Then we define our optimization problem as follows

$$\max \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} RA_{ij} ER_{uij}$$

Total risk capital = $\sum_{i=1}^{n}\sum_{j=1}^{m} RA_{ij} \leq R_b$, i = 1, 2, ..,n , j = 1, 2, …, m

Group Constraint: Let A be the set of departments, such that there is a limitation on total risk capital that can be allocated to them. There can be more than one such set. Let RG_a be the total risk capital, then we have

$\sum_{i} \sum_{j=1}^{m} RA_{ij} \leq RG_a$, where i ε A for all such groups.

The other constraints are

$\sum_{j=1}^{m} x_{ij} \leq 1$ for all i and all RA_{ij} are non negative integers.

In double auction formats two set of bids are submitted for the same department by independent persons. The bid format remains the same; however, the competing bid is submitted by another person or can be obtained from VAR approach. The bid by departmental manager is treated as buyer's bid, while that from another source is treated as seller's ask. Then the remaining formulation can be carried out as stated in chapter 3. As such this formulation is not stated here.

Adjustments: The bids are subject to certain adjustments. These adjustments are described in the next few sentences. The expected return here is basically risk adjusted return described later. The risk capital unit is treated as quantity and expected return as price. The risk capital is divided into common size bid units so that a bid consisting of pairs of number of units required and the yield per unit can be submitted (Type 2 bid described earlier). The appropriate size of units is related to the size of the department, the nature of its business, and/or whether capital is allocated on a transaction or portfolio level basis or a departmental level. If the latter is adopted, the minimum bid size may be, say, Rs. 10 million in risk capital. The larger the minimum bid unit, the more difficult to track risk capital allocations to actual positions, and this would work against the key assumption that risk capital utilization is verifiable after allocation. The bid information refers to specific time horizon. The risk capital is viewed as an insurance against unexpected losses. So it can be considered to be allocated for the residual life or maturity of the transaction or portfolio. Such a risk capital should absorb all unexpected losses over the life of the activity or portfolio. In general this means if activity or portfolio duration is longer, then the amount of risk capital has to be higher. The volatility of cumulative default rates is expected to be higher, when the time horizon is longer. This means, as time progresses, it may be necessary to incorporate a pro-rata reduction in risk capital. At the time of bidding one has to adjust risk capital suitably, taking into account the remaining time horizon. One has to ensure that with longer term, activities or portfolio require more risk capital. This does not mean that frequency of performance measurement depends upon the term of the capital – quarterly or monthly (or daily for trading activities) in order to assess earnings volatility. It may be as per the policies of the bank. The risk capital is viewed as insurance against that fraction of risk that is unavoidable only in the short-run. Rather than viewing capital as an insurance policy subscribed at origination and remaining linked with the facility until maturity, this approach sees risk and capital management as ongoing and proactive processes. The short-term focus of capital arises because corrective action can be taken to reduce risks or limit losses whenever risks increase (for example, an increase in earnings variability can be addressed with credit swaps or asset sales/securitizations). Further, capital can be raised to hedge unexpected risks as they appear. This approach would have the term of risk capital at one or two years and be more aligned with the frequency of performance measurement. Once problem formulation is done as above, developed algorithms are used to obtain optimum allocation. Then the risk capital to be allocated is obtained based on VCG mechanism or discriminatory mechanism proposed earlier. Then the capital to be allocated by bidding mechanism is worked out. A simple example is shown in the next section.

The risk capital (and funding for banking book positions) is allocated against those bids presenting the highest expected risk-adjusted returns, within constraints set by the overall operational plan and constraints of the bank. For example, a high-yielding loan portfolio may not be allocated funds because the bank is overexposed to this industry or customer segment, that is, the bank has a concentration of credit risk in this area. This mechanism guarantees a truth-revealing strategy for risk capital and expected return on the part of bidders.

This is due to the relation between risk capital and risk-adjusted return, expressed in the following simple identity:

risk-adjusted return = earnings / risk capital.

