QUANTUM HAMILTON-JACOBI STUDY OF WAVE-FUNCTIONS AND ENERGY SPECTRUM OF SOLVABLE AND QUASI-EXACTLY SOLVABLE MODELS

A Thesis

Submitted In Partial Fulfillment

For award of the degree of

DOCTOR OF PHILOSOPHY

By

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To

My dearest daughters

Amy *C.* **Geojo**

&

Swetha   Reddy   K.

DECLARATION

I hereby declare that the work embodied in this thesis entitled **Quantum Hamilton-Jacobi study of wave-functions and energy spectrum of solvable and quasi-exactly solvable models** carried out by me, under the supervision of Prof. A. K. Kapoor, School of Physics, University of Hyderabad, has not been submitted for any other degree or diploma either in part or in full to this or any other University or Institution.

Date: 31-12-2003　　　　　　　　　　K. G. GEOJO

CERTIFICATE

This is to certify that the work embodied in this thesis entitled **Quantum Hamilton-Jacobi study of wave-functions and energy spectrum of solvable and quasi-exactly solvable models** being submitted by K. G GEOJO is a record of work carried out under my supervision. This work has not been submitted to this or any other University or Institution for the award of a degree or diploma.
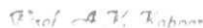
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Chapter 1

INTRODUCTION

In this thesis we present an alternative approach to the study of exactly solvable and quasi-exactly solvable (QES) problems **in** quantum mechanics. This approach, known as quantum Hamilton-Jacobi (QHJ) approach, [1] has been found to be an elegant and simple method to determine the energy spectrum of exactly solvable models in quantum mechanics. The advantage of this method is that it is possible to determine the energy eigen-values without having to solve for the eigen-functions. In this formalism, a quantum analog of classical action angle variables [2] is introduced. An exact quantization condition is formulated as a contour integral, representing the quantum action variable, in the complex plane. This exact quantization condition has been utilized for determining the energy eigen-values for one dimensional and separable systems. The quantization condition represents well known results on the number of nodes of the wave-function, translated in terms of logarithmic derivative, also called quantum momentum function (QMF). The equation satisfied by the QMF is a non-linear differential equation, called quantum Hamilton-Jacobi equation leads to two solutions. A boundary condition   in the **limit** $\hbar \rightarrow 0$ QMF tends to the classical momentum - is used to determine physically acceptable solutions for the QMF. The **application** of QHJ to eigen-values has been explored in great detail by

Bhalla et al [3,4].

In chapter 2 we review QHJ method and, by means of an example, we show how eigen-values are calculated without the need to obtain the full wave function. Briefly, this is possible because for the implementation of exact quantization condition one needs the knowledge of the singularities of QMF and the residues. The residues are easily computed by substituting only a few terms of the Laurent's expansion in the QHJ equation.

In chapter 3 we show how to calculate bound state wave functions in the QHJ formalism. For this purpose, again, one only requires knowledge of singularities of QMF and the corresponding residues. The technique to calculate residues is already available from earlier works and these are used for obtaining the bound state wave functions. In this process we clarify certain assumptions which are needed, and are found to be correct, for all the exactly solvable models which we have studied. As a by-product of the study of the bound state wave functions we have another way of obtaining the energy eigen-values. In this chapter we present details of our calculation for harmonic oscillator, hydrogen atom, Poschl Teller, Morse and Eckart Potentials, while details of some other potentials can be found in [5].

In chapter 4 we take up a study of the QES potential models [6] in one-dimension. These models have been extensively studied using Lie algebras. The QES models have the property that a part of the energy spectrum and corresponding wavefunctions can be computed exactly if the potential parameters satisfy a constraint known as the condition for quasi exact solvability. We study several QES models within the frame work of QHJ method. In each case we show that the condition for quasi exactly solvability follows from a very simple assumption about the behavior of QMF at infinity. Our assumption is equivalent to assuming that, after a suitable

transformation, the QMF reduces to a rational function of the independent variable. For all the known QES models, the quasi-exact solvability condition can be derived in this fashion [7].

In chapter 5 we study the wave function of quasi exact solvable models and present details of our calculation for sextic oscillator and hyperbolic potential. We find that obtaining eigen-values and eigen-functions does not require any new technique other than those given in chapter 3 for the exactly solvable models. However, this study reveals an interesting property of QMF for the bound states of quasi exact solvable models. This result concerns the zeros of the bound state wave-function **in** the complex plane. In the case of exactly solvable models, the moving poles of QMF appear only on the real line and all such poles correspond to the nodes of the wave-function. The number of such poles increases with energy in accordance with well known theorems on the number nodes of the wave-function. In the case of QES models all the bound state wave-functions, which are computable algebraically and also by our method, have complex zeros in addition to the real zeros corresponding to the nodes. **In fact we find that for a given QES potential all such wave-functions** have **the** same **number of zeros if we count all real and complex** zeros.

In the last chapter, we give a summary of our work as well as some directions for further investigations within the QHJ formalism.

Chapter 2

QUANTUM HAMILTON-JACOBI FORMALISM

In **this chapter,** we summarize the main results of the Hamilton-Jacobi theory in classical mechanics and the QHJ formalism to be used in this thesis. In section 3 a quantization condition is given, which is exact for one dimensional system and separable systems in higher dimensions. In section 4 the connection of QHJ formalism and Schrödinger quantum mechanics is spelled out and in the last section of this chapter an example of computation of eigen-values within the QHJ formalism, and without solving for wave-functions, is given.

2.1   Classical Hamilton-Jacobi Theory

The phase-space formalism of classical mechanics gives us freedom to introduce a pair of cannonical variables (Q_k, P_k) which are functions of a given starting set of variables (q_k, p_k). The Hamiltonian form of equations of motion is preserved if the transformation is cannonical in the sense of preserving Poisson brackets. This freedom is utilized in the Hamilton-Jacobi theory to give a formal solution of classical mechanical problems by making a transformation, so that the Hamiltonian becomes

constant.

In the Hamilton-Jacobi theory we look for a function $W(q_i, P_i)$, which generates the desired cannonical transformation making the Hamiltonian a constant. The transformation equation relating the old and new cannonical variables are

$$p_i = \frac{\partial W}{\partial q_i} \ , \ Q_i = \frac{\partial W}{\partial P_i} \tag{2.1.1}$$

and the requirement, that the Hamiltonian in terms of new variables be a constant α_1, gives a partial differential equation for $W(q_i, P_i)$:

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) = \alpha_1. \tag{2.1.2}$$

This equation is the Hamilton-Jacobi equation. The function $W(q_i, P_i)$ is known as the Hamilton's characteristic function. It is well known that a solution to the Hamilton-Jacobi equation is equivalent to full solution of Euler Lagrange equations of motion [2].

In general $W(q_i, P_i)$, which is a function of q_i and P_i, can be taken to be a function of q_i's and constants of motion $\alpha_1, \alpha_2; \bullet \bullet$, by identifying the new momenta P_i with the constants of motion a*, where α_1 is the total energy of the Hamiltonian.

For the purpose of finding the frequencies without solving the equation of motion completely, action variable J_i is introduced by

$$J_i = \oint \frac{\partial W(q_i, \alpha_1, \alpha_2, \cdots, \alpha_n)}{\partial q_i} dq_i \tag{2.1.3}$$

where the integral is over a periodic orbit. The action variable J_i are functions of the constants of motion α_i and one can eliminate α_i's in favor of the action variables J_i. In particular the Hamiltonian, $H = \alpha_1$, can now be expressed in terms of action variable J_i

$$\boldsymbol{H} = H\left(J_1, J_2, \cdots, J_n\right) \tag{2.1.4}$$

The generalized phase-space variable conjugate to J_i are known as the angle variable ω_i and are given by the transformation equations

$$\omega_i = \frac{\partial H(J_1, J_2, \cdots, J_n)}{\partial J_i} = \nu_i \, (J_1, J_2, \cdots, J_n) \tag{2.1.5}$$

where the ν_i's are a set of constant functions of the action variables. The equation has the solution

$$\omega_i = \nu_i t + \beta_i. \tag{2.1.6}$$

The constant ν_i are just the frequencies associated with the periodic motion, and *8i's* are constants of integration. This formalism (2.1.6) then gives, the frequencies of periodic motion. The semi-classical Bohr-Sommerfeld quantization rule is obtained if we require that the action-variables are integral multiples of Planck's constant.

We will now summarize the QHJ formalism and give an exact quantization rule and its relationship with Schrödinger formalism. We will first give the QHJ equation for one dimensional system which can be easily generalized for a separable system in several dimensions.

2.2 Quantum Hamilton-Jacobi Equation

In the quantum theory, one assumes the generating function *W(x, E)* satisfies

$$\frac{\hbar}{i} \frac{\partial^2 W(x, E)}{\partial x^2} + \left[\frac{\partial W(x, E)}{\partial x} \right]^z = E - V(x) \tag{2.2.1}$$

which will be called the quantum Hamilton-Jacobi (QHJ) equation. The momentum function

$$p(x, E) = \frac{\partial W(x, E)}{\partial x} \tag{2.2.2}$$

will be called the quantum momentum function (QMF). In the limit $\hbar \to 0$ the QHJ equation goes over to the classical Hamilton-Jacobi equation (2.1.2). Also the QMF

tends to classical momentum function

$$p(x, E) \xrightarrow{\hbar \to 0} p_{cl}(x, E) = [E - V(x)]^{\frac{1}{2}} \tag{2.2.3}$$

From (2.2.1) it is seen that the QMF satisfies the following equation

$$p^2(x, E) - i\hbar p'(x, E) - [E - V(x)] = 0 \tag{2.2.4}$$

This equation will also be referred to as the QHJ equation.

2.3 Exact Quantization

2.3.1 Boundary condition

The QHJ equation (2.2.4) is a non-linear differential and will give rise to two solutions. We need to establish a boundary condition to select physically acceptable solution. We first summarize the original boundary condition proposed by Leacock and Padgett. We state the boundary condition, which will complete the definition of the QMF $p(x,E)$ in terms of the classical momentum function $p_{cl}(x, E)$. The classical momentum function $p_{cl}(x, E)$ defined by (2.2.3) is a multi-valued function of p_{cl} is selected by the following rule:

The turning points x_1 and x_2 are defined by the vanishing of $p_{cl}(x, E)$ i.e by $p_{cl}(x_1, E) = p_{cl}(x_2, E) = 0$. The complex x plane on which $p_{cl}(x, E)$ is defined, is given a cut connecting the two branch points, i.e., a cut from x_1 to x_2. $p_{cl}(x, E)$ is defined as that branch of the square root, which is positive along the bottom of the cut.

With the above definition of the classical momentum function $p_{cl}(x, E)$, we state the physical condition which completes the definition of the QMF $p(x, E)$ as:

$$p(x, E) \xrightarrow{\hbar \to 0} p_{cl}(x, E) \tag{2.3.1}$$

Requirement (2.3.1) has two interpretations: (1) as a form of the correspondence principle and (2) as a boundary condition on $p(x, E)$.

The boundary condition given above is easy to implement only for very simple potentials because $p_{cl}(x, F,) = \sqrt{E - V(x)}$ will in general have several branch points, and the correct branch need to be selected. For this reason, we will impose other condition to select the solution. Several possibilities exists for an alternate condition. For example, the square integrability of the bound state wave-function is one such requirement. Some other conditions, useful in the context of super symmetric potentials models, can also be written down [4].

2.3.2 Exact Quantization Condition

Having introduced the QMF $p(x, E)$, we define the quantum action variable by generalizing the classical definition. The classical action variable can be defined as the integral

$$\frac{1}{2\pi} \oint_C p_{cl}(x, E) dx$$

where the integral is around a closed contour C. The contour C encloses the cut of $p_{cl}(x, E)$ which runs between the turning points x_1 and x_2.

Following the above definition, we define the quantum action variable by

$$J = J(E) \equiv \frac{1}{2\pi} \oint_C p(x, E) dx \qquad \text{(2.3.2)}$$

where $p(x, E)$ is the quantum momentum function, and C is the contour defined immediately above.

The definition (2.3.2) connects the action-variable eigen-value J to the energy eigen-value E. In order to use (2.3.2) it is necessary to obtain the eigen-values J. Equation (2.3.1) and (2.3.2) imply that $p(x, E)$ has poles of residue $— i\hbar$ on Re x-axis

between the turning points x_1 and x_2. For the ground state, first excited state, second excited state $\bullet\bullet\bullet$, $p(x, E)$ has zero, one, two $\bullet\bullet\bullet$, poles respectively in the potential well. The number of poles of $p(x, E)$ in the potential well gives the excitation level of the system. Since these poles of $p(x, E)$ are enclosed by the contour C, we have

$$J = n\hbar = J(E) \tag{2.3.3}$$

where n $= 0, 1, 2, \bullet\bullet\bullet$, and E is the energy eigen-value that is correlated with the values of nh for .7.

Equation (2.3.3) can be inverted. Thus one has

$$.7 = ,J(E) \text{ or } \qquad E = E(J) \tag{2,3.4}$$

2.4   Connection with Schrodinger Equation

To bring out an equivalence between the QHJ equation and the Schrodinger equation, Leacock defines the wave-function as

$$\psi(x, E) \equiv \exp\left(\frac{i}{\hbar}W(x, E)\right) \tag{2.4.1}$$

The wave-function $\psi(x, E)$ satisfies the correct Schrödinger equation and the appropriate physical boundary conditions. That the wave-function satisfies the correct Schrodinger equation can be seen from the above definition and the QHJ equation for $W(x,E)$. The wave-function for the bound states in one dimension has nodes whose number increases with energy; the wave-function for the nth excited state have n nodes in the classical region. Since the QMF is

$$p(x, E) = -i\hbar\frac{\psi'(x)}{\psi(x)} \tag{2.4.2}$$

these nodes are reflected as poles in the QMF and the residue of QMF at each pole is $-i\hbar$. Therefore, if we take a contour integral

$$\oint p(x, E)dx,$$

along a contour enclosing the poles of QMF corresponding to the nodes of the wave function, we will have

$$\oint p(x, E)dx = 2\pi i \ [\text{sum of the residues}] = nh \qquad \textbf{(2.4.3)}$$

This quantization condition along with the singularities and knowledge of residues of QMF is sufficient for obtaining energy eigen-values for exactly solvable models. For other models, approximation schemes can be developed.

2.5   Singularities of QMF

The QHJ equation(2.2.4) is of Riccati form. If $V(x)$ has a singular point, in the complex plane, $p(x, E)$ will also have singular point at that location. Such singular points are known as fixed singular points, and will be present in every solutions. On the other hand, other types of singular points with locations depending on the initial conditions, may also be present. These singular points are known as moving singular points. A well known theorem states that, the moving singular points of solutions of Riccati equation can only be poles. This pole will corresponds to a zero of the wave-function. Such a pole can only be a simple pole with residue $-i\hbar$. In fact if we substitute, assuming $b \neq 0$,

$$p(x, E) \sim \frac{b}{(x - x_0)^r} + \cdots \qquad (2.5.1)$$

in the QHJ equation

$$p^2(x, E) - i\hbar p'(x, E) - [E - V(x)] = 0 \qquad \textbf{(2.5.2)}$$

and if the potential is not singular at $x = x_0$ then r must be equal to one and $b = -i\hbar$. Thus the residues at each moving pole must be $-i\hbar$. This fact will be utilized throughout the thesis.

In the next section we show how to calculate the eigen-values by taking Morse oscillator as an example.

