DISCONJUGACY, CONJUGACY AND OSCILLATION CRITERIA FOR CONTINUOUS AND DISCRETE LINEAR HAMILTONIAN SYSTEMS

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This is to certify that **I, I. Sowjanya Kumari** have carried out the research embodied in the present thesis entitled **DISCONJUGACY, CONJUGACY AND OSCILLATION CRITERIA FOR CONTINUOUS AND DISCRETE LINEAR HAMILTONIAN SYSTEMS** for the **full** period prescribed **under** the **Ph.D.** ordinances of the University.

I declare to the best of my knowledge that no part of this thesis was earlier submitted for the award of research degree of any other University / Institute.

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CONTENTS

Chapter 1

Introduction

Conjugacy, disconjugacy and oscillation criteria for linear Hamiltonian continuous as well as discrete systems have been the subject of study by several authors for many years as can be seen from the references [9] to [16], [19], [21, 22, 23, 25, 26, 27, 29], [31] to [37], [39, 40], [42] to [47] and other references contained therein for the continuous case and from [1] to [5], [7, 8, 17, 20, 24, 30, 38] and other references contained therein for the discrete case.

Such criteria for continuous systems can be broadly classified into three classes, one type of criteria giving the sufficient conditions in terms of coefficient functions or some special solutions as for instance in [9] to [16] [19, 22, 23, 26, 27, 29, 31, 33, 34, 36, 37, 39, 40]. [42] to [47] and other references contained therein, the second type of criteria giving the sufficient, conditions in terms of the positive definiteness of a related quadratic functional or the existence of a positive solution of the Riccati equation as exemplified by [12, 23, 39, 40] and other references contained therein. The third type of criteria are comparison theorems giving sufficient, conditions for the conjugacy of a vector system in terms of the conjugacy of a related vector system or a scalar equation as for example in [21, 25, 32. 35, 39, 40].

Similar criteria for discrete systems have been discussed in the recent past by several authors, for instance as in [1] to [5], [17, 20, 24, 30, 38] and other references contained therein and further for singular systems in [7] and [8].

In this dissertation we obtain conjugacy and oscillation criteria for linear Hamiltonian continuous as well as discrete systems.

In the first part of the dissertation, we consider linear Hamiltonian continuous systems of which Sturm-Liouville systems are a special case. There is a vast amount of literature concerning the Sturm-Liouville systems especially their conjugacy, disconjugacy and oscillation criteria. A detailed historical account of the development of this theory including an extensive bibliography upto 1980 is given in Reid [40].

For more recent literature concerning the above topics, reference can be made to the papers [9] to [11], [19, 21, 27, 29, 31, 32, 33, 34, 42, 45] and other references contained therein for oscillation criteria and to [13, 14, 15, 1G, 36, 44, 46] and other references contained therein for conjugacy or disconjugacy criteria.

In the second part we consider the discrete scalar equation. Here we obtain conjugacy criteria for the discrete Sturm-Liouville equation based on the results given by Došlý in [16].

We now mention the details of the results obtained in this dissertation chapterwise.

In chapter 2 we give the definitions and several preliminary results from earlier literature which are useful for the discussion in the following chapters.

In chapter 3 we consider the linear Hamiltonian continuous system

$$\begin{aligned} u' &= a(x)u + b(x)v \\ v' &= c(x)u - a(x)v \end{aligned} \tag{1.1}$$

along with the associated unperturbed system

$$\begin{aligned} u' &= a(x)u + b(x)v \\ v' &= -a(x)v \end{aligned} \tag{1.2}$$

under the hypothesis

H : $a(x)$, $b(x) > 0$, $c(x)$ are real valued continuous functions on an open interval $I = (\alpha, \beta)$, $-\infty \leq \alpha < \beta \leq \infty$.

Here we obtain conjugacy criteria for the system (1.1) based on the results given by Došlý in [15] and [16] for the Sturm-Liouville equations

$$(p(x)u')' + q(x)u = 0, \tag{1.3}$$

$$u'' + q(x)u = 0 \tag*{$(1.3)_1$}$$

and

$$(p(x)u')' = 0, \tag{1.4}$$

where $p(x) > 0$ and $q(x)$ are continuous real valued functions on (α, β).

Our results are proved by extending to general linear Hamiltonian systems, the Riccati techniques used by Došlý for Sturm-Liouville systems.

In particular some of our results yield for the scalar equation (1.3) a conjugacy criterion which can be applied in some instances where Došlý's criterion fails. This fact is illustrated by means of an example (example 3.7).

In chapter 4 we obtain conjugacy criteria for the 2n-dimensional linear Hamiltonian system

$$\begin{aligned} u' &= A(x)u + B(x)v \\ v' &= C(x)u - A^{*}(x)v \end{aligned} \tag*{$(1.1)_v$}$$

under the hypothesis

H_v : $A(x), B(x) = B^{*}(x) > 0, C(x) = C^{*}(x)$ are $n \times n$ matrices of continuous real valued functions on the interval $I = (\alpha, \beta)$, $-\infty \le \alpha < \beta \le +\infty$ and for its equivalent second order form $(1.5)_v$

$$L[u] \equiv [P(x)u' + R(x)u]' - [R^*(x)u' - Q(x)u] = 0 \qquad (1.5)_v$$

under the transformation

$$v = P(x)u' + R(x)u$$

where

$$\begin{aligned} P(x) &= B^{-1}(x), \\ R(x) &= -B^{-1}(x)A(x) \\ \text{and } Q(x) &= -C(x) - A^*(x)B^{-1}(x)A(x). \end{aligned}$$

For the equation $(1.5)_v$ the hypothesis H_v translates into the hypothesis $\tilde{H}_v$: $P(x) = P^*(x) > 0, Q(x) = Q^*(x), R(x)$ are $n \times n$ matrices of continuous real valued functions on the interval $I = (\alpha, \beta)$, $-\infty \le \alpha < \beta \le +\infty$.

Conjugacy criteria given in this chapter are of two types. One type of results which give sufficient conditions for the equation $(1.5)_v$ to be conjugate are based on the results given by Etgen and Lewis [21] and Hartman [25] for the equation

$$(P(x)u')' + Q(x)u = 0 \qquad (1.3)_v$$

and are comparison theorems relating the conjugacy of the vector equation $(1.5)_v$ to the conjugacy of the scalar equation obtained by applying a positive linear functional g to the coefficient matrices of $(1.5)_v$. These theorems are proved by means of a generalized "Picone type identity" and the variational principle.

Such theorems are not in general true when applied directly to general linear Hamiltonian systems of the form $(1.1)_v$. This fact is illustrated by means of an example (example 4.8). Moreover another example (example 4.3) shows that if the hypotheses of such a comparison theorem are not satisfied, then the system $(1.5)_v$ need not be conjugate.

Second type of results concerning the conjugacy of the system $(1.1)_v$ are based on the results due to Došlý in [14] for the system

$$\begin{aligned} u' &= B(x)v \\ v' &= -C(x)u. \end{aligned} \qquad (1.6)_v$$

These results are obtained by using the Courant-Fischer min-max theorem and the extended variational principles for the system $(1.1)_v$.

In chapter 5 we consider the matrix Hamiltonian system

$$\begin{aligned} U' &= A(x)U + B(x)V \\ V' &= C(x)U - A^*(x)V \end{aligned} \qquad (1.1)_M$$

under the hypothesis H_1.

Oscillation criteria of several types for the systems of the form $(1.1)_M$ have been studied by several authors as can be seen from the references [9, 10, 11, 19, 21, 22, 23, 25, 26, 27, 29, 31, 32, 33, 34, 37, 39, 40, 42, 45, 47] and other references contained therein. Many of these criteria (for instance Etgen and Pawlowski [22], Erbe, Qingkai Kong and Shigui Ruan [19] and Fanwei Meng, Jizhong Wang and Zhaowen Zheng [34]) are generally modelled on either oscillation criteria due to Wintner [47] or Kamenev [29] for the self-adjoint scalar equation (1.3)

In this chapter we obtain oscillation criteria for the system $(1.1)_M$ which are (respectively) of Wintner or Kamenev types and include as special cases results of Etgen and Pawlowski [22] for the system

$$U'' + Q(x)U = 0 \qquad (1.3)_{1_M}$$

and of Erbe, Qingkai Kong and Shigui Ruan [19] for the system

$$(P(x)U')' + Q(x)U = 0. \qquad (1.3)_M$$

Further we generalize a more recent Kamenev type oscillation criterion due to Fanwei Meng, Jizhong Wang and Zhaowen Zheng [34] for the system $(1.3)_{1_M}$ to the general linear Hamiltonian system $(1.1)_M$.

We illustrate the significance of the hypotheses of each theorem of the chapter in the sense that if any one of the hypotheses is not satisfied, the conclusion may not hold.

In chapter 6 we consider the linear second order difference equation

$$\Delta(c_{n-1}\Delta x_{n-1}) + a_n x_n = 0 \qquad (1.7)$$

or the equivalent three term equation

$$c_n x_{n+1} + (a_n - c_n - c_{n-1})x_n + c_{n-1}x_{n-1} = 0 \qquad (1.8)$$

on an integer interval I which may be bounded or unbounded under one or the other of the following hypotheses:

H^d : $c_n > 0$ on $[M, N+1]$ (where $-\infty < M < N < \infty$) and a_n real on $[M+1, N+1]$,

H^d_+ : $c_n > 0$ on $[M, \infty)$ (where $-\infty < M < \infty$) and a_n real on $[M+1, \infty)$,

H^d_- : $c_n > 0$ on $(-\infty, N+1]$ (where $-\infty < N < \infty$) and a_n real on $(-\infty, N+1]$

and

Chapter 2

Preliminaries

In this chapter we fix the notation and state for the sake **of** convenience of reference some definitions and results **from earlier** literature which are useful for motivating the discussion in the following chapters. The reference cited for a definition or result is not necessarily the original source but is one of the sources. Illustrative examples are also given to explain some concepts.

We first recall the following basic result contained in theorems 2.2 and 2.3, p 223, [39] concerning the system

$$u' = a(x)u + b(x)v$$

$$v' = c(x)u - a(x)v \tag{1.1}$$

which can be directly verified.

Theorem 2.1. (i). If $(u(x), v(x))$, $(u_1(x), v_1(x))$ are two solutions of (1.1) then $\{u_1, u\}(x) \equiv v(x)u_1(x) - u(x)v_1(x)$ =constant.

(ii). If $(u_1(x), v_1(x))$ is a solution of (1.1) such that $u_1(x) \neq 0$ for $x \in I$, then $(u(x), v(x))$ is a solution of (1.1) on I if and only if

$$\begin{aligned} u(x) = u_1(x)h(x), \quad \text{where} \quad h(x) &= k_0 + k_1 \int_\gamma^x \frac{b(s)}{u_1^2(s)} ds, \\ v(x) &= v_1(x)h(x) + \frac{k_1}{u_1(x)}, \ \gamma \in I \end{aligned}$$

and k_0, k_1 are constants given by $k_0 = \frac{u(\gamma)}{u_1(\gamma)}$ and $k_1 = \{u_1, u\}$.

We now give the definitions and basic results needed in chapter 3 where we obtain conjugacy criteria for the system **(1.1).**

Definition 2.2 (p 233, [39]). The system (1.1) is said to be 'conjugate' on J

if there exist a pair of points x_1, x_2 in I, *X* < x_2 and a solution $(u(x)$, *V(X))* of (1.1) satisfying $u(x_1) = 0 = u(x_2)$ and $u(x) \not\equiv 0$ on *J*. Otherwise it is 'disconjugate' on /.

Note 2.3 (p 233, [39]). The pair of points x_1, x_2 in the above definition are said to be 'mutually conjugate' with respect to the system (1.1) and the solution $(u(x), v(x))$. x_1 is said to be 'left conjugate' to x_2 and x_2 is said to be 'right conjugate' to x_1 with respect to (1.1) and $(u(x), v(x))$.

Example 2.4. Consider the system

$$
\begin{aligned}
u' &= v \\
v' &= -u.
\end{aligned}
\tag{2.1}
$$

It, is conjugate on $[0, \pi]$ (and hence on any interval containing [0,7r]), since there exists a solution $(u(x), v(x))$ = (sin x, cos *x)* with *u(x)* $\not\equiv 0$ having two zeros $x_1 = 0$ and x_2 = 7r.

This system is disconjugate on [0,7r) (and hence on any subinterval of $[0, \pi)$). This follows from the fact that

$$(u(x), v(x)) = (c_1 \sin x + c_2 \cos x, c_1 \cos x - c_2 \sin x)$$

where c_1, c_2 are arbitrary constants, is the genera] solution of this system and $u(x_2) = 0 = u(x_1)$, $0 \leq x_1 < x_2 < \pi$ implies $c_1 \sin x_1 + c_2 \cos x_1 = 0$ and $c_1 \sin x_2 + c_2 \cos x_2 = 0$ and hence $c_1 = c_2 = 0$.

Now we give an example of the system (1.1) with *a(x)* $\neq 0$ which is conjugate on $[0, \pi]$ and disconjugate on $[0, \pi)$.

Example 2.5 The system

$$
\begin{aligned}
u' &= u + v \\
v' &= -2u - v
\end{aligned}
\tag{2.2}
$$

is conjugate on $[0, \pi]$ (since $(u(x), v(x)) = (\sin x, \cos x - \sin x)$ is a solution with $u(x) \not\equiv 0$ having two zeros $x_1 = 0$ and $x_2 = \pi$).

This system is disconjugate on $[0, \pi)$ (since

$$(u(x), v(x)) = (c_1 \sin x + c_2 \cos x, (c_1 - c_2) \cos x - (c_1 + c_2) \sin x)$$

where c_1, c_2 are arbitrary constants, is the general solution of this equation and $u(x_2) = 0 = u(x_1)$, $0 \le x_1 < x_2 < \pi$ implies $c_1 \sin x_1 + c_2 \cos x_1 = 0$ and $c_1 \sin x_2 + c_2 \cos x_2 = 0$ and hence $c_1 = c_2 = 0$.

Remark 2.6. It follows easily that the systems in the above two examples are conjugate on $[\alpha, \alpha + \pi]$ and disconjugate on $[\alpha, \alpha + \pi)$ for arbitrary α.

Note 2.7. Disconjugacy of (1.1) and $b(x) > 0$ on I imply the following result. $(u(x), v(x))$ is a solution and $u(x_1) = 0 = u(x_2)$, $x_1, x_2 \in I, x_1 < x_2$ $\Longrightarrow u(x) \equiv 0 \equiv v(x)$ on I.