It is assumed that the bank can verify expected earnings. If a manager expects before-tax earnings of Rs.1.4 million from a given activity and estimates the risk capital requirement for the position to be Rs. 10 million, the appropriate bid is a yield of 14% for risk capital of Rs. 10 million. The payment rule in our auction format ensures that it is non-optimal for the department's manager to understate the risk capital requirement (say to Rs. 9.5 million) and overstate the expected risk adjusted return (in this case the yield increases to 14.74%) in order to improve the probability of winning resources in the auction. Before proving this it is necessary to incorporate the bank's compensation mechanism into the auction design. Recall that under the Vickrey payment rule each successful bidder is offered a rebate that is just high enough to remove the incentive for bidders to misrepresent their respective valuations. The amount of this rebate corresponds to the difference in the marginal valuation curve of the successful bidder and the marginal valuation curve of the highest-losing bidder if they had instead been allocated the commodity under auction. In the bank case, department managers derive value from risk capital to the extent that risk-adjusted returns feed into the compensation mechanism. That is, the 'rebate' paid to managers must be a function of the actual return generated, the actual risk capital absorbed and the highest rejected yield lodged by another bidder in the auction. This can be expressed as follows:

$P_i = \Upsilon [Ka_i (r_i - c_i)]$

where P_i is the compensation payment associated with a given position i, Υ is a fixed coefficient, Ka_i is the actual level of risk capital utilized by position i, r_i is the actual risk-adjusted return on capital on position i, and c_i is the yield bid by a losing competitor at which ka_i was secured. In next section an example of our approach is presented.

Suppose a manager estimates a risk capital requirement of Rs. 10 million (non-verifiable ex-ante) to support a Rs. 100 million portfolio. The verifiable expected earnings on the portfolio are Rs. 1.4 million, resulting in an expected risk-adjusted return on capital, before tax, of 14 %. The manager is required to bid for risk capital under a sealed-bid Vickrey multiple-unit auction. Compensation will be paid to the departments based on the compensation formula expressed above, with the coefficient $\Upsilon = 1$. The manager will submit a demand curve detailing the volume of capital required and the expected return on that capital. The manager has three bidding options:

1. Understate risk capital and overstate the expected return. Here the manager bids for Rs. 9.5 million in capital at a yield of 14.74%. The higher yield increases the probability of winning the auction and securing the required funding.

2. Bid truthfully on risk capital and expected return. Here the manager bids for Rs. 10 million in capital at a yield of 14%.

3. Overstate risk capital and understate expected return. Here the manager bids for Rs. 10.5 million in capital at a yield of 13.34 %. The shading of the bid increases the potential gain should the funds be secured but reduces the probability of winning the auction.

Suppose the manager estimates that competitive bidders may bid yields of 15%, 14.5%, 14%, 13.5%, 13% and 12.50% with equal probability. What is the optimal bidding strategy for the manager?

Table 6.1 shows the expected payoff to the manager/departments if the first option is taken and a bid of 14.74% is placed (expected yield is overstated by understating risk capital). Only a competitor bid of 15% will result in this auction being lost. However if a competitor bids 14.5%, the manager will win the auction but incur a negative rebate given the actual yield realizable is 14%. The rebate is – Rs. 0.1 million, calculated using the compensation formula:

$P_i = \Upsilon\, [\, Ka_i\, (\, r_i - c_i\,)\,] = 1$ [RS. 10m (0.28 – 0.29)] = -RS. 0.1m

Table. 6.1 Understate Risk Capital – Overstate Expected Yield

Bid	Competitor Bid	Auction Result	Actual Requirements	Spread	Payoff in Rs.
14.74%	15.0 %	Lose	14.0%		
14.74%	14.5%	Win	14.0%	-1.00%	-0.1
14.74%	14.0%	Win	14.0%	0.00%	
14.74%	13.5%	Win	14.0%	1.00%	0.1
14.74%	13.0%	Win	14.0%	2.00%	0.2
14.74%	12.5%	Win	14.0%	3.0%	0.3
Expected Payoff					0.083

For any competitor bids below 14%, the manager receives a positive rebate as shown in table 1. The total expected payoff from this strategy is Rs. 83,000, and this payoff decreases for higher bids.