2.6 Energy Spectrum of Morse Oscillator

The potential energy of the Morse oscillator is

$$V(x) = A^2 + B^2 e^{-2\alpha x} - 2B(A + \frac{\alpha}{2})e^{-\alpha x} \tag{2.6.1}$$

with the super potential

$$W(x) = A - Be^{-\alpha x} \tag{2.6.2}$$

and $s = \frac{a}{\alpha}$

The quantum Hamilton-Jacobi equation is given by ($\hbar = 1 = 2m$)

$$p^2(x, E) - ip'(x, E) - \left[E - A^2 - B^2 e^{-2\alpha x} + 2B(A + \frac{\alpha}{2})e^{-\alpha x}\right] = 0 \tag{2.6.3}$$

We effect a transformation to a new variable

$$y = \frac{2B}{\alpha}e^{-\alpha x} \tag{2.6.4}$$

The quantum Hamilton-Jacobi equation in the new variable is

$$\tilde{p}^2(y, E) + i\alpha y \tilde{p}'(y, E) - \left[E - A^2 - \frac{\alpha^2}{4}y^2 + (A + \frac{\alpha}{2})\alpha y\right] = 0 \tag{2.6.5}$$

where $p(y) = p(x(y))$. We define $\phi(y, E)$ by

$$\tilde{p}(y, E) = i\alpha y \phi(y, E) \tag{2.6.6}$$

Therefore (2.6.5) transforms to

$$(\phi + \frac{1}{2y})^2 + \phi' - \frac{1}{4y^2} + \frac{1}{\alpha^2 y^2}[E - A^2 - \frac{y^2 \alpha^2}{4} + (A + \frac{\alpha}{2})y\alpha] = 0 \tag{2.6.7}$$

Let

$$\chi(y, E) = \phi(y, E) + \frac{1}{2y} \tag{2.6.8}$$

Therefore (2.6.7) transforms to

$$\chi^2 + \chi' + \frac{1}{4y^2} + \frac{1}{\alpha^2 y^2}[E - A^2 - \frac{y^2 \alpha^2}{4} + (A + \frac{\alpha}{2})y\alpha] = 0 \tag{2.6.9}$$

X has poles at $y = 0$ and there are moving poles between the classical turning points
. We assume that there are no more poles in the complex plane other than a pole of
finite order at infinity.

Residue at the fixed pole $y = 0$: For $y = 0$ we define

$$\chi = \frac{b_1}{y} + a_0 + a_1 y + .. \tag{2.6.10}$$

Using (2.6.10) in (2.6.9) and equating the coefficient of $\frac{1}{y^2}$, yields

$$b_1 = \frac{1}{2}[1 \pm i\frac{2}{\alpha}\sqrt{|E - A^2|}] \tag{2.6.11}$$

The residue has two values, and the correct value is selected by imposing the condi-
tion given below using the super potential viz.,

$$\lim_{E \to 0} p(x, E) = i\sqrt{2m}\, W(x)$$

In the y variable the above becomes (set $2m = 1$)

$$\lim_{E \to 0} \tilde{p}(y, E) = i\tilde{W}(y)$$

which yields the value of b_1 as in the $\lim_{E \to 0}$ as

$$b_1 = \frac{A}{\alpha} + \frac{1}{2}$$

Hence the correct sign of b_1 is to choose the negative sign in **(2.6.11)** and **the**
hence the value of b_1 is

$$b_1 = \frac{1}{2} \left[1 - i\frac{2}{\alpha}\sqrt{E - A^2|} \right] \tag{2.6.12}$$

Residue at y — oo: Now we determine the residue for the pole at infinity, for which we effect a transformation given by

$$y = \frac{1}{t} \tag{2.6.13}$$

With $\tilde{\chi}(t) = \chi(1/t)$ (2.6.9) transforms to

$$\tilde{\chi}^2(t) - t^2\tilde{\chi}'(t) + \frac{1}{4}t^2 + \frac{1}{\alpha^2}\left[(E - A^2)t^2 - \frac{\alpha^2}{4} + (A + \frac{\alpha}{2})\alpha t\right] = 0 \tag{2.6.14}$$

We assume an expansion for $\tilde{\chi}(t)$ as

$$\tilde{\chi}(t) = d_0 + d_1 t + d_2 t^2 + \cdots \tag{2.6.15}$$

The residue of $\tilde{\chi}(t)$ at $t = 0$ is obtained from the integral

$$\frac{1}{2\pi} \oint p(x, E)dx \tag{2.6.16}$$

which in the t variable yields the residue to be d_1. To determine the residue d_1 we use (2.6.15) in (2.6.14). Therefore (2.6.14) transforms to

$$[d_0 + d_1 t + d_2 t^2 + \cdots]^2 - t^2[d_1 + 2d_2 t + \cdots + \frac{1}{4}t^2 + \frac{1}{\alpha^2}\left[(E - A^2)t^2 - \frac{\alpha^2}{4} + (A + \frac{\alpha}{2})\alpha t\right] = 0 \tag{2.6.17}$$

Equating the constant term on both sides gives

$$d_0 = \pm\frac{1}{2} \tag{2.6.18}$$

Equating the power of t on both sides gives

$$d_1 = -\frac{1}{2\alpha d_0}\left[A + \frac{\alpha}{2}\right] \tag{2.6.19}$$

The correct sign for d_0 is chosen by the condition of square integrability on the wave function viz.,

$$\psi(x) = \exp\left(i \int p(x, E)dx\right)$$

The above integral is bounded at infinity only if $do = -\frac{1}{2}$

Summarizing we have the result that for large t, χ is given by (2.6.15) where $d_0 = -\frac{1}{5}$, d_1 by (2.6.19)

The Quantization Rule and Eigen-values:

We shall now obtain the eigen-values by enforcing the quantization rule

$$J(E) = \frac{1}{2\pi} \oint_C p(x, E)dx = n\hbar$$

where C is a contour enclosing the part of real axis between the turning points in the complex x-plane. Changing the variable to $y = \frac{2B}{\alpha}\exp(-\alpha x)$, the corresponding quantization condition in the y-plane becomes

$$J(E) = \frac{i}{2\pi} \oint_{C'} \left(\chi - \frac{1}{2y}\right)dy = n\hbar$$

where C is the image in the y-plane of the contour C in the x-plane, but with anti-clockwise orientation which compensates for the negative sign coming from the derivative. In addition to the moving poles, $p(y, E)$ has a fixed pole **at** $y = 0$. Let γ_1 be a small circle enclosing the singular point $y = 0$, and Γ_R is a circle of large radius R such that it encloses all the singularities of $p(x,E)$. See fig(2.1).

Hence

$$I_{\Gamma_R} = J(E) + I_{\gamma_1} \tag{2.6.20}$$

where I_{γ_1} is the contour integral for the contour γ_1 enclosing the pole $y = 0$ and I_{Γ_R} is the contour integral for the contour Γ_R.

We have evaluated the contour integral I_{γ_1} and its value is $-b_1$. The value of $J(E)$ is $-n$.
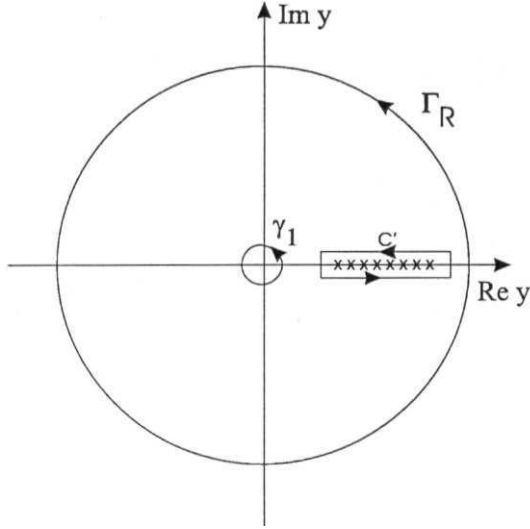
Figure 2.1: Contour for Evaluation of Action Integral

For evaluating the contour integral I_{Γ_R} we make a transformation of variable by $y = \backslash$ and hence the contour deforms to a new contour Γ_r which encloses the singular point at $y = \infty$ or $t = 0$. The value of this contour integral I_{Γ_r} has been evaluated and is $-d_1$.

Hence (2.6.20) transforms to

$$I_{\Gamma_r} = J(E) + I_{\gamma_1} \tag{2.6.21}$$

Hence to obtain the energy spectra of the Morse oscillator, we equate the residue of the fixed poles and the moving poles to those at infinity. Hence we have

$$-b_1 - n = -d_1 \tag{2.6.22}$$

Substituting the values of b_1 and d_1 we get

$$\frac{1}{2}\left(1 - i\frac{2}{\alpha}\sqrt{|E - A^2|}\right) + n = -\frac{1}{2\alpha d_0}\left[A + \frac{\alpha}{2}\right]$$

which on simplification gives the desired result for energy spectrum as

$$E = A^2 - (A - n\alpha)^2 \qquad\qquad (2.6.24)$$

Chapter 3

Calculation of Wave-Function for ES models

In this chapter we apply the QHJ formalism outlined in the previous chapter, to find bound state wave-functions for several exactly solvable potential problems in one dimension. We will show that, by making use of elementary theorems in complex variables, the form of QMF can be determined completely and hence the bound state wave-functions are easily obtained. To determine the form of QMF we begin with the QH.J equation (ft = 1 = *2m*)

$$p^2(x, E) - ip'(x, E) - [E\text{-}V(x)] = 0$$

where p(x,E) is the QMF continued in the complex x-plane, and is related to the wave-function by

$$p(x, E) = -i\frac{\psi'(x)}{\psi(x)}$$

The zeros of the wave-function will appear as poles in the QMF. According to the well known theorems about the nodes of wave-function, the nth excited state corresponds to *n* zeros on the real line, and there will be corresponding *n (moving)* poles in the QMF and the residue at each pole will be $-i$ as has been discussed in chapter 2. In addition to these moving poles, there are fixed poles corresponding to the

singularities of the potential. **We will make an assumption that QMF has no other singularities in the finite complex plane.** The QMF turns out to be meromorphic, and to fix its form one needs to know the behavior of QMF for large x in the complex x-plane. This information can be easily read from the QHJ equation and hence the form of QMF can then be fixed completely. In the next section we show how this strategy works for the harmonic oscillator. In the remaining sections of this chapter we give the details of the calculation of the bound state wave-functions for harmonic oscillator, Morse oscillator, Poschl Teller and Eckart potentials and hydrogen atom. For these potentials a change of variable becomes necessary and we always try to bring the QHJ equation in the new variable to a same form as the above equation. We also mention that several other potentials have been studied [5] and the bound state wave-functions in each case agree with the known results [8].

3.1   Harmonic Oscillator

The potential energy of the harmonic oscillator is

$$V(x) = \frac{1}{2} m \omega^2 x^2 \tag{3.1.1}$$

The quantum Hamilton-Jacobi equation is given by (ft = 1 = 2m)

$$p^2(x, E) - ip'(x, E) - [E - \frac{1}{4}\omega^2 x^2] = 0 \tag{3.1.2}$$

The QMF $p(x, E)$ has n poles corresponding to the zeros of the wave-function, and residue at each of these poles is $-$ i. It can be proved that $p(x, E)$ has no other poles except at infinity [1].

For large x

$$p(x, E) \approx \pm \frac{1}{2} i \omega x \tag{3.1.3}$$

and we write

$$p(x, E) \approx \pm \frac{1}{2} i \omega x + \phi(x) \tag{3.1.4}$$

where $\phi(x)$ is to be determined.

The sign of $p(x,E)$ is determined by the condition of square integrability of the wave-function.

The wave-function is expressed as

$$\psi(x) = \exp \left(i \int p(x, E) dx \right) \tag{3.1.5}$$

When the above value of $p(x, E)$ is substituted in the equation of wave-function, the wave-function is bounded at at large x if we choose the positive sign of $\frac{1}{2} i \omega x$.

Hence we write the quantum momentum function $p(x, E)$ as

$$p(x, E) = \sum_{k=1}^{n} \frac{-i}{x - x_k} + \frac{1}{2} i \omega x + \phi(x) \tag{3.1.6}$$

where $x_1, x_2, \bullet \bullet \bullet, x_n$ are the location of n poles on the x-axis and $\phi(x)$ is analytic every where and bounded at infinity. Therefore Liouville's theorem tells us, it has to be a constant. Hence let $\phi(x) = c$ a constant. Hence the above equation becomes

$$p(x, E) = \sum_{k=1}^{n} \frac{-i}{x - x_k} + \frac{1}{2} i \omega x + c \tag{3.1.7}$$

Substituting (3.1.7) in (3.1.2) we have

$$\left[\sum_{k=1}^{n} \frac{-i}{x - x_k} + \frac{1}{2} i \omega x + c \right]^2 + \sum_{k=1}^{n} \frac{1}{(x - x_k)^2} - [E - \frac{1}{4} \omega^2 x^2] = 0 \tag{3.1.8}$$

Equating the power of x, we have c = 0. Equating the constant term to zero on both sides in equation (3.1.8) gives

$$E = (n + \frac{1}{2}) \omega$$

which is the well known expression for energy of the harmonic oscillator in our notation (2m = 1)

The sum moving pole terms

$$\sum_{k=1}^{n} \frac{-i}{x - x_k}$$

can be expressed as $\frac{P'(x)}{P(x)}$ where $P(x)$ is the polynomial

$$P(x) = \prod_{k=1}^{n} (x - x_k). \tag{3.1.10}$$

The QMF (3.1.7) can be expressed as

$$p(x, E) = -i\frac{P'(x)}{P(x)} + \frac{1}{2}i\omega x, \tag{3.1.11}$$

Using (3.1.11) in (3.1.2) and on simplification yields

$$P''(x) - \omega x P'(x) + n\omega P(x) = 0 \tag{3.1.12}$$

We effect a transformation $\xi = ax$ where $\alpha^2 = \%$. Hence equation (3.1.12) changes to

$$P''(\xi) - 2\xi P'(\xi) + 2nP(\xi) = 0 \tag{3.1.13}$$

This above equation resembles the well known Hermite differential equation. Hence on comparison, we get

$$P(\xi) \equiv H_n(\alpha x) \tag{3.1.14}$$

and is the Herrnite polynomial.

The wave-function is expressed as

$$\psi(x) = \exp\left[i \int p(x, E)dx\right] = \exp\left[i \int \left(-i\frac{P'(x)}{P(x)} + i\frac{1}{\sqrt{2}}\omega x\right)dx\right].$$

and hence we have

$$\psi(x) = H_n(\alpha x)\exp\left(-\frac{1}{4}\omega x^2\right). \tag{3.1.15}$$

This is the desired wave-function for the harmonic oscillator.