Note that if (1.1) is disconjugate on $I = (\alpha, \beta)$ then it follows from a result due to Hartman and Wintner [26] (see also chapter IV section 3, [40] and for special cases theorem 6.4, p 355, [23] or theorem 5, p 7, [12]) that there exists a unique (up to a multiple by a nonzero real constant) pair of solutions $y_\alpha(x) = (u_\alpha(x), v_\alpha(x))$, $y_\beta(x) = (u_\beta(x), v_\beta(x))$ of (1.1) such that

$$\int_\alpha b(x) u_\alpha^{-2}(x) dx = \infty = \int^\beta b(x) u_\beta^{-2}(x) dx.$$

These are called the 'principal solutions' of (1.1) at α and β respectively.

The following examples illustrate disconjugate systems and their principal solution pairs on bounded and unbounded intervals.

Example 2.8. The principal solutions at 0 and π of the disconjugate system (2.1) on $(0, \pi)$ given in example 2.4 are

$$(u_0(x), v_0(x)) = (\sin x, \cos x) = (u_\pi(x), v_\pi(x))$$

since

$$\int_0^\gamma \sin^{-2} x dx = \infty = \int_\gamma^\pi \sin^{-2} x dx.$$

Example 2.9. The principal solutions at 0 and π of the disconjugate system (2.2) on $(0, \pi)$ given in example 2.5 are

$$(u_0(x), v_0(x)) = (\sin x, \cos x - \sin x) = (u_\pi(x), v_\pi(x))$$

since for $\gamma \in (0, \pi)$

$$\int_0^\gamma \sin^{-2} x dx = \infty = \int_\gamma^\pi \sin^{-2} x dx.$$

Example 2.10. The system

$$\begin{aligned} u' &= u + v \\ v' &= u - v \end{aligned} \tag{2.3}$$

is disconjugate on $(-\infty, \infty)$. This can be seen from the fact that the general solution is given by

$$(u(x), v(x)) = (c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x}, (\sqrt{2} - 1)c_1 e^{\sqrt{2}x} - (\sqrt{2} + 1)c_2 e^{-\sqrt{2}x})$$

where c_1, c_2 are arbitrary constants and $u(x_2) = 0 = u(x_1)$ $-\infty < x_1 < x_2 < \infty$ implies

$$c_1 e^{\sqrt{2}x_1} + c_2 e^{-\sqrt{2}x_1} = 0 \text{ and } c_1 e^{\sqrt{2}x_2} + c_2 e^{-\sqrt{2}x_2} = 0.$$

Hence $c_1 = 0 = c_2$. Further the principal solutions at $-\infty$ and $+\infty$ are respectively

$$\begin{aligned} (u_{-\infty}(x), v_{-\infty}(x)) &= (e^{\sqrt{2}x}, (\sqrt{2} - 1)e^{\sqrt{2}x}) \\ \text{and } (u_\infty(x), v_\infty(x)) &= (e^{-\sqrt{2}x}, -(\sqrt{2} + 1)e^{-\sqrt{2}x}) \end{aligned}$$

since for $\gamma \in (-\infty, \infty)$

$$\int_{-\infty}^{\gamma} e^{2\sqrt{2}x} dx = \infty = \int_{\gamma}^{\infty} e^{-2\sqrt{2}x} dx.$$

The following definition due to Došlý [16] which further classifies disconjugate systems is useful for our discussion in chapter 3.

Definition 2.11. Suppose the system (1.1) is disconjugate on $I = (\alpha, \beta)$. We say that it is '1-special'(1-general) on I if the principal solutions of (1.1) at α and β are linearly dependent (independent) on I.

Example 2.12. The principal solutions at 0 and π of the system (2.1) given in example 2.8 are

$$(u_0(x), v_0(x)) = (\sin x, \cos x) = (u_\pi(x), v_\pi(x))$$

and hence the system is 1-special on $(0, \pi)$.

Example 2.13. The principal solutions at 0 and π of the system (2.2) given in example 2.9 are

$$(u_0(x), v_0(x)) = (\sin x, \cos x - \sin x) = (u_\pi(x), v_\pi(x))$$

and hence the system is 1-special on $(0, \pi)$.

Example 2.14. The principal solutions of the system (2.3) given in example 2.10 at $-\infty$ and $+\infty$ are respectively

$$(u_{-\infty}(x), v_{-\infty}(x)) = (e^{\sqrt{2}x}, (\sqrt{2}-1)e^{\sqrt{2}x})$$

and

$$(u_{\infty}(x), v_{\infty}(x)) = (e^{-\sqrt{2}x}, -(\sqrt{2}+1)e^{-\sqrt{2}x})$$

which are linearly independent. Hence the system is 1-general on $(-\infty, \infty)$.

The following results given by Došlý in [16] and [15] for the equations

$$(p(x)u')' + q(x)u = 0, \tag{1.3}$$

$$u'' + q(x)u = 0, \tag{1.3$_1$}$$

and

$$(p(x)u')' = 0 \tag{1.4}$$

motivate our discussion in chapter 3.

Lemma 2.15 (remark 4 of [16]). Equation (1.4) is 1-special on I if and only if

$$\int_\gamma^\beta p^{-1}(x)dx = \infty = \int_\alpha^\gamma p^{-1}(x)dx. \tag{2.4}$$

Theorem 2.16 (theorem 2 of [15]). Suppose that there exist $\epsilon_1,\ \epsilon_2 > 0$ and $\gamma \in (\alpha, \beta)$ such that

$$\epsilon_1 \int_\gamma^\beta \exp\left\{2\int_\gamma^x \left[\int_\gamma^t q(s)ds - \epsilon_1\right] dt\right\} dx \ > A$$

and

$$\epsilon_2 \int_\alpha^\gamma \exp\left\{2\int_\gamma^x \left[\int_\gamma^t q(s)ds + \epsilon_2\right] dt\right\} dx \ > B$$

where

$$\epsilon_1 + \epsilon_2 - \frac{\pi(\epsilon_1 B + \epsilon_2 A)}{2AB} \geq 0.$$

Then $(1.3)_1$ is conjugate on (α, β).

Lemma 2.17 (corollary 1 of [16]). Suppose that (2.4) holds and there exists $c_i \in I,\ i = 1, 2$, such that

$$\liminf_{x\uparrow\beta} \frac{\int_\gamma^x p^{-1}(t)(\int_\gamma^t q(s)ds)dt}{\int_\gamma^x p^{-1}(t)dt} = c_1 > 0,$$

$$\limsup_{x\downarrow\alpha} \frac{\int_\gamma^x p^{-1}(t)(\int_\gamma^t q(s)ds)dt}{\int_\gamma^x p^{-1}(t)dt} = c_2 < 0.$$

Then (1.3) is conjugate on I.

Theorem 2.18 (theorem 2 of [16]). Let equation (1.4) be 1-general on I, and y_α, y_β be its positive principal solutions at α and β for which $p(x)(y'_\alpha y_\beta - y_\alpha y'_\beta)(x) = 1$. If

$$\liminf_{x_1\downarrow\alpha\ x_2\uparrow\beta} \int_{x_1}^{x_2} \left[4q(x)y_\alpha(x)y_\beta(x) - (p(x)y_\alpha(x)y_\beta(x))^{-1}\right] dx \geq 0$$

and

$$4q(x)y_\alpha(x)y_\beta(x) - (p(x)y_\alpha(x)y_\beta(x))^{-1} \not\equiv 0 \text{ on } (\alpha, \beta)$$

then (1.3) is conjugate on $I = (\alpha, \beta)$.

Theorem 2.19 (theorem 3 of [16]). Let one of the following assumptions be satisfied.

(i) $\int_\alpha^\beta p^{-1}(x)dx = A < \infty$, $\gamma \in (\alpha, \beta)$ is such that $\int_\gamma^\beta p^{-1}(x)dx = \int_\alpha^\gamma p^{-1}(x)dx = A/2$,

$$\int_\alpha^\beta \cos^2\left(\pi \int_\gamma^x p^{-1}(s)ds/A\right)\left[q(x) - \pi^2/(A^2 p(x))\right] dx \geq 0$$

and $q(x) \not\equiv \pi^2/(A^2 p(x))$.

(ii) $\int_\alpha^\gamma p^{-1}(x)dx < \infty$, $\int_\gamma^\beta p^{-1}(x)dx = \infty$, $\gamma \in (\alpha, \beta)$,

$$\int_\alpha^\beta \left[\frac{\left(\int_\alpha^x p^{-1}\right)^2}{\left(1 + \left(\int_\alpha^x p^{-1}\right)^2\right)}\right]\left[q(x) - \frac{3p^{-1}(x)}{\left(1 + \left(\int_\alpha^x p^{-1}\right)^2\right)^2}\right] dx \geq 0$$

and $q(x) \not\equiv 3p^{-1}(x)/[1 + (\int_\alpha^x p^{-1})^2]^2$.

Then (1.3) is conjugate on $I = (\alpha, \beta)$.

Remark 2.20 [16]. We get a result similar to (ii) of the above theorem if $\int_\gamma^\beta p^{-1}(x)dx < \infty$, $\int_\alpha^\gamma p^{-1}(x)dx = \infty$.

In chapter 3 we obtain generalizations of some of the above results for the systems (1.2) and (1.1).

Now we recall some definitions which are needed in chapter 4 wherein conjugacy criteria for the system

$$\begin{aligned} u' &= A(x)u + B(x)v \\ v' &= C(x)u - A^*(x)v \end{aligned} \qquad (1.1)_v$$

and

$$[P(x)u' + R(x)u]' - [R^*(x)u' - Q(x)u] = 0 \qquad (1.5)_v$$

are discussed.

Definition 2.21 [39]. The system $(1.1)_v$ is said to be 'identically normal' on $I = (\alpha, \beta)$ if $(0, v(x))$ is a solution of $(1.1)_v$ on I implies $v(x) \equiv 0$ on I.

Remark 2.22. All systems of the form $(1.1)_v$ with $B(x) > 0$ on I are identically normal on I.

Definition 2.23 [39]. The system $(1.1)_v$ $((1.5)_v)$ is 'conjugate' on I if there exist a pair of points $x_1, x_2 \in I$, $x_1 < x_2$ and a solution $(u(x), v(x))$ $(u(x))$ of $(1.1)_v$ $((1.5)_v)$ such that $u(x_1) = 0 = u(x_2)$ and $u(x) \not\equiv 0$ on I. Otherwise $(1.1)_v$ $((1.5)_v)$ is said to be 'disconjugate' on I.

Note 2.24. The pair of points x_1, x_2 in the above definition are said to be 'mutually conjugate' with respect to the system $(1.1)_v$ and the solution $(u(x), v(x))$. x_1 is said to be 'left conjugate' to x_2 and x_2 is said to be 'right conjugate' to x_1 with respect to the system $(1.1)_v$ and the solution $(u(x), v(x))$.

Remark 2.25. Disconjugacy and identical normality of $(1.1)_v$ on I imply the following result: $(u(x), v(x))$ is a solution of $(1.1)_v$ and $u(x_1) = 0 = u(x_2)$, $x_1, x_2 \in I \Longrightarrow u(x) \equiv 0 \equiv v(x)$ on I.

In the following example we consider the vector system, analogous to the scalar system (2.1).

Example 2.26. Consider the 4-dimensional vector system

$$\begin{aligned} u' &= v \\ v' &= -u \end{aligned} \qquad (2.1)_v$$

$u, v \in \mathbb{R}^2$. This system is conjugate on $[0, \pi]$ since

$$(u(x), v(x)) = \left(\begin{pmatrix} \sin x \\ 0 \end{pmatrix}, \begin{pmatrix} \cos x \\ 0 \end{pmatrix} \right)$$

is a vector solution with $u(0) = u(\pi) = 0 (\in \mathbb{R}^2)$ and $u(x) \not\equiv 0$ on $[0, \pi)$. Hence the system is conjugate on $[0, \pi]$.

The system is disconjugate on $[0, \pi)$ (since

$$(u(x), v(x)) = \left(\begin{pmatrix} c_1 \sin x + c_2 \cos x \\ c_3 \sin x + c_4 \cos x \end{pmatrix}, \begin{pmatrix} c_1 \cos x - c_2 \sin x \\ c_3 \cos x - c_4 \sin x \end{pmatrix} \right)$$

where c_1, c_2, c_3 and c_4 are arbitrary constants is the general vector solution of the system $(2.1)_v$ and $u(x_2) = 0 = u(x_1)$, $0 \le x_1 < x_2 < \pi$ implies

$$\begin{pmatrix} c_1 \sin x_1 + c_2 \cos x_1 \\ c_3 \sin x_1 + c_4 \cos x_1 \end{pmatrix} = 0 = \begin{pmatrix} c_1 \sin x_2 + c_2 \cos x_2 \\ c_3 \sin x_2 + c_4 \cos x_2 \end{pmatrix}$$

and hence $c_1 = 0 = c_2 = c_3 = c_4$.

The following example in which $A(x) \not\equiv 0$ also illustrates definition (2.23) .

Example 2.27. Consider the 4-dimensional system $(1.1)_v$ with

$$A(x) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \; B(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; C(x) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This system is conjugate on $[0, \pi]$ since

$$(u(x), v(x)) = \left(\begin{pmatrix} 0 \\ \sin x \end{pmatrix}, \begin{pmatrix} 0 \\ \cos x \end{pmatrix} \right)$$

is a vector solution with $u(0) = 0 = u(\pi)$ and $u(x) \not\equiv 0$.

The system is disconjugate on $[0,\pi)$. This follows from the fact that

$$(u(x),v(x)) = \left(\left(\begin{array}{c} c_1(1-x)+c_2 \\ c_3\sin x + c_4\cos x \end{array}\right), \left(\begin{array}{c} -c_1x+c_2 \\ c_3\cos x - c_4\sin x \end{array}\right)\right)$$

where c_1, c_2, c_3 and c_4 are arbitrary constants is the general solution of this system and $u(x_2) = 0 = u(x_1)$, $0 \le x_1 < x_2 < \pi$ implies

$$\left(\begin{array}{c} c_1(1-x_1)+c_2 \\ c_3\sin x_1 + c_4\cos x_1 \end{array}\right) = 0 = \left(\begin{array}{c} c_1(1-x_2)+c_2 \\ c_3\sin x_2 + c_4\cos x_2 \end{array}\right)$$

which in turn implies $c_1 = 0 = c_2 = c_3 = c_4$.

