GEOMETRIC AND ANALYTIC STUDIES OF SOME INTEGRABLE SYSTEMS

A THESIS SUBMITTED TO THE UNIVERSITY OF HYDERABAD FOR THE DEGREE OF **DOCTOR OF PHILOSOPHY** IN THE SCHOOL OF MATHEMATICS & COMPUTER/INFORMATION SCIENCES

By
S. Subbulakshmi

DEPARTMENT OF MATHEMATICS AND STATISTICS
SCHOOL OF MATHEMATICS AND
COMPUTER/INFORMATION SCIENCES
UNIVERSITY OF HYDERABAD
HYDERABAD - 500 046
INDIA

DECEMBER 1998

CERTIFICATE

DEPARTMENT OF MATHEMATICS AND STATISTICS
SCHOOL OF MCIS
UNIVERSITY OF HYDERABAD
HYDERABAD - 500 046.

Dated : DECEMBER 1998

This is to certify that **I,** S. **Subbulakshmi** have carried out the research embodied in the present thesis entitled **GEOMETRIC AND ANALYTIC STUDIES OF SOME INTEGRABLE SYSTEMS** for the full period prescribed under the Ph.D. ordinances of the University.

I declare to the best of my knowledge that no part of this thesis was earlier submitted for the award of research degree of any other University / Institute.

S. Subbulakshmi.
candidate
(S.Subbulakshmi)
Enrollment No : **92MMPM01**

K Vutt
Head of the Department
(Prof. K.VISWANATH)

Supervisor
(Prof. M. SITARAMAYYA) 31.12.1998

K. Vutt
Dean of the School
(Prof. K. VISWANATH)

ACKNOWLEDGEMENTS

I wish to express my deep sense of gratitude and indebtedness to my supervisor and guide ***Prof. M. SITARAMAYYA for*** *his invaluable guidance and motivation during the course of my thesis* ***work****. His lectures and the discussions with him have been a constant source of inspiration to* ***me****. I am* ***very*** *much grateful to him for his constant encouragement and immense help and for taking keen interest in this work, I am also thankful to him for the enormous patience with which he has gone through the manuscript and for the kindness that he has always shown towards me. I sincerely thank him for his encouragement and guidance without which the completion of this thesis would have been impossible.*

I sincerely thank Prof C *Musili,* ***Dean,*** *School of* ***Mathematics*** *and Computer Information Sciences and Prof* ***K. Viswanath,*** *Head, Department of Mathematics and Statistics for their encouragement and for providing the necessary facilities to carry out this work.*

I am highly indebted to Prof. ***T. Amarnath,*** *Dr. G. P.* ***Rajasekhar*** *and Dr. B. S.* ***Padmavathi*** *for procuring some of the reference articles which were not available in our library.*

I *am* ***grateful*** *to all the faculty members of the Department for their constant encouragement and advice during the course of my study here. Particularly, the inspiring lectures of Prof.* C ***Musili,*** *Prof. V. Kannan, and* ***Prof.K.*** *Viswanath gave an impetus to study and explore the fascinating world of mathematics.*

I wish to thank my ***seniors,*** *Dr. S.* ***Jayasree,*** *Dr. S. Durga* ***Bhavani,*** *Dr. R* ***S.*** *Padmavathi for their advice, support and for creating a nice atmosphere to* ***work*** *with and also Dr. P. Murali and Dr.* ***K. S.*** *S. Moosath for their kind help* ***and co-operation.***

My very special thanks are due to Sowjanya and Anuradha for extremely helping me in many ways and the pleasant times we had together will always remain in my memory.

Thanks are also due to my colleagues and friends , Rajasekhar, Datt, Arunkumar, Saradhi, Venkatlaxmi, Divya , Sreedevi and Chitra for their concern and help.

I am very much thankful to the non-teaching staff - School, Department, and other sections (specially, Smt. Harikumari, Smt. Shashikala and Smt. Cumudini) of the University - for their kind help and support during the tenure of my study here.

I thank Mr. K. Srinivas for his excellent job of typing this thesis and Ms. Grace for helping me in many ways.

I gratefully acknowledge the Department of Atomic Energy, Government of India for their financial support as a Senior Research Fellow under the BRNS - NBHM research project on the geometry of complex tori till October 1994. I also thank the UGC - SAP - DRS programme for providing the printouts for this thesis.

My sincere thanks to the University authorities for providing an excellent atmosphere (environment) to work.

Finally, I owe much to my family members and I find no words to express my gratitude, to my grandparents for their blessings, and to the affection and care of my parents, especially my father, who had a very special and keen interest in my studies, and to my uncle, aunt, their children - Subbu and Mani, my sisters and their daughters, Ramannya and Bhavani.

s.subbulakshmi

CONTENTS

CHAPTER 4

•ADLER - KOSTANT - SYMES (AKS) PRINCIPLE AND ITS APPLICATION TO SOME INTEGRABLE SYSTEMS

CHAPTER 5

•GEOMETRY OF THE **BOUSSINESQ** EQUATION AND ITS RELATION TO OTHER SYSTEMS

CHAPTER 6

•GEOMETRY OF BENNEY'S EQUATION AND HIGHER DIMENSIONAL EQUATIONS

INTRODUCTION AND SUMMARY

The breadth of integrable systems is that it ranges over mechanics, differential equations, global analysis, algebraic geometry and Lie theory and these are just some of their mathematical aspects, ignoring the vast intersection with physics. The subject accommodates a whole range of points of view from very "pure" to very "applied".

Integrable systems first appeared as mechanical systems for which the equations of motion could be solved by quadratures, i.e., by a sequence of operations which included only algebraic operations, integration and application of the inverse function theorem. Apart from some non-trivial examples which were constructed before, the first main result (due to Liouville, but essentially an application of a result due to Hamilton) was that if a mechanical system with n degrees of freedom of the form

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \qquad (i = 1, \cdots, n)$$

(H any function in the co-ordinates q_i, p_i) has n independent integrals in involution then it can be solved by quadratures. Two functions / and g are said to be in involution if their Poisson bracket

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$$

vanishes, $\{f, g\} = 0$ and / is called an integral of the system if / and H are in involution. Mechanical systems which satisfy the conditions of Liouville's theorem are called Liouville integrable or integrable in the sense of Liouville. A quite short - but important - list of (non-trivial) examples of Liouville integrable systems were found during the last century: a few integrable tops (the Euler top, the Lagrange top, the Kowalevski top, free motion of a particle on an ellipsoid (Jacobi), motion of a rigid body in an ideal fluid (Kirchhoff and Steklov case), motion in the field of a central potential (Newton) and few others. Both finding these systems (i.e., showing that enough integrals exist, which was clone by constructing them) and solving them explictly by quadratures required a lot of ingenuity and often quite long calculations. In the more complicated cases the solution was written down in terms of two-dimensional theta functions by a non-trivial use of the rich analytical properties of these functions. In turn it motivated the research in theta functions and Abelian varieties, which originated in the begining of that century in the works of Riemann and Abel.

Traditionally integrable systems are considered as differential geometric objects. The phase space is a smooth (or analytical) manifold, equipped with a symplectic structure and the functions in involution are smooth (or analytic) functions.

In the above classical definition of integrability in the sense of Liouville the existence of a sufficient number of integrals in involution among themselves and with a given function (the Hamiltonian) is demanded, sufficient meaning equal to the degrees of freedom of the system. For *integrable Hamiltonian systems*, it is better to consider the algebra of functions generated by the integrals and if the integrals are in involution, then this whole algebra is involutive. Giving only the function (Hamiltonian) does not suffice to determine the whole algebra, which confirms that the integrable system should consist of an algebra and not of a single function. Having a sufficient number of functions in involution corresponds to this algebra having maximal dimension. The algebra should be complete in the sense that every function which is in involution with all elements of the algebra actually belongs to the algebra.

Thus a completely integrable system is a Hamiltonian system that admits the maximum possible number of first integrals, i.e., if the notion of integrability means the existence of integrals of motion, then complete integrbility means that these integrals exist in sufficient number. For a system of n first order autonomous ordinary differential equations, sufficient means $n - 1$ time-independent (where the system can be reduced to a single quadrature) invariants or n time-dependent ones (in which case the solutions can be obtained by solving an algebraic problem). The study of completely integrable systems proceeds in three stages: (i) identification of the symplectic structure which gives the system its Hamiltonian character; (ii) identification of first integrals (or constants of motion or action variables); (iii) identification of a complementary set of variables, called angle variables and computation of their evolution under the various Hamiltonian flows associated to the first integrals if possible in terms of elementary functions. The symplectic manifolds on which the systems are defined are orbits of the co-adjoint action of a Lie group G on the dual of its Lie algebra, with the natural symplectic structure.

Complete integrability is strongly related to either Lie algebra or algebraic curve theory. For instance, in chapter 4, it is shown that both KdV equation and the Toda system can be viewed as Hamiltonian systems on the co-adjoint orbit of a Lie group with Kostant-Kirillov structure and the complete integrability of these systems can be traced to a single abstract Lie algebra theorem. The main thrust of the method is to associate with all the above Hamiltonian systems a Lax matrix differential equation which contains a parameter, i.e., equations of the form $A = [A, B] = AB - BA$, (†) where A and B are matrices whose entries depend on the phase space variable and are polynomials in the indeterminate h and h^{-1}. Then the curve in (h,z) space $X : det(A - z) = Q(z,h) = 0$ is formed whose coefficients are functions of the phase space. From (†), the curve $X : Q(z,h) = 0$ (of genus g) is time independent. (cf.section 3.3), i.e., its co-efficients are integrals of the motion (†). Then we linearise (†) and its associated flows on the $\mathrm{Jac}(X)$ of X. (cf.section 3.4)

Chapter 1 is concerned with certain partial differential equations and as such partial differential equations is a multi-faceted subject, created to describe the mechanical behaviour of objects such as vibrating strings and blowing winds, it has developed into a body of material that interacts with many branches of mathematics such as differential equations, complex analysis, harmonic analysis and a very important factor in the description and elucidation of problems in Mathematical Physics. This branch of partial differential equations in Mathematical Physics was studied by great mathematicians like Riemann, Jacobi,Weierstrass and Poincare etc.

For the last two decades certain special non-linear equations of Mathematical Physics like the Strum-Liouville equation, equations of non-linear string, the Korteweg-deVries(KdV) equation, the Kadomtsev-Petviashvilii(KP) equation for dispersive waves in shallow water theory were extensively investigated by the Soviet school[23,25,37] and the Courant School [40].

Several interesting connections were discovered in these works. For example, there are interlinks between Classical Mechanics, in particular infinite- dimensional integrable systems[37], spectral theory of differential operators[17,28,40] and algebraic geometry, in particular, geometry of complex tori [6,22,30,45,61].

In chapter 1, we discuss the solutions of the KdV and KP equations which are periodic and conditionally periodic and these solutions live on the Jacobian variety of a compact Riemann surface. Section 1.1 is concerned with basic definitions and results from complex analytic theory of a compact Riemann surface, divisors, abelian variety, Jacobian variety and the Abel map which gives the necessary and sufficient condition for a divisor of degree zero to be a principal divisor. We have also defined the Riemann-theta function and studied its properties. In section 1.2, we give a brief account of the KdV equation, the KP equation and the Boussinesq equation of which the KdV equation arose in the nineteenth century in connection with the theory of waves in shallow water and the KP equation which is a more general form of the KdV equation. In section 1.3, we introduce and study the Akhiezer functions and their relation with the KdV equations and Riemann-theta functions. We have proved the theorem (Theorem: 1.3.20) in which the solutions of the KP equation are expressed in terms of the Riemann-theta function.

Section 1.4 is like an inverse problem of section 1.3 where we construct the operators L_1, L_2 using the data (of an algebraic curve(CRS), Akhiezer function and special divisors). We have proved that for each Akhiezer function Ψ, $\exists$ unique pair of differential operators L_1 and L_2 of form (1.4.1) such that $L_1\Psi = \frac{\partial}{\partial t}\Psi, L_2\Psi = \frac{\partial}{\partial y}\Psi$ (Theorem 1.4.5). We have also discussed some special cases in the above two sections. In section 1.5, we give some remarks on the interconnection of the KP(KdV) equations theory with other topics in dynamical systems and deeper areas of algebraic geometry. In summary we proved here a common transcendental solution for the partial differential equations we considered (Theorem 1.3.20 and Theorem 1.4.5) in this chapter.

Subsequent chapters, starting with chapter 2, are concerned with the *complete integrability of some integrable (Hamiltonian) systems (CIHS).* Here we have introduced systems such as (a) The Toda Lattice (b) the Lagrange top (c) the geodesic motion on an ellipsoid in the finite dimensional case and (d) the KdV and the generalised KdV equations and (e) the Gel'fand- Dikii system in the infinite dimensional case, where we have given the description of these systems in the classical sense. The main purpose of chapter 2 is to introduce the above systems and describe them from classical view point and all these systems will be discussed in subsequent chapters from several other angles. In section 2.1, we have given an introduction to the various aspects of Poisson brackets where the purpose is to lay down the differential geometric basics for the use of Poisson brackets in the subsequent parts of the thesis, where the emphasis is on the integrable (Hamiltonian) systems. Section 2.2 deals with various forms of Hamiltonian structures via Poisson structures or sympletic structures used in the general theory of integrable Hamiltonian systems and also the natural sympletic structures on the co-tangent bundle and on the dual of a Lie algebra are given. In section 2.3, we have discussed the example of the Toda lattice, invented by M.Toda around 1968 and the Poisson bracket on it and the corresponding Hamiltonian structure are given.

Section 2.4 deals with the example **of** the three dimensional rigid body motion about a fixed point under the influence of gravity which has the Lagrange **top** as a special case. In section 2.5, we have discussed the KdV equations and the generalized KdV equations for which we have given the Hamiltonian structure and the Poisson bracket associated with it. Section 2.6 deals with the Gel'fand-Dikii systems which are generalizations of KdV equations to the n-th order partial differential operators. The higher order flows of these KdV hierarchy are also given (cf. 2.6.6).These have remarkable properties such as that they are infinite-dimensional Hamiltonian systems, having infinitely many constants of motion in involution, which are also integrals of local polynomial densities.

Finally in section 2.7 of chapter 2 we have introduced the differential equation for geodesics on an ellipsoid and also the confocal quadric family of this ellipsoid. First we considered an ellipsoid defined by a positive definite symmetric matrix A of order n x n and then passing to its isospectral matrix $L(x, y)$ got a special rational function $\Phi_z(x, y)$whose partial fraction expansion gives the n first integrals of the geodesic flow (cf. 2.7.6). The eigenvalues of L are related to the common tangent cone of a certain family of confocal quadrics. The method of rank 2 perturbation of matrix A to get $L(x, y)$is explained and the Hamitonian form and Lax form of geodesics equations are given. Then the Hamiltonian equations of geodesies for ellipsoid in $\mathbb{R}^n$ were extended to the cotangent bundle $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ and using the standard symplectic form on it, the integrals of motion of the geodesic were obtained and generalized (cf part(b) of (2.7)).

Chapter 3 is concerned with Lax representation , cohomological interpretation **and** linearization **of** flows of some completely integrable Hamiltonian systems (CIHS) **given in** chapter **2. In section 3.1,** we discuss the Lax equations of the KdV and **the KP** systems **from the view point of** isospectral deformations of ordinary differential operators **with formal power series co-efficients**. In otherwords, following the work **of Gel'fand and Dikii closely the general** isospectral deformation theory of general diffenrentialoperators in **the** larger set of pseudo-differential operators was developed first and then applied to special partial differential equations or operators arising out of them. The geometric way to understand the KP equation

$$\frac{3}{4}u_{yy} - (u_t - \frac{1}{4}u_{xxx} - 3uu_x)_x = 0 \tag{1}$$

is to introduce a large system of non-linear equations which contain the KP equation (1) as the first equation. This system of equations, called the KP system or KP hierarchy are the equations that describe the universal deformations of ordinary differential equations (cf. Remark 3.1.17). In remark (3.1.16), if we set $P = d^2 + 2u$, then we get the KdV equation

$$u_t - \frac{1}{4}u_{xxx} - 3uu_x = 0$$

In section 3.2, we discuss the Lax equations with a parameter associated with some of the completely integrable Hamiltonian systems discussed in chapter 2. The Euler equations of motion for the n-dimensional free rigid body about a fixed point as a Hamiltonian system on adjoint orbits of *so(n)* are given in this section, where we have proved the theorem (cf. Theorem 3.2.5). In this case if we put $n = 3$, then we get the usual Euler equations (cf. Remark 3.2.7). For the three dimensional heavy rigid body, it is noted that the Euler - Poisson equations can be written in Lax form [48] with variables formal matrix polynomials if and only if the equations describe the Lagrange top (two of the principal moments of inertia are equal and the center of mass is on the axis of symmetry of the body) or the heavy symmetric top (all three moments of inertia are equal) (cf. Theorem 3.2.10). We also discuss the examples of the Toda lattice (3.2.12) and the geodesic motion on an ellipsoid (3.2.13) in this section.

In section **3.3,** we discuss the spectral curves associated with the Lax equations with a parameter ξ, for the examples discussed in section 3.2 for which we have given some preparation from algebraic geometry ([6], [20],[26]). That is, suppose we have a Lax equation with a parameter ξ, $\frac{d}{dt}A_\xi = [A_\xi, B_\xi]$ where *A* and *B* are complex or real matrices which have entries in the ring of real or complex Laurent polynomials in the variable ξ, which is called the spectral parameter. This Lax equation has many first integrals since the matrix *A* stays in the same conjugacy class, its eigenvalues will be 'constants of motion'. In otherwords, the co-efficients of the characteristic polynomial of *A* are first integrals. Then the characteristic polynomial is a polynomial in two variables $Q(\xi,\eta) = det(\eta Id - A_\xi)$ which is a complex curve *C* of the equation $Q(\xi,\eta) = 0$ and which is called the spectral curve. It describes the eigenvalues, the spectrum of A_ξ. The co-efficients in the equation of *C* are the first integrals and so there are many spectral curves, one for each value of the set of integrals and $Q(\xi,\eta) = 0$ describes a family of curves corresponding to the curve *C*. For a general point $p = (\xi,\eta) \in C$, if we assume dim ker $\| \eta I - A(\xi,t) \| = 1$, then there is a uniquely determined vector $\nu(p,t) \in V$ (upto non-zero scalars) satisfying $A(\xi,t)\nu(p,t) = \eta\nu(p,t)$ (cf. 3.4.2) and the correspondence $p \to \mathbb{C}\nu(p,t) \subset V$ determines a family of holomorphic mappings depending holomorphically on t, $f_t : C \to \mathbb{P}V$, which are called the eigenvector mappings associated to the given Lax equation (cf. section 3.4). For the rigid body motion, the Lagrange top, the Toda lattice and the geodesics on ellipsoid the corresponding spectral curves are given along with their genus.

Section 3.4 deals with the cohomological interpretation of Lax equations and the linearization of flows. The aim is, given a Lax equation with a parameter, we associate to it an algebraic curve *C* (its spectral curve) together with a dynamical system or flow $\{L_t\}$ on its Jacobian *J(C)*. Then give the necessary and sufficient condition on *B* in the Lax pair (A,B) that the flow $t \to L_t$ be linear on *J(C)*. The answer to this lies in Theorem (3.4.13) and Theorem (3.4.20). The eigenvalues of $A(\xi,t)$ are fixed as time evolves **and** the eigenvectors of $A(\xi,t)$ will change with t. This leads to the eigenvector mappings $f_t : C \to \mathbb{P}V$ and we set $L_t = f_t^*(\vartheta_{\mathbb{P}V}(1)) \in Pic^d(C)$ where $Pic^d(C)$ is the set of line bundles of degree *d* on *C*. Choosing $L_o \in Pic^d(C)$, we have the map

$L \rightarrow L \otimes L_o^{-1}$ where under an isomorphism $Pic^d(C) = J(C)$ and the canonical isomorphism $T_{L_t}(Pic^d(C)) \models H^1(\vartheta_C)$. Then the condition on B becomes that the acceleration vector $\frac{d^2 L_t}{dt}$ be a multiple of $\frac{dL_t}{dt}$,i.e., $\frac{d^2 L_t}{dt^2} = \mu_t \frac{dL_t}{dt}$. The Lax equation is invariant under a substitution $B \rightarrow B + P(A, \xi)$, where $P(x, \xi) \in \mathbb{C}[x, \xi]$ and this suggests that it has invariant cohomology meaning. The necessary and sufficient condition for $\{L_t\}$ C $Pic^d(C)$ be linear is given in Corollary (3.4.21) and this is applied to the rigid body motion or Lagrange top; Toda lattice and geodesics on ellipsoid to conclude linearity of flows by showing the vanishing of the residue of B (cf. Examples 3.4.30-3.4.33).

In chapter 4, we discuss the geometric Adler - Kostant - Symes(AKS) principle and its application to some integrable systems such as, the Toda system, the Lagrange top, the geodesic motion on an ellipsoid, the KdV and the generalized KdV equations, and the Gel'fand-Dikii system. In section 4.2, we discuss some preliminaries from symplectic manifold theory and define the Kostant-Kirillov-Souriau orbit symplectic structure (cf. Definition 4.2.6). The KdV and the Toda systems and others as well are completely integrable Hamiltonian systems whose equations of motion are expressible in terms of the Lax isospectral equations. The splitting of a Lie algebra into a vector space direct sum of Lie algebras is responsible for the complete integrability of the above systems and the Lax isospectral equations associated with these systems.

Of the different approaches to the study of integrable systems, our approach here is that, a given non-linear system is written as a Hamiltonian dynamical system with respect to some Hamiltonian structure on the underlying space. (For finite dimensional manifolds, the term 'Poisson structure' is usually preferred, that of 'Hamiltonian structure' being more frequently applied to the infinite dimensional case). For finite dimensional Hamiltonian systems on a symplectic manifold of dimension 2n, integrability in the sense of Liouville(1855) and Arnold(1974) is defined by the requirement that 3 n conserved quantities that are functionally independent on a dense open set and in involution, i.e., whose pairwise Poisson brackets vanish and the geometric methods can be applied in various ways (cf.section 4.3).

In section 4.4, we dicuss the above'mentioned examples of which for the Toda lattice, the relevant group G is the group of lower triangular matrices with nonzero diagonal elements, as contained in $SL(n,\mathbb{R})$. We identify the dual algebra of (7, $\mathcal{L}^*$, with the upper triangular matrices through the trace form. The orbit Hamiltonian phase space θ_A is of the form $\theta_A = \{[U^{-1}AU]_+/U6\ G,\ A \in \mathcal{L}^*\}$, where $[B]_+$ denotes the matrix formed from B by setting its lower triangular entries equal to zero and A is subject to certain conditions (cf. 4.4.1). For the case of the generalized KdV equations, the relevant group G is the formal pseudo-differential symbols of negative type translated by the identity element 1, whose dual $\mathcal{L}^*$ is identified, through a trace form, with the differetial symbols of non-negative type, which are identified with formal differential operators. The algebra in which everything takes place is the algebra of formal pseudo-differential operators. The Hamiltonian orbit space is of the same form as θ_A (cf. 4.4.15).

We have also discussed the examples of the Lagrange top using the Kac-Moody Lie algebra (cf. 4.4.3) and the geodesic motion on ellipsoid based on the polynomials in the indeterminate h, h^{-1} with co-efficients in a Lie algebra (cf. 4.4.7). Thus in summary, in chapter 4 the geometric principle of Adler-Kostant-Symes is formulated in complete generality and is proved (cf. Theorem 4.3.1) and then applied to various systems of both finite and infinite dimensions and the corresponding set up and data for various systems is tabulated at the end of chapter 4.

In many problems of physical interest involving partial as well as ordinary differential equations, it is possible to find quantities that are invariants (which are equivalent to first integrals of equations of motion). This chapter 5, in which we study the motion of a non-linear string, that is concerned with the study of the Boussinesq equation and state its integrability in relation with a recursive scheme of Lenard and obtain variuos integrals of motion, gives a unified method for finding invariants for a class of equations that includes crystal lattices (the Toda system) and water waves (Boussinesq and KdV equations). The approach here is that at the center of the theory of integrable systems lies the notion of a Lax pair, describing the isospectral deformation of a linear operator. A Lax pair (L, M) is such that the time evolution of the Lax operator $L = [L, M]$, is equivalent to the given non-linear system. The study of the associated linear problem $L\psi = \lambda\psi$ can then be carried out by various methods.

Following the above approach, we have discussed the examples of Toda lattice and showed that its continuum limit the Boussinesq equation in partial differential equations. We have also discussed a common method for the construction of Lenard relations which along with Gel'fand - Dikii type operator trace formulae, yield an explicit recursive construction of the heirarchy of the integrable systems associated with each of the above systems. In section 5.1, we discuss a general method of construction to determine the Lenard relations (cf. 5.1.10-5.1.13) which can be applied to other systems (the Toda and Boussinesq systems) and in particular, we have discussed this here for the KdV equations (cf. relation 5.1.12). In section 5.2, we discuss the Boussinesq equation and derive the recursive relations of Lenard which is given by $A\nabla\lambda = J\nabla\lambda$ for appropriate A, J and $L\psi = \lambda\psi$ (cf. Theorem 5.2.20). Section 5.3 is concerned with the Hamiltonian structure of the Boussinesq equation where we have constructed the Hamiltonians $H_0, H_1, H_2, \cdots$, etc., using the trace functional approach of Adler[5]. Section 5.4 deals with the construction of an operator valued function which yields the infinitesimal generators of the Lax type isospectral deformations associated with the examples discussed in the previous section. Section 5.5 deals with the subhamiltonian system where the flows associated with the Boussinesq equation are restricted to the manifold r = 0. The subhamiltonian system is an integrable system in its own right specifically as $\phi = A_1 \nabla_\phi E_n$, $n = 1, 2, \cdots$, (cf. Theorem 5.5.6) where E_n satisfy a certain recursion scheme (cf.relation 5.1.12).

Finally following closely the works of B.A.Kupershmidt and Yu.I.Manin ([31], [32]) we discuss in this chapter 6, the 1-dimensional and 2-dimensional equations for long waves moving in long channel along a free surface with rigid bottom and various generalizations of these and also the associated Benney's equations for the moment functions of horizontal velocity component. We interpret them as completely integrable Hamiltonian systems by giving the Hamiltonian structure and its first integrals by using the AKS principle. Several variations of Benney's equations are also given as integrable systems including the super symmetric case. In the supersyrnmetric case, we have realized the super Poisson bracket as a non-trivial part of a commutator [,] in a ring of formal Laurent (differential) series with co-efficients in an algebra.

In section 6.1, we discuss the mathematical aspects of long non-linear wave propagation on a free surface and we have obtained some special solutions (cf. 6.1.16). The moment equations and the conservation laws for long non-linear waves are also discussed in this section (cf. 6.1.21). Then we define the moment function for the horizontal velocity function $u(x,y,t)$ of long waves by $A_n(x,t) = \int_0^l u(x,y,t)^n dy$ (cf. Definition 6.1.30). The conservation laws can be written in the form $\frac{\partial P_n}{\partial t} + \frac{\partial Q_n}{\partial x} = 0$ $(n = 1,2,3)$ (cf. 6.1.33), where P_n and Q_n are polynomials in $A_0, A_1, \cdots, A_n$. The equation (6.1.34) for the Benney's equations of long waves satisfied by the moment functions are also described. We also discuss a recursive method of Benney for constructing an infinite number of conservation laws for long waves (cf. Theorem 6.1.40) following the technique of generating function.

Section 6.2 deals with the Lax representation of Benney's equation and application of the geometric AKS principle to the Benney's system. The Benney's system admits a Lax representation given by $L_t = [L, P]$ where L and P are given by

$$L = (1+\Phi_\xi)\frac{\partial}{\partial x} - \Phi_x\frac{\partial}{\partial \xi},$$

$$P = \frac{\partial}{\partial x} - A_{o,x}\frac{\partial}{\partial \xi} \; . \tag{6.2.7}$$

The Benney's system satisfies the AKS principle, that is, they fit into a general scheme of constructing Hamiltonian systems with Lax representation and involutive conservation laws and that the relevant Hamiltonian structure could be identified with the canonical symplectic form on the orbits of the co-adjoint representation of a convenient Lie algebra. Then we have the Benney's system as the quassiclassical limit of the KP system or the generalized KdV equations. That is in the limit $e \to 0$, $[a,b]_o = \lim_{\epsilon\to 0}[a,b]_\epsilon$ and the Lie algebra $\mathcal{G}_o$ of Benney system is the quassiclassical limit of the Lie algebra $\mathcal{G}_1$ of the KdV algebra equation (cf. 6.2.14).

Section 6.3 discusses the Hamiltonian structure of Benney's system from analytical point of view. Here we have given the construction of an infinite number of polynomial conserved densities $H_i \in A_i + \mathbb{Z}[A_o, \cdots A_{i-1}], i \in \mathbb{Z}_+$ starting with $H_o, H_1, \cdots$, (cf. Theorem 6.3.3). Here the conservation laws are obtained from a non-linear integral equation involving **a** parameter. We also give a method of construction for higher Benney equations having an infinite set (cf. 6.3.2) of polynomial conserved densities. In particular, **the** higher flow equation is given by $A_{i,t} = A_{i+2,x} + A_o A_{i,x} + (i+1)A_i A_{o,x} + i A_{i-1} A_{1,x}$, $i \in \mathbb{Z}_+$ (cf. Theorem 6.3.9). We have also discussed the Hamiltonian flows with Hamiltonian structure

$$B_{ij} = iA_{i+j-1}\partial + \partial_j A_{i+j-1} \;, \quad i,j \in \mathbb{Z}_+ \;, \quad \partial = \frac{}{\partial x}$$

so **that the flow** $\#m$ can be written as

$$A_{i,t} = \sum_j B_{ij}\left(\frac{\partial \bar{H}}{\partial A_j}\right) \;, \quad \bar{H} = \frac{1}{m}H_m \;, \quad m \in \mathbb{N}$$

(cf. **Theorem 6.3.11**). The flows have a common Poisson structure given by $L_{,t} = \{P_+, L\} = \{L, P_-\}(*)$, where $L = \xi + \sum_{i=o}^{\infty} A_i \xi^{-(i+1)}$.

In **section** 6.4, we discuss **the supersymmetric** Benney's system. That is to understand, what happens with the flows (6.3.15) when the plane $T^*(\mathbb{R}^1 = \mathbb{R}^2$ is extended into the super plane $\mathbb{R}^{2N}$ equipped with the super Poisson bracket

$$\{F,G\} = F_{,\xi}G_{,x} - F_{,x}G_{,\xi} + \frac{1}{2\xi}\sum_{r=1}^{N}\mathcal{D}_r(F)\mathcal{D}_r(G) \ ,$$

(cf. 6.4.1). Then we have defined the super flows (cf. 6.4.2) and the supersymmetric Benney hierarchy is a **semi-integrable** system meaning that the extended flows do not commute between themselves but nevertheless, they have a common infinite set of polynomial conserved densities. That is, the new hierarchy of flows is not Hamiltonian, despite having a common infinite set of conserved densities (cf. Theorem 6.4.7). In section 6.2, we noted that the Benney's system is a quassiclassical limit of the KP or the KdV system. On the contrary, in the super symmetric Benney's system, the super Poisson bracket is realized as the first non-trivial term of the commutator followed by other symmetric expressions (cf. 6.4.13) and the relation between the super symmetric Benney's system and the super KP hierarchy needs to be explored. Also analogous formula to (6.4.13) are computed for lower dimensional superplanes.

In section 6.5, the Hamiltonian structure of Benney's long wave equations is discussed from a different view point than that was discussed in section 6.3. Here the algebraic approach is followed. The Hamiltonian structure here is defined in a different way by using a special Lie product and the first integrals of motion are obtained. Also, Benney's differential algebra and the operator *B* is defined in section (6.5.5). In otherwords, here we understand the Hamiltonian structure of Benney's system (6.3) as the formal analogue of the Kirillov structure on the orbits of the co-adjoint representation of Lie algebras(cf.6.5.15). Moreover the Poisson bracket defined here is invariant one rather than the one defined in section (6.3).

Now some comments are in order . This study embodies the results obtained **from our** attempt to understand the integrable Hamiltonian systems or partial differential equations by differential and algebraic geometric methods as well as by differential analytic methods and Lie methods. For example chapters 1 and 3 contain several results where algebraic geometric techniques were effectively used. Chapter 4 contains results obtained by us of analytic and Lie theoretic flavour. Chapter 2 simply explains various ways of understanding Poisson and Hamiltonian structures on systems. They were explained clearly from analytic point of view and also various Hamiltonian systems which we studied throughout to apply some unifying principles formulated were described from an elementary stand point in Chapter 2. The Lie theoretic AKS principle after proving was applied for the systems of chapter 2 and these results obtained are summarized in a tabular form in chapter 4. In chapter 5 we studied Boussinesq equation by analytic methods namely Lenard relations and determining certain invariant coefficients whose integral densities give the conservations. Then we have reinterpreted the Hamiltonian structure of Mckean of the Boussinesq equation in terms of the above method. Further this Lenard scheme is also applied to other systems because Boussinesq equation is the continuous limit of discrete Toda system which we studied throughout.

Finally chapter 6 gives the study of Benney's equations in a very systematic and also a thorough understanding of this topic. The classical flavour of this topic is given first. Then Lax type of understanding of this was given and the infinite family of conserved polynomial densities were computed by analytic methods. Then the Poisson structure and then the Hamiltonian structure of Benney's system were given. The AKS principle setup for Benney system was derived by us. Then the super Benney system was shown to be a semi-integrable system. Then we answered by explicitly relating the super Poisson Bracket and the commutator product on a differential ring Lie algebra **a** question of Kupershmidt - Manin. Also we computed the above relation for lower dimensional super plane. This section contains many new results. Finally in **the last section we have** given an invariant definition of Poisson bracket on a Benney differential algebra which we defined and made into a Lie algebra and related this to Kupershmidt-Manin Poisson bracket via an integral representation. In a sense this chapter is the heart of this thesis as it gives a complete understanding of the Benney system to date.

Throughout the thesis several new proofs were given and new interpretations and better understanding of known facts were inserted throughout. For example in chapter 5, our way of reinterpreting McKean's conserved quantities for Boussinesq is a correct procedure. Nevertheless it must be mentioned except for chapters 1 and 3 the rest of the chapters have computational flavour rather than usual mathematical theorem like results and this is justified by the fact after all we are studying properties of Hamiltonian systems arising out of physical contexts from classical mechanics, applied mathematics, theoretical physics etc. We have given reasonably good collection of references used even though it may not be complete as these topics were studied by theoretical physicists also. We have made every effort and taken care to minimize the typing mistakes and since this thesis contains lots of computational formulae, still some typing mistakes may exist which escaped our attention. Finally we have followed the usual indexing pattern of Proposition X.Y.Z means in chapter X, in section Y, the proposition Z.

Chapter 1

Integrability of some partial differential equations

In this chapter, we study certain non-linear partial differential equations such as the Korteweg-de Vries(KdV) equation, the Kadomtsev-Petviashvilii(KP) equation and the Boussinesq equation. We also study the Akhiezer functions and their relation with the KdV equations and Riemann theta functions. We also discuss these equations via the differential operator equations approach.

1.1 SOME GEOMETRIC PREPARATION:

1.1.1 Definition: Let M be a 2-dimensional manifold. A complex chart on M is a homeomorphism $\phi : U \to V$ of an open subset $U \subset M$ onto an open subset $V \subset \mathbb{C}$. Two complex charts $\phi_i : U_i \to V_i$, $i = 1,2$ are said to be *holomorphically compatible* if the map

is biholomorphic.

A *complex atlas* on M is a system $\mathcal{U} = \{\phi_i : U_i \to V_i, \ i \in I\}$ of charts which are holomorphically compatible and which cover M, i.e., $\bigcup_{i\in I} U_i = M$. Two complex charts $\mathcal{U}, \mathcal{U}'$ on M are called *analytically equivalent* if every chart of $\mathcal{U}$ is holomorphically compatible with every chart of $\mathcal{U}'$.

By a *complex structure* on a two-dimensional manifold M, we mean an equivalence class of analytically equivalent atlases on M.

1.1.2 Definition: A Riemann surface is a pair (M, E) where M is a connected two-dimensional manifold and E is a complex structure on M.

1.1.3 Remarks: (1) If M is a Riemann surface, then by a chart on M, we always mean a complex chart belonging to the maximal atlas of the complex structure on M.

(2) When M is compact, we call it a compact Riemann surface.

1.1.4 Definition: Let M be a compact Riemann surface. Then $g = \dim H^1(M, \vartheta)$ is called the genus of M where ϑ is the sheaf of germs of holomorphic functions on M.

1.1.5 Definition: By a complex torus T, we mean the quotient space $T = \mathbb{C}^g/\Gamma$ where Γ is a group of translations generated by $2g$ $\mathbb{R}$-linearly independent vectors in $\mathbb{C}^g$. Thus, T is a group (under addition modulo Γ) and is a complex analytic manifold with natural projection $p : \mathbb{C}^g \rightarrow T = \mathbb{C}^g/\Gamma$, which is a holomorphic local homeomorphism. The complex torus has a natural structure of a compact complex Lie group. The generators of Γ will be represented by $2g$ column vectors $\omega_1, \cdots, \omega_{2g}$, each ω_k € $\mathbb{C}^g$. The j-th component of the vector ω_k will be denoted by ω_{jk}. The g x $2g$ matrix,

$$\Omega = \begin{pmatrix} \omega_{1,1} & \cdots & \cdots & \omega_{1,2g} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ \omega_{g,1} & \cdots & \cdots & \omega_{g,2g} \end{pmatrix}$$

is called the *period matrix* associated with T = fl^/F.

1.1.6 Definition: The matrix Ω is called a *Riemann matrix* if there exists a non-singular, integral, skew-symmetric matrix Q of rank $2g$ such that

$$(1)\ \Omega A \Omega^t = 0 \qquad (2)\ -i\Omega\, Aft = A/>0$$

where $A = Q^{-1}$. The matrix A is called a *principal matrix* for the Riemann matrix Ω. The set (Ω, A) consisting of a Riemann matrix Ω and an associated principal matrix A

is called a Riemann matrix pair.

1.1.7 **Definition:** A complex torus whose period matrix is a Riemann matrix is called an *Abelian Variety.* All period matrices representing an abelian variety are necessarily Riemann matrices.

1.1.8: Let M be a compact Riemann surface of genus g, which is same topologically as a sphere with g handles. Hence M has first homology group $H_1(M, \mathbb{Z})$ as $TL + \cdots + TL$ $(2g$ copies). Choose a $\mathbb{Z}$-basis for $H_1(M, TL)$ $a_1, a_2, \cdots, a_g, b_1, b_2, \cdots b_g$ of $2g$ 1-cycles on M such that $a_i \; o \; a_j = b_i \; o \; bj = 0$ and $a, o \; b_j = Sij(i, j = 1, 2, \cdots, g)$ where o denotes the intersection index.

Let ω be a holomorphic differential on M and let $A_i = \int_a \omega$, $B_j = \int_b \omega$ for $1 < i, j < g$. Then $(A_1, \cdots, A_g) = A$ and $B = (B_1, —, B_g)$ are called the a-periods and b-periods of ω which are vectors in $\mathbb{C}^g$. Then for the space of holomorphic abelian differentials (=space $\mathcal{H}$), there exists a unique basis $\{\omega_1, \cdots, \omega_g\}$ with the property $\int_a w_k = \delta_{jk}(j, k = 1, 2, \cdots, g)$. Also for this basis, the matrix of b-periods $B = (b_{jk})$ with $b_{jk} = \int_b \omega_k$ $(1 < j, k < g)$ is symmetric with positive definite imaginary part. Thus, the vector space $\mathcal{H} — \mathcal{H}^1(M)$ of holomorphic differentials on M is a g-dimensional complex vector space with basis $\{\omega_1, \cdots, \omega_g\}$.

1.1.9 Definition: Let M be a compact Riemann surface of genus g. Let $L = L(M)$ denote the lattice (over TL) generated by the $2g$-columns of the $g \times 2g$ matrix $(/, B)$. The $2g$-column vectors $\{e_1, \cdots, e_g, e'_1, \cdots, e'_g\}$ are linearly independent over $\mathbb{R}$. Every element of $L = L(M)$ is of the form $\sum_{j=1}^{g} m_j e_j + \sum_{j=1}^{g} n_j e'_j, n_j, m_j \in \mathbb{Z}$ and $e'_k = (b_{k1}, \cdots, b_{kg})$ and $e_k = (0, \cdots, 1, 0, \cdots, 0)$ or of the vector form $IM + BN$, where $M — (m_1, \cdots, m_g)^t \in \mathbb{Z}^g$, $TV = (n_1, \cdots, n_g)^t \in TL^9$. The quotient space $J(M) = \mathbb{C}^g / L(M)$ is called the ***Jacobian*** Variety of M, and it is a compact, abelian, g-dimensional complex Lie group.

1.1.10 Definition: Let C_g denote the space of complex symmetric $g \times g$ matrices with positive definite imaginary part. This space is called the ***Siegel** generalized upper half* *plane* which is of dimension g. On the product $\mathbb{C}^g \times \mathcal{C}_g$, we define the Riemann-theta function $\theta(z_1, \cdots, z_g, B)$ to be the entire function on $\mathbb{C}^g$ defined by the Fourier series

$$\theta(z_1, \cdots, z_g, B) = \sum_{m_1,\cdots,m_g \in \mathbb{Z}} exp 2\pi i \{\frac{1}{2} \sum_{k,j=1}^{g} b_{kj} m_k m_j + \sum_{k=1} m_k z_k\} \quad (1.1.11)$$

where $m_k \in \mathbb{Z}$ and $(z_1, \cdots, z_g) \in \mathbb{C}^g$. The above θ-function can also be written in the form

$$\theta(z_1, \cdots, z_g, B) = \sum exp\{\pi i(Bm, m) + 2\pi i(m, z)\}$$

where $(Bm, m) = \sum_{k,j} b_{kj} m_k m_j$ and $(m, z) = m_1 z_1 + \cdots + m_g z_g$.

1.1.12 Remarks: (1) The summation in (1.1.11) extends over all integer vectors in $\mathbb{C}^g$.

(2) The series (1.1.11) converges absolutely and uniformly on compact subsets of $\mathbb{C}^g \times \mathcal{C}_g$.

(3) The θ-function has the properties:

(ii) $\theta(z_1 + b_{k1}, \cdots, z_g + b_{kg}) = exp(-\pi i \frac{b_{kk}}{2} - 2\pi i z_k)\theta(z_1, \cdots, z_g)$

(4) The second property above shows that the Riemann-theta function is related to the Jacobi lattice consisting of $2g$ vectors in an g-dimensional complex linear space $\mathbb{C}^g$ where the vectors are $e_k = (0, \cdots, 1, 0, \cdots, 0)$ and $e'_k = (b_{k1}, \cdots, b_{kg})$ (cf.definition (1.1.9)).

(5)The property (i) of (3) gives the periodicity of the theta function and the property (ii) of (3) gives the quasi periodicity of the Riemann theta function.

1.1.13 The **Abel map** and Special divisors on a Riemann surface:

1.1.14 Definition: Let M be a compact Riemann surface of genus g and $J(M)$ its Jacobian variety. Let $A : M \to J(M)$ be the map defined by $A(\underline{P}) = (A_1(\underline{P})), \cdots, A_g(\underline{P}))$ with $A_k(\underline{P}) = \int_{P_o}^{r} \omega_k (k = 1, 2, \cdots, g)$, where P_o is a fixed point of M and the path of integration is same for all k. A is a well defined map and is called the *Abel map.*

1.1.15 Remark: For $g = 1$, $A : M \to J(M) = T^2$ is an isomorphism.

1.1.16 Definition: A divisor D on a Riemann surface M is a formal sum , $D = \sum_{i=1}^{N} n_i P_i$

with n_i £ $\mathbb{Z}$ and the divisors on M form a group under addition, denoted by $\mathrm{Div}(M)$. Every meromorphic function on M has the same number of zeros as poles.

Now we state the basic

1.1.17 **Theorem(Abel):** For given points $P_1, \cdots, P_n, Q_1, \cdots, Q_n$ of M to be the zeros and poles of some meromorphic function on M, it is necessary and sufficient that on the torus $J(M)$,

$$\sum_{k=1}^{n} A(P_k) - \sum_{fc=i}^{n} A(Q_k) = 0$$ (congruence modulo the lattice) [57].

1.1.18 Example: Let ω be a meromorphic differential on M and (ω) be its divisor defined by its zeros and poles of ω. Then any two such meromorphic differentials on M are linearly equivalent and hence define an equivalence class $[(\omega)]$ of these divisors called the *canonical class of M*, denoted by C. Note that $\deg[(\omega)] = \deg C = 2g - 2, \forall (\omega) \in C$. We can extend the Abel map linearly to the group of divisors on M by, for $D = \sum$rc,\P,-, $A(D) = \sum n_i A(P_i)$, as $A : \mathrm{Div}(M) \to J\{M)$.

The Abel's theorem can be reformulated in terms of the above map A on divisors as

1.1.19 **Abel's** theorem: The divisors D, D' .G $\mathrm{Div}(M)$ are linearly equivalent iff (a) $\deg D = \deg D'$ (b) $A(D) = A(D')$ on $J\{M)$.

1.1.20 Definition: Let D G $\mathrm{Div}(M)$. Let $C(D) = \{/$ € $Mero(M)/(f) + D > 0\}$. Then $C(D)$ is a vector space with dimension denoted by $l(D)$. If $D \sim D'$ then $C(D) = \mathcal{L}(D')$ and hence $l(D) = l(D')$.

1.1.21 Remarks: (1) Let D be a positive divisor. If $\deg D < g$ then the meromorphic functions / with poles in D are only constants i.e., $l(D) = 1$.

(2) In particular if $D = nP_o, n \in N$ then $l(D) = 1$ for $n < g$ i.e., there are no meromorphic functions (non-constant) with a single pole at P_o of order $< g$.

(3) Those points $P_o \in M$ for which no such meromorphic functions / exist having a single pole at P_o of order $< g$, are called Weierstrass points of M.

1.1.22 Definition: The famous Riemann Roch theorem says that if D is any divisor and Z is any canonical divisor on a compact Riemann surface of genus g, then dim $C(D)$ = deg $D + 1 - g$ + dim $C(Z - D)$. We call the divisor D with deg $> g$ non-special if dim $C(Z{\sim}D) = 0$. That is, if $\deg D > 2g-2$, then deg $(Z-D) < 0$ and dim $C(Z-D) = 0$. Thus D is non-special. Otherwise, D is called a special divisor. That is, D € Div (M) with deg $D > g$ such that dim $\mathcal{L}(D) >$ deg $D - g + 1$ is a special divisor. That is, dim $C(Z - D) > 0$.

1.1.23 Remark: Let D be a special divisor with $D - \sum^{N} P_i$, iV $= \deg D > g$. Then consider the Abel map $A : S^n(M) \to J(M)$ where $S^n(M)$ is the n-th symmetric power of M (i.e., the set of all unordered sets of n points of M). Then the set of special divisors on M is precisely the set of critical points of the map A. i.e., those D at which the differential $A_*(D)$ has rank $< g$.

1.2 SOME PARTIAL DIFFERENTIAL EQUATIONS:

In this section, we give a brief account on the Korteweg de-Vries(KdV) equation, the Kadomtsev-Petviashvilii(KP) equation and the Boussinesq equation.

(i) The KdV equations arose originally in the studies of dissipative waves in shallow waters which was of interest in Mathematical Physics and has the form

$$4u_t = 6uu_x + u_{xxx} \tag{1.2.1}$$

(we give a general form of this in section 1.4 later, (cf. 1.4.12))

This equation has solution function $u(x,t)$ of two variables x and t, called respectively the spatial and time variables and the function $u(x,t)$ is called a 'potential function'. The integrability of (1.2.1) was studied by using the inverse scattering method [6,60]. This method puts limitations on the solutions to the class of potentials vanishing rapidly as the space variable tends to infinity resulting in solitons which are waves of definite form moving with different velocities. The most interesting case of (1.2.1) is those equations having solutions $u(x,t)$ which are periodic in x. It was Novikov who first introduced new ideas of integrating (1.2.1) in this case [46].

(ii) The KdV equation has a 2-dimensional analogue which arose in physics and was known now as the Kadomtsev-Petviashvilii equation (KP equation) [23] given by

$$3u_{yy} = \frac{\partial}{\partial x}[4u_t-(6uu_x + u_{xxx})] \qquad (1.2.2)$$

where $u(x,y,t)$ is a function of 2 + 1 (spatial and time) variables.

(iii) Finally we recall the equations of motion of a non-linear string, called the Boussinesq equation, namely

$$3u_{yy} + \frac{\partial}{\partial x}(6uu_x + u_{xxx}) = 0 \qquad (1.2.3)$$

1.2.4 Remark: We note that (1.2.2) reduces to (1.2.1) if the variable y is absent in the potential $u(x,y,t)$ and (1.2.2) reduces to (1.2.3) if the variable t is absent in the function $u(x,y,t)$. We give also the general forms of (1.2.2) and (1.2.3) in section 1.4.

1.3 INTEGRABILITY OF THE KP EQUATION:

In this section, we discuss a general scheme for constructing the periodic and almost periodic solutions of the general Zakharov-Shabat equation which makes it possible to express them explicitly in terms of the Riemann-theta function.

1.3.1 Definition (Akhiezer function): Let M be a compact Riemann surface of genus g. Let Q be a fixed point on M and $z = z(P)$ be a local coordinate in a neighbourhood of Q such that $z(Q) = 0$. Let $k = \frac{1}{z}$ so that $k(Q) = \infty$. Let $q(k)$ be a polynomial in the parameter k. Let $D = P_1 + \cdots + P_g$ be a positive divisor of degree g on M.

By an Akhiezer function on M corresponding to the data $\{ Q, k = \frac{1}{z}, q(k), D \}$, we mean a function $\Psi(\underline{P})$ on M such that

(a) $\Psi(\underline{P})$ is meromorphic on $M \setminus \{Q\}$ and has poles only at $P_1, \ldots, P_g$.

(b) The product $\Psi(\underline{P}) exp(-q(k))$ is analytic in a neighbourhood of Q.

1.3.2 Remarks: (1) Condition (b) says that Ψ has an essential singularity of the form

$\Psi(\underline{P}) \sim C\exp(q(k))$ at $P = Q$.

(2) For a given non-special divisor D of *degg* on M, such Akhiezer functions form a 1-dimensional (E-vector space *A(D)* and hence such $\Psi(\underline{P})$ is unique upto a constant. We state this as

1.3.3 Theorem: Let $D = \sum^{g} P_i$ be a non-special divisor of degree *g*. Then the space *A(D)* of Akhiezer functions for a given polynomial *q*, is 1-dimensional.

Proof: Note $\Psi(P)$ has *g* zeros on *M* and the divisor defined by these zeros is non-special. Suppose Ψ, Ψ' are two Akhiezer functions corresponding to the same divisor *D*. Then *ty(P)/W(P)* is a meromorphic function on *M* having poles on the divisor of zeros of Ψ'. Since such a divisor is non-special, Ψ/Ψ' is constant, and hence such Ψ is unique.

1.3.4: (a)Let P_o be a point in *M*. Then the mapping $A : M \to J(M)$ is defined (this is the Abel map defined in the previous section). The coordinates of the vector $A_k(P) = \int_{P_o}^{P} \omega_k$ where ω_k is a mermorphic differential on *M* which has a singularity at the fixed point *Q* in *M* (of the form $dq_j(z_j^{-1})$ in the local parameter z_j) and is normalized by the condition / $\omega = 0$. Let $U = (U_1, \cdots, U_g)$ be the vector of the *b*-periods of ω i.e., : $\int_b \omega$. Let *D* be a given non-special divisor, $D = \sum^{g} P_i$, of degree *g*. Then define the function

$$\Psi(\underline{P}) = \exp\left(\int_{P_o}^{P} \omega\right) \frac{\theta(A(P) - A(D) + U - K)}{\theta(A(\underline{P}) - A(D) - K)} \tag{1.3.5}$$

where *D* is the given non-special divisor and *A* is the Abel map and *K* is a vector on the Jacobian variety $J(M)$, called the Riemann vector where $K = (K_1, \cdots, K_g)$ is the vector of Riemann constants given by

$$K_j = \frac{1}{2} - \frac{1}{2}B_{jj} + \sum_{l=1, l\neq j}^{g} \int_{a_l} \left(\int_{P_o}^{P} \omega_j\right) \omega_l, \quad (j = 1, \cdots, g)$$

where the path of integration in $\exp\left(\int_{P_o}^{P} \omega\right)$ and $A(\underline{P})$ is the same. Then the function (1.3.5) is well-defined and Ψ is an Akhiezer function having an essential singularity of given type at Q.

(b) Special case $q(k) — kx + k^2y + kH$:

Let the **polynomial** $q(k)$ - $kx + k^2y + kH$ where x, y, t are parameters. Let $\Psi(x,y,t,\underline{P})$ denote the Akhiezer function corresponding to a non-special divisor D of degree g on M. Suppose Ψ is normalized in a neighbourhood of Q so that it has an expansion of the form

$$\Psi(x,y,t,\underline{P}) = e^{kx+k^2y+k^3t}\left(1 + \frac{\xi_1}{k} + \frac{\xi_2}{k^2} + \cdots\right) \tag{1.3.6}$$

where ξ_i are functions of x, y, t. Then Ψ in (1.3.6) satisfies

$$\left[-\frac{\partial}{\partial y} + \frac{\partial^2}{\partial x^2} + u\right]\Psi = O\left(\frac{1}{k}\right)e^{kx+k^2y+k^3t} \tag{1.3.7}$$

$$\left[-\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3} + \frac{3}{2}u\frac{\partial}{\partial x} + w\right]\Psi = O\left(\frac{1}{k}\right)e^{kx+k^2y+k^3t} \tag{1.3.8}$$

where u and w can be found from the condition for the vanishing of the co-efficients of k^{-n} for $n = 0, 1, 2, 3$.

In fact,

$$u = -2\frac{\partial \xi_1}{\partial x} \tag{1.3.9}$$

and

$$w = 3\xi_1\frac{\partial \xi_1}{\partial x} - 3\frac{\partial^2 \xi_1}{\partial x^2} - 3\frac{\partial \xi_2}{\partial x} \tag{1.3.10}$$

Consider the differential operators

$$L = \frac{\partial^2}{\partial x^2} + u \quad , \quad A = \frac{\partial^3}{\partial x^3} + \frac{3}{2}u\frac{\partial}{\partial x} + w \tag{1.3.11}$$

in x given by (1.3.7) and (1.3.8).