3.2   Morse Oscillator

The potential energy of the Morse oscillator is

$$V(x) = A^2 + B^2 e^{-2\alpha x} - 2B(A + \frac{\alpha}{2})e^{-\alpha x} \tag{3.2.1}$$

with the super potential

$$W(x) = A - Be^{-\alpha x} \tag{3.2.2}$$

and

$$s = \frac{a}{\alpha}. \tag{3.2.3}$$

The quantum Hamilton-.]acobi equation is given by ($\hbar = 1 = 2m$)

$$p^2(x, E) - ip'(x, E) - \left[E - A^2 - B^2 e^{-2\alpha x} + 2B(A + \frac{\alpha}{2})e^{-\alpha x} \right] = 0 \tag{3.2.4}$$

We effect a transformation to the variable

$$y = \frac{2B}{\alpha} e^{-\alpha x} \tag{3.2.5}$$

The quantum Hamilton-Jacobi equation in the new variable is

$$\tilde{p}^2(y, E) + i\alpha y \tilde{p}'(y, E) - \left[E - A^2 - \frac{\alpha^2}{4} y^2 + (A + \frac{\alpha}{2})\alpha y \right] = 0 \tag{3.2.6}$$

We define

$$\tilde{p}(y, E) = i\alpha y \phi(y, E) \tag{3.2.7}$$

Then (3.2.6) transforms to

$$(\phi + \frac{1}{2y})^2 + \phi' - \frac{1}{4y^2} + \frac{1}{\alpha^2 y^2}[E - A^2 - \frac{y^2 \alpha^2}{4} + (A + \frac{\alpha}{2})y\alpha] = 0 \tag{3.2.8}$$

Lei

$$\chi(y, E) = \phi(y, E) + \frac{1}{2y} \tag{3.2.9}$$

Therefore (3.2.8) transforms to

$$\chi^2 + \frac{1}{4y^2} + \chi' + \frac{1}{\alpha^2 y^2}[E - A^2 - \frac{y^2\alpha^2}{4} + (A + \frac{\alpha}{2})y\alpha] = 0 \qquad (3.2.10)$$

This equation suggests that χ has a pole at $y = 0$. It will also have n moving poles corresponding to the nodes of the wave-function. We assume that there are no other poles in the finite complex plane. For large y the behavior of χ has already been obtained in section (3.1) and is seen to be bounded for large y. Hence we get using Liouville's theorem

$$\chi = \frac{b_1}{y} + \sum_{k=1}^{n}\left(\frac{1}{y - y_k}\right) + c \qquad (3.2.11)$$

where b_1 and c are constants to be fixed. The residue of χ at $y = 0$ is b_1 and has been obtained in section (3.1)and

$$b_1 = \frac{A}{\alpha} + \frac{1}{2} \qquad (3.2.12)$$

We write, once again,

$$\sum_{k=1}^{n}\frac{1}{y - y_k} = \frac{P'(y)}{P(y)}$$

where

$$P(y) = \prod_{k=1}^{n}(y - y_k)$$

Substituting (3.2.11) in (3.2.10) gives

$$\frac{P''}{P} + 2\frac{b_1}{y}\frac{P'}{P} + 2\frac{P'}{P}c + 2\frac{b_1}{y}c + c^2 - \frac{1}{4} + \frac{1}{\alpha y}(A + \frac{\alpha}{2}) = 0 \qquad (3.2.13)$$

In order to proceed further we look at the behavior of each term for large y. Using the leading terms

$$\frac{P''(y)}{P(y)} \sim \frac{n(n-1)}{y^2}, \qquad \frac{P'(y)}{P(y)} \sim \frac{n}{y}$$

in equation (3.2.13) and equating the constant term on both sides gives, $c = \pm\frac{1}{2}$. The correct sign for c is chosen by the condition of square integrability on the wave function which fixes $c = -\frac{1}{2}$.

Comparing the coefficient of - for large y on both sides we get

$$2b_1 c + 2nc + \left(A + \frac{\alpha}{2}\right)\frac{1}{\alpha} = 0 \tag{3.2.14;}$$

which on using the values of b_1 and c and on simplification gives the energy eigen-value

$$E = A^2 - (A - n\alpha)^2 \tag{3.2.15}$$

Substituting the value of b_1 and c in equation (3.2.13) we have

$$yP''(y) + \{1 - y + 2(s - n)\}P'(y) + nP(y) = 0 \tag{3.2.16}$$

Compare this with the standard Laguerre differential equation

$$xy'' + (\beta + 1 - x)y' + ny = 0$$

we have $P(y) = L_\beta^n(y)$.

The wave-function for the Morse oscillator is given by

$$\psi(x) = \exp\left(i \int p(x, E)dx\right) \tag{3.2.17}$$

In terms of y variable we have

$$\psi(y) = \exp\left(i \int \left[\frac{b_1}{y} + \frac{P'(x)}{P(x)} - \frac{1}{2} - \frac{1}{2y}\right]dy\right)^{\cdot} \tag{3.2.18}$$

On integrating and simplifying we get

$$\psi_n(y) = y^{s-n} \exp(-\frac{1}{2}y)P(x) \tag{3.2.19}$$

Replacing the value of $P(y)$ we have

$$\psi_n(y) = y^{s-n} \exp(-\frac{1}{2}y)L_\beta^n(y) \tag{3.2.20}$$

3.3   Poschl-Teller Potential

The Poschl-Teller potential is

$$V(x) = A^2 + (B^2 + A^2 + A\alpha) \ \text{csch}^2 \alpha x - B \ (2A + a) \ \text{coth} \ ax \ \text{csch} \alpha x \qquad (3.3.1)$$

with the super potential given by

$$W(x) = A \coth ax - B \ \text{csch} \ ax \ (A < B) \qquad (3.3.2)$$

and

$$s = \frac{A}{\alpha}, \qquad \lambda = \frac{\beta}{\alpha} \qquad \textbf{(3.3.3)}$$

The quantum Hamilton-Jacobi equation is given by ($\hbar = 1 = 2$m)

$$p^2(x, E) - ip'(x, E)$$

$$- \left[E - A^2 - (B^2 + A^2 + Aa) \ \text{csch}^2 \alpha x + B(2A + a) \coth ax \ \text{csch} \ \alpha x \right] = 0 \quad \textbf{(3.3.4)}$$

We effect a transformation to a new variable

$$y = \cosh ax \qquad \textbf{(3.3.5)}$$

The quantum Hamilton-Jacobi equation in the new variable becomes

$$\tilde{p}^2(y, E) - i\alpha\sqrt{y^2 - 1}\tilde{p}'(y, E)$$

$$- \left[E - A^2 - (B^2 + A^2 + A\alpha)\frac{1}{y^2 - 1} + B(2A + \alpha)\frac{y}{y^2 - 1} \right] = 0 \qquad \textbf{(3.3.6)}$$

We define

$$\tilde{p}(y, E) = -i\alpha\sqrt{y^2 - 1}\phi(y). \qquad \textbf{(3.3.7)}$$

Therefore (3.3.6) transforms to

$$(\phi + \frac{1}{2}\frac{y}{y^2 - 1})^2 + \phi' - \frac{1}{4}\frac{y^2}{(y^2 - 1)^2}$$

$$+\frac{1}{\alpha^2(y^2-1)}\left[E-A^2-(B^2+A^2+A\alpha)\frac{1}{y^2-1}+B(2A+\alpha)\frac{y}{y^2-1}\right]=0 \quad (3.3.8)$$

Let

$$\chi=\phi+\frac{1}{2}\frac{y}{y^2-1}. \tag{3.3.9}$$

Therefore (3.3.8) transforms to

$$\chi^2+\chi'+\frac{3}{4}\frac{y^2}{(y^2-1)^2}-\frac{1}{2}\frac{1}{y^2-1}$$

$$+\frac{1}{\alpha^2(y^2-1)}\left[E-A^2-(B^2+A^2+A\alpha)\frac{1}{y^2-1}+B(2A+\alpha)\frac{\dot{y}}{y^2-1}\right]=0 \quad (3.3.10)$$

X has poles at $y=\pm 1$ and there are moving poles between the classical turning points. We assume that there are no more poles in the complex line. We now determine the residue at each of these poles.

For $y=+1$, we define

$$\chi=\frac{b_1}{y-1}+a_0+a_1(y-1)+\cdots \tag{3.3.11}$$

Using (3.3.11) in (3.3.10) and equating the coefficient of , $\frac{1}{-1)^2}$ yields

$$b_1=\frac{1}{2}\left[1\pm\frac{1}{2\alpha}[2(B-A)-\alpha]\right] \tag{3.3.12}$$

The correct value of $b\backslash$ is selected by imposing the condition

$$\lim_{E\to 0}\operatorname{Res}\tilde{p}(y,E)=i\operatorname{Res}\tilde{W}(y,E)$$

on super potential as explicitly shown for Morse oscillator gives

$$b_1=\frac{1}{4}-\frac{1}{2\alpha}(A-B) \tag{3.3.13}$$

For $y=-\ 1$, we define

$$\chi=\frac{b_1'}{y+1}+a_0'+a_1'(y+1)+\cdots \tag{3.3.14}$$

Using (3.3.14) in (3.3.10) and equating the coefficient of $, \frac{1}{+1)^2}$ and following the above procedure yields

$$b_1' = \frac{1}{4} - \frac{1}{2\alpha}(A + B) \tag{3.3.15}$$

The residue at a moving pole is seen from (3.3.10) to be 1. Hence we arrive at the form

$$\chi = \frac{b_1}{y - 1} + \frac{b_1'}{y + 1} + \frac{P'(y)}{P(y)} + c \tag{3.3.16}$$

for χ where c is a constant to be determined

Substituting (3.3.16) in (3.3.10) gives the following equation.

$$c^2 + \frac{2b_1 b_1'}{y^2 - 1} + \frac{2b_1'}{y + 1}\frac{P'}{P} + 2\frac{P'}{P}c + \frac{2b_1 c}{y - 1} + \frac{2b_1}{y - 1}\frac{P'}{P} + \frac{2b_1'}{y + 1}c - \frac{1}{2}\frac{1}{y^2 - 1} + \frac{P''}{P}$$

$$\frac{3}{8}\frac{1}{y^2 - 1} + \frac{1}{\alpha^2}(E - A^2)\frac{1}{y^2 - 1} + \frac{1}{2\alpha^2}(B^2 + A^2 + A\alpha)\frac{1}{y^2 - 1} = 0 \tag{3.3.17}$$

Now we look at different terms in equation (3.3.17) for large y and equate their coefficient to zero. Equating the constant term to zero gives c = 0 . With c = 0 the above equation becomes

$$\frac{P''}{P} + \frac{P'}{P}\left[\frac{2b_1}{y - 1} + \frac{2b_1'}{y + 1}\right] + \frac{2b_1 b_1'}{y^2 - 1} - \frac{1}{2}\frac{1}{y^2 - 1}$$

$$\frac{3}{8}\frac{1}{y^2 - 1} + \frac{1}{\alpha^2}(E - A^2)\frac{1}{y^2 - 1} + \frac{1}{2\alpha^2}(B^2 + A^2 + A\alpha)\frac{1}{y^2 - 1} = 0 \tag{3.3.18}$$

For large y, $P(y)$ behaves as $P(y) \sim y^n + \cdots$ and $\frac{P''(y)}{P(y)} \sim \frac{n(n-1)}{y^2}$, $\frac{P'(y)}{P(y)} \sim \frac{n}{y}$

Using these in (3.3.18) and equating the coefficient of $\frac{1}{y^2}$ gives

$$2b_1 b_1' + 2nb_1' + 2nb_1 + n(n - 1) + \frac{3}{8} + \frac{1}{\alpha^2}(E - A^2) + \frac{1}{2\alpha^2}(B^2 + A^2 + A\alpha) - \frac{1}{2} = 0$$

and substituting the values of b_1 and b_1' gives the expression for energy as

$$E = A^2 - (A - no)^2 \tag{3.3.19}$$

Substituting the value of E, b_1 and b_1' in (3.3.18) we get

$$(1 - y^2)P''(y) + [(2s - 1)y - 2\lambda]P'(y) + [n(n - 2s)]P(y) = 0 \tag{3.3.20}$$

The above equation resembles the standard Jacobi polynomial, such that

$$P(y)\, a P_n^{(\alpha,\beta)}(y) = P_n^{(\lambda-s-\frac{1}{2},\,-\lambda-s-\frac{1}{2})}(y) \tag{3.3.21}$$

The wave function for the Poschl Teller potential is obtained on the same line as that for the Morse oscillator and is given by

$$\psi(y) = (y - 1)^{\frac{(\lambda-s)}{2}} y + 1)^{\frac{-(\lambda+s)}{2}} P_n^{(\lambda-s-\frac{1}{2},\,-\lambda-s-\frac{1}{2})}(y) \tag{3.3.22}$$

which agrees well with the values given in literature. [5]

3.4   Eckart Potential

The potential energy of the Eckart potential is

$$V(x) = A^2 + \frac{B^2}{A^2} - 2B \coth \alpha x + A(A - \alpha) \operatorname{csch}^2 \alpha x \tag{3.4.1}$$

with the super potential given by

$$W(x) = -A \coth \alpha x + \frac{B}{A}, \qquad (B > A^2) \tag{3.4.2}$$

and

$$s = \frac{A}{\alpha}, \qquad \lambda = \frac{B}{\alpha^2}, \qquad \alpha = \frac{\lambda}{n + s} \tag{3.4.3}$$

The quantum Hamilton-Jacobi equation is given by (ft = 1 = 2m)

$$p^2(x, E) - ip'(x, E) - \left[E - A^2 - \frac{B^2}{A^2} + 2B \coth \alpha x - A(A - \alpha) \operatorname{csch}^2 \alpha x \right] = 0 \tag{3.4.4}$$

We effect a transformation by the variable

$$y = \coth ax \qquad (3.4.5)$$

The quantum Hamilton-Jacobi equation in the new variable is

$$\tilde{p}^2(y, E) - i\alpha(1 - y^2)\tilde{p}'(y, E) - \left[E - A^2 - \frac{B^2}{A^2} + 2By - A(A - \alpha)(y^2 - 1)\right] = 0$$

(3.4.6)

We define

$$\tilde{p}(y, E) = -i\alpha(1 - y^2)\phi(y) \qquad \textbf{(3.4.7)}$$

Hence equation $(3.4.6)$ simplifies to

$$[\phi - \frac{y}{1 - y^2}]^2 + \phi' - \frac{y^2}{(1 - y^2)^2}$$

$$\frac{1}{\alpha^2(1 - y^2)^2}\left[E - A^2 - \frac{B^2}{A^2} + 2By - A(A - \alpha)(y^2 - 1)\right] = 0 \qquad . (3.4.8)$$

Let

$$\chi = \phi - \frac{y}{1 - y^2} \qquad (3.4.9)$$

Therefore the above equation changes to

$$\chi^2 + \chi' + \frac{y^2}{(1 - y^2)^2} + \frac{1}{1 - y^2}$$

$$+\frac{1}{\alpha^2(1 - y^2)^2}\left[E - A^2 - \frac{B^2}{A^2} + 2By - A(A - \alpha)(y^2 - 1)\right] = 0 \qquad (3.4.10)$$

X has poles at $y - \pm 1$ and there are moving poles between the classical turning points. We assume that there are no more poles in the complex plane. We determine the residue at each of these poles.