Table 6.2 shows the expected payoff if the second option is taken and the manager bids on the basis of true expected return (28%). Despite the lower probability of winning the auction relative to option one, the expected payoff is larger at Rs. 100,000.

Table. 6.2: Truthful Bid on Risk Capital and Expected Yield

Bid	Competitor Bid	Auction Result	Actual Requirements	Spread	Payoff in Rs.
14.0%	15.0 %	Lose	14.0%		
14.0%	14.5%	Lose	14.0%		
14.0%	14.0%	Tie	14.0%	0.00%	
14.0%	13.5%	Win	14.0%	1.00%	0.1
14.0%	13.0%	Win	14.0%	2.00%	0.2
4.0%	12.5%	Win	14.0%	3.0%	0.3
Expected Payoff					0.1

Table 6.3 shows that the third option involving bid shading does not increase the expected payoff. At a bid yield of 13.34% the expected payoff is Rs. 83,000.

Table 6.3: Overstate Risk Capital – Understate Expected Yield

Bid	Competitor Bid	Auction Result	Actual Requirements	Spread	Payoff in Rs.
13.34%	15.0%	Lose	14.0%		
13.34%	14.5%	Lose	14.0%		
13.34%	14.0%	Lose	14.0%	0.00%	
13.34%	13.5%	Lose	14.0%	1.00%	0.1
13.34%	13.0%	Win	14.0%	2.00%	0.2
13.34%	12.5%	Win	14.0%	3.0%	0.3
Expected Payoff					0.1

In fact the best outcome that can be achieved is an expected payoff of Rs. 100,000, which matches that of the truth-revealing strategy but embodies a lower probability of winning the auction.

The dominant bidding strategy is to bid truthfully. All three strategies result in a maximum payoff equivalent to $(r_i - c_i)$. However the strategy of understating risk capital in order to improve expected return is dominated because it exposes the bidder to a negative rebate should the highest losing bid exceed the expected return of the bid in question (that is, $c_i > r_i$). The strategy of bid shading is also dominated because it reduces the probability of winning the auction without increasing the size of the rebate that would be secured should the auction be won. Consequently misrepresenting risk capital and expected return on risk capital is a non-optimal strategy. The auction mechanism is also efficient allocation because risk capital is allocated to those departments that value it the greatest; meaning those units that can generate the highest risk-adjusted returns on the bank's capital.

Limitations: In this approach a part of risk capital is valued by departments managers because their compensation structure is directly influenced by the actual capital utilized and the realized return on risk capital. Managers are offered a rebate based on the difference between the realized yield per unit of risk capital and the opportunity cost of assigning that risk capital to the departments. The auction mechanism induces truthful revelation of information because the size of the rebate is independent of the manager's bid on expected yield. A truthful bid on yield corresponds to a truthful bid on risk capital because expected earnings can be verified at a later stage. There are certain limitations to this analysis. First it was assumed that managers were risk-neutral (that is, winning and losing the auction are viewed symmetrically). Risk averse managers will place excessive weight on the fear of losing the auction. Risk-averse bidders would thus be inclined to submit excessively high bids to mitigate the likelihood of not acquiring capital in the auction. Under the auction design developed, there is a positive probability that these managers would incur negative rebates through the compensation mechanism. As a consequence, there is the potential for post-contractual moral hazard if these managers subsequently take on extra risks in order to increase actual returns and subvert the potential negative rebate. These managers may also be inclined to misrepresent the earnings of their departments in order to influence the measurement of risk capital utilized. The consequences of risk-aversion on the part of department managers and the design/modification of the allocation mechanism to deal with moral hazard are the subject of further research by the author. The allocation mechanism also relied on the assumption that risk capital utilization was verifiable. The standard deviation of earnings or prices may be readily measured, but a complication arises to the extent that the appropriate level of risk capital is not just some multiple of earnings or price volatility. The earnings-at-risk represents the amount of expected earnings that management is willing to forego in any one period without causing a change in business plans. That is, the role of capital is to provide a buffer against future unexpected losses while still leaving the bank or departments able to operate at the same level of capacity. Unexpected losses in any period absorb risk capital. A bank in this situation would need to scale down operations in order to ensure that the remaining level of risk capital is appropriate to support its high risk activities. Alternatively the bank could raise more capital to replace that absorbed by losses, but it is doubtful that this could be raised on favorable terms following a period of large un-provisioned losses. The implications are that while it may be possible to measure earnings or price volatility, it may not always be possible for the centre to measure accurately the adequacy of risk capital held by a department over a certain period. However constraints can be imposed so that no department is missed out. Alternatively only part of risk capital can be allocated through bidding mechanism. A mortgage insurance business line, for example, may not generate high returns relative to other businesses when measured on a stand-alone basis. The business may, however, provide a gateway to new mortgage business or provide diversification benefits across the organization. A problem arises to the extent that the managers of such business lines can only bid in the auction for risk capital on the basis of expected yields on the stand-alone business. In fact line managers could not be expected to know that their businesses provide ancillary benefits to the organization, as this information is typically only observable at the centre. If expected returns on the stand-alone business are low (or possibly lower than the internal hurdle rate) these managers may face a disincentive to bid for capital in the current period. At the same time the centre is likely to be aware of these benefits and may encourage bids by signaling an intention to gross-up yields for these businesses to incorporate these expected benefits. However incorporating these adjustments into the compensation mechanism may be problematic and threaten the credibility of the system if other managers perceive this as interference or favoritism from the centre. A final consideration is the frequency of auctions. The risk capital allocation process must be sufficiently flexible to ensure that departments have the risk capital necessary to support their day-to-day operations. Without this flexibility it is conceivable that a department would be placed in a situation where it had to reject important business acquisition opportunities because it could not be guaranteed of securing risk capital to support the business in the next ‘scheduled’ auction. From this perspective, the auction mechanism for allocating risk capital should perhaps only be considered when there are large competing demands for capital at the beginning of the planning period. While the mechanism has attractive truth-revealing properties and is proved to be efficient allocation in a theoretical setting, too rigid an application may result in costs that outweigh benefits.