1.3.12 Theorem: Let $\Psi(x,y,t,\underline{P})$ be the Akhiezer function for the polynomial $q(k) = kx + k^2y + k^3t$ corresponding to D. Then Ψ is a solution of the system of equations

$$\frac{\partial \Psi}{\partial y} = L\Psi \tag{1.3.13}$$

and

$$\frac{\partial\Psi}{\partial t} = A\Psi \tag{1.3.14}$$

Proof: The function $\Phi_1 = \left(-\frac{\partial}{\partial y} + L\right)\Psi$ and $\Phi_2 = \left(-\frac{\partial}{\partial t} + A\right)\Psi$ are Akhiezer functions and $\Phi_i e^{-(kx+k^2y+k^3t)}$ vanishes at Q and hence by the uniqueness, $\Phi_i = 0$ on $M, i = 1,2$.

1.3.15 Corollary: The function u and w given above satisfy the system of non-linear equations

$$\frac{3}{2}u_y + \frac{3}{2}u_{xx} - 2w_x = 0 \quad w_y - u_t + u_{xxx} + \frac{3}{2}uu_x - w_{xx} = 0 \tag{1.3.16}$$

Proof: The compatibility condition for $\frac{\partial\Psi}{\partial y} = L\Psi$, $\frac{\partial\Psi}{\partial t} = A\Psi$ is the commutator equation $\left[-\frac{\partial}{\partial y} + L, -\frac{\partial}{\partial t} + A\right] = 0$ which gives (1.3.16).

1.3.17 Remarks: (1) Eliminating w from (1.3.16) we get

$$\frac{3}{4}u_{yy} = \frac{\partial}{\partial x}\left[u_t - \frac{1}{4}(6uu_x + u_{xxx})\right] \tag{1.3.18}$$

which is called the Kadomtsev-Petviashvilii equation (KP equation) [23].

(2) For each Riemann surface M of genus g, for each point Q on M and for each local parameter k^{-1} in a neighbourhood of Q we can find a family of solutions $u(x,y,t)$ of the Kadomtsev-Petviashvilii equation parametrized by the non-special divisors of degree g on M (or by points in general position of the Jacobian variety $J(M)$ of M).

(3) Using the formula (1.3.5) connecting the Akhiezer function Ψ and the Riemann theta function we can express solutions of Kadomtsev-Petviashvilii equation in terms of theta function of M.

Let $\Omega_1, \Omega_2, \Omega_3$ denote normalized meromorphic differentials on M with poles only at Q and with principal parts at Q of the form $\Omega_1 = dk + \cdots \Omega_2 = d(k^2) + \cdots$, $\Omega_3 = d(k^3) + \cdots$. Then (1.3.5) gives

$$\Psi(x,y,t,\underline{P}) = \exp\left(x\int_{P_o}^{P}\Omega_1 + y\int_{P_o}^{P}\Omega_2 + t\int_{P_o}^{P}\Omega_3\right) \times \frac{\theta(A(P)-A(D)+xU+yV+tW-K)}{\theta(A(\underline{P})-A(D)-K)} \quad (1.3.19)$$

where U, V, W are the b-periods of $\Omega_1, \Omega_2, \Omega_3$ respectively.

1.3.20 Theorem: Then the solutions of the Kadomtsev-Petviashvilii equation given above by $u(x,y,t) = -2\frac{\partial}{\partial x}\xi_1$ can be expressed as $u(x,y,t) = (-2\frac{\partial^2}{\partial x^2}\xi_1)\log\theta(xU + yV + tW + Z_o) + C$ (1.3.21) where $\theta = \theta(z)$ is the theta function on M, $Z_o = A(D) - K$ is an arbitrary vector of $J(M)$ and C is a constant.

Proof: Recall $u(x,y,t)$ is $-2\frac{\partial\xi_1}{\partial x}$ where ξ_1 is the coefficient of $\frac{1}{k}$ in the expression of the Akhiezer function Ψ in a neighbourhood of Q. Note that $\log\Psi = kx + k^2y + k^3t + \xi_o + \frac{\xi_1+ax+by+ct}{k} + \cdots$ where ξ_o is a function of x, y, t and a, b, c are constants. Hence $\xi_1 + ax + by + ct$ is the coefficient of $\frac{1}{k}$ in the expression of the function log $\frac{\theta(A(P)-A(D)-K+xU+yV+tW)}{\theta(A(P)-A(D)-K)}$ in a neighbourhood of $P = Q$. Since $A(P)$ in a neighbourhood of Q has the expression $A(P) = A(Q) - \frac{1}{k}U + 0\left(\frac{1}{k^2}\right)$ and taking Q as the base point in the Abel map (i.e. $A(Q) = 0$), we get $\xi_1 + ax + by + ct = -\frac{\partial}{\partial x}\log\theta(xU + yV + tW - A(D) - K) + C$. Then $u(x,y,t) = -2\frac{\partial\xi_1}{\partial x} = 2\frac{\partial^2}{\partial x^2}\log\theta(xU + yV + tW - A(D) - K) + C$.

1.3.22 Remarks: (1) Relation (1.3.21) tells us that the solutions $u(x,y,t)$ of the Kadomtsev-Petviashvilii equation are conditionally periodic functions of the variables x, y and t. Note that $\frac{\partial^2}{\partial x^2}\log\theta(z)$ is a meromorphic function on $J(M)$ and $u(x,y,t)$ is obtained by restricting this meromorphic function to the linear winding by (x,y,t) on $J(M)$ spanned by the vectors U, V, W respectively.

(2) *Special cases:* (a) Suppose that for the Riemann surface M and for the point Q on M there exists a meromorphic function $\lambda(P)$ on M with a unique double pole at Q. Choose $k^{-1}(\underline{P}) = \frac{1}{\sqrt{\lambda}}$ as the local parameter $k^{-1} = k^{-1}(\underline{P})$ in a neighbourhood of Q. Then the Akhiezer function $\Psi(x,y,t,\underline{P})$ with an essential singularity $\exp(kx + k^2y + k^3t)$ at

Q has the form $\Psi(x,y,t,P) = e^{y\lambda(P)}\Phi(x,t,\underline{P})$ where Φ is the Akhiezer function with the same divisor of poles as Ψ and with an essential singularity $\Phi \sim e^{kx+k^3t}$ at Q. Then the function Φ satisfies $L\Phi(x,t,\underline{P}) = \lambda(P)\Phi(x,t,P)$ and $\frac{\partial\Phi}{\partial t} = A\Phi$ i.e., Φ is the eigenfunction of the Schrodinger operator L with eigenvalue $\lambda(\underline{P})$, depending on the parameter *t*. The coefficients of the operators L and A are independent of *y* and the Kadomtsev-Petviashvilii equation reduces to the KdV equation whose solution is given by $u(x,t) = 2\frac{\partial^2}{\partial x^2}\log\theta(xU + tW + Z_o) + C$.

(b) Suppose for a Riemann surface M and for a point Q on M there exists a meromorphic function $\mu(\underline{P})$ with a single pole at Q of third order. Then as in (a) above the dependence in *t* disappears and the Kadomtsev-Petviashvilii equation reduces to the Boussinesq equation for a non-linear string whose solution is given by

$$u(x,y) = 2\frac{\partial^2}{\partial x^2}\log \theta(xU + yV + Z_o) + C\,.$$

1.4 NOVIKOV-DUBROVIN-KRICHEVER-LAX SCHEME:

In this approach we concentrate more on the study of differential operators and operator equations rather than handling partial differential equations satisfied by the coefficient functions of these operators. In other words, we reverse the roles of the Kadomtsev-Petviashvilii equations and the operator pairs L, A in the above section. Let us start with a pair of differential operators L_1 and L_2 given by

$$L_1 = \sum_{k=0}^{n} u_k(x,y,t)\frac{\partial^k}{\partial x^k} \quad \text{and} \qquad L_2 = \sum_{l=0}^{m} v_l(x,y,t)\frac{\partial^l}{\partial x^l} \tag{1.4.1}$$

with co-efficients as functions of the parameters x, y and t. The scheme is to study the non-linear partial differential equations for the coefficient functions of the operators L_1 and L_2. We concentrate on the equivalent operator equations

$$\left[L_1 - \frac{\partial}{\partial t}\ ,\ L_2 - \frac{\partial}{\partial y}\right] = 0 \tag{1.4.2}$$

These equations are called Zakharov-Shabat equations [63]. Note that (1.4.2) can be written also as Lax type equation $L_{2t} - L_{1y} = [L_1, L_2]$.

1.4.3 Definition (general Akhiezer functions): For each Riemann surface M of genus g and for a point $Q \in M$ and for each non-special divisor D of degree g, there exists a unique function $\Psi(x,y,t,P)$ on M satisfying

(a) Ψ is meromorphic in $M \setminus \{Q\}$ with only poles at points of D

(b) Near Q, Ψ has expression

$$\Psi = e^{kx+q(k)y+r(k)t}\left(1+\sum_{s=1}^{\infty}\frac{\xi_s(x,y,t)}{k^s}\right) \tag{1.4.4}$$

where $q(k) = q_m k^m + \cdots + q_o$ and $r(k) = r_n k^n + \cdots + r_o$ are given polynomials of degrees m and n respectively in the local parameter k with $k^{-1}(Q) = 0$. Now generalizing Theorem (1.3.12) we get

1.4.5 Theorem: For each Akhiezer function Ψ as in (1.4.4) there exist unique differential operators L_1 and L_2 of form (1.4.1) such that $L_1\Psi = \frac{\partial}{\partial t}\Psi$, $L_2\Psi = \frac{\partial}{\partial y}\Psi$.

Proof: Note that for Ψ of (1.4.4) as formal series, there exists a unique operator L_1 such that

$$L_1\Psi \equiv \frac{\partial}{\partial t}\Psi \left(\mathrm{mod}\ 0\left(\frac{1}{k}\right)e^{kx+q(k)y+r(k)t}\right) \tag{1.4.6}$$

The point now is, the congruence equation (1.4.6) for Ψ becomes an exact equation on the compact Riemann surface M. Then the function $(L_1 - \frac{\partial}{\partial t})\Psi(x,y,t,\underline{P})$ satisfies the definition of the Akhiezer function and the function $(L_1 - \frac{\partial}{\partial t})\Psi e^{-(kx+q(k)y+r(k)t)}$ vanishes at Q and hence by the uniqueness of the Akhiezer function $L_1\Psi = \frac{\partial}{\partial t}\Psi$ on M. Similarly the L_2 operator can be found such that $L_2\Psi = \frac{\partial}{\partial y}\Psi$.

1.4.7 Corollary: The operators L_1 and L_2 of (1.4.5) satisfy the operator equation

Proof: By the above theorem, the kernel of the operator $[L_1 - \frac{\partial}{\partial t}, L_2 - \frac{\partial}{\partial y}]$ contains a 1-parameter family of functions $\Psi(x,y,t,\underline{P})$. But this operator involves only differentiation in x and hence, its kernel, if it is non-zero is of finite dimension and hence the operator must be identically zero operator.

1.4.8 Remarks: (1) This Zakharov-Shabat equation (1.4.2) is equivalent to the non-linear system (1.3.13) which in its turn is equivalent to the Kadomtsev-Petviashvilii equation (1.3.18) (cf. proof of Lemma 1.3.12).

(2) Note that $L_2 = A = \frac{\partial^3}{\partial x^3} + \frac{3}{2}u\frac{\partial}{\partial x}$+ w and L_1 - L — $\frac{}{\partial x^2}$ + u are as in section 1.3. Now we derive the general Kadomtsev-Petviashvilii equation following this scheme with $q(k) = q_2k^2+q_o$, $r(k) = r_3k^3+r_1k+r_o$. Then we get the operators L_1 - $q_2(\frac{\partial^2}{\partial x^2}+v_o(x,y,t))$ and $L_2 = r_3\left(\frac{\partial^3}{\partial x^3} + u_1(x,y,t)\frac{\partial}{\partial x}u_o(x,y,t)\right)$ satisfying (1.4.2).

Now as in section 1.3 eliminating u_1 and u_o from the equivalent system of equations to (1.4.2) we get the equation for $v(x,y,t)$ as

$$-\frac{3}{4}\beta^2 v_{yy} + \frac{3}{\partial x}\left\{\alpha v_t + hv_x - \frac{1}{4}(v_{xxx} + 6vv_x)\right\} = 0 \qquad (1.4.9)$$

where β, a, h are certain constants $(\alpha = \frac{1}{q_2}, \beta = \frac{1}{r_3}, h = r_1)$

Note that $v_o = q_o - 2\frac{\partial \xi_1}{\partial x_1}$ as in section 1.3. We call (1.4.9) the generalized Kadomtsev-Petviashvilii equation.

Now we can proceed as in section 1.3 to solve the generalized Kadomstev-Petviashvilli equation (1.4.9) for $v(x,y,t)$ in terms of the theta function.

1.4.10 Theorem: The solutions of the generalized KP equation (1.4.9) is given by $v(x,y,t) = q_o + 2\frac{\partial^2}{\partial x^2}\log\theta(xU + yV + tW + Z)$ where Z is an arbitrary point in the $J(M)$.

Proof: Proceed the proof as that of Theorem (1.3.20) by taking Ω_1 as the normalized meromorphic differential having a unique pole at Q with principal part as $-\frac{dz}{z^2}$ at Q, Ω_2 as that one with $dq(\frac{1}{z})$,Ω_3 as that one with $dr(\frac{1}{z})$ at pole Q in the local parameter $z(P)$.

1.4.11 Remarks: (1) **If** $V = 0$, i.e., there exists on M a meromorphic function with a unique pole of order 2 at Q, then $v(x,y,t) = u(x,t)$ satisfies the generalized KdV equation

$$\alpha v_t + h v_x - \frac{1}{4}(v_{xxx} + 6 v v_x) = 0. \tag{1.4.12}$$

(2) If $W = 0$, i.e., there exists a meromorphic function with a unique pole of order 3 at Q, then $v(x, y, t) = v(x, y)$ satisfies the generalized Boussinesq equation

$$v_{yy} - v_{xx} + \frac{1}{4} v_{xxxx} + \frac{3}{2}\frac{\partial}{\partial x}(v v_x) = 0 \tag{1.4.13}$$

$(h = \frac{3}{4},\ \beta^2 = 1)$ Finally we note that (1.4.9), (1.4.12) reduce to the usual Kadomtsev-Petviashvilii and the KdV equations (1.3.18), and (1.3.16).

(3) Since $v(x, y, t) = q_o + 2\frac{\partial^2}{\partial x^2}\log\theta(xU_1 + yU_2 + tU_3)$ and since the vectors U_i of b-periods (as defined above) determine the rectilinear parts of the $J(M)$, it follows that all the solutions of the Zakharov-Shabat equations are conditionally periodic functions.

1.5 CONCLUDING REMARKS:

We close this chapter by indicating some interesting inter connections and applications of Kadomtsev-Petviashvilii equations theory to other topics.

(1) We proved above that the general Zakharov-Shabat equations $L_t - A_y = [A, L]$ (or the generalized Kadomtsev-Petviashvilii equations) have solutions $u(x, y, t)$ which can be expressed in terms of Riemann theta functions. Using this solution function $u(x, y, t)$ we can explicitly solve the non-stationary Schrodinger equation

$$i\Psi_t + \Psi_{xx} - u(x, y, t)\Psi = 0$$

by giving explicit formula for Ψ. This method of construction of solutions of Kadomtsev-Petviashvilii equation also classifies the commutative rings of differential operators in one (and several) variables [16,17,28].

(2) The works on periodic solutions of KdV and Kadomtsev-Petviashvilii equations are essentially those of Novikov, Dubrovin, Its, Matveev and Lax [34,46]. It was Novikov [46] who first discovered the deep algebraic geometric nature of the finite zone periodic and

conditionally periodic solutions of the KdV equation and also by Mckean and Moerbeke later [40]. It was Dubrovin and Novikov who introduced the concept of finite zone linear differential operators. The general philosophy is with each linear differential operator L (such as the Sturm-Liouville operator) we can associate a special eigenfunction Ψ called the Bloch or Floquet function (which depends on a spectral parameter A of I) and there exists a Riemann surface M on which Ψ becomes single-valued and we say L is of finite zone if the associated Riemann surface M is of finite genus. More generally, certain important partial differential equations can be expressed as commutator equations of Novikov type $[L_1, L_2] = 0$ where $L_1 = \sum_\alpha u_\alpha(x)\frac{\partial^\alpha}{\partial x^\alpha}$, $L_2 = \sum_\alpha v_\beta(x)\frac{\partial^\beta}{\partial x^\beta}$.Then there exists a polynomial $\underline{P}(w, E)$ of two variables w, E such that $P(L_2, L_1) = 0$ which gives a Riemann surface M defined by $P(x, y) = 0$ and the Bloch eigenfunction Ψ makes sense on M .

(3) One can study the Hamiltonian structure of equations of Lax type $L_t = [L, A]$ and more generally for Zakharov-Faddeev type equations also. Some other applications are: Euler's equations of motion for a rigid body, as Novikov type operator equation [37] and a two-dimensional Schrodinger operator and its algebraic analogue and more generally the study of Integrable Hamiltonian systems [40], the relation between motion of n particles in a straight line and the singularities of rational type and elliptic solutions of the KdV equation [37] and finally a cohomological interpretation of Lax type equations [19]. These will be investigated in the subsequent chapters wherein we find (not just one transcendental solution as given in the above Theorems 1.3.20 and 1.4.10) all the first integrals of these systems.

(4) There are deeper connections between holomorphic vector bundles over a compact Riemann surface M and certain non-linear equations. This set up can be understood in terms of a pair of commuting differential operators. This whole theory of KdV and Kadomtsev-Petviashvilii equations, we have dealt with in sections 1.3 and 1.4 can be

thought of as Lax type equations $[L_1, L_2] = 0$ of "rank 1". For higher rank cases of Lax equation, holomorphic vector bundles over compact Riemann surface will arise having some special parameter called "Tyurin parameters" [23,29].

Chapter 2

Complete Integrability of some integrable systems (CIHS)

In this chapter, in the first two sections all the basics on Poisson structure on Euclidean spaces and general manifolds and Hamiltonian structures are developed and the properties of them needed for later use are given. In particular, these will be used in chapter 4, chapter 5 and chapter 6. Then, we study various integrable systems such as the Toda lattice, the Lagrange top, the geodesic motion on an ellipsoid, the KdV equations and the generalized KdV equations (or the Gel'fand-Dikii system). We describe the Hamiltonian function, the Poisson structure and the first integrals of motion of these systems. We explain in chapter 4 how these systems obey the Adler-Kostant-Symes geometric principle.

2.1 SOME PRELIMINARIES: In this section, we give an introduction to the various aspects of Poisson brackets used in different parts (contexts) of the thesis.

2.1.1: The basic geometry of a physical system is described by a configuration space M, a C^∞ manifold. Elements q of M represent instantaneous configurations of the physical system. For instance, for k particles in 3-dimensional space $\mathbb{R}^3$, the configuration space is $M = (\mathbb{R}^3)^* \cong \mathbb{R}^{3k}$. Let / be an interval in IR. A curve $\mu : t \to q(t)$ (where t runs over the time interval /) with values in M describes the motion, i.e., change in time, of the physical system. The velocity $v(t_o)$ at time t_o (also denoted by $\dot{q}(t_o)$) is defined as: $v(t_o) = \frac{d}{dt}|_{t=t_o} q(t)$. Thus $v(t_o)$ belongs to the tangent space $T_{q(t_o)}M$ at $q(t_o)$. The collection of spaces T_qM for q in M is the tangent bundle TM over M. Hence to describe the positions and velocities of the various parts of the system at a certain time t_o, we need to specify a point in TM. We can interpret TM as the kinematical space.

Let N be the dimension of M. Every system of local coordinates $q^1, \cdots, q^N$ on M gives rise to a system of local coordinates $q^1, \cdots q^N, v^1, \cdots v^N$ on the tangent bundle TM by differentiation. If the transition functions between two coordinate systems q and q are given by

$$\bar{q}^j = \phi^j(q^1, \cdots, q^N) \ , \ 1 \le j \le N \ ,$$

the corresponding transition functions between v^j and $\bar{v}^j$ are

$$\bar{v}^j = \sum_{i=1}^{N} \partial_i \phi^j(q^1, \cdots q^N) v^i \ ,$$

where $\partial_i = \frac{\partial}{\partial q^i}$. The coordinate changes on TM corresponding to coordinate changes on M are linear in the v_i's; in more abstract terms, TM is not only a fibration over A/, but even a vector bundle. Velocity vectors at a point can be added. For q in A/, we denote by (q, v) a point in $T_q M$, and by (q, ξ) a point in $T_q^* M$, where $T_q^* M = (T_q M)^*$ is the algebraic linear dual of $T_q M$. The collection of the vector spaces $T_q^* M$, for q running over M, is another vector bundle T^*M over A/, called the co-tangent bundle.

A force field F is called conservative if the work integral

$$\int_a^b \mathbf{F} d\mathbf{S} = \int_{t_a}^{t_b} \mathbf{F}(q(t)).\dot{q}(t) dt$$

only depends on the end points a and b, and not on the particular choice of the curve $q(t)$ joining them. In the case of a conservative field there exists a function V, unique upto an additive constant, such that

$$\mathbf{F}_i = -\partial_i V \quad \text{or} \quad F = -dV$$

In fact, we define $V(a) - V(b) = \int_{t_a}^{t_b} F(q(t)).q(t)dt$ for an arbitrary curve $q(t)$ between a and b. In summary, the potential V is a function on M, the force field F is a differential form of degree one on M, that is, a section of the cotangent bundle T^*M over M.

2.1.2: In the case of a conservative force field, acting on a particle of mass m, the law of conservation of energy

$$\frac{1}{2}m|v_a|^2 + V(a) = \frac{1}{2}m|v_b|^2 + V(b)$$

is a consequence of the differential equation (Newton's 2nd law)

$$F = m\mathbf{a}$$

Here V is the potential energy and v_a (resp. v_b) is the velocity at time t_a (resp. t_b), where the particle is at the point a (resp. b).

Given a system of various particles with positions $q(i)$, velocities $v(i)$ and masses $m(i)$, we can verify that this expression $\sum_i \frac{1}{2}m(i)|v(i)|^2$defines a differential quadratic form on M, called the kinetic energy T. This is a function $T : TM \longrightarrow \mathbb{R}$ and T restricted to the vector space T_qM is a quadratic form for every q in M. The potential $V : M \rightarrow \mathbb{R}$ can be lifted to a function $V : TM \rightarrow \mathbb{R}$. The total energy $E = T + V$ is then a function defined on TM.

The discovery of Lagrange is that the laws of motion can be formulated entirely in terms of the function $L = T - V$ on M called the Lagrangian. The equations governing the motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q^i} \quad \text{(Euler-Lagrange)}$$

Here we denote the local coordinates on TM by $q^1, \cdots q^N, \dot{q}^1, \cdots \dot{q}^N$where $\dot{q}^i$ is used as an alternative notation for v^i. The Euler-Lagrange equations are derived from the variational principle of Hamiltonian $\delta \int L(q, \dot{q})dt = 0$. Another picture of the above was discovered by Hamilton which we give below.

Assume that a Lagrangian function $L : TM \rightarrow \mathbb{R}$ is given, not necessarily of the form $L = T - V$ as above. We define the Legendre transformation $\Lambda : TM \rightarrow T^*M$ by

$\Lambda(q,v) = (q, \Lambda_q(v))$ where $\Lambda_q(v) \in T_q^*M$ is defined by

$$< \Lambda_q(v), w >= \frac{d}{d\epsilon}|_{\epsilon=0} L(q, v + \epsilon w) \text{ for } q \in M , v , w \in T_qM$$

. The energy function $E: TM \rightarrow \mathbb{R}$ is then defined by

$$E(q,v) =< \Lambda_q(v), v > -L(q,v).$$

In terms of local coordinates, $\dot{q}^1, \cdots, \dot{q}^N$ are a set of linear coordinates on the vector space T_qM for fixed $q = (q^1, \cdots q^N)$; we define $p_1, \cdots p_N$ as the dual system of coordinates on the vector space T_q^*M dual to T_qM. Then $q^1, \cdots, q^N, p_1, \cdots, p_N$ form a set of local coordinates on T^*M associated in a canonical way to the local coordinates $q^1, \cdots, q^N$ on M. The Legendre transformation is then given by $p_i = \frac{\partial L}{\partial \dot{q}^i}$, and the energy function is $E = \sum_i p_i \dot{q}^i - L(q, \dot{q})$. The Lagrangian is non-degenerate iff the map Λ is a local diffeomorphism of TM into T^*M, that is, iff the determinant det $\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right)$ is nowhere 0 on TM. The coordinates $p_1, \cdots, p_N$ are called the generalized momenta. The Hamiltonian is the function H on T^*M such that $H \circ \Lambda = E$. i.e., $\Lambda^* H = E$. In coordinates:

$$H(q^1, \cdots, q^N, p_1, \cdots p_N) = \sum_{i=1}^{N} p_i \dot{q}^i - L(q, \dot{q})$$

where the relation between p_i and $\dot{q}^i$ is given by $p_i = \frac{\partial L}{\partial \dot{q}^i}$ as above.

2.1.3: According to Hamilton, the **Euler-Lagrange** equations are equivalent, via the Legendre transformation, to the following system

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} \quad , \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} . \tag{2.1.4}$$

This is a system of first order differential equations on the cotangent bundle with symmetry.

Consider a mechanical quantity represented by a function $F : T^*M \rightarrow \mathbb{R}$ on the phase space. Suppose **that a** motion $\mu : I \rightarrow \mathbb{R}$ is given, where $\mu(t) = q(t)$ is the moving point in M. Lift μ to TM, with value $(q(t),\dot{q}(t))$ at time t and then define $\tilde{\mu} : I \longrightarrow T^*M$ by

$$\tilde{\mu}(t) = \Lambda(q(t), \dot{q}(t)) \quad .$$

The time derivative $\frac{d}{dt}(F o \tilde{\mu})(t)$ can be written as $\dot{F} o \tilde{\mu}$ where $\dot{F}$ is a new function on T^*M, independent of the motion μ. Using Hamilton's equations (2.1.4), we obtain the dynamical law

$$\dot{F} = \sum_i \left(\frac{\partial F}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q^i} \right)$$

This can be expressed as follows. On the cotangent bundle, we can introduce in an invariant way the Poisson bracket $\{F,G\}$ of two functions F, G in $C^\infty(T^*M)$ by

$$\{F,G\} = \sum_i \left(\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} - \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} \right) \qquad \textbf{(2.1.5)}$$

This Poisson bracket is independent of the chosen Lagrangian L and of the corresponding Hamiltonian H. The dynamical law can be written in a concise way as

$$\dot{F} = \{H, F\} \quad \text{for every} \quad F \ \text{in} \ C^\infty(T^*M) \qquad (2.1.6)$$

This Poisson bracket $\{F,G\}$ is a bilinear expression of F, G and enjoys the following properties:

(1) $\{F,G\} = -\{G,F\}$ (Skew symmetry)

(2) $\{F,G_1G_2\} = \{F,G_1\}G_2 + G_1\{F,G_2\}$ (Leibniz rule)

(3) $\{F_1,\{F_2,F_3\}\} + \{F_2,\{F_3,F_1\}\} + \{F_3,\{F_1,F_2\}\} = 0$ (Jacobi identity)

For the coordinate functions we obtain

Hamilton's equations (2.1.4) are recovered from the dynamical law (2.1.6) by using the relations

$$\{q^i, F\} = -\frac{\partial F}{\partial p_i} \ , \ \{p_i, F\} = \frac{\partial F}{\partial q^i}$$

2.1.7 Definition: In general, a Poisson bracket on a manifold M is a bilinear mapping $\{.,.\}$ on the function space $C^\infty(M)$ of smooth functions satisfying the following conditions:

(1) $\{F, G\} = -\{G, F\}$ (Skew symmetry)

(2) $\{F, G_1G_2\} = \{F, G_1\}G_2 + G_1\{F, G_2\}$ (Leibniz rule)

(3) $\{F_1, \{F_2, F_3\}\} + \{F_2, \{F_3, F_1\}\} + \{F_3, \{F_1, F_2\}\} = 0$ (Jacobi identity)

A Poisson manifold is by definition a manifold equipped with a Poisson bracket.

2.1.8 Remarks: (1) Let P be a Poisson manifold. The mapping ξ_F defined for F in $C^\infty(P)$ by $\xi_F(G) = \{F, G\}$ is a derivation of $C^\infty(P)$ and therefore can be viewed as a vector field on P. It is called the Hamiltonian vector field with Hamiltonian F. From the Leibniz rule, we have the derivation, $\xi_{F_1F_2} = F_1\xi_{F_2} + F_2\xi_{F_1}$. Hence we can define a mapping $\tilde{J} : T^*P \to TP$ by

$$J(dF) = \xi_F \qquad \text{or} \qquad < dG, J(dF) > = \{F, G\} .$$

For local coordinates $x_1, \cdots, x_n$ on P, J is given by a matrix $(J_{\alpha\beta})$ of functions, namely, $J_{\alpha\beta} = \{x_\alpha, x_\beta\}$. The Poisson bracket then becomes:

$$\{F, G\} = \sum_{\alpha,\beta} J_{\alpha\beta}\partial^\alpha F \partial^\beta G$$

where $\partial^\alpha = \frac{\partial}{\partial x_\alpha}$. Hence the vector field ξ_F has components $\xi_F^\beta = \sum_\alpha J_{\alpha\beta}\partial^\alpha F$.

(2) In the canonical case where P is the cotangent bundle of a manifold M and the Poisson bracket is the usual one, we can choose a local coordinate system such that the matrix J is of a simple form (in block form):

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

In fact, any coordinate system $(q^1, \cdots, q^N, p_1, \cdots, p_N)$ on $P = T^*M$ corresponding to a coordinate system $(q^1, \cdots, q^N)$ on M does the trick.

(3) The Poisson bracket can be recovered from the mapping $\tilde{J}$ on the matrix $(J_{\alpha\beta})$ and we can define a Poisson structure in terms of $J_{\alpha\beta}$. The various conditions on the bracket become;

(i) $J_{\alpha\beta} = -J_{\beta\alpha}$ (Skew symmetry)

(ii) $\sum_\lambda (J_{\alpha\lambda}\partial^\lambda J_{\beta\gamma} + J_{\beta\lambda}\partial^\lambda J_{\gamma\alpha} + J_{\gamma\lambda}\partial^\lambda J_{\alpha\beta}) = 0$ (Jacobi identity)

(4) Given a function H in $C^\infty(P)$, we obtain the Hamiltonian vector field ξ_H and hence a differential equation $\dot{x}(t) = \xi_H(x(t))$ or, in local coordinates

2.1.9 Definition: A symplectic manifold is a manifold P equipped with a differential 2-form ω such that

(i) *to* is closed, i.e., $d\omega$ — 0 and

(ii) ω is non-degenerate, i.e., if $v \in T_xP$ is such that $\omega(v,w) = 0$ V $w \in T_xP$, then $v = 0$.

(2) Using local coordinates x_α, define the matrix A = $(\Lambda_{\alpha\beta})$ as the inverse of the matrix J = $(J_{\alpha\beta})$. The non-degeneracy of ω proves that the mapping A : $TP \rightarrow T^*P$ defined by $\omega(v,w) = < \Lambda w, v >$ is invertible. The Poisson bracket on P is defined by

$$\{F, G\} = < dG, \Lambda^{-1}(dF) > = \omega(\Lambda^{-1}(dF), \Lambda^{-1}(dG))$$

and every symplectic manifold is a Poisson manifold (that is, a symplectic manifold is nothing but a Poisson manifold P for which the associated map $J : T^*P \rightarrow TP$ is invertible),

2.1.10: We discuss below how the Poisson bracket is defined in a few examples:

(1) On *the cotangent bundle:* Let M be an arbitrary manifold of dimension n and let T^*M be the cotangent bundle of M, which is of dimension 2n. The Liouville form a on T^*M is defined in local coordinates $q^1, \cdots q^n, p_1, \cdots, p_n$ as a = $\sum_i p_i dq^i$. The global definition is as follows: We write $P = T^*M$ and $\pi_p : TP \rightarrow P$ for the canonical

projection. Let $\pi^*_M : T^*M \to M$ be the canonical projection of the cotangent bundle of M into M. Then $T\pi^*_M : TP \to TM$ is the differential of π^*_M. For $v \in TP$, set $\alpha(v) = <\pi_p(v), T\pi^*_M(v)>$ and this is the local expression of the differential 1-form α on $P = T^*M$. The canonical symplectic form ω on T^*M is defined by $\omega = d\alpha = \sum dp_i \wedge dq^i$. The Poisson bracket is expressed as

$$\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q^i} - \frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p_i} \right) \quad \text{(cf. 2.1.5)}$$

(2) On *the dual* of a *Lie algebra:* Let $\mathcal{G}$ be a Lie algebra of finite dimension over 1R and $\mathcal{G}^*$ be its dual as a vector space. The space Q can be embedded as a subspace of $C^\infty(\mathcal{G}^*)$, namely to X in $\mathcal{G}$ we associate the linear function F_X . $\xi \mapsto <\xi, X>$ on $\mathcal{G}^*$. Then there exists a unique Poisson bracket on $\mathcal{G}^*$ such that

$$\{F_X, F_Y\} = F_{[X,Y]} \qquad \text{for} \qquad X, Y \in \mathcal{G} \tag{2.1.11}$$

In otherwords, the map $X \mapsto F_X$ from $\mathcal{G}$ into $C^\infty(\mathcal{G}^*)$ is a homomorphism of Lie algebras. There are three descriptions of this Poisson bracket:

(a) Let $e_1, \cdots, e_n$ be a basis of $\mathcal{G}$. The functions $\xi_1 = F_{e_1}, \xi_2 = F_{e_2}, \cdots, \xi_n = F_{e_n}$ form a system of linear coordinates on $\mathcal{G}^*$. Introduce the structure constants $c^{\alpha\beta}_\gamma$ of the Lie algebra $\mathcal{G}$ by $[e_\alpha, e_\beta] = \sum_\gamma c^{\alpha\beta}_\gamma e_\gamma$.

Hence the first expression of the Poisson bracket on $\mathcal{G}^*$:

$$\{F_1, F_2\} = \sum_{\alpha,\beta,\gamma} c^{\alpha\beta}_\gamma \frac{\partial F_1}{\partial \xi_\alpha} \frac{\partial F_2}{\partial \xi_\beta} \cdot \xi_\gamma \tag{2.1.12}$$

It is the unique Poisson bracket for which formula (2.1.11) holds for X, Y running over the given basis of $\mathcal{G}$, that is, $\{\xi_\alpha, \xi_\beta\} = \sum_\gamma c^{\alpha\beta}_\gamma \xi_\gamma$.

(b) For any function F in $C^\infty(\mathcal{G}^*)$ we can consider its gradient ∇Fas a function on $\mathcal{G}^*$ with values in $\mathcal{G}$ characterized by the relation

$$< \eta, \nabla F(\xi) > = \left. \frac{d}{dt} \right|_{t=0} F(\xi + t\eta) \quad \text{for} \quad \xi, \eta \in \mathcal{G}^* .$$

With the previous notations

$$\nabla F = \sum_{\alpha} \frac{\partial F}{\partial \xi_\alpha} e_\alpha$$

The invariant version of formula (2.1.12) is read as follows:

$$\{F_1, F_2\}(\xi) = < \xi, [\nabla F_1(\xi), \nabla F_2(\xi)] > \qquad (2.1.13)$$

(c) Let G be a Lie group with Lie algebra $\mathcal{G}$. For every g in G, the left translation $\gamma_g : g' \rightarrow gg'$ is an automorphism of the mainfold G, hence induces an automorphism ρ_g of the cotangent bundle T^*G. We can identify $\mathcal{G}^*$ with the fibre of T^*G at the unit element of G. The map defined by $\rho(g,\xi) = \rho_g.\xi$, $g \in G$, $\xi \in \mathcal{G}^*$ is a diffeomorphism ρ of $G \times \mathcal{G}^*$ with T^*G. We define the projection $\pi^* : T^*G \rightarrow \mathcal{G}^*$ by $\pi^*(\rho(g,\xi)) = \xi$. On the cotangent bundle T^*G there is defined a canonical Poisson bracket, invariant under the automorphism ρ_g of T^*G. The map $F \mapsto F \circ \pi^*$ identifies $C^\infty(\mathcal{G}^*)$ with the subspace $C^\infty(T^*G)^G$ consisting of the functions f on T^*G such that $f \circ \rho_g = f$ for every g in G. This space is closed under Poisson brackets. The Poisson bracket in $C^\infty(\mathcal{G}^*)$ is characterized by the property

$$\{F_1, F_2\} \circ \pi^* = \{F_1 \circ \pi^*, F_2 \circ \pi^*\}$$

for F_1, F_2 in $C^\infty(\mathcal{G}^*)$ (i.e., π^* is a Poisson map from T^*G to $\mathcal{G}^*$).

2.2 **HAMILTONIAN** STRUCTURE: We discuss in this section the various forms of Hamiltonian structures used in different contexts (or parts) of the thesis.

2.2.1: A Hamiltonian structure in an even dimensional phase space M of dimension n is defined by the Poisson bracket. The Poisson bracket $\{,\}$ is a mapping $\{,\} : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$ which puts each pair of functions $f, g \in C^\infty(M)$ into correspondence a third function $h = \{f,g\}$. If $\xi_1, \cdots, \xi_n$ are local coordinates of a point $\xi \in M$, then the operation $\{,\}$, by definition, implies

$$\{f,g\} = \sum_{i,k=1}^{n} \omega^{ik}(\xi) \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \xi_k} \qquad (2.2.2)$$

where ω is an n x n non-degenerate matrix. The conditions of Poisson bracket given in the previous section imply that the matrix $\omega^{ik}(\xi)$ is skew-symmetric:

The condition of Jacobi identity is equivalent to a certain system of equations for $\omega^{ik}(\xi)$ that are identical with the first pair of Maxwell's equations for the inverse matrix of a;, i.e., W: $W_{ik} = (\omega^{-1})_{ik}$:

$$W_{ik,\!t} \; W_{li,k} + W_{kl,i} = 0 \tag{2.2.4}$$

It is defined in terms of a differential form as:

$W = W_{ik} d\xi^{\text{'}} \text{A}\, \xi^k$, where rff $\text{A}\, d\xi^k = -d\xi^k \text{A}\, d\xi^{\text{'}}$.

The Poissori bracket maps a linear space of functions on a phase space into a Lie algebra. Having chosen some function $h(\xi)$ and terming it a Hamiltonian function, we can define a map (dependent on *t)* of this algebra into itself $f(\xi) \to f(\xi, t), f(\xi, 0) = f(\xi)$ by means of the following differential equation:

$$- = \{f, h\} \tag{2.2.5}$$

In particular, taking / in the form $f(\xi, t) = \prod_{i=1}^{n} \delta(\xi_i - \xi_i(t))$, for $\xi_i(t)$, weobtain:

$$\frac{d\xi_i}{dt} = \sum_k \omega^{ik} \frac{\partial h}{\partial \xi_k}$$

which are Hamiltonian equations.

Using a linear transformation in the phase space, we can reduce every skew-symmetric non-degenerate constant matrix ω to a block form:

$$\omega = \begin{pmatrix} J & 0 & & \vdots & & \vdots \\ 0 & J & 0 & \vdots & & \vdots \\ 0 & 0 & J & \vdots & & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & J \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Denoting the coordinates in M space as $\xi_{2k-1} = p_k, \xi_{2k} = q_k$, $k = i, \cdots n/2$ where i = 1, • • •, n, and from (2.2.6) we obtain

$$dh \qquad dh$$

i.e., Hamiltonian equations in the conventional form. The pair of variables p_k and q_k are canonical conjugates. In these variables the Poisson bracket is written as follows:

$$\{J, g\} \quad \sum_k \overline{\partial q_k} \, \overline{\partial p_k} \quad \partial p_k \, \partial q_k \; .$$

2.2.7: Any function $H \in C^\infty(M)$ defines a derivation $\{H, .\}$,it defines a vector field X_H the Hamiltonian vector field, by definition, $X_H f — \{H, f\} — df(X_H)$. The function H (the Hamiltonian) defines a vector field X_H (the Hamiltonian vector field) which in turn defines a differential equation, $x = X_H(x) — J_x dH_x$, the Hamiltonian system generated by the Hamiltonian *H.*

By skew-symmetry, $\{H, H\} = 0$ so that $dH(X_H) = 0$ and H is constant along the trajectories of *H.* Any function / having that property (remaining constant along the trajectories), that is such that $df(X_H)$= 0 (or $\{H, f\}$= 0) is called a first integral.

Symplectic structures provide a special instance of Poisson structure. On a symplectic manifold (M, ω), the Poisson bracket $\{f, g\}$ of two functions is defined in terms of their Hamiltonian vector fields by $\{f, g\} = \omega(X_f, X_g)$= $dg(X_f)$, for f, g € $C^\infty(M)$. The Hamiltonian system generated by / can be written as $\dot{q}_j = \{q_j, f\}, p_j — \{p_j, f\}$.

2.2.8 Examples: (a) The vector triple product defines a Poisson bracket in $\mathbb{R}^3$ by $\{f, g\}(\mathbf{r})$= r.(V/(r) x $\nabla g(\mathbf{r})$)

(b) A Poisson bracket on $\mathbb{R}^2$ is defined by

$$\{f, g\}(x, y) = y\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)$$

(c) Poisson bracket in $\mathcal{G}^*$: Let $\mathcal{G}$ be a Lie algebra, and let $\mathcal{G}^*$ be the dual vector space, with < .,. > the pairing between them. For ξ € $\mathcal{G}$, let / ∈ $C^\infty(\mathcal{G}^*)$ be the function

$< \xi, \cdot >$. Define $\{f_\xi, f_\eta\} = f_{[\xi,\eta]}$. This is a Poisson bracket on linear functions on $\mathcal{G}^*$. For $f \in C^\infty(\mathcal{G}^*)$, we introduce its gradient

$$\nabla f : \mathcal{G}^* \to \mathcal{G} \text{ by } \lim_{\epsilon \to 0} \frac{f(\mu + \epsilon\nu) - f(\mu)}{\epsilon} = < \nu, \nabla f(\mu) >$$

The Lie-Poisson bracket is given by

$$\{f, g\}(\mu) = < \mu, [\nabla f(\mu), \nabla g(\mu)] >$$

The invariant functions, f on $\mathcal{G}^*$ are such that $\{f, g\} = 0 \; \forall g$ and these functions satisfy $f(Ad_g^*\mu) = f(\mu)$.

2.3 THE TODA LATTICE:

The Toda lattice is a system of unit masses connected by non-linear springs governed by an exponential restoring force. The equations of motion of the system can be derived from the Hamiltonian function

$$H = H(x, y) = \frac{1}{2}\sum_{i=1}^{n} y_i^2 + \sum_{i=1}^{n} e^{x_i - x_{i+1}} \quad , \quad x_o = x_{n+1} = 0$$

where x_i is the displacement of n^{th} mass from equilibrium (or x_i denotes the position of the mass points, $i = 1, \cdots n$) and y_i is the corresponding momentum. The equations of motion are given by

$$\dot{x}_i = y_i$$

The corresponding flow is given by

$$\dot{x}_i = \frac{\partial H}{\partial y_i} \; , \; \dot{y}_i = -\frac{\partial H}{\partial x_i} \; , \; i = 1, \cdots n$$

By the Flaschka's transformation,

$$\begin{aligned} a_i &= \frac{1}{2} e^{\frac{1}{2}(x_i - x_{i+1})} \; , \; i = 1, \cdots n-1 \\ b_i &= -\frac{1}{2} y_i \; , \; i = 1, \cdots n, \end{aligned}$$

we can write the system as

$$\begin{aligned} \dot{a}_i &= a_i(b_{i+1} - b_i) \\ \dot{b}_i &= 2(a_i^2 - a_{i-1}^2)\ ,\ i = 1,\cdots n-1\ ,\ a_o = a_n = b_{n+1} = 0 \end{aligned} \qquad (2.3.1)$$

The above system (2.3.1) is a Hamiltonian system. It can be written in the form, $\dot{z} = J\nabla_z H$, (2.3.2) where $z = \begin{pmatrix} a \\ b \end{pmatrix}$, $a \in \mathbb{R}^{n-1}$, $b \in \mathbb{R}^n$ and J defines a Poisson bracket through the formula

$$\{f(z), g(z)\} = (\nabla f, J\nabla g)$$

where (,) is the standard dot product in $\mathbb{R}^{2n-1}$, ∇ is the standard gradient. We take **for** J, the $(2n-1) \times (2n-1)$ matrix

$$J = J(a) = \begin{bmatrix} O_{n-1} & S \\ -S^t & O_n \end{bmatrix}$$

where O_{n-1}, O_n are the $(n-1) \times (n-1)$ and $n \times n$ zero matrices respectively. $S = S(a)$ is a $(n-1) \times n$ matrix function defined by

$$\begin{aligned} S_{ij} &= (-\delta_{ij} + E_{ij})a_i\ ,\ i = 1,\cdots n-1,\ \ j = 1,\cdots n,\ . \\ E_{ij} &= 1 \text{ if } j = i+1,\ \ 0 \text{ if } j \neq i+1 \text{ and} \\ \delta_{ij} &= 1 \text{ if } i = j\ ,\ \ 0 \text{ if } i \neq j \end{aligned}$$

Then the Poisson bracket is given by

$$\{g, f\} = \sum_{i=n}^{n-1} \{a_i f_{a_i}(g_{b_i} - g_{b_{i+1}}) + a_i g_{a_i}(f_{b_{i+1}} - f_{b_i})\}$$

The function H of (2.3.2) is given by

$$H = H(z) = \sum_{i=1}^{n} b_i^2 + \sum_{i=1}^{n-1} a_i^2$$

2.4 THE LAGRANGE TOP:

A Lagrange top is an axially symmetric rigid body with centre of mass on the axis of symmetry, moving about a fixed point (the origin of $\mathbb{R}^3$) under the influence of gravity. Let S be a rigid body moving about a fixed point $0 \in V = \mathbb{R}^3$. Let G be its centre of gravity, and μ be the total mass. Let 7 be the unitary vector field on S in the direction of gravity, say z-axis. The (S, γ) is a dynamical system.

Let M be the angular momentum of S with respect to body coordinates. Let $\Omega = (p, q, r)$ be the angular velocity of the body S (or the rotation vector of S or the variable position vector of 5). Let / be the interia matrix of S which can be regarded as a positive definite symmetric automorphism of $V = \mathbb{R}^3$. Then $M = I(\Omega) = (I_1 p, I_2 q, I_3 r)$ (2.4.1)

where I_1, I_2, I_3 denote the principal moments of inertia, when the body frame is principal. The total derivative with respect to time is given by

$$\frac{d}{dt}(.)|_{\text{total}} = \frac{d}{dt}(.) + \Omega \times (.) \qquad (2.4.2)$$

Then the torque exerted on the body by the (vertical) force of gravity is $l \times$ gravity where / is the centre of gravity in the body coordinates and $\mu g \gamma$ is the downward gravity force. The rotation version of Newton's equations

$\frac{d}{dt}$ (angular momentum)$|_{\text{total}}$=torque and $\frac{d}{dt}(z\text{-axis})|_{\text{total}} = 0$

give the Euler-Poisson equations

Using the Lie algebra isomorphism,

$$\lambda : (\mathbb{R}^3, \times) \rightarrow (so(3), [,])$$

$$x - (x_1, x_2, x_3) \mapsto \hat{x} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$$

we get the Lie algebra version of (2.4.2) as

$$\dot{M} = \frac{dM}{dt} = [M, \Omega] + \mu g[\gamma, l], \quad \frac{d\gamma}{dt} = [\gamma, \Omega]. \tag{2.4.4}$$

In the absence of gravity, we have the quadratic differential equation,

$$\frac{dM}{dt} = [M, \Omega] \tag{2.4.5}$$

where $[M, \beta] = [\Omega, a]$, $[\alpha, \beta] = 0$, $a = \beta^2$ with $fi = \frac{1}{2}(I_1 + I_2 + I_3)I$ - $\text{diag}(I_1, I_2, I_3)$. Lagrange top corresponds to the case where $I_1 = I_2$, and where the centre of gravity and fixed point of rotation belong to the principal axis of inertia. Let z_o be their respective distance and let $/ = (0, 0, z_o)$. Adjoin the relation

$$[M, \beta] = [\Omega, a], \ \beta = \mu g l, \ \ a = I_1\beta, \text{ and hence } [a, \hat{\beta}] = 0 \tag{2.4.6}$$

The equations (2.4.4) and (2.4.6) are the equations of the Lagrange top.

2.5 THE KdV EQUATION AND THE GENERALIZED KdV EQUATION:

The KdV equation for $u \in C_o^\infty(\mathbb{R})$, $u_t = 6uu_x - 2u_{xxx}$where u describes the amplitude of waterwaves in a narrow channel of finite depth has been studied over recent years. The KdV equation can be written as

$$u_t = \frac{\partial}{\partial x}(3u^2 - u_{xx}) = \frac{\partial}{\partial x}\left(\frac{\delta H}{\delta u(x)}\right) \tag{2.5.1}$$

where the functional H is

$$H[u] = \int_{-\infty}^{\infty} \left(\frac{u_x^2}{2} + u^3\right) dx \tag{2.5.2}$$

and the symbol — denotes the variational derivative. Expressing $\frac{\partial}{\partial x}\frac{\delta H}{\delta u(x)}$as

$$-\int_{-\infty}^{\infty} \delta'(x - x') \frac{\delta H}{\delta u(x')} dx' \ , \tag{2.5.3}$$

we note that equation (2.5.1) takes the form of (2.2.6). We interpret the coordinates of a function u (points in the phase space) as the set of its values at the points on x-axis, so that x and x' are the suffixes in (2.2.6): $\omega(x,x')$ = ~8{x - x'). That is, ω is skew-symmetric and on functions, it is non-degenerate as the operator d/dx is reversible on such functions. Also, since ω is not dependent on the point u in the phase space, the Poisson bracket

$$\{S,R\} = \int_{-\infty}^{\infty} \frac{\partial S}{\delta u(x)} \frac{\partial}{\partial x} \frac{\delta R}{\delta u(x)} dx \qquad \textbf{(2.5.4)}$$

satisfies the Jacobi identity . Thus, the KdV equation is a Hamiltonian equation. The functional // (2.5.2) is a Hamiltonian function, the phase space consists of sufficiently smooth functions $u\{x)$ which decrease at infinity, and the Hamiltonian structure is defined by the Poisson bracket (2.5.4), i.e., by the skew-symmetric operator $\frac{\partial}{\partial x}$in $L_2(\mathbb{R})$.

2.5.5 Fact: P. Lax [34] discovered that the KdV equation is identical to the operator equation $L = [A, L]$ where $L = -\frac{d^2}{dx^2}+u$, $A = \frac{4d^3}{dx^3} - 3\left(u\frac{d}{dx}+\frac{d}{dx}u\right)$.P. Lax [34] and Gardner [14] have shown that the known polynomial integrals $I_n(u) = \int_{-\infty}^{\infty} P_n(u,\cdots, u^{(n)})dx$ of the KdV equation (the I_n are expressed in terms of the spectrum of the operator L) all determine equations

$$\dot{u} = \frac{d}{dx}\frac{\delta I_n}{\delta u(x)}$$

admitting the Lax representation $L = [A_n, L]$ where $L = \frac{d^2}{dx^2} + u$ and the A_n are certain skew-symmetric operators of order $2n + 1$,

$$\begin{aligned}
I_o &= \int u^2 dx \ , \ I_1 = \int \left(\frac{u_x^2}{2} + u^3\right) dx \ , \ I_2 = \int \left(\frac{u_{xx}^2}{2} - \frac{5}{2}u^2 u_{xx} + \frac{5}{2}u^4\right) dx, , \\
A_o &= \frac{d}{dx} \ , \ A_1 = \frac{4d^3}{dx^3} - 3\left(u\frac{d}{dx} + \frac{d}{dx}u\right) \\
A_2 &= \frac{d^5}{dx^5} - \frac{5}{2}u\frac{d^3}{dx^3} - \frac{15}{4}u_x\frac{d^2}{dx^2} + \frac{15u^2 - 25u_{xx}}{8}\frac{d}{dx} + \frac{15}{8}\left(uu_x - \frac{u_{xxx}}{2}\right)
\end{aligned}$$

These equations are called "higher KdV equations".

2.5.6: By the above fact (2.5.5) there exists a denumerable set of local polynomial integrals of the KdV equation. The KdV equation is a completely integrable system, for, the substitution $u(x) \rightarrow$ 'scattering data' is a transformation in the phase space to "action-angle" type variables and each polynomial $I_j[u]$, which is an integral of the KdV, gives a completely integrable Hamiltonian system defined by the non-linear partial differential equation:

$$u_t = \frac{\partial}{\partial x}\frac{\delta I_j}{\delta u(x)} \tag{2.5.7}$$

Equations of type (2.5.7) for $j > 3$ are called higher order KdV equations. They can be integrated with the same substitution of variables and have the same integrals of motion as the KdV equation.

2.6 THE GEL'FAND-DIKII SYSTEM: P. Lax [34] described the KdV equation in the form:

$$\frac{dL}{dt} = [B, L] \quad \text{where} \quad L = -\partial_x^2 + q(x,t) \ , \quad B = -4\partial_x^3 + \partial(q\partial_x + \partial_x q)$$

Gel'fand-Dikii generalized the Lax form of the KdV equation with

$$L = (-i\partial_x)^n + \sum_{j=0}^{n-2} q_j(-i\partial_x)^j \ , \quad q_i \in C_o^\infty(\mathbb{R}) \ , \ i = 0, \cdots, n-2 \ ,$$

then the Lax equations $\frac{dL}{dt} = [B, L]$ for appropriate choice of B, is a Hamiltonian system with an infinite sequence of involutive polynomial integrals, in analogy with $L = -\partial_x^2 + q(x,t)$. In the case of the generalized KdV equation or the Gel'fand-Dikii system, the Lax operator is a differential operator of the form

$$\begin{aligned} L_n &= \partial^n + u_{n-2}\partial^{n-2} + \cdots + u_o \ , \quad \partial = \frac{d}{dx} \quad or \quad \left(\partial = \sqrt{-1}\frac{d}{dx}\right) \\ &= \partial^n + \sum_{k=0}^{n-2} u_k D^k \left(D = -\sqrt{-1}\frac{d}{dx}\right) \end{aligned}$$

(These systems **are** infinite dimensional Hamiltonian systems having infinitely many constants of motion in involution, which are integrals of local polynomial densities. These systems are also (formal) infinitesimal isospectral deformations of (formal) linear ordinary differential operators. A formal linear ordinary differential operator or linear differential expression is a polynomial in $\frac{d}{dx}$ (or in $D = -\sqrt{-1}\frac{d}{dx}$) with coefficients which are functions of x, which are infinitely differentiable in some open interval on the real line).