For $y = +1$, we define

$$\chi = \frac{b_1}{y - 1} + a_0 + a_1(y - 1) + \cdot \qquad (3.4.11)$$

Using (3.4.11) in (3.4.10) and equating the coefficient of $\frac{1}{(y-1)^2}$ yields

$$b_1 = \frac{1}{2}\left[1 \pm \frac{1}{\alpha}\sqrt{\left(A - \frac{B}{A}\right)^2 - E}\,\right] \qquad (3.4.12)$$

As the residue has two values, the correct value is selected by imposing the condition on the super potential as done in Morse oscillator and the correct value is

$$b_1 = \frac{1}{2}\left[1 + \frac{1}{\alpha}\sqrt{\left(A - \frac{B}{A}\right)^2 - E}\,\right] \qquad (3.4.13)$$

Similarly the residue for $y = -1$ is determined and is given as

$$b_1' = \frac{1}{2}\left[1 - \frac{1}{\alpha}\sqrt{\left(A + \frac{B}{A}\right)^2 - E}\,\right] \qquad (3.4.14)$$

We assume χ to have the form

$$\chi = \frac{b_1}{y-1} + \frac{b_1'}{y+1} + \frac{P'(y)}{P(y)} + c \qquad (3.4.15)$$

where c is a constant to be determined

Using (3.4.15) in (3.4.10) and following the similar lines as that of Morse and Poschl Teller potential one gets the value of c $= 0$ and the resulting equation becomes

$$\frac{P''(y)}{P(y)} + \frac{P'(y)}{P(y)}\left[\frac{2b_1}{y-1} + \frac{b_1'}{y+1}\right] + \frac{2b_1 b_1'}{y^2-1}$$

$$-\frac{1}{2}\frac{1}{y^2-1} - \frac{1}{2\alpha^2}\frac{1}{y^2-1}[E - A^2 - \frac{B^2}{A^2}] - \frac{1}{\alpha^2}\frac{1}{y^2-1}A(A-\alpha) = 0 \qquad (3.4.16)$$

For large y assuming $P(y) \sim y^n + \cdots$ and equating the coefficient of $\frac{1}{y^2}$ gives

$$2b_1 b_1' + 2b_1' n + 2b_1 n + n(n-1) - \frac{1}{2} - \frac{1}{2\alpha^2}(E - A^2 - \frac{B^2}{A^2}) - \frac{1}{\alpha^2}A(A-\alpha) = 0$$

Substituting the values of b_1 and b_1' gives the energy expression as

$$E = A^2 - [A + n\alpha]^2 - \frac{B^2}{(A+n\alpha)^2} + \frac{B^2}{A^2} \qquad (3.4.17)$$

Using the values of b_1, b'_1 and E in (3.4.16) one gets the differential equation for Eckart potential as

$$(1 - y^2)P''(y) + [-2a - 2\{-n - s + 1\}y]\, P'(y) + \{-2ns)P\{y) = 0 \qquad (3.4.18)$$

Equation (3.4.18) resembles the standard Jacobi polynomial and

$$P(y)\; s\; P_n^{(\alpha,\beta)}(y)\; =\; P_n^{(s_3,s_4)}(y) \qquad (3.4.19)$$

The wave function for the Eckart Potential is obtained from

$$\psi(x) = e^{i \int p(x,E)dx}$$

and is given by

$$\psi(y) = (y-1)^{\frac{s_3}{2}}\,(y+1)^{\frac{s_4}{2}}\,P_n^{(s_3,s_4)}(y) \qquad (3.4.20)$$

The values for energy and wave function agree with those found in the literature.[5]

3.5 Hydrogen Atom

In this section we obtain the bound state wave functions of the radial part of the Schrödinger equation ($\hbar = 1m = 11$

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left(E + \frac{Ze^2}{r} - \frac{\lambda}{r^2}\right)R = 0 \qquad (3.5.1)$$

where $A = l(l+1)$. Using the transformation $R(r) = \phi(r)/r$, the Schrödinger equation becomes

$$\frac{d^2\phi}{dr^2} + (E + \frac{Ze^2}{r} - \frac{\lambda}{r^2})\phi = 0. \qquad (3.5.2)$$

The QHJ equation in terms of

$$q = \frac{d}{dr}\ln(\phi(r)) \qquad (3.5.3)$$

is given by

$$q^2 + \frac{dq}{dr} + \left(E + \frac{Ze^2}{r} - \frac{\lambda}{r^2} \right).$$ (3.5.4)

The range of r is from 0 to ∞, and the wave function $\phi(r)$ should vanish at $r = 0$. Thus q has a fixed pole at $r = 0$, along with the n moving poles with residue equal to one on the real line. Like harmonic oscillator there are no other singularities in the finite complex plane. Thus we can write $q,$ in a similar fashion as for harmonic oscillator, as

$$q(r) = \frac{P'}{P} + \frac{b_1}{r} + C$$ **(3.5.5)**

where b_1 is the residue at $r = 0$ which can be obtained by doing a Laurent expansion of q around the pole at the origin. The two values of residues obtained are

$$b_1 = -l, \ b_1 = l + 1.$$ (3.5.6)

One chooses the right residue by using the square integrability property of the wave function 6 and obtain

$$b_1 = l + 1$$ (3.5.7)

as the right choice. Substituting (3.5.5) for q in (3.5.4) and expanding different terms of the resulting equation for large r and comparing the leading terms we get

$$C^2 = -E, \ E = -\frac{Z^2 e^4}{(2n')^2}$$ (3.5.8)

where $n' = n + l + 1$ and one is left with the differential equation

$$rP'' + 2P' \left(l + 1 - \frac{Ze^2 r}{2n'} \right) + \frac{(n' - l - 1)Ze^2}{n'} P = 0.$$ **(3.5.9)**

Now defining

$$\frac{Ze^2}{n'} r = \rho$$ (3.5.10)

(3.5.8) becomes

$$pP'' + ((2l + 1) + 1 - \rho)P' + (n' - l - 1)P = 0 \qquad (3.5.11)$$

which is the associated Laguerre differential equation where P is the Laguerre poly-
nomial denoted by L. The bound state wave function obtained from (3.5.5) and
(3.5.3) is

$$\psi_n(\rho) = \rho^{l+1} \exp(-\rho/2) L_{n'+l}^{2l+1}(\rho) \qquad (3.5.12)$$

which is seen to be identical with known correct answer.

Chapter 4

CONDITIONS FOR QUASI-EXACT SOLVABILITY

4.1   Introduction to QES

In **this** chapter we study QES model in one dimension. These are the models for which a part of the bound state energy spectrum and corresponding wave-functions can be obtained exactly. These models have been constructed and studied extensively by means of Lie algebraic approach. For a review we refer to the book by Ushveridze et al [6]. In order that a part of the spectrum be obtained exactly, the potential parameters appearing in the potential must satisfy a condition known as the condition for quasi-exact solvability. Within the QHJ approach, used for exactly solvable models, it is not clear how such a condition can arise and why only a part of the spectrum is exactly solvable. In this chapter we report a study of these aspects of QES models.

In order to study QES models within QHJ formalism one needs to have information of singularities of QMF. This in general is not very easy to obtain except for some simple cases like harmonic oscillator and hydrogen atom problems. In the limit $\hbar \longrightarrow 0$ the QMF $p(x, E)$ goes over to $p_{cl}(x, E) = \sqrt{E - V(x)}$ which will, in general, have several branch points. This is an indication that in general, the singularity structure of $p(x, E)$ will be very complicated. **In order to make progress, we**

make a simplifying assumption **that** the **point at infinity is an isolated singular point** and more **specifically it** is a **pole of some finite order.** Thus this amounts to saying that $p(x, E)$ has fixed poles, and a finite number of moving poles and a pole at infinity. Using these requirements, we can proceed as in the case of exactly solvable models and work out the consequences of exact quantization condition given below.

$$\oint p \, dq = nh. \tag{4.1.1}$$

We find that for all the QES potential models studied by us, (4.1.1) and our **assumptions,** imply that potential parameter must satisfy a condition which turns **out** to be identical **with** the condition of quasi-exact solvability of the potential. A list of potentials studied and the condition of quasi-exact solvability in each case are given below.

The potentials are:

1. Sextic oscillator:

$$V(x) = \alpha x^2 + \beta x^4 + \gamma x^6, \qquad \gamma > 0. \tag{4.1.2}$$

2. Sextic oscillator with centrifugal barrier:

$$V(x) = 4(s - \tfrac{1}{4})(s - \tfrac{3}{4})\tfrac{1}{x^2} + [b^2 - 4a(s + \tfrac{1}{2} + \mu)]x^2 + 2abx^4 + a^2 x^6. \tag{4.1.3}$$

3. Circular potential:

$$V(x) = \frac{A}{\sin^2 x} + \frac{B}{\cos^2 x} + C \sin^2 x - D \sin^4 x, \tag{4.1.4}$$

with

$$A = 4(s_1 - \tfrac{1}{4})(s_1 - \tfrac{3}{4}), \tag{4.1.5}$$

$$B = 4(s_2 - \tfrac{1}{4})(s_2 - \tfrac{3}{4}), \tag{4.1.6}$$

$$C = q_1^2 + 4q_1(s_1 + s_2 + \mu), \tag{4.1.7}$$

$$D = q_1^2. \tag{4.1.8}$$

4. Hyperbolic potential:

$$V(x) = -\frac{A}{\cosh^2 x} + \frac{B}{\sinh^2 x} - C\cosh^2 x + D\cosh^4 x, \tag{4.1.9}$$

with

$$A = 4(s_1 - \frac{1}{4})(s_1 - \frac{3}{4}), \tag{4.1.10}$$

$$B = 4(s_2 - \frac{1}{4})(s_2 - \frac{3}{4}), \tag{4.1.11}$$

$$C = [q_1^2 + 4q_1(s_1 + s_2 + \mu)], \tag{4.1.12}$$

$$D = q_1^2. \tag{4.1.13}$$

5.

$$V(x) = A\sinh^2\sqrt{\nu}x + B\sinh\sqrt{\nu}x + C\tanh\sqrt{\nu}x\,\text{sech}\sqrt{\nu}x + D\text{sech}^2\sqrt{\nu}x$$

$$\tag{4.1:14}$$

(i.

$$V(x) = A\cosh^2\sqrt{\nu}x + B\cosh\sqrt{\nu}x + C\coth\sqrt{\nu}x\,\csc h\sqrt{\nu}x + D\text{csch}^2\sqrt{\nu}x$$

$$\tag{4.1.15}$$

7.

$$V(x) = Ae^{2\sqrt{\nu}x} + Be^{\sqrt{\nu}x} + Ce^{-\sqrt{\nu}x} + De^{-2\sqrt{\nu}x} \tag{4.1.16}$$

The conditions for quasi exact solvability for these potentials are:

1.

$$\frac{1}{\sqrt{\gamma}}\left(\frac{\beta^2}{4\gamma} - \alpha\right) = 3 + 2n, \qquad n = \text{integer} \tag{4.1.17}$$

2.

$$\mu = \text{integer} \qquad (4.1.18)$$

3. Taking

$$A = 4\left(s_1 - \frac{1}{4}\right)\left(s_1 - \frac{3}{4}\right) \qquad (4.1.19)$$

$$B = 4\left(s_2 - \frac{1}{4}\right)\left(s_2 - \frac{3}{4}\right) \qquad (4.1.20)$$

$$C = q_1^2 + 4q_1(s_1 + s_2 + \mu) \qquad (4.1.21)$$

$$D = q_1^2 \qquad (4.1.22)$$

the condition is

$$\mu = \text{integer} \qquad (4.1.23)$$

4. Taking

$$A = 4\left(s_1 - \frac{1}{4}\right)\left(s_1 - \frac{3}{4}\right), \qquad (4.1.24)$$

$$B = 4\left(s_2 - \frac{1}{4}\right)\left(s_2 - \frac{3}{4}\right), \qquad (4.1.25)$$

$$C = [q_1^2 + 4q_1(s_1 + s_2 + \mu)], \qquad (4.1.26)$$

$$D = q_1^2, \qquad (4.1.27)$$

the condition is

$$\mu = \text{integer} \qquad (4.1.28)$$

5.

$$\left[B \pm 2(n+1)\sqrt{\nu A}\right]^4 + A(4D - \nu)\left[B \pm 2(n+1)\sqrt{\nu A}\right]^2 - 4A^2C^2 = 0 \qquad (4.1.29)$$

6.

$$\left[B \pm 2(n+1)\sqrt{\nu A}\right]^4 - A(4D + \nu)\left[B \pm 2(n+1)\sqrt{\nu A}\right]^2 + 4A^2C^2 = 0 \qquad (4.1.30)$$

7.

$$2(n+1)\sqrt{\nu AD} = \pm B\sqrt{D} \pm C\sqrt{A} \qquad \text{[4.1.31]}$$

The calculations for these potentials are given in the next few sections.

4.2   A Representation of QES Quantization Rule

We now bring out some common features of the exactly solvable models, that have been studied in this thesis and those reported in the paper [5]. For the exactly solvable model the QMF written in terms of suitable variables y takes the form

$$p(y) = \frac{b_1}{y - \xi_1} + \frac{b'_1}{y - \xi_2} + \cdots + \sum_{k=1}^{n} \frac{-i}{y - y_k} + R(y) \qquad (4.2.1)$$

where $\xi_1, \xi_2, \bullet\bullet\bullet$ are fixed poles, the summation term corresponds to n moving poles at $y_1, y_2, \bullet\bullet\bullet, y_n$ and $R(y)$ is atmost a polynomial in y. The residues $b_1, b'_1, \bullet\bullet\bullet$ have been calculated using the QHJ equation and demanding a condition such as the one proposed by Leacock and Padgett, or

$$\lim_{E \to 0} p(x, E) = i\ W(x),$$

or the square integrability of the wave-function. Thus in all the cases studied we are lead to a rational expression for the QMF.

Under an assumption about the behavior of QMF at infinity, even for QES models, the QMF turns out to be a rational function. The quantization rule

$$\oint p(x, E)dx = nh \qquad \textbf{(4.2.2)}$$

is then easily seen to be equivalent to the well know result, that for a rational function, sum of residues at all poles, including the one at infinity vanishes. Written explicitly for a rational form of QMF that we have, this requirement becomes

$$\sum_{\text{fixed poles}} (\text{Res of QMF}) + n + (\text{Res of QMF at infinity}) = 0 \qquad \textbf{(4.2.3)}$$

where Res stand for the residue and the middle term n, corresponds to the contri-
bution of moving poles to the residue.

In this and the next chapter, we will use this condition (4.2.3) as a substitute for
quantization rule.

4.3   Sextic Oscillator

The potential for the sextic oscillator is:

$$V(x) = \alpha x^2 + \beta x^4 + \gamma x^6, \qquad 7 > 0 \tag{4.3.1}$$

The QHJ equation is (ft $= 1 = 2$m)

$$p^2(x, E) \quad - \quad ip'(x, E) - (E - V) = 0 \tag{4.3.2}$$

For the nth excited state, the QMF has n poles on the real axis and we assume
that there are no other moving poles. We shall use the quantization condition viz.,

$$\frac{1}{2\pi} \oint_C p(x, E) dx = n\hbar \tag{4.3.3}$$

in the form (4.2.3) as given above.

To evaluate the integral in (4.3.3) a Laurent expansion of $\bar{p}(y)$ in powers of $y = 1/x$, is made

$$\bar{p}(y) = \frac{b_3}{y^3} + \frac{b_2}{b^2} + \frac{b_1}{y} + a_0 + a_1 y + \cdots \tag{4.3.4}$$

Substituting this in (4.3.2) and integrating term by term we get

$$J(E) = ia\backslash \tag{4.3.5}$$

The quantization condition gives

$$a_1 = \textbf{\textit{-in}} \tag{4.3.6}$$

It only remains to compute the coefficient a_1 of the Laurent expansion given in (4.3.4). To do this we start from the **QHJ** equation

$$\tilde{p}^2(y) + iy^2\tilde{p}'(y) - E + \frac{\alpha}{y^2} + \frac{\beta}{y^4} + \frac{\gamma}{y^6} = 0 \qquad \textbf{(4.3.7)}$$

Substituting the Laurent expansion and equating the coefficients of different powers of y on both sides of the equation we get

$$b_3 = \pm i\sqrt{\gamma} \qquad (4.3.8)$$

$$b_1 = -\frac{\beta}{2}b_3 \qquad (4.3.9)$$

$$a_1 = \frac{(-\alpha - b_1^2 + 3ib_3)}{2b_3} \qquad (4.3.10)$$

It is important to know that, we would get two solutions for b_1 corresponding to the two solutions of $b_3 = \pm i\sqrt{\gamma}$.This happens due to the fact that the QHJ is quadratic in the QMF. Thus one needs a boundary condition to pick the correct solution. We propose to use the square integrability of the wave-function instead of the original boundary condition, explained in chapter 2, which was proposed by Leacock and Padgett. This is because the original boundary condition is difficult to implement in the present case due to the presence of six branch points in the p_{cl}. In order to find the restrictions coining from the square integrability, we compute the wave-function

$$\psi(x) = \exp\left(\int ip(x)dx\right) \qquad \textbf{(4.3.11)}$$

for large x as follows. The most important term in the Laurent expansion (4.3.4) for small $y \approx 0$, corresponding to large x is

$$\tilde{p}(y) \approx \frac{b_3}{y^3} \qquad (4.3.12)$$

and the wave-function for large x becomes

$$\psi(x) \approx \exp\left(i\frac{b_3 x^4}{4}\right) \qquad (4.3.13)$$

Out of the two solutions, $b_3 = \pm i\sqrt{\gamma}, \psi(x)$ is square integrable only for $b_3 = i\sqrt{\gamma}$. Using this value of b_3 and from (4.3.9) and (4.3.10), equating a_1 to $-in$ we get

$$\frac{1}{\sqrt{\gamma}}\left(\frac{\beta^2}{4\gamma} - \alpha\right) = 3 + 2n \tag{4.3.14}$$

In order to compare the results in (4.3.14) with the well known condition, we write

$$\gamma = a^2, \ \beta = 2ab \tag{4.3.15}$$

Thus we get $\alpha = b^2 - a(3 + 2n)$ which agree with the result given in [6].