Remark 2.28. It is obvious from the above reasoning that the systems in the above examples are conjugate on $[\alpha, \alpha+\pi]$ and disconjugate on $[\alpha, \alpha+\pi)$ for arbitrary α.

We now recall some definitions and properties concerning the matrix system

$$\begin{aligned} U' &= A(x)U + B(x)V \\ V' &= C(x)U - A^*(x)V \end{aligned} \qquad (1.1)_M$$

which are needed for further discussion of conjugacy criteria of the system $(1.1)_v$.

Proposition 2.29. If $(U(x), V(x))$, $(U_1(x), V_1(x))$ are two $n \times k$ matrix pairs satisfying $(1.1)_M$ on I then

(i) $(U^*V_1 - V^*U_1)(x) \equiv C$, a constant $k \times k$ matrix on I, C depending only on (U, V), (U_1, V_1).

(ii) In particular $(U^*V - V^*U)(x) = D$, a constant $k \times k$ matrix on I, D depending only on (U, V).

Note 2.30. $(U(x), V(x))$ is a solution of $(1.1)_M$ on I if and only if $(u(x), v(x))$

$= (U(x)c, V(x)c)$ is a solution of $(1.1)_v$ for every constant $n \times 1$ vector c.

Definition 2.31. If $(U(x), V(x))$ is an $n \times n$ matrix pair satisfying $(1.1)_M$ on I such that

$$(U^\star V - V^\star U)(x) \equiv \text{zero matrix}$$

then $(U(x), V(x))$ is called 'prepared solution' of $(1.1)_M$ on I. In particular if $U(x_0) = E_n$ and $V(x_0) = 0_n$ for some $x_0 \in I$, then $(U(x), V(x))$ is a prepared solution of $(1.1)_M$ on I.

Note 2.32. This terminology is due to Hartman [23]. Such a solution is called 'isotropic' according to the terminology of Coppel [12] and 'self conjoined' according to Reid [39].

Definition 2.33 [19]. A solution $(U(x), V(x))$ of $(1.1)_M$ is said to be 'nontrivial' if $\det U(x) \neq 0$ for atleast one $x \in I$.

Example 2.34. Consider the 4-dimensional matrix Hamiltonian system $(1.1)_M$ with

$$A(x) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; B(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; C(x) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \tag{2.5}$$

on $(-\infty, \infty)$. The matrix pair

$$(U(x), V(x)) = \left(\begin{bmatrix} 1 & \sin x \\ 0 & -\cos x \end{bmatrix}, \begin{bmatrix} 0 & \cos x \\ -1 & 0 \end{bmatrix} \right)$$

is a nontrivial solution of the system (2.5) (since $\det U(x) = -\cos x \not\equiv 0$ on $(-\infty, \infty)$ and it is prepared since

$$(U^\star V)(0) = (V^\star U)(0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

so that

$$(U^\star V - V^\star U)(0) = \text{zero matrix}.$$

The following theorem is the matrix analogue of the result (ii) of theorem 2.1.

Theorem 2.35 (theorem 2.2 p 308 [39]). Suppose that $(U_1(x), V_1(x))$ is a prepared solution of $(1.1)_M$ and $U_1(x)$ is nonsingular on a subinterval I_0 of I. If $\gamma \in I_0$ then $(U(x), V(x))$ is a solution of $(1.1)_M$ on I_0 if and only if on this interval

$$U(x) = U_1(x)H(x), \quad V(x) = V_1(x)H(x) + U_1^{\star^{-1}}(x)K_1$$

where

$$H(x) = K_0 + \left[\int_\gamma^x U_1^{-1}(s)B(s)U_1^{\star^{-1}}(s)ds\right] K_1$$

and $K_0,\ K_1$ are constant matrices with $K_1 \equiv V_1^\star(x)U(x) - U_1^\star(x)V(x)$.

The following basic result from [39] relating the disconjugacy of $(1.1)_v$ to the positive definiteness of an associated functional is crucial to our discussion in chapter 4. We state this result using the following definitions given in [39].

Definition 2.36 [39]. For a given compact subinterval $[\alpha, \beta]$ we shall denote by $D[\alpha, \beta]$ the class of n-dimensional vector-valued functions η which are absolutely continuous and for which there exists a $\zeta \in \mathcal{L}^\infty[\alpha, \beta]$ such that

$$\eta'(x) - A(x)\eta(x) - B(x)\zeta(x) = 0 \text{ a.e. on } [\alpha, \beta].$$

The subclass of $D[\alpha, \beta]$ on which $\eta(\alpha) = 0 = \eta(\beta)$ is denoted by $D_0[\alpha, \beta]$.

Definition 2.37 [39]. For $\eta \in D[\alpha, \beta]$ we shall denote by $J[\eta; \alpha, \beta]$ the functional

$$J[\eta; \alpha, \beta] = \int_\alpha^\beta \{\zeta^\star B\zeta + \eta^\star C\eta\}(x)dx \tag{2.6}$$

corresponding to the system $(1.1)_v$.

The functional corresponding to the equation $(1.5)_v$ is given by

$$J[\eta; \alpha, \beta] = \int_\alpha^\beta \{[P\eta' + R\eta].\eta' + [R^\star\eta' - Q\eta].\eta\}(x)dx \tag{2.7}$$

Theorem 2.38 (theorem 5.1, p 337 [39]). Assume the hypothesis H_v on $[\alpha,\beta]$. Then each of the following conditions is necessary and sufficient for $(1.1)_v$ to be disconjugate on $[\alpha,\beta]$.

(i) $J[\eta;\alpha,\beta]$ (defined by equation (2.6)) is positive definite on $D_0[\alpha,\beta]$.

(ii) There is no point on $(\alpha,\beta]$ conjugate to $x=\alpha$.

(iii) There is no point on $[\alpha,\beta)$ conjugate to $x=\beta$.

(iv) There exists a conjoined basis $(U(x),V(x))$ for $(1.1)_v$ with $U(x)$ nonsingular on $[\alpha,\beta]$.

(v) There exists an $n\times n$ Hermitian matrix function $W(x)$, $x\in[\alpha,\beta]$, which is a solution of the Riccati matrix differential equation

$$W'(x)+W(x)A(x)+A^*(x)W(x)+W(x)B(x)W(x)-C(x)=0,\ x\in[\alpha,\beta].$$

It may be noted that we will be using in our later discussion the equivalence of (i), (ii) (iii) and (v) only.

The following theorems motivate the definition of a principal solution and give the existence of principal solutions for a disconjugate system $(1.1)_v$.

Theorem 2.39 (theorem 10.4 p 392, [23]). Assume the hypothesis H_v on $J=[\alpha_1,\beta_1)$ $(\beta_1\le\infty)$ and that $(1.1)_v$ is disconjugate on J. Let $(U(x),V(x))$ be a prepared solution of $(1.1)_M$ such that $\det U(x)\neq 0$ on $\gamma\le x<\beta_1$ for some $\gamma\in J$. Then

$$S(x,\gamma/U)=\left(\int_\gamma^x U^{-1}(s)B(s)U^{*^{-1}}(s)ds\right)$$

is nonsingular for $\gamma<x<\beta_1$ and

$$\lim_{x\to\beta_1} S^{-1}(x,\gamma/U)=M_1 \tag{2.8}$$

exists, where M_1 depends on γ and the matrix function $U(x)$. In particular if $M_1=0$ the solution $(U(x),V(x))$ is called a 'principal solution' of $(1.1)_M$

at β_1.

Theorem 2.40 (theorem 10.5, p 393 [23]). Assume the hypothesis H_v on $J = [\alpha_1, \beta_1)$ $(\beta_1 \leq \infty)$ and that $(1.1)_v$ is disconjugate on J. Then

(i) the matrix system $(1.1)_M$ possesses a principal solution $(U_0(x), V_0(x))$.

(ii) another solution $(U(x), V(x))$ is a principal solution if and only if $(U(x), V(x)) = (U_0(x)K_1, V_0(x)K_1)$, where K_1 is a constant nonsingular matrix.

(iii) let $(U(x), V(x))$ be a solution of $(1.1)_M$. Then the constant matrix $K_0 = U^\star V_0 - V^\star U_0$ is nonsingular if and only if $\det U(x) \neq 0$ for x near β_1 and

$$U^{-1}(x)U_0(x) \longrightarrow 0 \text{ as } x \longrightarrow \beta_1$$

in which case M_1 in (2.8) is nonsingular.

Note 2.41. The principal solution $(U_{\alpha_1}, V_{\alpha_1})$ of $(1.1)_M$ at α_1 is defined analogously.

Example 2.42. Consider the system $(1.1)_v$ with

$$A(x) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \ B(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ C(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This system is disconjugate on $(-\infty, \infty)$ (since

$$(u(x), v(x)) = \left(\begin{pmatrix} c_1 e^{-x} + \dfrac{c_2 e^x}{2} \\ c_3 + c_4 x \end{pmatrix}, \begin{pmatrix} c_2 e^x \\ c_4 \end{pmatrix} \right)$$

where c_1, c_2, c_3 and c_4 are arbitrary constants is the general vector solution of the system and $u(x_2) = 0 = u(x_1)$, $-\infty \leq x_1 < x_2 < \infty$ implies

$$\begin{pmatrix} c_1 e^{-x_1} + \dfrac{c_2 e^{x_1}}{2} \\ c_3 + c_4 x_1 \end{pmatrix} = 0 = \begin{pmatrix} c_1 e^{-x_2} + \dfrac{c_2 e^{x_2}}{2} \\ c_3 + c_4 x_2 \end{pmatrix}$$

and hence $c_1 = 0 = c_2 = c_3 = c_4$. Moreover

$$(U_{-\infty}(x), V_{-\infty}(x)) = \left(\begin{bmatrix} e^x/2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} e^x & 0 \\ 0 & 0 \end{bmatrix} \right)$$

and

$$(U_\infty(x), V_\infty(x)) = \left(\begin{bmatrix} e^{-x} & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

are its principal solutions at $-\infty$ and ∞ respectively.

The following definition due to Došlý [14] gives a further classification of an identically normal and disconjugate system.

Definition 2.43 [14]. Assume that the system $(1.1)_v$ is identically normal and disconjugate and (U_α, V_α), (U_β, V_β) are principal solutions of $(1.1)_v$ at α and β respectively. The system $(1.1)_v$ is said to be k-general $(0 \leq k \leq n)$ on I if the rank of the $2n \times 2n$ solution matrix

$$\begin{pmatrix} U_\alpha(x) & U_\beta(x) \\ V_\alpha(x) & V_\beta(x) \end{pmatrix}$$

which is always independent of x (from the theory of linear differential systems) is equal to $n + k$.

Remark 2.44 (p.90 [14]). The system $(1.1)_v$ is k-general on I if and only if the rank of the constant matrix $U_\beta^* V_\alpha - V_\beta^* U_\alpha$ is equal to k.

Example 2.45. Consider the 4-dimensional system $(2.1)_v$ given in example 2.26 which is disconjugate on $(0, \pi)$. Its pricipal solutions at 0 and π are respectively

$$(U_0(x), V_0(x)) = \left(\begin{bmatrix} \sin x & 0 \\ 0 & \sin x \end{bmatrix}, \begin{bmatrix} \cos x & 0 \\ 0 & \cos x \end{bmatrix} \right) = (U_\pi(x), V_\pi(x)).$$

Hence the system is 0-general.

Example 2.46. Consider the system as in example 2.27. and

$$(U_{-\infty}(x), V_{-\infty}(x)) = \left(\begin{bmatrix} \frac{e^x}{2} & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} e^x & 0 \\ 0 & 0 \end{bmatrix} \right)$$

and

$$(U_\infty(x), V_\infty(x)) = \left(\begin{bmatrix} e^{-x} & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

are its principal solutions at $-\infty$ and ∞ respectively. Hence the system is 1-general.

Example 2.47. Consider the system $(1.1)_v$ with

$$A(x) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\ B(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ C(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This system is disconjugate on $(-\infty, \infty)$ (since

$$(u(x), v(x)) = \left(\begin{pmatrix} c_1 e^{-x} + \dfrac{c_2 e^x}{2} \\ c_3 e^{-x} + \dfrac{c_4 e^x}{2} \end{pmatrix}, \begin{pmatrix} c_2 e^x \\ c_4 e^x \end{pmatrix} \right)$$

where c_1, c_2, c_3 and c_4 are arbitrary constants is the general vector solution of the system and $u(x_2) = 0 = u(x_1)$, $0 \le x_1 < x_2 < \pi$ implies

$$\begin{pmatrix} c_1 e^{-x_1} + \dfrac{c_2 e^{x_1}}{2} \\ c_3 e^{-x_1} + \dfrac{c_4 e^{x_1}}{2} \end{pmatrix} = 0 = \begin{pmatrix} c_1 e^{-x_2} + \dfrac{c_2 e^{x_2}}{2} \\ c_3 e^{-x_2} + \dfrac{c_4 e^{x_2}}{2} \end{pmatrix}$$

and hence $c_1 = 0 = c_2 = c_3 = c_4$. Moreover

$$(U_{-\infty}(x), V_{-\infty}(x)) = \left(\begin{bmatrix} \dfrac{e^x}{2} & 0 \\ 0 & \dfrac{e^x}{2} \end{bmatrix}, \begin{bmatrix} e^x & 0 \\ 0 & e^x \end{bmatrix} \right)$$

and

$$(U_{\infty}(x), V_{\infty}(x)) = \left(\begin{bmatrix} e^{-x} & 0 \\ 0 & e^{-x} \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

are its principal solutions at $-\infty$ and ∞ respectively. Hence the system is 2-general.

The following notation, definitions and results from Etgen and Lewis [21] are useful for discussing further conjugacy criteria of the system $(1.5)_v$.

Let $H_0'[\alpha, \beta]$ be the space of real valued absolutely continuous functions vanishing at $x = \alpha, \beta$ and having $L^2[\alpha, \beta]$ first derivatives, $\mathcal{M}$ be the linear space of $n \times n$ matrices with real entries and $\mathcal{S} \subset \mathcal{M}$ be the subspace of $n \times n$ symmetric matrices.