Further Work: An important limitation of this work is that the bids from manager are subjective. On the other hand estimates of value at risk are objective. This aspect of minimizing effect of subjectivity can be an important future work.

Chapter 7

Conclusion and Future Work

In this work the electronic auctions with different types of constraints have been studied. Two basic problems addressed in the work are

(1) determining optimum assignment of bids and ask when there are different types of constraints

(2) determining how much each participant has to pay or receive,

in the context of electronic auctions with different types of constraints. We have developed set of algorithms to obtain optimum assignment of bids/asks. Further payment mechanisms with desirable properties have been developed.

In chapter 3, we studied Double Auctions under different types of constraints. Double Auctions provide an efficient mechanism to implement bidding based many to many negotiations. In these auctions, sellers and buyers submit asks and bids respectively, which are matched and cleared periodically. The matching problems in case of indivisible demand have been formulated as generalized assignment or multiple knapsack problems, which require solution of NP-Hard optimization problems. In our work we have investigated the problem of matching under different types of constraints. The main contributions and advantages are as follows.

(1) We formulated this problem as integer programming problem and develop set of new results, which form the basis of our algorithm.

(2) We developed two algorithms to obtain optimum assignments of bids and asks in case of different types of constraints like assignment and indivisibility constraints. Our algorithms can also handle usual Double Auction formulation.

(3) We showed that our algorithms always obtain optimum assignment with polynomial. time complexity. Our algorithms can easily be implemented.

(4) We have developed algorithm to compute VCG Payoff from the optimum solution without solving set of optimization problems.

(5) Our algorithms can be applied in case there are no constraints and can obtain VCG Payoff without solving set of optimization problems.

(6) We have designed an efficient payment mechanism. It is also budget balanced and individually rational. It has been shown that group of buyers cannot affect our system by changing their respective bid price. The gains of the buyers are always bounded.

The future work includes handling the cases where attributes have significant influence on price. The impact of changes in prices by buyers and sellers, especially in case number buyers or sellers are large can also be studied. Future work also includes extending this work to cover Combinatorial Double Auctions.