4.4   Sextic Oscillator with a Centrifugal Barrier

The potential is given as

$$V(x) = 4(s - \frac{1}{4})(s - \frac{3}{4})\frac{1}{x^2} + [b^2 - 4a(s + \frac{1}{2} + \mu)]x^2 + 2abx^4 + a^2x^6 \tag{4.4.1}$$

We shall consider only the case $s > |$ so that the coeffi cient of the centrifugal term, $\frac{1}{-}$ is positive. The Q.H.J equation is (ft $= 1 = 2m$)

$$p^2(x, E) - ip'(x, E) - (E - V) = 0 \tag{4.4.2}$$

Substituting the potential the QHJ equation is

$$p^2(x, E) - ip'(x, E) - [E - 4(s - \frac{1}{4})(s - \frac{3}{4})\frac{1}{x^2} - [b^2 - 4a(s + \frac{1}{2} + \mu)]x^2 - 2abx^4 - a^2x^6 = 0 \tag{4.4.3}$$

$p(x, E)$ has poles at $x = 0$ and as the potential is symmetric there are moving poles on either side of the origin. We assume that there are no more poles in the complex plane. We assume the infinity is a pole. We find below the residues for each of these pole.

We expand $p(x, E)$ as

$$p(x, E) = \frac{b_1}{x} + a_0 + a_1 x + \cdots \tag{4.4.4}$$

Using (4.4.4) in (4.4.3) and equating the power of $\frac{1}{x^2}$ we get

$$b_1 = -\frac{i}{2}\left[1 \pm (4s - 2)\right] \tag{4.4.5}$$

Demanding that the wave-function remains finite, for $x \to 0$, gives

$$b_1 = -\frac{i}{2}\left[4s - 1)\right] \tag{4.4.6}$$

To find residue for the pole at infinity, we use the mapping $x = \frac{1}{t}$. Therefore equation (4.4.3) transforms to

$$\tilde{p}^2(t, E) + it^2 \tilde{p}'(t, E)$$

$$-\left[E - 4(s - \frac{1}{4})(s - \frac{3}{4})t^2 - [b^2 - 4a(s + \frac{1}{2} + \mu)]\frac{1}{t^2} - 2ab\frac{1}{t^4} - a^2\frac{1}{t^6}\right] = 0 \tag{4.4.7}$$

We expand $\tilde{p}(t, E)$ as

$$\tilde{p}(t, E) = \frac{d_3}{t^3} + \frac{d_2}{t^2} + \frac{d_1}{t} + c_0 + c_1 t + c_2 t^2 + \cdots \tag{4.4.8}$$

Using (4.4.8) in (4.4.7) and equating different power of t we get

$$d_3 = \pm ia \tag{4.4.9}$$

$$d_2 = \frac{-ab}{d_3} \tag{4.4.10}$$

$$c_0 = 0 \tag{4.4.11}$$

$$c_1 = \frac{2a(s + \frac{1}{2} + \mu)}{d_3} + \frac{3i}{2} \tag{4.4.12}$$

The correct sign of d_3 is fixed by the condition of square integrability and is given **by**

$$d_3 = -\ ia \tag{4.4.13}$$

Now equating the sum of all residues to zero, we get

$$ib_1 + 2n = c_1 \tag{4.4.14}$$

Substituting the values of b_1 and c_1 in the above relation yields the required condition, viz

$$n = \mu \tag{4.4.15}$$

The above condition agrees with those given in [6]

4.5   Circular Potential

The potential is given as

$$V(x) = \frac{A}{\sin^2 x} + \frac{B}{\cos^2 x} + C \sin^2 x - D \sin^4 x \tag{4.5.1}$$

where

$$A = 4(s_1 - \frac{1}{4})(s_1 - \frac{3}{4}) \tag{4.5.2}$$

$$B = 4(s_2 - \frac{1}{4})(s_2 - \frac{3}{4}) \tag{4.5.3}$$

$$C = q_1^2 + 4q_1(s_1 + s_2 + \mu) \tag{4.5.4}$$

$$D = q_1^2 \tag{4.5.5}$$

We effect a change of variable by

$$y - \sin^2 x \tag{4.5.6}$$

The Q.H.J equation is

$$p^2(x, E) - ip'(x, E) - (E - V) = 0 \tag{4.5.7}$$

In the new variable the QH.I equation is

$$\bar{p}^2(y, E) - 2i\sqrt{y}\sqrt{1 - y}\bar{p}'(y, E) - \left[E - \frac{A}{y} - \frac{B}{1 - y} - Cy + Dy^2\right] = 0 \tag{4.5.8}$$

Let

$$\tilde{p}(y, E) = -2i\sqrt{y}\sqrt{1 - y}\phi \tag{4.5.9}$$

In terms of the above transformation the QHJ becomes

$$\phi^2 + \phi' + \frac{1}{2}\frac{(1 - 2y)}{y(1 - y)}\phi + \frac{1}{4}\frac{1}{y(1 - y)}\left[E - \frac{A}{y} - \frac{B}{1 - y} - Cy + Dy^2\right] = 0 \tag{4.5.10}$$

ϕ has poles at $y = 0$ and at $y = +1$ and there are a finite number of moving poles in the complex plane. We assume that there are no more poles in the complex plane. We find the residues for each of these pole below.

For $y = 0$ we consider an expansion in ϕ as

$$\phi = \left(\frac{b_1}{y} + a_0 + a_1y + \cdots\right) \tag{4.5.11}$$

Using (4.5.11) in (4.5.10) and equating the various powers of y we get the following. The power of $\frac{1}{y^2}$ gives

$$b_1 = \frac{1}{2}\left[\frac{1}{2} \pm (2s_1 - 1)\right] \tag{4.5.12}$$

The correct value of $b\backslash$ is fixed by the condition of square integrability of the wave-function and is given below as

$$b_1 = \frac{1}{2}\left[\frac{1}{2} + (2s_1 - 1)\right] \tag{4.5.13}$$

For $y = 1$ we consider an expansion in ϕ as

$$\phi = \left(\frac{b_1'}{y - 1} + a_0' + a_1'(y - 1) + \cdots\right) \tag{4.5.14}$$

Using (4.5.14) in (4.5.10) and equating the various powers of y we get the following. The power of $\frac{1}{(y-1)^2}$ gives

$$b_1' = \frac{1}{2}\left[\frac{1}{2} \pm (2s_2 - 1)\right] \tag{4.5.15}$$

The correct value of b_1' is fixed by the condition of square integrability of the wave-function and is given below as

$$b_1' = \frac{1}{2}\left[\frac{1}{2} + (2s_2 - 1)\right] \tag{4.5.16}$$

To find residue for the pole at infinity, we use the mapping $y = \frac{1}{t}$. Therefore equation (4.5.10) transforms to

$$\tilde{\phi}^2(t, E) - t^2\tilde{\phi}'(t, E) + \frac{1}{2}\frac{t(t-2)}{t-1}\tilde{\phi}(t, E)$$

$$\left[\frac{E}{4}\frac{t^2}{t-1} - \frac{A}{4}\frac{t^3}{t-1} - \frac{B}{4}\frac{t^3}{(t-1)^2} + \frac{C}{4}\frac{t}{t-1} + \frac{D}{4}\frac{1}{t-1}\right] \tag{4.5.17}$$

We expand $\tilde{\phi}(t, E)$ as

$$\tilde{\phi}(t, E) = \frac{d_1}{t} + c_0 + c_1 t + \cdots \tag{4.5.18}$$

Using (4.5.18) in (4.5.17) and equating the power of $\frac{1}{t^2}$ we get

$$d_1 = 0 \tag{4.5.19}$$

Equating the constant term we have

$$c_0 = \pm\frac{q_1}{2} \tag{4.5.20}$$

Equating the power of t gives the residue at infinity, the coefficient of $\frac{1}{t}$ as

$$c_1 = \frac{q_1(s_1 + s_2 + \mu)}{2c_0} - \frac{1}{2} \tag{4.5.21}$$

The correct sign of c_0 is fixed by the condition of square integrability and is given by

$$c_0 = \frac{q_1}{2} \tag{4.5.22}$$

Now equating the sura of all residues of fixed poles **and** the moving poles and the pole at infinity, we have the following relation.

$$b_1 + b_1' + n = c, \tag{4.5.23}$$

Substituting the values of b_1, b_1', and $C\backslash$ in the above relation yields the required condition

$$\mu = n \tag{4.5.24}$$

The above condition agrees with those given in [6]

4.6   Hyperbolic Potential

The potential for the hyperbolic potential is:

$$V(x) = -\frac{A}{\cosh^2 x} + \frac{B}{\sinh^2 x} - C\cosh^2 x + D\cosh^4 x \tag{4.6.1}$$

where

$$A = 4(s_1 - \frac{1}{4})(s_1 - \frac{3}{4})$$

$$B = 4(s_2 - \frac{1}{4})(s_2 - \frac{3}{4})$$

$$C = [q_1^2 + 4q_1(s_1 + s_2 + \mu)]$$

$$D = q_1^2$$

We will consider the case $s_2 > |$.

The Q.H.J equation is (ft $= 1 = 2m$)

$$p^2(x, E) - ip'(x, E) - (E - V) = 0 \tag{4.6.2}$$

We use a mapping is $y = \cosh x$

The Q.H.J equation in the new variable is:

$$\bar{p}(y, E) - i\hbar\sqrt{y^2 - 1}\, p'(y, E) - [E + \frac{A}{y^2} - \frac{B}{y^2 - 1} + Cy^2 - Dy^4] = 0$$

Let

$$\bar{p}(y, E) = -i\sqrt{y^2 - 1}\,\phi(y, E) \tag{4.6.3}$$

and hence

$$\bar{p}\,'(y, E) = -i\{\sqrt{y^2 - 1}\,\phi' + \phi\frac{y}{\sqrt{y^2 - 1}}\} \tag{4.6.4}$$

Therefore the Q.H.J. equation becomes

$$\phi^2 + \phi' + \frac{y}{y^2 - 1}\phi + \frac{1}{y^2 - 1}[E + \frac{A}{y^2} - \frac{B}{y^2 - 1} + Cy^2 - Dy^4] = 0 \tag{4.6.5}$$

The Q.H.J equation is written as:

$$\phi^2 + \phi' + [\frac{1}{2}\frac{1}{y+1} + \frac{1}{2}\frac{1}{y-1}]\phi + E[\frac{1}{2}\frac{1}{y-1} - \frac{1}{2}\frac{1}{y+1}] + A[-\frac{1}{y^2} - \frac{1}{2}\frac{1}{y+1} + \frac{1}{2}\frac{1}{y-1}]$$

$$-B[-\frac{1}{4}\frac{1}{y-1} + \frac{1}{4}\frac{1}{(y-1)^2} + \frac{1}{4}\frac{1}{y+1} + \frac{1}{4}\frac{1}{(y+1)^2}]$$

$$+C[1 + \frac{1}{2}\frac{1}{y-1} - \frac{1}{2}\frac{1}{y+1}] - D[1 + y^2 + \frac{1}{2}\frac{1}{y-1} - \frac{1}{2}\frac{1}{y+1}] = 0 \tag{4.6.6}$$

For $y = 0$

Let

$$\phi = \frac{b_1}{y} + \sum_{n=0}^{\infty} a_n y^n$$

$$-\frac{i}{2\pi}\oint \phi\, dy = -\frac{i}{2\pi}\oint (\frac{b_1}{y} + \sum_{n=0}^{\infty} a_n y^n)\, dy = -\frac{i}{2\pi}2\pi i b_1 = b_1$$

The QHJ equation (4.6.6) becomes

$$(\frac{b_1}{y} + a_0 + a_1 y + \ldots)^2 + (-\frac{b_1}{y^2} + a_1 + \ldots) + [\frac{1}{2}\frac{1}{y+1} + \frac{1}{2}\frac{1}{y-1}]\phi + E[\frac{1}{2}\frac{1}{y-1} - \frac{1}{2}\frac{1}{y+1}]$$

$$+A[-\frac{1}{y^2} - \frac{1}{2}\frac{1}{y+1} + \frac{1}{2}\frac{1}{y-1}] - B[-\frac{1}{4}\frac{1}{y-1} + \frac{1}{4}\frac{1}{(y-1)^2} + \frac{1}{4}\frac{1}{y+1} + \frac{1}{4}\frac{1}{(y+1)^2}]$$

$$+C[1 + \frac{1}{2}\frac{1}{y-1} - \frac{1}{2}\frac{1}{y+1}] - D[1 + y^2 + \frac{1}{2}\frac{1}{y-1} - \frac{1}{2}\frac{1}{y+1}] = 0 \qquad (4.6.7)$$

The coefficient of $\frac{1}{.z}$ gives

$$b_1^2 - b_1 - A = 0 \qquad (4.6.8)$$

and hence

$$b_1 = \frac{1}{2} \pm (2s_1 - 1) \qquad (4.6.9)$$

For $y = 1$, let

$$\phi = \frac{b_1'}{y-1} + \sum_{n=0}^{\infty} a_n'(y-1)^n \qquad ;4.6.10)$$

Therefore

$$-\frac{i}{2\pi} \oint \phi dy = -\frac{i}{2\pi} \oint [\frac{b_1'}{(y-1)} + \sum_{n=0}^{\infty} a_n'(y-1)^n] dy = -\frac{i}{2\pi} 2\pi i b_1' = b_1'$$

Substituting (4.6.10) in (4.6.6) and equating the coefficient of $\frac{1}{(y-1)^2}$ we get

$$b_1'^2 - \frac{1}{2}b_1' - B\frac{1}{4} = 0 \qquad (4.6.11)$$

which yields

$$b_1' = \frac{1}{4} \pm \frac{1}{2}(2s_2 - 1) \qquad (4.6.12)$$

For $y = -1$, let

$$\phi = \frac{b_1''}{y+1} + \sum_{n=0}^{\infty} a_n''(y+1)^n \qquad (4.6.13)$$

Therefore

$$-\frac{i}{2\pi} \oint \phi dy = -\frac{i}{2\pi} \oint [\frac{b_1''}{(y+1)} + \sum_{n=0}^{\infty} a_n''(y+1)^n] dy = -\frac{i}{2\pi} 2\pi i b_1'' = b_1''$$