Definition 2.48 ([21]). A matrix function $U : [\alpha, \beta] \to \mathcal{M}$ is 'L-admissible' if each of U and $(PU' + RU)$ is continuously differentiable on $[\alpha, \beta]$ and

$$(U^*[PU' + RU] - [PU' + RU]^*U)(x) \equiv 0.$$

Example 2.49. Let

$$P(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } R(x) = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}.$$

Then $U(x) = \begin{bmatrix} e^{-x} & 0 \\ -e^{-x} & 1 \end{bmatrix}$ is L-admissible since

$$U^*[PU' + RU](x) = \begin{bmatrix} -e^{-2x} & 0 \\ 0 & 0 \end{bmatrix} = [PU' + RU]^*U(x).$$

Definition 2.50 [21]. A linear functional $g : \mathcal{M} \to \mathbb{R}$ is positive if $g(A) > 0$ whenever $A \in \mathcal{S}$ and $A > 0$.

The following identities and results concerning the conjugacy of the equation

$$(P(x)u')' + Q(x)u = 0 \qquad (1.3)_v$$

are given in [21].

Theorem 2.51 (theorem 4.1 of [21]) ("Picone type" identity). Let g be a positive linear functional and let $f : [\alpha, \beta] \rightsquigarrow \mathbb{R}$ be piecewise continuously differentiable. If $U : [\alpha, \beta] \to \mathcal{M}$ is an L-admissible function which is nonsingular on an interval $J \subseteq \mathbb{R}$, then

$$\begin{aligned} & g\left\{(f'E_n - fU'U^{-1})^*P(f'E_n - fU'U^{-1})\right\} + \left\{f^2 g[PU'U^{-1}]\right\}' \\ = \ & f'^2 g[P] - f^2 g[Q] + f^2 g\{L[U]U^{-1}\} \text{ on } J. \end{aligned}$$

Theorem 2.52 (theorem 4.2 of [21]). If for each $\alpha \in \mathbb{R}$ there is a $\beta > \alpha$, a positive linear functional $g \neq 0$ and a piecewise continuously differentiable function $f : [\alpha, \beta] \to \mathbb{R}$ such that $f(\alpha) = f(\beta) = 0$, $f \neq 0$ on $[\alpha, \beta]$ and

$$\int_\alpha^\beta \left\{ f'^2(x) g[P(x)] - f^2(x) g[Q(x)] \right\} dx \leq 0,$$

then equation $(1.3)_v$ is conjugate on $[\alpha, \beta]$.

Theorem 2.53 (theorem 4.3 of [21]). If there is a positive linear functional g such that the scalar equation

$$(g[P(x)]y')' + g[Q(x)]y = 0$$

is conjugate, then equation $(1.3)_v$ is conjugate on $[\alpha, \beta]$.

In chapter 4 we obtain conjugacy criteria for the system $(1.5)_v$ which are motivated by the above results.

We now recall some definitions and properties concerning nonlinear functionals given by Hartman in [25] using the same notation as in [25].

Definition 2.54. A nonlinear functional $q : \mathcal{S} \to \mathbb{R}$ is called "superadditive" or "(positively) superhomogeneous" according as

$$q(A + B) \geq q(A) + q(B) \tag{2.9}$$

or

$$q(\lambda A) \geq \lambda q(A) \qquad \text{for } \lambda \geq 0 \text{ and } A, B \in \mathcal{S} \tag{2.10}$$

holds. A functional q having both the above properties is said to be "concave".

Correspondingly, a nonlinear functional $p : \mathcal{S} \to \mathbb{R}$ is called "subadditive" or "(positively) subhomogeneous" according as

$$p(A + B) \leq p(A) + p(B) \tag{2.11}$$

or

$$p(\lambda A) \leq \lambda p(A) \qquad \text{for } \lambda \geq 0 \text{ and } A, B \in \mathcal{S} \tag{2.12}$$

holds. A functional p having both the above properties is said to be "convex".

The following remark from [25] is needed for further discussion in chapter 4.

Remark 2.55. (i) A convex (concave) functional : $\mathcal{S} \to \mathbb{R}$ is continuous.
(ii) (Jensen's inequality) If $q : \mathcal{S} \to \mathbb{R}$ is a concave functional and $f : [\alpha, \beta] \to \mathbb{R}$ is an arbitrary continuous function then it follows that

$$q\left[\int_\alpha^\beta f^2 Q dx\right] \geq \int_\alpha^\beta f^2 q[Q] dx \tag{2.13}$$

where $Q \in \mathcal{M}$.

The following results concerning conjugacy or disconjugacy for the system $(1.3)_v$ are given in [25].

Lemma 2.56 (lemma 2.2 of [25]). Equation $(1.3)_v$ is disconjugate on $[\alpha, \beta]$ if there exists a real-valued function $f \in H_0^1[\alpha, \beta]$, $f \not\equiv 0$, such that the symmetric (constant) matrix

$$Z \equiv \int_\alpha^\beta (f'^2 P - f^2 Q)(x) dx$$

is not positive definite.

Theorem 2.57 (theorem 1.1 of [25]). Let $p, q : \mathcal{S} \to \mathbb{R}$ satisfy (2.9)-(2.12) and

$$q(A) \leq p(A) \ \text{ for } \ 0 \leq A \in \mathcal{S}.$$

If the scalar differential equation

$$[p(P(x))u']' + q(Q(x))u = 0$$

is conjugate on $[\alpha, \beta]$, then $(1.3)_v$ is conjugate on $[\alpha, \beta]$.

Based on the the above result we obtain in chapter 4 a conjugacy criterion for the equation $(1.5)_v$.

We also obtain a different type of conjugacy criterion for the system $(1.1)_v$ based on the following results due to Došlý [14]. For the statements of these results we use the same notation as in [14]. We also include here for the sake of convenience of reference the Courant-Fischer min-max principle which can be found in [6].

Denote the eigen values of an $n \times n$ Hermitian matrix P (in the increasing order) by $\lambda_{min}[P] = \lambda_n[P] \leq ... \leq \lambda_1[P] = \lambda_{max}[P]$.

Theorem 2.58 (p,115, [6]). The eigen values λ_i, $i = 1, 2, ..., n$ of an Hermitian matrix P may be defined as follows:

$$\begin{aligned} \lambda_1 &= \max_x (x, Px)/(x, x) \\ \lambda_2 &= \min_{(y,y)=1} \max_{(x,y)=0} (x, Px)/(x, x) \\ &\vdots \\ \lambda_k &= \min_{(y^i,y^i)=1} \max_{(x,y^i)=0} (x, Px)/(x, x) \; i = 1, 2...k-1 \\ &\vdots \\ \text{equivalently, } \lambda_{n-1} &= \max_{(y,y)=1} \min_{(x,y)=0} (x, Px)/(x, x) \\ \lambda_n &= \min_x (x, Px)/(x, x). \end{aligned}$$

The following lemma and the conjugacy criterion for the system

$$\begin{aligned} u' &= B(x)v \\ v' &= -C(x)u \end{aligned} \qquad (1.6)_v$$

are given in [14].

Lemma 2.59 (lemma 1 of [14]). Let the differential system

$$u' = B(x)v, \quad v' = 0 \tag{2.14}$$

be k-general on $I = (\alpha, \beta)$, for some fixed $k \in \{0, ..., n\}$ and let
$\lim_{x \to \beta-} \lambda_1 \left(\int_\gamma^x B(s)ds \right) = \infty$ for some (and hence for every) $\gamma \in I$. Then there exists an $(n-k)$-dimensional linear space $V_{n-k} \subset \mathbb{R}^n$ such that
$\lim_{x \to \alpha+} \left(\int_x^\gamma u^\star B(s)uds \right) = \infty$ for every $0 \neq u \in V_{n-k}$.

Theorem 2.60 (theorem 1 of [14]). Let the system (2.14) be k-general on $I = (\alpha, \beta)$, $0 \leq k \leq n-1$, $\lim_{x \to \beta-} \lambda_1 \left(\int_\gamma^x B(s)ds \right) = \infty$ and let there exist $\alpha_1, \beta_1 \in I$ such that the matrix $C(x)$ is nonnegative definite for $x \in (\alpha, \alpha_1) \cup (\beta, \beta_1)$. If there exists a $(k+1)$-dimensional space $V_{k+1} \subset \mathbb{R}^n$ such that

$$\liminf_{\substack{t \to \alpha+ \\ z \to \beta-}} \int_t^z w^\star(x)C(x)w \, dx > 0$$

$\forall\ (0 \neq)\ w \in V_{k+1}$, then system $(1.6)_v$ is conjugate on (α, β).

In chapter 5 we obtain oscillation criteria for the system $(1.1)_M$. For this discussion we need the following definitions and known results from the earlier literature.

Definition 2.61 [19]. The system $(1.1)_M$ is said to be 'oscillatory' on $[\alpha, \infty)$ if one nontrivial prepared solution $(U(x), V(x))$ of $(1.1)_M$ has the property that $\det U(x)$ vanishes on $[T, \infty)$ for every $T > \alpha$. Otherwise it is said to be 'nonoscillatory' on $[\alpha, \infty)$.

In the following example we consider the matrix analogue of the system (2.1).

Example 2.62. Consider the 4-dimensional matrix system

$$U' = V$$

$$V' = -U \tag{2.1$_M$}$$

where U, V are 2×2 matrices. Note that

$$(U(x), V(x)) = \left(\begin{bmatrix} \sin x & \cos x \\ \cos x & \sin x \end{bmatrix}, \begin{bmatrix} \cos x & -\sin x \\ -\sin x & \cos x \end{bmatrix} \right)$$

is a nontrivial prepared solution of the system $(2.1)_M$ since
$\det U(x) = \sin^2 x - \cos^2 x \not\equiv 0$

$$(U^\star V)(0) = (V^\star U)(0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This system is oscillatory on $[0, \infty)$ since $\det U(x)$ has infinitely many zeros on $[0, \infty)$.

The following example gives an oscillatory system with $A(x) \not\equiv 0$.

Example 2.63. Consider the system (2.5) as in example 2.34. This system is oscillatory since

$$(U(x), V(x)) = \left(\begin{bmatrix} 1 & \sin x \\ 0 & -\cos x \end{bmatrix}, \begin{bmatrix} 0 & \cos x \\ -1 & 0 \end{bmatrix} \right)$$

is a nontrivial prepared solution with $\det U(x) = -\cos x$ having infinitely many zeros on $(-\infty, \infty)$.

The following theorem (known as Morse's separation theorem) given by Morse in [35] for the matrix system $(1.3)_M$ is analogous to the Sturm's separation theorem for the scalar case and is stated here in the form given in [45].

Theorem 2.64 [45]. The number of zeros of the determinant of any nontrivial prepared matrix solution of $(1.3)_M$ on a given interval (open or closed) differs from that of any other nontrivial prepared matrix solution by at most n.

Note 2.65. It follows from Morse's separation theorem that if the system $(1.3)_M$ is oscillatory on $[\alpha, \infty)$ then every nontrivial prepared solution $(\tilde{U}(x), \tilde{V}(x))$ has the property that $\det \tilde{U}(x)$ vanishes on $[T, \infty)$ for every $T > \alpha$.

Note 2.66. In the definition 2.61 the hypothesis of the solution $(U(x), V(x))$ of $(1.3)_M$ being prepared is needed in order to obtain an analog of the classical theory of oscillation of the scalar equation (1.3).

The following example given by Noussair and Swanson [37] illustrates this point.

Example 2.67 [37]. Consider the 2×2 matrix differential equation

$$U'' + U = 0 \quad \text{on} \quad [0, \infty) \tag{2.15}$$

as an analog of the oscillatory scalar equation $u'' + u = 0$, all of whose solutions are of the form $u(x) = a \sin(x + b)$ (a, b constants). Note that the equation (2.15) can be put in the form $(1.6)_M$ and the matrix-valued function

$$U(x) = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$$

is a nontrivial solution of the equation (2.15) which is not prepared since $\det U(x) \equiv 1$ and

$$(U^{\star}V - V^{\star}U)(0) = (U^{\star}U' - U'^{\star}U)(0) = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \neq \text{zero matrix.}$$

Clearly $\det U(x) \equiv 1$ has no zeros on $[0, \infty)$. However

$$U(x) = \begin{bmatrix} \cos x & 0 \\ 0 & \sin x \end{bmatrix}$$

is a nontrivial prepared solution of (2.15) since $\det U(x) \equiv \sin x \cos x$

$$\text{and} \quad (U^{\star}V - V^{\star}U)(0) = (U^{\star}U' - U'^{\star}U)(0) = \text{zero matrix}$$

with $\det U(x) = \cos x \sin x$ having infinitely many zeros on $(-\infty, \infty)$.

Thus the equation (2.15) becomes oscillatory by virtue of definition (2.61) whereas the omission of the assumption of the solution being 'prepared' makes the system nonoscillatory.

Now we state a separation theorem for the more general system $(1.1)_M$ given by Reid in [40]. For stating this separation theorem we need the following notations.

For $p = 1, 2$ we have the respective differntial systems $(1.1)^1_M$ and $(1.1)^2_M$ where A^p, B^p, C^p satisfy hypotisis H_v and the difference functional

$$J_{1,2}[\eta : \alpha, \beta] = J_2[\eta : \alpha, \beta] - J_1[\eta : \alpha, \beta]$$

(where $J[\eta; \alpha, \beta]$ is as in equation (2.6)) is well-defined for $\eta \in D[\alpha, \beta]$ where $D[\alpha, \beta]$ is the common value of $D^1[\alpha, \beta]$ and $D^2[\alpha, \beta]$.

Theorem 2.68 (theorem 8.1, p 303, [40]). Suppose that for $p = 1, 2$ the matrix functions A^p, B^p, C^p satisfy hypotisis H_v and each of the systems $(1.1)^p_M$ is identically normal. Moreover, $D^1[\alpha, \beta] = D^2[\alpha, \beta]$ and for arbitrary compact interval $[\alpha, \beta]$ the difference functional $J_{1,2}[\eta, \alpha, \beta]$ is non-negative on $D^1_0[\alpha, \beta] = D^2_0[\alpha, \beta]$. If $x^+_{\nu p}(\gamma)$ and $x^-_{\nu p}(\gamma)$, $(p = 1, 2, \ \nu \geq 1)$,denote the sequences of right and left hand conjugate points to $x = \gamma$ relative to the respective system $(1.1)^p_M$, then whenever the conjugate point $x^+_{\nu 1}(\gamma)$, $\{x^-_{\nu 1}(\gamma)\}$ exists, the conjugate point $x^+_{\nu 2}(\gamma)$, $\{x^-_{\nu 2}(\gamma)\}$, also exists and

$$x^+_{\nu 2}(\gamma) \leq x^+_{\nu 1}(\gamma), \quad \{x^-_{\nu 2}(\gamma) \leq x^-_{\nu 1}(\gamma)\}. \tag{2.16}$$

Moreover, if $J_{1,2}[\eta, \alpha, \beta]$ is positive definite on $D^1_0[\alpha, \beta] = D^2_0[\alpha, \beta]$ for arbitrary compact intervals $[\alpha, \beta]$, then strict inequalities hold in (2.16).