We consider the coefficients u_i's as symbols to which the operator *d* can be applied: $\partial u_k = u'_k$, $\partial u'_k = u''_k, \cdots, \partial u_k^{(i-1)} = u_k^{(i)}$, $u_k^{(o)} = u_k$. Differential polynomials, i.e., expression of the form / = $\sum a^{i_1 \cdots i_r}_{k_1 \cdots k_r} u^{(i_1)}_{k_1} \cdots u^{(i_r)}_{k_r}$ form a differential algebra $\mathcal{A}$, i.e., an algebra with a given derivation *d*. Here the coefficients *a* can be either complex or real numbers or smooth functions of *x*.

The ring of pseudo-differential operators *R* consists of formal sums of the form $X = \sum_{-\infty}^{i} a_i \partial^i$, a, $\in A$ with the commutation law, $\partial^i f = \sum_{k=0}^{\infty} \binom{i}{k} f^{(k)} \partial^{i-k}$, $\forall\ i$, where $f^{(k)}$ is the k-th derivative of the function *f(x)* and $\binom{i}{k} = {}^iC_k = \frac{i(i-1)\cdots(i-k+1)}{k!}$. There is a decomposition $R = R_+ + R_-$ where R_+ is the subring of differential operators $\{X = \sum_{i=0}^{r} a_i \partial^i\}$ and R_- that of integral' or 'Volterra' operators $R_- = \{X = \sum_{i=-1}^{-\infty} a_i \partial^i\}$. We remark that the operators can also be written in the 'left' form $\sum_i \partial^i a_i$ using the given commutation law.

For a pseudo-differential operator of the form $X = \sum a_i \partial^i$ we define its residue (res *X)* as the coefficient of the ∂^{-1} term, namely, Res $X = a_{-1}$. The Trace is defined by $\mathrm{Tr}X = \int \mathrm{Res}X\,dx$ (2.6.1). Tr satisfies the property *Tr(XY)* = $\mathrm{Tr}(YX)$ (2.6.2) and it follows that $\mathrm{Tr}[X, Y] = 0$ (2.6.3) for any two pseudo-differential operators *X* and *Y*.

If $X = \sum_{-\infty}^{N} a_i \partial^i$ and $a_N = 1$, then the following operators exist:

$X^{-1} = \sum_{-\infty}^{-N}$fc-^" and $X^{1/N} = \sum_{-\infty}^{1} c_i \partial^i$ where $b_{-N} = c_1 = 1$ and hence $X^{m/N}$ also exists.

These operators commute with *X*.

A formal isospectral deformation of *L* is a specification

$$\dot{L} = \sum_{k=0}^{n-2} u_k D^k = [P, L] = P \circ L - L \circ P \qquad \textbf{(2.6.4)}$$

where P is a linear differential expression whose coefficients depend polynomially on $u_o, \cdots u_{n-2}$ and their derivatives, having the property that the commutator appearing

on the right hand side of (2.6.4) is of order *(n* - 2) or less. Equation (2.6.4) can be **regarded** as a collection of *(n* — 1)-partial differential equations for the coefficients u_k, **where** the dot is interpreted to mean differentiation with respect to a (time) parameter t, **(for** example for n - 2, choosing $L = D^2 + u$, $P = D^3 + \frac{3}{2}uD + \frac{3}{4}Du$, equation (2.6.4) is equivalent to the KdV equation for the coefficient u, $-iu = \frac{1}{4}D^3u + \frac{3}{4}uDu$). Let m be any positive integer. Then $L^{m/n}$ is a u-symbol of order m whose homogeneous pieces have integral degree.

2.6.2 Lemma: $[L_+^{m/n}, L]$ where $(L^{m/n})_+ = L_+^{m/n}$ is the differential part of $L^{m/n}$ is a differential operator of order < *n* — 2.

Proof: $[L^{m/n}, L] = 0$, hence $[L_+^{m/n}, L] = -[L_-^{m/n}, L]$. The right hand side is of order < - 1 + n - 1 = n - 2 .

Suppose u_k depends on the parameter *t*. Then the equation L — $[L_+^m \quad , L]$ $(L = \partial L/\partial t)$ makes sense because L is an operator of order *n* — 2 as in the right hand side. The set of all these equations, for all m, is called the n-th KdV hierarchy (for $n = 2, m = 3$, we obtain the KdV equation).

The k-th flow of the hierarchy is defined by the Lax equation,

$$\partial_{t_k} L_n = [(L_n^{k/n})_+ \ , \ L_n] = [L_n \ , \ (L_n^{k/n})_-], \ \partial_{t_k} = \partial/\partial t_k \qquad (2.6.6)$$

From (2.6.6), it can also be written as

$$\begin{aligned} 0 &= \partial_{t_k} L_n - [(L_n^{k/n})_+ , \ L_n] \\ &= \sum_{i=0}^{n-1} L_n^{i/n} \left(\partial_{t_k} L_n^{1/n} - [(L_n^{k/n})_+ \ , \ L_n^{1/n}]\right) L_n^{(n-i-1)/n} \ , \end{aligned}$$

from which it follows that

$$\partial_{t_k} L^{1/n} = \left[(L_n^{k/n})_+ \ , \ L_n^{1/n}\right] \ .$$

Alternately, we can write $\partial_{t_k} L^{l/n} = [(L_n^{k/n})_+, L^{l/n}]$ (2.6.7) **for** any arbitrary integer /. **From equation** (2.6.6) and (2.6.7), we have that $\partial_{t_l}\partial_{t_k} L_n = \partial_{t_k}\partial_{t_l} L_n$. In other **words,** any two operators in the hierarchy commute. Therefore, we can solve all these equations simultaneously, obtaining $L(t_1, t_2; \cdot \cdot)$.

Now taking the trace of equation (2.6.7) and using the relation (2.6.3), we obtain $\partial_{t_k} Tr(L_n^{l/n}) = 0$. If we define $H_l = \frac{n}{l} Tr(L_n^{l/n})$ V /, these are conserved under any flow and these are the constants of motion (or first integrals) of all the equations of the n-th hierarcy.

We now describe the Lax pair associated with $L^{m/n}$. We set $L^{m/n} = P_m + N_m$ where

(A_p is defined as follows: Let R(\) be the resolvent symbol for L defined by R(\) o (L — A) = L We define the symbol L^s, for complex s, by $L^s = \frac{1}{2\pi i} \oint \lambda^s R(\lambda) d\lambda$ where 0 is the contour from ReX = $\frac{1}{2}$ in the semicircle |A| = $\frac{1}{2}$ in the counterclock wise direction to ReX = $-\frac{1}{2}$. We obtain $L^s = \sum_{p=o}^{\infty} A_p(s)$ by evaluating the above integral where $ord A_p = n$ Res s — p,

$$A_p(s) \equiv (\xi^n)^s (\xi^n)^{l-m} \times \sum_{m=2}^{p} B_{p+n,m}(-1)^{m-1} \frac{1}{(m-1)!} \sum_{j=0}^{m-2} (s-j)$$

Here L is defined by $L(q,\xi) = \xi^n + \sum_{k=0}^{n-2} u_k \xi^k$ and $A(x,\xi) = \sum_{l=o}^{\infty} A_l(x,\xi)$.

Thus P_m is a polynomial u symbol and $ord N_m < -1$. If m is not divisible by n, then $A_p(\frac{m}{n}) \neq 0$ for $p > m+1$. Since [I, L^{m+n}] = 0, we have $[P_m$, I] = [I, $N_m]$. Since the left hand side of this equation is polynomial u symbol, so is the right hand side . Also, since the right hand side has order < n - 2, so does the left hand side. Thus $L = [P_m, L]$ is a Lax equation for each positive integer m. This is **Gel'fand-Dikii's** construction of Lax pairs [15,16].

2.7 THE GEODESICS ON AN ELLIPSOID: (a) Let A be a positive definite symmetric n x n matrix with distinct eigenvalues and x 6 $\mathbb{R}^n$ be a vector. Then the (n − 1)-dimensional ellipsoid has equation

$$< A^{-1}x, x >= 1. \tag{2.7.1}$$

Also the differential equation of the geodesics is given by

$$\frac{d^2x}{dt^2} = -\nu A^{-1}x \text{ where } \nu = \frac{< A^{-1}y, y >}{|A^{-1}x|^2}, \ y = \frac{dx}{dt} \tag{2.7.2}$$

Further the family of confocal quadrics related to the ellipsoid (2.7.1) is given by the equation

$$< (z - A)^{-1}x, x> +1 = 0 \tag{2.7.3}$$

Introduce the notation

$$Q_z(x, y) =< (z - A)^{-1}x, y>, \quad Q_z(x) = Q_z(x, x) \tag{2.7.4}$$

Note that $Q_0(x)+1= 0$ is the ellipsoid (2.7.1) and (2.7.3) can be written as $Q_z(x,x)+1 = 0$. Let $L(x,y)$ be an isospectral symmetric matrix of A obtained by a process of rank 2 pertubation (which will be explained later) which depends on two vectors $x, y \in \mathbb{R}^n$. Define

$$\frac{|y|^2}{z}\frac{\det(zI - L)}{\det(zI - A)} =: \Phi_z(x, y) \tag{2.7.5}$$

which is a rational function of z with poles at the eigenvalues $\alpha_1, \cdots, \alpha_n$ of A and zeros at $\lambda_1, \cdots, \lambda_{n-1}$, the non-trivial eigenvalues of $L(x,y)$. In fact, one eigenvalue of L is $\lambda_n = 0$ with corresponding eigenvector $y = \frac{dx}{dt}$ and these eigenvalues $\lambda_1, \cdots, \lambda_n$ are preserved by the geodesic flow (2.7.2) (here note that we denote by x the position vector on the ellipsoid and its velocity by $y = \frac{dx}{dt}$). As a function of x, y, Φ_z is a quartic polynomial. The partial fraction expansion of $\Phi_z(x,y)$ corresponding to its poles $\alpha_1, \cdots, \alpha_n$ is

$$\Phi_z = \sum_{j=1}^{n} \frac{G_j(x, y)}{z - \alpha_j}, \tag{2.7.6}$$

$G_j(x,y)$ $(j = 1, \cdots, n)$ are quartic polynomials of x and y which axe *the integrals* of the flow (2.7.2). In fact only $n - 1$ of them are independent on the ellipsoid, since there is a relation $\Phi_0 = -\sum_{j=1}^{n} \alpha_j^{-1} G_j(x, y) = 0$ among them. For a given parameter z we denote the quadric $Q_z(z)+1=0$ in the confocal family of (2.7.1) (i.e., that particular member of (2.7.3)) by $\mathcal{U}_z$.

Consider the eigenvalue equation $\Phi_z(x,y)=0$ defined by (2.7.5). We have the following identities connecting Φ_z and Q_z as

$$\Phi_z(x,y)=Q_z(y)(1+Q_z(x))-Q_z^2(x,y), \tag{2.7.7}$$

so that for a fixed z and x, this represents a quadratic form in y. The equation $\Phi_z(x,y)=0$ represents the quadratic cone of tangents to the quadric $\mathcal{U}_z$, passing through the point x, after the point x is translated to the origin. Also we have,

$$\Phi_z(x+sy,y)=\Phi_z(x,y),$$

so that Φ_z is constant along any line $x=x_0+sy,\ y\neq 0$ (*). Hence we get that for a given line $x=x_0+sy$, the roots $z=\lambda_1,\cdots,\lambda_{n-1}$, of the equation $\Phi_z(x_0,y)=0$ are such that the above line (*) is tangent to the confocal quadrics $\mathcal{U}_{\lambda_j}$ $(j=1,\cdots,n-1)$ (†). That is the equation

$$Q_z(y)(1+Q_z(x))-Q_z^2(x,y)=0$$

is the equation of tangency of the confocal quadric family. We consider the Hamiltonian system

$$dy \qquad\qquad dx$$

resticted to the surface $\Phi_z=0$. Then the differential equation (2.7.2) can be expressed as

$$\frac{d^2(x+sy)}{dt^2}=k\nabla Q_x$$

at the point x_0+sy, k constant. That is, this differential equation governs the motion of the tangents to the hyperquadric $Q_z(x)+1=0$, i.e., $\mathcal{U}_z$ along the geodesics by (†). In otherwords, the geodesic flow is obtained by just following the motion of the point of tangency, $x_0 + sy$ by reducing the system by $|y|^2$ by an integral process. If we put $z = 0$ in this, we get the geodesic flow on the ellipsoid $Q_0(x) + 1 = 0$.

2.7.8 Remarks: (1) Geometric preparation: First we note that, a given line in $\mathbb{R}^n$ touches exactly $n - 1$ con focal quadrics. The set of all lines tangent to these $n - 1$ quadrics $\mathcal{U}_{\lambda_1}, \mathcal{U}_{\lambda_2}, \cdots, \mathcal{U}_{\lambda_{n-1}}$ is called a normal congruence. Then the spectrum of $L(x, y)$ can be given the following geometrical interpretation: The "isospectral manifold" of matrices $L(x, y)$ with a fixed distinct spectrum $\lambda_1, \cdots, \lambda_{n-1}$ is identified with the normal congruence of common tangents to $n - 1$ confocal quadrics $\mathcal{U}_{\lambda_j}$ $(j = 1, \cdots, n - 1)$.

(2) The eigenvalue $\lambda_n = 0$ corresponds to the eigenvector $\phi_n = y = \frac{dx}{dt}$ and the other eigenvalues ϕ_j (corresponding to λ_j) are the normals of $\mathcal{U}_{\lambda_j}$ at the point of contact of the line $x = x_0 + sy$. Since $L = L(x, y)$ is a symmetric matrix, these n vectors are pairwise orthogonal. Then under the geodesic flow (2.7.2) the orthogonal frame $(\phi_j)_{j=1}^n$ will undergo a motion given by a skew- symmetric matrix B so that

$$\dot{\phi}_j = B\phi_j, \quad \dot{L} = [B, L], \tag{2.7.9}$$

which is the Lax representation of the geodesic flow where

$$B = -(\alpha_i^{-1}\alpha_j^{-1}(x_i y_j - x_j y_i)). \tag{2.7.10}$$

(3) Rank 2 perturbation of a given symmetric matrix A (Adler-Moser approach):
Let A be a fixed symmetric matrix and x, y, ξ, η be four n-vectors. We call $A + x \otimes \xi + y \otimes \eta$ a rank 2 perturbation of A. Take $\xi = ax + by$, $\eta = cx + dy$, where a, b, c, d are reals with $\Delta = ad - bc \neq 0$. Then define

$$L(x, y) = A + ax \otimes x + bx \otimes y + cy \otimes x + dy \otimes y \tag{2.7.11}$$

which is a matrix depending on two n-vectors, x and y. By the isospectral manifold of $L(x, y)$, we mean the algebraic manifold $\mathcal{M}(\lambda_1, \cdots, \lambda_n)$ consisting of those $x, y \in \mathbb{R}^n$

for which $L(x, y)$ has the fixed spectrum $\lambda_1, \cdots, \lambda_n$. Let $\omega = \sum_{j=1}^n dy_j \wedge dx_j$, be the symplectic 2-form on $\mathbb{R}^{2n}$. Then the eigenvalues of $L(x, y)$of (2.7.11) are in involution with respect to ω, i.e., $\{\lambda_j, \lambda_k\} = 0$ where the corresponding Poisson bracket is defined by $\{F, G\} = \sum_j (F_{x_j} G_{y_j} - F_{y_j} G_{x_j})$ the strandard one. Again we can consider, even the symmetric functions of the eigen values λ_j or even the more general function of $G_j's$ in

$$\Phi_z(x, y) = \sum_{j=1}^{n} \frac{G_j(x, y)}{z - \alpha_j} = 1 - \frac{\det(zI - L)}{\det(zI - A)},$$

which are quartic polynomials in x, y. Hence, as in (a) above , we have n quartic polynomials G_j which are in involution.

(4) Let $H = \phi(G_1, \cdots, G_n)$ be any Hamiltonian function in these or even any Hamiltonian function depending on the spectrum of L only. Then the corresponding Hamiltonian vector field X_H is tangential to the isospectral manifold $\mathcal{M}_\lambda$ and

$$X_H = \sum_{j=1}^{n} \frac{\partial H}{\partial G_j} . X_{G_j},$$

where we also have $[X_{G_j}, X_{G_k}] = -X_{\{G_j, G_k\}} = 0$ and hence all these vector fields commute. Thus $\mathcal{M}_\lambda$ is a Lagrange manifold and all these Hamiltonian systems are integrable. i.e., the corresponding vector field X_H of this system admits n integrals G_j in involution for which dG_j $(j = 1, \cdots, n)$ are independent over a dense open subset.

(b) We close this section by giving the integrals for the geodesic flow on the ellipsoid. Let the setup be as above where we have , $\mathbb{R}^n$, $< \, , \, >$, A, a positive symmetric matrix with distinct eigenvalues. Assume $A = \text{diag}(\alpha_1, \cdots, \alpha_n)$ with $0 < \alpha_1 < \cdots < \alpha_n$. Then $< A^{-1}x, x >= 1$ defines an ellipsoid. The quadrics $\mathcal{U}_z$ confocal to this ellipsoid are given by the equation

$$< (z - A)^{-1} x, x > +1 = 0. \tag{2.7.12}$$

Consider the bilinear form

$$Q_z(x,y) =< (z - A)^{-1}x, y >, \quad Q_z(x) = Q_z(x,x) \tag{2.7.13}$$

so that $\mathcal{U}_0$ is defined by

$$Q_z(x) + 1 = 0 \tag{2.7.14}$$

and note that (2.7.12) is simply $\mathcal{U}_0$. Through any point $x = (x_1, \cdots, x_n)$ with $x_1 x_2 \cdots x_n \neq 0$ there pass exactly n confocal quadrics which intersect each other perpendicularly. For any given point $x_0 \in \mathbb{R}^n$ we ask for the cone of lines which are tangent to a quadric $Q_z(x) + 1 = 0$. The equation of this cone is given by

$$\begin{aligned}\det\begin{bmatrix} 1+Q(x) & 1+Q(x,x_0) \\ 1+Q(x,x_0) & 1+Q(x_0)\end{bmatrix} &= Q(x) - 2Q(x,x_0) + Q(x_0) \\ &\quad + Q(x)Q(x_0) - Q^2(x,x_0) \\ &= 0.\end{aligned}$$

In otherwords, if we set $y = x - x_0$ this equation becomes

$$\begin{aligned}\det\begin{bmatrix} Q(y) & Q(x_0,y) \\ Q(x_0,y) & 1+Q(x_0)\end{bmatrix} &= Q(y) + Q(x_0)Q(y) - Q^2(x_0,y) \\ &= 0,\end{aligned}$$

which for fixed x_0 describes a cone with vertex at the origin. Now we note that this equation agrees with $\Phi_z(x_0, y) = 0$ of the above paragraph (a) with $a = 0$, $b = -c = 1$, $d = -1$ (or $a = 0$, $b = -c = i$, $d = 1$). Hence we can geometrically understand this equation as the set of lines $x = x_0 + sy$ tangent to $\mathcal{U}_z$. In particular, this equation $\Phi_z(x_0, y) = 0$ describes the tangents to the ellipsoid $\mathcal{U}_0$. Then the Hamiltonian differential equations are

$$\dot{x} = \frac{\partial}{\partial y}\Phi_0(x,y), \quad \dot{y} = -\frac{\partial}{\partial x}\Phi_0(x,y) \tag{2.7.15}$$

which when restricted to $\Phi_0 = 0$ describes the motion of such tangent lines and the point of contact with $\mathcal{U}_0$ moves along a geodesic while the point x moves perpendiclar to this tangent. In fact, if the line through x in the direction $y \neq 0$ has the point of contact $x + sv = £$ with $\mathcal{U}_0$, then we have

$$Q(x + sy, y) = 0, \text{ or } s = -\frac{Q(x, y)}{Q(y)}.$$

Then

$$\begin{aligned} \frac{d}{dt}\xi &= \frac{d}{dt}(x + sy) = \dot{x} + s\dot{y} + \dot{s}y \\ &= \frac{2\Phi_0(x, y)}{Q(y)}A^{-1}y + \dot{s}y = \dot{s}y, \end{aligned}$$

since $\Phi_0(x, y) = 0$, and

$$\frac{dy}{dt} = -2Q(y)A^{-1}x + 2Q(x, y)A^{-1}y = -2Q(y)A^{-1}\xi.$$

Introduce τ by $d\tau/dt — \dot{s}$, then

$$\frac{d\xi}{d\tau} = y, \quad \frac{d^2\xi}{d\tau^2} = \frac{dy}{d\tau} = -\frac{2Q(y)}{\dot{s}}A^{-1}\xi$$

and so the point of contact $\xi = \xi$ (r) moves on a geodesic. From the equations

$$< \dot{x}, y > = 2\Phi_0(x, y) = 0, \quad < y, y > = 0,$$

we get that x is perpendicular to y, i.e., the direction of this line and that $< y, y >$ is a constant. Thus (2.7.15) can be viewed as an extension of the geodesic flow on the ellipsoid to a flow in $T^*\mathbb{R}^n = \mathbb{R}^{2n}$. Putting the other way, the geodesic flow is obtained by constraining (2.7.15) to the symplectic manifold $Q_0(x, y) - 0$, $|y|^2 =$ constant > 0, and to the energy manifold $\Phi_0(x, y)$. Then the relations $\Phi_0 = 0$, $Q_0(x, y) = 0$ **are** equivalent to

$$Q_0(x) + 1 = 0, \quad Q_0(x, y) = 0, \quad \text{if } Q_0(y) \neq 0$$

which is the tangent bundle description of the ellipsoid (cf. paragraph (a) above). This constrained flow takes place on this tangent bundle. To establish the geodesic flow as an integrable system, it suffices to show that the extended flow (2.7.15) is integrable. This follows as was done in above paragraph with

$$\Phi_0(x,y) = \sum_{j=1}^{n} \frac{G_j}{z-\alpha_j}$$

and hence these $G_1, \cdots, G_n$ are the integrals of motion which are in involution.

2.7.16 Remark: Note that the n - 1 roots of $\Phi_z(x,y)$ are the eigenvalues of L and the n-th eigenvalue $\lambda_n = 0$ corresponds to the eigenvector y. The matrix L undergoes isospectral deformation under the flow (2.7.15) in Lax form $\frac{d}{dt}L$ = [2?, L] with an appropriate matrix B. More generally, define

$$H = \frac{1}{2}\sum_j \beta_j y_j^2 + \frac{1}{2}\sum_{i<j} \frac{\beta_i - \beta_j}{\alpha_i - \alpha_j}(x_i y_j - x_j y_i)^2 = -\frac{1}{2}\sum_{j=1}^{n} \beta_j G_j \qquad (2.7.17)$$

with arbitrary constants fi\, $\cdots, \beta_n$. Note that $\beta_j = 2\alpha_j^{-1}$ we get H — Φ_0 the one considered above. Then for this function H, we get the Hamiltonian system $x = H_y$, $y = -H_x$ with H given in (2.7.17) can be put in the matrix form $\frac{d}{dt}L = [B, L]$ (2.7.18) where L is given by

$$L(x,y) = \left\{I - \frac{y \otimes y}{<y,y>}\right\}(A - x \otimes x)\left\{I - \frac{y \otimes y}{<y,y>}\right\}$$

and

$$B = -\left(\frac{\beta_i - \beta_j}{\alpha_i - \alpha_j}(x_i y_j - x_j y_i)\right)$$

with zeros on the diagonal. This L and B can be put in tensor product notation which will be done in chapter 3 with details by introducing suitable notations.

Chapter 3

Lax representation, Cohomological Interpretation and Linearization of flows of some CIHS

3.1 LAX EQUATIONS (ISOSPECTRAL DEFORMATION):

3.1.1 Definition (Lax Equation): A Lax equation is a differential equation of the form $\frac{d}{dt}A = [A, B]$ where A and B are real or complex matrices depending on time and have entries in the ring of real or complex Laurent polynomials in a variable ξ, which will be called the spectral parameter. The bracket is the usual Lie bracket of matrices, so that such an equation expresses at the infinitesimal level the fact that the matrix A remains in the same conjugacy class. In otherwords, that the solutions have the form $A(t) = U(t)A(0)U(t)^{-1}$ for some unknown invertible matrix $U(t)$. We note that, in general, the unknown functions that are entries of A also appear in the entries of B. That is, B will be a function of A.

3.1.2 The Eigen Value Problem: Let $R = \mathbb{C}[[x]]$ be the ring of formal power series in x over $\mathbb{C}$. We consider an ordinary differential operator with co-efficients in R of the form

$$P = \partial^n + a_1(x)\partial^{n-1} + \cdots + a_n(x),$$

where $a_i(x) \in R$ and $3 = \frac{d}{dx}$. The eigenvalue problem $Pf(x) = \lambda f(x)$ has n linearly independent solutions in the function space R for any given A $\in \mathbb{C}$. The solution space is called the eigenspace. Let E_λ denote the A-eigenspace which is an n-**dimensional subspace** of R over $\mathbb{C}$. The spectrum of P (that is, the collection $\{E_\lambda\}_{\lambda \in \mathbb{C}}$) is an n-sheeted **covering** space of $\mathbb{C}$. If we extend the eigenvalue problem for A = ∞, then the corresponding *n*-sheeted covering is a compact Riemann surface. Thus, for every differential operator P over R, its spectrum can be regarded as a compact Riemann surface X_p which is an n-sheeted covering space of $\mathbb{C} = \mathbb{C} \cup \{\infty\}$.

3.1.3 Definition: Let $\{P(t)|t \in M\}$ be a parametrized family of differential operators $P(t)$ by a space $M = \mathbb{R}^n$ or $\mathbb{C}^n$. Let P = P(0) at time $t = 0$. We say that this family $\{P(t)|t \in M\}$ is an isospectral deformation of $P =$ P(0) if there exist differential operators $Q_i(t), i = 1, \cdots N$ depending analytically on t such that the following system of differential equations

$$P(t)f(x,t) = \lambda f(x,t), \ \lambda \in \mathbb{C} \tag{3.1.4}$$

$$\frac{\partial}{\partial t_i} f(x,t) = Q_i(t) f(x,t), \ i = 1, \cdots N \tag{3.1.5}$$

has nontrivial solutions for every complex number A.

3.1.6 Remarks: (1) A in the above definition is independent of the parameter t so that equation (3.1.4) says that, the family of operators $\{P(t)|t \in M\}$ have the same spectrum as that of P = P(0).

(2) Equations (3.1.5) are N boundary conditions imposed on the solutions $f(x,t)$ involving the operators $Q_i(x), i = 1, \cdots N$.

3.1.7 Compatibility condition for the above system:

We have

$$\begin{aligned}
0 &= \frac{\partial}{\partial t_i}(P(t)f(x,t) - \lambda f(x,t)) \\
&= \frac{\partial}{\partial t_i}\{(P(t)f(x,t)\} - \lambda \frac{\partial}{\partial t_i} f(x,t)) \\
&= \left\{\frac{\partial}{\partial t_i} P(t)\right\} f(x,t) + P(t)\frac{\partial}{\partial t_i} f(x,t) - \lambda Q_i(t) f(x,t)) \quad \text{by (3.1.5)} \\
&= \left(\frac{\partial}{\partial t_i} P(t)\right) f(x,t) + P(t)Q_i(t) f(x,t) - \lambda Q_i(x,t) f(x,t)) \quad \text{by (3.1.5)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\partial}{\partial t_i}P(t)f(x,t) + P(t)Q_i(t)f(x,t) - Q_i(x,t)P(t)f(x,t)) \quad \text{by } (3.1.4)\\
&= \frac{\partial}{\partial t_i}P(t)f(x,t) - [Q_i(t), P(t)]f(x,t)\\
\Rightarrow\ & \frac{\partial}{\partial t_i}P(t)f(x,t) = [Q_i(t),\ P(t)]f(x,t) \quad i = 1,\cdots N
\end{aligned} \tag{3.1.8}$$

This is the Lax equation of isospectral deformations of the operator P(0). Thus, every isospectral deformation of *P* gives rise to, in a natural way, a Lax equation.

3.1.9 Compatibility conditions for the solution function *f(x,t)* with respect to t:

Suppose that the solution function $f(x,t)$ of (3.1.4) satisfies the smoothness condition of order 2 with respect to *t:*

$$\frac{\partial^2 f(x,t)}{\partial t_j \partial t_i} = \frac{\partial^2 f(x,t)}{\partial t_i \partial t_j}$$

We have for the left hand side:

$$\begin{aligned}
\frac{\partial^2 f(x,t)}{\partial t_j \partial t_i} &= \frac{\partial}{\partial t_j}\frac{\partial}{\partial t_i}f(x,t)\\
&= \frac{\partial}{\partial t_j}\{Q_i(t)f(x,t)\}\\
&= \frac{\partial}{\partial t_j}Q_i(t)f(x,t) + Q_i(t)\frac{\partial}{\partial t_j}f(x,t)\\
&= \frac{\partial}{\partial t_j}Q_i(t)f(x,t) + Q_i(t)Q_j(t)f(x,t)
\end{aligned}$$

Right hand side:

$$\begin{aligned}
\frac{\partial^2 f(x,t)}{\partial t_i \partial t_j} &= \frac{\partial}{\partial t_i}\frac{\partial}{\partial t_j}f(x,t)\\
&= \frac{\partial}{\partial t_i}\{Q_j(t)f(x,t)\}\\
&= \frac{\partial}{\partial t_i}Q_j(t)f(x,t) + Q_j(t)\frac{\partial}{\partial t_i}f(x,t)\\
&= \frac{\partial}{\partial t_i}Q_j(t)f(x,t) + Q_j(t)Q_i(t)f(x,t)
\end{aligned}$$

By smoothness condition,

$$\begin{aligned}
\frac{\partial}{\partial t_j}Q_i(t)f(x,t) + Q_i(t)Q_j(t)f(x,t) &= \frac{\partial}{\partial t_i}Q_j(t)f(x,t) + Q_j(t)Q_i(t)f(x,t) \\
\Rightarrow \frac{\partial}{\partial t_j}Q_i(t)f(x,t) - \frac{\partial Q_j}{\partial t_i}f(x,t) &= [Q_j, Q_i]f(x,t) \\
\Rightarrow \frac{\partial Q_j}{\partial t_i} - \frac{\partial Q_i}{\partial t_j} &= [Q_i, Q_j] \qquad (3.1.10)
\end{aligned}$$

Thus, compatibility condition on / of order 2 with respect to t gives rise to the differential equation $\frac{\partial Q_j}{\partial t_i} - \frac{\partial Q_i}{\partial t_j} = [Q_i, Q_j]$, $i,j = 1,2, \cdots N$, which is the general Zakhrov-Shabat equation.

3.1.11 Definitions: (1) Let V denote the set of all formal ordinary differential operators with coefficients in the ring R. By a pseudo-differential operator, we mean an expression of the form

$$D = a_m(x)\partial^m + \cdots + a_1(x)\partial + a_o(x) + a_{-1}(x)\partial^{-1} + a_{-2}(x)\partial^{-2} + \cdots$$

with $a_m(x) \neq 0$, $a_j(x) \in R \; \forall j$. m is called the order of the differential operator.

(2) D is called monic if $a_m(x) = 1$ and is called normalized if $a_o(x) = 0$. We denote by $\mathcal{E}$, the ring of all formal pseudo-differential operators with coefficients in the ring R.

3.1.12 Proposition: (a) If $P \in S$ is a monic element, then there exists a unique monic element $Q \in S$ of order 1 such that $QoP = P \, o \, Q = l$.

(b) If $P \in S$ is a monic element of order ra, then there exists a unique monic element $Q \in S$ of order one such that Q^m — P. Moreover, if P is normalized, then its ra-th root Q is also normalized.

(c) For every normalized monic first order pseudo-differential operator $L = \partial + u_2(x)\partial^{-1} + u_3(x)\partial^{-2} + \cdots$, there is a monic zero-th order operator S such that $S^{-1}LS = d$. Such an S is unique upto a pseudo-differential operator with constant co-efficients.

3.1.13 Remarks: (1) Suppose that in the family $\{P(t) \backslash t \in M\}$ each $P(t)$ is a monic differential operator and is normalized of order n, then taking the n-th root , we get a corresponding family of pseudo-differential operators *{L(t)\t* 6 $M\}$ of order 1.

(2) By (3.1.8) of the isospectral deformation of $L = L(0)$ applied to *{L(t)\t* € A/}, we get $\frac{\partial}{\partial t_i} L(t) = [Q_i(t), L(t)]$,*i* = 1, • • • TV (3.1.14). If the parametrized family *{P(t)\t* $\in M\}$ satisfies the boundary conditions (3.1.5), then the family *{L(t)\t* $\in M\}$ also satisfies **the** same boundary conditions.

(3) By taking the n-th power of a solution of (3.1.14), we get a solution of (3.1.8) and the converse is also true.

(4) Since the left hand side of (3.1.14) is a pseudo-differential operator of order at most -1, the right hand side $[Q_i(t), L(t)]$ must also be a pseudo-differential operator of order almost -1 and we denote by $\mathcal{E}^{-1}$, the set of all pseudo-differential operators of order at most -1.

3.1.15 Theorem (Gel'fand-Dikii) [16]: Let $L = d + u_2(x)\partial^{-1} + u_3(x)\partial^{-2} + \cdots$ be an arbitrary normalized monic pseudo-differential operator. Then the set *ty[^l^* = $\{Q \in \mathcal{D} | [Q, L] \in \mathcal{E}^{-1}\}$ is a vector space generated (over *R)* by a collection $\{(L^m)_+, m = 0, 1, 2, \cdots\}$ where $L(t)^m_+$ denotes the differential operator part of $L(t)^m$.

3.1.16 Remark: Using this theorem, we can rewrite the equations (3.1.8) as $\frac{\partial}{\partial t} P(t) = [P(t)^{m/n}_+, P(t)]$. This family of equations is called the n-th order KdV hierarchy.

3.1.17 Remark: The following system of infinitely many non-linear differential equations with infinitely many variables,

$$\frac{\partial}{\partial t_i} L(t) = [(L^i(t))_+, \ L(t)], \quad i = 1, 2, \cdots,$$

where $L(t) = \partial + u_2(x,t)\partial^{-1} + u_3(x,t)\partial^{-2} + \cdot$, $t = (t_1, t_2, \cdots)$ and $u_j(x,t)$, $j = 2, 3, \cdots$ are the unknown functions are called the KP hierarchy. It governs the universal isospectral deformations of ordinary differential operators, because by making constraints on the unknown functions $u_j(x,t)'s$,one can recover the deformation equations of any given differential operators. If we assume that the rc-th power of $L(t)$ is a differential operator, then the KP hierarchy specializes to the n-th KdV hierarchy.

3.2 LAX REPRESENTATION OF SOME COMPLETELY INTEGRABLE HAMILTONIAN SYSTEMS:

Some of the completely integrable systems discussed in Chapter 2 have an associated Lax equation

$$\frac{dA}{dt}(\xi,t)= [B(\xi,t),A(\xi,t)] \tag{3.2.1}$$

with a rational parameter ξ. The following systems can be written in Lax form with a parameter ξ as in (3.2.1), with $A(\xi,t) = \sum_{h=-p}^{q} A_k(t)\xi^k$ where $A(\xi,t)$ are matrix functions of finite Laurent series in the variable ξ and whose coefficients lie in a linear Lie algebra $\mathcal{G}$ C $\mathcal{G}l(n)$:

(i) The Euler equations of a free rigid body M moving about a fixed point (M is co-adjoint orbit in $so(n)^*$).

(ii) The Euler-Poisson equations for a symmetric heavy rigid body. In this case M is a co-adjoint orbit in the semi-direct product of $SO(n)$with its Lie algebra.

(iii) The Toda lattice and its generalizations (M is co-adjoint orbit in a Kac-Moody Lie algebra)

(iv) The geodesics on an ellipsoid (M — T^*E) and the Neumann's problem of Newtonian motion on a sphere S^n with a quadratic potential ($M = T^*S^n$).

3.2.2 Definition: By a Lax equation with a parameter ξ, we mean an equation of the form

$$\dot{A}(\xi) = [B(\xi),A(\xi)] \quad (\cdot = \frac{-}{dt}) \tag{3.2.3}$$

where $B(\xi)$ is a finite Laurent series in ξ and whose coefficients are in $\mathcal{G}$. We now discuss the form of the Lax equation with **a** parameter ξ, associated to each **of the examples** above.

3.2.4 The Euler equations of a free rigid body: We consider the free rotation of a free rigid body about a fixed point, which we assume to be the origin in $\mathbb{R}^n$. "Rigid" means that the distances between the points of the body are unchanged during the motion. Let *f(t,x)* denote the position of the particle of the body at time *t* which was at x at time zero. Rigidity means that $f(t,\mathbf{x})= A(t)x$ where *A(t)* is an orthogonal matrix. We assume the motion to be smooth. Since /(0, x) $= A(0)\mathbf{x} =$ x, for every x $\in \mathbb{R}^n$, A(0) is the identity matrix so that $A(t) \in SO(n)$. The kinetic energy of the body is given by $K(t) = \frac{1}{2}\int_{\mathbb{R}^n} ||f(\mathbf{x},t)||^2 d\mu(\mathbf{x})$ where $||.||$ denotes the Euclidean norm on $\mathbb{R}^n$ and μ is a positive measure on $\mathbb{R}^n$ whose support is not in a 1-dimensional subspace. We have $f(t,\mathbf{x}) = \Omega_s(t)f(t,\mathbf{x})$ where $\Omega_s(t) = \rho(A(t)) \in so(n)$ is the vector $A(t) \in T_{A(t)}SO(n)$ expressed in space coordinates. The integrand of *K(t)* is $||A(t)^{-1}\Omega_s(t)A(t)\mathbf{x}||^2$.But $A(t)^{-1}\Omega_s A(t) = Ad_{A(t)^{-1}}\Omega_s(t) =$ \(A(t)) $= \Omega(t)$ is the expression of the vector $A(t)$ in body coordinates. Thus the kinetic energy has the form:

$$K(t) = \frac{1}{2}\int_{\mathbb{R}^n} ||\Omega(t)\mathbf{x}||^2 d\mu(\mathbf{x}) \ .$$

For $A, B \in so(n)$, we define an inner product on *so(n)* by

$$< A, B >= \int_{\mathbb{R}^n} A\mathbf{x} \ . \ B\mathbf{x} \ d\mu(\mathbf{x})$$

where x.y denotes the usual dot product on $\mathbb{R}^n$ (i.e., x.x $= ||\mathrm{x}||^2$). Thus $K(t) = \frac{1}{2} < \Omega(t), \Omega(t) >$. We define on *so(n),* the following bilinear Ad-invariant form:

$$(A, B) = -\frac{1}{2}Tr(AB)$$

(The motivation for this is the adjoint action of 50(3) on $so(3)$ under the **standard** isomorphism **of $\mathbb{R}^3$ with $so(3)$** : $\mathbf{x} = (x_1, x_2, x_3) \leftrightarrow$ x $= \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$ and x.y = $-\frac{1}{2}Tr(\hat{\mathbf{x}}\hat{\mathbf{y}}))$. Now, to write Euler's equations, we have to determine the operator $J : so(n) \rightarrow so(n)$ given by the condition

$$(\tilde{J}A, B) = -\frac{1}{2}Tr((\tilde{J}A)B) =< A, B >= \int_{\mathbb{R}^n} A\mathbf{x} \,.\, B\mathbf{x}\, d\mu(\mathbf{x})$$

Let e_{ij} denote the matrix all of whose entries are zero except the (i, j)-th entry which is 1. The above condition can be rewritten as

$$\frac{1}{2}Tr((\tilde{J}A)B) = \int_{\mathbb{R}^n} AB\mathbf{x} \,.\, \mathbf{x}\, d\mu(\mathbf{x})$$

Take $B = e_{ij} - e_{ji}$. Then by using the antisymmetry of JA, we get for the left hand side,

$$\frac{1}{2}Tr((\tilde{J}A)(e_{ij} - e_{ji})) = \frac{1}{2}((\tilde{J}A)_{ji} - (\tilde{J}A)_{ij}) = (\tilde{J}A)_{ji} \ ,$$

where $(JA)_{ji}$ denotes the (j, i)-thentry of JA.

Now, for the right hand side, we first have that

$$\begin{aligned} Ae_{ij}\mathbf{x} &= (A_{1_i}x_j, \cdots, A_{n_i}x_j) \qquad \text{so that denoting} \\ J_{jk} &= \int_{\mathbb{R}^n} x_j x_k d\mu(\mathbf{x}), \qquad \cdot\ \mathbf{x} = (x_1, \cdots x_n) \ , \end{aligned}$$

we have $J_{jk} = J_{kj}$ and

$$\int_{\mathbb{R}^n} Ae_{ij}\mathbf{x} d\mu(\mathbf{x}) = \int_{\mathbb{R}^n} \left(\sum_{k=1}^{n} A_{k_i} x_j x_k \right) d\mu(\mathbf{x}) = \sum_{k=1}^{n} J_{jk} A_{ki}$$

Similarly $\int_{\mathbb{R}^n} A\ e_{ji}\ \mathbf{x}\ d\mu(\mathbf{x}) = \sum_{k=1}^{n} J_{ik} A_{kj}$ and finally $(\tilde{J}A)_{ji} = \sum_{k=1}^{n} J_{jk} A_{ki} - \sum_{k=1}^{n} J_{ik} A_{kj}$

$$= \sum_{k=1}^{n} (J_{jk} A_{ki} + A_{jk} J_{ki}) = (JA + AJ)_{ji} \text{ where } J = (J_{ji}).$$

Thus JA $= AJ + JA$.

Since J is symmetric, there is a $g \in SO(n)$ such that $D = gJg^{-1}$ is diagonal. We define a measure ν in $\mathbb{R}^n$ by $\nu(\mathbf{x}) = \mu(g^{-1}\mathbf{x})$ and an operator J : $so(n) \rightarrow so(n)$ by $\tilde{J}(A) = gJ(g^{-1}Ag)g^{-1}$. J is a linear, (,)-symmetric, positive isomorphism. We have

~$j(A)=AD + DA$ and $\tilde{d}(A)$, $B) \int_{\mathbb{R}^n}$ Ax . $Bx(\mathbf{x})$.

Thus there is a new orthonormal basis of $\mathbb{R}^n$ having the same orientation as the initial one in which the operator j is diagonal. Hence, we choose this coordinate system which is completely determined by the mass distribution of the body as the initial one in $\mathbb{R}^n$ and we obtain JA— $AJ + J A$, for $J = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$. Since $e_{ij} - e_{ji}$ for $i < j$ is a basis of $so(n)$, the relations $J(e_{ij} - e_{ji}) = (\lambda_i + \lambda_j)(e_{ij} - e_{ji})$ imply that the canonical basis of $so(n)$ is a basis of eigenvectors of J. In particular, $\lambda_i + \lambda_j > 0$ since J is positive definite. For $n = 3$, the $\frac{n(n-1)}{2}$ numbers $\lambda_i + \lambda_j$, $i \neq j$ are called the principal moments of inertia. Thus, we have proved the following theorem,

3.2.5 Theorem: Given a rigid body in $\mathbb{R}^n$, there exists an orthonormal basis in $\mathbb{R}^n$ completely determined by the mass-distribution of the body in which the equations of motion of the free rigid body about a fixed point (the origin) have the form

$$\dot{M} = [M, \Omega] \tag{3.2.6}$$

where $M, \Omega \in so(n)$, $M = \Omega J + J\Omega$, for J a diagonal matrix $J = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ satisfying $\lambda_i + \lambda_j > 0$ for $i \neq j$. These equations are Hamiltonian on each adjoint orbit of $so(n)$ defined by initial conditions with Hamiltonian

$$H(M) = \frac{1}{2}(M, \Omega) = -\frac{1}{4}Tr(M\Omega)$$

The complete integrability of (3.2.6) was done by Manakov [37] who observed that (3.2.6) can be written as

$$(M + J^2\xi)^{\cdot} = [M + J^2\xi, \Omega + J\xi] \tag{3.2.7}$$

for any parameter ξ. The functions $\frac{1}{2k}\mathrm{Tr}(M + J^2\xi)^k$, $k = 2, \cdots, n$ are constant on the flow of (3.2.7), for, if $\dot{X} = [X, Y]$, $\mathrm{Tr}X^k$ are constant on its flow. Denoting by $t \rightarrow M(t)$ the flow of (3.2.6), $t \rightarrow M(t) + J^2\xi$ is the flow of (3.2.7). Hence the coefficients of ξ in the expansion of $\frac{1}{2k}\mathrm{Tr}(M + J^2\xi)^k$ will be constant on the flow of

(3.2.6).

Let c_{kj} be the co-efficient of ξ^j in $\frac{1}{2k}\mathrm{Tr}(M + J^2\xi)^k$, $k = 2, \cdots, n$. The constants of motion $\{c_{kj}\}$are called the Manakov integrals and their number equals half the dimension of the generic orbit making them candidates for the generically independent integrals in involution of (3.2.6).

3.2.7 Remark: For $n = 3$, we get the usual Euler equations. If x,y $\in \mathbb{R}^3$, and x x y denote their cross product, then (x x $\mathbf{y}$)$\hat{}$ = $[\hat{\mathbf{x}}, \hat{\mathbf{y}}]$. Let / : $\mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by

$$(/x) = Jx = xJ + Jx .$$

If $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$, then $e_1 = -(e_{23} - e_{32})$, $e_2 = (e_{13} - e_{31})$, $e_3 = -(e_{12} - e_{21})$, so that

$$(Ie_1) = (\mathrm{A}, + \lambda_3)\hat{e}_1 , (Ie_2)\hat{} = (\mathrm{A}, + \lambda_2)\hat{e}_2 , (Ie_3) = (\mathrm{A}, + \lambda_2)\hat{e}_3$$

and hence / : $\mathbb{R}^3 \to \mathbb{R}^3$ is a 3 x 3 diagonal matrix

$$1 = \mathrm{diag}(\lambda_2 + \lambda_3 , \lambda_1 + \lambda_3 , \lambda_1 + \lambda_2) = \mathrm{diag}(I_1, I_2, I_3)$$

Thus, the Euler's equations become $(I\mathbf{x})^{\cdot} = (I\mathbf{x}) \mathrm{x}\, \mathrm{x}$ or $I_1\dot{x}_1 - (I_2 - I_3)x_2x_3 = 0$, $I_2\dot{x}_2 - (I_3 - I_1)x_3x_1 = 0$, $I_3\dot{x}_3 - (I_1 - I_2)x_1x_2 = 0$ with I_1, I_2, I_3 principal moments of inertia.

3.2.8 The **Euler-Poisson** equations for a symmetric heavy rigid body (or the Lagrange top): As a physical system, the Lagrange top is a symmetric rigid body with a constant vertical gravitational force acting on its centre of mass and leaving the base point of its body symmetry axis fixed. The problem is to describe the motion of the top. The space of all positions of the Lagrange top is 50(3), the Lie group of all proper rotations of $\mathbb{R}^3$. Phase space, the space of all positions and momenta of the top is the cotangent bundle $T^*SO(3)$ of $SO(3)$ with its canonical symplectic form $ft = -d\theta$. On the Lie algebra $so(3)$, which is the tangent space at the identity e $\in$ 50(3), the moment of inertia tensor is a non-degenerate inner product <, >: $so(3)$ x $so(3) \to \mathbb{R}$ defined by $A, B \mapsto \kappa(L(A), B)$, where κ is the killing form defined by $\kappa(A,B) = -\frac{1}{2}Tr(AB)$, where $L : so(N) \to so(N)$ is the positive κ- symmetric isomorphism with inverse $L^{-1}(C) = CJ + JC$, $J = \text{diag}(\lambda_1, \cdots \lambda_N)$ a fixed diagonal matrix satisfying $\lambda_i + \lambda_j > 0$ for $i \neq j$. The Hamiltonian $M \to \frac{1}{2}\kappa(M, L(M))$ defines on adjoint orbits in $so(N)$ the equations of motion $\dot{M} = [M, L(M)]$ which is the Euler-vector field of the geodesic spray of the metric in $SO(N)$. These equations represent the motion of the free N-dimensional rigid body about a fixed point, the origin of $\mathbb{R}^N$. The $\frac{N(N-1)}{2}$ positive numbers $\lambda_i + \lambda_j$, $i \neq j$ are the principal moments of interia of the body (discussed in the previous example). If $N = 3$, the centre of mass χ is not the origin, the fixed point about which the body of unit weight moves, a potential $V(A) = \kappa(\chi Ad_{A^{-1}}\epsilon)$ must be added to the kinetic energy $\frac{1}{2}\kappa(M, L(M))$ in the expression of the total energy and it represents the height of X over the horizontal plane; ϵ is the unit vector of the OZ-axis. Since gravity in $\mathbb{R}^N$ is meaningless, we define the rigid body Hamiltonian in $SO(N)$x $so(N)$ by

$$E(A, M) = \frac{1}{2}\kappa(M, \Omega) + \kappa(\chi, Ad_{A^{-1}}\epsilon) \quad L^{-1}(\Omega) = M ,$$

for $\chi, \epsilon \in so(N)$ fixed. The Euler-Poisson equations in $so(N)$ x $so(N)$ induced by this Hamiltonian are

$$\dot{\Gamma} = [\Gamma, \Omega] , \dot{M} = [M, \Omega] + [\Gamma, \chi] \tag{3.2.9}$$

for $M = \Omega J + J\Omega$, χ a fixed matrix in $so(N)$, Γ, $\Omega \in so(N)$. The Hamiltonian of (3.2.9) is given by

$$H(\Gamma, M) = \frac{1}{2}\kappa(M, \Omega) + \kappa(\Gamma, \chi)$$

The Lagrange top is defined by $\alpha = \lambda_1 = \lambda_2$, $\beta = \lambda_3 = \cdots = \lambda_n$, and $\chi_{12} \neq 0$, $\chi_{ij} = 0 \ \forall i, j \neq 1,2, \ i < j$. The heavy symmetric top is defined by $\lambda_1 = \cdots = \lambda_n = \alpha$, , χ arbitrary. In this case, $A = \alpha Id$, $M = \alpha \times \Omega$ and (3.2.9) becomes

$$\dot{\Gamma} = [\Gamma, \Omega] \ , \ 2\alpha\Omega = [\Gamma, \chi] .$$

3.2.10 Theorem [49]: Assume $\chi_{12} \neq 0$. The Euler-Poisson equations (3.2.9) can be written in the form

$$(\Gamma + M\xi + c\xi^2)^{\cdot} = [\Gamma + M\xi + c\xi^2, \Omega + \lambda\xi]$$

if and only if (3.2.9) describe the N-dimensional Lagrange or heavy symmetric top, in which case $C = (\alpha + \beta)\chi$.

3.2.11 Remark: If $\chi = 0$, then $C = 0$ and we get the free rigid body which is a completely integrable system.

3.2.12 The Toda Lattice: In $\mathbb{R}^{2n}$ with coordinates $(x_1, \cdots x_n, y_1, \cdots y_n)$ and symplectic form $\omega = \sum_i dx_i \wedge dy_i$, we consider the Hamiltonian function

$$H(x,y) = \frac{1}{2}\sum_{i=1}^{n} y_i^2 + \sum_{i=1}^{n} e^{x_i - x_{i+1}} \ , \ x_{n+1} = x_1$$

whose corresponding flow is given by

$$\dot{x}_k = \frac{\partial H}{\partial y_k} \ , \ \dot{y}_k = -\frac{\partial H}{\partial x_k} \tag{3.2.13}$$

By Flaschka's transformation,

$$a_k = \frac{1}{2}(e^{x_k - x_{k+1}}) \ , \ b_k = -\frac{y_k}{2} \ ,$$

the equations (3.2.13) are transformed into

$$\begin{aligned} \dot{b}_k &= 2(a_k^2 - a_{k-1}^2) \ , \ a_o = a_n \\ \dot{a}_k &= a_k(b_{k+1} - b_k) \ , \ a_{n+1} = b_1 \end{aligned} \tag{3.2.14}$$

We set

$$A = \begin{bmatrix} b_1 & a_1 & \cdots & a_n\xi^{-1} \\ a_1 & b_2 & & \\ & & & \\ & & b_{n-1} & a_{n-1} \\ a_n\xi & & a_{n-1} & b_n \end{bmatrix} \quad B = -\begin{bmatrix} 0 & a_1 & \cdots & -a_n\xi^{-1} \\ -a_1 & 0 & & \\ & & & \\ & & 0 & a_{n-1} \\ a_n\xi & & -a_{n-1} & 0 \end{bmatrix}$$

From the first equation in (3.2.14) we have $\sum b_k = 0$. If we normalize the b_k's such that $\sum_k b_k = 0$, then (3.2.14) are equivalent to the Lax equations with a parameter, $A = [B, A]$ where A, B are given above.