Substituting (4.6.13) in (4.6.6) and equating the coefficient of $\frac{1}{(y+1)^2}$ we get

$$b_1''^2 - \frac{1}{2}b_1'' - \frac{B}{4} = 0 \qquad (4.6.14)$$

On simplification yields the same value as b_1'. Hence

$$b_1'' = \frac{1}{4} \pm \frac{1}{2}(2s_2 - 1) \tag{4.6.15}$$

To calculate the residue at $y = \infty$, we apply the mapping $y = -$

The QHJ equation (4.6.6) becomes

$$\tilde{\phi}^2(u, E) + (-u^2)\tilde{\phi}'(u, E) + [\frac{1}{2}\frac{u}{1+u} + \frac{1}{2}\frac{u}{1-u}]\tilde{\phi}(u, E)$$

$$+E[\frac{1}{2}\frac{u}{1+u} - \frac{1}{2}\frac{u}{1+u}] + A[-u^2 - \frac{1}{2}\frac{u}{1+u} + \frac{1}{2}\frac{u}{1-u}]$$

$$-B[-\frac{1}{4}\frac{u}{1-u} + \frac{1}{4}\frac{u^2}{(1-u)2} + \frac{1}{4}\frac{u}{1+u} + \frac{1}{4}\frac{u^2}{(1+u)^2}] + C[1 + \frac{1}{2}\frac{u}{1-u} - \frac{1}{2}\frac{u}{1+u}]$$

$$-D[1 + \frac{1}{u^2} + \frac{1}{2}\frac{u}{1-u} - \frac{1}{2}\frac{u}{1+u}] = 0 \tag{4.6.16}$$

Lei

$$\tilde{\phi}(u, E) = \frac{c_1}{u} + d_0 + d_1 u + \cdots \tag{4.6.17}$$

The residue at infinity is the coefficient of d_1 and is given as

$$d_1 = -\frac{2q_1(s_1 + s_2 + \mu)}{c_1} - 1 \tag{4.6.18}$$

with

$$c_1 = \pm q_1 \tag{4.6.19}$$

Equating the sum of all residues of fixed poles, the moving poles to the pole at infinity, we have the following relation.

$$d_1 = 2n + b_1 + b_1' + b_1'' \tag{4.6.20}$$

The equation (4.6.6) does not change when replacements $y \to -y$ and $\phi \to -\phi$ is made. Therefore we select the residue $b_1' = b_1''$ and the condition of finiteness of the wave-function at $x = 0$ requires that positive sign be selected in (4.6.12) and (4.6.15). Therefore

$$b_1' = b_1'' = s_2 - \frac{1}{4} \qquad (4.6.21)$$

The point $y = 0$ corresponds to complex value of x and therefore one cannot insist on finiteness of the wave-function at $x = 0$. One must fall back on the boundary condition given in chapter 2 section 2.3.1. We will simply note that selecting positive sign in (4.4.19) leads us to the correct condition

$$\mu = n \qquad (4.6.22)$$

for quasi exact solvability [6]. In this chapter, our objective has been to show that QES conditions follows from our assumption that point at inifnity is an isolated singular point. In the cases where for certain ranges of potential parameters, both the residues are acceptable, one must accept all such answers and work out the consequences. This may lead to some new and interesting results as is evidenced by the investigations on phases of super symmetry [13] and periodic potentials [15].

Besides the above QES potentials, we now take up three classes of QES potentials [9,10] and find the conditions for quasi exact solvability within our approach.

$$V(x) = A\sinh^2\sqrt{\nu}x + B\sinh\sqrt{\nu}x + C\tanh\sqrt{\nu}x\,\text{sech}\sqrt{\nu}x + D\text{sech}^2\sqrt{\nu}x$$

The potential is given as

$$V(x) = A\sinh^2\sqrt{\nu}x + B\sinh\sqrt{\nu}x + C\tanh\sqrt{\nu}x\,\text{sech}\cdot\sqrt{\nu}x + D\sec h^2\sqrt{\nu}x \qquad (4.7.1)$$

We effect a change of variable by

$$y = \sinh\sqrt{\nu}x \qquad \textbf{(4.7.2)}$$

The QHJ equation is $(\hbar = 1 = 2m)$

$$p^2(x, E) - ip'(x, E) - (E - V) = 0 \qquad (4.7.3)$$

In the new variable the QHJ equation is

$$\tilde{p}^2(y, E) - i\sqrt{\nu}\sqrt{1 + y^2}\tilde{p}'(y, E) - \left[E - Ay^2 - By - C\frac{y}{1 + y^2} - D\frac{1}{1 + y^2} \right] = 0 \quad (4.7.4)$$

Let

$$\tilde{p}(y, E) = -i\sqrt{\nu}\sqrt{1 + y^2}\phi \qquad (4.7.5)$$

In terms of the above transformation the QHJ becomes

$$[\phi + \frac{1}{2}\frac{y}{1 + y^2}]^2 + \phi' - \frac{1}{4}\frac{y^2}{(1 + y^2)^2}$$

$$+ \frac{1}{\nu(1 + y^2)}\left[E - Ay^2 - By - C\frac{y}{1 + y^2} - D\frac{1}{1 + y^2} \right] = 0 \qquad (4.7.6)$$

Lei

$$\chi = \phi + \frac{1}{2}\frac{y}{1 + y^2} \qquad (4.7.7)$$

Therefore the above equation becomes

$$\chi^2 + \chi' + \frac{3}{4}\frac{y^2}{(1 + y^2)^2} - \frac{1}{2}\frac{1}{1 + y^2}$$

$$+ \frac{1}{\nu(1 + y^2)}\left[E - Ay^2 - By - C\frac{y}{1 + y^2} - D\frac{1}{1 + y^2} \right] = 0 \qquad (4.7.8)$$

X has poles at $y = \pm i$ and there are moving poles between the turning points. We assume that there are no more poles in the complex line. We find the residues for each of these pole below.

For $y = i$ we consider an expansion in χ as

$$\chi = \left(\frac{b_1}{y - i} + a_0 + a_1(y - i) + \cdots \right) \qquad (4.7.9)$$

Using (4.7.9) in (4.7.8) and equating the various powers of y we get the following. The power of $\frac{1}{(y-i)^2}$ gives

$$b_1 = \frac{1}{2}\left[1 \pm \frac{1}{2}\sqrt{1 - 4(\frac{D}{\nu} + \frac{iC}{\nu})}\right] \tag{4.7.10}$$

Fory $= -\ i$ we consider an expansion in χ as

$$\chi = \left(\frac{b'_1}{y+i} + a'_0 + a'_1(y+i) + \cdots\right) \tag{4.7.11}$$

Using (4.7.11) in (4.7.8) and equating the power of $,\ \frac{1}{+i)^2}$ gives

$$b'_1 = \frac{1}{2}\left[1 \pm \frac{1}{2}\sqrt{1 - 4(\frac{D}{\nu} - \frac{iC}{\nu})}\right] \tag{4.7.12}$$

To find the pole at infinity, we use the mapping $y = \frac{1}{t}$ Therefore equation (4.7.8) transforms to

$$\chi^2(t, E) - t^2\chi'(t, E) + \frac{3}{4}\frac{t^2}{(t^2+1)^2} - \frac{1}{2}\frac{t^2}{t^2+1}$$
$$+\frac{1}{\nu(t^2+1)}[Et^2 - A - Bt - C\frac{t^5}{t^2+1} - D\frac{t^6}{t^2+1}] = 0 \tag{4.7.13}$$

For the point at infinity we expand χ as

$$\tilde{\chi}(t, E) = d_0 + d_1 t + d_2 t^2 + \cdots \bullet \tag{4.7.14}$$

Using (4.7.14) in (4.7.13) and equating the constant term we get

$$d_0 = \pm\sqrt{\frac{A}{\nu}} \tag{4.7.15}$$

and equating the term in i we have

$$d_1 = \frac{B}{2\nu d_0} \tag{4.7.16}$$

Now equating the sum of residues due to fixed poles moving poles and that at infinity,to zero we have the following relation.

$$b_1 + b'_1 + n = d_1 \qquad (4.7.17)$$

Substituting the values of b_1, b'_1 and d_1 in the above relation yields the required condition given below

$$B^4 + 16A^2\nu^2(n+1)^4 + 24AB^2\nu(n+1)^2 + 4AB^2D - AB^2\nu(n+1)^2$$

$$\pm 32AB\nu\sqrt{A\nu}(n+1)^3 \pm 4AB\nu\sqrt{A\nu}(n+1) - AB^2\nu - 4A^2\nu^2(n+1)^2$$

$$+16A^2D\nu(n+1)^2 \pm 16ABD\sqrt{A\nu}(n+1) \pm 8B^3\sqrt{A\nu}(n+1) - 4A^2C^2 \qquad (4.7.18)$$

This can be written in the compact form as:

$$\left[B \pm 2(n+1)\sqrt{\nu A}\right]^4 + A(4D - \nu)\left[B \pm 2(n+1)\sqrt{\nu A}\right]^2 - 4A^2C^2 = 0 \qquad (4.7.19)$$

The above condition agrees with those given in [9,10]

4.8 $V(x) = A\cosh^2\sqrt{\nu}x + B\cosh\sqrt{\nu}x + C\coth\sqrt{\nu}x\,\mathrm{csch}\sqrt{\nu}x + D\,\mathrm{csch}^2\sqrt{\nu}x$

The potential is given as

$$V(x) = A\cosh^2\sqrt{\nu}x + B\cosh\sqrt{\nu}x + C\coth\sqrt{\nu}x\,\mathrm{csch}\sqrt{\nu}x + D\,\mathrm{csch}^2\sqrt{\nu}x \qquad (4.8.1)$$

We effect a change of variable by

$$y = e^{\sqrt{\nu}x} \qquad (4.8.2)$$

The QHJ equation is $(\hbar = 1 = 2m)$

$$p^2(x, E) - ip'(x, E) - (E - V) = 0 \qquad (4.8.3)$$

In the new variable the QHJ equation is

$$\tilde{p}^2(y, E) - i\sqrt{\nu}\tilde{p}'(y, E)$$

$$-\left[E - A\frac{1}{4y^2}(y^2 + 1)^2 - B\frac{1}{2y}(y^2 + 1)^2 - C\frac{2y(y^2 + 1)}{(y^2 - 1)^2} - D\frac{4y^2}{(y^2 - 1)^2}\right] = 0 \quad (4.8.4)$$

Let

$$\tilde{p}(y, E) = -i\sqrt{\nu}y\phi \qquad (4.8.5)$$

In terms of the above transformation the QHJ becomes

$$[\phi + \frac{1}{2y}]^2 + \phi' - \frac{1}{4y^2}$$

$$+\frac{1}{\nu y^2}\left[E - A\frac{1}{4y^2}(y^2 + 1)^2 - B\frac{1}{2y}(y^2 + 1)^2 - C\frac{2y(y^2 + 1)}{(y^2 - 1)^2} - D\frac{4y^2}{(y^2 - 1)^2}\right] = 0$$

$$(4.8.6)$$

Let

$$\chi = \phi + \frac{1}{2y} \qquad (4.8.7)$$

Therefore the above equation becomes

$$\chi^2 + \chi' + \frac{1}{4y^2}$$

$$+\frac{1}{\nu y^2}\left[E - A\frac{1}{4y^2}(y^2 + 1)^2 - B\frac{1}{2y}(y^2 + 1)^2 - C\frac{2y(y^2 + 1)}{(y^2 - 1)^2} - D\frac{4y^2}{(y^2 - 1)^2}\right] = 0$$

$$(4.8.8)$$

χ has poles at $y = 0$ and at $y = \pm1$ and there are moving poles between the turning points. We assume that there are no more poles in the complex line. We find the residues for each of these pole below.

For $y = 0$ we consider an expansion in χ as

$$\chi = \left(\frac{b_2}{y^2} + \frac{b_1}{y} + a_0 + a_1 y + \cdots\right) \qquad (4.8.9)$$

Using (4.8.9) in (4.8.8) and equating the power of $\frac{1}{.4}$ gives

$$b_2 \doteq \pm\sqrt{\frac{A}{4\nu}} \tag{4.8.10}$$

The power of $\frac{1}{.3}$ gives

$$b_1 = 1 + \frac{B}{4\nu b_2} \tag{4.8.11}$$

For $y = 1$ we consider an expansion in χ as

$$\chi = \left(\frac{b_1'}{y-1} + a_0' + a_1'(y-1) + \cdots \right) \tag{4.8.12}$$

Using (4.8.12) in (4.8.8) and equating the power of $\frac{1}{(y-1)^2}$ gives

$$b_1' = \frac{1}{2}\left[1 \pm \sqrt{1 + 4(\frac{D}{\nu} + \frac{C}{\nu})} \right] \tag{4.8.13}$$

For $y = -\ 1$ we consider an expansion in χ as

$$\chi = \left(\frac{b_1''}{y+1} + a_0'' + a_1''(y+1) + \cdots \right) \tag{4.8.14}$$

Using (4.8.14) in (4.8.8) and equating the power of $\frac{1}{(y+1)^2}$ gives

$$b_1'' = \frac{1}{2}\left[1 \pm \sqrt{1 + 4(\frac{D}{\nu} - \frac{C}{\nu})} \right] \tag{4.8.15}$$

To find the pole at infinity, we use the mapping $y = \frac{1}{t}$. Therefore equation (4.8.8) transforms to

$$\tilde{\chi}^2(t, E) - t^2\tilde{\chi}'(t, E) + \frac{1}{4}t^2$$

$$+\frac{1}{\nu}t^2[E - \frac{A}{4}\frac{(1+t^2)^2}{t^2} - \frac{B}{2}\frac{1+t^2}{t} - 2C\frac{(1+t^2)t}{(1-t^2)^2} - 4D\frac{t^2}{(1-t^2)^2}] = 0 \tag{4.8.16}$$

we expand $x(t, E)$ as

$$\tilde{\chi}(t, E) = d_0 + d_1 t + d_2 t^2 + \cdots \tag{4.8.17}$$

Using (4.8.17) in (4.8.16) and equating the constant term we get

$$d_0 = \pm\sqrt{\frac{A}{4\nu}}$$

(4.8.18)

Equating the term in t we have

$$d_1 = \frac{B}{4\nu d_0}$$

(4.8.19)

Now equating the sum of residues due to fixed poles, the moving poles, and that at infinity, to zero we have the following relation.

$$b_1 + b'_1 + b''_1 + 2n = d_1$$

(4.8.20)

Substituting the values of b_1, b'_1, b''_1 and d_1 in the above relation yields the required condition This can be written in the compact form as:

$$\left[B \pm 2(n+1)\sqrt{\nu A}\right]^4 - A(4D + \nu)\left[B \pm 2(n+1)\sqrt{\nu A}\right]^2 + 4A^2C^2 = 0$$

(4.8.21)

The above condition agrees with those given in [9,10]

4.9 $V(x) = Ae^{2\sqrt{\nu}x} + Be^{\sqrt{\nu}x} + Ce^{-\sqrt{\nu}x} + De^{-2\sqrt{\nu}x}$

The potential is

$$V(x) = Ae^{2\sqrt{\nu}x} + Be^{\sqrt{\nu}x} + Ce^{-\sqrt{\nu}x} + De^{-2\sqrt{\nu}x}$$

(4.9.1)

The QHJ equation is given by $(\hbar = 1 = 2m)$

$$p^2(x, E) - ip'(x, E) - [E - Ae^{2\sqrt{\nu}x} - Be^{\sqrt{\nu}x} - Ce^{-\sqrt{\nu}x} - De^{-2\sqrt{\nu}x} = 0$$

(4.9.2)

We use a change of variable by

$$y = e^{\sqrt{\nu}x}$$

(4.9.3)

Therefore the QHJ transforms to

$$\tilde{p}^2(y, E) - i\sqrt{\nu}y\tilde{p}'(y, E) - [E - Ay^2 - By - C\frac{1}{y} - D\frac{1}{y^2}] = 0 \qquad (4.9.4)$$

Let

$$\tilde{p}(y, E) = -i\sqrt{\nu}y\phi \qquad (4.9.5)$$

Therefore the above equation becomes

$$(\phi + \frac{1}{2y})^2 + \phi' - \frac{1}{4y^2} + \frac{1}{\nu y^2}\left[E - Ay^2 - By - C\frac{1}{y} - D\frac{1}{y^2}\right] = 0 \qquad (4.9.6)$$