The following corollory to the theorem 2.68, not explicitly stated in [40] yields a separation theorem for the system $(1.1)_M$.

Corollary 2.69. Let the system $(1.1)_M$ be identically normal and $(U_1(x), V_1(x)), (U_2(x), V_2(x))$ be two solutions of $(1.1)_M$. If $x^+_{\nu p}(\gamma)$ and $x^-_{\nu p}(\gamma)$, $(p = 1,2,\ \nu \geq 1)$, denote the sequences of right and left hand conjugate points to $x = \gamma$ relative to the respective solutions $(1.1)^p_M$, then whenever the conjugate point $x^+_{\nu 1}(\gamma)$, $\{x^-_{\nu 1}(\gamma)\}$ exists, the conjugate point $x^+_{\nu 2}(\gamma)$, $\{x^-_{\nu 2}(\gamma)\}$, also exists and

$$x^+_{\nu 2}(\gamma) \leq x^+_{\nu 1}(\gamma), \quad \{x^-_{\nu 2}(\gamma) \leq x^-_{\nu 1}(\gamma)\}.$$

We use the above result in proving that the system given in the following example is disconjugate.

Example 2.70. Consider the 4-dimensional matrix Hamiltonian system $(1.1)_M$ with

$$A(x) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ B(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ C(x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{2.17}$$

on the interval $(-\infty, \infty)$. It is nonoscillatory on $(-\infty, \infty)$ since there exists a solution $(U(x), V(x))$ where

$$(U(x), V(x)) = \left(\begin{bmatrix} e^{-x} & 0 \\ -e^{-x} & 1 \end{bmatrix}, \begin{bmatrix} -e^{-x} & 0 \\ 0 & 0 \end{bmatrix} \right)$$

such that it is prepared

$$\left(\text{since } (U^\star V)(0) = (V^\star U)(0) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

and $\det U(x) = e^{-x} \neq 0$ on $(-\infty, \infty)$.

Remark 2.71. By virtue of corollory 2.69 and definition 2.61 it follows that $(1.1)_M$ is nonoscillatory if there exists one nontrivial prepared solution $(U(x), V(x))$ such that $\det U(x)$ doesnot vanish for all $x > T$ for some $T > \alpha$.

Now we state for the sake of convenience of reference Weyl's inequality and a result which specializes to the case of the space $\mathcal{M}$ the more general result on positive linear functionals stated in [41]. These statements will be followed by the oscillation criteria given by Etgen and Pawlowski [22], Erbe, Qingkai Kong and Shigui Ruan [19] and Fanwei Meng, Jizhong Wang and Zhaowen Zheng [34] for the systems $(1.3)_M$ and $(1.3)_{1_M}$ with minor changes of notation.

Lemma 2.72 [28]. Let $P,\ Q \in C^{n\times n}$ be Hermitian and let the eigen values $\lambda_i[P],\ \lambda_i[Q]$ and $\lambda_i[P+Q]$ be arranged in increasing order as stated above. For each $i = 1, 2, ..., n$ we have,

$$\lambda_i[P] + \lambda_n[Q] \leq \lambda_i[P+Q] \leq \lambda_i[P] + \lambda_1[Q].$$

Lemma 2.73 [41]. If g is a positive linear functional on $\mathcal{M}$ then for all $A,\ B \in \mathcal{M}$ $|g(A^*B)|^2 \leq g(A^*A)g(B^*B)$.

Theorem 2.74 (theorem 1 of [22]). If there exists a positive linear functional g such that

$$\lim_{x\to\infty} g\left[\int_0^x P(t)dt\right] = \infty$$

then the equation $(1.3)_{1_M}$ is osillatory.

Theorem 2.75 (theorem 2 of [22]). If there exists a function $a \in \mathcal{C}[0,\infty)$ such that $a > 0$ on $[0,\infty)$ and $\lim\limits_{x\to\infty}\int_0^x a^{-1}(t)dt = \infty$ and if the operator J defined by

$$J(x) = \int_0^x \left\{a(t)P(t) - \left(\frac{a'(t)]^2}{4a(t)}\right)E_n\right\}dt + \frac{a'(t)}{2}E_n$$

has the property that there exists a positive linear functional g such that $\lim\limits_{x\to\infty} g[J(x)] = \infty$, then the equation $(1.3)_{1_M}$ is oscillatory.

Theorem 2.76 (theorem 1 of [19]). Let $g(x,s)$ and $h(x,s)$ be continuous on $D = \{(x,s) : x \geq s \geq \alpha \geq 0\}$ such that $g(x,x) = 0$ for $x \geq \alpha$ and $g(x,s) > 0$ for $x \geq s \geq \alpha$. We assume further that the partial derivative

$\frac{\partial g}{\partial s}(x,s) \equiv g_s(x,s)$ is nonpositive and continuous for $x \geq s \geq \alpha$ and $h(x,s)$ is defined by

$$g_s(x,s) = -h(x,s)g^{\frac{1}{2}}(x,s) \text{ for all } (x,s) \in D.$$

Finally assume that

$$\limsup_{x\to\infty} \frac{1}{g(x,\alpha)}\lambda_1 \left[\int_\alpha^x \left(g(x,s)Q(s) - \frac{1}{4}h^2(x,s)P(s)\right) ds\right] = \infty.$$

Then the equation $(1.3)_M$ is oscillatory.

Theorem 2.77 (theorem 1 of [34]. Let $g(x,s)$ and $h(x,s)$ be as in the above theorem. If there exists a function $f \in \mathcal{C}^1[\alpha,\infty]$ such that

$$\limsup_{x\to\infty} \frac{1}{g(x,\alpha)}\lambda_1 \left[\int_\alpha^x \{g(x,s)T(s) - \frac{1}{4}b(s)h^2(x,s)\}ds\right] = \infty$$

where

$$\begin{aligned} b(x) &= \exp\{-2\int_\alpha^x f(s)ds\} \text{ and} \\ T(x) &= b(x)[Q(x) + f^2(x)E_n - f'(x)E_n], \end{aligned}$$

then system $(1.3)_{1_M}$ is oscillatory.

Based on the above results we obtain oscillation criteria for the more general system $(1.1)_M$.

In chapter 6, we discuss conjugacy criteria for the discrete system

$$\Delta(c_{n-1}\Delta x_{n-1}) + a_n x_n = 0. \tag{1.7}$$

So we recall some definitions and basic results needed exclusively for the discussion in chapter 6. We assume hereafter equation (1.7) is such that the hypothesis H^d is satisfied.

Definition 2.78 [30]. A scalar sequence x_n is said to be a 'solution of equation (1.7)' on $[M, N+2]$ if x_n is defined on $[M, N+2]$ and satisfies the equation (1.7) for all n, $M+1 \leq n \leq N+1$.

Example 2.79. Consider the equation (1.7) with $M = 0$, $N = 1$,

$$c_n = 1 \text{ on } [0,2] \text{ and } a_n = 1 \text{ on } [1,2]. \tag{2.18}$$

It can be directly verified that $x_n = \sin(n\pi/3)$ is a solution of this equation on $[0,3]$.

Definition 2.80. A solution x_n of (1.7) is said to be 'trivial' on $[M, N+2]$ if $x_n = 0$ for all n, $M \leq n \leq N+2$.

Note 2.81. $x_n = 0$ for all n, $0 \leq n \leq 3$ is a solution of (2.18).

The following result given in [30] is basic to the theory of second order linear difference equations. This is analogous to the result that a nontrivial solution of a linear second order differential equation cannot have a double zero.

Lemma 2.82. If x_n is a nontrivial solution of the equation (1.7) such that $x_{n_0} = 0$, $M < n_0 < N+2$, then $x_{n_0-1}x_{n_0+1} < 0$.

Note 2.83. If x_n, y_n are two solutions of (1.7) on $[M, N+2]$ then $w(x,y,n) = c_n[x_{n+1}y_n - x_n y_{n+1}]$ = constant, independent of n, $M \leq n \leq N+1$. This can be directly verified by showing that $\Delta w(x,y,n) \equiv 0$.

We now recall the definitions of a 'generalized zero' of a solution and the 'disconjugacy' of equation (1.7) from [30] and of a 'recessive solution at $\infty(-\infty)$' from [2]. (The concept of a generalized zero was introduced by Hartman in [24].)

Definition 2.84 [30]. A solution x_n of (1.7) is said to have a 'generalized zero' at $p \in I = [M, N+2]$ provided $x_p = 0$ in case $p = M$ and $x_p = 0$ or $x_{p-1}x_p < 0$ in case $M < p \leq N+2$.

Example 2.85. Consider equation (1.7) with $M = 0$, $N = 2$,

$$c_n = \begin{cases} (3/2) & n = 0 \\ 2 & n = 1 \\ 1 & n = 2 \\ (1/4) & n = 3 \end{cases}$$

and

$$a_n = \begin{cases} 3 & n = 1 \\ (3/2) & n = 2 \\ (1/2) & n = 3. \end{cases} \tag{2.19}$$

Equation (1.7) is equivalent to

$$x_{n+1} = \frac{-(a_n - c_n - c_{n-1})}{c_n} x_n - \frac{c_{n-1}}{c_n} x_{n-1}.$$

If x_n is the solution of the given system with $x_0 = 0$ and $x_1 = 1$ then

$$\begin{aligned} x_2 &= \frac{-(a_1 - c_1 - c_0)}{c_1} x_1 - \frac{c_0}{c_1} x_0 = \frac{1}{4} > 0, \\ x_3 &= \frac{-(a_2 - c_2 - c_1)}{c_2} x_2 - \frac{c_1}{c_2} x_1 = \frac{-13}{8} < 0. \end{aligned}$$

Hence x_n has a zero at $n = 0$ and a generalized zero at $n \gtrsim 3$.

Note 2.86. In the definition 2.84 the conditions corresponding to $p = M$ or $p = N + 2$ or both have to be omitted according as $I = (-\infty, N + 2]$ or $[M, \infty)$ or $(-\infty, \infty)$ respectively.

Definition 2.87 [30]. Equation (1.7) is said to be 'disconjugate on I' provided no nontrivial solution of (1.7) has two or more generalized zeros on I. Otherwise it is said to be 'conjugate' on I.

Example 2.88. Equation (2.19) in example 2.85 is conjugate on [0,4].

The following separation theorem is a discrete version of the Sturm separation theorem.

Theorem 2.89 (theorem 6.5, p 261, [30]. Two linearly independent solutions of equation (1.7) can not have a common zero. If a nontrivial solution of equation (1.7) has a zero at t_1 and a generalized zero at $t_2 > t_1$, then any second linearly independent solution has a generalized zero in $(t_1, t_2]$. If a nontrivial solution of equation (1.7) has a generalized zero at t_1 and a generalized zero at $t_2 > t_1$, then any second linearly independent solution has a generalized zero in $[t_1, t_2]$.

We can use the above theorem to show that the equation in the following example is disconjugate.

Example 2.90. Consider equation (1.7) with $I = (-\infty, \infty)$

$$c_n \equiv 1, \quad a_n \equiv 0 \tag{2.20}$$

Clearly $x_n = 1$ is a solution of (2.20) on I which has no genaralized zeros on I. Hence by theorem 2.89 equation (2.20) is disconjugate on I.

Now we state a theorem due to Ahlbrandt and Hooker [2] which gives existence of the 'recessive solutions' for the correspoding vector difference equation

$$\Delta(C_{n-1}\Delta x_{n-1}) + A_n x_n = 0. \qquad (1.7)_v$$

For stating this theorem we need the following definitions.

Definition 2.91. A vector sequence x_n is said to be a 'solution of equation $(1.7)_v$' on $[M, N+2]$ if x_n is defined on $[M, N+2]$ and satisfies the equation $(1.7)_v$ for all n, $M+1 \leq n \leq N+1$.

Definition 2.92 [20]. For positive integers M and N with $M < N$, equation $(1.7)_v$ is called 'disconjugate' on $[M, N+2]$ if there exists at most one integer $p \in [M-1, N]$ such that

$$x^{*}_{p-1} C_{p-1} x_p \leq 0$$

for any nontrivial solution x_n of $(1.1)_v$.

Definition 2.93. A matrix sequence X_n is said to be a 'solution of equation

$$\Delta(C_{n-1}\Delta X_{n-1}) + A_n X_n = 0. \qquad (1.7)_M$$

on $[M, N+2]$ if X_n is defined on $[M, N+2]$ and satisfies the equation $(1.7)_M$ for all n, $M+1 \leq n \leq N+1$.

Note 2.94 [2]. If X_n, Y_n are two solutions of $(1.7)_M$ on $[M, N+2]$ then

$$W(X,Y,n) = X^\star_{n-1}C_{n-1}Y_n - X^\star_n C_{n-1}Y_{n-1} = \text{constant (independent of } n),$$

$M \leq n \leq N+1$. This can be directly verified by showing that $\Delta W(X,Y,n) \equiv 0$.

Definition 2.95 [2]. A matrix solution X_n of $(1.7)_M$ is said to be prepared on $[M, N+2]$ if

$$W(X,X,n) \equiv X^\star_{n-1}C_{n-1}X_n - X^\star_n C_{n-1}X_{n-1} \equiv \text{zero matrix}$$

Definition 2.96 [2]. Suppose equation $(1.7)_v$ is disconjugate on $(-\infty, \infty)$. A solution x_n of $(1.7)_v$ is called 'recessive at ∞' (recessive at $-\infty$) if there exists an integer P (Q) such that

(i) $x_n x_{n+1}$ is positive for $n \geq P$ ($n \leq Q$)

(ii) $\sum_{i=P}^{n}(x_i C_i x_{i+1})^{-1} \longrightarrow \infty$ as $n \longrightarrow \infty$

$$\left(\sum_{i=n}^{Q}(x_i C_i x_{i+1})^{-1} \longrightarrow \infty \text{ as } n \longrightarrow -\infty\right).$$

Theorem 2.97 (theorem 4.1, [2]). Assume hypothesis H^d and $(1.7)_v$ is disconjugate. Then

(i) every prepared solution X_n is such that $X^\star_{n-1}C_{n-1}X_n$ is positive definite for some M;

(ii) there exists a solution Z_n which is 'recessive' at ∞;

(iii) if Y_n is any prepared solution with $W(Y,Z,n)$ nonsingular, then Y_n is 'dominant' and

$$Y_n^{-1}Z_n \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty;$$

(iv) Z_n is essentially unique, that is, if X_n and Z_n are recessive at ∞, then there exists a nonsingular matrix H such that $X_n = Z_n H$ for all n;
(v) if M is such that $(1.7)_v$ is disconjugate on $[M-2, \infty)$ and $X_n(M, N)$ is defined as the solution of the two point boundary value problem

$$X_{M-1}(M,N) = E_n, \quad X_N(M,N) = 0$$

then the recessive solution at ∞ is determined up to nonsingular constant multiples as $\lim_{N\to\infty} X_n(M,N)$.