3.2.15: The geodesies on an ellipsoid and Neumann's forced harmonic motion are discussed in many places (e.g.[25]). Here we shall follow [2], and explain how they may be written as a Lax equation with a parameter. For this we let $W = \mathbb{R}^n$ or $\mathbb{C}^n$, make the identification $W \otimes W = \mathrm{Hom}(W, W)$,and define maps

$$\Gamma_{xy} : \quad W \times W \mapsto \mathrm{Hom}(W, W)$$

$$\Delta_{xy} : W \times W \mapsto \mathrm{Hom}(W, W)$$

by

$$\Gamma_{xy} = x \otimes y - y \otimes x \quad (\text{i.e., } (\Gamma_{xy})_{ij} = x_i y_j - x_j y_i)$$

$$\Lambda_{xx} \text{ — } x®x \qquad (\text{i.e., } (\Lambda_{xx})_{ij} = x_i x_j)$$

We note that

$$\text{Image } \Gamma = \text{skew-symmetric matrices of rank } 2$$

$$\text{Image A} = \text{matrices of rank } 1$$

$$\text{Image A} = \text{symmetric matrices of rank} < 2$$

and that the rank of a matrix is invariant when it moves on an adjoint orbit. We also set

$$\alpha = \mathrm{diag}(\alpha_1, \cdots, \alpha_n), \quad \alpha_i > 0,$$

$$\beta = \mathrm{diag}(\beta_1, \cdots, \beta_n)$$
$$\Gamma = (\beta\alpha^{-1})\Gamma_{xy}(\beta\alpha^{-1})^{-1}.$$

Then the equations for geodesics on an ellipsoid and for Neumann's system are respectively

$$(-\Lambda_{xx} + \xi\dot{\Gamma}_{xy} + \xi^2\alpha) = [-\Lambda_{xx} + \xi\Gamma_{xy} + \xi^2\alpha, \Gamma + \quad tf] \qquad (3.2.16)$$

$$(\Delta_{xy} - \alpha + \xi\,\dot{\Gamma}_{xy} + \xi^2\alpha) = [\Delta_{xy} - \alpha + \xi\Gamma_{xy} + \xi^2\alpha, \Gamma + \xi\beta] \qquad (3.2.17)$$

Equation (3.2.16) is equivalent to

$$(i)\quad \dot{\Lambda}_{xx} = [\Lambda_{xx}, \mathrm{r}], \qquad (ii)\quad \Gamma_{xy} = [\Gamma_{xy}, \mathrm{r}] - [\Lambda_{xx}, \beta]$$

Since $\Gamma^t = -\Gamma$, $\Lambda^t_{xx} = \Lambda_{xx}$ and so Λ_{xx} moves in the space of matrices of rank 1. In (ii)

$$\dot{\Gamma}^t_{xy} = -\Gamma_{xy}, \quad (\dot{\Gamma}_{xy}v, w) = 0 \quad \text{if} \quad (v,x) = (v,y) = (w,x) = (w,y) = 0,$$

from which it follows that Γ_{xy} moves in the space of skew-symmetric matrices of rank 2. The relationship between the geodesics on an ellipsoid and Neumann's system to algebraic curves is dicussed in [25].

3.3 SPECTRAL CURVES:

3.3.1 Definition: Given Lax equation (3.2.3) with $A(\xi) = \sum_{k=-p}^{q} A_k(t)\xi^k$ and $B(\xi) = \sum_{-p}^{q} B_k(t)\xi^k$ we can associate its spectral curve C whose affine equation is given by

$$Q(\xi,\eta) = \{(\xi,\eta) \in \mathbb{C}^* \times \mathbb{C} / \det(\eta I - A(\xi,t)) = 0\}$$

Here $A(\xi, t)$ in $\mathcal{G}[\xi, \xi^{-1}]$ is an isospectral family of matrices, i.e., the matrices in the family are Laurent polynomials in ξ and all of them have the same spectrum. This characteristic polynomial defines a spectral curve. The spectral curve C is the normalization of the complete algebraic curve C_o whose affine equation is given by $Q(\xi,\eta)$ and we explain this concept below.

3.3.2: Let X be a smooth variety. Let ϑ_X denote the structure sheaf of X and Θ_X the tangent sheaf. Let $\vartheta_X(1)$ be the hyperplane bundle for $X \subset \mathbb{P}^N$. Let C be a smooth curve and $D = \sum_i n_i p_i$, $n_i > 0$ be an effective divisor on C. Let $C(D) = H^0(\vartheta_C(D)) = \{f$ meromorphic in $C : (f) + D > 0\}$ be the vector space corresponding to the divisor D. Let Ω_C be the sheaf of holomorphic 1-forms on C. Then the Jacobian variety, $J(C) = \mathrm{Pic}^o(C)$ = line bundles of degree zero = $H^1(\vartheta_C)/H^1(C,\mathbb{Z}) = H^0(\Omega_C)^*/H_1(C,\mathbb{Z})$.

Let V be a complex vector space of dimension m. Let $P = \mathbb{P}^1$ with homogeneous coordinates $[\xi_0,\xi_1]$ and affine coordinate $\frac{\xi_1}{\xi_0} = \xi$. Let $\vartheta_P(1)$ be the standard line bundle over P and $\vartheta_P(k) = \vartheta_P(1) \otimes \cdots \otimes \vartheta_P(1)$ (k times). We set $V = V \otimes_{\mathbb{C}} \vartheta_P(1)$ and $V(k) = \underline{V} \otimes_{\mathbb{C}} \vartheta_P(k)$ where V is the sheaf of germs of holomorphic sections of $\vartheta_P(1)$ with values in V and $V(k)$ is the sheaf of germs of holomorphic sections of $\vartheta_P(k)$ with values in V.

We assume the following data:

(i) $A(t,\xi) = \sum_{k=0}^{n} A_k\xi^k = \xi_0^n A_0 + \xi_0^{n-1} A_1\xi_1 + \cdots + A_n\xi_1^n \in H^0(P,\mathrm{Hom}(V, V(n)))$

(ii) $B(t,\xi) = \sum_{l=0}^{N} B_l\xi^l \in H^0(P,\mathrm{Hom}(V, V(N)))$ and

(iii) a Lax equation $\dot{A} = [B, A]$ $\quad (\cdot = \frac{d}{dt})$.

Let Y be the bundle space of the line bundle $\vartheta_P(n)$. i.e., $Y \to P$. Points of Y are pairs $(\xi,\nu) \in Y$, where $\xi \in P$, $\nu \in \vartheta_P(n)$ such that $\eta(\xi,\nu) = \nu$. Then there is defined over Y the section $\eta \in H^0(Y,\pi^*\vartheta_P(n))$.

We note that $Y \overset{def}{=} \{(\xi,\nu)/\xi \in P,\ \nu \in \vartheta_P(n)\}$. The characteristic polynomial $Q(\xi,\eta) = \det(\eta I - A(\xi,t))$ where $A(\xi,t)$ in $H^0(P,\mathrm{Hom}(V,V(n)))$, $V = \mathbb{C}^m$ can be interpreted as an element in $H^0(Y,\pi^*\vartheta_P(n))$. Therefore, $Q(\xi,\eta) = det\|\eta I - A(\xi,t)\| \in H^0(Y,\pi^*\vartheta_P(mn))$. The divisor of Q will be a complete curve $C_o \subset Y$ and $\pi : C_o \subset Y \to P$ where $C_o = D_Q = \{(\xi,\nu) \in P \times \vartheta_P(n)/Q(\xi,\nu) = 0\}$. Then the divisor locus of D_Q of Q in

Y is a complete curve C_o in Y and the projective map $\pi : C_o \rightarrow P = \mathbb{P}^1(\mathbb{C})$ is a branched covering with $\pi^{-1}(\xi) \in Y = \{(\xi,\eta_1), \cdots (\xi, \eta_m)\}$ where $\eta_1, \cdots, \eta_m$ are the roots (or eigenvalues) of the characteristic equation. We assume that C_o is irreducible and its normalization is $C \rightarrow C_o$ which is a non-singular curve. Consider $\pi^{-1}(\mathbb{P}^1\backslash\{\infty\}) \cong \mathbb{C}^2$ with coordinates (ξ,η). Then C_o is the compactification of the affine curve determined by det $||\eta I - A(\xi,t)|| = 0$ in $\mathbb{C}^2$. The curve C is called the *spectral curve* associated to the Lax equation (3.2.3). We discuss below the spectral curves associated with each of the systems given in the previous section.

3.3.3: For the free rigid body in $\mathbb{R}^3$, we have

$$\begin{aligned} Q(\xi,\eta) &= det||\eta I - \xi J^2 - M|| \\ &= \eta^3 + \delta\xi^3 + \cdots \end{aligned}$$

where $\delta = -\det J^2$. Thus, C_o is the compactification of a cubic curve in $\mathbb{C}^2$.

Let $\xi \in P$ be a general point and we write

$$Q(\xi,\eta) = \prod_{\nu=1}^{m}(\eta - \eta_\nu(\xi)) \tag{3.3.4}$$

We assume that the curve $C = C_o$ is smooth.

Claim: The spectral curve above associated with the free rigid body has genus g, given by formula

$$g = \frac{mn(m-1)}{2} - m + 1 \qquad \text{where} \qquad m = \dim_{\mathbb{C}} V = 3 \text{ and } n = 1$$

We give below a general outline of the proof of the claim. Consider the discriminant of this polynomial in η of (3.3.4) given by $\Delta(\xi) = \prod_{\mu<\nu}(\eta_\nu(\xi) - \eta_\mu(\xi))^2$. Then $\Delta(\xi)$ is **a well-defined** function of $\xi = (\xi_o, \xi_1) \in \mathbb{C}^2$, and $\Delta(\lambda\xi) = \lambda^{2n\binom{m}{2}}\Delta(\xi)$. Thus, $\Delta \in H^0(P, \vartheta_P(mn(m-1)))$. Suppose at some point $\xi \in P$, k **of** the eigenvalues, say $\eta_1, \cdots \eta_k$ **coincide** and cyclically permute as ξ moves around ξ_o. For a suitable local coordinate t, centered at ξ_o, we will have $\eta_k = \zeta^\nu t^{1/k}$, $\zeta = e^{2\pi i/k}$, $1 \leq \nu \leq k$.

Since $\Delta(\xi) = \prod_{1\leq\mu\leq\nu\leq k} (\eta_\nu(\xi) - \eta_\mu(\xi))^2 = \prod_{1\leq\mu\leq\nu\leq k} (\zeta^\mu t^{1/k} - \zeta^\nu t^{1/k})^2 = ct^{k-1}, c \neq 0$,

the order of vanishing of $\Delta(\xi)$ gives the sum of the ramification indices of the points of C lying over ξ. Since C — C_o, we can have $\eta_\nu(\xi) = \eta_\mu(\xi)$, $\nu \neq \mu$, only as a branching point. Hence $C \to P$ is an n-sheeted covering space whose ramified divisor has degree given by r = deg(A) = $mn(m-1)$. i.e., A $\in \Gamma(P, \vartheta_P(mn(m1))$. The Riemann-Hurwitz formula given by $2g - 2$ = -2m + r = -2m + mn(m - 1) gives $g = 1 - m + \frac{mn(m-1)}{2}$.

3.3.5 Remark: We can understand the linearization of the Lax flow on the Prym variety (C/C').

3.3.6: Referring to the Lagrange top in the case n = 3 discussed in Example (3.2.8) in section 3.2 above, we set

$$A(\xi) = \Gamma + M\xi + C\xi^2 \in so(3)[\xi] \tag{3.3.7}$$

From

$$\det \begin{Vmatrix} \eta & -a_1 & -a_3 \\ a_1 & \eta & -a_2 \\ a_3 & a_2 & \eta \end{Vmatrix} = \eta(\eta^2 + (a_1^2 + a_2^2 + a_3^2))$$

we observe that

$$Q(\xi, \eta) = \det ||\eta I - A(\xi)|| = \eta(\eta^2 + |A|^2)$$

where $|A|^2$ is the sum of the squares of the entries of A. By (3.3.7), we have

$$|A|^2 = \gamma_0 + \gamma_1\xi + \gamma_2\xi^2 + \gamma_3\xi^3 + \gamma_4\xi^4 \tag{3.3.8}$$

where $\gamma_o = |\Gamma|^2$, $\gamma_4 = |C|^2$. The spectral curve is reducible with one component ($\eta = 0$) corresponding to the zero eigenvalue ($\eta = 0$) of any matrix in $so(3)$. The other component $\eta^2 + |A(\xi)|^2 = 0$ is by (3.3.8) an elliptic curve, which is smooth and which can be realized by $(\xi, \eta) \to \xi$, as a 2-sheeted branched covering of $\mathbb{P}^1$ with sheet interchange given by

3.3.9: **Referring** to the Toda lattice given by Example (3.2.12) in section 3.2 above, we have

$$A(\xi) = A_{-1}\xi^{-1} + A_o + A_1\xi$$

where

$$A_{-1} = \begin{pmatrix} 0 & \cdots & a_n \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \quad A_1 = A_{-1}^t$$

$$A_0 = \begin{pmatrix} b_1 & a_1 & \cdots & \cdots & 0 \\ a_1 & b_2 & \cdots & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & \cdots & b_{n-1} & a_{n-1} \\ 0 & \cdots & \cdots & a_{n-1} & b_n \end{pmatrix}$$

We then have

$$\begin{aligned} Q(\xi,\eta) &= \det ||\eta I - A(\xi)|| \\ &= a_1 \cdots a_{n-1}(\xi + \xi^{-1}) + \tilde{P}(\eta) \end{aligned}$$

where $\tilde{P}(\eta) = \eta^n + c_1\eta^{n-1} + \cdots + c_n$ is a polynomial in η. Multiplying by $a_1 \cdots a_{n-1} \neq 0$, we get the **affine** curve in $\mathbb{C}^* \times \mathbb{C}$ given by

$$R(\xi,\eta) = \xi + \xi^{-1} + P(\eta) = 0 .$$

This affine curve will be smooth and its completion C_o in $\mathbb{P}^2$ will be singular for $n > 4$. To compute the genus of the normalization C of C_o, we observe that the involution of the spectral curve given *by* $j : C \to C$ *by* $j(\xi,\eta) = (\xi^{-1},\eta)$ realizes C as a 2-sheeted covering $C \to \mathbb{P}^1$ = η-sphere (3.3.10).

The fixed points **of** j, which coincide with the branch points of (3.3.10), occur when

$\xi^2 = 1$. **i.e.,** when $\xi = \pm 1$. In general, there are n of these for each value, $\xi = +1$, $\xi = -1$. Thus there are 2n branch points in all, and by the **Riemann-Hurwitz** formula, the genus of C is given by $g = n - 1$. This is the number of required additional integrals of motion.

3.3.11: The spectral curves associated to **geodesics** on an ellipsoid and Neumann's problem are given by

$$\begin{aligned} Q_1(\xi,\eta) &= \det\|\eta I - \xi^2\alpha - \xi\Gamma_{xy} + \Gamma_{xx}\| = 0 \\ Q_2(\xi,\eta) &= \det\|\eta I - \xi^2\alpha - \xi\Gamma_{xy} - \Delta_{xy} + \alpha\| = 0 \end{aligned} \tag{3.3.12}$$

Each of these is of the form

$$\det\|\eta I - \xi^2\alpha - P\| = 0 \tag{3.3.13}$$

where P is a rank 2 matrix. We set $T - \eta I - \xi^2\alpha$ so that (3.3.13) becomes

$$\begin{aligned} 0 = \det\|T - P\| &= \det T \det\|I - T^{-1}P\| \\ &= \det T(1 - \text{Trace}\, T^{-1}P + \text{Trace}\,(\Lambda^2 T^{-1}P)) \end{aligned}$$

since $T^{-1}P$ has all $k \times k$ minors equal to zero for $k > 3$. To compute the genus of the spectral curve, we can use the reparametrization of the spectral curves given by (cf. Adler's paper [2]),

$$\begin{aligned} Q_1(\xi,\eta) &= \xi^2 + \langle (\eta-\alpha)^{-1}x, y \rangle = 0 \\ Q_2(\xi,\eta) &= \xi^2 - 1 + (-2\langle (\eta-\alpha)^{-1}x, y\rangle + \langle (\eta-\alpha)^{-1}x, x\rangle \\ &\quad \times \langle (\eta-\alpha)^{-1}y, y\rangle - \langle (\eta-\alpha)^{-1}x, y\rangle^2) = 0 \end{aligned}$$

These are **hyperelliptic** curves of the form (in $\mathbb{C} \times \mathbb{C}^*$) $\xi^2 = R(\eta)$ where $R(\eta)$ is a rational function of the form

$$R(\eta) = \frac{c(\eta^{n-1} + \cdots)}{(\eta^n + \cdots)}$$

In each case, the genus of the curve is $n - 1$ and we need $n - 1$ commuting integrals of motion, in addition to the total energy for the system to be completely integrable.

3.4: **COHOMOLOGICAL** INTERPRETATION OF LAX EQUATIONS AND LINEARIZATION OF FLOWS:

Following [19] we consider Lax equations with a parameter ξ,

$$\dot{A}(\xi) = [B(\xi), A(\xi)] \tag{3.4.1}$$

where $A(\xi, t) = \sum_{k=-r}^{q} A_k(t)\xi^k$ and $B(\xi, t) = \sum_{k=-l}^{m} B_k(t)\xi^k$ are finite Laurent series in the parameter ξ and $A_k(t), B_k(t) \in \mathcal{G} \subset \mathcal{G}l(m)$. Its spectral curve C (after normalization of C_o) is given by $Q(\xi,\eta) = \det \| \eta I - A(\xi,t) \| = 0$ with $p = (\xi,\eta) \in C$. As the flow $t \mapsto A(\xi,t)$ is isospectral, the polynomial $Q(\xi,\eta)$ is independent of t. In otherwords, the eigenvalues η are fixed as time evolves whereas the eigenvectors of $A(\xi,\eta)$ will change with t. The problem we want to study is "to find necessary and sufficient conditions on B such that the flow on the Jacobian variety $J(C)$ of the spectral curve C of (3.4.1) corresponding to $t \mapsto A(\xi,t)$,be linear".

First we note that the Lax equation is invariant when B is replaced by $B + P(A,\xi)$where $P(x,\xi) \in \mathbb{C}[x,\xi]$ and hence B lives in some quotient space. Now we follow the notations of section (3.3) of spectral curves. Assume that for a general point $p = (\xi,\eta) \in C$, $\dim \ker \| \eta I - A(\xi,t) \| = 1$. Hence there is a uniquely determined vector $\nu(p,t) \in V$ (upto non-zero scalars) satisfying $A(\xi,t)\nu(p,t) = \eta\nu(p,t)$ (3.4.2). Then the map $p \to \mathbb{C}\nu(p,t) \subset V$ determines a family of holomorphic maps $f_t : C \to \mathbb{P}V$ (3.4.3) depending holomorphically on t, which is called the *eigenvector mappings* associated to the Lax equation (3.4.1). Let $\Theta_{\mathbb{P}V}(1)$ denote the hyperplane bundle of $\mathbb{P}V$ and let

$$L_t = f_t^*(\vartheta_{\mathbb{P}V}(1)) \in \mathrm{Pic}^d(C) \quad (\cong \quad J(C)) \ , \ L = L_o.$$

3.4.4: Some preparation from algebraic geometry: ([6], [20], [26])

Let $f : C \to X$ (†) be a non-constant holomorphic map where C is a given smooth algebraic curve and X is a complex manifold. Define the *normal sheaf* of C in X (or of /) TV/, by the exact sequence

$$0 \to \Theta_C \xrightarrow{f_*} f^*\Theta_X \to N_f \to 0 \qquad (3.4.5)$$

with Θ_C, Θ_X are tangent sheaves and f_* is the differential of f. Then the Kodaira-Spencer tangent space to the moduli space of the map / (in (†)) is given by $H^o(C, N_f)$, i.e., $f_t : C_t \to X$ maps where C and / are both varying with t. i.e., if $f_t : C_t \to X$, $f_o =$ / is a deformation of / in (†), then $f \in H^o(C, N_f)$ (3.4.6) is the corresponding infinitesimal deformation at t — 0. i.e., in local product co-ordinates (z, t)on $\cup_t C_t$ and $w = (w^1, w^2, \cdots, w^m)$ of X, f_t is given by $(z, t) \mapsto w(z, t)$, then the section / of N_f over C in (3.4.6) is locally given by ∂t $/_{t=o}$ modulo ∂z . The corresponding cohomological sequence of (3.4.5) is

$$H^o(\Theta_C) \to H^o(f^*\Theta_X) \to H^o(N_f) \xrightarrow{\delta} H^1(\Theta_C) \text{ over } C$$

Here $H^1(\Theta_C)$ is the tangent space to the moduli space of C as an abstract curve and $\delta(f)$=: $C \in H^1(\Theta_C)$ is the tangent to the family of curves $\{C_t\}$. Thus the tangent space to / in (†) where the curve C remains fixed, is given by $H^o(f^*\Theta_X)/H^o(\Theta_C) \subset H^o(N_f)$. Since the isospectral curve C is independent of t, this is the situation we have to consider. As in section (3.3), V is a m-dimensional $\mathbb{C}$-vector space and take $X = \mathbb{P}V$ and consider the exact sequence of sheaves over *JPV:*

$$0 \to \vartheta_{\mathbf{P}V} \to V \otimes \vartheta_{\mathbf{P}V}(1) \to \Theta_{\mathbf{P}V} \to 0 \qquad (3.4.5').$$

Using / to pull back this sequence to C and combining this with (3.4.5), we get $0 \to \Theta_C \xrightarrow{f_*} f^*\Theta_{\mathbf{P}V} \to N_f \to 0$ and $0 \to \vartheta_C \xrightarrow{\nu} V \otimes L \to f^*\Theta_{\mathbf{P}V} \to 0$ where $L = f^*\vartheta_{\mathbf{P}V}(1)$

which gives the piece of the cohomology diagram

$$\exists\, \omega \in H^o(V \otimes L)$$
$$\downarrow \tau$$
$$H^o(\Theta_C) \to \nu \in H^o(f^*\Theta_{\mathbb{P}V}) \xrightarrow{j} f \in H^o(N_f) \xrightarrow{\delta} H^1(\Theta_C) \qquad (3.4.7)$$
$$\downarrow \delta$$
$$L \in H^1(\vartheta_C)$$

Consider the family of holomorphic maps $f_t : C \to \mathbb{P}V$. Locally choose a coordinate z on C and also a position vector mapping $(z,t) \mapsto \nu(z,t) \in V \setminus \{0\}$ (i.e. a local lift of f_t to $V \setminus \{0\}$) such that $f_t(z) = \mathbb{C}\nu(z,t) \subset V$. Set $\dot\nu(z) = \left.\frac{\partial\nu(z,t)}{\partial t}\right|_{t=0}$ modulo $\nu(z,t)$.

Since the inclusion $\vartheta_C \hookrightarrow V \otimes L$, $L = f^*\vartheta_{\mathbb{P}V}(1)$ is locally given by $\varphi \mapsto \varphi.\nu$, $\nu \in H^o(C, V \otimes L/\vartheta_C) = H^o(C, f^*\Theta_{\mathbb{P}V})$ (3.4.8) is well defined of the representative position mapping of ν.

Then we have $j(\dot\nu) = f$ (3.4.9). We are interested in the tangent vector $L = (\frac{d}{dt}L_t)_{t=0}$ in $H^o(\vartheta_C)$ (as $t \mapsto L_t$ is linear iff $\frac{d^2}{dt^2}L_t = \mu_t \frac{d}{dt}L_t$). Then we have $L = \delta(\dot\nu)$ **(3.4.10)** where $\dot\nu$ is the infinitesimal variation of $f_t : C \to \mathbb{P}V$. In particular, $L = 0$ **(3.4.11)** iff $\exists\, \omega \in H^o(V \otimes L)$ such that $\dot\nu = \tau(\omega)$, ω depending on B. Recall from section 3.3 that $B(\xi, t) \in H^o(C, \mathrm{Hom}(V, V(N))$.

Let $D = (\xi_o^N)$ be the divisor $N.\pi^{-1}(\infty)$. Then

$$B/\xi_o^N \in H^o(\mathrm{Hom}(V, V(D)) = (Hom(V, V) \otimes \vartheta_C(D)) \qquad (3.4.12)$$

and $\nu \in H^o(V \otimes L)$ where $V(D) \cong V(N)$. Hence

$$\left(\frac{B}{\xi_o^N}.\nu\right) \in H^o(V \otimes L(D)).$$

The cohomological interpretation of Lax equation (3.4.1) is given by

3.4.13 Theorem: In the above notation $\dot\nu = \tau\left(\frac{B\nu}{\xi_o^N}\right)$ (3.4.14)

(cf. (3.4.11) with $\omega = \frac{B\nu}{\xi_o^N}$).

We now give the more precise meaning of the formula (3.4.14) as follows:

We have the following diagram of exact sequence of sheaves on the curve C :

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & & \vartheta_C & \to & \vartheta_C(D) & \to & \vartheta_D(D) & \to & 0 \\ 0 \to & & V\otimes L & \to & V\otimes L(D) & \to & V\otimes L\otimes\vartheta_D(D) & \to & 0 \\ 0 \to & & f^*\Theta_{\mathbb{P}V} & \to & f^*\Theta_{\mathbb{P}V}(D) & \to & f^*\Theta_{\mathbb{P}V}\otimes\vartheta_D(D) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{3.4.15}$$

whose part of cohomological information is

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & \downarrow & & \downarrow \\ & & \exists\, \varphi \in H^\circ(\vartheta_C(D)) \to & & H^\circ(\vartheta_D(D)) \\ & & \downarrow \nu & & \downarrow \sigma \\ \exists\, b \in H^\circ(V\otimes L) \xrightarrow{i} & & \frac{B\nu}{\xi_o^N} \in H^\circ(V\otimes L(D)) \xrightarrow{j} & & H^\circ(V\otimes L\otimes\vartheta_D(D)) \\ \downarrow \tau & & \downarrow \tau & & \downarrow \tau \\ \dot\nu \in H^\circ(f^*\Theta_{\mathbb{P}V}) \xrightarrow{i} & & H^\circ(f^*\Theta_{\mathbb{P}V}(D)) \xrightarrow{j} & & H^\circ(f^*\Theta_{\mathbb{P}V}\otimes\vartheta_D(D)) \\ \downarrow \delta & & & & \\ \mathbf{A} \in H^\circ(\vartheta_D(D)) \xrightarrow{\delta_1} \dot L \in H^1(\vartheta_C) & & & & \end{array} \tag{3.4.16}$$

Then the meaning of formula (3.4.14) is (a) $\frac{B\nu}{\xi_o^N} \in H^\circ(V\otimes L(D))$ and (b) $\tau(\frac{B.\nu}{\xi_o^N}) = i(\dot\nu)$.

Proof: Working in $\mathbb{C}^2$ with **co-ordinates** (ξ,η), $A(\xi,t)$ and $B(\xi,t)$ are polynomials in $\xi \in \mathbb{C}$ whose **co-efficients are** holomorphic functions of t. Write $B(\xi,t) — \frac{B}{\xi_o^N}$ where $B \in H^o(V, V(N))$ considered as a homogeneous polynomial in ξ_o, ξ_1. For a general point $P = (\xi,\eta) \in \mathbb{C}$ we have by (3.4.2), $A(\xi,t)\nu(p,t) = \eta\ \nu(p,t)$. Then $Av + Av = \eta\nu$ and so $(BA - AB)v + A\dot\nu = \eta\dot\nu$, i.e., $A(\dot\nu - B\nu) = \eta(\dot\nu - B\nu)$, **i.e., if ν is an eigenvector of A with**

eigenvalue 77 then v — Bv is also an eigenvector with same eigenvalue. Since eigenspace E_η is 1-dimensional generically, we have $Bv = v + \lambda v$ for some $\lambda(p)$. **(3.4.17)**

Hence $T(BV) = v \in V \otimes L/\mathbb{C}.\nu$.

3.4.18 Corollary: $L = 0 \Leftrightarrow \exists$ a meromorphic function $\varphi \in H^o(\vartheta_C(D))$ such that $\frac{B\nu}{\xi^N} + \varphi\nu \in H^o(V \otimes L(D))$ is holomorphic.

Proof: Note that there exists $\varphi \in H^o(\vartheta_C(D))$such that $\frac{B\nu}{\xi_o^N} + \varphi\nu \in H^o(V \otimes L(D))$ is holomorphic (i.e., belongs to $H^o(V \otimes L) \Leftrightarrow \exists b \in H^o(V \otimes L)$ such that $i(b) = \frac{B\nu}{\xi_o^N} + \varphi\nu \in H^o(V \otimes L(D))$. Then by above commutative diagram (3.4.16) and (3.4.10), we get $L = \delta(\dot{\nu}) = \delta(\tau b) = 0$.

3.4.19 Definition: Let C be a smooth curve of genus g and $D = \sum_i n_i p_i$, $n_i > 0$ be an effective divisor of C. If z_i is a local co-ordinate centred at p_i, then $\varphi_i = \sum_{k=1}^{n_i} \frac{a_{ik}}{z_i^k}$ is called the *Laurent tail* at p_i.

3.4.20 Remarks: (1) The Mittag-Leffler problem is, given Laurent tails φ_i when does there exist a meromorphic function $\varphi \in H^o(\vartheta_C(D))$(i.e. $(\varphi) + D > 0$) such that $\varphi - \varphi_i$ is holomorphic near p_i?

(2) Such φ exists iff $\sum_i \mathrm{Res}_{p_i}(\varphi_i.\omega) = 0$ for all holomorphic 1-forms $\omega \in H^o(\Omega_C)$.

(3) We have by (3.4.17), $Bv = \dot{\nu} + \lambda(p)\nu$ and $\tau(B\nu) = \dot{\nu} \in V \otimes L/\mathbb{C}\nu$ and since v and $\dot{\nu}$ are holomorphic around D, the function A on C induces a section of $\vartheta_D(D)$ over C.

(4) A fundamental invariant of the Lax equation (3.4.1) given by *the residue p(B)* of B is the section of $\vartheta_D(D)$ induced by A on C.

3.4.21 Theorem: Referring to Lax equation (3.4.1) we have

$$\dot{L} = \frac{d}{dt}L_t\bigg|_{t=0} = \partial(\rho(B)) \tag{3.4.22}$$

Proof: By commutative diagram (3.4.16), let $E \in H^o(V \otimes L(D))$ satisfy $\tau(E) = i(\omega)$ for some $\omega \in H^o(f^*\Theta_{\mathbf{P}V})$. In particular, take $E = \frac{B\nu}{\xi_o^N}$ and $\omega = \dot{\nu}$ as in Theorem (3.4.13). Then by commutativity $TJ(E) = J\tau(E) = j(i(\omega)) = 0$ and so $\exists$ A $\in H^o(\vartheta_D(D))$

such that $\sigma(\lambda) = j(E)$ and since $H^o(\vartheta_D(D))$ occurs in the top right corner as well as bottom left corner we get

$$L = \delta(\dot{\nu}) = \delta_1(\lambda) = \delta(\rho(B)).$$

3.4.23 Corollary: Let $\mathcal{L}(H^o(\vartheta_D(D))$ be the set of all Laurent tails of all functions in $H^o(\vartheta_C(D))$ and *p(B)* the residue of *B* is a function of *t*. Then $\{L_t\}$ C $\text{Pic}^d(C)$ is linear $\Leftrightarrow$ *p(B)* = 0 mod span $\{\mathcal{L}, \rho(B)\}$ in $H^o(\vartheta_D(D))$**(3.4.24).**

Proof: Note that (3.4.24) takes place in a fixed vector space $H^o(\vartheta_D(D)) \cong \mathbb{C}^k$, $k = \deg D$. By (3.4.22) $\{L_t\}$ is linear $\Leftrightarrow$ $\ddot{L}_t = \mu_t L_t$ is equivalent to $\ddot{L} = \partial \dot{\rho}(B)) = \partial\{\mu\rho(B) + \varphi_i\} = \mu L$.

3.4.25 Definition: $\rho_i(B) = \text{Res}_{pi} B = \rho_{p_i}(B)$ and $J(C) = H^o(\Omega_C)^*/H_1(C, \mathbb{Z})$. Hence a linear flow on *J(C)* is given by, upto a fixed translation a bilinear map

$(t, \omega) \to t <\lambda, \omega >$ (3.4.26) where A $\in H^o(\Omega_C)^*$.

3.4.27 Corollary: The condition (3.4.24) is equivalent to

$\sum_i \text{Res}(\dot{\rho}_i(B)\omega) = \mu \sum_i \text{Res}_{p_i}(\rho_i(B)\omega)$(3.4.28) for all ω € $H^o(\Omega_C)$. Then the linear flow is given by $(t, \omega) \mapsto t \sum_i \text{Res}_{p_i}(\rho_i(B)\omega)$ (3.2.29).

We apply this to our systems in examples:

3.4.30 Example: Motion of rigid body about a fixed point we have seen above, can be put in Lax form with $B - -(\Omega + J\xi)$. Then $D = \sum_{i=1}^m p_i$ is the divisor with p_i being the *n* distinct points lying over ξ = oo. If z_i is a local co-ordinate around p_i (say $z_i = \xi^{-1}$), then from (3.4.17), $Bv = v + \lambda(p)\nu$ and taking $B = -(\Omega + J\xi)$ we get the residue *p(B)* = - $\sum_i \frac{\lambda_i}{z_i}$ where $J = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$. Then $\dot{\rho}(B) = 0$ so that the flow is linearized on $J(C)$.

3.4.31 Example: Consider the Lagrange top. Then its spectral curve is a 2-sheeted covering $\pi : C \to$ IP $= \mathbb{P}^1(\xi)$ branched over 4 points ξ_ν with all $\xi_\nu \neq$ oo. Since $B(\xi) = \Omega + \chi\xi$,we have $D = \pi^{-1}(\infty) = p + q$ where $\eta = \frac{\alpha+\beta}{\xi} + \cdots$ **near** *p* and

$\eta = -\frac{(\alpha+\beta)}{} + \ldots$ near q. Then the residue $p(B) \in H^o(\vartheta_D(D))$ is given by $p(B) = \frac{1}{\xi} + \cdots$ near p and $\rho(B) = --\!\!\dashv$——near q. Hence $p(B) = 0$ and hence the flow is linearized on $J(C)$.

3.4.32 Example: The Toda lattice has in Lax form a Laurent parameter. Consider the hyperelliptic curve C which is a 2-sheeted covering $C \rightarrow \mathbb{P}^1 = \eta-$ sphere. Set $\eta^{-1}(\infty) = p + q$. From the affine equation of C, $\xi + \xi^{-1} + P(\eta) = 0$, it follows that the divisor $D = np - nq$ (the poles of ξ occur on one sheet lying over a neighbourhood of ∞ and the zeros on the other sheet, and each has multiplicating n). Hence, setting $D = np + nq$, it follows that $B \in H^o(C, \text{Hom}(V, V(D))$. Then we may define the residue $p(B) \in H^o(\vartheta_D(D))$. Then we have, $p(B) = \frac{\eta}{\xi}$ and so $p(B) = 0$. Hence the Toda system is linearized on $J(C)$ where dimension $J(C) = g - n - \mathbf{1}$.

3.4.33 Example: Geodesics on Ellipsoid: The spectral curve C_o in the affine form is given by $Q(\xi, \eta) = \det \| \eta I - \xi^2\alpha - \xi\Gamma_{xy} + \Gamma_{xx} \| = 0$ which is smooth over $\xi = \infty$. Hence the map $C \rightarrow \mathbb{P}$ from the normalization of C_o is not modified over $\xi = \infty$. Let $D = \sum_{i=1}^{n} p_i = \pi^{-1}(\infty)$ and $\eta_i = \frac{\lambda_i}{z_i^2} + \frac{1}{z_i}$+holomorphic terms near p_i. Then near p_i, $A = A_o + \frac{A_1}{z_i} + \frac{A_2}{z_i^2}$, $B = -\frac{1}{2}A_1 - \frac{A_2}{z_i}$. Then the eigenvector ν_i satisfies $A\nu_i = \eta_i\nu_i$ and so $A_2\nu_i(p_i) = \lambda_i\nu_i(p_i)$ and $p(B) = -\sum_i \frac{\lambda_i}{z_i}$ and hence $p(B) = 0$ and hence the flow on $J(C)$ is linear.

3.4.34 Remarks: (1) This technique applies for systems which are represented in Lax form over finite dimensional Lie algebras.

(2) There is a different approach for linearizing flows on $J(C)$ of these systems in [2], [3].

Chapter 4

Adler-Kostant-Symes (AKS) principle and its application to some integrable systems

4.1 INTRODUCTION:

Among general dynamical systems there are some distinguished ones called Hamiltonian systems, namely, symplectic manifolds with a Hamiltonian vector field. Such a system is a completely integrable Hamiltonian system (CIHS) if there are many Hamiltonian functions which are in involution with respect to the Poisson structure defined by the symplectic form and these functions are independent over a dense subset of the manifold. In nature, there are several completely integrable Hamiltonian systems. For example, in [52], we discussed the periodic solutions of certain partial differential equations occurring in mathematical physics, mechanics and geometry and there is an associated Lax type equation and these periodic solutions live on the Jacobian variety *J(C)* of the spectral curve *C* associated with the Lax equations.

Kostant-Kirillov-Souriau discovered the canonical symplectic structure ω_Γ, on the orbits Γ of the co-adjoint representation of a Lie algebra [24,27]. On the otherhand, while studying certain special (partial) differential equations such as KdV, Toda **systems,** KP equation etc., it was noted that they define a certain Hamiltonian structure [2,5] **which** can be identified with the Kostant-Kirillov structure on the orbits of a suitable Lie algebra. There is a scheme developed over the years for constructing Hamiltonian systems having Lax type representations and also for finding an algebra of first integrals of this system (thus giving complete integrability of it) from the corresponding orbit structure of Lie algebras. For certain physical systems like, motion of rigid body about a fixed point, finite Toda systems in $\mathbb{R}^{2n}$ etc., this scheme works for finite dimensional Lie algebras. For certain other systems like KdV, general Toda systems, etc., for this scheme to work, infinite dimensional Lie algebras (even Kac-Moody algebras) are needed [2,16,48,61]. The main difficulty here in applying this technique for complete integrability is to find a suitable Lie algebra and direct sum decomposition, known as Adler-Kostant-Symes (AKS) geometric principle [16,27,60]. We formulate and discuss this principle in as general a form as possible and also give its application to several systems such as KdV equations, Toda system [2], Gel'fand-Dikii systems [16] and Lagrange Top [48].

The material in this chapter is arranged as follows: In section 4.2, some preliminary definitions are given. In section 4.3, we discuss the main geometric principle of Adler-Kostant-Symes. In section 4.4, we discuss its application to various systems, both finite and infinite dimensional systems like the Toda system,the Lagrange Top and the generalized KdV equations.

4.2 SOME PRELIMINARIES:

4.2.1 Definition: By a dynamical system, we mean a C^∞-manifold with a smooth vector field (M, X). A symplectic structure on M^{2n} is given by a 2-form ω on M such that $d\omega = 0$ and $\omega^n \neq 0$ on M. Then (M^{2n}, ω) is called a symplectic manifold.

4.2.2 Remark: We can associate with every $H \in C^\infty(M)$, a vector field X_H defined by $\omega(X_H, v) = dH(v)$, $v \in T(M)$, i.e., $i_{X_H}\omega = dH$.

4.2.3 Definition: By a Hamiltonian dynamical system, we mean a symplectic manifold (M^{2n}, ω) with a Hamiltonian vector field X_H, i.e., (M^{2n}, ω, X_H). The 2-form ω defines a Poisson structure on M by $\{F,G\} = \omega(X_F, X_G)$ for $F, G \in C^\infty(M)$.

4.2.4 Remark: $(C^\infty(M), \{\ \}) \rightarrow (X(M), [,])$ gives a Lie algebra homomorphism by $\{F, G\} \mapsto [X_F, X_G]$, $F, G \in C^\infty(M)$.

4.2.5 Definition: A Hamiltonian dynamical system (M^{2n}, ω, X_H) is called *completely integrable* if $\exists$ n functions $H_1, \cdots, H_n$ on M such that $\{H_i, H_j\} = 0$ $\forall$ $i,j = 1,2, \cdots n$. That is, they are in involution with respect to $\{,\}$ and $dH_1 \wedge \cdots \wedge dH_n \neq 0$ on a dense open subset S of M or the corresponding Hamiltonian vector fields are linearly independent on a dense open subset S of M.

4.2.6 Definition: (Kostant-Kirillov-Souriau symplectic structure): Let G be a Lie group and $\mathcal{G}$ its Lie algebra. Let P be a C^∞-manifold and $\Phi : G \times P \to P$ be a smooth action of G on P. For $\xi \in \mathcal{G}$, let $\xi_P(p) = \frac{d}{dt}\Phi(\exp t\xi, p)|_{t=o}$, $p \in P$, $t \in \mathbb{R}$, denote the infinitesimal generator of this action corresponding to ξ. Let $G.p = \{\Phi(g,p)/g \in G\}$ denote the G-orbit through $p \in P$ and $T_p(G.p) = \{\xi_P(p)|\xi \in \mathcal{G}\}$, the tangent space to the orbit $G.p$ at p.

(a) We define the Adjoint action: $G \times \mathcal{G} \to \mathcal{G}$. It is given by $g \to Ad_g = (R_{g^{-1}} o L_g)$, $\eta \mapsto (R_{g^{-1}} o L_g)\eta$ and the infinitesimal generator is $\xi_{\mathcal{G}} = (ad\ \xi) : \eta \mapsto (ad\ \xi)\eta = [\xi, \eta]$, $\xi \in \mathcal{G}$.

(b) We define the co-adjoint action $G \times \mathcal{G}^* \to \mathcal{G}^*$: It is given by $g \mapsto Ad^*_{g^{-1}} : \mathcal{G}^* \to \mathcal{G}^*$ and the infinitesimal generator is $\xi_{\mathcal{G}^*} = -(ad\ \xi)^*$, i.e., $G \times \mathcal{G}^* \to \mathcal{G}^*$ given by $g \mapsto Ad^*_{g^{-1}}$: $\mathcal{G}^* \to \mathcal{G}^*$, $\beta \mapsto (Ad^*_{g^{-1}}\beta)\eta = \beta(Ad_g\eta)$, $g \in G, \beta \in \mathcal{G}^*, \eta \in \mathcal{G}$.

For $\mu \in \mathcal{G}^*$, the co-adjoint orbit $G.\mu$ of μ is $\{Ad^*_{g^{-1}}\mu | g \in G\} \subset \mathcal{G}^*$. Then the Kostant-Kirillov-Souriau symplectic structure on the co-adjoint orbit $G.\mu$ is defined by

$\omega_\mu(\beta)(ad(\xi_1)^*\beta, ad(\xi_2)^*\beta) = \beta([\xi_1, \xi_2])$, $\xi_1, \xi_2 \in \mathcal{G}$, $\beta = Ad^*_{g^{-1}}\mu \in G.\mu$. This ω_μ is a non-degenerate closed 2-form which is $\mathcal{G}$-invariant and defines a symplectic structure on $G.\mu$ called the Kostant-Kirillov-Souriau symplectic structure.

4.2.7 Remarks: (1) Note that the tangent space at $\beta \in G.\mu$ is given by

$T_\beta(G.\mu) = \{(ad\xi)^*\beta | \xi \in \mathcal{G}\}$

(2) The most important property of this structure is that the natural Poisson structure on $\mathcal{G}^*$ restricted to the orbit O agrees with the Poisson structure restricted to the orbit O

with respect to ω_μ. i.e., $\{f,g\}|_\theta = \{f|_\theta, g|_\theta\}_{\omega_\mu}$ where $f, g \in C^\infty(U, \mathcal{G}), U \subset \mathcal{G}^*$ is an open subset.

4.3 ADLER-KOSTANT-SYMES (AKS) GEOMETRIC PRINCIPLE:

In this section we discuss the AKS geometric principle for completely integrable Hamiltonian systems.

4.3.1 Theorem: Let L be a Lie algebra equipped with a non-degenerate, ad-invariant (symmetric) bilinear form <, > so that L and L^* can be paired. Suppose $L = K \oplus N$ has a direct sum decomposition (as a vector space) where K and N are Lie subalgebras. L and L^* can be identified via <, >. Then $L^* = K^\perp \oplus N^\perp$ where $K^\perp$ and $N^\perp$ are the orthogonal complements of K and N in L with respect to <, >. By the non-degeneracy of <, >, we identify $K^\perp \approx N^*$ = dual of N and $N^\perp \approx K^*$ =dual of A'. Thus, $K^\perp \sim N^*$ gets co-adjoint N-action and hence gets the Kostant-Kirillov-Souriau orbit symplectic structure on $K^\perp$ with respect to N. Let $\Gamma \subset K^\perp$ be the orbit invariant manifold under the co-adjoint action of N and hence Γ has the Kostant-Kirillov symplectic structure. On the otherhand, we have the co-adjoint action of the connected Lie group associated with L on L^* and hence we have L-invaraint functions defined on L^* or on neighbourhoods of $\Gamma \subset K^\perp$ on L^*. Let $\mathcal{A}(\Gamma)$ be the algebra of functions defined on an neighbourhood of Γ on L^* which are invariant when restricted to Γ under the co-adjoint action of L. Then this principle asserts that

(a) The algebra $\mathcal{A}(\Gamma)$, forms a system of commuting integrals on Γ and on the orbits of Γ themselves via the orbit symplectic structure, i.e., there are functions H on L^* such that $[\nabla H(A), A] = 0 \ \forall A \in \Gamma \subset K^\perp$ where ∇H is the gradient function $\nabla H : L^* \to$ I, $H : U \subset L^* \to L$. That is, these functions H regarded as functions on Γ by restriction to Γ form an involutive system of Γ-invariant functions with respect to orbit symplectic structure induced on $\Gamma \subset K^\perp$ by the N-action.

(b) Moreover, for such functions $H \in \mathcal{A}(\Gamma)$, the associated Hamiltonian vector field X_H

is given by the formula,

$$X_H(A) = (ad_B)^* A = [B, A] = -[A, B] = [A, P_K \nabla H(A)],$$

$$B = -P_K \nabla H(A) = -\nabla_{N^\perp} H\Big|_{N^\perp}(A) \in \Gamma \qquad (4.3.2)$$

where P_K is the projection onto K along N. The dual $*$ is taken with respect to $<,>$.
(c) the Hamiltonian equation $A = X_H(A)$has the following Lax form, $A = X_H(A) = (ad_B)^* A$, $B = -P_K \nabla H(A)$In the above, we have in general that P_K, P_N, $P_{K^\perp}$, $P_{N^\perp}$ are respectively the projections onto $K, N, K^\perp, N^\perp$along $N, K, N^\perp, K^\perp$respectively.

Proof: We give an outline of the proof of the above theorem. We observe that if H is a function on $L^*(\approx L)$, and if $\nabla_{K^\perp} H$, $\nabla_{N^\perp} H$, ∇H are the gradients of H in the $K^\perp$, $N^\perp$, L^* directions, respectively, then we have,

$$\nabla_{K^\perp} H = P_N \nabla H, \quad \nabla_{N^\perp} H = P_K \nabla H.$$

By the above formula, an L^* function being L invariant on Γ is equivalent to

$$[\nabla H(A), A] = 0, \; A \in \Gamma, \text{ or equivalently, } [A, \nabla_{K^\perp} H] = -[A, P_K \nabla H]. \qquad (4.3.3)$$

If // and F are functions on TV*, then the Kostant-Kirillov Poisson bracket has the form

$$\{H, F\}(A) =< A, [\nabla_{N^*} H, \nabla_{N^*} F] >, \quad A \; e \; NT,$$

where $\ll \, , \gg$ is the natural pairing between N and N^*, and where $\nabla_{N^*} H \in N$ is the natural gradient of H defined by $dH(X) \equiv \ll dX, \nabla_{N^*} H >$; so in our case $K^\perp \approx N^*$ and $< , > = < , > |_{K^\perp \times N}$. Hence

$$\{H, F\}(A) =< A, [\nabla_{K^\perp} H, \nabla_{K^\perp} F] > . \qquad \textbf{(4.3.4)}$$

Suppose H, F € $\mathcal{A}(\Gamma)$, then

$$\{H, F\}(A) = < A, [\nabla_{K^\perp} H, \nabla_{K^\perp} F] >$$

$$= \ < [A, \ \nabla_{K^\perp} H], \ \nabla_{K^\perp} F> \quad \text{(by the ad-invariance of } < \ , \ > \text{)}$$
$$= \ - < [A, \ P_K \nabla H], \nabla_{K^\perp} F > \quad \text{(by (4.3.3))}$$
$$= \ - < A, \ [P_K \nabla H, \ \nabla_{K^\perp} F] >$$
$$= \ < A, \ [P_K \nabla H, \ P_K \nabla F]> \quad \text{(by repeating the argument for F)}$$
$$= \ 0$$

since $A \in K^\perp$ and K is a Lie algebra. The Hamiltonian vector field equals

$$X_H(F) = \{F, H\} = < [\nabla_{K^\perp} H, \ A], \quad \nabla_{K^\perp} F >,$$

from which it follows that $X_H(A) = P_{K^\perp}[\nabla_{K^\perp} H, A]$. Hence the corresponding Hamiltonian flow

$$\begin{aligned} \dot{A} &= P_{K^\perp}[\nabla_{K^\perp} H, \ A] \quad (\text{for } H \in \mathcal{A}(\Gamma)) \\ &= P_{K^\perp}[A, \ P_K \nabla H] \qquad (\text{by} \quad (4.3.3)) \\ &= [A, \ P_K \nabla H], \qquad \text{as } [K^\perp, \ K] \subset K^\perp, \end{aligned}$$

which proves (4.3.2) and thus the theorem is proved.

4.3.5 Remark: If the bilinear form <, > is symmetric also, for example, trace or killing form, then $(ad)_{B^*} = -(ad)_B$ and $A = X_H(A) = -(ad)_B A = -[B, A] = [A, B] = [A, -P_K \nabla H(A)]$ which is the usual Lax equation.

This principle was noticed in several special cases before and we have stated above, this principle in the most general set up of a Lie algebra $(\mathcal{G}, K$, TV, <>).

4.4 EXAMPLES:

In this section, we discuss the application of the above AKS geometric principle to **the** finite-dimensional systems, (a) the Toda system (b) the Lagrange Top, (c) the geodesic motion on an ellipsoid and the infinite dimensional systems, (d) the generalized KdV equations and (e) the Gel'fand-Dikii system.

4.4.1 The Toda System: The Toda system is given in Chapter 2. Here we discuss the

AKS setup for this system.

Let $\mathcal{G} = sl_n(\mathbb{R})$ be the Lie algebra of real traceless matrices. Let a be the Lie subalgebra of lower triangular matrices and b that of skew-symmetric matrices:

$$\begin{aligned} \mathsf{a} &= \{a \in sl_n(\mathbb{R}) | a_{i,j} = 0 \text{ for } j > i\} \\ \mathsf{b} &= \{b \in sl_n(\mathbb{R}) | b_{j,i} = -b_{i,j}\} \end{aligned}$$

The vector space $\mathcal{G}$ is the direct-sum of a and b: if $X = (x_{i,j}) \in sl_n(\mathbb{R})$, then $X = X_+ + X_-$, where $X_+ = (y_{i,j}) \in$ a and $X_- = (z_{i,j}) \in$ b are defined by

$$y_{i,j} = \begin{cases} x_{i,j} + x_{j,i} & j < i \\ x_{i,j} & j = i \\ 0 & j > i \end{cases}$$

and

$$z_{i,j} = \begin{cases} -x_{i,j} & j < i \\ 0 & j = i \\ x_{i,j} & j > i \end{cases}$$

The non-degenerate symmetric bilinear form $< X, Y >= \mathrm{tr}(XY)$is invariant under conjugation (this is the killing form). We have that $\mathsf{a}^\perp$ is the subspace of strictly lower-triangular matrices and that $\mathsf{b}^\perp$ is that of symmetric matrices.

We now consider the invariant function $f(L) = \frac{1}{2}\mathrm{tr}(L^2)$, which is defined on $\mathcal{G}$, and the Hamiltonian system it defines on a* $= \mathsf{b}^\perp$. Since $df_L(X) = \mathrm{tr}(LX)$, $\nabla_L f = L$. For $L = (x_{i,j}) \in \mathsf{b}^\perp$, a symmetric matrix, the projection $(\nabla_L f)_- = L_-$ onto the subspace of skew-symmetric matrices is

$$(L_-)_{i,j} = \begin{cases} -x_{i,j} & j < i \\ 0 & i = j \\ x_{j,i} & j > i \end{cases}$$

By AKS theorem, the system is $L = [L, M]$ where L is a symmetric traceless matrix and M is **the** skew-symmetric matrix whose **entries above the diagonal are those of L,** which can be understood as a Hamiltonian system and admits all the functions $L \mapsto \mathrm{tr}(L^k)(2 < k < n)$ as pairwise commuting first integrals. We thus have $n - 1$ integrals. We now consider the space $\Gamma \subset \mathbf{b}^{\perp}$. The Lie group A is that of invertible lower-triangular matrices and its Lie algebra a is the group of lower triangular matrices. By using the bilinear form, $< X, Y >= \mathrm{tr}(XY)$, we can identify the dual of a = $\mathbf{a}^*$ with the upper triangular matrices. A acts on a by conjugation and on a* by duality i.e., $A \times a^* \longrightarrow a^*$, $g, l^* \mapsto [g^{-1}l^*g]_+$ where $[\ ]_+$ is the operator of setting all terms below the diagonal equal to zero. Then the orbit $\theta = \theta_{l^*} = \{[g^{-1}l^*g]_+/g \in A\}$ and its tangent space is $T_{l^*}\theta_{l^*} = \{[l^*, l]_+/l \in \mathrm{a}\}$. Then the Kostant-Kirillov Souriau symplectic structure on θ_{l^*} is given by

$$\begin{aligned} \omega([l^*, l_1]_+ , [l^*, l_2]_+) &= < l^*, [l_1, l_2] > \\ &= < [l^*, l_1]_+, l_2 > \end{aligned}$$

Let $A = [A_{ij}]$ be a variable on a* and let $H = H(A) = H(A_{ij})$ be a function on a*. We can identify the gradient at A of $H = H(A)$ with respect to $<, >$, $\nabla H/A$ as an element of a $(\nabla H: \mathbf{a}^* \rightarrow \mathrm{a})$. The Hamiltonian vector field Y_H associated to H is given by $Y = Y_H = [\nabla H, A]_+$. Then $\mathcal{A}(\Gamma) = \{H : \mathbf{a}^* \rightarrow \mathbf{a}/[A, \nabla H]_+ = 0\ \forall A \in \theta_B\}$ is the algebra of a-invariant functions defined on the orbit θ_B, $B \in$ a. Then the functions H in A are the integrals of motion for the Toda system, i.e., they are in involution with respect to ω as

$$\begin{aligned} \left\{H^{(1)}, H^{(2)}\right\}(A) &= \omega(Y_{H^{(1)}}, Y_{H^{(2)}})(A) \\ &= (A, [\nabla H^{(1)}, \nabla H^{(2)}]) \ . \end{aligned}$$

4.4.2 Remark: This can be generalized to m-ogonal (m > 1) systems having Lax type equations and these systems will also fit into the AKS setup.