In chapter 4, we studied truthful double auctions under different types of constraints. In electronic auctions, property of incentive compatibility can be very important. Incentive compatibility ensures that truthful bidding is the dominant strategy. Other important properties in electronic auctions are (BB) and (IR). The former ensures that auction does not run in loss, whereas later ensures voluntary participation. However these can be achieved only after sacrificing efficiency. In this chapter we considered double auctions under different types of constraints, where truthful bidding is the dominant strategy. We designed a mechanism which strategy proof, individually rational and budget balanced. In this case a part of bid and ask is public information, remaining part is private. The main contributions in chapter are as follows.

(1) We generalized multi unit double auction (MDA) mechanism to handle different types of constraints. It was shown that mechanism is budget balance, strategy proof and individually rational.

(2) The bounds on efficiency loss were also been established. We further showed that efficiency loss tends to 0 as number of buyers or sellers became large.

(3) Then we developed discriminatory mechanism. It was shown that mechanism is budget balanced, strategy proof and individually rational. It was shown that in this mechanism also efficiency loss was bounded and asymptotically goes to 0.

(4) It was shown that this mechanism is false-name-poof. It meant that buyer could not improve his gains by submitting bids with false identification. This property is very important property in electronic auctions. It was shown that buyer does not have incentive to reduce the demand.

(5) The conditions for the existence of desirable mechanism were worked out. Then the case where, mechanism is strategy proof, individually rational and efficient but not

budget balanced, was considered. We attempted to introduce participation fees to cover up deficit. The cases of efficiency loss were also worked out. This can be helpful in cases where demand is far excess of supply or vice versa.

In the future work includes analyzing effects of false-name-proof bids. Our mechanism does not distinguish between buyers. It can be extended to cases where honest buyers are not punished. The future work also includes designing mechanism where buyers or sellers have minimum incentive to misreport volumes. It also includes the cases where number of buyers and sellers are large.

In chapter 5, auction mechanism for privatization of state owned enterprises, a very important application of electronic auctions in many countries, in the financial domain was studied. Traditionally auction based mechanisms have been used for leasing of mining rights, bandwidth allocation etc. In this work, we considered different types of auctions, which could be used in financial domains for Initial Public Offering (IPO) of Government owned companies. We also considered the scenario with different types of constraints. The main contributions of this chapter were as follows.

(1) We formulated an optimization problem that can handle single object multi unit auctions with different types of constraints. It was shown that some of the single object multi unit auction formulations, which are studied earlier were particular cases of our formulation.

(2) We derive new results to obtain optimum assignment. An algorithm, which computed the optimum solution, was developed. The algorithm generated optimum solution with polynomial time complexity.

(3) We then developed algorithm to compute VCG Payoff from the optimum solution without solving set of optimization problems.

(4) Then we generalized uniform price mechanism to handle different types of constraints. It has been shown that our mechanism was efficient, strategy proof and individually rational.

(5) Then we worked out discriminatory mechanism. It was shown that mechanism is efficient, strategy proof and individually rational.

(4) It was shown that this mechanism is false-name-proof. It meant that buyer cannot improve his gains by submitting bids with false identification. So this property is very

important in electronic auctions. It was shown that the buyer also did not have incentive to reduce the demand.

The future work includes extension of Ansubel auction to handle auctions with constraints. It also includes studying effect of false name bids.

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Publication List

(1) Continuous Call Double Auctions with Indivisibility Constraints, A. R. Dani, V. P. Gulati Arun K. Pujari – Accepted for presentation in 2005 IEEE International Conference on E-Technology, E-Commerce and E-Services at Hongkong March 29- April 1, 2005

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(4) Multi Unit Auctions with Constraints In Financial Domain, A. R. Dani, V. P. Gulati Arun K. Pujari – Accepted for presentation in 2006 IEEE International Conference on E-Technology, E-Commerce and E-Services at Sanfransico June 26-29, 2006

(5) A Framework for Modeling Electronic Negotiation - A. R. Dani, Dr. V. P. Gulati, Accepted at 8th IASTED International Conference on Software Engineering and Application Applications, MIT, Cambridge, USA, November 9-11, 2004

Communicated

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(1) Efficient Double Auctions Under Different Types of Constraints
(2) Strategy Proof Double Auctions with different types of constraints
(3) Multi Unit Auctions Under Constraints