Note 2.98. It follows from the above theorem specialized to the case $n = 1$ that if equation (1.7) is disconjugate on $(-\infty, \infty)$ then 'recessive' solutions at $+\infty$ and $-\infty$ exist in the following sense.

Definition 2.99 [2]. Suppose equation (1.7) is disconjugate on $(-\infty, \infty)$. A solution x_n of (1.7) is called 'recessive at ∞' (recessive at $-\infty$) if there exists an integer P (Q) such that

(i) $x_n x_{n+1}$ is positive for $n \geq P$ $(n \leq Q)$

(ii) $\sum_{i=P}^{n} (x_i c_i x_{i+1})^{-1} \longrightarrow \infty$ as $n \longrightarrow \infty$

$\left(\sum_{i=n}^{Q} (x_i c_i x_{i+1})^{-1} \longrightarrow \infty \text{ as } n \longrightarrow -\infty \right)$.

We now define what is meant by equation (1.7) being '1-special' or '1-general' on $(-\infty, \infty)$ based on similar definitions by Došlý [16] for the continuous case.

Definition 2.100. Suppose equation (1.7) is disconjugate on $(-\infty, \infty)$. We say that it is '1-special' (1-general) if the recessive solutions at $+\infty$ and $-\infty$ are linearly dependent (linearly independent).

Example 2.101. Consider the system (1.7) on $(-\infty, \infty)$ with

$$a_n = 0 \text{ and } c_n = 1 \text{ on } (-\infty, \infty).$$

This system is disconjugate on $(-\infty,\infty)$ since $x_n \equiv 1$ is a solution not having any generalized zeros on $(-\infty,\infty)$. Further it follows from definition 2.100 that

$$x_\infty(n) = 1 \text{ and } x_{-\infty}(n) = 1$$

are the recessive solutions at $+\infty$ and $-\infty$ repectively. Hence the system is 1-special.

The following result from [1] shows the equivalence of the disconjugacy of the scalar difference equation (1.7) to the existence of a special solution to the associated Riccati equation and to the positive definiteness of an associated quadratic form.

Theorem 2.102 (theorem 2.1, [1]). Assume the hypothesis H^d. Then the following conditions are equivalent:

(i) If x_n is a solution of (1.7) such that

$$x_M c_M x_{M+1} \leq 0 \text{ and } x_{M+1} \neq 0,$$

then

$$x_n c_n x_{n+1} > 0 \text{ for } n = M+1, ..., N.$$

(ii) If y_n is a solution of (1.7) such that

$$y_N c_N y_{N+1} \leq 0 \text{ and } y_N \neq 0,$$

then

$$y_n c_n y_{n+1} > 0 \text{ for } n = M, ..., N.$$

(iii) (1.7) is disconjugate on $[M, N+1]$.

(iv) There exists a solution x_n of (1.7) with $x_n c_n x_{n+1} > 0$ for $n = M, ..., N$.

(v) There exists a sequence $w_n = \{w_n\}$, $n = M+1, ..., N+2$, with

$w_n + c_{n-1} > 0$, $n = M+1, ..., N+2$ and

$$\Delta w_n + a_n + \frac{w_n^2}{w_n + c_{n-1}} = 0 \quad n = M+1, ...N+1.$$

(vi) The quadraticform $J_2[\eta]$ defined by

$$J_2[\eta] = \sum_{M+1}^{N+1} c_{n-1}(\Delta\eta_{n-1})^2 - a_n\eta_n^2$$

is positive definite on the class of η with

$$\eta_M = 0 = \eta_{N+1}.$$

The following result in the paper [18] (yet to appear in print) is a conjugacy criterion for the equation

$$\Delta^2 x_{n-1} + a_n x_n = 0. \qquad (1.7)_1$$

Theorem 2.103 (theorem 1 of [18]). Suppose that there exist $\epsilon_1 > 0$, $\epsilon_2 > 0$ such that

$$\limsup_{n\to\infty} \sum_{k=0}^{n} \arctan \frac{\epsilon_1}{\alpha_k(a,\epsilon_1)} > \frac{\pi}{2},$$

$$\limsup_{n\to-\infty} \sum_{k=n}^{1} \arctan \frac{\epsilon_2}{\beta_k(a,\epsilon_2)} > \frac{\pi}{2},$$

where

$$\alpha_0 = 1 + \epsilon_1, \quad \beta_1 = 1 + \epsilon_2,$$

$$\alpha_k = \alpha_k(a,\epsilon_1) = \left(\epsilon_1 - \sum_{i=0}^{k-1} a_i + 1\right) \prod_{j=0}^{k-1} \left(\epsilon_1 - \sum_{i=0}^{j-1} a_i + 1\right)^2 \text{ for } k \geq 1$$

$$\beta_k = \beta_k(a,\epsilon_2) = \left(\epsilon_2 - \sum_{i=k-1}^{-1} a_i + 1\right) \prod_{j=k+1}^{1} \left(\epsilon_2 - \sum_{i=j-1}^{-1} a_i + 1\right)^2 \text{ for } k \leq 0.$$

Then $(1.7)_1$ is conjugate in $\mathbb{Z}$.

In chapter 6 we obtain a conjugacy criterion for the equation (1.7) in terms of the coefficient functions. This criterion is analogous to the one given by Došlý (theorem 2.16) for the continuous case.

Chapter 3

Conjugacy Criteria for a Two Dimensional Linear Hamiltonian System

In this chapter we consider the linear Hamiltonian system

$$\begin{aligned} L_2[u,v] &\equiv u' - a(x)u - b(x)v = 0 \\ L_1[u,v] &\equiv v' - c(x)u + a(x)v = 0 \end{aligned} \tag{1.1}$$

along with the associated unperturbed system

$$\begin{aligned} u' &= a(x)u + b(x)v \\ v' &= -a(x)v \end{aligned} \tag{1.2}$$

under the hypothesis

H : $a(x)$, $b(x) > 0$, $c(x)$ are real valued continuous functions on an open interval $I = (\alpha, \beta)$, $-\infty \le \alpha < \beta \le \infty$.

We are particularly interested in obtaining sufficient conditions for the system (1.1) to be conjugate on I in terms of its coefficient functions. Our results are motivated by the conjugacy criteria given by Došlý [15, 16], Tipler [43] and Müller-Pfeiffer [36].

Conjugacy criteria for the equation

$$(p(x)u')' + q(x)u = 0 \tag{1.3}$$

with $p(x) > 0$ and $q(x)$ continuous on $(-\infty, \infty)$ and for its special form with $p(x) \equiv 1$, that is,

$$u'' + q(x)u = 0 \tag*{$(1.3)_1$}$$

have been obtained by Müller -Pfeiffer [36] and Tipler [43] respectively.

In [43] Tipler has shown using Riccati techniques that the equation $(1.3)_1$ is conjugate on $(-\infty, \infty)$ if

$$\liminf_{t_1 \downarrow -\infty, t_2 \uparrow \infty} \int_{t_1}^{t_2} q(x)\ dx > 0$$

whereas in [36] Müller-Pfeiffer has shown using variational techniques that the more general equation (1.3) is conjugate on I if for some (and hence for every) γ in I,

$$\int_{\alpha}^{\gamma} p^{-1}(x)dx = \infty = \int_{\gamma}^{\beta} p^{-1}(x)dx \tag{3.1}$$

and

$$\liminf_{t_1 \downarrow -\infty, t_2 \uparrow \infty} \int_{t_1}^{t_2} q(x)\ \ dx > 0.$$

In [13] Došlý proved that the 2n-th order equation

$$(-1)^{(n)} \left(p(x)u^{(n)}\right)^{(n)} + q(x)u = 0$$

where $p(x) > 0$, $q(x)$ are continuous on $\mathbb{R}$ is conjugate if there exists an integer m, $0 \leq m \leq n-1$ such that the following conditions hold:

(*i*) $\displaystyle\int_{-\infty}^{0} x^{2m} p^{-1}(x)dx = \infty = \int_{0}^{\infty} x^{2m} p^{-1}(x)dx.$

(*ii*) there exists a real-valued polynomial

$$Q_k(x) = a_k x^k + ... + a_1 x + a_0$$

of degree k, $0 \leq k \leq n-m-1$, such that

$$\limsup_{t_1 \downarrow -\infty, t_2 \uparrow \infty} \int_{t_1}^{t_2} Q_k^2(x) q(x)\ dx = c < 0.$$

More recently Došlý has shown in [16] that the conditions (3.1) are merely equivalent to the unperturbed disconjugate equation

$$(p(x)u')' = 0 \tag{1.4}$$

being 1-special on I (see lemma 2.15, chapter 2). He then established by means of the associated Riccati equation and transformation techniques, that irrespective of whether the equation (1.4) is 1-special or otherwise the two inequalities

$$\epsilon_1 \int_\gamma^\beta p^{-1}(x) \exp\left\{2\int_\gamma^x p^{-1}(t)\left[\int_\gamma^t q(s)ds - \epsilon_1\right] dt\right\} dx \quad > \quad \pi/2$$

$$\epsilon_2 \int_\alpha^\gamma p^{-1}(x) \exp\left\{2\int_\gamma^x p^{-1}(t)\left[\int_\gamma^t q(s)ds + \epsilon_2\right] dt\right\} dx \quad > \quad \pi/2$$

holding together for some γ in I and for some $\epsilon_1 > 0$, $\epsilon_2 > 0$ constitute a conjugacy criterion for (1.3) on I. It is to be remarked here that the above criterion is significant in the sense that if any one of the inequalities is not satisfied then there is no guarantee that the equation (1.3) is conjugate on I. The disconjugacy of the equation $u'' = 0$ on $(-\infty, \infty)$ illustrates this remark.

In this chapter following Došlý [15] we obtain under the hypothesis H a conjugacy criterion (theorem 3.3) for the system (1.1) irrespective of the consideration whether the associated unperturbed disconjugate system (1.2) is 1-special or not (see proposition 3.1 for the disconjugacy of the system (1.2)). As a consequence of this theorem we obtain (corollary 3.4) conjugacy criteria for the system (1.1) in the particular cases when the associated unperturbed system (1.2) is 1-general or 1-special. These criteria in turn yield for the scalar equation (1.3) in the 1-general case a conjugacy criterion which can be applied in some instances where Došlý's criterion (theorem 2 of [16]) (see theorem 2.18, chapter 2) fails. Moreover it will be clear in these instances that our criterion is easier to apply than Došlý's another criterion (theorem 3 of [16]) (see theorem 2.19, chapter 2). However in the 1-special case our criterion (corollary 3.6) reduces to that given by Došlý in corollary 1 of [16] (see lemma 2.17, chapter 2). Lastly, we note that even though criteria as in theorem 3.3 can be obtained by transforming the system (1.1) into the equation (1.3) by means of the well known transformations (see p 220, [39]),

$$g(x) = \exp^{\int_{x_0}^{x} a(s)ds}, \quad \tilde{u} = (1/g)u, \quad \tilde{v} = gv$$

so that

$$\begin{aligned} gL_1[u,v] &= \tilde{L}_1[\tilde{u},\tilde{v}] \\ (1/g)L_2[u,v] &= \tilde{L}_2[\tilde{u},\tilde{v}] \end{aligned}$$

where

$$\begin{aligned} \tilde{L}_1[\tilde{u},\tilde{v}] &= -\tilde{v}' + \tilde{c}(x)\tilde{u}, \\ \tilde{L}_2[\tilde{u},\tilde{v}] &= \tilde{u}' - \tilde{b}(x)\tilde{v} \end{aligned}$$

with

$$\tilde{b}(x) = \frac{b(x)}{g^2(x)}, \quad \tilde{c}(x) = c(x)g^2(x),$$

the direct method gives the criterion in a form analogous to that given by Došlý in [15].

Now we shall prove some preliminary results which will be useful in our further discussion.

We first state the following proposition which gives the disconjugacy of the unperturbed system (1.2).

Proposition 3.1. Under the hypothesis H, the unperturbed system (1.2) associated with (1.1) is disconjugate on I.

Proof. This follows from the fact that if $(u(x), v(x))$ is the general solution of the system (1.2) then by variation of constants $u(x)$ is given by

$$u(x) = \left\{\exp\left(\int_{x_0}^{x} a(s)ds\right)\right\} \left[c_2 + c_1 \int_{x_0}^{x} b(t)\exp\left(-2\int_{x_0}^{t} a(s)ds\right) dt\right]$$

where c_1, c_2 are arbitrary constants and $x_0 \in I$ is arbitrary but fixed. Now $u(x_2) = 0 = u(x_1)$, $x_1 < x_2$ implies $c_1 = c_2 = 0$. Hence the proposition.

In the rest of the chapter for the sake of convenience we shall denote

$$e(x) = c(x) + a^2(x)/b(x)$$

$$(3.2)_{a,b,c}$$

$$l_1 = \int_\alpha^\gamma b(t)\exp(-2\int_\gamma^t a(s)ds)\ dt \ \text{ and }\ l_2 = \int_\gamma^\beta b(t)\exp(-2\int_\gamma^t a(s)ds)\ dt$$

for some $\gamma \in I$, where $0 < l_1,\ l_2 \leq \infty$.

The following proposition generalizes to the system (1.2) the result of Došlý (see lemma 2.15, chapter 2) that the conditions (3.1) are equivalent to equation (1.4) being 1-special on I.

Proposition 3.2. Under the hypothesis H the system (1.2) is 1-special on I if and only if

$$l_1 = \infty = l_2 \tag{3.3}$$

for some (and hence for every) γ in I.