4.4.3 The Lagrange top: Motion of a 3-dimensional rigid body about a fixed point under the influence of gravity. This example is given in Chapter 2 and we discuss here its Lie algebra setup and the AKS setup.

(i) **Lie** algebra setup: Let $\mathcal{G}$ be the Lie algebra of dimension 6 which is a semi-direct product of $V(=\mathbb{R}^3)$ and $so(3)$($i.e.$,$so(3)$ x $so(3)$ = $\mathcal{G}$). Associating to the unitary vector field 7, the corresponding instantaneous rotation vector $\gamma(t)$ for time t, we get a pair $(M(t),\gamma(t))$ $\forall$ t and $M(t)\in V$. We denote this as an element $\gamma+$ eM $(\epsilon^2 = 0)$ in $\mathcal{G}$ — $V\oplus so(3)$. Then the equations of motion, (4.4.4) and (4.4.5),

$$\begin{aligned} \dot{M} &= M\times\Omega+\mu g\gamma * l \\ \dot{\gamma} &= \gamma\times\Omega \end{aligned} \tag{4.4.4}$$

and

$$\begin{aligned} \dot{M} &= [M\times\Omega]+[\gamma,l] \\ \dot{\gamma} &= [\gamma,l] \end{aligned} \tag{4.4.5}$$

can be written as

$$(a)\ \ \dot{\overline{\gamma+\epsilon M}} = [\gamma+\epsilon M,\Omega+\epsilon l],\quad (b)\ \ M = I(\Omega) \tag{4.4.6}$$

Since $\gamma\neq 0$, the orbits of 7 $+\epsilon M$, θ_μ under the adjoint $\mathcal{G}$-action are subvarieties of dimension 4 in $\mathcal{G}$ defined by the equations:

$$\theta_c \quad : \quad \begin{cases} <\gamma,\gamma>=1 \\ <\gamma,M>=c\ ;\ \text{constant} \end{cases}$$

Then the functions (7 + eM) $\mapsto<$ 7,7 $>$ and (7 + eM) $\mapsto<$ 7, M $>$ are constants on the orbits of the trajectories called the first integrals. The total energy of the system is given by

$$H_1(\gamma,M)=\frac{1}{2}<\Omega,M>+<\gamma,l>$$

(ii) Three special cases:

(a) Centre of rotation O is at the centre of gravity G of S. In this case, we get the classical Euler top (1750) which is completely integrable and $H_2 < 7, M > = < M, M >$; $I — \lambda_{id_V}$.

(b) S possesses a symmetric axis of rotation passing through the centre of rotation. Then we get the classical Lagrange top (1788) with complete integrability and $H_2 = < M, l >$.

(c) There exists an orthonormal basis for V where $/ = \mathrm{diag}(2\lambda, 2\lambda, \mathrm{A})$, $/ = (x_o, 0, 0)$. Then we get the classical Kowalesvsky top (1881) which is also completely integrable and

$$H_2(\gamma, M) = ||(m_1 + im_2)^2 - 4\lambda x_o(\gamma_1 + i\gamma_2)||^2 .$$

(iii) AKS setup for Lagrange top:

Let $L = so(V)[h^{-1}, h] = \left\{ \sum_{-\infty}^{m} A_i h^i / A_i \in so(V),\ m \text{ arbitrary, finite} \right\}$ be a Lie algebra with bracket defined by

$$\left[\sum_i A_i h^i, \sum_j B_j h^j \right] = \sum_p \left(\sum_{i+j=p} [A_i, B_j] \right) h^p .$$

Then $(L, [,])$ is called the Kac-Moody algebra associated to $so(V)$. Let

$$L^+ = K = so(V)[h] = \left\{ \sum_{i=o}^{m} A_i h^i / A_i \in so(V),\ m \text{ arbitrary} \right\}$$

$$L^- = N = \left\{ \sum_{-\infty}^{-1} A_i h^i / A_i \in so(V) \right\} .$$

Then $L = K \oplus N$. Define bilinear form $<,>$ on L by $\langle \sum_{-\infty}^{m} A_i h^i , \sum_{-\infty}^{l} B_j h^j \rangle = \sum_{i+j=-1} < A_i, B_j >$. Then this bilinear form $<,>$ is symmetric, non-degenerate, Ad_L-invariant bilinear form on L. Then with respect to $<,>$, we have $K = K^\perp$, $N = N^\perp$ and the dual of $K = K^*$ can be identified with N (i.e., $K^* \sim N$). Let the projection $L \to K = L^+$ be denoted by $X \to X_+$ onto K along N. Then the subalgebra N acts on $K^\perp = K$ by $(n, X) \mapsto [n, X]_+$. The subspace $E = \{\sum_{i=o}^{2} A_i h^i / \text{polynomial of degree 2 over } so(V)\} \subset K$ is a vector subspace of K of dimension

9 which is stable under the above action of N.

The subalgebra $N_3 = \left\{\sum_{-\infty}^{-3} A_i h^i\right\}$ of N acts trivially on E. Passing to the quotient algebra $N = N/N_3$ action on E (N is 6-dimensional nilpotent Lie algebra), let Γ be the orbit in E under N which is a 4-dimensional variety of E. Then the algebra of integrals of motion can be found as follows: Imbed L as a subalgebra of $\mathrm{End}(V)[h^{-1}h]$. Then V $n \in N$, V $A \in L$, define $A^n = \sum_{-\infty}^{\infty} c_{n,k}(A)h^k$, $c_{n,k}(A)$ € End(V). The functions $A \mapsto f_{n,k}(A) = tr(c_{n,k}(A))$ are Ad_L-invariant. Consider the orbit of A under N such that $A_2 = XL$ i.e., $A = A_0 + A_1 h + A_2 h^2 \in E$. The first integrals are given by

$$
\begin{aligned}
f_{2,0}(A) &= -2 < \gamma, \gamma > \\
f_{2,1}(A) &= -4 < M, \gamma > \\
f_{2,2}(A) &= -2 < M, M > -4\lambda < \gamma, l > \\
f_{2,3}(A) &= -4\lambda < M, l >
\end{aligned}
$$

4.4.7 Geodesics on an ellipsoid: This example is given in Chapter 2. Here we discuss the AKS setup for this system.

Let

$$\mathcal{L} = \{A = \sum_{-\infty < i \leq N} A_i h^i / N \ \text{arbitrary}, A_i \in \mathcal{M}\}$$

be the affine Lie algebra where $\mathcal{M}$ is the algebra of n x n matrices. The elements of £ are viewed as Laurent series in the indeterminate h and h^{-1}. The bracket in £ is defined as follows:

$$[\sum_i A_i h^i \, , \, \sum_j B_j h^j] = \sum_{i,j} [A_i, B_j] h^{i+j} \tag{4.4.8}$$

The ad-invariant, non-degenerate, symmetric bilinear form on £ is given by

$$\langle \sum_i A_i h^i \, , \, \sum_j B_j h^j \rangle = \sum_{i+j=0} (A_i, B_j) \tag{4.4.9}$$

where (,) is the killing form on $\mathcal{M}$ with the property

$$< A, B >_k = < A, Bh^k > = \sum_{i+j=-k} (A_i, B_j) \tag{4.4.10}$$

and $< A, B >_0 = < A, B >$ and $\mathcal{A} = \{\sum_{j \leq i \leq k} A_i h^i / A_i \in \mathcal{M}\}$. In Theorem 4.3.1, we set $L = \mathcal{L}$, $\mathcal{M} = \mathcal{G}l(n, \mathbb{R})$ or $\mathcal{G}l(n, \mathbb{C})$ and $<,>_L = <,>_1$ of (4.4.10). Also, let

$$K = K^\perp = \mathcal{A}_{0,\infty} = \sum_{j=0}^{\infty} \mathcal{L}^j, \quad N = N^\perp = \mathcal{A}_{-\infty,-1} = \sum_{j=-\infty}^{\infty} \mathcal{L}^j$$

and

$$\mathcal{A}_{0,\infty} = \{ \sum_{0 \leq i \leq \infty} A_i h^i / A_i \in \mathcal{M}\} \tag{4.4.11}$$

so that

$$P_K(\sum_i A_i h^i) = \sum_{i \geq 0} A_i h^i \equiv (\sum_i A_i h^i)_+, \quad P_N(\sum_i A_i h^i) = \sum_{i<0} A_i h^i \equiv (\sum_i A_i h^i)_-.$$

For the invariant manifold Γ, we take

$$\Gamma = \Gamma_m(\alpha, \gamma) = \alpha h^m + \gamma h^{m-1} + \mathcal{A}^0_{0,m-1} \tag{4.4.12}$$

with $\alpha = \text{diag}(\alpha_1, \cdots, \alpha_n)$, $\prod_{i<j}(\alpha_i - \alpha_j) \neq 0$, γ a diagonal matrix and $\mathcal{A}^0_{0,m-1} = \{\sum_{0 \leq i \leq m-1} A_i h^i / \text{diag}(A_{m-1}) = 0\}$. The terms α, γ are the orbit invariants or parameters. For the algebra $\mathcal{A}(\Gamma)$ we shall take functions of the form $H = < f(Ah^{-j}), h^k >_1$ so that equation (4.3.2) becomes

$$\dot{A} = [A, (f'(Ah^{-j})h^{k-j})_+], \quad A = \alpha^m + \gamma h^{m-1} + \sum_{0 \leq i \leq m-1} A_i h^i \tag{4.4.13}$$

The group G_N is the affine group, $G_N = I + N$, so that if $I - x \in G_N$, then $(I - x)^{-1} = I + x + x^2 + \cdots$. Then N is given by the Lie algebras $N_k = \mathcal{A}_{-k-1}$ with the bracket defined by

$$[\sum_{i=-k}^{-1} A_i h^i, \sum_{j=-k}^{-1} B_j h^j] = \sum_{i+j=-k}^{-1} [A_i, B_j] h^{i+j},$$

and the corresponding Lie group is $G_{N_k} = I + N_k$, with multiplication

$$\sum_{i=-k}^{0} A_i h^i . \sum_{j=-k}^{0} B_j h^j = \sum_{i+j=-k}^{-1} A_i B_j h^{i+j}, \quad A_0 = B_0 = I$$

The orbits θ_A through $A \in \Gamma_m$ are of the form

$$\theta_A = \{(g^{-1}Ag)_+ / g \in I + N\},$$

since only the non-negative terms in $g^{-1}Ag$ register in $(g^{-1}Ag)_+$, the orbit is equal to

$$\theta_A = \{(g^{-1}Ag)_+ / g \in G_{N_{m+1}}\} = \{(g^{-1}Ag)_+ / g \in G_{N_m}\}, \tag{4.4.14}$$

and

$$T\theta_A = \{[A, B]_+ / B \in N_m\}.$$

Since $A \in \mathcal{A}_{0,m} \approx N^*_{m+1}$, under $<\ ,\ >_1$, we can take A to be in the dual of N_{m+1} and so θ_A is identified with co-adjoint orbit of the finite dimensional Lie group $G_{N_{m+1}}$. Using $\nabla_{K^\perp} H = \sum_{j\geq 0} h^{-j-1} \nabla_{A_j} H$, the Poisson bracket is given by

$$\{H, F\} = \sum_{i=j+k+1} (A_i, [\nabla_{A_j} H, \nabla_{A_k} F]), \quad \text{with } (A, B) = \text{trace } A.B.$$

4.4.15 Adler-Kostant-Symes setup for the generalized KdV Equations: Let R be a commutative ring over complex numbers equipped with a derivation D. That is, R is a differential ring. We define the infinite integrals / as / = R/DR. Let $\pi : R \longrightarrow R/DR$ — I denote the projection associated with /, defined by $\phi \mapsto \phi$ (the /-equivalence class of ϕ) and the equality in / is denoted by $=$.

Let $R[b_o, b_1, \cdots, b_n]$ C R denote the ring of polynomials in $b_o, \cdots, b_n$ and their derivatives. Similarly for $I[b_o, b_1, \cdots, b_n]$. We define Φ, the ring of formal pseudo-differential operators to be the formal Laurent series in the variable ξ over the differential ring R. i.e., $\Phi = \left\{ \phi = \sum_{-\infty < i \leq N < \infty} a_i \xi^i / a_i \in R, N \text{ arbitrary} \right\}$ with multiplication defined by

$$\phi_1 \circ \phi_2 = \sum_{\nu \geq 0} \frac{1}{\nu!} (\partial\xi)^\nu \phi_1 \circ (-iD)^\nu \phi_2, \quad \text{for} \quad \phi_1, \phi_2 \in \Phi$$

We define $A_{l_j} = \left\{ \sum_{s=l}^{j} a_s \xi^s / a_s \in R \right\}$ and projections $P_{l,j}$ into $\mathcal{A}_{l_j}$ by $P_{lj}\left(\sum_{-\infty \le i < N < \infty} a^i \xi^i \right) = \sum_{l \le i \le j} a_i \xi^i$. We also define the trace functional $<,>: \Phi \to I$ by

$$\text{tr} \;\; \phi \;\equiv < \phi > \;\doteq \bar{a}_{-1} \in I \;,\;\; \phi = \sum a_i \xi^i \;.$$

The trace functional satisfies $\text{tr}[a,b] \doteq 0$ where $[a,b] = a \circ b - b \circ a$ and with this, **we have** the symmetric inner product

$$< a, b > \equiv < a \circ b > = < b \circ a > = < b, a > \quad .$$

Let $K = \mathcal{A}_{o,\infty} = \left\{ \phi = \sum_{i=0}^{\infty} a_i \xi^i / a_i \in R \right\}$ and $N = \mathcal{A}_{-\infty,-1} = \left\{ \phi = \sum_{-\infty}^{-1} a_i \xi^i / a_i \in R \right\}$. Then from the above two properties of trace, we have $K^\perp = K$, $N^\perp = N$ and $\Phi = K + N$. The maps $K \hookrightarrow \text{Hom}(N, I)$, $N \hookrightarrow \text{Hom}(K, I)$ defined by $K \mapsto < K, . >$ are injections and K, N are dual with respect to $<,>$ and Φ being self-dual, we have $K^* = N$, $N^* = K$ and $\Phi^* = \Phi$. We have $\Phi = K + N$, K and N are dually paired, Φ self dual, $K^\perp = K$, $N^\perp = N$. The formal Lie algebra Φ acts on $\Phi^*(\sim \Phi)$ by the co-adjoint action and we can speak of Φ-invariant functions in $\Phi^*(\sim \Phi)$. Considering $\Phi^{(n)} = \left\{ \phi = \xi^n + \sum_{0 \le i \le n-2} a_i \xi^i / a_i \in R \right\}$ as being parametrized by $a_o, a_1, \cdots, a_{n-2}$, we let $G = 1 + A_{-\infty,-1}$ be the formal Lie group with formal Lie algebra $\mathcal{L} = \mathcal{A}_{-\infty,-1}$. Let $\mathcal{L} = \left\{ \sum_{j \ge 0} (\xi - iD)^{-j-1} a_j / a_j \in R \right\}$ where

$$(\xi - iD)^{-j} b \equiv \sum_{\nu \ge 0} \xi^{-j-1-\nu} \begin{pmatrix} j + v \\ v \end{pmatrix} (iD)^v b \;,\; j \ge 0.$$

Let the dual of $\mathcal{L}$, $\mathcal{L}^* \hookrightarrow \text{Hom}(\mathcal{L}, I)$ be the differential operators

$$\mathcal{L}^* \;=\; \left\{ \sum_{\infty > n \ge i \ge 0} a_i \xi^i / a_i \in R \right\} = \mathcal{A}_{o,\infty} \qquad \text{and if} \qquad A = \sum_{i \ge 0} a_i \xi^i \in \mathcal{L}^* \;,$$

$$B \;=\; \sum_{j \ge 0} (\xi - iD)^{-j-i} b_j \in \mathcal{L} \qquad \text{then} \qquad < A, B > \doteq \sum_{i \ge 0} \overline{a_i b_i} \;.$$

Let θ_B be the orbit through B of the co-adjoint action of G on $\mathcal{L}^*$, i.e., $\theta_B = \{[g^{-1}Bg]_+/g \in G\}$, where + denotes the projection into $\mathcal{A}_{o,\infty}$. If $B \in \mathcal{A}_{o,m}$, then so also does A (i.e., $A \in \theta_B$) and we write $A = \sum^{m} a_i \xi^i$. Then the (formal) Kostant-Kirillov symplectic form ω on θ_B at the point A is given by $\omega([A, l_1]_+, [A, l_2]_+) \doteq < A, [l_1, l_2] > = < [A, l_1]_+, l_2 >$ with $l_1, l_2 \in \mathcal{L}$, and so $[A, l_1]_+ \in T_A\theta_B$. The tangent space of θ_B at B is given by $T_B\theta_B = \{[B, l]_+ | l \in \mathcal{L}\}$.

Let $H = H(A) = P(a)$, where $P(a)$ is a polynomial in a_i, $i = 0, 1, \cdots, m$ and its derivatives. Then the Hamiltonian vector field X_H induced by ω is given by $X_H = [\nabla H, A]^{m-2}(*)$ where $[\ \ ]^m = P_{o,m}[\ \ ]$ and $VH = \sum^{m}(\xi - iD)^{-j-1}\frac{DH}{Da_i}$ Here $\frac{DH}{Da_j}$ is the formal variational derivative of $H = //(a)$ with respect to a_j and the Poisson bracket $\{,\}$ based on ω is given by $\{H, F\}_{(A)} = < A, (\nabla H, \nabla F) >$. Moreover, for $A \in \Phi^{(m)}$, $(*)$ becomes the Lax isospectral equation $A = [\nabla H, A]^{m-2} = [A, P_N]$ where $P_N = (A^{N/n})_+$ and the fractional powers of A, $\{< A^{N/n} > /N\}$ form an involutive system with respect to $\{,\}$.

4.4.16 The **Gel'fand-Dikii** System: I.M. Gel'fand and L.A. Dikii generalized the Lax form of the KdV equation. That is, let $L = (-i\partial_x)^n + \sum_{j=0}^{n-2} q_j(-i\partial_x)^j$, $q_i \in C_o^\infty(R)$, i = $0, 1, 2, \cdots, n-2$. Then the Lax equations $— = [L, B_j], j = 1, 2, \cdots$ with appropriate differential operators B_j are equivalent to the Hamiltonian equations $q_t = X_{H_j}(q) = I\frac{DH}{Dq}$ with $q = (q_o, q_1, \cdots, q_{n-2})^t$, $\frac{DH}{} = (\frac{DH}{Dq_o}, \cdots \frac{DH}{Dq_{n-2}})^t$ and $H = \int_R P dx$ where t denotes transpose and P is the polynomial in q_i's and their derivatives with respect to x. In the above equation, I is a $(n-1) \times (n-1)$ matrix differential operator with co-efficients polynomials in q and its derivatives. / defines a Poisson bracket $\{,\}$ via $\{H, F\} = \int_R(\frac{DH}{Dq}, I(\frac{DF}{Dq}))$ with the $\mathbb{R}^{n-1}$ scalar product, $(,)$. The coefficients of B_j's are polynomials in q and its derivatives. Also, the H_j's form an involutive system, i.e., $\{H_j, H_k\} = 0 \ \forall\ j, k \in \mathbb{Z}$.

For the AKS setup for this system, we let $\Phi = \Phi^* = L$, $K = \mathcal{A}_{o,\infty}$, $N = \mathcal{A}_{-\infty,-1}$ **with inner** product $<, >$ defined as $< a, b > = < a \circ b > = < b \circ a > = < 6, a >$ for a, $b \in \Phi$. We have A'[1] = A', $N^{\perp}$ = AT, $(ad)^* = -(ad)$ and $\Gamma = \Phi_n = \{\xi^n + \mathcal{A}_{o,n-1}\}$ C K and $H^\nu = \mathrm{tr}L^{\nu/n} \in \mathcal{A}(\Gamma)$, $\nu = 0,1,2,\cdots$. Let $A \in \Phi_n$, $H^\nu = \mathrm{tr}A^{\nu/n}$ € $\mathcal{A}(\Gamma)$, $\nu = 0,1,2,\cdots$. Then $A = X_H(A) = [B_\nu, A]$ where $B_\nu = \frac{\nu}{n}[A^\mu]_+ = P_K \nabla H$, $\mu = \frac{\nu}{n} - 1$, where $X_H = X_H(A) = [\nabla_K H, A]_+ = [P_{(n-1),-1}(\nabla H), A]_+$. The Poisson bracket is given by $\{H, F\}(A) = < A, [\nabla_K H, \nabla_K F] >$. The functions $H^\nu (= trA^{\nu/n})$ defined on $T = \Phi_n$ are constants, i.e., they are orbit invariants.

4.4.17 Remark: In chapter 6, we discuss the application of AKS principle to Benney's **long** wave equations.

(a) Completely Integrable Hamiltonian Systems/PDE	(b) Associated Lie algebra (with its decomposition)	(c) Kostant Kirillov symplectic structure ω	(d) Orbits in Γ under co-adjoint representation
	FINITE DIMENSIONAL SYSTEMS		
1. The Toda Systems $H = H_2(x, y)$ $= \frac{1}{2}\sum_{i=1}^{n} y_i^2 + \sum_{i=1}^{n} e^{x_i - x_{i+1}}$ $x_o = x_{n+1} = 0$ with $\dot{x}_i = \frac{\partial H}{\partial y_i}$, $\dot{y}_i = -\frac{\partial H}{\partial x_i}$, $i = 1, \cdots n$	$\mathcal{L}$ = lower triangular matrices	$\omega(l^*, l_1]_+, [l^*, l_2]_+)$ $=< l^*, [l, l_2] >$ $=< [l^*, l_1]_+, l_2 >$	$\Theta = \Theta_{l^*} = \{[g^{-1}l^*g]_+/g \in G\}$, $l^* \in \mathcal{L}^*$
2. Lagrange Top $H_2 =< M, l >$	$L = so(V)[h^{-1}, h]$ $= \{\sum_{-\infty}^{m} A^i h^i / A_i \in so(V), m, arb.\}$ $K = so(V)[h]$ $= \{\sum_{o}^{m} A^i h^i / A_i \in so(V), m, arb.\}$ $N = \{\sum_{-\infty}^{-1} A^i h^i / A_i \in so(V)\}$	$\omega_\Theta(\pi_x(X), \pi_x(Y))$ $=< x, [X, Y] >$, $x \in \Theta, X, Y \in L^*$	θ is a subvariety of dimension 4 in E, $E \subset K^\perp$ of dimension 9
3. Geodesic motion on an ellipsoid, $Q_o(x) + 1 = 0$ with Hamiltonian system $\dot{x} = \partial_y \Phi_z$, $\dot{y} = -\partial_x \Phi_z$ with $\Phi_z = 0$	$\mathcal{L} = \{A = \sum_{-\infty < i \leq N} A_i^o h^i / N arb.,$ $finite, A_i \in \mathcal{M}\}$, $\mathcal{M}$-algebra of $n \times n$ matrices $K = K^\perp = \mathcal{A}_{o,\infty} = \sum_{j=o}^{\infty} \mathcal{L}^j$ $N = N^\perp = \mathcal{A}_{-\infty,-1} = \sum_{-\infty}^{-1} \mathcal{L}^i$	$\omega(\xi_X, \xi_Y)(u) =< u, [X, Y] >$, $\forall, X, Y \in \mathcal{L}, u \in \Omega$, Ω- orbit of u in $\mathcal{L}^*$ under co-adjoint reprn.	$\Theta_A = \{(g^{-1}Ag)_+/g \in G_{N_m}\}$, $A \in \Gamma_m$ where $\Gamma = \Gamma_m(\alpha, \gamma)$ $= \alpha h^m + \gamma h^{m-1} + \mathcal{A}^o_{o,m-1}$, where $\mathcal{A}^o_{o,m-1} =$ $\{\sum_{o}^{m-1} A_i h^i / deg(A_{m-1}) = 0\}$
	INFINITE DIMENSIONAL SYSTEMS		
4. Generalized KdV Equation	$\mathcal{L} = \Phi =$ $\{\phi = \sum_{-\infty \leq i < N < \infty} a_i \xi^i / a_i \in R\}$ $K = \mathcal{A}_{o,\infty} = \mathcal{L}^* = \sum_{o}^{\infty} a_i \xi^i / a_i \in R\}$ $N = \mathcal{A}_{\infty,-1}$ with $G = 1 + \mathcal{A}_{-\infty,-1}$	$\omega([A, l_1]_+, [A, l_2]_+)$ $\doteq < A, [l_1, l_2] >$ $\doteq < [A, l_1]_+, l_2 >$, $l_1, l_2 \in \mathcal{L}$, $[A, l_1]_+ \in T_A \theta_B$	$\Theta_B = \{[g^{-1}Bg]_+/g \in G_N\}$, where + denotes projection onto $\mathcal{A}_{o,\infty}$
5. Gel'fand-Dikii system	$\Phi = \Phi^* = L$ $K = \mathcal{A}_{o,\infty}$ $N = \mathcal{A}_{-\infty,-1}$		$\Theta_B = \{[g^{-1}Ag]_+/g \in 1 + N\}$ under $\Gamma = \Phi_n = \{\xi^n + \sum_{o}^{n-1} a_i \xi^i\}$

(e) Hamiltonian function, it gradient and its Poisson structure	(f) $\mathcal{A}(\Gamma)$-the algebra of commuting integrals	(g) Lax equation
FINITE DIMENSIONAL SYSTEMS		
$H = H(A) = H([A_{ij}])$ where $A = [A_{ij}] \in \mathcal{L}^*$ and $X_H = [\nabla H, A]_+$ $\{H^{(1)}, H^{(2)}\}(A) = \omega(X_{H^{(1)}}, X_{H^{(2)}})(A)$ $=< A, [\nabla H^{(1)}, \nabla H^{(2)}] >$	$\mathcal{A}(\Gamma) = \{H/[\nabla, \nabla H]_+ = 0$ $\forall\, A \in \Theta_B\}, B \in \mathcal{L}^*$	$\dot{L} = [P, L]$ where $P = \begin{bmatrix} 0 & a_1 & 0 \\ -a_1 & & 0 \\ & & a_{n-1} \\ 0 & a_{n-1} & 0 \end{bmatrix}$ $L = \begin{bmatrix} b_1 & a_1 & 0 \\ -a_1 & b_2 & \\ & & a_{n-1} \\ 0 & a_{n-1} & b_n \end{bmatrix}$
$Ham(f)(x) = -CoAdj\left(\frac{\partial f}{\partial x}, x\right)$, $x \in \Theta$ $\{f, g\}(x) =< x, \left[\frac{\partial f}{\partial x}, \frac{\partial g}{\partial x}\right] >$, $x \in U \subset L^*$	$\mathcal{A}(\Gamma) = \{A \mapsto f_{n,k}(A) = Tr(c_{n,k}(A))\}$, with $A \in L$ s.t. $A^n = \sum_{-\infty}^{\infty} c_{n,k}(A)h^k$, where $c_{n,k}(A) \in End(V)$	$\dot{A} = [A, \varphi(A)]$, where A is a polynomial in h with co-efficients in $L = so(V)$.
$H(A) =< f\varphi(A)h^k >$ where φ is a represen- -tation of $\mathcal{L}$, $\nabla_{K^\perp} H = \sum_{j \geq 0} h^{-j-1}\nabla_{A_j} H$, $\{H, F\} = \sum_{i=j+k+1}(A_i, [\nabla_{A_j} H, \nabla_{A_k} F])$, with (A,B)=tr A.B	$\mathcal{A}(\Gamma) = \{H/H =< f(Ah^{-j}), h^k >$ where $< A, B >_1 =< A, Bh >$ $= \sum_{i+j=-1}(A_i, B_j)$	$\dot{A} = [A, f'(Ah^{-j}, h^{k-j})_+]$ with $A = \alpha h^m + \gamma h^{m-1} + \sum_o^{m-1} A_i h^i$
INFINITE DIMENSIONAL SYSTEMS		
$H = H(A) \doteq P(a)$, $P(a)$-polynomial in $a_i, i = 0, 1, \cdots m$. $X_H = [\nabla H, A]^{m-2}$ $\nabla H = \sum_{j=o}^{m}(\xi - iD)^{j-1}\frac{DH}{Da_j}$, where $\frac{DH}{Da_j}$ =formal variational derivative of $H(a)$ w.r.t. a_j, $\{H, F\}(A) =< A, [\nabla H, \nabla F] >$	$\mathcal{A}(\Gamma) = \{H \in \mathcal{L}^*/[\nabla H(A), A] = 0\}$, $A \in \Gamma \subset K^\perp$	$\dot{A} = X_H(A) = (ad)^*_B A$, $B = -P_K \nabla H$, $A \in \Gamma \subset K^\perp$, $(ad)^* = -(ad)$.
$H^\nu = trL^{\nu/n}$, $\nu = 0, 1, \cdots$ $\{H, F\}(A) =< A, [\nabla_K H, \nabla_K F] >$	$\mathcal{A}(\Gamma) = \{H/H^\nu = trL^{\nu/n}, \nu = 0, 1, \cdots\}$	$\dot{A} = X_H(A) = [B_\nu, A]$, $B_\nu = \frac{\nu}{n}[A^\mu]_+ = P_K \nabla H$, $\mu = \frac{\nu}{n-1}$

Chapter 5

Geometry of the Boussinesq's equation and its relation to other systems

In this chapter, we discuss the geometry of the Boussinesq's equation and its relation with other systems, such as the Toda system and the KdV equations. In section 5.1, we give a general method of construction of the Lenard relations. In sections 5.2 and 5.3, we discuss the Hamiltonian structure of the Boussinesq's equation. In section 5.4, we give some Lax-type formulae associated with the above mentioned systems and in section 5.5, we discuss the sub-hamiltonian system which arises when the flows of the Bousssinesq hierarchy are restricted to the manifold, r = 0.

5.1 GENERAL METHOD OF CONSTRUCTION:

We describe in this section a general method to derive the Lenard relations which can be applied to systems like the Toda system, the Kac-Moerbeke system and the Boussinesq equation. We describe the method of construction in the case of the Korteweg- de Vries(KdV) equation. The KdV equation is given by

$$q_t = \partial_x(3q^2 - q_{xx}), \qquad \textbf{(5.1.1)}$$

where the domain of q is $\mathbb{R}^+$ x $\mathbb{R}$. We have the Lax isospectral deformation equation associated with the KdV equation (5.1.1), $L_t — [B, L]$, where

$$L = \text{-fig} + q(x,t), \quad B = -4\partial_x^3 + 3(q\partial_x + \partial_x q.) \qquad \textbf{(5.1.2)}$$

with L defined in some appropriate domain. As q evolves according to (5.1.1), the spectrum of L remains unchanged. We consider the formal series

$$a(\lambda) = -\sum_i log(\lambda - \lambda_i), \tag{5.1.3}$$

where $\lambda_i's$ are the discrete eigenvalues of the operator $L = -d_x^2 + q(x)$, for some fixed $q(x)$. We define $L_0 = -d_x^2$. Then

$$\frac{\partial a}{\partial \lambda} = a_\lambda = \sum_i \frac{1}{\lambda_i - \lambda} = tr\left\{(L-\lambda)^{-1} - (L_o - \lambda)^{-1}\right\} = \lambda^{1/2}\sum_{\nu=1}^{\infty}(\nu - \frac{1}{2})C_\nu\lambda^{-\nu} ,$$

$$C_\nu = \int c_\nu(q, q_x, \cdots \partial_x^{2(\nu-1)}q)dx \ , \ C_1 = -\frac{1}{4}\int qdx \ , \tag{5.1.4}$$

where C_ν is a polynomial in its arguments. We thus have

$$a(\lambda) = \sum_{\nu=1}^{\infty} C_\nu \ \lambda^{-(\nu-\frac{1}{2})} \ + \ F_1(q), \tag{5.1.5}$$

where $F_1(q)$ is a functional of q and C_ν, a polynomial in the arguments $(q, q_x, \cdots, \partial_x^{2(\nu-1)}q)$. Since under equation (5.1.1), the operator $L = L(t)$ has fixed spectrum, the functionals $C_\nu, \nu = 1, 2, \cdots$ of q remain constant in time. We compute the $C_\nu's$ below:
Let L be a formally self-adjoint operator with respect to the inner product (,) and let L depend on the vector-valued function $\tau = (\tau_1, \tau_2, \cdots, \tau_n)$ of $\mathbb{R}^n$. We have

$$(L-\lambda)\psi = 0, \quad (\psi, \psi) = 1. \tag{5.1.6}$$

Then thinking of r as an independent variable, and taking small increments of r, we **have from** (5.1.6),

$$0 = S[(L - \lambda)\psi] = (L - A)\ \delta\psi + \delta L\psi - (\delta\lambda)\psi$$

Taking inner product with ψ, we have

$$0 = ((L - \lambda)\delta\psi \ , \psi) + (\delta L\psi, V) - \delta\lambda(\psi \ , \psi)$$

Since

$$((L - \lambda)\delta\psi\,,\,\psi) = (\delta\psi,\,(L - \lambda)\psi) = 0,\ (\psi,\,\psi) = 1$$

we have,

$$\delta\lambda = (\delta L\psi,\,\psi) = \sum_{i=1}^{n}(F_i,\,\delta\tau_i) \tag{5.1.7}$$

We then have, assuming boundary conditions,

$$\frac{\partial\lambda}{\partial\tau_i} = F_i, \quad i.\,e.\,, \ \nabla\lambda = \lambda_\tau = F = (F_1,\cdots,F_n), \tag{5.1.8}$$

where λ_τ is the variational derivative of A with respect to r. In the above general principle, if we take $L = -\partial_x^2 + q$, $(v,w) = \int vwdx$ and $\tau = q$, then

$$\nabla\lambda = \psi^2. \tag{5.1.9}$$

We consider $A\psi^2 = \lambda J\psi^2$, where

$$A = \frac{-1}{4}(\partial_x^3 - 2(q\partial_x + \partial_x q\,.\,)),\ \ J = \partial_x.$$

Then we have

$$A\nabla\lambda = \lambda J\nabla\lambda,\ A^* + A = 0,\ J^* + J = 0. \tag{5.1.10}$$

This relation is called the Lenard relation. In (5.1.10), if we employ *a(\)* defined in (5.1.3), then we get

$$\begin{aligned} A\nabla a &= A\sum_i\frac{\nabla\lambda_i}{\lambda - \lambda_i} = J\sum_i\frac{\lambda_i\nabla\lambda_i}{\lambda - \lambda_i} \\ &= \lambda J\nabla a - J\nabla(\sum\lambda_i), \\ \text{i.e.,}\quad A\nabla a &= \lambda J\nabla a + F_2(q)\,, \end{aligned} \tag{5.1.11}$$

where F_2 is a functional of q. Hence we have

$$A\nabla C_s = J\nabla C_{s+1},\ s = 1,\,2,\,\cdots,\ A^* + A = 0,\ J^* + J = 0 \tag{5.1.12}$$

This equation (5.1.12) is the Lenard recursion scheme associated with the Lenard relation (5.1.10). By (5.1.9), using $(\psi, \psi) = 1$, we have,

$$\int \nabla a = \int \sum \frac{\nabla \lambda_i}{\lambda - \lambda_i} = \frac{1}{\lambda - \lambda_i} \int \sum \psi^2 = \frac{1}{\lambda - \lambda_i} \int 1 dx = -a_\lambda(x).$$

$$\text{Then} \quad \int (\nabla C_{s+1}) dx = (s - \frac{1}{2}) C_s, \quad s = 1, 2, \cdots \tag{5.1.13}$$

$$\text{i. e.,} \quad D(C_s) \cong c_s \cong (s - \frac{1}{2})^{-1} \nabla C_{s+1}.$$

where $D(C_\nu)$ is the integral density associated with the functional C_ν. With (5.1.12) and (5.1.13), we have a recursion relation for the gradients of the trace quantities C_ν, which also computes representatives of the associated integral densities. Relations (5.1.12), (5.1.13) give a procedure for the computation of the $C_\nu^\prime s$.

5.2 THE BOUSSINESQ'S EQUATION:

We want to apply the above method of construction in the case of the Boussinesq equation in this section. But we explain first, how the Boussinesq system is the continuum limit of the discrete Toda system. The Boussinesq equation is given by

$$\frac{\partial^2 q}{\partial t^2} = 3 \frac{\partial^2}{\partial x^2} \left(\frac{\partial^2 q}{\partial x^2} + 4q^2 \right)$$

The associated isospectral equation, discussed by Zakharov[64], is in the following form:

$$L = L(\tau) = i\partial_x^3 + i(\phi \partial_x + \partial_x \phi .) + r, \quad \tau = (\tau_1, \tau_2) = (\phi, r) \tag{5.2.1}$$

$$B = B(\phi) = i(\partial_x^2 + \frac{4}{3}\phi)$$

And L = [B,L] is equivalent to

$$(a) \ \phi_t = r_x, \quad r_t = \partial_x [\frac{4}{3}(\phi^2 + \frac{1}{4}\phi_{xx})]$$

We explain **below** the discrete Toda system and its continuum limit as the Buossinesq equation. The Toda system in the discrete case is given by

$$\dot{a}_i = a_i(b_{i+1} - b_i)$$

$$\dot{b}_i = 2(a_i^2 - a_{i-1}^2), \qquad i = 1, \cdots, n$$

where $z = \begin{pmatrix} a \\ b \end{pmatrix}$, $a \in \mathbb{R}^{n-1}$, $b \in \mathbb{R}_n$, and $a_0 = a_n = 0$.

This can be written equivalently as

$$\begin{aligned} \frac{1}{2}\left(\frac{\dot{a}_i}{a_i} + \frac{\dot{a}_{i-1}}{a_{i-1}}\right) &= \frac{1}{2}(b_{i+1} - b_i) \\ \frac{1}{2}(b_{i+1} + b_i)^{\cdot} &= (a_{i+1}^2 - a_{i-1}^2) \quad i = 1, \cdots, n. \end{aligned}$$

Taking $a_i = \frac{1}{2}e^{-\frac{1}{2}z_i}$, $i = 1, \cdots, n-1$ in the above relations, we get

$$\begin{aligned} \frac{1}{2}(z_i + z_{i-1})^{\cdot} &= -(b_{i+1} - b_{i-1}), \\ -(b_{i+1} + b_i)^{\cdot} &= -\frac{1}{2}(e^{z_{i-1}} - e^{-z_{i+1}}), \quad i = 1, \cdots n \end{aligned}$$

Developing all the exponentials in a power series and truncating approximately, we arive, after letting,

$$\begin{aligned} b_i \to r(x), \quad z_i &\to \phi(x) \quad z_i \to q(x), \\ (a) \quad \phi_t &= -2r_x \quad \left(\frac{1}{2}(z_i + z_{i-1})^{\cdot} = -(b_{i+1} - b_{i-1}), \quad \frac{1}{2}\phi_t = -r_x\right) \quad (5.2.3) \\ -2r_t &= \partial_x\left(\phi + \frac{1}{6}\phi_{xx} - \frac{1}{2}\phi^2\right) \end{aligned}$$

The Toda system is a Hamiltonian system, i.e., it can be written in the form,

$$\dot{z} = J\nabla_z H, \tag{5.2.4}$$

where J defines a Poisson bracket through the formula

$$\{f(z), g(z)\} = (\nabla f, J\nabla g), \tag{5.2.5}$$

where (,) is the standard dot product in $\mathbb{R}^{2n-1}$, and V the standard gradient. The matrix J is a $(2n - 1)$ x $(2n - 1)$ matrix given by

$$J = J(a) = \begin{bmatrix} 0_{n-1} & S \\ -S^t & 0_n \end{bmatrix} \tag{5.2.6}$$

where 0_{n-1}, 0_n the $(n-1)\times(n-1)$ and $n \times n$ zero matrix respectively, $S = S(a)$ is a $(n-1) \times n$ matrix function defined by

$$S_{ij} = (-\delta_{ij} + E_{ij})a_i\,,\ i = 1, \cdots n-1\,,\ j = 1, \cdots n \tag{5.2.7}$$

$$E_{ij} = 1 \text{ if } j = i+1\,, 0 \text{ if } j \neq *,\ \delta_{ij} = 1 \text{ if } i = j\,, 0 \text{ if } i \neq j \tag{5.2.8}$$

Then by (5.2.5), (5.2.6), (5.2.7),

$$\{g, f\} = \sum_{i=1}^{n-1} \{a_i f_{a_i}(g_{b_i} - g_{b_i+1}) + a_i g_{a_i}(f_{b_i+1} - f_{b_i})\}\ ,$$

the H of (5.2.4) being

$$H = H(z) = \frac{1}{2}\sum_{i=1}^{n} b_i^2 + \sum_{i=1}^{n-1} a_i^2$$

The above matrix J which gives the Poisson structure associated with (5.2.2) is expressed

$$J = \begin{bmatrix} 0 & C \\ -C^t & 0 \end{bmatrix}\ ,\ C_{ij} = (E_{ij} - \delta_{ij})\ ,\ i = 1, \cdots n-1\ ,\ j = 1, \cdots n,$$

These constant coefficient matrices have the continuum limits $J \rightarrow \begin{bmatrix} 0 & \partial_x \\ \partial_x & 0 \end{bmatrix}$! . Eliminating ϕ_x in (5.2.3), and using the substitutions $\phi \rightarrow \phi + 1, q \rightarrow q + 1$ and after rescaling, we get

$$\phi_t = r(x),\ r_t = \partial_x \left[\frac{4}{3}(\phi^2 + \frac{1}{4}\phi_{xx})\right]\ , \text{and} \tag{5.2.9}$$

$$J \rightarrow \begin{bmatrix} 0 & \partial_x \\ \partial_x & 0 \end{bmatrix} \tag{5.2.10}$$

This is the form of the Boussinesq equation and thus, we have showed that the discrete Toda system in the continuous case gives the Boussinesq system. We now describe a method to compute the a_λ's in the case of the Boussinesq equation.

We first consider the Sturm-Liouville case:

For a **formally** self-adjoint $L = L(T) = e_o\partial_x^n + h(\tau)\partial_x^{n-2} + \cdots + g(\tau)$, where $e_o = \sqrt{\pm 1}$, $L_o = L(o) = p(\partial_x)$, $\delta = \delta(\tau) = L - L_o$ and $\mathbb{R}_o$ is the resolvent operator associated **with** $L_o - \mathbf{A}$ and R_o its associated integral kernel. We think of L, L_o as operators defined on an approximately restricted domain of $L^2(\mathbb{R})$, and compute the following, thinking of 6 as a perturbation:

$$\begin{aligned}
&\left[(L-\lambda)^{-1} - (L_o-\lambda)^{-1}\right] f = \left[(L_o-\lambda)^{-1}(1+\delta(L_o-\lambda)^{-1})^{-1} - (L_o-\lambda)^{-1}\right] f \\
&= (-\mathbb{R}_o \circ \delta \circ \mathbb{R}_o + \mathbb{R}_o \circ \delta \circ \mathbb{R}_o \circ \delta \circ \mathbb{R}_o \cdots) f \qquad (5.2.11)\\
&= \left(\sum_{i=1}^{\infty} \mathbb{R}_i\right) f = \mathbb{R} f , \\
(\mathbb{R}_o f)(x) &= \int \mathbb{R}_o(x-y) f(y) dy \ , \ \mathbb{R} f(x) = \int R(x,y) f(y) dy \\
R_o(x) &= R_o(x,\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikx} dk}{p(ik)-\lambda} \ , \ \text{since } L_o = p(\partial_x) \qquad (5.2.12)
\end{aligned}$$

For the case $L = -\partial_x^2 + q, L_o = -\partial_x^2$, Gel'fand Dikii[15] take for the resolvent

$$R_o(x) = e^{\frac{-\sqrt{\lambda}|x|}{2\sqrt{\lambda}}} \qquad \left(\text{i.e., } R(x,y,\zeta) = \frac{e^{-\sqrt{\zeta}|x-y|}}{2\sqrt{\zeta}}\right)$$

We express $\delta(\tau(x)) = \sum_{n=0}^{\infty}(x-x_o)^n \delta_n(x_o)$ and substitute this in (5.2.11).Evaluating

$$\mathrm{tr}[(L-\lambda)^{-1} - (L_o-\lambda)^{-1}] = \int R(x,x) dx = a_\lambda(\lambda) \quad \mathrm{Im}\ \lambda > 0 \ , \qquad (5.2.13)$$

$R(x,x)$ as in (5.2.11), $a(\lambda)$ as in (5.1.3), but associated with $L = L(\tau)$ in (5.2.1). Using (5.2.11), (5.2.13), we conclude

$$a_\lambda(\lambda) = \sum_{\alpha,\beta} C_{\alpha,\beta} D^\alpha \tau_\beta \ , \ \mathrm{Im}\ \lambda > 0 \ , \qquad (5.2.14)$$

where $D^\alpha\tau_\beta = \prod_{i=1}^{\ell-1}(\partial_x^{\alpha_i}\tau_{\beta_i}(x))$, $\alpha = (\alpha_1, \cdots \alpha_{\ell-1})$, $\beta = (\beta_1, \cdots \beta_{\ell-1})$,

$$C_{\alpha,\beta} = C_{\alpha,\beta}(\lambda) = \int_{\mathbb{R}^{(2\ell-1)}} \int \cdots \int \left\{\frac{P_\ell(\kappa) Q_{\ell-1}(\eta)}{T_\ell(\kappa,\lambda)}\right\} \times \exp[i(\kappa, E\eta)] d\kappa^{(\ell)} d\eta^{(\ell-1)} \ ,$$

with $P_\ell(\kappa)$ **a** sum of positive integral powers of $\kappa_1, \cdots \kappa_\ell$, of degree atmost $n-2$, $Q_{\ell-1}(\eta)$ a monomial in $\eta_1, \cdots \eta_{\ell-1}$, E an $\ell \times (\ell-1)$ constant matrix-all depending on α, β, **while**

$$T_\ell(\kappa,\lambda) = \prod_{i=1}^{\ell}(p(i\kappa_j)-\lambda) \ , \ d\kappa^{(\ell)} = d\kappa_1, \cdots d\kappa_\ell$$

etc., and the $d\kappa^{(\ell)}$ integration is performed first.

We label the n roots of the monic polynomial equation in x,

$$p(ix) - \lambda = 0 \ , \ r_\nu = r_\nu(\lambda) \ , \ \nu = 1, 2, \cdots n \ .$$

Then for large λ

$$(r_\nu)^{-1} = \left(\lambda^{1/n}\xi^{\nu-1}\right)^{-1} + \sum_{\tau=2}^{\infty}\left(\lambda^{1/n}\xi^{\nu-1}\right)^{-\tau} D_\tau \ \ \nu = 1, \cdots n \ , \tag{5.2.15}$$

ξ a primitive n^{th} root of unity, where $\lambda = re^{i\theta}$, $r > 0$, $0 < \theta < \pi$, $\lambda^{1/n} = r^{1/n}e^{i\frac{\theta}{n}}$, $r^{1/n} > 0$.

Substituting (5.2.15) into (5.2.14) and using residue thorem in (5.2.14), we **conclude that** $a_\lambda(\lambda)$, for large λ, is of the form

$$a_\lambda(\lambda) = -\sum_{\nu=1}^{n}\left\{\left(\lambda^{1/n}\right)^{-\nu}\left(\sum_{\eta=1}^{\infty}\left(\eta-1+\frac{\nu}{n}\right)C_\eta^\nu\lambda^{-\eta}\right)\right\} = \sum_{\nu=1}^{n}\left\{a_\lambda^{(\nu)}\right\} \ , \tag{5.2.16}$$

with $C_\eta^\nu = \int C_\eta^\nu(\tau, \tau_x, \cdots, (\partial_x^m \tau))dx$, m a positive integer depending on ν, η, C_η^ν a polynomial, $\nu = 1, \cdots n$, $\eta = 1, \cdots$. Relation (5.1.11), holding for $a(\lambda)$, automatically **holds** for each $a^{(\nu)}(\lambda)$ separately, by the series representation **of** $a^\nu(\lambda)$ (cf.(5.2.16)), **and hence** we have analogous to (5.1.12), that

$$A\nabla_\tau C_\eta^\nu = J\nabla_\tau C_{\eta+1}^\nu \ , \ \eta = 1, \cdots, \nu = 1, 2, \cdots n \ . \tag{5.2.17}$$

Now, if $L = \sum_{i=1}^{n}(f_i\partial_x^i + \partial_x^i f_i\cdot) + \tau_m$, $f_i = f_i(\tau_1, \cdots \tau_{m-1})$, then if the inner product associated with L is $(v, w) = \int v\bar{w}dx$, then we **find** if $L\xi = \lambda\xi$, $(\xi, \xi) = 1$ **that**

$\nabla_{\tau_m}\lambda = \psi\bar{\psi}.$

By the same reasoning as in the case of **KdV** equation, we have $/\nabla_{\tau_m} a = -a_\lambda$ and as in relation (5.1.13), we have

$$\int \nabla_{\tau_m} C^{\nu}_{\eta+1} = \left(\eta + 1 - \frac{\nu}{n}\right) C^{\nu}_{\eta}\,, \eta = 1, \cdots, \nu = 1, 2, \cdots n \tag{5.2.18}$$

i.e., $D(C^{\nu}_{\eta}) \cong c^{\nu}_{\eta} \cong \left(\eta - 1 + \frac{\nu}{n}\right)^{-1} \nabla_{\tau_m} C^{\nu}_{\eta+1}$ which is the corresponding integral density formula.

In the case of the Boussinesq equation, r = $(\tau_1, \tau_2) = (\phi, r)$. Hence τ_m = r in (5.2.18). Since *P(ix)* = x^3, n = 3, we have three gradient series $\nabla^{(1)}_a, \nabla^{(2)}_a, \nabla^{(3)}_a$. But these series cannot be linearly independent viewed as baseless, infinite dimensional vectors $\nabla^{(i)}_a = (\nabla C^i_o, \nabla C^i_1, \cdots)$, i = 1,2,3, assuming a relation of the form (5.1.10), and hence (5.2.17), with *J* as in (5.2.10).

From (5.1.8), with *L* of the Boussinesq equation, (5.2.1), we compute $\nabla_\tau \lambda$ with *(v, w)* = *J vwdx,*

$$\nabla_\tau \lambda = \begin{pmatrix} \nabla_{\tau_1}\lambda \\ \nabla_{\tau_2}\lambda \end{pmatrix} = \begin{pmatrix} \nabla_{\phi}\lambda \\ \nabla_{r}\lambda \end{pmatrix} = \begin{pmatrix} i(\psi_x\bar{\psi} - \psi\bar{\psi}_x) \\ \psi\bar{\psi} \end{pmatrix} \tag{5.2.19}$$

where $(L - \lambda)\psi = 0$, $(\psi, \psi) = 1$.

5.2.20 Theorem (Lenard relation for the Boussinesq system):

If $L\psi = \lambda\psi$, L of (5.2.1), then $A\nabla\lambda = J\nabla\lambda, A + A^* = 0, J + J^* = 0$, with $\nabla\lambda$ as computed in (5.2.19),

$J = \begin{bmatrix} 0 & \partial_x \\ \partial_x & 0 \end{bmatrix}$, which yields the Poisson structure and

$$\begin{aligned} A &= A(\phi, r) = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \text{ where} \\ A_1 &= \frac{1}{3}\left[\partial^3_x + \phi\partial_x + \partial_x\phi.\right] \\ A_2 &= r\partial_x + \frac{2}{3}r_x\,,\ A_3 = r\partial_x + \frac{1}{3}r_x \end{aligned}$$

$$A_4 = \frac{1}{9}[\partial_x^5 + 5\phi\partial_x^3 + \partial_x^3(5\phi). + (8\phi^2 - 3\phi_{xx})\partial_x + \partial_x(8\phi^2 - 3\phi_{xx})]. \quad (5.2.21)$$

Proof: Since $A_1 = -A_1^*$, $A_4^* = -A_4$, $A_3^* = -A_2$, with respect to $(v, w) = \int v\bar{w}dx$, we have $A + A^* = 0$. Similarly for J, with respect to the inner product $\left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right] = \int(v_1\bar{w}_1 + v_2\bar{w}_2)dx$. Since $A(\phi, r + \lambda) = \lambda J$, it is sufficient to verify the theorem for the case $\lambda = 0$.

That is we prove $A\nabla\lambda = 0$.

$$\text{i.e.} \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} \nabla_\phi\lambda \\ \nabla_r\lambda \end{bmatrix} = 0 \qquad \text{i.e., to prove}$$

$$A_1(\nabla_\phi\lambda) + A_2(\nabla_r\lambda) = 0$$

$$A_3(\nabla_\phi\lambda) + A_4(\nabla_r\lambda) = 0$$

Using $L\psi = 0$, we compute

$$(a) \qquad \partial_x^2(\psi\bar{\psi}) = \psi_{xx}\bar{\psi} + 2\psi_x\bar{\psi}_x + \psi\bar{\psi}_{xx}$$

$$(b) \qquad \partial_x^3(\psi\bar{\psi}) = 3(\psi_x\bar{\psi}_{xx} + \psi_{xx}\bar{\psi}_x) - 2[\phi(\psi\bar{\psi})]_x\,, \text{ using } L\psi = 0$$

$$(c) \qquad \partial_x^4(\psi\bar{\psi}) = 3(\psi_x\bar{\psi}_{xxx} + \psi_{xxx}\bar{\psi}_x) + 6(\psi_{xx}\bar{\psi}_{xx}) - 2[\phi(\psi\bar{\psi})]_{xx}$$

$$(d) \qquad \partial_x^5(\psi\bar{\psi}) = 3(\psi_x\bar{\psi}_{xxxx} + \psi_{xxxx}\bar{\psi}_x)$$
$$+9(\psi_{xx}\bar{\psi}_{xxx} + \psi_{xxx}\bar{\psi}_{xx}) - 2[\phi(\psi\bar{\psi})]_{xxx}$$

$$(e) \qquad \partial_x^2(\bar{\psi}\psi_x - \psi\bar{\psi}_x) = (\bar{\psi}_x\psi_{xx} - \psi_x\bar{\psi}_{xx})$$
$$+2ir(\bar{\psi}\psi) + 2\phi(\psi\bar{\psi}_x - \bar{\psi}\psi_x)\,, \text{ using } L_\psi = 0\,,$$

$$(f) \qquad \partial_x^3(\bar{\psi}\psi_x - \psi\bar{\psi}_x) = 2ir_x(\psi\bar{\psi}) + 3ir(\psi\bar{\psi})_x + 2\phi(\psi\bar{\psi}_x - \bar{\psi}\psi_x)_x$$
$$+3\phi_x(\psi\bar{\psi}_x - \bar{\psi}\psi_x)\,, \text{ using } L_\psi = 0\,, \qquad (5.2.22)$$

Equation (5.2.22)f is essentially, $-3(A_1(\nabla_\phi\lambda) + A_2(\nabla_r\lambda)) = 0$.