Let

$$\chi = \phi + \frac{1}{2y} \qquad (4.9.7)$$

With this transformation we get

$$\chi^2 + \chi' + \frac{1}{4y^2} + \frac{1}{\nu y^2}\left[E - Ay^2 - By - C\frac{1}{y} - D\frac{1}{y^2}\right] = 0' \qquad (4.9.8)$$

X has poles at $y = 0$ and there are moving poles between the turning points. We assume that there are no more poles in the complex line other than a pole at infinity. We compute the residue for $y = 0$

For $y = 0$ we define

$$\chi = \frac{b_2}{y^2} + \frac{b_1}{y} + a_0 + a_1 y + \cdots \qquad (4.9.9)$$

Substituting (4.9.9) in (4.9.8) we get

$$(\frac{b_2}{y^2} + \frac{b_1}{y} + a_0 + a_1 y + \cdots)^2 + (-2\frac{b_2}{y^3} - \frac{b_1}{y^2} + a_1 + \cdots) + \frac{1}{4y^2}$$

$$+ \frac{1}{\nu y^2}\left[E - Ay^2 - By - C\frac{1}{y} - D\frac{1}{y^2}\right] = 0 \qquad (4.9.10)$$

Equating the coefficient of $\frac{1}{t^4}$ we get

$$b_2 = \pm\sqrt{\frac{D}{\nu}} \tag{4.9.11}$$

Equating the coefficient of $\frac{1}{t^3}$ we get

$$b_1 = 1 + \frac{C}{2b_2\nu} \tag{4.9.12}$$

We assume that infinity is a pole and compute the residue at infinity, we which e use the mapping

$$y = \frac{1}{t} \tag{4.9.13}$$

$$\tilde{\chi}^2(t, E) - t^2\tilde{\chi}'(t, E) + \frac{t^2}{4} + \frac{t^2}{\nu}\left[E - A\frac{1}{t^2} - B\frac{1}{t} - Ct - Dt^2\right] = 0 \tag{4.9.14}$$

We use a Laurent's expansion of the form

$$\tilde{\chi}(t, E) = d_0 + d_1 t + d_2 t^2 + \cdots \tag{4.9.15}$$

Using (4.9.15) in (4.9.14) and equating the of t the constant term yields

$$d_0 = \pm\sqrt{\frac{A}{\nu}} \tag{4.9.16}$$

and the coefficient of t yields

$$d_1 = \frac{B}{2d_0\nu} \tag{4.9.17}$$

Now equating the sum of residues due to fixed poles, the moving poles, and that at infinity, to zero we have the following relation.

$$b_1 + n = d_1 \tag{4.9.18}$$

Substituting the values of b_1 and d_1 in the above equation we get.

$$\pm \frac{C\sqrt{\nu}}{2\nu\sqrt{D}} + (n+1) = \pm \frac{B\sqrt{\nu}}{2\nu\sqrt{A}} \tag{4.9.19}$$

The above on simplification gives the desired condition given below which agrees with that given in [9,10].

$$2(n+1)\sqrt{\nu AD} = \pm B\sqrt{D} \pm C\sqrt{A} \tag{4.9.20}$$

4.10   Quartic Oscillator

We end this chapter with a short analysis of quartic an-harmonic oscillator and give some remarks on polynomial potentials of degree different from six.

First we consider the x^4 oscillator with

$$V(x) = \alpha x + \beta x^2 + \gamma x^3 + \delta x^4, \qquad \delta > 0 \tag{4.10.1}$$

We ask whether this model is QES for any choice of parameters. We repeat the analysis given for sextic oscillator, assuming that the point at infinity is an isolated singular point, a pole of some order m. We therefore substitute

$$p(x, E) = b_m x^m + b_{m-1} x^{m-1} + \cdots \tag{4.10.2}$$

in the QHJ equation and determine the constants b_m. We find that $b_m = 0$ for $m > 2$ and $b_2 = \pm i\sqrt{\delta}$. Thus the corresponding bound state wave-function for large x will behave as

$$\psi(x) \sim \exp\left(i\frac{b_2}{3}x^3\right) \tag{4.10.3}$$

and for both the choices $\pm i\sqrt{\delta}$ for b_2, one gets wave-function which grows either at + infinity or — infinity. Hence the assumptions, that the point at infinity is an isolated singular point of QMF, is inconsistent with QHJ for real parameter α, β, γ and δ. Thus, x^4 oscillator does not lead to any choice of real parameter. However for complex parameters, one can get the known results for QES quartic model [18].

4.11 Summary and Observations

From our study in this chapter we arrive at the following conclusions.

1. For the QES models, the QMF corresponding to the algebraic part of the spectrum has singularity structure very similar to the exact solvable models.

2. The integer n appearing in the exact quantization condition is just the number of moving poles of QMF in the complex plane. In the case of exact solvable models the moving poles are in a one to one correspondence with the real nodes of the wave-functions, but a corresponding statement for QES model is not true. This result and some other interesting properties will be explicitly demonstrated in the next chapter for bound state wave-functions of QES models.

3. The integer n in the right hand side of the quantization condition appears in the condition of quasi-exact solvability of the potential.

4. The condition of quasi-exact solvability is equivalent to our assumption about the behavior of QMF at infinity, reflecting the simplification of the singularity structure for the QES bound states.

Chapter 5

CALCULATION OF WAVE-FUNCTIONS FOR QES MODELS

In the previous chapter we have seen that the condition for exact solvability arises from a simple requirement on the behavior of QMF at infinity. We continue our study of QES models and we take up an investigation of wave-functions. We find that the wave-functions can be computed by proceeding as in the case of exactly solvable models. We begin with our simplifying assumption mentioned in the previous chapter for the QES models and proceed in the same fashion as for the case of exactly solvable models in chapter 3. Thus the QMF is meromorphic and the corresponding residues at the poles are known, and also the behavior at infinity is known, with this information the bound state wave-functions can be obtained as in chapter 3. We give our results for two potential models viz. the sextic oscillator and the hyperbolic potential. This study reveals a new interesting feature of the zeros of the wave-functions, which will be discussed at the end of this chapter.

5.1 Sextic Oscillator

The potential for the sextic oscillator is:

$$V(x) = ax^2 + \beta x^4 + \gamma x^6, \qquad 7 > 0 \tag{5.1.1}$$

with the following values for α, β, γ and the condition for μ, n, p where p stands for parity

$\alpha = h^2 - a(3 + 2n),\ fi = 2ab,\ 7 = a^2,\ 4\mu + 2p = 2n$ with $p = 0\ or\ 1$

The QHJ equation is $(\hbar = 1 = 2m)$

$$p(x,\ E) - ip'(x,\ E) - \{E - \alpha x^2 - \beta x^4 - \gamma x^6\} = 0 \tag{5.1.2}$$

We assume that the point at infinity is a pole. Therefore $p(x, E)$ behaves as x^n for some n

$$p(x, E) \sim x^n$$

for large *x*. Hence $p(x, E)$ takes the form for large *x*.

$$p(x,\ E) = a_3 x^3 + a_2 x^2 + a_1 x + a_0 + O(\tfrac{-}{-}) \tag{5.1.3}$$

where $a_0, a_1 \cdots, a_3$ are constants, on the assumption that $p(x, E)$ have no other singular points and substitute (5.1.3) in equation (5.1.2). Next we equate the coefficient of powers of x^6 to zero, gives

$$a_n^2 + 7 = 0 \tag{5.1.4}$$

Since $7 = a^2$, we have

$$a_3 = \pm ia \tag{5.1.5}$$

As a_3 has two values, the correct value is fixed by the condition of square integrability on the wave function.

$$\psi(x) = \exp\left(i \int p(x, E) dx\right) = \exp\left(i \int (a_3 x^3 + a_1 x) dx\right)$$

If the above integral have to bounded at infinity, the we require that

$$a_3 = +ia \tag{5.1.6}$$

Next equating the coefficient of successive powers x^5, x^4,... to zero we get

$$a_2 = 0 \tag{5.1.7}$$

$$a_1 = -\frac{ab}{a_3} \tag{5.1.8}$$

$$a_0 = 0 \tag{5.1.9}$$

Hence

$$a_1 = ib \tag{5.1.10}$$

Therefore $p(x,E)$ becomes

$$p(x, E) = \sum_{k=1}^{n} \frac{-i}{x - x_k} + iax^3 + ibx + c \tag{5.1.11}$$

To determine x_k, or equivalently $P(x) = \prod_{k=1}^{n}(x - x_k)$, we substitute (5.1.11) in (5.1.2) and get

$$-[E - ax^2 - fix^4 - \gamma x^6] = 0 \tag{5.1.12}$$

Therefore, the above equation becomes

$$c^2 + 2ax^3\frac{P'}{P} + 2ibcx - 2i\frac{P'}{P}c + 2bx\frac{P'}{P} + 2iacx^3 - \frac{P''}{P} + b - E - 2anx^2 = 0 \tag{5.1.13}$$

Equating the coefficient of x^3 term we have

$$2iacx^3 = 0 \implies c = 0$$

and hence we have

$$2ax^3\frac{P'}{P} + 2bx\frac{P'}{P} - \frac{P''}{P} + b - E - 2anx^2 = 0 \qquad (5.1.14)$$

The above equation thus gives the following differential equation in $P(x)$

$$P'' - P'(2ax^3 + 2bx) - P(b - E - 2anx^2) = 0 \qquad (5.1.15)$$

We get the expression for energies and wave functions for various values of **n** as follows:

We will derive explicit form of wave-functions for $n = 0, 1$ and 2. Later we will discuss the general form of the wave-function for arbitrary n. The general strategy for obtaining the wave-functions is the same as discussed for exactly solvable models in chapter 3.

Wave-function for $n=0$: Only one energy level can be solved in this case. Since the number n, representing the number of moving poles is zero (5.1.11), with $c=0$ as already found, becomes

$$p(x, E) = iax^3 + ibx \qquad (5.1.16)$$

and hence the wave-function is given by

$$\psi(x) = \exp\left(i\int p(x)dx\right) = \exp\left(i\int [iax^3 + ibx]dx\right) = \exp\left(-a\frac{x^4}{4} - b\frac{x^2}{2}\right)$$

and the corresponding energy is obtained from (5.1.15) by equating the constant term and is given as

$$E = b. \qquad (5.1.18)$$

Wave-function for n=1: In this case we take $P(x, E)$ to be a first degree polynomial, $(x - x_0)$. There (5.1.15) gives, $x_0 = 0$ and the energy is given as

$$E = 3b. \qquad (5.1.19)$$

Therefore the wave-function comes out to be

$$\psi(x) = Nx \exp\left(-a\frac{x^4}{4} - b\frac{x^2}{2}\right). \tag{5.1.20}$$

Wave-function for $n=2$: We seek a solution of (5.1.15) with $P(x)$ as a second degree polynomial. Substituting $P(x)$ as

$$P(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2. \tag{5.1.21}$$

Using the above equation in (5.1.15) and comparing different powers of x gives

$$\alpha_1 = 0 \tag{5.1.22}$$

$$4a\alpha_0 - \alpha_2(5b - E) = 0 \tag{5.1.23}$$

$$\alpha_0(b - E) - 2\alpha_2 = 0. \tag{5.1.24}$$

The last two equation have non-trivial solution for α_0 and α_2 only if the

$$\begin{vmatrix} 4a & (5b - E) \\ (b - E) & -2 \end{vmatrix} = 0.$$

This gives two energy eigen-values

$$E = 3b \pm 2\sqrt{b^2 + 2a}. \tag{5.1.25}$$

To get the wave function we compute α_1 and α_2 from equation (5.1.23) and (5.1.24) and use

$$p(x) = -i\frac{2\alpha_2 x}{\alpha_0 + \alpha_2 x^2} + iax^3 + ibx. \tag{5.1.26}$$

Therefore the wave function is given by:

$$\psi(x) = N \exp\left(i \int p(x)dx\right) = N \exp\left(i \int [-i\frac{2\alpha_2 x}{\alpha_0 + \alpha_2 x^2} + iax^3 + ibx]dx\right) \tag{5.1.27}$$

$$\psi(x) = N(\alpha_0 + \alpha_2 x^2) \exp\left(-a\frac{x^4}{4} - b\frac{x^2}{2}\right) \tag{5.1.28}$$

where N is the normalizing factor. The value of α_0 is given bv

$$\alpha_0 = \frac{5b - E}{4a}.$$ (5.1.29)

Replacing the value of α_0 and energy value E in the above equation one gets the expression for wave-function as

$$\psi(x) = N \frac{E - 5b}{4a} \left[b \pm \sqrt{b^2 + 2ax^2} - 1 \right] \exp\left(-a\frac{x^4}{4} - b\frac{x^2}{2} \right).$$ (5.1.30)

The wave-functions and eigen-values explicitly obtained for the cases $n = 0, 1$ and 2 agree with the known results [6].

For an arbitrary value of n the polynomial $P(x)$ will be obtained by solving (5.1.15). If we take $P(x)$ to be of the form

$$P(x) = \sum_{k=0}^{n} \alpha_k x^n,$$ (5.1.31)

then the differential equation (5.1.15) leads to a set of homogenous equations for the corresponding coefficients $\alpha_0, \alpha_1, \cdots, \alpha_n$. These equations will have a non-trivial solution only if determinant of the coefficients vanishes. This condition will determine the energy eigen-value, corresponding to each eigen-value we can find the coefficients $\alpha_0, \alpha_1, \cdots, \alpha_n$. Thus we get n independent wave-function each having the form

$$\psi(x) \sim P(x) \exp\left(-a\frac{x^4}{4} - b\frac{x^2}{2} \right).$$ (5.1.32)

Notice that all these eigen-functions corresponding to a fixed value of n have a polynomial of the same degree n as a factor. Thus for a fixed value of n, and hence for a given set of potential parameters, wave-functions for all the states which can be solved have the same number of zeros equal to n. If these levels are arranged according to increasing energy, the number of zeros on the real axis (nodes) will increase. Hence the number of complex zero will decrease with increasing energy. This feature appears to be a general property of quasi-exactly solvable models.

5.2 Hyperbolic Potential

The hyperbolic potential is

$$V(x) = -\frac{A}{\cosh^2 x} + \frac{B}{\sinh^2 x} - C \cosh^2 x + D \cosh^4 x, \tag{5.2.1}$$

where

$$A = 4(s_1 - \frac{1}{4})(s_1 - \frac{3}{4}),$$

$$B = 4(s_2 - \frac{1}{4})(s_2 - \frac{3}{4}),$$

$$C = [q_1^2 + 4q_1(s_1 + s_2 + \mu)],$$

$$D = q_1^2.$$

The QHJ equation is ($\hbar = 1 = 2m$)

$$p^2(x, E) - ip'(x, E) - [E + \frac{A}{\cosh^2 x} - \frac{B}{\sinh^2 x} + C \cosh^2 x - D \cosh^4 x] = 0. \tag{5.2.2}$$

We effect a transformation by

$$y = \cosh x \tag{5.2.3}$$

The QHJ equation in the new variable is

$$\tilde{p}(y, E) - i\hbar\sqrt{y^2 - 1}\, p'(y, E) - [E + \frac{A}{y^2} - \frac{B}{y^2 - 1} + Cy^2 - Dy^4] = 0 \tag{5.2.4}$$

Let

$$\tilde{p}(y, E) = -i\sqrt{y^2 - 1}\,\phi(y, E) \tag{5.2.5}$$

Therefore the QHJ equation becomes

$$\left[\phi + \frac{1}{2}\frac{y}{y^2 - 1}\right]^2 - \frac{1}{4}\frac{y^2}{(y^2 - 1)^2} + \phi' + \frac{1}{y^2 - 1}[E + \frac{A}{y^2} - \frac{B}{y^2 - 1} + Cy^2 - Dy^4] = 0. \tag{5.2.6}$$

Let

$$\chi = \phi + \frac{1}{2}\frac{y}{y^2 - 1}. \tag{5.2.7}$$

Therefore the above equation becomes

$$\chi^2 + \chi' + \frac{3}{4}\frac{y^2}{(y^2-1)^2} - \frac{1}{2}\frac{1}{y^2-1} + \frac{1}{y^2-1}[E + \frac{A}{y^2} - \frac{B}{y^2-1} + Cy^2 - Dy^4] = 0 \quad (5.2.8)$$

X has poles at $y = 0$, and $y = \pm 1$, and there are moving poles between the turning points. We assume that there are no more poles in the complex plane other than a pole at infinity. We will first compute the residues at $y = 0, \pm 1$ and then in the general form of χ (5.2.19) the constants b_1, b_1' and b_1'' will be known and then we give the general form of the wave-function.