Proof. Suppose (3.3) holds. Then we can directly verify that the solution

$$y(x) = (u(x), v(x)) = \left(\exp\ (\int_\gamma^x a(t)\ dt), 0\right)$$

satisfies the system (1.2) as well as the definition of principal solutions at α and β by virtue of the assumption (3.3). Hence (1.2) is 1-special on I.

Conversely, assume that (1.2) is 1-special on I and (3.3) does not hold. If $0 < l_1 < \infty$ and $l_2 = \infty$ we can show that

$$\begin{aligned} y_\alpha(x) &= (u_\alpha(x), v_\alpha(x)) \\ &= \left(\{\exp(\int_\gamma^x a(s)ds)\}\int_\alpha^x b(t)\exp(-2\int_\gamma^t a(s)ds)dt, \exp(-\int_\gamma^x a(t)dt)\right) \end{aligned}$$

and

$$\begin{aligned} y_\beta(x) &= (u_\beta(x), v_\beta(x)) \\ &= \left(\exp(\int_\gamma^x a(t)\ dt), 0\right) \end{aligned}$$

are the principal solutions of (1.2) at α and β respectively.

On the other hand if $l_1 = \infty$ and $0 < l_2 < \infty$ we can show that

$$\begin{aligned} y_\alpha(x) &= (u_\alpha(x), v_\alpha(x)) \\ &= \left(\exp(\int_\gamma^x a(t)dt), 0\right) \end{aligned}$$

and

$$\begin{aligned} y_\beta(x) &= (u_\beta(x), v_\beta(x)) \\ &= \left(-\{\exp(\int_\gamma^x a(s)ds)\} \int_x^\beta b(t)\exp(-2\int_\gamma^t a(s)ds)dt, \exp(-\int_\gamma^x a(t)dt)\right) \end{aligned}$$

are the principal solutions of (1.2) at α and β respectively.

Lastly, if $0 < l_1, l_2 < \infty$ we can show that $y_\alpha(x)$ as in case (i) and $y_\beta(x)$ as in case (ii) are the principal solutions of (1.2) at α and β respectively.

Thus in all the three possible cases, the principal solutions at α and β are linearly independent which is a contradiction to (1.2) being 1-special on I. This completes the proof of the proposition.

In the following theorem we obtain sufficient conditions for the system (1.1) to be conjugate on I using Riccati and transformation techniques. This theorem generalizes for the system (1.1) the conjugacy criterion for $(1.3)_1$ of Došlý (see theorem 2.16, chapter 2).

Theorem 3.3. Suppose there exist $\gamma \in I, \epsilon_1 > 0$ and $\epsilon_2 > 0$ such that

$$\epsilon_1 \int_\gamma^\beta b(x)\exp\left\{-2\int_\gamma^x \left[a(t) + b(t)\left(\epsilon_1 + \int_\gamma^t e(s)ds\right)\right]\ dt\right\}\ dx \ > \ A$$

and $(3.4)_{a,b}$

$$\epsilon_2 \int_\alpha^\gamma b(x) \exp\left\{-2\int_\gamma^x \left[a(t) + b(t)\left(-\epsilon_2 + \int_\gamma^t c(s)ds\right)\right] dt\right\} dx \quad > \quad B.$$

If

$$\epsilon_1 + \epsilon_2 - \frac{\pi(\epsilon_1 B + \epsilon_2 A)}{2AB} \geq 0, \tag{3.5}$$

then (1.1) is conjugate on (α, β).

Proof. Denote by $(u_i(x), v_i(x))$, $i = 1, 2$, the solutions of (1.1) given by the conditions $u_1(\gamma) = 1$, $v_1(\gamma) = c_R$, $u_2(\gamma) = 1$, $v_2(\gamma) = \epsilon_1$ where $c_R = \epsilon_1(1 - \pi/2A)$.

We now claim $u_1(x)$ has a zero on (γ, β). Otherwise we have (i) $u_1(x) > 0$ on (γ, β) (ii) the function $\theta(x) = \arctan(u_2(x)/u_1(x))$ is well defined on $[\gamma, \beta)$ (iii) $\theta(\gamma) = \pi/4$ and (iv) $\theta(\beta-) \leq \pi/2$.
Further on (γ, β) we have

$$\begin{aligned}
\theta'(x) &= (u_1^2(x) + u_2^2(x))^{-1}(u_1u_2' - u_2u_1')(x) \\
&= (u_1^2(x) + u_2^2(x))^{-1}b(x)(u_1v_2 - u_2v_1)(x) \text{ (by system (1.1))} \\
&= (u_1^2(x) + u_2^2(x))^{-1}b(x)(u_1v_2 - u_2v_1)(\gamma) \text{ (by } (i) \text{ of theorem 2.1)} \\
&= (u_1^2(x) + u_2^2(x))^{-1}b(x)(\epsilon_1 - c_R)
\end{aligned}$$

that is,

$$\theta'(x) = (\epsilon_1\pi/2A)\ b(x)\ (u_1^2(x) + u_2^2(x))^{-1}. \tag{3.6}$$

This implies $\theta'(x) > 0$ and hence $u_2(x) > u_1(x) > 0$ on (γ, β). Further the function $w(x) = v_2(x)/u_2(x)$ is a solution of the Riccati equation

$$w'(x) + 2a(x)w(x) + b(x)w^2(x) - c(x) = 0$$

associated with the system (1.1) on (γ, β). Consequently,

$$w(x) = \epsilon_1 + \int_\gamma^x (c(t) - 2a(t)w(t) - b(t)w^2(t))dt$$

implying,

$$\begin{aligned} u_2'(x) &= a(x)u_2(x) + b(x)v_2(x) = u_2(x)(a(x) + b(x)w(x)) \\ &= u_2(x)\left[a(x) + b(x)\left(\epsilon_1 + \int_\gamma^x (c(t) - 2a(t)w(t) - b(t)w^2(t))dt\right)\right] \end{aligned}$$

on (γ, β).

Thus

$$u_2(x) = \exp\left\{\int_\gamma^x \left[a(t) + b(t)\left(\epsilon_1 + \int_\gamma^t (c - 2aw - bw^2)(s)ds\right)\right] dt\right\}$$

implying,

$$u_2(x) \leq \exp\left\{\int_\gamma^x \left[a(t) + b(t)\left(\epsilon_1 + \int_\gamma^t e(s)ds\right)\right]\ dt\right\} \tag{3.7}$$

since

$$c(s) - 2a(s)w(s) - b(s)w^2(s) = e(s) - b(s)(w(s) + a(s)/b(s))^2 \leq e(s) \text{ (by } (3.2)_a).$$

Now

$$\begin{aligned} (\pi/2) &\geq \theta(\beta-) = \theta(\gamma) + \int_\gamma^\beta \theta'(x)\ dx \\ &= \pi/4 + (\epsilon_1\pi/2A)\int_\gamma^\beta b(x)(u_1^2(x) + u_2^2(x))^{-1}dx \quad \text{(by (3.6))} \\ &\geq \pi/4 + (\epsilon_1\pi/2A)\int_\gamma^\beta b(x)(2u_2^2(x))^{-1}dx \\ &\geq \pi/4 + (\epsilon_1\pi/4A)\int_\gamma^\beta b(x)\exp\left\{-2\int_\gamma^x [a(t) + b(t)\right. \\ &\qquad \left.(\epsilon_1 + \int_\gamma^t e(s)ds)]\ dt\right\} dx \text{ (by (3.7))} \\ &> \pi/2 \quad \text{(by (3.7) and } (3.4)_a). \end{aligned}$$

This contradiction proves our claim that $u_1(x)$ has a zero on (γ, β).

Similarly assuming $(u_i(x), v_i(x))$, $i = 3, 4$ to be solutions of (1.1) determined by the initial conditions $u_3(\gamma) = 1, v_3(\gamma) = c_L; u_4(\gamma) = 1, v_4(\gamma) = -\epsilon_2$ where $c_L = \epsilon_2(\pi/2B - 1)$ we can show by means of $(3.4)_b$ that $u_3(x)$ has a zero on (α, γ).

Note that $c_L \leq c_R$ by virtue of (3.5) and in the case $c_L = c_R$ we have $(u(x), v(x)) \equiv (u_1(x), v_1(x)) \equiv (u_3(x), v_3(x))$ is a solution of (1.1) satisfying that $u(x)$ has two zeros on (α, β) and we are done.

In case $c_L < c_R$ let $(u(x), v(x))$ be the solution of (1.1) satisfying $u(\gamma) = 1$, $v(\gamma) = \mu$ where $c_L < \mu < c_R$. Then from (1.1) we have $u_3'(\gamma) < u'(\gamma) < u_1'(\gamma)$. The second inequality implies either (i) $(u - u_1,\ v - v_1)$ is a solution of (1.1) having two zeros on $[\gamma, \beta)$ or (ii) (u, v) is a solution of (1.1) such that $u(x)$ has a zero on (γ, β). The first inequality implies either (i) $(u - u_3, v - v_3)$ is a solution of (1.1) having two zeros on $(\alpha, \gamma]$ or (ii) (u, v) is a solution of (1.1) such that $u(x)$ has a zero on (α, γ). In any case it follows that (1.1) is conjugate on (α, β). This completes the proof of the theorem.

In the remaining part of this chapter we shall use the following notation:

$$g(x) = b(x)\ \exp(-2\int_\gamma^x a(t)dt) \tag{3.8}$$

(so that from equations $(3.2)_{b,c}$ we have

$$l_1 = \int_\alpha^\gamma g(t)dt\ \ ,\ \ l_2 = \int_\gamma^\beta g(t)dt),$$

$$c_1 = \limsup_{x\downarrow\alpha} \frac{2\int_x^\gamma\ b(t)\ [\ \int_t^\gamma\ e(s)\ ds\ ]\ dt - \log[\int_\alpha^x g(t)\ dt]}{\int_x^\gamma (b(t) + g(t))\ dt} \qquad (3.9)_a$$

provided $0 < l_1 < \infty$,

$$c_2 = \liminf_{x\uparrow\beta} \frac{-2\int_\gamma^x\ b(t)\ [\ \int_\gamma^t\ e(s)\ ds\ ]\ dt + \log[\int_x^\beta g(t)\ dt]}{\int_\gamma^x\ (b(t) + g(t))\ dt} \qquad (3.9)_b$$

provided $0 < l_2 < \infty$,

$$c_3 = \liminf_{x \uparrow \beta} \frac{-2\int_\gamma^x b(t)\,[\int_\gamma^t e(s)\,ds]\,dt}{\int_\gamma^x (b(t)+g(t))\,dt}, \tag{3.10)$_a$ ($$

$$c_4 = \limsup_{x \downarrow \alpha} \frac{2\int_x^\gamma b(t)[\int_t^\gamma e(s)\,ds]\,dt}{\int_x^\gamma (b(t)+g(t))dt}. \qquad (3.10)_b$$

The conclusions $(i), (ii)$ and (iii) of the following corollory give conjugacy criteria for the system (1.1) in the case the unperturbed system (1.2) is 1-general on I and the conclusion (iv) gives a conjugacy criterion for (1.1) in the case (1.2) is 1-special on I.

Corollary 3.4. The system (1.1) is conjugate on I if exactly one of the following holds.

(i) $l_1 < \infty,\ l_2 < \infty,\ 0 < c_2 \le \infty$ and $-\infty \le c_1 < 0$.

(ii) $l_1 < \infty,\ l_2 = \infty,\ -\infty \le c_1 < 0$ and $0 < c_3 \le \infty$.

(iii) $l_1 = \infty,\ l_2 < \infty,\ 0 < c_2 \le \infty$ and $-\infty \le c_4 < 0$.

(iv) $l_1 = \infty = l_2,\ 0 < c_3 \le \infty$ and $-\infty \le c_4 < 0$.

We shall give the proof for case (i) only, since the proofs for cases (ii), (iii) and (iv) are along the same lines as that of (i).

Proof (i). We shall first assume $0 < c_2 < \infty$. Let $\epsilon_1 = (1/4)c_2$, then by $(3.9)_b$ there exists $T_1 \in (\gamma, \beta)$ such that

$$\frac{-2\int_\gamma^x b(t)\,[\int_\gamma^t e(s)\,ds]\,dt + \log[\int_x^\beta g(t)\,dt]}{\int_\gamma^x (b(t)+g(t))\,dt} > 2\epsilon_1$$

whenever $x \in (T_1, \beta)$. That is,

$$-2\int_\gamma^x b(t)\left[\int_\gamma^t e(s)ds\right]dt + \log\left[\int_x^\beta g(t)dt\right] > 2\epsilon_1 \int_\gamma^x (b(t)+g(t))dt$$

for every $x \in (T_1, \beta)$. Hence

$$-2\int_\gamma^x b(t)\left[\epsilon_1(1+g(t)/b(t)) + \int_\gamma^t e(s)ds\right]dt > -\log\left[\int_x^\beta g(t)dt\right]$$

implying,

$$\exp\left\{-2\int_\gamma^x b(t)\left[\epsilon_1(1+g(t)/b(t))+\int_\gamma^t e(s)ds\right]dt\right\} > \left[\int_x^\beta g(t)dt\right]^{-1} \tag{3.11}$$

for every $x \in (T_1, \beta)$.