Now, we verify the other half of the Lenard relation.

We compute, $-3A_3(\nabla_\phi\lambda) = -(3r(\nabla_\phi\lambda)_x + r_x(\nabla_\phi\lambda)) = -i[3r(\psi\bar{\psi}_{xx} - \bar{\psi}\psi_{xx}) + r_x(\psi\bar{\psi}_x - \bar{\psi}\psi_x)]$.

Using $L\psi = 0$, $(L\psi)_x = 0$ on $r\psi, r\psi_x$ respectively, we must verify that

$$\begin{aligned}
3A_4(\nabla_r\lambda) &= -3A_3(\nabla_\phi\lambda) \\
&= (\psi_x\bar{\psi}_{xxxx} + \bar{\psi}_x\psi_{xxxx}) + 3(\bar{\psi}_{xx}\psi_{xxx} + \bar{\psi}_{xxx}\psi_{xx}) + 8\phi(\bar{\psi}_{xx}\psi_x + \psi_{xx}\bar{\psi}_x) \\
&\quad +3\phi_x(\psi\bar{\psi}_{xx} + \bar{\psi}\psi_{xx} + 2\psi_x\bar{\psi}_x) + \phi_{xx}(\psi\bar{\psi})_x \\
&= \left\{\left[\frac{1}{3}\partial_x^5 + \left(2\phi_{xx}\partial_x + \frac{2}{3}\phi_{xxx} + 2\phi_x\partial_x^2 + \frac{2}{3}\phi\partial_x^3\right)\right]\right. \\
&\quad \left. + \left[\frac{8}{3}\phi\partial_x^3 + \frac{16}{3}\phi^2\partial_x + \frac{16}{3}\phi\phi_x\right] + \left[3\phi_x\partial_x^2 + \phi_{xx}\right]\right\}(\nabla_r\lambda)\ .
\end{aligned}$$

(using (5.2.22) a,b,d).

But the operator in brackets is $3A_4$ expanded out and hence the theorem is proved.

5.2.23 Remarks: (1) We can prove similarly as in Theorem (5.2.20) that the Toda system satisfies the Lenard relation given in (5.1.10)

(i.e., $A\nabla\lambda = \lambda J\nabla\lambda$, $A^* + A = 0$, $J^* + J = 0$) for appropriate A, J and $\nabla\lambda$ where A is defined by the $(2n-1)\times(2n-1)$ matrix function $A = A(z)$ given by

$$\begin{aligned}
A &= \begin{bmatrix} A_1 & A_3 \\ A_3^t & A_2 \end{bmatrix} \quad \text{where} \\
(A_1)_{ij} &= \frac{a_i a_{i+1}}{2}\left(E_{ij} - E_{ij}^t\right), \quad i,j = 1,2,\cdots,n-1 \\
(A_2)_{ij} &= 2a_i^2(E_{ij} - E_{ij}^t), \qquad i,j = 1,\cdots,n \\
(A_3)_{ij} &= a_i(b_{i+1}E_{ij} - b_i\delta_{ij}), \quad i = 1,\cdots,n-1,\ j = 1,\cdots,n
\end{aligned}$$

with E_{ij} defined as in (5.2.8).

(2) The Kac-Moerbeke system is defined as follows: Let $\dot{z} = J\nabla C_3$, where z and J are as in the of Toda system defined in the begining of the section 5.2 and $C_i = C_i(a,b) = \frac{\mathrm{tr}L^i}{i}$, $i = 1,2,\cdots$. This reduces to, on the manifold $b_i = 0$, $i = 1,\cdots,n$, along with $\dot{b}_i = 0$,

$$\dot{a}_i = a_i(a_{i+1}^2 - a_{i-1}^2), \quad i = 1,\cdots,n-1.$$

This is the system of Kac-Moerbeke. This system can also be written as a Hamiltonian system **and** this system also satisfies the Lenard relations of **the** form (5.1.10) (similar to the systems of KdV, Toda and Boussinesq).

5.3 BOUSSINESQ'S EQUATION AS A HAMILTONIAN SYSTEM:

We give below the Hamiltonian system of Boussinesq equation following H.P. McKean [38]. The Boussinesq equation is given by

$$\frac{\partial^2 q}{\partial t^2} = 3\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 q}{\partial x^2} + 4q^2\right) \tag{5.3.1}$$

The Hamiltonian formalism is based upon the Poisson bracket

$$[A, B] = \int_o^1 \nabla A J \nabla B dx = \int_0^1 \left(\frac{\partial A}{\partial q} D \frac{\partial B}{\partial p} - \frac{\partial B}{\partial q} D \frac{\partial A}{\partial p}\right) dx$$

where ∇ denotes the function space gradient $\nabla H = (\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p})$ of smooth functions, $H : C_1^\infty \times C_1^\infty \to \mathbb{R}$ and J is the skew-symmetric operator, $J = \begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix}$ $(D = \frac{\partial}{\partial x})$.

Equation (5.3.1) is euqivalent to

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 3p' \\ q''' + 8qq' \end{pmatrix} = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} \begin{pmatrix} \partial H_2/\partial q \\ \partial H_2/\partial p \end{pmatrix} \tag{5.3.2}$$

with

$$H_2 = \int_o^1 \left(\frac{3}{2}p^2 + \frac{4}{3}q^3 + \frac{1}{2}q'^2\right) dx \tag{5.3.3}$$

Zakharov [64] proved the existence of infinitely many first integrals of (5.3.2):

$$\begin{aligned}
H_{-1} &= \int_o^1 q dx \,,\ H_o = \int_o^1 \frac{3}{2} p dx \,,\ H_1 = \int_o^1 pq dx \\
H_2 &= \int_o^1 q\left(\frac{3}{2}p^2 + \frac{4}{3}q^3 + \frac{1}{2}(q')^2\right) dx \\
H_3 &= \int_o^1 \left[-\frac{1}{2}(p')^2 + 2p^2 q + \frac{1}{6}(q'')^2 + \frac{8}{9}q^4 - 2q(q')^2\right] dx \\
H_4 &= \int_o^1 \left[\frac{5}{2}p3 + \frac{20}{3}pq^3 + \frac{15}{2}p(q')^2 + 10pqq'' + p''(q'')\right] dx
\end{aligned}$$

These are involutive integrals of motion.

H.P. Mckean in his paper [38] has proved that the Hamiltonians given above $H_{-1}, H_o, \cdots$ are in involution, using the eigenfunctions of the operator L,

$$L = \sqrt{-1}\left(D^3 + qD + Dq\right) + p \ .$$

These Hamiltonians $H_{-1}, H_o, H_1, \cdots$ can also be obtained using the trace functional approach.

We mentioned in the previous section that for the Boussinesq equation, r $= (r_1, r_2) = (\phi, r)$ and since $P(ix) = x^3$, $n = 3$, we have three gradient series $\nabla a^{(1)}, \nabla a^{(2)}, \nabla a^{(3)}$ and these series cannot be linearly independent viewed as baseless, infinite dimensional vectors $\nabla a^{(i)} = (\nabla C_o^i, \nabla C_1^i, \cdots), i = 1, 2, 3$. For example, suppose that the first non-zero term of $\nabla a^{(1)}$ is $\nabla C'_{s_o}$, then even if $s_o = 0$, we can adjoin ∇C^o_{-1} to the series. Then $J\nabla C'_{s_o} = 0$ and so we must have $\nabla C'_{s_o} = (d_1, d_2)$, where d_1, d_2 are two constants, since J is a differential operator with no constant term. Also, $\nabla C'_s$, $s > s_o$ will contain no constant terms. Hence the first non-zero terms of the series $\nabla a^{(i)}$, $i = 1,2,3$ must be of the form (d_1, d_2) and then the next terms in the series are uniquely determined by (5.2.16). Thus, the first non-zero terms respectively, of the three series $v_i, i = 1,2,3$ must be linearly dependent and by the linearity of (5.2.16), that relation will continue to hold for the remainder of the series. In otherwords, we can have atmost two linearly independent series. Thus, we have that the first non-zero terms of the series are of the form $\int(d_1\phi + d_2 r)$ and assuming the v_i $i = 1,2$ span a two dimensional subspace, we can take

$$\int r = H_1 \ , \ \int \phi = F_1 \tag{5.3.4}$$

as the first terms, respectively, of the trace series H_i, F_i, i $= 1,2, \cdots$. The v_i, $i = 1,2$ span a two-dimensional subspace and there exists no v_3, i.e., $\nabla a^{(3)} = 0$.

With the L of the Boussinesq equation (5.2.1), we compute using $(v, w) = f v\bar{w}dx$, that

$$\nabla_\tau = \begin{pmatrix} \nabla_{\tau_1}\lambda \\ \nabla_{\tau_2}\lambda \end{pmatrix} = \begin{pmatrix} \nabla_\phi \lambda \\ \nabla_r \lambda \end{pmatrix} = \begin{pmatrix} i(\psi_x\bar{\psi} - \psi\bar{\psi}_x) \\ \psi\bar{\psi} \end{pmatrix}$$

where $(L - \lambda)\psi = 0$, $(\psi, \psi) = 1$.

In the case of r = $(\phi,$ r) evolving via the **Boussinesq** flow, (5.2.9), $L = [B, L]$ implies that $\psi_t = B\psi = i(\partial_x^2 + \frac{4}{3}\phi)\psi$, from which we compute, ft(V_rA) $= \partial_x(\nabla_x\lambda)$.

This gives ft(V_rA^5) $= \partial_x(\nabla_\phi\lambda^s)\ s = 1, 2, \cdots$. Hence $\partial_t(\nabla_r(trL^s)) = \partial_x(\nabla_\phi(trL^s))$.

By using formula (5.2.9), (5.2.10) for $a(\lambda)$, we have $\partial_t(\nabla_r a_\lambda) = \partial_x(\nabla_a a_\lambda)$ which gives the 'local' conservation laws of the Boussinesq equation,

$$\partial_t(\nabla_r C_\eta^\nu) = \partial_x(\nabla_\phi C_\eta^\nu) \ , \ \eta = 1, \cdots \ , \ \nu = 1, \cdots n$$

By using (5.3.4), we have

$$\begin{aligned} \partial_t(\nabla_r H_i) &= \partial_x(\nabla_\phi H_i) \ , \\ \partial_t(\nabla_r F_i) &= \partial_x(\nabla_\phi F_i) \ , \ i = 1, 2, \cdots \end{aligned}$$

Using the formula $A\nabla C_s = J\nabla C_s + 1$, s — $1, 2, \cdots$ for H_i, F_i we can compute the traces,

$$\begin{aligned} H_1 &= \int r dx \\ H_2 &= \frac{2}{3}\int \left\{\frac{1}{2}r^2 + \frac{4}{3}\left(\frac{\phi^3}{3} - \frac{\phi_x^2}{8}\right)\right\} dx \\ H_3 &= \frac{1}{3}\int \frac{5}{6}r^3 + \frac{4}{3}r\left[(\phi^2 + \frac{1}{4}\phi_{xx})_{xx} + \frac{5}{3}\phi^3 + \frac{1}{2}(\phi\phi_x)_x - \frac{5}{8}\phi_x^2\right] dx \\ &\vdots \\ F_1 &= \int \phi dx \\ F_2 &= \frac{1}{3}\int \{r\phi\} dx \end{aligned}$$

We observe here that $\frac{3}{2}H_2$ is the Hamiltonian associated with the Boussinesq flow, (5.2.9). i.e., $\partial_t \begin{pmatrix} \phi \\ r \end{pmatrix} = J\nabla(\frac{3}{2}H_2)$ is equivalent to (5.2.9). Thus we have calculated the Hamiltonian H_2 of the Boussinesq equation using trace formula.

5.4 SOME LAX-TYPE FORMULAS:

In this section we construct an operator valued function which yields the infinitesimal generators of the Lax type isospectral deformations associated with the following examples: The Toda and Kac-Moerbeke systems in ordinary differential equations and their respective continuum limits, the Boussinesq and KdV equations in partial differential equations.

(i) For the KdV system discussed in Section 1:

Given the following partial differential equation,

$$q_t = J\nabla C_\nu = A\nabla C_{\nu-1} \tag{5.4.1}$$

with $L = L(q)$ defined in (5.1.2) and the following definition,

$$D_v L(\tau) \equiv L(Jv)\ ,\ \hat{D}_v L(\tau) \equiv L(Av)\ , \tag{5.4.2}$$

$v = v(\tau)$, $\tau = q$, a vector field, we find

$$D_{\nabla C_\nu} L = \hat{D}_{\nabla C_{\nu-1}} L = [B_\nu, L]\, , \nu = 1, 2, \cdots \tag{5.4.3}$$

with

$$B_\nu = \sum_{\mu=1}^{\nu-1} \Phi_\mu L^{\nu-1-\mu}\ \ ,\ \ \Phi_\mu = \Phi(\nabla C_\mu)\ , \tag{5.4.4}$$

where $\Phi(.)$ is a linear operator from vector fields to operators defined by

$$\Phi(v) = \frac{1}{4}(2v\partial_x - v_x)\ ,\ v = v(q)\ . \tag{5.4.5}$$

having the property,

$$[\Phi(v), L] = D_v L - (D_v L).L \tag{5.4.6}$$

This implies (5.4.3) is seen by the following:

$$\begin{aligned}
[B_\nu, L] &= \left[\sum_{\mu=1}^{\nu-1} \Phi_\mu L^{\nu-1-\mu} \ , \ L\right] \\
&= \sum_{\mu=1}^{\nu-1} [\Phi_\mu, L] L^{\nu-1-\mu} \ , \\
&= \sum_{\mu=1}^{\nu-1} \left\{ (\hat{D}_{\nabla C_\mu} L) L^{\nu-1-\mu} - (D_{\nabla C_\mu} L) L^{\nu-\mu} \right\} \\
&= \sum_{\mu=1}^{\nu-1} \left\{ (D_{\nabla C_{\mu+1}} L) L^{\nu-(\mu+1)} - (D_{\nabla C_\mu} L) L^{\nu-\mu} \right\} \ \text{by}(5.1.12) \\
&= D_{\nabla C_\nu} L - (D_{\nabla C_1} L) L^{\nu-1} = D_{\nabla C_\nu} L \qquad \text{by (5.1.4)}
\end{aligned}$$

Equation (5.4.3) implies that the vector fields $J\nabla C_\nu, \nu - 1, 2, \cdots$ preserve the spectrum of L and in particular all the C_s, $s = 1, 2, \cdots$.

Since under (5.4.1),

$$0 = \frac{dC_s}{dt} = \int (\nabla C_s).(J\nabla C_\nu) dx = (\nabla C_s, J\nabla C_\nu) = \{C_s, C_\nu\} \tag{5.4.7}$$

(this defines {,}), we must have

$$\begin{aligned}
(\nabla C_\nu).(J\nabla C_\tau) &= \frac{dF_{\nu\tau}}{dx}(q, q_x, \cdots \partial_x^\beta q) \\
\text{i.e.,} \quad (\nabla C_s).(J\nabla C_\nu) &\cong 0 \ , \ s, \nu = 1, 2, \cdots
\end{aligned} \tag{5.4.8}$$

Thus, (5.4.7) says that C_ν under the bracket {,} are pairwise in involution or equivalently, the vector fields $J\nabla C_\nu$ pairwise commute.

We note that given L and J, (5.4.6) may be thought of as a recipe for discovering $\Phi(.)$, which yields another technique for discovering A and thus formula (5.1.12).

We also have a more direct proof that the $C_\nu's$ are pairwise in involution with respect to {,} of (5.4.7), based on (5.1.12), which is as follows:

$$\begin{aligned}
\{C_m, C_n\} &= (\nabla C_m, J\nabla C_n) = (\nabla C_m, A\nabla C_{n-1}) \\
&= -(A\nabla C_m, \nabla C_{n-1}) = -(J\nabla C_{m+1}, \nabla C_{n-1}) \\
&= (\nabla C_{m+1}, J\nabla C_{n-1}) = \{C_{m+1}, C_{n-1}\} \\
\text{Hence} \quad \{C_m, C_n\} &= \{C_{m+1}, C_{n-1}\} = \cdots = \{C_{m+m-1}, C_1\} \\
&= \{\nabla C_{m+n-1}, J\nabla C_1\} = 0 \ ,
\end{aligned}$$

since $J\nabla C_1 = 0$ and thus $\{C_m, C_n\} = 0$, $m, n = 1, 2, \cdots$.

(ii) For the Toda system discussed in section 2, we define the linear operator $\Phi(.)$, which takes vector fields into *nxn* matrices and which satisfies (5.4.6), with L, A, J defined for the Toda system.

If $w = (u_1, \cdots u_{n-1}, v_1, \cdots v_n) = (u, v)$, then we define

$$[\Phi(w)]_{ij} = \frac{1}{2}\delta_{ij}(a_i u_i - a_{i-1}u_{i-1}) - a_i(E_{ij}v_i - E^t_{ij}v_{i+1}) \ i, j = 1, \cdots n \ . \qquad (5.4.9)$$

and $\Phi(.)$ satisfies the commutator relation (5.4.6).

(iii) For the Boussinesq equation discussed in section 3, we define the function $\Phi(.)$, which takes vector fields of the form $\begin{pmatrix} v(x) \\ w(x) \end{pmatrix} = V(x)$ into operators and satisfies (5.4.6), with L, A, J and vector fields defined for the Boussinesq equation. This insures an isospectral formula of the form (5.4.3) holding with the $H_i, F_i, i = 1, 2, \cdots$, in place of the C_i in (5.4.3). We define

$$\Phi(V) = \Phi\begin{pmatrix} v(x) \\ w(x) \end{pmatrix} = \frac{1}{3}\left\{(2iw\partial_x^2 + (v - iw)\partial_x + \frac{8}{3}i\phi w + \frac{i}{3}(w_{xx} - v_x)\right\}$$

Then we can prove that

$[\Phi(v), L] = \hat{D}_v L - (D_v L).L$, i.e., Φ satisfies the relation (5.4.6).

(iv) As in the case of the above systems, we define for the Kac-Moerbeke system, $\hat{\Phi}$, a linear operator from vectors $u = (u_1, \cdots, u_{n-1})$ into matrices by $\hat{\Phi}(u) = \Phi(u_0) \,.\, L_0$ $+\ \Phi(J^{-1}A_0(u_0))$ with Φ, J, A and L defined as in the case of Toda system and the subscript 0 indicates the evaluation on the manifold $b = 0$, $u_0 = \begin{pmatrix} u \\ 0 \end{pmatrix}$, and 0 - the null n-vector and J^{-1} maps vector functions homogeneous in a into vectors homogeneous in a. Then the operator $\hat{\Phi}(.)$ satisfies the following commutator relation

$$[\hat{\Phi}(u), L_0] = \tilde{D}_{u_0} L_0 - (D_{u_0} L_0) L_0^2$$

where $\tilde{D}_{u_0} L_0 = L_0(A_0 J^{-1} A_0 u_0)$, which is analogous to (5.4.6).

5.5 THE SUB-HAMILTONIAN SYSTEM (THE INVARIANT MANIFOLD, r = 0):

In this section, we consider the flows associated with the Boussinesq hierarchy when restricted to the manifold, r = 0. We consider flows of the form

$$\text{(a)} \qquad \partial_t \begin{pmatrix} \phi \\ r \end{pmatrix} = (J\nabla H_{2n+1})|_{r=0} \quad i.e., \quad \phi_t = (\partial_x \nabla_r H_{2n+1})|_{r=o}$$

$$n = 1, 2, \cdots \qquad (5.5.1)$$

$$\text{(b)} \qquad \partial_t \begin{pmatrix} \phi \\ r \end{pmatrix} = (J\nabla F_{2n})|_{r=0} \quad i.e., \quad \phi_t = (\partial_x \nabla_r F_{2n})|_{r=o}$$

To show that r = 0 is an invariant manifold of the above flows, we introduce the following terminology. Let *Q[r]* denote a function of r (and its spatial derivatives), which is even in r (and its spatial derivatives) and atleast quadratic in them. Let $L_o[r]$ be a function which is odd in r (and its spatial derivatives), *L[r]* is defined as $L_o[r]$ which is cubic in r (and its derivatives) or identically zero.

Let $\{\phi\}$, $\{r\}$, refer to a function or operator, solely containing ϕ, r respectively and **their** derivatives. The following lemma, using the form of J as in (5.2.9) guarantees that $r = 0$ is an invariant manifold of the above flows.

Let $D(H_i)$ denote the integral density of the functional H_i, defined as in section 5.1 (cf. 5.1.13). For the functionals H_i, F_i, we have

5.5.2 Lemma: For the functionals H_i, F_i, $i = 1,2,\cdots$,

$$\begin{aligned} &\text{(a)} \quad (i) \quad D(H_{2s}) = Q[r] + \{\phi\}_s, \quad (ii) \quad D(H_{2s+1}) \cong L[r] + r\{\phi\}_{-s}, \\ &\text{(b)} \quad (i) \quad D(F_{2s-1}) \cong Q[r] + \{\phi\}'_s, \quad (ii) \quad D(F_{2s}) \cong L[r] + r\{\phi\}'_{-s}, \; 3 = 1,2,\cdots \end{aligned} \tag{5.5.3}$$

Proof: We prove (a) by induction and (b) can be proved similarly. By the calculation of trace formula in the previous section, the case $s = 1$ is clear.

That is, for $s = 1$, $D(H_2) \simeq Q[r] + \{\phi\}_1$, $D(H_3) \simeq L[r] + r\{\phi\}_{-}D(F_1) \cong Q[r] + \{\phi\}'$, $D(F_2) = L[r] + r\{\phi\}'_{-1}$ and comparing the trace formulas H_2, H_3, F_1, F_2 in section 5.3, the above is true for $s = 1$.

Now, we assume (a) for $s_o - 1 < s$, verify (a) (i) for $s = s_o$ and verify (a) (ii) for $s = s_o$. By the above notation and (5.2.20), we have

$$A = \begin{bmatrix} A_1 & A_3 \\ A_2 & A_4 \end{bmatrix} = \begin{bmatrix} \{\phi\} & L_o(r) \\ L_o[r] & \{\phi\} \end{bmatrix} \tag{5.5.4}$$

By induction hypothesis,

$$\begin{aligned} J^{-1}A\nabla(H_{2s}) &= J^{-1}A\begin{pmatrix} Q[r] + \nabla\int\{\phi\}_s dx \\ L_o[r] \end{pmatrix}, \text{ and by (5.5.3)} \\ &= J^{-1}\begin{pmatrix} Q[r] + A_1\nabla\int\{\phi\}_s dx \\ L_o[r] \end{pmatrix} = \begin{pmatrix} \nabla_\phi H_{2s+1} \\ \nabla_r H_{2s+1} \end{pmatrix} \end{aligned}$$

via the recursion relation. Therefore, by the form of J, (5.2.20) and the above relation, we must have simultaneously,

$$\begin{aligned} D(H_{2s+1}) &= L[r] + r(\partial_x^{-1}(A_1\nabla\int\{\phi\}_s dx)) + \{\phi\} \\ D(H_{2s+1}) &= L_o[r] + \{r\} \qquad \text{and so} \\ D(H_{2s+1}) &= L[r] + r(\partial_x^{-1}(A_1\nabla\{\phi\}_s)) \end{aligned}$$

and thus proving a(ii) with $\{\phi\}_{-s} = (\partial_x^{-1}(A_1\int\{\phi\}_s dx))$

The induction step from a(ii), $s_o = s$ to a(i), $s_o = s + 1$ can be proved similarly and this completes the proof of the lemma.

From the above lemma we note that, (5.5.1)(a) takes the form

$$\phi = \partial_x\{\phi\}_{-s}\ ,\ \dot{r} = L_o[r]|_{r=o} = 0. \tag{5.5.5}$$

We now define

$$E_n = E_n(\phi) = \int\{\phi\}_n dx\ ,$$

where $\{\phi\}_n$ is defined as in (5.5.3).

5.5.6 Theorem: The subsystem (5.5.1)(a)

$$\begin{pmatrix} \phi \\ r \end{pmatrix}^{\cdot} = (J\nabla H_{2n+1})|_{r=o}\ ,\quad n = 1, 2, \cdots$$

may be written as an integrable Hamiltonian system in its own right, specifically as

$$\dot{\phi} = A_1\nabla_\phi E_n\ ,\quad n = 1, 2, \cdots, \tag{5.5.7}$$

where the E_n satisfy the (formal) recursion scheme (cf. 5.1.12),

$$A_1\nabla_\phi E_n = B\nabla_\phi E_{n+1}\ ,\ A_1 + A_1^* = B + B^* = 0\ ,\ B = \partial_x A_4^{-1}\partial_x \tag{5.5.8}$$

Proof: Equation (5.5.7) is an immediate consequence of (5.5.5) and $\{\phi\}_{-s} = \partial_x^{-1}A_1\nabla_\phi E_s$, a formula which is proved in Lemma (5.5.2). While the above formula and $\nabla E_{s+1} =$

$\partial_x^{-1}(A_4\{\phi\}_{-s})$, yield (5.5.8).

We show that A_1 defines a Hamiltonian structure, i.e., the bracket,

$$\{F,G\} = \int (\nabla F)(A_1\nabla G) = (\nabla F, A_1\nabla G)\ , \tag{5.5.9}$$

satisfies the Jacobi identity, as (5.5.8) guarantees $\{E_i, E_j\} = 0$, i.e., the integrability **of** the E_i hierarchy.

5.5.10 Lemma: $\{.,.\}$ as defined above is a Poisson bracket.

Proof: We show that all constant coefficient, skew-symmetric differential operators define a Poisson bracket. As A_1 does not have constant coefficients, we first compute $G_{\{F,H\}} = \nabla\{F,H\}$ (dropping the subscript on A_1)

$$\begin{aligned}(G_{\{F,H\}}, v) &= \frac{d}{d\epsilon}[\{F,H\}(\phi+\epsilon v)]|_{\epsilon=o}\ ,\quad v\in C_o^\infty(\mathbb{R})\\ &= (G'_F v, AG_H) + (G_F, AG'_H v) + (G_F, A'G_H)\\ &= (v, (G'_F AG_H - G'_H AG_F)) + (G_F, (2v\partial_x + v_x)G_H)\ ,\end{aligned}$$

where we have used the skew-symmetry and symmetry respectively of the differential operators $A, G'_F(G'_H)$.

The operator $G'_F(G'_H)$ is symmetric,

$$\left.\frac{\partial^2 F}{\partial t\partial s}\right|_{s,t=o}(\phi + sv + tw) = (G'_F v, w) = \left.\frac{\partial^2 F}{\partial s\partial t}\right|_{s,t=o} = (G'_F w, v)$$

From the above, we conclude,

$$\begin{aligned} G_{\{F,H\}} &= [G_F, G_H]_1 + [G_F, G_H]_2, \qquad [G_F,G_H]_1 = (G'_F AG_H - G'_H AG_F)\\ [G_F,G_H]_2 &= (G_F\partial_x G_H - G_H\partial_x G_F)\end{aligned}$$

We have to show the Jacobi identity,

$$\{\{F,H\},K\} + \{\{H,K\},F\} + \{\{K,F\}H\} = 0\ ,\ \text{i.e.,}$$

$$\{([OF, G_H]_1, AG_K) + ([G_H, G_K]_1, AG_F) ([G_K, G_F]_1, G_K)\}$$
$$+\{([G_F, G_H]_2, AG_K) + ([G_H, G_K]_2, AG_F) + ([G_K, G_F]_2, G_K)\} = 0$$

i.e., $\{/\} + \{//\} = 0$

Here / refers to the term in the first bracket and similarly //. By the symmetry, skew-symmetry of $G'_F(G'_H$ and $G'_K), A,$ respectively, / is zero, which implies that skew-symmetric, diflferential operators with constant coefficients automatically defines a Poisson bracket. We rewrite // as

$$// = ([G_H, G_F]_3, G_K),$$
$$[v, w(x)]_3 = A(vw_x - wv_x) + 2(w_x Av - v_x Aw) + (w(Av)_x - v(Aw)_x).$$

$[\cdots, \cdot\cdot]_3 = 0$ follows from a straightforward computation from $A = (\partial_x^3 + \phi\partial_x + \partial_x\phi.)$. Thus the lemma is proved and hence the Theorem 5.5.6. We note that the $E_s, s = 1, 2, \cdots$, give an integrable hierarchy which is essentially a subhierarcy of the KdV hierarchy.

Chapter 6

Geometry of Benney's Equation and higher dimensional equations

In this chapter we systematically understand the geometry of long wave equations of Benney. Though this topic is a quarter century old (Benney 1973 [10]) classically, after the recent modern attempts to understand Hamiltonian mechanics in general and some completely integrable Hamiltonian system in particular of mathematical physics and applied mathematics, Benney's equations were studied extensively during last 15 years in the Russian school at Moscow, Courant school and Italian school. References used are given at the end.

6.1 CLASSICAL RESULTS:

6.1.1: Consider the 2-dimensional time dependent motion of an inviscid homogeneous fluid under the action of gravity g. Let $y = 0$ be the rigid bottom and $y = h(x,t)$ be the free surface. Then the equations governing the motion of such fluid are

$$u_x + v_y = 0 \qquad (6.1.2) \; ; \qquad u_t + uu_x + vu_y = \frac{1}{\rho}p_x \qquad (6.1.3)$$

$$\mu^2\{v_t + uv_x + vv_y\} = -\frac{1}{\rho}p_y - g \qquad (6.1.4)$$

where $u(x,y,t)$ and $v(x,y,t)$ are the horizontal and vertical velocity components respectively, h_o is the mean depth, 1 is the horizontal scale of the wave, $\mu^2 = (h_o/l)^2$ is the long wave parameter. The boundary conditions are $v = 0, y = 0$ (6.1.5); $p = p_o$, $y =$

$h(x,t)$ (6.1.6); $h_t + uh_x - v = 0, y = h(x,t)$ (6.1.7). Here p_o is the constant atmospheric pressure.

6.1.8: In the case of *long waves* the long wave paramter μ^2 is very small. In fact we take $\mu^2 = 0$ and in this case the pressure is hydrostatic given by $p = p_0 - \rho g(y - h)$ and the equations of motion take simpler form $u_x + v_y = 0$ (6.1.9), $u_t + uu_x + vu_y = -gh_x$ (6.1.10) subject to $v = 0, y = 0$ (6.1.11), $h_t + uh_x - v = 0$ (6.1.12).

6.1.13: In the case of *Shallow water wave theory*, the motion is an irrotational motion. In fact in this case we have $u = u(x,t), v = -yu_x$ and the subsequent wave motion is determined by the well known non-linear Shallow water wave equations

$$h_t + uh_x + hu_x = 0 \quad (6.1.14), \qquad u_t + uu_x + gh_x = 0 \quad (6.1.15).$$

We are interested here only in the evolutionary properties of long waves relative to time.

6.1.16 Special solutions: We want to find a special nonlinear solution of long waves (6.1.7) to (6.1.12) by finding a decomposition of the form $u = u(y,h)$ (6.1.17) where h satisfies $h_t = -c(h)h_x$(6.1.18) so that the free surface deforms and propagates with speed $c(h)$. From (6.1.9) and (6.1.11) by integration, we get $v = -h_x\int_o^y u_h(y,h)dy$ (6.1.19). Then using (6.1.18) and (6.1.19) in (6.1.9) and (6.1.12) on calculation by eliminating v we get

$\int_o^y u_h dy = -g(u - c)\int_o^y \frac{dy}{(u-c)^2}$ (6.1.20) where $\int_o^h \frac{dy}{(u-c)^2} = \frac{1}{g}$ (6.1.21). The equations (6.1.20) and (6.1.21) contain the velocity structure and speed of the wave motion.

6.1.21 Moment equations and classical conservation laws: The nonlinear long-wave equations have some interesting properties as well as some formal aspects which we will discuss in this paragraph. More explicitly these equations are

$$u_x + v_y = 0 \quad (6.1.22), \qquad u_t + uu_x + vu_y = -gh_x \quad (6.1.23)$$

$$v = 0, y = 0 \quad (6.1.24), \qquad h_t + uh_x - v = 0, \; y = h \quad (6.1.25).$$

Suppose at time $t = 0$, $u(x,y,0)$and $h(x,0)$ are prescribed, then equations (6.1.22)-(6.1.25) pose an intial value prolem for the unknown functions $u(x,y,t)$, $h(x,t)$.

6.1.26 Remark: Suppose the motion is of persistent irrotationality. Then this corresponds to $\frac{\partial u}{\partial y}(x,y,0) = 0$ and hence $\frac{\partial u}{\partial v}(x,y,t) = 0$ for a small deformation around $t = 0$ and hence $u(x,t), h(x,t)$ satisfy equations (6.1.14) and (6.1.15) (Shallow water wave equations justifiably). The conservation laws of mass, momentum and energy are well known for fluids. Now we want to derive them for the long wave free surface problem.

(a) Conservation law of mass for a long wave:

Equation (6.1.22) gives $v = -\int_o^h u_x dy$ and substituting this in (6.1.25), we get $h_t + uh_x + \int_o^h u_x dy = 0$ which can be put in the form $h_t + \frac{\partial}{\partial x}\left(\int_o^h u dy\right) = 0$ (6.1.27) which is the *law of conservation of mass.*

(b) Conservation law of momentum for long wave:

Integrating (6.1.23) with respect to y from $y = 0$ to $y = h$ and using (6.1.22), (6.1.24) and (6.1.25) to eliminate v, h_t and h_x we get

$$\frac{\partial}{\partial t}\left(\int_o^h u dy\right) + \frac{\partial}{\partial x}\left(\int_o^h u^2 dy + \frac{gh^2}{2}\right) = 0 \qquad (6.1.28)$$

which is the *conservation law for momentum.*

(c) Conservation law of energy for long wave:

By multiplying (6.1.23) with u and integrating from $y = 0$ to $y — h$ with respect to y and using (6.1.22), (6.1.24) and (6.1.25) we get

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\int_o^h u^2 dy + \frac{gh^2}{2}\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\int_o^h u^3 dy + gh\int_o^h u dy\right) = 0 \qquad (6.1.29)$$

which is the *conservation law of energy.*

We **note** that from these three equations that $\int_o^h u^k dy (k = 1,2,3)$ are respectively unsolved.

6.1.30 Definition: For the horizontal velocity function $u(x,y,t)$ **of** a long wave we define its moment functions by

$$A_n = \int_0^h u^n(x,y,t)dy \tag{6.1.31}$$

Thus we have the sequence $A_o = h,\ A_1 = \int_o^h udy,\ A_2 = \int_o^h u^2dy,\ A_3 = \int_o^h u^3dy$ etc., $(n = 0,1,2,\ldots)$

6.1.32 Remarks: (1) From (6.1.27) to (6.1.29) we notice that the conservation laws can be written in the form $\frac{\partial P_n}{\partial t} + \frac{\partial Q_n}{\partial x} = 0 \quad (n = 1,2,3)$ (6.1.33)
where $P_1 = A_0,\ Q_1 = A_1$ (6.1.27); $P_2 = A_1,\ Q_2 = A_2 + \frac{1}{2}gA_o^2$ (6.1.28) and
$P_3 = \frac{1}{2}A_2 + \frac{1}{2}gA_o^2;\ Q_3 = \frac{1}{2}A_3 + gA_oA_1$ (6.1.29).
In general we expect P_n and Q_n to be polynomials in $A_o, A_1, ..A_n$ and also an infinite number of conservation laws for long wave equations as in (6.1.33) for $n = 1,2,$___
(2) More generally, as we derived (6.1.28) and (6.1.29) by induction, multiplying (6.1.23) by u^{n-1} and integrating with respect to y from $y = 0$ to $y = h$ and using (6.1.22), **(6.1.24)** and (6.1.25) and previous laws, we get an equation satisfied by the moment functions A_n as

$$\frac{\partial A_n}{\partial t} + \frac{\partial}{\partial x}A_{n+1} + ngA_{n-1}\frac{\partial A_o}{\partial x} = 0 \quad (n = 0,1,2,\ldots) \tag{6.1.34}$$

which we call the Benney's equations for long waves satisfied by the moment functions or simply Benney's moment equations. Note that these equations are not of conservation type (that is not of the form (6.1.33)). We give another proof of (6.1.34) in a later section.

6.1.35: Benney's recursive method for constructing an infinite number of conservation laws for long waves. We have the **Benney's** moment equations

$$\frac{\partial A_n}{\partial t} + \frac{\partial A_{n+1}}{\partial x} + ngA_{n-1}A_{o,x} = 0 \quad (n = 0,1,2,\ldots)\ .$$

Instead of handling the above infinite system of equations directly we introduce a generating function with $A_n's$ as co-efficients of power series

$$f(x,t,z) = \sum_{n=0}^{\infty} A_n(x,t)z^n \tag{6.1.36}$$

Consider the functional equation

$$zf_t + f_x = (1 - gz^2(zf)_z)f_x(0) \tag{6.1.37}$$

or more explicitly

$$\left(\frac{\partial}{\partial t} + \frac{1}{z}\frac{\partial}{\partial x}\right) f(x,t,z) = \left(\frac{1}{z} - gz\frac{\partial}{\partial z}(zf(x,t,z))\right)\frac{\partial f}{\partial x}(x,t,0) = 0\ . \tag{6.1.38}$$

Then the equation (6.1.33) are obtained from (6.1.38) by equating the coefficients of z^n on both sides in general case and the special case by taking $/ = \frac{n}{1-uz}$ (6.1.39).

6.1.40 Proposition: There exist an infinite number of conservation laws for the Benney system (6.1.33).

Proof: Multiply (6.1.39) by *zf* and then differentiate with respect to *z* and then multiply by gz^2 to get

$$\left(gz^2\left(\frac{z^2f^2}{2}\right)_z\right)_t + \left(gz^2\left(\frac{zf^2}{2}\right)_z\right)_x = \left(gz^2(zf)_z - gz^2\left(gz^2\left(\frac{z^2f^2}{2}\right)_z\right)_z\right)f_x(0)\ . \tag{6.1.41}$$

Adding (6.1.37) and (6.1.41) we get

$$\left(zf + gz^2\left(\frac{z^2f^2}{2}\right)_z\right)_t + \left(f + gz^2\left(\frac{zf^2}{2}\right)_z\right)_x = \left(1 - gz^2\left(gz^2\left(\frac{z^2f^2}{2}\right)_z\right)_z\right)f_x(0). \tag{6.1.42}$$

Note that (6.1.42) is in the conservation form except for the last term which involves a higher power of *z* than others. For example the next two steps we get after continuing this formal procedure (taking (6.1.37) as **step-1,** (6.1.42) as **step-2** and for n = 3 and

$n = 4$ the steps are :)

$$\mathbf{n=3:}\quad \left(zf + gz^2\left(\frac{z^2f^2}{2!}\right)_z + gz^2\left(gz^2\left(\frac{z^3f^3}{3!}\right)_z\right)_z\right)_t$$
$$+\left(f + gz^2\left(\frac{zf^2}{2!}\right)_z + gz^2\left(gz^2\left(\frac{z^2f^3}{3!}\right)_z\right)_z\right)_x$$
$$= \left(1 - gz^2\left(gz^2\left(gz^2\left(\frac{z^3f^3}{3!}\right)_z\right)_z\right)_z\right) f_x(0)\,. \qquad \textbf{(6.1.43)}$$

$$\mathbf{n=4:}\quad \left(zf + gz^2\left(\frac{z^2f^2}{2!}\right)_z + gz^2\left(gz^2\left(\frac{z^3f^3}{3!}\right)_z\right)_z\right.$$
$$\left. +gz^2\left(gz^2\left(gz^2\left(\frac{z^4f^4}{4!}\right)_z\right)_z\right)_z\right)_t$$
$$+\left(f + gz^2\left(\frac{zf^2}{2!}\right)_z + gz^2\left(gz^2\left(\frac{z^2f^3}{3!}\right)_z\right)_z\right.$$
$$\left. +gz^2\left(gz^2\left(gz^2\left(\frac{z^3f^4}{4!}\right)_z\right)_z\right)_z\right)_x$$
$$= \left(1 - gz^2\left(gz^2\left(gz^2\left(gz^2\left(\frac{z^4f^4}{4!}\right)_z\right)_z\right)_z\right)_z\right) f_x(0) \qquad (6.1.44)$$

and can be continued for $n = 5, 6\ldots$ formally.

Let L denote the differential operator $L = gz^2\frac{\partial}{\partial z}$. Then continuing (6.1.44) infinitely we get

$$\frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty} L^n\left(\frac{z^{n+1}f^{n+1}}{(n+1)!}\right)\right) + \frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty} L^n\left(\frac{z^nf^{n+1}}{(n+1)!}\right) - f(0)\right) = 0 \qquad \textbf{(6.1.45)}$$

Then in the equation (6.1.45) each power of z yields a distinct conservation law of the form $\frac{\partial P_n}{\partial t} + \frac{\partial Q_n}{\partial x} = 0$ and hence there are an infinite number $(n = 0, 1, 2, \ldots)$ of conserved laws for the Benney's system.

6.2 LAX REPRESENTATION AND APPLICATION OF ADLER-KOSTANT-SYMES (AKS) PRINCIPLE TO BENNEY'S SYSTEM:

The Benney's system is a system of non-linear long wave equations describing the motion of a two-dimensional inviscid, incompressible heavy fluid with a free surface in a gravitational field given in the long wave approximation by

$$\begin{aligned} u_t + uu_x - u_y \int_o^y u_x(x,\eta,t)d\eta + h_x &= 0 \\ h_t + \left(\int_o^h u dy\right)_x &= 0 \end{aligned} \tag{6.2.1}$$

Here $-\infty < x < \infty$ is the horizontal coordinate; $0 < y$ the vertical coordinate and t the time: $y = h(x,t)$ is the height of the free surface over a flat rigid bottom $y = 0$ (i.e., the free surface above $(x,0)$ at time t); $u(x,y,t)$ is the horizontal velocity component at (x,y) at time t. The notation u_t means $\frac{\partial}{\partial t}u(x,y,t)$ and likewise for the other derivatives. The units are so chosen that the gravitational constant and the density are unity.

6.2.2 Benney's lemma [31]: We define the moments $A_n(x,t) = \int_o^h u^n(x,y,t)dy$, $n > 0$. Then equation (6.2.1) implies the following equations for these moments:

$$A_{n,t} + A_{n+1,x} + n\, A_{n-1} A_{o,x} = 0 \tag{6.2.3}$$

6.2.4 Remarks: (1) The system of equations (6.2.3) possesses an infinite number of conservation laws of (6.2.1) in the form $H_{n,t} + F_{n,x} = 0$, $n > 0$ where H_n and F_n are two sequences of polynomials in $Q[A_o, \cdots A_n]$ and $Q[A_o, \cdots A_{n+1}]$ respectively.

(2) The Benney's system (6.2.1) admits a standard Lax representation widely used in 'the inverse scattering method' of solving wave equations, and the conserved densities H_n are given by the formula (6.2.6) described below.

Let ξ be a formal 'spectral parameter' and we set

$$\begin{aligned} \Phi(\xi) &= \int_o^h \frac{dy}{\xi - u(x,y,t)} = \int_o^h \frac{dy}{\xi(1-\frac{u}{\xi})} = \int_o^h \frac{dy}{\xi}\left(1 - \frac{u}{\xi}\right)^{-1} \\ &= \int_o^h \frac{dy}{\xi}\left(1 + \frac{u}{\xi} + \frac{u^2}{\xi^2} + \cdots + \frac{u^i}{\xi^i} + \cdots\right) \\ &= \int_o^h dy\left(\frac{1}{\xi} + \frac{u}{\xi^2} + \frac{u^2}{\xi^3} + \cdots + \frac{u^i}{\xi^{i+1}} + \cdots\right) \\ &= \sum_{i=o}^{\infty} \int_o^h \left(\frac{u^i}{\xi^{i+1}}\right) dy = \sum_{i=o}^{\infty} \int_o^h \frac{u^i}{\xi^{i+1}} dy \end{aligned} \tag{6.2.5}$$

$$= \sum_{i=o}^{\infty} A_i \xi^{(i+1)} \left(\text{since } A_n = \int_o^h u^n(x,y,t)dy \ , \ n \geq 0 \right) .$$

The conserved densities H_n are given by the formula

$$H_n = \text{res}_\xi(\xi + \Phi(\xi))^n \ , \tag{6.2.6}$$

where res_ξ means the coefficient of ξ^{-1} in the corresponding formal series. We define two first order differential operators L and P by

$$\begin{aligned} L &= (1 + \Phi_\xi)\frac{\partial}{\partial x} - \Phi_x \frac{\partial}{\partial \xi} \ , \\ P &= \frac{\partial}{\partial x} - A_{o,x} \frac{\partial}{\partial \xi} \ . \end{aligned} \tag{6.2.7}$$

Then system (6.2.3) is equivalent to the following Lax type equation:

$$L_t = [L, P] \ . \tag{6.2.8}$$

6.2.9 Remark: The Benney's system satisfies the AKS principle, that is, they fit into a general scheme of constructing Hamiltonian systems with Lax representation and involutive conservation laws and that the relevant Hamiltonian structure could be identified with the canonical symplectic form on the orbits of the co-adjoint representation of a convenient Lie algebra.

To be more specific we review the essential features of the finite dimensional situation. Let $\mathcal{G}$ be a Lie algebra equipped with an invariant, non-degenerate scalar product $(,)$. Suppose $\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-$ where $\mathcal{G}_\pm$ are Lie subalgebras and $(\mathcal{G}_+, \mathcal{G}_+) = (\mathcal{G}_-, \mathcal{G}_-) = 0$. For $u \in \mathcal{G}$, we denote by $u_\pm$ the components of u in $\mathcal{G}_\pm$. For a differentiable function f on $\mathcal{G}_-$, we define its gradient $df : \mathcal{G}_- \to \mathcal{G}_-$ by $(v, df(u)) = \frac{d}{dt} f(u + tv)|_{t=o} \quad \forall u, v \in \mathcal{G}$.

We identify $\mathcal{G}_-$ with the dual space to $\mathcal{G}_+$ via the scalar product $(,)$, i.e., $\mathcal{G}_- = (\mathcal{G}_+)^*$. Then we can introduce the symplectic structure on the orbits of the co-adjoint representation of $\exp(\mathcal{G}_+)$ in $\mathcal{G}_-$ following Kirillov [24]. The Poisson bracket of functions on $\mathcal{G}_-$

is defined by the formula $\{h, f\}(u) = (u, [dh(u), df(u)])$, $\forall u \in G_-$ and the flow on $\mathcal{G}_-$ corresponding to the Hamiltonian h is defined by the vector field $u_t = [u, dh(u)]_-$.

We describe below a very special case of Hamiltonians which pairwise commute and also give rise to Lax type flows in G_-.

Suppose that $\exists$ a vector $w \in G$ s.t. $[w, \mathcal{G}_+] \subset \mathcal{G}_+$ and $[w, \mathcal{G}_-] \subset \mathcal{G}_-$. A function f is called invariant on G if $[u, df(u)] = 0$ $\forall u \in \mathcal{G}$. For an invariant function f, set $f_w(u) = f(u + w) \forall u \in \mathcal{G}_-$. Then $\{f_w, g_w\} = 0$ on $\mathcal{G}_-$ if f and g are invariant and the Hamiltonian flow $u_t = [u, df_w(u)]_-$ can be written in the equivalent Lax form $(u + w)_t = [u + w, df_w(u)]$.

i.e., $\mathcal{H} = \{f_w / w$ is fixed vector such that $[w, G_+] \subset G_+$ and $[w, \mathcal{G}_-] \subset G_-\}$ is an algebra of first integrals depending on w and the corresponding flows are of Lax type. This family of first integrals obeys the AKS principle but something more, that it is parametrized by a fixed vector $w \in \mathcal{G}$.

To apply the above formalism to partial differential equations, we work with infinite dimensional Lie algebras.

Let $\mathcal{B}$ be a differentiable ring. Let $\mathcal{B}((\xi^{-1}))$ denote the ring of formal Laurent series $\sum_i X_i \xi^{-i}$ over B with a finite number of positive powers. In this ring there are two differentiations: $\partial_\xi = \frac{\partial}{\partial \xi}$ and $\partial_i : \sum X_i \xi^{-i} \rightarrow \sum_i \partial X_i \xi^{-i}$. By means of this there is defined on $\mathcal{B}(\xi^{-1})$ an operation of multiplication $aob = \sum_{k>0} \frac{1}{k!} \partial_\xi^k \; a \; \partial^k \; b$. (†).

This operation is associative and therefore converts $\mathcal{B}((\xi^{-1}))$ into a ring. We define the bracket $[a, b]_o$ as $[a, b]_o = \partial_\xi \; a \; \partial_x \; b - \partial_\xi \; b \; \partial_x \; a$ and scalar product from $/ : B \rightarrow \frac{B}{\partial_x B}$ by setting $(a, b) \mapsto \int \text{res}_\xi(ab) dx$.

Now, we set

$$\mathcal{G}_{o+} = (\text{polynomials in } \xi)$$

$$\mathcal{G}_{o-} = (\text{formal series in } \xi^{-1} \text{ without constant term})$$

We consider two invariant functionals on $\mathcal{G}_{o-}$, $f_k : u \to \int \text{res } u^k$ and $\xi = w$. Then $df_{k,\xi}(u) = k(\xi + u)_+^{k-1}$ on $\mathcal{G}_{o-}$. (i.e., $f_{k,\xi}(u) = (u+\xi)^k$, $df_{k,\xi}(u) = k(u+\xi)_+^{k-1}$).
We take $u = \sum_{i=0}^{\infty} A_i \xi^{-(i+1)} \in \mathcal{G}_{o-}$.

6.2.10 Proposition: We prove the following:

(1) The Lax equation $(\xi + u)_t = [\xi + u, \frac{1}{2}(\xi + u)_+^2]$ (6.2.11)

is equivalent to Benney's flow (6.2.3) $(A_{n,t} + A_{n+1,x} + nA_{n-1}A_{o,x} = 0)$ on the coefficients A_i.

(2) The conserved densities are given by $res(u + \xi)^k$.

(3) To get equations (6.2.7) and (6.2.8) from the Lax equation, by replacing u by the corresponding vector field of u, $(\partial_\xi u)\partial_x - (\partial_x u)\partial_\xi$, where $u \in g_{o-}$.

Proof: The Benney's flow (6.2.3) is given by

$$A_{n,t} + A_{n+1,x} + nA_{n-1}A_{o,x} = 0 \ , \ n \geq 0 \ .$$

To get this flow equation (6.2.3) from the Lax equation (6.2.11),

$$(\xi + u)_t = [\xi + u, \frac{1}{2}(\xi + u)_+^2]_o$$

where $L = \xi + u$, $P = \frac{1}{2}(\xi + u)_+^2$ and where $u \to (\partial_\xi u)\partial_x - (\partial_x u)\partial_\xi$.

Proof of (3): If $u = \xi + \Phi(\xi)$, then replacing u by $(\partial_\xi u)\partial_x - (\partial_x u)\partial_\xi$, we get $(1 + \Phi_\xi)\partial_x - \Phi_x \partial_\xi$ which is L.

Calculating $\frac{1}{2}(\xi + u)_+^2$:

$$\begin{aligned}
(\xi + u) &= \xi + \sum_{i=o}^{\infty} A_i(x,t)\xi^{-(i+1)} \\
&= \xi + A_o(x,t)\xi^{-1} + A_1(x,t)\xi^{-2} + A_2(x,t)\xi^{-3} + \cdots \\
\frac{1}{2}(\xi + u)_+^2 &= \xi^2 + \left(\sum_{i=o}^{\infty} A_i(x,t)\xi^{-(i+1)}\right)^2 + 2\xi\left(\sum_{i=o}^{\infty} A_i(x,t)\xi^{-(i+1)}\right) \\
&= \xi^2 + [(A_o(x,t)\xi^{-1} + A_1(x,t)\xi^{-2} + A_2(x,t)\xi^{-3} + \cdots)
\end{aligned}$$

$$
\begin{aligned}
& \times \left(A_o(x,t)\xi^{-1} + A_1(x,t)\xi^{-2} + \cdots\right)] \\
& +2\left(A_o(x,t) + A_1(x,t)\xi^{-1} + A_2(x,t)\xi^{-2} + \cdots\right) \\
= \; & \xi^2 + A_o^2\xi^{-2} + A_oA_1\xi^{-3} + A_oA_2\xi^{-4} + \cdots + 2A_o + 2A_1\xi^{-1} + 2A_2\xi^{-2} + \cdots \\
= \; & \xi^2 + A_o^2\xi^{-2} + A_oA_1\xi^{-3} + A_oA_2\xi^{-4} + \cdots + 2A_o + 2A_1\xi^{-1} + 2A_2\xi^{-2} + \cdots
\end{aligned}
$$

Out of this, we collect the terms which is a polynomial in ξ, namely, $\frac{1}{2}(\xi^2 + 2A_o)$. Now replacing $u \rightarrow (\partial_\xi u)\partial_x$ - $(\partial_x u)\partial_\xi$,we get

$$\xi\partial_x - A_{o,x}\frac{\partial}{\partial\xi} = P \ .$$

Therefore, by replacing $u \mapsto (\partial_\xi u)\partial_x — (\partial_x u)\partial_\xi$ in the Lax equation (6.2.11), we have obtained the operators L and P given in equations (6.2.7) and (6.2.8). This proves **part (3)** of Lemma.