Computation of residues: For $y = 0$ let

$$\chi = \frac{b_1}{y} + a_0 + a_1 y + \cdots \quad (5.2.9)$$

Therefore equation (5.2.8) becomes

$$[\frac{b_1}{y} + a_0 + a_1 y + \cdots]^2 + [-\frac{b_1}{y^2} + a_1 + \cdots] + \frac{3}{4}\frac{y^2}{(y^2-1)^2} - \frac{1}{2}\frac{1}{y^2-1}$$

$$+ \frac{1}{y^2-1}[E + \frac{A}{y^2} - \frac{B}{y^2-1} + Cy^2 - Dy^4] = 0 \quad (5.2.10)$$

Equating the coefficient of 4 on both sides gives

$$b_1 = \frac{1}{2}[1 \pm (4s_1 - 2)]$$

By the condition of square integrability, the positive sign has to be taken. Hence the value of b_1 is

$$b_1 = 2s_1 - \frac{1}{2} \quad (5.2.12)$$

For $y = 1$ let

$$\chi = \frac{b_1'}{y-1} + a_0' + a_1'(y-1) + \cdot \quad (5.2.13)$$

Therefore equation (5.2.8) becomes

$$[\frac{b_1'}{y-1} + a_0' + a_1'(y-1) + \cdots]^2 + [-\frac{b_1'}{(y-1)^2} + a_1' + \cdots] + \frac{3}{4}\frac{y^2}{(y^2-1)^2} - \frac{1}{2}\frac{1}{y^2-1}$$

$$+ \frac{1}{y^2 - 1}[E + \frac{A}{y^2} - \frac{B}{y^2 - 1} + Cy^2 - Dy^4] = 0$$

Equating the coefficient of $\frac{1}{(y-1)2}$ on both sides gives

$$b'_1 = \frac{1}{2}[1 \pm (2s_2 - 1)\,]\tag{5.2.14}$$

By the condition of square integrability, the positive sign has to be taken. Hence the value of b'_1 is

$$b'_1 = s_2\tag{5.2.15}$$

For $y = -1$ let

$$\chi = \frac{b''_1}{y + 1} + a''_0 + a''_1(y + 1) + \cdots\tag{5.2.16}$$

Therefore equation (5.2.8) becomes

$$[\frac{b''_1}{y + 1} + a''_0 + a''_1(y + 1) + \cdots]^2 + [-\frac{b''_1}{(y + 1)^2} + a''_1 + \cdots] + \frac{3}{4}\frac{y^2}{(y^2 - 1)^2} - \frac{1}{2}\frac{1}{y^2 - 1}$$

$$+ \frac{1}{y^2 - 1}[E + \frac{A}{y^2} - \frac{B}{y^2 - 1} + Cy^2 - Dy^4] = 0$$

Equating the coefficient of $\frac{1}{(y+1)2}$ on both sides gives

$$b''_1 = \frac{1}{2}[1 \pm (2s_2 - 1)\,]\tag{5.2.17}$$

By the condition of square integrability, the positive sign has to be taken. Hence the value of b''_1 is

$$b''_1 = s_2.\tag{5.2.18}$$

For the fixed poles at $y = 0, \pm 1$ let χ have the form

$$\chi = \frac{P'(y)}{P(y)} + \frac{b_1}{y} + \frac{b'_1}{y - 1} + \frac{b''_1}{y + 1} + c_1 y + c_2,\tag{5.2.19}$$

where c_1 and c_2 are constants to be determined. This form of χ is because the equation (5.2.8) has a y^2 term.

Form of wave-function: Using (5.2.19), equation (5.2.8) transforms to

$$\left[\frac{P'(y)}{P(y)} + \frac{b_1}{y} + \frac{b_2}{y-1} + \frac{b_3}{y+1} + c_1 y + c_2\right]^2 + \left[\frac{P''(y)}{P(y)} - \left(\frac{P'(y)}{P(y)}\right)^2\right]$$

$$-\frac{b_1}{y^2} - \frac{b_2}{(y-1)^2} - \frac{b_3}{(y+1)^2} + c_1 + \frac{3}{4}\frac{y^2}{(y^2-1)^2} - \frac{1}{2}\frac{1}{y^2-1}$$

$$+\frac{1}{y^2-1}[E + \frac{A}{y^2} - \frac{B}{y^2-1} + Cy^2 - Dy^4] = 0 \qquad (5.2.20)$$

and for large y equating the coefficients of y^2 to zero gives,

$$c_1^2 - D = 0,$$

hence

$$c_1 = \pm\sqrt{D} = \pm q_1. \qquad (5.2.21)$$

For large y equating the coefficients of y gives

$$2c_1 c_2 = 0,$$

hence

$$c_2 = 0. \qquad (5.2.22)$$

The correct sign of c_1 is fixed by square integrability and is found to be $c_1 = -q\backslash$. As the potential is symmetric, there are moving poles on either side and hence we take $P(y)$ to have the form given below

$$P(y) = \prod_{k=1}^{n}(y^2 - y_k^2). \qquad (5.2.23)$$

The wave-function for this model is computed as follows

$$\psi(y) = \exp\int [\chi - \frac{1}{2}\frac{y}{y^2-1}]dy. \qquad (5.2.24)$$

$$= \exp\int [\frac{P'(y)}{P(y)} + \frac{b_1}{y} + \frac{b_1'}{y-1} + \frac{b_1''}{y+1} + c_1 y]dy. \qquad (5.2.25)$$

On integrating and substituting the values of b_1, b_2 and c_1 we get the expression for the wave-function in the x variable as

$$\psi(x) = (\cosh^2 x)^{s_1 - \frac{1}{4}} (\sinh^2 x)^{s_2 - \frac{1}{4}} \exp\left(-\frac{q_1}{2}\cosh^2 x\right) \prod_{k=1}^{n} (\cosh^2 x - y_k^2). \quad (5.2.26)$$

Computation of energy-eigenvalue: We shall now show how our analysis leads to the correct answer for energy spectrum. With $c_2 = 0$ and substituting the values of b_1, b_2 and c_1 (5.2.20) takes the form

$$\frac{P''(y)}{P(y)} + \frac{P'(y)}{P(y)}\left[\frac{(4s_1 - 1)}{y} + \frac{2s_2}{y-1} + \frac{2s_2}{y+1} - 2q_1 y\right]$$

$$+[q_1^2 y^2 + \frac{4s_1 s_2}{y-1} - \frac{4s_1 s_2}{y+1} + \frac{s_2}{y+1} - \frac{s_2}{y-1} + \frac{s_2^2}{y-1} - \frac{s_2^2}{y+1} - 2s_2 q_1 \frac{y}{y+1} - 2s_2 q_1 \frac{y}{y-1}$$

$$-4s_1 q_1 - \frac{1}{8}\frac{1}{y^2 - 1} + \frac{E+A}{y^2-1} + \frac{B}{2}\frac{1}{y^2-1} + C\frac{y^2}{y^2-1} - D\frac{y^4}{y^2-1}] = 0 \quad (5.2.27)$$

Using (5.2.23) in (5.2.27) we get the following equation.

$$\left[\sum_{k=1}^{n}\frac{2y}{y^2 - y_k^2}\right]^2$$

$$-\sum_{k=1}^{n}\left[\frac{1}{(y+y_k)^2} + \frac{1}{(y-y_k)^2}\right] + \sum_{k=1}^{n}\frac{2y}{y^2 - y_k^2}\left[\frac{(4s_1 - 1)}{y} + \frac{2s_2}{y-1} + \frac{2s_2}{y+1} - 2q_1 y\right]$$

$$+[q_1^2 y^2 + \frac{4s_1 s_2}{y-1} - \frac{4s_1 s_2}{y+1} + \frac{s_2}{y+1} - \frac{s_2}{y-1} + \frac{s_2^2}{y-1} - \frac{s_2^2}{y+1} - 2s_2 q_1 \frac{y}{y+1} - 2s_2 q_1 \frac{y}{y-1}$$

$$-4s_1 q_1 - \frac{1}{8}\frac{1}{y^2 - 1} + \frac{E+A}{y^2-1} + \frac{B}{2}\frac{1}{y^2-1} + C\frac{y^2}{y^2-1} - D\frac{y^4}{y^2-1}] = 0 \quad (5.2.28)$$

Multiplying the above throughout by ^{l}- and integrating along a closed contour enclosing $y = 0$ we get the expression for energy as

$$E = -4\left[s_1 + s - 2 - \frac{1}{2}\right]^2 - 8s_1\left[\frac{q_1}{2} + \sum_{k=1}^{n}\frac{1}{y_k^2}\right] \quad (5.2.29)$$

Using $y'i = \xi_k$ the above equations for energy become

$$E = -4\left[s_1 + s - 2 - \frac{1}{2}\right]^2 - 8s_1\left[\frac{q_1}{2} + \sum_{k=1}^{n}\frac{1}{\xi_k}\right] \tag{5.2.30}$$

Changing to $y^2 = \xi$ and $y_k^2 = \xi_k$ (5.2.28) becomes

$$4\xi\left[\sum_{k=1}^{n}\frac{1}{\xi - \xi_k}\right]^2 - 2\sum_{k=1}^{n}\left[\frac{\xi + \xi_k}{(\xi - \xi_k)^2}\right] + 2\left[(4s_1 - 1) + 4s_2\frac{\xi}{\xi - 1} - 2q_1\xi\right]\sum_{k=1}^{n}\left[\frac{1}{\xi - \xi_k}\right]$$

$$[q_1^2\xi + 8s_1s_2\frac{1}{\xi - 1} - 2s_2\frac{1}{\xi - 1} + 2s_2^2\frac{1}{\xi - 1} - 4s_2q_1\frac{\xi}{\xi - 1} - 4s_1q_1 - \frac{1}{8}\frac{1}{\xi - 1}$$

$$+\frac{(E + A)}{\xi - 1} + \frac{B}{2}\frac{1}{\xi - 1} + C\frac{\xi}{\xi - 1} - D\frac{\xi^2}{\xi - 1}] = 0 \tag{5.2.31}$$

and integrating (5.2.31) over ξ around a closed contour, enclosing only one of the points ξ_i and repeating for $(i = 1, 2, \bullet\bullet\bullet, n)$ we get the following result

$$\prod_{k=1}^{n}\frac{1}{\xi_i - \xi_k} - \frac{s_1}{\xi_i} + \frac{s_2}{\xi_i - 1} - \frac{q_1}{2} = 0 \tag{5.2.32}$$

$$i = 1, 2, \cdots n$$

The results (5.2.26), (5.2.30) and (5.2.32) and in agreement with those given in [6]

The general feature of the zeros of the wave-function for the sextic oscillator are also true for the QES hyperbolic potential. In particular, for a given potential it is correct that all the exactly solvable wave-functions have the same total number, (real and complex) of zeros. This feature is found to be correct for all QES model studied so far including the QES periodic potentials [12].

5.3   Concluding Remarks

Our study of bound state wave-functions in this chapter shows the following similarities and differences between the exactly solvable and QES models.

1. In both the models, the "QMF" turns out to be a rational function after a suitable change of variables.

2. In both the cases, the integer n in the right hand side of quantization condition coincides with the number of moving poles.

3. For every bound state in one dimension the kth excited state wave-function have fc-nodes on the real axis. This statement is a general one and is true for all models including exactly solvable and QES potentials. the study in chapter 3 shows that QMF for exactly-solvable models has moving poles which are in correspondence with the nodes of the wave-function. There are no poles off the real axis. However this property fails to be true for QES potentials where the QMF has poles off the real axis, in addition to the poles on the real axis corresponding to the nodes of the wave-function.

4. For the QES potentials only a part of the energy spectrum and the corresponding wave-functions can be computed exactly. An interesting property of the QMF for all these levels is that the total number of moving poles is the same and equal to the integer n of the quantization.

5. Different values of integer n correspond to different QES potentials within a family, and it does not refer to different excited state of a single potential, as was the case for exact-solvable model.

Chapter 6

CONCLUSIONS AND OUTLOOK

In this thesis, we have studied exactly solvable and QES potentials in one dimensional quantum mechanics **with** in the frame work of QHJ formalism. The following results have been obtained.

1. The eigen-values and eigen-functions of the exactly solvable models can be obtained by a very simple and elegant method, which makes use of elementary results from theory of complex variables.

2. The non-trivial input in this analysis is the singularity structure of the QMF. Besides this, a change of variable is needed to transform the QHJ equation to a Riccati form with rational functions as coefficients. Having done this, it is very easy to identify the fixed poles and the corresponding residues. It is the location and the number of moving poles which present some difficulty. For a large class of exactly solvable and QES models studied by us, the number of moving poles for the bound states turns out to be finite. In addition, the behavior of large values of independent variables has been very simple to read from QHJ equation. All these observations can be summarized in one sentence by saying that the QMF is a rational function after a suitable change of variables.

3. The quantization condition as given by Leacock and Padgett, is applicable to separate systems which can be reduced to one dimensional problems. It will be interesting to formulate an exact quantization condition for non-separable systems in higher dimensions, and investigate its relation to the well known existing semi-classical schemes and to see applications to chaotic systems.

4. For other models which are not exactly solvable, one has to device and approximation scheme. Here again some idea about the knowledge of location of moving poles has any relation to classical trajectories.

5. We have tried to study the QHJ formalism for an-harmonic oscillator which is a test case of any computational scheme. One can compute the asymptotic value of the QMF for large x and one can use this answer as an input for numerical integration of QHJ. Detailed investigation is in progress and interesting approximation scheme for an-harmonic oscillator is expected from the preliminary results. The simple form of QHJ equation offers a possibility of several analytic approximation schemes also.

6. When computing bound states for some potentials such as Rosen Morse hyperbolic potential, it is found that applying boundary condition p(x, E) $\xrightarrow{\hbar \to 0}$ $p_{cl}(x,E)$, carefully leads one to select different residues for different ranges of potential parameters. Excepting this and proceeding, further analysis leads to different set of energy spectrum and wave-functions for such different ranges of parameters in the potential. This is consistent with the known result on phases of super-symmetry in Rosen Morse potential 13]. Other such potentials, for example trigonometric Scarf potential [14], which exhibit different phases for different ranges of potential parameters, can also be investigated within our

frame work.

7. It must be remarked that, the QHJ formalism as presented in this thesis is applicable to bound states only. Modifications will be needed to apply this formalism to continuous energy solutions. The requirements such as the quantization rule, square integrability etc., are no longer applicable. For such cases we must accept all possible combination of residues, consistent with the other equations of the theory, and proceed to analyse the consequences. In fact analysis of this type has been performed for some of the QES and exactly solvable periodic potentials, and QHJ formalism leads to the full set of band-edge wave-function and corresponding energy eigen-values [12,15].

8. The QHJ formalism offers advantages from the pedagogical point of view. The understanding of method and results requires only the basic understanding of the theory of complex variables. In this connection we mention periodic potentials, both exactly solvable and QES, and PT symmetric complex potentials [16] which can be handled with equal ease within the QHJ approach [12,15, 17].

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