Proof of (2): By (6.2.6) $(H_n = \text{res}_\xi(\xi + \Phi(\xi))^n)$, we have,

$$H_n = \text{res}_\xi(\xi + \Phi(\xi))^n = \text{res}_\xi(\xi + u)^n \ \ (\text{since} \ \ x = \Phi(\xi) \in \mathcal{G}_{o-})$$

Proof of (1): To show that the Lax equation (6.2.11) is equivalent **to** Benney's **flow (6.2.3).**

$$
\begin{aligned}
(\xi + u) &= \xi + A_o\xi^{-1} + A_1\xi^{-2} + A_2\xi^{-3} + A_3\xi^{-4} + \cdots \\
(\xi + u)_t &= A_{o,t}\xi^{-1} + A_{1,t}\xi^{-2} + A_{2,t}\xi^{-3} + A_{3,t}\xi^{-4} + \cdots \qquad (6.2.12) \\
\left[(\xi+u), \frac{1}{2}(\xi+u)_+^2\right]_o &= \frac{\partial}{\partial\xi}(\xi+u)\frac{\partial}{\partial x}\left(\frac{1}{2}(\xi+u)_+^2\right) \\
& \quad -\frac{\partial}{\partial\xi}\left(\frac{1}{2}(\xi+u)_+^2\right)\frac{\partial}{\partial\xi}(\xi+u) \\
&= \frac{\partial}{\partial\xi}(\xi + A_o\xi^{-1} + A_1\xi^{-2}A_2\xi^{-3} + \cdots)\frac{\partial}{\partial x}(\xi+u)) \\
& \quad -\frac{1}{2}.2.\xi\frac{\partial}{\partial x}(\xi + A_o\xi^{-1} + A_1\xi^{-2} + A_2\xi^{-3} + \cdots) \\
&= (1 - A_o\xi^{-2} - 2A_1\xi^{-3} - A_2\xi^{-4})
\end{aligned}
$$

$$-\xi(A_{o,x}\xi^{-1} + A_{1,x}\xi^{-2} + A_{2,x}\xi^{-3} + \cdots)$$

$$(1 - A_o\xi^{-2} - 2A_1\xi^{-3} - 3A_2\xi^{-4}\cdots)$$

$$1 - A_o(A_{o,x}\xi^{-2} - A_1A_{o,x}\xi^{-3} - 3A_2A_{o,x}\xi^{-4}\cdots$$

$$-A_{o,x} - A_{1,x}\xi^{-1} + A_{2,x}\xi^{-2}\cdots$$

$$\begin{aligned}
\frac{1}{2}(\xi+u)_+^2 &= \frac{1}{2}(\xi^2 + u^2 + 2\xi u)_+ \\
&= \frac{1}{2}(\xi^2 + A_o^2\xi^{-2} + A_oA_1\xi^{-3} + A_oA_2\xi^{-4} + A_oA_3\xi^{-5} + \cdots) \\
&\quad +A_oA_2\xi^4 + A_1A_2\xi^{-5} + \cdots + A_oA_4\xi^{-6} + A_1A_4\xi^{-7} + \cdots \\
&\quad +2A_o\xi^o + 2A_1\xi^{-1} + 2A_2\xi^{-2} + \cdots)_+ \\
&= \frac{1}{2}(\xi^2 + 2A_o) = \frac{1}{2}\xi^2 + A_o\ . \\
\left[\xi+u, \frac{1}{2}(\xi+u)_+^2\right]_o &= \frac{\partial}{\partial\xi}(\xi+u)\frac{\partial}{\partial x}\left(\frac{1}{2}(\xi+u)_+^2\right) \\
&\quad -\frac{\partial}{\partial\xi}\left(\frac{1}{2}(\xi+u)_+^2\right)\frac{\partial}{\partial x}(\xi+u) \\
&= (1+u)A_{o,x} - \xi(A_{o,x}\xi^{-1} + A_{1,x}\xi^{-2} + \cdots) \\
&= 1 - (A_o\xi^{-2} + 2A_1\xi^{-3} + 3A_2\xi^{-4} + \cdots)A_{o,x} \\
&\quad -(A_{o,x} + A_{1,x}\xi^{-1} + A_{2,x}\xi^{-2}\cdots) \\
&= 1 - (A_oA_{o,x}\xi^{-2} + 2A_{o,x}A_1\xi^{-3} + 3A_2A_{o,x}\xi^{-4}\cdots) \\
&\quad -(A_{o,x} + A_{1,x}\xi^{-1} + A_{2,x}\xi^{-2} + \cdots)
\end{aligned} \tag{6.2.13}$$

Comparing coefficients on both sides of (6.2.12) and (6.2.13) of powers of ξ, we get:

$$\begin{aligned}
\text{Coefficient of}\quad \xi^{-1} : A_{o,x} &= A_{1,x} \\
\xi^{-2} : A_{1,t} &= -A_oA_{o,x}A_{1,x} - A_{2,x} \\
\xi^{-3} : A_{2,t} &= -2A_{o,x}A_1 - A_{3,x} \\
\xi^{-4} : A_{3,t} &= -3A_{o,x}A_2 - A_{4,x}
\end{aligned}$$

i.e., $A_{n,t} = -nA_{n-1}A_{o,x} - A_{n+1,x}$

i.e., $A_{n,t} + nA_{n-1}A_{o,x} + A_{n+1,x} = 0$

i.e., $A_{n,t} + A_{n+1} + nA_{n-1}A_{o,x} = 0$ which is Benney's flow (6.2.3). This proves part (1) of lemma and hence the lemma is proved.

6.2.14 Classical Case: (Benney's system as quasiclassical limit of the KP system or the generalized KdV equations:) Now, suppose the Lie algebra $\mathcal{G}_1 = \mathcal{B}((\xi^{-1}))$ is defined with a new bracket $[a, b]_1 = a \circ b - b \circ a$ where $a \circ b = (a \circ_\epsilon b)_{\epsilon=1}$, $a \circ_\epsilon b = \sum_{k>0} \frac{1}{k!}\partial_\xi^k a \, (\epsilon\partial_x^k)b$, and considering flows on $\mathcal{G}_{1+}$ instead of $\mathcal{G}_o$. Then we can introduce the Lie algebra $\mathcal{G}_\epsilon$ with bracket $[a, b]_\epsilon = \epsilon^{-1}(a \circ_\epsilon b - b \circ_\epsilon a)$. Then $[a, b]_o = \lim_{\epsilon \to 0}[a, b]_\epsilon$. Therefore, the Lie algebra $\mathcal{G}_o$ is called the quasiclassical limit of $\mathcal{G}_1$.

i.e., Benney's system is the quasiclassical limit of the generalized KdV equations, because for $\epsilon = 1$, $a \circ b = a \circ_1 b = \sum_{k>0} \frac{1}{k!}\partial_\xi^k \, a \, \partial_x^k \, b$ coincides with the operation of multiplication ($\dagger$) defined for Benney's system. Here, $\mathcal{G}_1 = \mathcal{B}((\xi^{-1}))$ with bracket $[a, b]_1$ is the Lie algebra defined for the generalized KdV equations or KP equations.

6.2.15 Remark: The Poisson bracket {, } for Benney's system is the standard one on $T^*(\mathbb{R}^1)$:

$$\{F, G\} = F_{,\xi}G_{,x} - F_{,x}G_{,\xi}$$

This Poisson bracket on $T^*(\mathbb{R}^1)$ is the quasiclassical limit (=zero dispersion) of the commutator

$$[F, G] = F \circ G - G \circ F$$

where

$$F \circ G = \sum_{n \geq 0} \frac{1}{n!} \frac{\partial^n F}{\partial \xi^n} \partial^n(G)$$

i.e., The Poisson bracket coincides with the commutator defined for the generalized KdV equations (or KP hierarchy).

Thus, the Benney hierarchy is the quasiclassical limit of the KP hierarchy.

6.3 THE HAMILTONIAN STRUCTURE OF BENNEY'S SYSTEM (ANALYTIC APPROACH):

The Benney's system is given in section 6.2. The Benney's system has the following moment representation:

$$A_{i,t} = A_{i+1,x} + iA_{i-1}A_{o,x} \quad i \in \mathbb{Z}_+ \tag{6.3.1}$$

where A_i **=** ***Ai(x,t).***

This system has many remarkable properties:

(i) There exists an infinite number of polynomial conserved densities $H_i \in A_i + \mathbb{Z}[A_o, \cdots A_{i-2}]$, $i \in \mathbb{Z}_+$ starting with

$$\begin{aligned} H_o &= A_o, \quad H_1 = A_1, \quad H_2 = A_2 + A_o^2 \\ H_3 &= A_3 + 3A_oA_1, \quad H_4 = A_4 + 4A_oA_2 + 2A_1^2 + 2A_o^3 \\ H_5 &= A_5 + 5A_oA_3 + 5A_1A_2 + 10A_o^2A_1, \text{etc.,} \end{aligned} \tag{6.3.2}$$

We give below a method of construction of the $H_i's$.

6.3.3 Theorem: We set

$$\begin{aligned} \Phi(\lambda) &= \sum_{i=o}^{\infty}(-1)^i A_i \lambda^{-(i+1)} \\ &= \int_o^h (\lambda + u)^{-1} dy \ . \end{aligned}$$

Then there is a unique solution of the equation

$$\mu(\lambda) + \Phi(\mu(\lambda)) = \lambda \tag{6.3.4}$$

(i) in the class of formal series of the form

$\lambda + Q[A_o, \cdots A_i, \cdots][[\lambda^{-1}]]$;

(ii) in the class functions of A which are of the form A + $0(\lambda^{-1})$ and analytic in some neighbourhood of oo (depending on $u(x, y, t)h(x, t)$).

Proof: Set

$$\mu(\lambda) = \lambda - \sum_{i=-1}^{\infty} (-1)^i H_i \lambda^{-(i+1)} \tag{6.3.5}$$

We determine the coefficients H_i from Equation (6.3.4) which we write in the form:

$$\sum_{i=-1}^{\infty} (-1)^i H_i \lambda^{-(i+1)} - \sum_{i=o}^{\infty} \lambda^{-(i+1)} A_i (-1)^i \left[1 - \sum_{j=-1}^{\infty} (-1)^j H_j \lambda^{-(j+2)}\right]^{-(i+1)} = 0 \tag{6.3.6}$$

i.e., we have Equation(6.3.4): $\mu(\lambda) + \Phi(\mu(\lambda)) = \lambda$.

Substituting $\mu(\lambda)$ defined by Equation (6.3.5) in this Equation (6.3.4), we get

$$\begin{aligned}
\lambda - \sum_{i=-1}^{\infty} (-1)^i H_i \lambda^{-(i+1)} + \Phi\left(\lambda - \sum_{i=-1}^{\infty} (-1)^i H_i \lambda^{-(i+1)}\right) &= \lambda \\
\Rightarrow -\sum_{i=-1}^{\infty} (-1)^i H_i \lambda^{-(i+1)} + \sum_{i=o}^{\infty} (-1)^i A_i \left(\lambda - \sum_{j=-1}^{\infty} (-1)^j H_j \lambda^{-(j+1)}\right)^{-(i+1)} &= 0 \\
\Rightarrow -\sum_{i=-1}^{\infty} (-1)^i H_i \lambda^{-(i+1)} + \sum_{i=o}^{\infty} (-1)^i A_i \left(\lambda\left(1 - \frac{1}{\lambda}\sum_{j=-1}^{\infty} (-1)^j H_j \lambda^{-(j+1)}\right)\right)^{-(i+1)} &= 0 \\
\Rightarrow -\sum_{i=-1}^{\infty} (-1)^i H_i \lambda^{-(i+1)} + \sum_{i=o}^{\infty} (-1)^i A_i \lambda^{-(i+1)} \left(1 - \sum_{j=-1}^{\infty} (-1)^j H_j \lambda^{-(j+2)}\right)^{-(i+1)} &= 0 \\
\Rightarrow \sum_{i=-1}^{\infty} (-1)^i H_i \lambda^{-(i+1)} - \sum_{i=o}^{\infty} (-1)^i A_i \lambda^{-(i+1)} \left(1 - \sum_{j=-1}^{\infty} (-1)^j H_j \lambda^{-(j+2)}\right)^{-(i+1)} &= 0
\end{aligned}$$

which is equation (6.3.6).

From this equation (6.3.6), for various values of $i = -1, 0, 1, 2, \cdots$, etc, we get $H_o, H_1, H_2, \cdots$

(i) <u>$i = -1$:</u> $-H_{-1}\lambda = 0 \Rightarrow H_{-1} = 0$

(ii) <u>$i = 0$:</u> $H_o\lambda^{-1} - (A_o\lambda^{-1}(1 - (-H_{-1}\lambda^{-3} + H_o\lambda^{-2})^{-1}) = 0$

$\underline{i=0:}$

$$\begin{aligned}
H_o\lambda^{-1} &= A_o\lambda^{-1}(1-x)^{-1} \\
&= A_o\lambda^{-1}(1+x+x^2+\cdots) \\
&= A_o\lambda^{-1}(1+(-H_{-1}\lambda^{-1}+H_o\lambda^{-2}-H_1\lambda^{-3}+H_2\lambda^{-4}+\cdots) \\
&\quad +(H_{-1}\lambda^{-1}+H_o\lambda^{-2}\cdots)^2+\cdots \\
&= A_o\lambda^{-1}+A_oH_o\lambda^{-3}+\cdots \\
\Rightarrow H_o &= A_o
\end{aligned}$$

where

$$\begin{aligned}
x &= \sum_{j=-1}^{\infty}(-1)^j H_j\lambda^{-(j+2)} \\
&= -H_{-1}\lambda^{-1}+H_o\lambda^{-2}-H_1\lambda^{-3}+H_2\lambda^{-4}+\cdots \\
&= H_o\lambda^{-2}-H_1\lambda^{-3}+H_2\lambda^{-4}+\cdots(\text{ since } H_{-1}=0)
\end{aligned}$$

$\underline{i=1:}$

$$\begin{aligned}
H_o\lambda^{-1}-H_1\lambda^{-2} &= A_o\lambda^{-1}(1-x)^{-1}-A_1\lambda^{-2}(1-x)^{-2} \\
&= A_o\lambda^{-1}(1+x+x^2+\cdots)-A_1\lambda^{-2}(1+2x+3x^2+\cdots) \\
&= A_o\lambda^{-1}(1-H_1\lambda^{-1}+H_o\lambda^{-2}-H_1\lambda^{-3}+\cdots) \\
&\quad -A_o\lambda^{-2}(1-2H_{-1}\lambda^{-1}+2H_o\lambda^{-2}-2H_1\lambda^{-3}+\cdots)
\end{aligned}$$

$\underline{i=2:}$ $\Rightarrow -H_1 = -A_1 \qquad \Rightarrow H_1=A_1$

$$\begin{aligned}
H_o\lambda^{-1}-H_1\lambda^{-2}+H_2\lambda^{-3} &= A_o\lambda^{-1}(1-x)^{-1}-A_1\lambda^{-2}(1-x)^{-2}+A_2\lambda^{-3}(1-x)^{-3} \\
&= A_o\lambda^{-1}(1+x+x^2+\cdots)-A_1\lambda^{-2}(1+2x+3x^2+\cdots) \\
&\quad +A_2\lambda^{-3}(1+3x+6x^2+\cdots) \\
&= A_o\lambda^{-1}(1+H_o\lambda^{-2}-H_1\lambda^{-3}+H_2\lambda^{-4}+\cdots) \\
&\quad -A_1\lambda^{-2}(1+2H_o\lambda^{-2}-2H_1\lambda^{-3}+2H_2\lambda^{-4}+\cdots) \\
&\quad +A_2\lambda^{-3}(1+3H_o\lambda^{-2}-3H_1\lambda^{-3}+3H_2\lambda^{-4}+\cdots) \\
\Rightarrow H_2 &= A_o^2+A_2 \qquad \Rightarrow H_2=A_2+A_o^2
\end{aligned}$$

and similarly continuing this, we can compute

$$\begin{aligned}
H_3 &= A_3 + 3A_oA_1 \\
H_4 &= A_4 + 4A_oA_2 + 2A_1^2 + 2A_o^3 \\
H_5 &= A_5 + 5A_oA_3 + 5A_1A_2 + 10A_o^2A_1 \\
H_6 &= A_6 + 6A_oA_4 + 6A_1A_3 + 3A_2^2 + 15A_o^2A_2 + 15A_oA_1^2 + 5A_o^4 \\
H_7 &= A_7 + 7A_oA_5 + 7A_1A_4 + 7A_2A_3 + 21A_o^2A_3 + 42A_oA_1A_2 + 7A_1^3 + 35A_o^3A_2 \\
H_8 &= A_8 + 8A_oA_6 + 8A_1A_5 + 8A_2A_4 + 56A_oA_1A_3 + 28A_oA_2^2 + 4A_o^2 \\
&\quad +28A_o^2A_4 + 28A_1^2A_2 + 56A_o^3A_2 + 84A_o^3A_1^2 + 14A_o^5 \text{ etc..,}
\end{aligned}$$

Thus, $H_n = A_n + P_n$, where $P_n \in \mathbb{Z}[A_o, \cdots, A_{n-2}, H_o, \cdots H_{n-2}]$ for $n \geq 2$. Hence we have that $H_n \in A_n + \mathbb{Z}[A_o, \cdots A_{n-2}]$ exists and is unique. For example, we can get H_6, H_7, H_8 as given above.

(b) The function $\mu(\lambda)$ as analytic function:

Consider equation (6.3.4),

$$\mu(\lambda) + \int_o^h (\mu(\lambda) + u)^{-1} dy = \lambda \ .$$

In this we set $\mu(\lambda) = \lambda + \epsilon(\lambda)$. The uniqueness of an analytic function μ of λ satisfying equation (6.3.4) and the estimate $\epsilon = 0(\lambda^{-}1)$ follows from the uniqueness of the formal series.

To prove existence, we set $\epsilon_o = 0$, $\epsilon_N = -\int_o^h (u + \lambda + \epsilon_{N-1})^{-1} dy$ and show that the limit $\epsilon = \lim_{N\to\infty} \epsilon_N$ exists (for given x, t) uniformly in λ, when $|\lambda|$ is so large that

$$h\left(|\lambda| - U\right)^{-1}\left(1 - 4h(|\lambda| - U)^{-2}\right)^{-1} \leq \frac{(|\lambda| - U)}{2}$$

$$|\lambda| - U > 2\sqrt{h}\ ,\ U = \sup\{u/o \leq y \leq h\}\ . \qquad (6.3.7)$$

In particular, if u and h are **bounded,** the region of analyticity of μ does not depend on x and t. We establish the following inequalities by induction on x:

$$\begin{aligned} |\epsilon_n| &\leq \frac{(|\lambda|-U)}{2} \\ |\epsilon_n - \epsilon_{n-1}| &\leq \theta|\epsilon_{n-1} - \epsilon_{n-2}| \ , \ \theta = \frac{4h}{(|\lambda|-U)^2} < 1 \end{aligned} \qquad (6.3.8)$$

We have

$$\begin{aligned} |\epsilon_1| &= \left|\int_o^h (u+\lambda)^{-1} dy\right| \\ &\leq h \sup |u+\lambda|^{-1} \\ &\leq h(2\sqrt{h})^{-1} \\ &< \frac{(|\lambda|-U)}{4} < \frac{(|\lambda|-U)}{2} \end{aligned}$$

Also, for any $N \geq 1$, we have $|\epsilon_{N+1} - \epsilon_N| \leq h \sup |(u+\lambda+\epsilon_N)(u+\lambda+\epsilon_{N-1})|^{-1} |\epsilon_N - \epsilon_{N-1}|$.
Setting $N = 1$ and using (6.3.8), for $N = 1$ we get by (6.3.7):
$|u+\lambda|^{-1} \leq (|\lambda|-U)^{-1}$,
$|u+\lambda+\epsilon_1|^{-1} \leq 2(|\lambda|-U)^{-1}$, hence $|\epsilon_2 - \epsilon_1| \leq 2h(|\lambda|-U)^{-2}|\epsilon_1| < \forall\, h(|\lambda|-U)^{-2}|\epsilon_1|$
This is the induction step. Suppose that (6.3.8) holds $\forall\, n \leq N$. Then

$$\begin{aligned} |\epsilon_{N+1} - \epsilon_N| &\leq h \sup |(u+\lambda+\epsilon_N)(u+\lambda+\epsilon_{N-1}|^{-1} |\epsilon_N - \epsilon_{N-1}| \\ &\leq \theta|\epsilon_N - \epsilon_{N-1}| \ \text{by (6.3.8)} \quad \text{for} \ n = N-1, N \ . \end{aligned}$$

Hence

$$\begin{aligned} |\epsilon_{N+1}| &\leq \sum_{n=1}^{N+1} |\epsilon_N - \epsilon_{N-1}| \\ &< (1-\theta)^{-1}|\epsilon_1| \\ &\leq (1-\theta)^{-1} h(|\lambda|-U)^{-1} \\ &\leq \frac{(|\lambda|-U)}{2} \ . \end{aligned}$$

This completes the proof of theorem.

(ii) There exists an infinite number of "higher" Benney equations having the same infinite set (6.3.2) of polynomial conserved densities. In particular, the next flow has the form $A_{i,t} = A_{i+2,x} + A_o A_{i,x} + (i+1) A_i A_{o,x} + i A_{i-1} A_{1,x}, i \in \mathbb{Z}_+$. **This is given the following** theorem (Theorem 6.3.9).

Let $\mathcal{A} = Q[A_i^{(j)}]$ be a differential ring with differentiation operators $\partial = \frac{\partial}{\partial x}$: $A_i^{(j)} \to A_i^{(j+1)}$ and partial variational derivatives

$$\frac{\delta}{\delta u_i} = \sum_{j=o}^{\infty} (-i)^j \partial^j \frac{\partial}{\partial u_i^{(j)}} ,$$

$$\frac{\delta}{\delta \bar{u}} = \left(\frac{\delta}{\delta u_1} , \cdots \frac{\delta}{\delta u_q} \right)^t : \mathcal{A} \to \mathcal{A}^q$$

6.3.9 Theorem [32] : Let $H \in \mathcal{A}$ be any element. Then from a system of equations for u and h of the form

$$u_t = \left(\sum_{j=o}^{\infty} u^j H_{(j)} \right)_x - u_y \int_o^y \left(\sum_{j=o}^{\infty} u^{j-1} H(j) \right)_x \Big|_{y=\eta} dy ,$$

$$h_t = \left(\sum_{j=o}^{\infty} j A_{j-1} H_{(j)} \right)_x \qquad \text{where } H(j) = \frac{\delta H}{\delta A_j} , \qquad (6.3.10)$$

one gets a system of equations $\bar{A}_t = \frac{B \delta H}{\delta A}$ with Hamiltonian H.

(iii) All these flows commute between themselves. All these flows are Hamiltonian, with Hamiltonian structure

$$B_{ij} = i A_{i+j-1} \partial + \partial_j A_{i+j-1} \ , \ i, j \in \mathbb{Z}_+ \ , \ \partial = \frac{\partial}{\partial x}$$

so that the flow $\#m$ can be written as

$$A_{i,t} = \sum_j B_{ij} \left(\frac{\partial \bar{H}}{\partial A_j} \right) \ , \ \bar{H} = \frac{1}{m} H_m \ , \ m \in \mathbb{N}$$

To prove the above two properties, we consider the following:

We prove that the operator $B = B_1\partial + \partial o B_1^t$, $B_{1,ij} = iA_{i+j-1}, i, j > 0$ is Hamiltonian and the elements $H_i \in A$ defined by

$$\mu(\lambda) = \lambda - \sum_{i=-1}^{\infty}(-1)^i H_i \lambda^{-(i+1)}$$

commute with respect to the Hamiltonian structure with operator B.

Let $A = Q[A_i^{(j)}]$ be a differential ring with differentiation operators $d = \frac{\partial}{\partial x}$: $A_i^{(j)} \to A_i^{(j+1)}$ and partial variational derivative $\frac{\delta}{\delta u_i} = \sum_{j=o}^{\infty}(-1)^j \partial^j \frac{\partial}{\partial u^j}$, $\frac{\delta}{\delta \bar{u}} = (\frac{\delta}{\delta u_1}, \cdots \frac{\delta}{\delta u_q})$: $\mathcal{A} \to \mathcal{A}^q$. In this ring $\mathcal{A}$, P denotes an infinite column vector $P_o, P_1, P_2, \cdots$ and P^t denotes the corresponding infinite row vector. Also, $\frac{\delta}{\delta A}$: $A \longrightarrow \mathcal{A}^{\infty}$ is the operator $P \mapsto (\frac{\delta P}{\delta A_o}, \frac{\delta P}{\delta A_1}, \cdots)^t$. $\bar{P}$ is finite if it has only a finite member of non-zero components.

Let $B_o, \cdots B_d$ be matrices over A and $B = \sum_{i=1}^{d} B_i \partial^i - (-1)^i \partial^i o B_i^t$ be a formally antisymmetric differential operator. If B is given, then any element $P \in A$ uniquely defines a differentiation $X_P : A \to A$ with the properties $[X_P, \partial] = 0$ and $X_P A = (X_P A_o, \cdots X_P A_i, \cdots)^t = \frac{B\delta P}{\delta A}$. The differentiation X_P is a formal model of an evolutionary system of equations for A_i of the form $A_t = \frac{B\delta P}{\delta A}$. The operator B is called Hamiltonian if for any $P, Q \in A$, we have $[X_Q, X_P] = X_R$, where $R = X_Q P$. If B is Hamiltonian, the equations $A_t = \frac{B\delta P}{\delta A}$ are called Hamiltonian with Hamiltonian P and operator B.

6.3.11 Theorem: Let $B = B_1\partial + \partial o B_1^t$, where $B_{1,ij} = iA_{i+j-1}$, $i, j > 0$. Then the operator B is Hamiltonian.

We prove this theorem using a few lemmas.

(Let $k \subset A$ be a commutative ring with an identity of characteristic 0 and $d : A \to A$ be a derivation with the condition $dk < k$. The triplet (k, A, d) is called a differential k-algebra. Let the symbol $\Omega^1 A$ denote the module of k-differentials of the algebra A and the symbol $S : A \to \Omega^1 A$ denote the mapping of the universal differential).

Let $\delta : A \to \Omega^1 \mathcal{A}$ be the universal differentiation in the ring A with values in the

$\mathcal{A}$-module of differentials. The $\mathcal{A}$-module is freely generated over A by the elements $\delta A_i^{(j)}, i > 0, j > 0$ and over $A[d]$ by the elements $\delta A_i, i > 0$. We denote by D_{A_i} : $A \rightarrow A[d]$ the partial Frechet derivative: $D_{A_i}(P) = \sum_{j=o}^{\infty} \frac{\partial P}{\partial A^{(j)}} \partial^j$, and by definition $SP = \sum_{i>0} D_{A_i}(p)\delta A_i$. If $P \in \mathcal{A}^{\infty}$ is a vector, we denote by $D(P)$ the infinite matrix of differential operators, where the operator $D_{A_i}(P_j)$ is in the $(ij)-$th place. We have $6P = (\cdots \delta P_i \cdots)^t = D(P)6A$. For any differentiation $X : A \rightarrow N$ (N is an $\mathcal{A}$-module), we have $XP = D(P)XA$. For any R € A, the operator $D(\frac{\delta R}{\delta A})$ is formally symmetric and for any two vectors P, Q, only one of which is finite, we have $\bar{P}^t D(\frac{\delta R}{\delta A})\bar{Q} \sim$ where $\bar{P}^t D(\frac{\delta R}{\delta A})\bar{Q} = \sum_{i,j\geq 0} P_i D_{A_i}(\frac{\delta R}{\delta A_i})Q_j$ and $\sim$ denotes congruence modulo Im ∂ and Im ∂ C $\ker(\delta/\delta A)$.

6.3.12 Lemma: The operator $B = \sum_{i>0}(B_i\partial^i - (-1)^i\partial^i o B_i^t)$ is Hamiltonian, iff for any $P, Q \in \mathcal{A}$, we have

$$B\frac{\delta}{\delta \bar{A}}\left(\frac{\delta P}{\delta \bar{A}^t}B\frac{\delta Q}{\delta \bar{A}}\right) = D\left(B\frac{\delta P}{\delta \bar{A}}\right)B\frac{\delta Q}{\delta \bar{A}} - D\left(B\frac{\delta Q}{\delta \bar{A}}\right)B\frac{\delta P}{\delta \bar{A}}$$

where $\frac{\delta P}{\delta \bar{A}^t} = \left(\frac{\delta P}{\delta \bar{A}}\right)^t$.

Proof: We have by definition that the operator B is Hamiltonian if for any P, $Q \in \mathcal{A}$, we have $[X_P, X_Q] = X_R$, where $R = X_Q P$. Therefore, we verify that (6.3.13) is equivalent to the identity $[X_Q, X_P] = X_R$, where $R = X_Q P$ for any $P, Q \in$ A Since X_R and $[X_Q, X_P]$ commute with ∂, and A_i^j are free generators of .4, the above is equivalent to the condition $[X_Q, X_P]\bar{A} = X_R \bar{A}$. Also, we have

$$\begin{aligned} X_R\bar{A} &= B\frac{\delta R}{\delta \bar{A}} = B\frac{\delta}{\delta \bar{A}}\sum_{i,j\geq 0}\frac{\partial P}{\partial A_i^{(j)}}(X_Q A_i)^{(j)} \\ &= B\frac{\delta}{\delta \bar{A}}\sum_{i\geq 0}\frac{\delta P}{\delta A_i}X_Q A_i \\ &\left(\text{since} \quad \sum_{j\geq 0}\frac{\partial P}{\partial A_i^{(j)}}, S^{(j)} \sim \frac{\delta P}{\delta A_i}S \quad \text{and} \quad \text{Im}\,\partial \subset \ker\frac{\delta}{\delta \bar{A}}\right) \end{aligned}$$

$$= B\frac{\delta}{\delta \bar{A}}\left(\frac{\delta P}{\delta \bar{A}^t}B\frac{\delta Q}{\delta \bar{A}}\right)$$

Therefore,

$$\begin{aligned}[X_Q, X_P]\bar{A} &= X_Q\left(B\frac{\delta P}{\delta \bar{A}}\right) - X_P\left(B\frac{\delta Q}{\delta \bar{A}}\right) \\ &= D\left(B\frac{\delta P}{\delta \bar{A}}\right)B\frac{\delta Q}{\delta \bar{A}} - D\left(B\frac{\delta Q}{\delta \bar{A}}\right)B\frac{\delta P}{\delta \bar{A}}\end{aligned} \quad (6.3.14)$$

i.e., we have proved that $[X_Q, X_P]\bar{A} = X_R\bar{A}$ where $X_R\bar{A} = B\frac{\delta}{\delta \bar{A}}\left(\frac{\delta P}{\delta \bar{A}^t}B\frac{\delta Q}{\delta \bar{A}}\right)$ which is the left hand side of (6.3.13) and $[X_Q, X_P]\bar{A} = D\left(B\frac{\delta P}{\delta \bar{A}}\right)B\frac{\delta Q}{\delta \bar{A}} - D\left(B\frac{\delta Q}{\delta \bar{A}}\right)B\frac{\delta P}{\delta \bar{A}}$ which is the right hand side of (6.3.13).

Thus left hand side = right hand side and the lemma is proved.

(iv) All these flows have a common Poisson representation:

$L_{,t} = \{P_+, L\} = \{L, P_-\}$ **(6.3.15)**, where $L = \xi + \sum_{i=o}^{\infty} A_i\xi^{-(i+1)}$.

Here P is an element of the Poisson centralizer $Z(L)$ of L in the ring $\bar{\theta} = \bar{\mathcal{A}}((\xi^{-1}))$, $\bar{\mathcal{A}}$ being the minimal differential $\mathbb{Q}$-algebra generated by ∂ and the A_i's: $\bar{A} = \mathbb{Q}[A_i^{(j)}]$, $i, j \leq \mathbb{Z}_+$ with a derivation ∂ acting on the polynomial generators of $\bar{\mathcal{A}}$ by the standard rule, $\partial(A_i^{(j)}) = A_i^{(j+1)}$. Thus $Z(L)$ is generated over $\mathbb{Q}$ by $\{L^l / l \in \mathbb{Z}\}$. For an element $\sum_l p_l\xi^l \in \bar{\theta}$, define

$$\left(\sum p_l\xi^l\right)_+ = \sum_{l\geq 0} p_l\xi^l \ , \ \left(\sum p_l\xi^l\right)_- = \sum_{l\leq 0} p_l\xi^l \ .$$

The Poisson bracket {,} given in formula (6.3.15) is the standard one in $T^*(\mathbb{R}^1)$: $\{F, G\} = F_{,\xi}G_{,x} - F_{,x}G_{,\xi}$.

6.3.16 Remarks: (1) The flow $\#m$ written as $A_{i,j} = \sum_j B_{ij}\left(\frac{\partial \bar{H}}{\partial A_j}\right), \bar{H} = \frac{1}{m}H_m$, m € IN has the Poisson representation (6.3.15) with $P = \frac{1}{m}L^m$. The flows #2 and #3 are given by

$A_{i,t} = A_{i+1,x} + iA_{i-1}A_{o,x}$ i £ $\mathbb{Z}_+$ and

$A_{i,t} = A_{i+2,x} + A_o A_{i,x} + (i+1) A_i A_{o,x} + i A_{i-1} A_{i,x}$, $i \in \mathbb{Z}_+$ **respectively.**

(2) The properties (i)-(iv) given above are not logically independent. That is, the flows commute (iii), since they are Hamiltonian (iii) and all the Hamiltonians are in involution (ii).

(3) The properties (i)-(iii) follow from the single Poisson representation property (iv), even when L is taken to be of the general form

$$\mathcal{L} = \xi^M + \sum^{M-2} u_l \xi^l \ , \quad M \in \mathbb{N} \ , \ Q = 0 \text{ or } Q = -\infty \ .$$

In a later section, we give an alternative understanding of the Hamiltonian structure of the Benney's system.

6.4 THE SUPERSYMMETRIC BENNEY'S SYSTEM:

We want to understand what happens with the flows (6.3.15) when we extend the plane $T^*(\mathbb{R}^1) = \mathbb{R}^2$ into the super plane $\mathbb{R}^{2N}$ equipped with the super Poisson bracket

$$\{F, G\} = F_{,\xi} G_{,x} - F_{,x} G_{,\xi} + \frac{1}{2\xi} \sum_{i=1}^{N} \mathcal{D}_r(F) \mathcal{D}_r(G) \ ,. \tag{6.4.1}$$

where F and G are even and $\mathcal{D}_r = \frac{\partial}{\partial \theta_r} + \theta_r \frac{\partial}{\partial x}$, $1 < r < N$ are odd supercommuting derivations satisfying $[\mathcal{D}_r, \mathcal{D}_{\bar{r}}] = \mathcal{D}_r \mathcal{D}_{\bar{r}} + \mathcal{D}_{\bar{r}} \mathcal{D}_r = 2\delta_{r\bar{r}} \partial$, $\theta_1, \cdots \theta_N$ being the generators of the Grassmann algebra $\Lambda(N)$.

The supersymmetry destroys integrability, but not entirely; i.e., the super extended flows do not commute between themselves, but nevertheless, all these flows do have a common infinite set of polynomial conserved densities. That is, the supersymmetric Benney heirarchy is a *semi-integrable* system where **semi-integrable** means that it describes a system of non-commuting flows with a common set of conserved densities.

6.4.2 Super flows:

Let $C = \xi^M + \sum_{l=-\infty}^{M-2} u_l \xi^l$, $M \in \mathbb{N}$, where $u_l = u_l(x, 0, t)$ is even V /. (6.4.3)

Let P = $\mathcal{L}^{m/M} = \xi + \cdots$, m € IN (6.4.4) be a $\mathbb{Q}$-**generator** of positive ξ-**degree** of the Poisson centralizer *Z(L)* of I in the ring *0* = $\mathcal{A}((\xi^{-1}))$, where *A* is generated over $\mathbb{Q}$ by the $\partial, \mathcal{D}_r$'s and u_l's. We consider an evolutionary derivation ∂_p of *A* defined by the rule

$$\textit{dp}(\pounds) \quad = \quad \{\mathcal{P}_+, \mathcal{L}\} \qquad \textbf{(6.4.5)}$$

$$= \quad \{\mathcal{L}, \mathcal{P}_-\} \qquad (6.4.6)$$

with the super Poisson bracket { } defined by the formula (6.4.1) and with the understanding that *dp* acts trivially on ξ. Thus, the action of *dp* on *A* is given by:

$$\partial_{\mathcal{P}}(u_l) \;=\; \mathrm{res}_l[\partial_{\mathcal{P}}(\mathcal{L})] \;= \mathrm{res}_l(\{\mathcal{P}_+, \mathcal{L}\}) \;= \mathrm{res}_l(\{\mathcal{L}, \mathcal{P}_-\})\,.$$

The expressions $\{\mathcal{P}_+, \mathcal{L}\}$ and $\{\mathcal{L}, \mathcal{P}_-\}$ agree between themselves, since

$$0 \;=\; \{\mathcal{P}, \mathcal{L}\} \;= \{\mathcal{P}_+ + \mathcal{P}_-, \mathcal{L}\} \;= \{\mathcal{P}_+, \mathcal{L}\} - \{\mathcal{L}, \mathcal{P}_-\}.$$

We verify that the derivation $\partial_{\mathcal{P}}$ is correctly defined:

From the form of $\mathcal{L}$ (6.4.3), we see that $\partial_{\mathcal{P}}(\mathcal{L})$ must belong to $\theta_{\leq M-2}$ where

$$\theta_{\leq r} = \left\{ \sum_{l \leq r} p_l \xi^l \;|p_l \in \mathcal{A} \right\}\,.$$

By formula (6.4.1), $\{\theta_{<r}, \theta_{<l}\}$ C θ_{r+l-1}. Hence by formula (6.4.6), $\partial_{\mathcal{P}}(\mathcal{L})$ = $\{\mathcal{L}, \mathcal{P}_-\}$*e* $\{\theta_{\leq M}, \theta_{\leq -1}\}$C $\theta_{<M-2}$ as required.

Now suppose that $\mathcal{R}$ is another element of *Z(L)* of positive ξ-degree: $\mathcal{R}$ = $\mathcal{L}^{\tilde{m}/M}$, m € IN, $\tilde{m} \neq m$. Let $\partial_{\mathcal{R}}$ be the corresponding evolutionary derivation of $\mathcal{A}$, given by the formulas $\partial_{\mathcal{R}}(\mathcal{L})$— $\{\mathcal{R}_+, \mathcal{L}\} = \{\mathcal{L}, \mathcal{R}_-\}$. We show that, in contrast to the purely even case (when the $\mathcal{D}_r$'s are absent), the derivations $\partial_{\mathcal{P}}$ and $\partial_{\mathcal{R}}$ do not commute.

6.4.7 Theorem: $[\partial_{\mathcal{P}}, \partial_{\mathcal{R}}](z)$= $\{(2\xi)^{-1}\mathcal{D}_r[\mathrm{res}_o(\mathcal{R})]\mathcal{D}_r[\mathrm{res}_o(\mathcal{P})]$, *£]* (6.4.8)

Proof: Since the Poisson bracket (6.4.1) is a derivation with respect to each argument, formulas (6.4.5), (6.4.6) imply that

$$\partial_{\mathcal{P}}(\mathcal{R}) \;=\; \{\mathcal{P}_+, \mathcal{R}\} \qquad = \{\mathcal{R}, \mathcal{P}_-\}$$

In particular, $\partial_{\mathcal{P}}(\mathcal{R}_+) = [\partial_{\mathcal{P}}(\mathcal{R})]_+ = \{\mathcal{R}, \mathcal{P}_-\}_+$

Hence,

$$\begin{aligned}\partial_{\mathcal{P}}\partial_{\mathcal{R}}(\mathcal{L}) &= \partial_{\mathcal{P}}(\{\mathcal{R}_+, \mathcal{L}\}) = \{\partial_{\mathcal{P}}(\mathcal{R}_+), \mathcal{L}\} + \{\mathcal{R}_+, \partial_{\mathcal{P}}(\mathcal{L})\} \\ &\quad \text{(Since } \partial_{\mathcal{P}} \text{ commutes with } \partial, \mathcal{D}_r\text{'s, } \xi) \\ &= \{\{\mathcal{R}, \mathcal{P}_-\}_+, \mathcal{L}\} + \{\mathcal{R}_+, \{\mathcal{P}_+, \mathcal{L}\}\} \qquad (6.4.9)\end{aligned}$$

Interchanging *V* and $\mathcal{R}$ in the above formula (6.4.9), we obtain

$\partial_{\mathcal{R}}\partial_{\mathcal{P}}(\mathcal{L}) = \{\{\mathcal{P}, \mathcal{R}_-\}_+, \mathcal{L}\} + \{\mathcal{P}_+, \{\mathcal{R}_+, \mathcal{L}\}\}$ (6.4.10).

Substracting (6.4.10) from formula (6.4.9) and using Jacobi identity, we get, $[\partial_{\mathcal{P}}, \partial_{\mathcal{R}}](\mathcal{L})=$ $\{\Delta, \mathcal{L}\}$ where

$$\begin{aligned}\Delta &= \{\mathcal{R}, \mathcal{P}_-\}_+ + \{\mathcal{R}_-, \mathcal{P}\}_+ + \{\mathcal{R}_+, \mathcal{P}_+\} \\ &\quad (\text{i.e., } \{\{\mathcal{R}, \mathcal{P}_-\}_+, \mathcal{L}\} + \{\mathcal{R}_+, \{\mathcal{P}_+, \mathcal{L}\}\} \\ &\quad -\{\{\mathcal{P}, \mathcal{R}_-\}_+, \mathcal{L}\} - \{\mathcal{P}_+, \{\mathcal{R}_+, \mathcal{L}\}\}) \\ &= \{\{\mathcal{R}, \mathcal{P}_-\}_+, \mathcal{L}\} - \{\{\mathcal{P}, \mathcal{R}_-\}_+, \mathcal{L}\} \\ &\quad +\{\mathcal{R}_+, \{\mathcal{P}_+, \mathcal{L}\}\} + \{\mathcal{P}_+, \{\mathcal{L}, \mathcal{R}_+\}\} \\ &= \{\{\mathcal{R}, \mathcal{P}_-\}_+, \mathcal{L}\} + \{\{\mathcal{R}_-, \mathcal{P}\}_+, \mathcal{L}\} \\ &\quad +\{\{\mathcal{R}_+, \mathcal{P}_+, \}\mathcal{L}\} \text{ (using J.id. of } \mathcal{R}_+, \mathcal{P}_+, \ \mathcal{L}) \\ &= \{\{\mathcal{R}, \mathcal{P}_-\}_+ + \{\mathcal{R}_-, \mathcal{P}\}_+ + \{\mathcal{R}_+, \mathcal{P}_+\}, \mathcal{L}\} \ = \{\Delta, \mathcal{L}\})\end{aligned}$$

Now, by formula (6.4.1), $\{\theta_{<0}, \theta_{<0}\}_+ = \{0\}$, so that

$$\begin{aligned}\Delta &= \{\mathcal{R}_+, \mathcal{P}_-\}_+ + \{\mathcal{R}_-, \mathcal{P}_+\}_+ + \{\mathcal{R}_+, \mathcal{P}_+\}_+ \\ &\quad +\{\mathcal{R}_+, \mathcal{P}_+\} - \{\mathcal{R}_+, \mathcal{P}_+\}_+ \qquad (\text{since } \ \{\mathcal{R}_-, \mathcal{P}_-\}_+ = 0)\end{aligned}$$

By definition of $\mathcal{P}$,and ft, $\{\mathcal{R}, \mathcal{P}\} = 0$.

Hence

$$0 = \{\mathcal{R},\mathcal{P}\}_+ = \{\mathcal{R}_+ + \mathcal{R}_-, \mathcal{P}_+ + \mathcal{R}_-\}_+ = \{\mathcal{R}_+,\mathcal{P}_-\}_+ + \{\mathcal{R}_-,\mathcal{P}_+\}_+ + \{\mathcal{R}_+,\mathcal{P}_+\}_+ \qquad (\text{ since } \{\mathcal{R},\mathcal{P}_-\}_+ = 0)$$

and

$$\Delta = \{\mathcal{R}_+,\mathcal{P}_+\} - \{\mathcal{R}_+,\mathcal{P}_+\}_+ = \{\mathcal{R}_+,\mathcal{P}_+\} = \frac{1}{2\xi}\mathcal{D}_r[\mathrm{res}_o(\mathcal{R})]\mathcal{D}_r[\mathrm{res}_o(\mathcal{P})] \quad (6.4.11)$$

Thus,

$$[\partial_{\mathcal{P}},\mathcal{P}_{\mathcal{R}}](\mathcal{L}) = \{\Delta,\mathcal{L}) = \{(2\xi)^{-1}\mathcal{D}_r[\mathrm{res}_o(\mathcal{R})]\mathcal{D}_r[\mathrm{res}_o(\mathcal{P})],\mathcal{L}].$$

6.4.12 Remarks: (1) Since $m \neq m$, $\mathrm{res}_o(\mathcal{P})$ and $\mathrm{res}_o(\mathcal{R})$ are two polynomials in the entries u_l's not all of which are the same, the expression (6.4.11) does not vanish unless one of the $\mathrm{res}_o(\mathcal{R})$ and $\mathrm{res}_o(\mathcal{P})$ does, which happens when either m or m equals one. In this case, when m = 1, $\partial_{\mathcal{P}}$ — d and this commutes with all $\partial_{\mathcal{R}}$'s. In general, when m nor m equals one, $[\partial_{\mathcal{P}},\partial_{\mathcal{R}}] \neq 0$.

(2) The classical Poisson bracket on $T^*(\mathbb{R}^1)$ is the quasiclassical (=zero dispersion) limit of the commutator $[F,G]$ = *F o G - G o F*, where *FoG* = $\sum_{n>0}\frac{1}{n!}\frac{\partial^n F}{\partial\xi^n}\partial^n(G)$, i.e., the Benney hierarchy is the quasiclassical limit of the KP hierarchy (in the classical case). But in the supersymmetric case, the super Poisson bracket is realized as part of the commutator resulting from an associated product of *F o G* extended by the $\mathcal{D}_r$'s. That is, if the multiplication *F o G* is defined with an additional term involving the XVs, then the super Poisson bracket is realized as part of the commutator defined by $[F,G]$ = $F \circ G - G \circ F$.

We verify this below:

Define

$$FoG = \sum_{n\geq 0}\frac{1}{n!}\frac{\partial^n F}{\partial\xi^n}\partial^n(G) + \frac{1}{4\xi}\mathcal{D}_n(F)(1+\mathcal{D}_n(G))$$

$$GoF = \sum_{n\geq 0}\frac{1}{n!}\frac{\partial^n G}{\partial \xi^n}\partial^n(F) + \frac{1}{4\xi}\mathcal{D}_n(G)(1+\mathcal{D}_n(G))$$

For $N = 1;\ n = 0, 1$: super $P.B : \{F, G\} = F_\xi G_x - F_x G_\xi + \frac{1}{2\xi}\mathcal{D}_1(F)\mathcal{D}_1(G)$

$$\begin{aligned}
FoG &= FG + F_\xi G_x + \frac{1}{4\xi}\mathcal{D}_1(F)(1+\mathcal{D}_1(G_1)) \\
GoF &= GF + F_x G_\xi + \frac{1}{4\xi}\mathcal{D}_1(G)(1+\mathcal{D}_1(F)) \\
[F,G]_1 &= F\ o\ G - G\ o\ F \\
&= FG - GF + F_\xi G_x - F_x G_\xi + \frac{1}{4\xi}(\mathcal{D}_1(F) + \mathcal{D}_1(F)\mathcal{D}_1(G) \\
&\quad -\mathcal{D}_1(G) - \mathcal{D}_1(G)\mathcal{D}_1(F)) \\
&= F_\xi G_x - F_x G_\xi + \frac{1}{2\xi}\mathcal{D}_1(F)\mathcal{D}_1(G) + FG - GF + \frac{1}{4\xi}(\mathcal{D}_1(F) - \mathcal{D}_1(G))
\end{aligned}$$

$N = 2; n = 0, 1, 2$:

$$\begin{aligned}
\{F,G\} &= F_\xi G_x - F_x G_\xi + \frac{1}{2\xi}(\mathcal{D}_1(F)\mathcal{D}_1(G) + \mathcal{D}_2(F)\mathcal{D}_2(G)) \\
FoG &= FG + F_\xi G_x + \frac{1}{4\xi}\mathcal{D}_1(F)(1+\mathcal{D}_1(G)) \\
&\quad +\frac{1}{4\xi}\mathcal{D}_2(F)(1+\mathcal{D}_2(G)) + \frac{1}{2}\frac{\partial^2 F}{\partial \xi^2}\frac{\partial^2 G}{\partial x^2} \\
GoF &= GF + F_x G_\xi + \frac{1}{4\xi}\mathcal{D}_1(G)(1+\mathcal{D}_1(F)) \\
&\quad +\frac{1}{4\xi}\mathcal{D}_2(G)(1+\mathcal{D}_2(F)) + \frac{1}{2}\frac{\partial^2 G}{\partial \xi^2}\frac{\partial^2 F}{\partial x^2}
\end{aligned}$$

$$\begin{aligned}
[F,G]_1 &= F\ o\ G - G\ o\ F \\
&= FG - GF + F_\xi G_x - F_x G_\xi + \frac{1}{4\xi}(\mathcal{D}_1(F) + \mathcal{D}_1(F)\mathcal{D}_1(G) - \mathcal{D}_1(G) - \mathcal{D}_1(G)\mathcal{D}_1(F)) \\
&\quad +\frac{1}{4\xi}(\mathcal{D}_2(F) + \mathcal{D}_2(F)\mathcal{D}_2(G) - \mathcal{D}_2(G) - \mathcal{D}_2(G)\mathcal{D}_2(F)) \\
&\quad +\frac{1}{2}\frac{\partial^2 F}{\partial \xi^2}\frac{\partial^2 G}{\partial x^2} - \frac{1}{2}\frac{\partial^2 G}{\partial \xi^2}\frac{\partial^2 F}{\partial x^2} \\
&= F_\xi G_x - F_x G_\xi + \frac{1}{2\xi}(\mathcal{D}_1(F)\mathcal{D}_1(G) + \mathcal{D}_2(F)\mathcal{D}_2(G))
\end{aligned}$$

$$
\begin{aligned}
&+FG - GF + \frac{1}{4\xi}(\mathcal{D}_1(F) - \mathcal{D}_1(G) + \frac{1}{4\xi}(\mathcal{D}_2(F) - \mathcal{D}_2(G)) \\
= \;& \frac{1}{2}\frac{\partial^2 F}{\partial \xi^2}\frac{\partial^2 G}{\partial x^2} - \frac{1}{2}\frac{\partial^2 G}{\partial \xi^2}\frac{\partial^2 F}{\partial x^2}
\end{aligned}
$$

$N = 3; n = 0, 1, 2, 3$:

$$
\begin{aligned}
\{F, G\} &= F_\xi G_x - F_x G_\xi + \frac{1}{2\xi}(\mathcal{D}_1(F)\mathcal{D}_1(G) + \mathcal{D}_2(F)\mathcal{D}_2(G) + \mathcal{D}_3(F)\mathcal{D}_3(G))) \\
FoG &= FG + F_\xi G_x + \frac{1}{4\xi}\mathcal{D}_1(F)(1 + \mathcal{D}_1(G)) + \frac{1}{2}\frac{\partial^2 F}{\partial \xi^2}\frac{\partial^2 G}{\partial x^2} \\
&\quad + \frac{1}{4\xi}\mathcal{D}_2(F)(1 + \mathcal{D}_2(G)) + \frac{1}{6}\frac{\partial^3 F}{\partial \xi^3}\frac{\partial^3 G}{\partial x^3} + \frac{1}{4\xi}\mathcal{D}_3(F)(1 + \mathcal{D}_3(G)) \\
GoF &= GF + F_x G_\xi + \frac{1}{4\xi}\mathcal{D}_1(G)(1 + \mathcal{D}_1(F)) + \frac{1}{2}\frac{\partial^2 G}{\partial \xi^2}\frac{\partial^2 F}{\partial x^2} \\
&\quad + \frac{1}{4\xi}\mathcal{D}_2(G)(1 + \mathcal{D}_2(F)) + \frac{1}{6}\frac{\partial^3 G}{\partial \xi^3}\frac{\partial^3 F}{\partial x^3} + \frac{1}{4\xi}\mathcal{D}_3(G)(1 + \mathcal{D}_3(F))
\end{aligned}
$$

$$
\begin{aligned}
[F, G]_1 &= FoG - GoF \\
&= F_\xi G_x - F_x G_\xi + \frac{1}{2\xi}(\mathcal{D}_1(F)\mathcal{D}_1(G) + \mathcal{D}_2(F)\mathcal{D}_2(G) + \mathcal{D}_3(F)\mathcal{D}_3(G)) \\
&\quad + FG - GF + \frac{1}{4\xi}(\mathcal{D}_1(F)) - \mathcal{D}_1(G)) + \frac{1}{4\xi}(\mathcal{D}_2(F) - \mathcal{D}_2(G)) \\
&\quad + \frac{1}{4\xi}(\mathcal{D}_3(F) - \mathcal{D}_3(G)) + \frac{1}{2}\frac{\partial^2 F}{\partial \xi^2}\frac{\partial^2 G}{\partial x^2} - \frac{1}{2}\frac{\partial^2 G}{\partial \xi^2}\frac{\partial^2 F}{\partial x^2} \\
&\quad + \frac{1}{6}\frac{\partial^3 F}{\partial \xi^3}\frac{\partial^3 G}{\partial x^3} - \frac{1}{6}\frac{\partial^3 G}{\partial \xi^3}\frac{\partial^3 F}{\partial x^3}
\end{aligned}
$$

$N = 4; n = 0, 1, 2, 3, 4$:

$$
\begin{aligned}
\{F, G\} &= F_\xi G_x - F_x G_\xi + \frac{1}{2\xi}(\mathcal{D}_1(F)\mathcal{D}_1(G) \\
&\quad + \mathcal{D}_2(F)\mathcal{D}_2(G) + \mathcal{D}_3(F)\mathcal{D}_3(G) + \mathcal{D}_4(F)\mathcal{D}_4(G)) \\
FoG &= FG + F_\xi G_x + \frac{1}{4\xi}\mathcal{D}_1(F)(1 + \mathcal{D}_1(G)) + \frac{1}{2}\frac{\partial^2 F}{\partial \xi^2}\frac{\partial^2 G}{\partial x^2} \\
&\quad + \frac{1}{4\xi}\mathcal{D}_2(F)(1 + \mathcal{D}_2(G)) + \frac{1}{6}\frac{\partial^3 F}{\partial \xi^3}\frac{\partial^3 G}{\partial x^3} \\
&\quad + \frac{1}{4\xi}\mathcal{D}_3(F)(1 + \mathcal{D}_3(G)) + \frac{1}{24}\frac{\partial^4 F}{\partial \xi^4}\frac{\partial^4 G}{\partial x^4} + \frac{1}{4\xi}\mathcal{D}_4(F)(1 + \mathcal{D}_4(G))
\end{aligned}
$$

$$
\begin{aligned}
GoF &= GF + F_x G_\xi + \frac{1}{4\xi}\mathcal{D}_1(G)(1+\mathcal{D}_1(F)) + \frac{1}{2}\frac{\partial^2 G}{\partial \xi^2}\frac{\partial^2 F}{\partial x^2} \\
&= \frac{1}{4\xi}\mathcal{D}_2(G)(1+\mathcal{D}_2(F)) + \frac{1}{6}\frac{\partial^3 G}{\partial \xi^3}\frac{\partial^3 F}{\partial x^3} + \frac{1}{4\xi}\mathcal{D}_3(G)(1+\mathcal{D}_3(F)) \\
&+ \frac{1}{24}\frac{\partial^4 G}{\partial \xi^4}\frac{\partial^4 F}{\partial x^4} + \frac{1}{4\xi}\mathcal{D}_4(G)(1+\mathcal{D}_4(F))
\end{aligned}
$$

$$
\begin{aligned}
[F,G]_1 &= FoG - GoF \\
&= F_\xi G_x - F_x G_\xi + \frac{1}{2\xi}(\mathcal{D}_1(F)\mathcal{D}_1(G) + \mathcal{D}_2(F)\mathcal{D}_2(G) \\
&+ \mathcal{D}_3(F)\mathcal{D}_3(G) + \mathcal{D}_4(F)\mathcal{D}_4(G)) \\
&+ FG - GF + \frac{1}{4\xi}(\mathcal{D}_1(F) - \mathcal{D}_1(G)) + \frac{1}{4\xi}(\mathcal{D}_2(F) - \mathcal{D}_2(G)) \\
&+ \frac{1}{4\xi}(\mathcal{D}_3(F) - \mathcal{D}_3(G)) + \frac{1}{4\xi}(\mathcal{D}_4(F) - \mathcal{D}_4(G)) \\
&+ \frac{1}{2}\frac{\partial^2 F}{\partial \xi^2}\frac{\partial^2 G}{\partial x^2} - \frac{1}{2}\frac{\partial^2 G}{\partial \xi^2}\frac{\partial^2 F}{\partial x^2} + \frac{1}{6}\frac{\partial^3 F}{\partial \xi^3}\frac{\partial^3 G}{\partial x^3} \\
&- \frac{1}{6}\frac{\partial^3 G}{\partial \xi^3}\frac{\partial^3 F}{\partial x^3} + \frac{1}{24}\frac{\partial^4 F}{\partial \xi^4}\frac{\partial^4 G}{\partial x^4} - \frac{1}{24}\frac{\partial^4 G}{\partial \xi^4}\frac{\partial^4 F}{\partial x^4}
\end{aligned}
$$

We give a general formula, $[F,G]_1 = \{F,G\} + [F,G]_o + \frac{1}{4\xi}\sum_{n=1,2,\cdots}(\mathcal{D}_n(F) - \mathcal{D}_n(G))$ where $[F,G]_o = FoG - GoF$ with $FoG = \sum_{n=0,,2,3,\cdots}\frac{1}{n!}\frac{\partial^n F}{\partial \xi^n}\partial^n(G)$

2nd method:

Define

$$
\begin{aligned}
FoG &= \sum_{n\geq 0}\frac{1}{n!}\frac{\partial^n F}{\partial \xi^n}\partial^n(G) + \frac{1}{4\xi}\mathcal{D}_n(F\mathcal{D}_n(G)) \\
GoF &= \sum_{n\geq 0}\frac{1}{n!}\frac{\partial^n G}{\partial \xi^n}\partial^n(F) + \frac{1}{4\xi}\mathcal{D}_n(G\mathcal{D}_n(F))
\end{aligned}
$$

For $N = 1; n = 0, 1$:

$$\{F,G\} = F_{,\xi}G_{,x} - F_{,x}G_{,\xi} + \frac{1}{2\xi}\mathcal{D}_1(F)\mathcal{D}_1(G)$$

$$FoG = \frac{\partial F}{\partial_\xi}\frac{\partial G}{\partial x} + \frac{1}{4\xi}\mathcal{D}_1(F\mathcal{D}_1(G)) \text{ , } GoF = \frac{\partial G}{\partial_\xi}\frac{\partial F}{\partial x} + \frac{1}{4\xi}\mathcal{D}_1(G\mathcal{D}_1(F))$$

$$\begin{aligned}
[F,G]_1 &= F\ o\ G - G\ o\ F\ = FG - GF + \frac{\partial F}{\partial_\xi}\frac{\partial G}{\partial x} + \frac{1}{4\xi}\mathcal{D}_1(F\mathcal{D}_1(G)) \\
&\quad -\frac{\partial G}{\partial_\xi}\frac{\partial F}{\partial x} - \frac{1}{4\xi}\mathcal{D}_1(G\mathcal{D}_1(F)) \\
&= F_\xi G_x - F_x G_\xi + \frac{1}{4\xi}(\mathcal{D}_1(F\mathcal{D}_1(G)) - (\mathcal{D}_1(G\mathcal{D}_1(F)) + FG - GF \\
&= F_\xi G_x - F_x G_\xi + \frac{1}{4\xi}(\mathcal{D}_1(F)\mathcal{D}_1(G) + F\mathcal{D}_1\mathcal{D}_1(G) \\
&\quad -(\mathcal{D}_1(G)\mathcal{D}_1(F) - G\mathcal{D}_1\mathcal{D}_1(F)) + FG - GF \\
&= F_\xi G_x - F_x G_\xi + \frac{1}{4\xi}(2\mathcal{D}_1(F)\mathcal{D}_1(G) + F\mathcal{D}_1\mathcal{D}_1(G) \\
&\quad -G\mathcal{D}_1\mathcal{D}_1(F)) + FG - GF \\
&= F_\xi G_x - F_x G_\xi + \frac{1}{2\xi}\mathcal{D}_1(F)\mathcal{D}_1(G) + \frac{1}{4\xi}(F\mathcal{D}_1\mathcal{D}_1(G) - G\mathcal{D}_1\mathcal{D}_1(F)) + FG - GF
\end{aligned}$$

For $N = 2; n = 0, 1, 2$:

$$\{F,G\} = F_{,\xi}G_{,x} - F_{,x}G_{,\xi} + \frac{1}{2\xi}\mathcal{D}_1(F)\mathcal{D}_1(G) + \mathcal{D}_2(F)\mathcal{D}_2(G))$$

$$\begin{aligned}
FoG &= 1 + \frac{1}{4\xi}\mathcal{D}_o(F\mathcal{D}_o(G)) + \frac{1}{4\xi}\mathcal{D}_1(F\mathcal{D}_1(G)) \\
&\quad +\frac{1}{4\xi}\mathcal{D}_2(F\mathcal{D}_2(G)) + \frac{\partial F}{\partial \xi}\frac{\partial G}{\partial x} + \frac{\partial^2 F}{\partial \xi^2} + \frac{\partial^2 G}{\partial x^2} \\
GoF &= 1 + \frac{1}{4\xi}\mathcal{D}_o(G\mathcal{D}_o(F)) + \frac{1}{4\xi}\mathcal{D}_1(F\mathcal{D}_1(G)) \\
&\quad +\frac{1}{4\xi}\mathcal{D}_2(G\mathcal{D}_2(F)) + \frac{\partial G}{\partial \xi}\frac{\partial F}{\partial x} + \frac{\partial^2 G}{\partial \xi^2}\frac{\partial F^2}{\partial x^2} \\
[F,G]_1 &= F\ o\ G - G\ o\ F \\
&= \frac{\partial F}{\partial \xi}\frac{\partial G}{\partial x} + \frac{1}{4\xi}(\mathcal{D}_1(F)\mathcal{D}_1(G)) + (F\mathcal{D}_1\mathcal{D}_1(G)) + \frac{\partial^2 F}{\partial_\xi^2}\frac{\partial^2 G}{\partial x^2} \\
&\quad +\frac{1}{4\xi}(\mathcal{D}_2(F)\mathcal{D}_2(G)) + (F\mathcal{D}_2\mathcal{D}_2(G)) - \frac{\partial G}{\partial \xi}\frac{\partial F}{\partial x} \\
&\quad -\frac{1}{4\xi}(\mathcal{D}_1(G)\mathcal{D}_1(F)) + (G\mathcal{D}_1\mathcal{D}_1(F)) - \frac{\partial^2 G}{\partial_\xi^2}\frac{\partial^2 F}{\partial x^2} \\
&\quad -\frac{1}{4\xi}(\mathcal{D}_2(G)\mathcal{D}_2(F)) + (G\mathcal{D}_2\mathcal{D}_2(F)) + FG - GF
\end{aligned}$$

$$
\begin{aligned}
&= F_\xi G_x - G_\xi F_x - \frac{1}{4\xi}(2\mathcal{D}_1(F)\mathcal{D}_1(G) + 2\mathcal{D}_2(F)\mathcal{D}_2(G)) \\
&\quad + \frac{1}{2}\frac{\partial^2 F}{\partial \xi^2}\frac{\partial^2 G}{\partial x^2} - \frac{1}{2}\frac{\partial^2 G}{\partial \xi^2}\frac{\partial^2 F}{\partial x^2} + \frac{1}{4\xi}(F\mathcal{D}_2\mathcal{D}_2(G)) - G\mathcal{D}_2\mathcal{D}_2(F)) + FG - GF \\
&= F_\xi G_x - F_x G_\xi + \frac{1}{2\xi}(\mathcal{D}_1(F)\mathcal{D}_1(G) + \mathcal{D}_2(F)\mathcal{D}_2(G) \\
&\quad + \frac{1}{2}\frac{\partial^2 F}{\partial^2_\xi}\frac{\partial^2 F}{\partial x^2} - \frac{1}{2}\frac{\partial^2 G}{\partial \xi^2}\frac{\partial^2 F}{\partial x^2} + \frac{1}{4\xi}(F\mathcal{D}_2\mathcal{D}_2(G)) - (G\mathcal{D}_2\mathcal{D}_2(F)) \\
&\quad + \frac{1}{4\xi}(F\mathcal{D}_1\mathcal{D}_1(G)) - (G\mathcal{D}_1\mathcal{D}_1(F)) + FG - GF
\end{aligned}
$$

$N = 3; n = 0, 1, 2, 3$:

$$
\{F, G\} = F_\xi G_x - F_x G_\xi + \frac{1}{2\xi}(\mathcal{D}_1(F)\mathcal{D}_1(G) + \mathcal{D}_2(F)\mathcal{D}_2(G) + \mathcal{D}_3(F)\mathcal{D}_3(G))
$$

$$
\begin{aligned}
FoG &= FG + F_\xi G_x + \frac{1}{4\xi}\mathcal{D}_1(F)(1 + \mathcal{D}_1(G)) + \frac{1}{2}\frac{\partial^2 F}{\partial \xi^2}\frac{\partial^2 G}{\partial x^2} \\
&\quad + \frac{1}{4\xi}\mathcal{D}_2(F)(1 + \mathcal{D}_2(G)) + \frac{1}{6}\frac{\partial^3 F}{\partial \xi^3}\frac{\partial^3 G}{\partial x^3} + \frac{1}{4\xi}\mathcal{D}_3(F)(1 + \mathcal{D}_3(G)) \\
GoF &= GF + F_x G_\xi + \frac{1}{4\xi}\mathcal{D}_1(G)(1 + \mathcal{D}_1(F)) + \frac{1}{2}\frac{\partial^2 G}{\partial \xi^2}\frac{\partial^2 F}{\partial x^2} \\
&\quad + \frac{1}{4\xi}\mathcal{D}_2(G)(1 + \mathcal{D}_2(F)) + \frac{1}{6}\frac{\partial^3 G}{\partial \xi^3}\frac{\partial^3 F}{\partial x^3} + \frac{1}{4\xi}\mathcal{D}_3(G)(1 + \mathcal{D}_3(F)) \\
[F, G]_1 &= F \, o \, G - G \, o \, F \\
&= FG - GF + F_\xi G_x - F_x G_\xi + \frac{1}{4\xi}[\mathcal{D}_1(F)(1 + \mathcal{D}_1(G)) - (\mathcal{D}_1(G)(1 + \mathcal{D}_1(F)] \\
&\quad + \frac{1}{2}\frac{\partial^2 F}{\partial \xi^2}\frac{\partial^2 G}{\partial x^2} - \frac{1}{2}\frac{\partial^2 G}{\partial \xi^2}\frac{\partial^2 F}{\partial x^2} + \frac{1}{4\xi}[\mathcal{D}_2(F)(1 + \mathcal{D}_2(G)) - \mathcal{D}_2(G)(1 + \mathcal{D}_2(F))] \\
&\quad + \frac{1}{6}\frac{\partial^3 F}{\partial \xi^3}\frac{\partial^3 G}{\partial x^3} - \frac{1}{6}\frac{\partial^3 G}{\partial \xi^3}\frac{\partial^3 F}{\partial x^3} + \frac{1}{4\xi}[\mathcal{D}_3(F)(1 + \mathcal{D}_3(G)) - \mathcal{D}_3(G)(1 + \mathcal{D}_3(F))] \\
&= FG - GF + F_\xi G_x - F_x G_\xi + \frac{1}{4\xi}[\mathcal{D}_1(F) + \mathcal{D}_1(F)\mathcal{D}_1(G) - \mathcal{D}_1(G) - \mathcal{D}_1(G)\mathcal{D}_1(F)] \\
&\quad + \frac{1}{2}\frac{\partial^2 F}{\partial \xi^2}\frac{\partial^2 G}{\partial x^2} - \frac{1}{2}\frac{\partial^2 G}{\partial \xi^2}\frac{\partial^2 F}{\partial x^2} + \frac{1}{4\xi}[\mathcal{D}_2(F) + \mathcal{D}_2(F)\mathcal{D}_2(G) - \mathcal{D}_2(G) - \mathcal{D}_2(G)\mathcal{D}_2(F)] \\
&\quad + \frac{1}{6}\frac{\partial^3 F}{\partial \xi^3}\frac{\partial^3 G}{\partial x^3} - \frac{1}{6}\frac{\partial^3 G}{\partial \xi^3}\frac{\partial^3 F}{\partial x^3} + \frac{1}{4\xi}[\mathcal{D}_3(F) + \mathcal{D}_3(F)\mathcal{D}_3(G) - \mathcal{D}_3(G) - \mathcal{D}_3(G)\mathcal{D}_3(F)]
\end{aligned}
$$

$$
\begin{aligned}
&= F_\xi G_x - F_x G_\xi + \frac{1}{2\xi}[\mathcal{D}_1(F)\mathcal{D}_1(G) + \mathcal{D}_2(F)\mathcal{D}_2(G) + \mathcal{D}_3(F)\mathcal{D}_3(G)] \\
&\quad + FG - GF + \frac{1}{2}\frac{\partial^2 F}{\partial \xi^2}\frac{\partial^2 G}{\partial x^2} - \frac{1}{2}\frac{\partial^2 G}{\partial \xi^2}\frac{\partial^2 F}{\partial x^2} \\
&\quad + \frac{1}{6}\frac{\partial^3 F}{\partial \xi^3}\frac{\partial^3 G}{\partial x^3} - \frac{1}{6}\frac{\partial^3 G}{\partial \xi^3}\frac{\partial^3 F}{\partial x^3} + \frac{1}{4\xi}[\mathcal{D}_1(F) - \mathcal{D}_1(G)] \\
&\quad + \frac{1}{4\xi}[\mathcal{D}_2(F) - \mathcal{D}_2(G)] + \frac{1}{4\xi}[\mathcal{D}_3(F) - \mathcal{D}_3(G)] \\
&= F_\xi G_x - F_x G_\xi + \frac{1}{2\xi}[\mathcal{D}_1(F)\mathcal{D}_1(G) + \mathcal{D}_2(F)\mathcal{D}_2(G) + \mathcal{D}_3(F)\mathcal{D}_1(G)] \\
&\quad + FG - GF + \frac{1}{4\xi}[\mathcal{D}_1(F) - \mathcal{D}_1(G)] + \frac{1}{4\xi}[\mathcal{D}_2(F) - \mathcal{D}_2(G)] + \frac{1}{4\xi}[\mathcal{D}_3(F) - \mathcal{D}_3(G)] \\
&\quad + \frac{1}{2}\frac{\partial^2 F}{\partial \xi^2}\frac{\partial^2 G}{\partial x^2} - \frac{1}{2}\frac{\partial^2 G}{\partial \xi^2}\frac{\partial^2 F}{\partial x^2} + \frac{1}{6}\frac{\partial^3 F}{\partial \xi^3}\frac{\partial^3 G}{\partial x^3} - \frac{1}{6}\frac{\partial^3 G}{\partial \xi^3}\frac{\partial^3 F}{\partial x^3}
\end{aligned}
$$

$N = 4; n = 0, 1, 2, 3, 4$:

$$
\{F, G\} = F_\xi G_x - F_x G_\xi + \frac{1}{2\xi}(\mathcal{D}_1(F)\mathcal{D}_1(G) + \mathcal{D}_2(F)\mathcal{D}_2(G) + \mathcal{D}_3(F)\mathcal{D}_3(G) + \mathcal{D}_4(F)\mathcal{D}_4(G))
$$

$$
\begin{aligned}
FoG &= FG + F_\xi G_x + \frac{1}{4\xi}\mathcal{D}_1(F)(1 + \mathcal{D}_1(G)) + \frac{1}{2}\frac{\partial^2 F}{\partial \xi^2}\frac{\partial^2 G}{\partial x^2} \\
&\quad + \frac{1}{4\xi}\mathcal{D}_2(F)(1 + \mathcal{D}_2(G)) + \frac{1}{6}\frac{\partial^3 F}{\partial \xi^3}\frac{\partial^3 G}{\partial x^3} + \frac{1}{4\xi}\mathcal{D}_3(F)(1 + \mathcal{D}_3(G)) \\
&\quad + \frac{1}{24\xi}\frac{\partial^4 F}{\partial \xi^4}\frac{\partial^4 G}{\partial x^4} + \frac{1}{4\xi}\mathcal{D}_4(F)(1 + \mathcal{D}_4(G)) \\
GoF &= GF + F_x G_\xi + \frac{1}{4\xi}\mathcal{D}_1(G)(1 + \mathcal{D}_1(F)) + \frac{1}{2}\frac{\partial^2 G}{\partial \xi^2}\frac{\partial^2 F}{\partial x^2} \\
&\quad + \frac{1}{4\xi}\mathcal{D}_2(G)(1 + \mathcal{D}_2(F)) + \frac{1}{6}\frac{\partial^3 G}{\partial \xi^3}\frac{\partial^3 F}{\partial x^3} + \frac{1}{4\xi}\mathcal{D}_3(G)(1 + \mathcal{D}_3(F)) \\
&\quad + \frac{1}{24}\frac{\partial^4 G}{\partial \xi^4}\frac{\partial^4 F}{\partial x^4} + + \frac{1}{4\xi}\mathcal{D}_4(G)(1 + \mathcal{D}_4(F)) \\
[F, G]_1 &= F \text{ o } G - G \text{ o } F \\
&= F_\xi G_x - F_x G_\xi + \frac{1}{2\xi}[\mathcal{D}_1(F)\mathcal{D}_1(G) + (\mathcal{D}_2(F)\mathcal{D}_2(G) \\
&\quad + (\mathcal{D}_3(F)\mathcal{D}_3(G) + (\mathcal{D}_4(F)\mathcal{D}_4(G)] \\
&\quad + FG - GF + \frac{1}{4\xi}[\mathcal{D}_1(F) - \mathcal{D}_1(G)] + \frac{1}{4\xi}[\mathcal{D}_2(F) - \mathcal{D}_2(G)]
\end{aligned}
$$

$$+\frac{1}{4\xi}[\mathcal{D}_3(F)-\mathcal{D}_3(G)]+\frac{1}{4\xi}[\mathcal{D}_4(F)-\mathcal{D}_4(G)]$$
$$+\frac{1}{2}\frac{\partial^2 F}{\partial\xi^2}\frac{\partial^2 G}{\partial x^2}-\frac{1}{2}\frac{\partial^2 G}{\partial\xi^2}\frac{\partial^2 F}{\partial x^2}+\frac{1}{6}\frac{\partial^3 F}{\partial\xi^3}\frac{\partial^3 G}{\partial x^3}$$
$$-\frac{1}{6}\frac{\partial^3 G}{\partial\xi^3}\frac{\partial^3 F}{\partial x^3}+\frac{1}{24}\frac{\partial^4 F}{\partial\xi^4}\frac{\partial^4 G}{\partial x^4}-\frac{1}{24}\frac{\partial^4 G}{\partial\xi^4}\frac{\partial^4 F}{\partial x^4}$$

The general formula is given by

$$\begin{aligned}[F,G]_1 &= \{F,G\}+\sum_{n=0,2,3,\cdots}\frac{1}{n!}\left(\frac{\partial^n F}{\partial\xi^n}\partial^n(G)-\frac{\partial^n G}{\partial\xi^n}\partial^n(F)\right)+\frac{1}{4\xi}\sum_{n=1,2,..}(\mathcal{D}_n(F)-\mathcal{D}_n(G))\\ &= \{F,G\}+[F,G]_o+\frac{1}{4\xi}\sum_{n=1,2,..}(\mathcal{D}_n(F)-\mathcal{D}_n(G))\end{aligned}$$

where

$$[F,G]_o = F\ o\ G - G\ o\ F \qquad , \qquad F\ o\ G=\sum_{n=0,2,3..}\frac{1}{n!}\left(\frac{\partial^n F}{\partial\xi^n}\partial^n G\right)$$

In summary we have proved

6.4.13 Theorem: In the super symmetric Benney setup, the commutator product can be defined realizing it as a non-trival extension of the super Poisson bracket.

6.4.14 Remarks: (1) This answers a question raised by Kuperschmidt in lower dimensions [33].

(2) This is in contrast to the classical Benney case where the Lie algebra $B[[\xi^{-1}]]$ with [,] coincides with the Lie algebra associated to the KdV system as quasiclassical limit as explained above.

6.5 THE HAMILTONIAN STRUCTURE OF BENNEY'S LONG WAVE E-QUATIONS (ALGEBRAIC APPROACH):

6.5.1 Introduction: Benney [10] gave the following equations of motion of an incompressible nonviscous fluid with a free surface as approximations of long waves

$$u_t+uu_x-u_y\int_o^y u_x\Big|_{y=\eta}d\eta+h_x=0, \qquad h_t+\left(\int_o^h u dy\right)_x=0\ . \qquad \textbf{(6.5.2)}$$

Here $h = h(x,t)$ denotes the height of **the** free surface **from the bottom** $y = 0; u(x,y,t)$ is **the** horizontal component **of** velocity **and** u_z denotes **partial derivative of** u with z. He also defined the moment functions $A_n(x,t) = \int_0^l u(x,y,t)^n dy$ **and he proved** that **the** A_n**'s** satisfy a system of evolution equations

$$A_{n,t} + A_{n+1,x} + nA_{n-1}A_{o,x} = 0 \quad , \quad n > 0 \quad . \tag{6.5.3}$$

Then (6.5.3) can be written in the Hamiltonian form as $A_t = B(\delta H/\delta A)$ where $A = (A_o, A_j, \ldots)^t$, $H = -\frac{1}{2}(A_2 + A_o^2)$ and
$\delta H/\delta A = (\delta H/\delta A_o, \ldots, \delta H/\delta A_j, ..)^t$ and B is a matrix differential operator

$$B = B_1\partial + \partial o B_1^t \text{ where } (B_1)_{ij} = iA_{i+j-1}\ , \partial = \frac{}{\partial x} \tag{6.5.4}$$

Then this operator B is Hamiltonian. In other words this means that $\{P,Q\} = \frac{\delta r}{\delta A^t} B \frac{\delta q}{\delta A}$ defines Poisson bracket on the space of polynomials in A_i^j modulo exact ∂-derivatives where $P = \int Pdx$ with $\int : \mathcal{A} = \mathcal{B}[A_i^j] \rightarrow A/SA$. This was proved in [31], [32], [55] (by directly checking the Jacobi Identity). Since the entries of B involve the unknown functions A_k this direct check was complicated and obscured the actual understanding of $\{,\}$. In this section we will give an invariant description of the operator B and also understand the Hamiltonian structure [55] of the Benney's system as the formal analogue of the Kirillov structure [54] on the orbits of the co-adjoint representations of Lie groups. A thorough and systematic study of Benney's system as a completely integrable Hamiltonian system was done in [55].

6.5.5 Benney's differential algebra and *B* operator: Let $\mathbb{R}^2$ be the plane with coordinates x and ξ and $\mathcal{C}^\infty(\mathbb{R}^2) = \{X(x,\xi) \in C^\infty(\mathbb{R}^2)\}$. Let $H_2(\mathbb{R}^2)$ denote

$$\left\{ D_X \mid D_X = X_\xi(x,\xi)\frac{\partial}{\partial x} - X_x(x,\xi)\frac{\partial}{\partial \xi} : X \in \mathcal{C}^\infty(\mathbb{R}^2) \right\}$$

the set of formal Hamiltonian vector fields on $\mathbb{R}^2$. Define

$$[D_X, D_Y] = D_{\{X,Y\}} \text{ where } \{X,Y\} = X_\xi Y_x - Y_\xi X_x. \tag{6.5.6}$$

Then $(H_2(\mathbb{R}^2), [\ ,\ ])$ becomes a Lie algebra called the Lie algebra of formal **Hamiltonian** vector fields on $\mathbb{R}^2$. Then [,] induces a Lie algebra structure on $C^\infty(\mathbb{R}^2)$ via $\{,\}$.

6.5.7 Definition: Let $\mathcal{K}$ be a field of characteristic 0. By a differential ring B we mean an $\mathcal{K}$-algebra B endowed with a **derivation** $d : B \to B$ which is trivial on AC C $\mathcal{B}$. If A € $\mathcal{B}$, $X^{(j)} = \partial^j X$. Let $\mathcal{B}^\infty$ denote the vector space of infinite columns $X = (X_o, X_1, \ldots, \ldots)^t$ **with** $X_i \in \mathcal{B}$.

Let $\mathcal{B}[[\xi]]$ be the formal power series ring in ξ over B. That is $\{X \setminus X = \sum_{i>0} X_i \xi^i \mid X_i \in B\}$. Note that there are two derivations on $\mathcal{B}[[\xi]]$ namely ∂_ξ and d. In fact $\partial_\xi (\sum_{i=0}^\infty X_i \xi^i) = \sum_{i=1}^\infty i X_i \xi^{i-1}$ and $\partial (\sum_{i=0}^\infty X_i \xi^i) = \sum_{i=0}^\infty X_i^{(1)} \xi^i$.

Consider $\mathcal{B}[[\xi]]$ and define $\{X, Y\} = X_\xi Y^{(1)} - Y_\xi X^{(1)}$ (6.5.8)

where $X, Y \in \mathcal{B}[[\xi]]$ or more explicitly

$$\begin{aligned}
\{X,Y\} &= \sum_{i,j} \left[(i+1)X_{i+1}Y_j^{(1)} - (i+1)Y_{i+1}X_j^{(1)}\right] \xi^{i+j} \\
&= \sum_{l\geq 0} \left[\sum_{j=0}^{l} (l-j+1)X_{l-j+1}Y_j^{(1)} - (l-j+1)Y_{l-j+1}X_j^{(1)}\right] \xi^l \\
&= \sum_{l\geq 0} \{X,Y\}_l \xi^l \qquad (6.5.9)
\end{aligned}$$

which makes $\mathcal{B}[[\xi]]$ a Lie algebra.

There is a vector space isomorphism: $\mathcal{B}^\infty = \mathcal{B}[[\xi]]$ given by $X = (X_0, X_1, ..)^t \leftrightarrow X = \sum_{i=0}^\infty X_i \xi^i$ which can be used to transfer the Lie algebra structure on $\mathcal{B}[[\xi]]$ given by {, } to $\mathcal{B}^\infty$ by defining for X, Y € $\mathcal{B}^\infty$, $X * Y = \vec{Z}$€ $\mathcal{B}^\infty$ where $Z \leftrightarrow Z = \sum_{l>0} \{X, Y\}_l \xi^l \in \mathcal{B}[[\xi]]$ where $\{X, Y\}_l$ is defined by (6.5.9).

6.5.10 Definition: The Lie algebra $(\mathcal{B}^\infty, *)$ is called the Benney differential algebra which is a differential algebra of infinite rank and which has a subalgebra consisting of finite columns denoted by $\mathcal{B}^\infty_{\mathbf{fin}}$.

6.5.11 Definition: Let $X, Y \in \mathcal{B}^\infty$. Define the scalar product on $\mathcal{B}^\infty$ by $X^t.Y = \sum_{i=0}^\infty X_i Y_i$ where we assume that B is equipped with a topology in which this **above**

series converges. We denote the canonical map $B \to B/dB$ by /. Then we define the map / : $\mathcal{B}^\infty \to B/dB$ by $X = (X_i) \in \mathcal{B}^\infty$ maps to $\int A^t X \in \mathcal{B}/\partial\mathcal{B}$ where $A = (A_0, A_1, \ldots)^t \in \mathcal{B}^\infty$ is the column vector.

6.5.12 Remarks: (1) This map / is the formal analogue of a linear functional on a Lie algebra in the finite dimensional case.

(2) The vector $A \in \mathcal{B}^\infty$ is a fixed vector corresponding to the one having Benney's moment functions as components. Now we give an invariant understanding of the B operator.

6.5.13 Theorem: For all $X, Y \in \mathcal{B}^\infty$ we have the formula

$$\int \vec{A}^t[\vec{X} * \vec{Y}] = \int \vec{X}^t B \vec{Y} \text{ where } B = B_1\partial + \partial o B_1^t$$

and $(B_1)_{ij} = iA_{i+j-1}$ and $\partial = \frac{\partial}{\partial x}$ and $\vec{A}$ is Benney's infinite column vector.

Proof: We have $\int \vec{A}^t[\vec{X} * \vec{Y}] = \int \sum_{l \geq 0} A_k[\vec{X} * \vec{Y}]_k$

$$= \int \sum_{k \geq 0} \sum_{j=0}^{k} \left[X_{k+1-j}^{(k+1-j)} A_k Y_j^{(1)} - Y_{k+1-j}^{(k+1-j)} A_k X_j^{(1)}\right]$$

$$= \int \sum_{k \geq 0} \sum_{j=0}^{k} \left[X_{k+1-j}^{(k+1-j)} A_k Y_j^{(1)} - \sum_{k \geq 0} \sum_{j=0}^{k} Y_{k+1-j}^{(k+1-j)} A_k \frac{\partial}{\partial x} X_j\right] (\text{using (6.5.9)})$$

Integrating by parts the second term and using J modulo Im d we get

$$= \int \sum_{k \geq 0} \sum_{j=0}^{k} \left[X_{k+1-j}^{(k+1-j)} A_k Y_j^{(1)} + X_j \partial\{(k+1-j) A_k Y_{k+1-j}\}\right]$$

$$= \int \sum_{i,j} \left[X_i i A_{i+j-1} Y_j^{(1)} + X_j \partial\{i A_{i+j-1} Y_i\}\right]$$

(by substituting $i = k + 1 - j$)

(interchanging i & j in 2nd term)

$$= \int \sum_{i,j} \left[X_i i A_{i+j-1} Y_j^{(1)} + X_i \partial\{i A_{i+j-1} Y_i\}\right]$$

$$= \int \vec{X}^t B_1 \vec{Y}^{(1)} + X^t \partial(B_1^t \vec{Y}) = f X^t B Y$$

From the above proof we get

6.5.14 Corollary: The commutator *[X * Y]* in $\mathcal{B}^\infty$ can also be given by the formula

$$[\vec{X} * \vec{Y}] = \vec{X}^t \frac{\partial B_1}{\partial \vec{A}} \vec{Y}^{(1)} - \vec{X}^{t(1)} \frac{\partial B_1^t}{\partial \vec{A}} \vec{Y} \ ,$$

where $X^t(\partial B_1/\partial \vec{A})\vec{Y}^{(1)}$ is the column in $\mathcal{B}^\infty$ having its fc-th component $X^t \frac{\partial B_1}{\partial A_k} \vec{Y}^{(1)}$.

6.5.15 Relation with Kirillov structure on orbits:

We recall briefly the

6.5.16 General set **up of Kirillov:** Let *G* be a connected Lie group and £ be its Lie algebra and £* its dual space. Let $u \in \mathcal{L}^*$ and let Ω denote the orbit of *u* in $\mathcal{L}^*$ under the co-adjoint representation. Since *G* acts transitively on ft, every element *X* ∈ £ **defines** a vector field ∂_X on Ω which gives a map A : £ —> $\mathcal{X}(\Omega)$between Lie algebras where $\mathcal{X}(G)$ is the Lie algebra of vector fields on *ft* and $\lambda(\mathcal{L})$generates the whole Lie algebra $\mathcal{X}(\Omega)$. Then the Kirillov symplectic structure on the orbit *ft* is given by the symplectic form ω on *ft* as

$$\omega(\xi_X, \xi_Y)(u) =< u, [X, Y] > \qquad \forall\, X, Y \in \mathcal{L}\,,\ u \in \Omega \tag{6.5.17}$$

Consider the ring $C^\infty(\Omega)$ of C^∞-functions on Ω and let $f, h \in C^\infty(\Omega)$ and let u € ft. Then $df(u)$ and *dh(u)* can be considered as elements of $\mathcal{L}$. Then the Poisson bracket on $C^\infty(\Omega)$ is defined by the formula

$$\{/, h\}(u) =< u,\ [df(u), dh(u)] > \quad . \tag{6.5.18}$$

Now we want to understand the formal analogue of Kirillov structures on Ω to $\mathcal{B}^\infty$. We assume $A_o, A_1, ..$ are all differentiably **independent** over $\mathcal{B}$, that is, the family $\{A_i^{(j)} \backslash i > 0, j > 0\}$ is algebraically independent over *B*. Let $A = \mathcal{B}[A_i^{(j)}]$ and $\mathcal{A} = \mathcal{A}/\partial\mathcal{A}$. Then formally the vector space $\mathcal{A}$ is the analogue of the ring of C°°-functions on the orbit

Ω of the point $u \in \mathcal{L}^*$ in the finite dimensional case.

Consider the Lie algebra

$$\mathrm{Der}(\mathcal{A}) = \left\{ \partial_{\vec{X}} \mid \partial_{\vec{X}} = \sum_{i,j} X_i^{(j)} \frac{\partial}{\partial A_i^{(j)}} \ , \ X_i \in \mathcal{A} \right\} \tag{6.5.19}$$

of continuous derivations of A commuting with d and trivial on B. The subalgebra

(6.5.20)

with A the moment vector, B the Benney matrix, is the formal analogue of the algebra of $\mathcal{X}(\Omega)$ of vector fields on the orbit Ω. Similar to the above paragraph, define the 2-form on $\mathrm{Der}_{\mathcal{A}}$ by $\omega(\partial_{B\vec{X}}, \partial_{B\vec{Y}})(A) = \int A^t[X * Y] = \int \vec{X}^t B\vec{Y}$ by above Theorem (6.5.13). Using ω we define the Poisson bracket on $\mathcal{A}$ by

$$\left\{ \tilde{P}, \tilde{Q} \right\} (A) = \omega \left(\partial_{B\frac{\delta P}{\delta \vec{A}}}, \partial_{B\frac{\delta Q}{\delta \vec{A}}} \right) \tag{6.5.21}$$

where $\tilde{P} = \int P$, $\tilde{Q} = \int Q$ **with** $P, Q \in \mathcal{A}$ **.**

Now we prove that {,} indeed defines a Poisson structure on $\mathcal{A}$.

6.5.22 Theorem: For all $P, Q, R \in \mathcal{A}$ we have

(i) $\{\tilde{P}, \tilde{Q}\} = -\{\tilde{Q}, \tilde{P}\}$

(ii) $\{\tilde{P}, \{\tilde{Q}, \tilde{R}\}\} + \{\tilde{Q}, \{\tilde{R}, \tilde{P}\}\} + \{\tilde{R}, \{\tilde{P}, \tilde{Q}\}\} = 0.$

Proof: (i) is trivial by (6.5.21) as ω is skew symmetric.

(ii) Let $\vec{X} = \frac{\delta P}{\delta \vec{A}}$, $\vec{Y} = \frac{\delta Q}{\delta \vec{A}}$, $\vec{Z} = \frac{\delta R}{\delta \vec{A}}$. Then we have the following identity [32]:

$$\frac{\delta}{\delta \vec{A}}(\vec{X}^t B\vec{Y}) = D(\vec{X})B\vec{Y} - D(\vec{Y})B\vec{X} + [\vec{X} * \vec{Y}] \tag{6.5.23}$$

where $D(\vec{X})$ is an infinite matrix with the diffrential operator $D_{A_j}(X_i) = \sum_{k=0}^{\infty} \frac{\partial X_i}{\partial A_j^{(k)}} \partial^k$ in the (i,j)-th place. Hence

$$\{\tilde{P}, \{\tilde{Q}, \tilde{R}\}\} = \int \vec{X}^t BD(\vec{Y})B\vec{Z} - \vec{X}^t D(\vec{Z})B\vec{Y} + \vec{X}^t B[\vec{Y} * \vec{Z}] \tag{6.5.24}$$

$$\{\tilde{Q},\{\tilde{R},\tilde{P}\}\} = \int \vec{Y}^t BD(\vec{Z})B\vec{X} - \vec{Y}^t BD(\vec{X})B\vec{Z} + \vec{Y}^t B[\vec{Z} * \vec{X}] \quad (6.5.25)$$

$$\{\tilde{R},\{\tilde{P},\tilde{Q}\}\} = \int \vec{Z}^t BD(\vec{X})B\vec{Y} - \vec{Z}^t BD(\vec{Y})B\vec{X} + \vec{Z}^t B[\vec{X} * \vec{Y}] \quad (6.5.26)$$

Using self adjointness of B and $D(X)D(Y)$,$D(Z)$ we have the sum of the left hand side of (6.5.24), (6.5.25), (6.5.26) as

$$\int \vec{X}^t B[\vec{Y} * \vec{Z}] + \int \vec{Y}^t B[\vec{Z} * \vec{X}] + \int \vec{Z}^t B[\vec{B} * \vec{Y}]$$
$$= \int \vec{A}^t \left([\vec{X} * [\vec{Y} * \vec{Z}]] + [\vec{Y}[\vec{Z} * \vec{X}]] + [\vec{Z} * [\vec{X} * \vec{Y}]]\right) = 0 .$$

6.5.27 Remark: This theorem is equivalent to the Hamiltonian condition for the operator B. The equations $F_t = \{\tilde{H},\tilde{F}\}$ or $A_t = B(SH/SA)$ (6.5.28) are the formal analogues of the Hamiltonian flows on the finite dimensional orbit Ω.

6.5.29 Linear functionals on $\mathcal{B}^{\infty}_{\text{fin}}$:

Let $\mathcal{B}((\xi^{-1}))$= $\left\{X \mid X = \sum_{i=0}^{N} X_i\xi^i + \sum_{j=0}^{\infty} A_j\xi^{-(j+1)}, X_i, A_j \in \mathcal{B}\right\}$. We can put a Lie algebra structure on $\mathcal{B}((\xi^{-1}))$by $[X,Y] = X_\xi Y^{(1)}$ - $Y_\xi X^{(1)}$ = $\{X,Y\}$ (6.5.30) and we denote this Lie algebra by $\mathcal{L}_1$. Let $\mathcal{L}_{1+}$ = $\{X \in \mathcal{L}_1 \mid X = \sum_{i=0}^{N} X_i\xi^i\}$ and $\mathcal{L}_{1-} = \{A \in \mathcal{L}_1 \mid \sum_{j=0}^{\infty} A_j\xi^{-(j+1)}\}$([16]). Then $\mathcal{L}_{1\pm}$ are Lie subalgebras of $\mathcal{L}_1$and $\mathcal{L}_1 = \mathcal{L}_{1+} \oplus \mathcal{L}_{1-}$ as vector space. Note that $\mathcal{L}_{1+}$ is isomorphic with $\mathcal{B}^{\infty}_{?}$.

6.5.31 Definition: For all $X,Y \in B((\xi^{-1}))$ define a scalar product by (X,Y)= / Res $X\,Y$ (6.5.32) where Res(XY) means the coefficients ξ^{-1} in the formal series ([16]). Note that for $A \in \mathcal{L}_{1-}, X \in \mathcal{L}_{1+}$ we have $(A,X) = \int_l \sum_{l>0} A_l X_l$ and $(\mathcal{L}_{1+},\mathcal{L}_{1+}) = 0$ and $(\mathcal{L}_{1-},\mathcal{L}_{1-}) = 0$. Hence we can identify the linear functionals on $\mathcal{B}^{\infty}_{\text{fin}}$ with $\mathcal{L}_{1-}$ by means of this scalar product (,).

6.5.33 Remark: The Lie algebra $(\mathcal{B}((\xi^{-1})), [,\ ]) = C\backslash$ can be regarded as an extension of $\mathcal{B}^{\infty}_{\text{fin}}$ and $\mathcal{L}_1$was considered by Gel'fand-Dikii [16], Adler [5], Kostant [27], Frenkel.I.E., Reyman.A.G., Semenov-Tian-Shansky.M.A. [13]).

6.5.34 Proposition: The scalar product (,) on $\mathcal{L}_1 = \mathcal{B}((\xi^{-1}))$ is invariant. That is, for all $X, Y, Z \in \mathcal{B}((\xi^{-1}))$, $(Z, [X, Y]) = ([Z, X], Y)$.

Proof: We first note that

Res $(ZX^{(1)}Y)_\xi = 0$ and so $0 = \text{Res } (ZX^{(1)}Y)_\xi = \text{Res } (Z_\xi X^{(1)}Y + ZX^{(1)}_\xi Y + ZX^{(1)}Y_\xi)$.

Then $(Z, [X, Y]) = \int \text{Res } (Z[X, Y])$

$$
\begin{aligned}
&= \int \text{Res } (Z(X_\xi Y^{(1)} - Y_\xi X^{(1)})) = \int \text{Res } (ZX_\xi Y^{(1)} - ZY_\xi X^{(1)}) \\
&= \int \text{Res } \{-(ZX_\xi)^{(1)}Y - ZY_\xi X^{(1)}\} \quad \text{(on integration by parts of first term).} \\
&= \int \text{Res } \left\{-Z^{(1)}X_\xi Y - ZX^{(1)}_\xi Y - ZY_\xi X^{(1)}\right\} \\
&= \int \text{Res } \{-Z^{(1)}X_\xi Y - Z(X^{(1)}Y)_\xi\} \quad \text{(which on integration by parts of } 2nd \text{ term).} \\
&= \int \text{Res } \{-Z^{(1)}X_\xi Y + Z_\xi X^{(1)}Y\} \\
&= \int \text{Res}([Z, X]Y) = ([Z; X], Y) .
\end{aligned}
$$

Let $Z = \sum_{i>0} A_i \xi^{-(i+1)} \in \mathcal{L}_{1-}$. For any $F \in \mathcal{A}$, $F_{\overline{A}} = \sum_{j\geq 0} \left(\frac{\delta F}{\delta A_j}\right) \xi^j$. Then the formula (6.5.17) can be rewritten as $\{\hat{P}, \hat{Q}\}(Z) = (Z, [P_A, Q_A]) = ([Z, P_A]_-, Q_A)$ and the Hamiltonian flow with Hamiltonian $H = \int H, H \subset A$ is given by $Z_t = [Z, H_A]_-$. Hence the Benney's equation (6.5.3) can be rewritten in the form (6.5.28) above with Hamiltonian $H = \frac{1}{2}(A_2 + A_1)$.

6.5.35 Remarks: (1) In fact all higher order Hamiltonian flows having the same infinite conserved densities H_n which are in involution relative to the Poisson bracket can be determined and also all such H_n's can be determined answering a question of Lebedev [36].

References

[1] **Abraham.R and Marsden.J** : Foundations of Mechanics, Benjamin/Cummings, 1978.

[2] **Adler.M and van Moerbeke.P:** Completely Integrable Systems, Euclidean Lie algebras and Curves, Adv. in Math., 38, (1980), 267-317.

[3] - : Linearization of Hamiltonian systems, Jacobian varieties and representation theory, Adv. in Math,, 38, (1980), 318-379.

[4] **Adler.M** :On a trace functional for formal pseudo- differential operators and the Hamiltonian structure of the KdV type equations, in Global Analysis, Lect. Notes in Math., Spriger-Verlag, 755, (1978), 1-16.

[5] **Adler.M** : On a trace functional for formal pseudo- differential operators and the symplectic structure of the KdV type equations, Inventiones Math., 50, (1979), 219-248.

[6] **Arbarello.E, Cornalba.M, Griffiths.P.A and Harris.J** : Geometry of Algebraic curves, Vol.1, Springer-Verlag, 1985.

[7] **Arbarello.E and De** Concini.C : On a set of equations characterizing Riemann matrices, Ann. Math., 120, (1984), 119-140.

[8] **Arnold. V.I :** Mathematical methods of Classical Mechanics, Springer-Verlag, New York, 1978.

[9] **Audin.M** : Spinning tops, Cambridge studies in Adv. Math. 51, Cambridge University Press, 1996.

[10] **Benney.D.J** : Some properties of long nonlinear waves, Studies in Appl. Math., L11, No.1, (1973), 45-50.

[11] **Cartier.P, Babelon.O and Kosmann-Schwarzbach.Y** : Lectures on Integable Systems, in memory of J.L.Verdier, Proceedings of the CIMPA School, Nice, France, World Scientific, 1991.

[12] **Farkas.H.M and Kra.I** : Riemann Surfaces, Springer-Verlag, 1980.

[13] **Frenkel.I, Reyman.A.G and Semenov-Tian-Shansky.M** : Graded Lie algebras and completely integrable dynamical systems, Soviet Math. Dokl., 20, No.4, (1979), 811-814.

[14] **Gardner.C.S, Miura.R.M and Kruskal.M :** Korteweg-de Vries equation and generalizations, Jour. Math. Phys., 9, No.8, (1968), 1202-1209.

[15] **Gel'fand.I.M and Dikii.L.A** : Asymptotic behaviour of the resolvent of Strum-Liouville equations and the algebra of the KdV equations, Russ. Math. Surv., 30, No.5, (1975), 77-113.

[16] **Gel'fand.I.M and Dikii.L.A** : Fractional powers of operators and Hamiltonian systems, Funct. Anal. Appli., 10, No.4 (1976), 259-273.

[17] **Gel'fand.I.M and Dikii.L.A** : The Resolvent and Hamiltonian systems, Funct. Anal. Appli., 11, No.2 (1977), 93-105.

[18] **Gel'fand.I.M, Manin.Yu.I and Shubin.M.A** : Poisson brackets and the kernel of the variational derivative in the formal calculus of variations, Funct. Anal. Appli., 10, No.4 (1976), 274-278.

[19] **Griffiths.P.A :** Linearizing flows and a cohomological interpretation of Lax equations, Non-linear PDE's M.S.R.I. Publications 2 (edited by S.S.Chern) (1984), 37-46.

[20] **Griffths.P.A and Harris.J** : Principles of Algebraic geometry, John Wiley and Sons, 1978.

[21] **Gunning.R.C** : Lectures on Riemann surfaces, Princeton Univ. Press, New Jersey, 1966.

[22] **Gunning.R.C** : Some curves in abelian varieties, Inv. Math., 66, (1982). 377-389.

[23] **Kadomstev.B. and Petviashvilli.V.I** : On the stability of soliton waves in weakly dispersing media, Soviet Phys. Dokl., 15, No.6 (1970), 539-541.

[24] **Kirillov.A.A** : The Elements of the theory of Representations, Springer-Verlag, 1976.

[25] **Knörrer.H** : Geodesics on the ellipsoid, Inv. Math., 59, (1980), 119-144.

[26] **Kodaira.K** : Complex manifolds and Deformation of complex structures, Springer-Verlag, New York, 1986.

[27] **Kostant.B** : The Solution to a generalized Toda Lattice and Representation theory, Adv. in Math., 34, (1979), 195-338.

[28] **Krichever.I.M** : Algebraic curves and commuting matrix differential operators, Funct. Anal. Appli., 10, No.2 (1976),144-146.

[29] **Krichever.I.M and Novikov.S.P** : Holomorphic vector bundles over algebraic curves and non-linear equations, Russ. Math. Surveys, 35, (1980), 53-79.

[30] **Krichever.I.M** : Methods of algebraic geometry in the theory of non-linear equations, Russ. Math. Surveys, 32, No.6 (1977), 185-213.

[31] **Kupershmidt.B.A and Manin.Yu.I** : Long-wave equations with free boundaries I, Conservation laws and solution, Funct. Anal. Appli., 11, No.3 (1977), 188-197.

[32] **Kupershmidt.B.A and Manin.Yu.I** : Equations of long-waves with a free surface II, Hamiltonian Structure and Higher equations, Funct. Anal. Appli., 12, No.1 (1978), 20-29.

[33] **Kupershimdt.B.A** : A super symmetric Benney hierarchy : A semi-integrable system, J. Math. Phy., 31, (1990), 459-469.

[34] **Lax.P** : Integrals of non-linear equations of evolution and solitary waves, Comm. in Pure and Applied Math., 21, No.2, (1968), 467-490.

[35] **Lebedev.D.R** : Benney's long wave equations : Hamiltonian formalism, Lett. in Math. Phys., 3, (1979), 481-488.

[36] **Lebedev.D.R** : Conservation laws and Lax representation of Benney's long wave equations, Phy. letters, 74, No.3,4 (1979), 154-156.

[37] **Manakov.S.V** : Note on the integration of the Euler equations for the dynamics of an n-dimensional rigid body, Funct. Anal. Appli., 10, (1976), 328-329.

[38] **McKean.H.P** : Boussinesq's equation as a Hamiltonian system, Topics in Funct. Analy., Adv. in Math. Suppl. Studies, 3, (1978), 217-226.

[39] **McKean.H.P** : Boussinesq's equation on the circle, Comm. in Pure and Applied Math., 34, (1981), 599-691.

[40] **McKean.H.P** : Integrable systems and algebraic curves, Lecture Notes in Math., 755, (1979), 83-200.

[41] **McKean.H.P and van Moerbeke.P** : The spectum of Hill's equation, Inv. Math., 30, (1975), 217-274.

[42] **Moosath.K.S.S** : Geometry of subvarieties of complex tori and fibrations over complex tori, Ph.D thesis, University of Hyderabad, (1994), 150 pages.

[43] **Moser. J** : Geometry of quadrics and spectral theory, Proceedings of the International Symposium on Differential Geometry in honour of S.S.Chern, (The Chern Symposium 1979), Berkeley, California, (1979),147-187.

[44] **Moser. J** : Finitely many mass points on the line under the influence of an exponential potential - an integrable system, Lect. Notes in Phys., Springer-Verlag, 38, (1975), 467-497.

[45] **Mulase.M** : Cohomological srtucture of KP equation and Jacobian varieties, Jour. of Diffl. Geom., 19, (1984), 403-430.

[46] **Novikov.S.P** : The periodic problem for the KdV equation, Funct. Anal. Appli., 8, No.3 (1974), 236-246.

[47] **Rama.K** : Geometry of Moishezon spaces, Ph.D thesis, University of Hyderabad, (1984), 126 pages.

[48] **Ratiu.T and van Moerbeke.P** : The Lagrange rigid body motion, Ann. Inst. Fourier, Grenoble 31, No.1 (1982), 211-234.

[49] **Ratiu.T** : Euler-Poisson equations on Lie algebras and the N-dimensional heavy rigid body, Amer. Jour. Math., 104, No.2, (1982), 409-448.

[50] **Ratiu.T** : Involution theorems, Lect. Notes in Math., Springer-Verlag, 775, (1979), 219-257.

[51] **Reyman.A.G and Semenov-Tian-Shansky.M.A** : Reduction of Hamiltonian Systems, Affine Lie algebras and Lax equations I and II, Inv. Math., 54, (1979), 81-100 and Inv. Math. 63, (1981), 423-432.

[52] **Sitaramayya.M and Subbulakshmi.S** : Integrability of certain partial differential equations, preprint, University of Hyderabad, (1996), 25 pages.

[53] **Sitaramayya.M** : Geometry of classical Benney long wave equations and generalizations, Tech. Report No. DMS/1998/08, Dept. of Math. & Stat., University of Hyderabad, (1998), 29 pages.

[54] **Sitaramayya.M** : Geometric AKS principle and applications to some integrable systems, paper presented in National Symposium on Dynamical systems and Ergodic theory, Dec. 6-8, (1997), 13 pages.

[55] **Sitaramayya.M and Subbulakshmi.S** : Geometry of Benney's equations and Higher equations, Tech. Report No. DMS/1998/05, Dept. of Math. & Stat., University of Hyderabad, (1998), 22 pages.

[56] **Souriau.J.M** : Structure des systemes dynamiques, Dunod, Paris, 1969.

[57] **Springer.G** : Introduction to Riemann surfaces, Chelsea Publ. Co., N.Y., 1981.

[58] **Subbulakshmi.S and Sitaramayya.M** : Geometry of Boussinesq equation, Lenard relations and applications, Tech. Report No. DMS/1998/07, Dept. of Math. & Stat., University of Hyderabad, (1998), 21 pages.

[59] **Symes.W** : Relations among generalized Korteweg-deVries systems, J. Math. Phys., 20, Xo.4 (1979), 721-725.

[60] **Symes.W** : Systems of Toda type, Inverse spectral problems and Representation theory, Inv. Math., 59, (1980), 13-51.

[61] **Verdier.J.L** : Algèbres de Lie, Systemes Hamiltoniens, Courbes Algébriques, Séminaire Bourbaki, 33, No.566 (1980/81), 85-94.

[62] **Wells.R.O,Jr.** : Differential Analysis on Complex manifolds, Springer-Verlag, New York. 1980.

[63] **Zakharov.V.E and Shabat.A.B** : A scheme for integrating the nonlinear equations of Mathematical Physics by the method of the inverse scattering problem-I, Funct. Anal. Appli., 8, No.3 (1978), 226-235.

[64] **Zakharov.V.E** : On the problem of stochastication of 1-dimensional non-linear oscillators, Sov. Phy. JETP., 38, No.1 (1974), 108-110.