Now replacing β by λ $(T_1 < \lambda < \beta)$ in the inequality $(3.4)_a$, splitting the integral in $(3.4)_a$ as $\int_\gamma^\lambda = \int_\gamma^{T_1} + \int_{T_1}^\lambda$ and adding and subtracting $\int_\gamma^x g(t)dt$ inside the exponential function we have

$$\begin{aligned}
&\epsilon_1\int_\gamma^\lambda b(x)\exp\left\{-2\int_\gamma^x\left(a(t)+b(t)\left[\epsilon_1+\int_\gamma^t e(s)ds\right]\right)dt\right\}\;dx = \\
&\quad K+\epsilon_1\int_{T_1}^\lambda b(x)\exp\left\{-2\int_\gamma^x\left(a(t)+b(t)\left[\epsilon_1\left(1+\frac{g(t)}{b(t)}-\frac{g(t)}{b(t)}\right)\right.\right.\right. \\
&\quad \left.\left.\left.+\int_\gamma^t e(s)ds\right]\right)dt\right\}\;dx.
\end{aligned} \tag{3.12}$$

(where $K = \epsilon_1\int_\gamma^{T_1} b(x)\;\exp\{-2\int_\gamma^x\;(a(t)+b(t)\;[\epsilon_1+\int_\gamma^t\;e(s)ds])\;dt\}\;dx$.)

$$\begin{aligned}
&= K+\epsilon_1\int_{T_1}^\lambda b(x)\exp\left\{-2\int_\gamma^x a(t)dt\right\}\exp\left\{-2\int_\gamma^x b(t)\left[\epsilon_1\left(1+\frac{g(t)}{b(t)}\right)\right.\right. \\
&\quad\left.\left.+\int_\gamma^t e(s)ds\right]dt\right\}\;\exp\left\{2\epsilon_1\int_\gamma^x g(t)dt\right\}dx \\
&> K+\epsilon_1\int_{T_1}^\lambda g(x)\;\exp\left\{2\epsilon_1\int_\gamma^x g(t)dt\right\}\left(\int_x^\beta g(t)dt\right)^{-1}dx
\end{aligned}$$

(by (3.11) and the definition of $g(x)$ in (3.8))

$$\begin{aligned}
&> K+\epsilon_1\int_{T_1}^\lambda g(x)\left(\int_x^\beta g(t)dt\right)^{-1}dx \\
&> K+\epsilon_1\int_{\tilde\lambda}^{\tilde T_1} 1/\tau\;\;d\tau
\end{aligned}$$

where $\displaystyle\int_x^\beta g(t)dt = \tau,\; \tilde T_1 = \int_{T_1}^\beta g(t)dt > 0,\; \tilde\lambda = \int_\lambda^\beta g(t)dt > 0$

with $\tilde{\lambda} \to 0$ as $\lambda \to \beta-$, by (3.8) and $(3.2)_c$.
Consequently, the integral on the L.H.S. of $(3.4)_a$ is divergent and $= +\infty$ and thus the inequality $(3.4)_a$ of theorem 3.3 is satisfied.

If $c_2 = \infty$ by suitably modifying the above argument with $\epsilon_1 = M/2$ where M is any positive constant, we can show again that the integral on the L.H.S. of $(3.4)_a = \infty$.

Similarly we can show by using the hypothesis $-\infty \leq c_1 < 0$ and $l_1 < \infty$ that the integral on the L.H.S. of $(3.4)_b = \infty$. Further by choosing $A = B = \pi/2$, we see that the inequalities $(3.4)_{a,b}$ and (3.5) of theorem 3.3 are satisfied. This completes the proof of case (i) of the corollary.

Recall that the equation $u'' = 0$ is disconjugate on $(-\infty, \infty)$ and for this equation we have $a(x) = 0,\ b(x) = 1,\ c(x) = 0,\ e(x) = 0,\ g(x) = 1,$

$$l_1 = \int_{-\infty}^{\gamma} dt = \infty = l_2 = \int_{\gamma}^{\infty} dt$$

and $c_3 = 0 = c_4$. Hence the bound 0 for the constants c_3, c_4 in case (iv) of the above corollary are sharp. However the sharpness of these bounds for the conjugacy of the system (1.1) in cases (i),(ii) and (iii) remains an open question.

The following example shows that if the criteria of corollary 3.4 are not satisfied then the system need not be conjugate.

Example 3.5. Consider the system

$$\begin{aligned} u' &= u + v \\ v' &= u - v \end{aligned}$$

on $(-\infty, \infty)$. This system does not satisfy any of the criteria of corollary 3.4 since $e(x) = 2,\ g(x) = e^{-2(x-\gamma)}$,

$$l_1 = \int_{-\infty}^{\gamma} e^{2(\gamma-t)} dt = \infty,\ \ l_2 = \int_{\gamma}^{\infty} e^{2(\gamma-t)} dt = 1/2,$$

$$c_2 = \liminf_{x\uparrow\infty} \frac{-4(x-\gamma)^2 - 4(x-\gamma) - 2\log 2}{2(x-\gamma) - 2\exp\{-2(x-\gamma)\} + 1} = -\infty$$

$$\text{and } c_4 = \limsup_{x\downarrow-\infty} \frac{-4(x-\gamma)^2}{2(x-\gamma) - 2\exp\{-2(x-\gamma)\} + 1} = 0.$$

However the system is disconjugate on $(-\infty, \infty)$.This can be seen from the fact that the general solution is given by

$$(u(x), v(x)) = (c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x}, (\sqrt{2}-1)c_1 e^{\sqrt{2}x} - (\sqrt{2}+1)c_2 e^{-\sqrt{2}x})$$

where c_1, c_2 are arbitrary constants and $u(x_2) = 0 = u(x_1)$ $-\infty \leq x_1 < x_2 \leq \infty$ implies

$$c_1 e^{\sqrt{2}x_1} + c_2 e^{-\sqrt{2}x_1} = 0 \text{ and } c_1 e^{\sqrt{2}x_2} + c_2 e^{-\sqrt{2}x_2} = 0.$$

Hence $c_1 = 0 = c_2$.

We shall now apply corollary 3.4 to the equation (1.3) in which $a(x) \equiv 0$, $b(x) = r^{-1}(x)$ and $c(x) = -p(x)$.

It will be convenient to use the following notations for the statement of this result.

Denote

$$\int_\alpha^\gamma r^{-1}(x)dx = A, \quad \int_\gamma^\beta r^{-1}(x)dx = B, \qquad (3.13)_{a,b}$$

$$a_1 = \limsup_{x\downarrow\alpha} \frac{2\int_x^\gamma r^{-1}(t)\,[\,\int_t^\gamma p(s)\,ds]\,dt + \log[\int_\alpha^x r^{-1}(t)dt]}{-2\int_x^\gamma r^{-1}(t)dt} \qquad (3.14)_a$$

provided $0 < A < \infty$,

$$a_2 = \liminf_{x\uparrow\beta} \frac{2\int_\gamma^x r^{-1}(t)\,[\,\int_\gamma^t p(s)\,ds]\,dt + \log[\int_x^\beta r^{-1}(t)dt]}{2\int_\gamma^x r^{-1}(t)dt} \qquad (3.14)_b$$

provided $0 < B < \infty$,

$$a_3 = \liminf_{x\uparrow\beta} \frac{\int_\gamma^x r^{-1}(t)\,[\,\int_\gamma^t p(s)\,ds]\,dt}{\int_\gamma^x r^{-1}(t)dt}, \qquad (3.15)_a$$

$$a_4 = \limsup_{x \downarrow a} \frac{\int_x^\gamma r^{-1}(t) \left[\int_t^\gamma p(s)\, ds \right] dt}{-\int_x^\gamma r^{-1}(t) dt} \tag{3.15$_b$}$$

Corollary 3.6. The equation (1.3) is conjugate on I if exactly one of the following holds.

(*i*) $0 < A < \infty$, $0 < B < \infty$, $0 < a_2 \leq \infty$ and $-\infty \leq a_1 < 0$.

(*ii*) $0 < A < \infty$, $B = \infty$, $-\infty \leq a_1 < 0$ and $0 < a_3 \leq \infty$.

(*iii*) $A = \infty$, $0 < B < \infty$, $0 < a_2 \leq \infty$ and $-\infty \leq a_4 < 0$.

(*iv*) $A = \infty = B$, $0 < a_3 \leq \infty$ and $-\infty \leq a_4 < 0$.

The above corollary yields (*a*) corollary 1 of [16] (see lemma 2.17, chapter 2) in the case the associated equation (1.4) is 1-special and (*b*) a conjugacy criterion which works in some instances whereas that given in theorem 2 of [16] (see theorem 2.18, chapter 2) fails. We illustrate this claim by means of the following example.

Example 3.7. Consider the differential equation

$$(e^{\mu x} y')' + \lambda e^{\nu x} y = 0, \qquad -\infty < x < \infty \tag{3.16}$$

under the assumption $0 < \mu < \nu (\nu < \mu < o)$ and $\lambda > 0$. To be specific we shall assume $0 < \mu < \nu$ since the arguments in the other case are similar.

Note that the unperturbed equation $(e^{\mu x} y')' = 0$ is disconjugate and 1-general on $(-\infty, \infty)$ (by propositions 3.1 and 3.2) since $l_1 = \infty$ and $l_2 = 1/\mu$. Moreover $y_{-\infty}(x) = 1$ and $y_{+\infty}(x) = e^{-\mu x}$ are the principal solutions at $-\infty$ and $+\infty$ respectively since $\int_{-\infty}(e^{-\mu x} \times 1)dx = \infty = \int^{\infty}(e^{-\mu x} \times e^{2\mu x})dx$. Further they satisfy as in theorem 2 of [16] (see theorem 2.18, chapter 2), $r(y'_{-\infty} y_{+\infty} - y_{-\infty} y'_{+\infty}) = 1$.

However, the integral in theorem 2 of [16] with $t_1 = k$ and $t_2 = \log \log k$ becomes

$$\int_{-k}^{\log \log k} [4p(x) y_{-\infty}(x) y_{+\infty}(x) - (r(x) y_{-\infty}(x) y_{+\infty}(x))^{-1}]\, dx$$

$$= \int_{-k}^{\log\log k} [(4\lambda/\mu)\, e^{(\nu-\mu)x} - \mu\,]\, dx$$

$$= \frac{4\lambda}{\mu(\nu-\mu)}\,[\,(\,\log k)^{\nu-\mu} - e^{-(\nu-\mu)k}] - \mu\,(\log\log k + k)$$

$\longrightarrow -\infty$ as $k \longrightarrow +\infty$.

Consequently,

$$\liminf_{t_1\downarrow -\infty\ t_2\uparrow\infty} \int_{t_1}^{t_2} [4p(x)y_{-\infty}(x)y_{+\infty}(x) - (r(x)y_{-\infty}(x)y_{+\infty}(x))^{-1}]\, dx = -\infty$$

thus making theorem 2 of [16] inapplicable.

On the other hand for applying theorem 3 of [16] (see theorem 2.19, chapter 2) to this example in view of the remark 5(ii) following that theorem (see remark 2.20, chapter 2), we have to consider the integral

$$I(c,d) = \int_c^d \frac{1}{1+\mu^2 e^{2\mu x}}\,[\lambda e^{\nu x} - \frac{3\mu^4 e^{3\mu x}}{(1+\mu^2 e^{2\mu x})^2}]\, dx$$

and show that

$$\liminf_{c\downarrow -\infty\ d\uparrow\infty} I(c,d) = \limsup_{c\downarrow -\infty\ d\uparrow\infty} I(c,d) \geq 0.$$

However, we show that case (iii) of corollary 3.6 can be applied more easily for this example. For this we have

$$A = \int_{-\infty}^{0} e^{-\mu x} dx = \infty, \quad B = \int_0^\infty e^{-\mu x} dx = 1/\mu,$$

$$\log\left(\int_x^\beta r^{-1}(t)dt\right) = \log\left((1/\mu) - \int_0^x e^{-\mu t}dt\right) = -\mu x - \log\mu,$$

$$2\int_0^x e^{-\mu t}\left(\int_0^t \lambda e^{\nu s} ds\right) dt = (2\lambda/\nu)\ \left[\frac{e^{(\nu-\mu)x}}{\nu-\mu} + \frac{e^{-\mu x}}{\mu} - \frac{\nu}{\mu(\nu-\mu)}\right]$$

and hence

$$\begin{aligned} a_2 &= \liminf_{x\uparrow\infty} \frac{(2\lambda/\nu)(e^{(\nu-\mu)x}/(\nu-\mu) + e^{-\mu x}/\mu - \nu/\mu(\nu-\mu)) - \mu x - \log\mu}{(2/\mu)\,(1-e^{-\mu x})} \\ &= +\infty \end{aligned}$$

and

$$\begin{aligned} a_4 &= \limsup_{x\downarrow-\infty} \frac{(\lambda/\nu)\,(e^{(\nu-\mu)x}/(\nu-\mu)+(e^{-\mu x}/\mu)-(\nu/\mu(\nu-\mu)))}{(1-e^{-\mu x})/\mu} \\ &= \limsup_{x\downarrow-\infty} \frac{(\lambda/\nu)(e^{\nu x}/(\nu-\mu)+(1/\mu)-(\nu e^{\mu x}/\mu(\nu-\mu)))}{(e^{\mu x}-1)/\mu} \\ &= -\lambda/\nu < 0. \end{aligned}$$

Hence the equation (3.16) is conjugate on $(-\infty,\infty)$ by case (iii) of corollary 3.6.

Chapter 4

Conjugacy Criteria For A 2n-dimensional Linear Hamiltonian System

In this chapter we present conjugacy criteria for the 2n-dimensional linear Hamiltonian system $(1.1)_v$

$$\begin{aligned} u' &= A(x)u + B(x)v \\ v' &= C(x)u - A^\star(x)v \end{aligned} \qquad (1.1)_v$$

under the hypothesis

$H_v:\ A(x), B(x) = B^\star(x) > 0, C(x) = C^\star(x)$ are $n \times n$ matrices of continuous functions on the interval $I = (\alpha, \beta),\ -\infty \leq \alpha < \beta \leq +\infty$

and for its equivalent second order form $(1.5)_v$

$$L[u] \equiv [P(x)u' + R(x)u]' - [R^\star(x)u' - Q(x)u] = 0 \qquad (1.5)_v$$

under the hypothesis

$\tilde{H}_v:\ P(x) = P^\star(x) > 0, Q(x) = Q^\star(x), R(x)$ are $n \times n$ matrices of continuous functions on the interval $I = (\alpha, \beta),\ -\infty \leq \alpha < \beta \leq +\infty$.

The conjugacy criteria given here are of two types. One type of results concern the system $(1.5)_v$ and are motivated by the works of Etgen and Lewis [21], Hartman [25] and some related references contained therein. More specifically these are comparison theorems giving sufficient conditions for the conjugacy of the vector system $(1.5)_v$ in terms of the conjugacy of a related scalar equation. The second type of results concern the system $(1.1)_v$ and are motivated by the work of Došlý [14] and other related references contained therein.

As for the first type of results we have obtained by means of a generalized Picone type identity (theorem 4.1) a conjugacy criterion (theorem 4.2) which says that the equation $(1.5)_v$ is conjugate if the scalar differential equation, obtained by applying a positive linear functional g to the coefficient matrices of $(1.5)_v$ is conjugate. This is an extension to systems of the form $(1.5)_v$ of the conjugacy criterion (theorem 4.2 of [21]) (see theorem 2.52, chapter 2) to systems of the form

$$(P(x)u')' + Q(x)u = 0. \tag{1.3$_v$}$$

An example (example 4.3) of a 4-dimensional conjugate system of the form $(1.5)_v$ to which theorem 4.2 applies but theorem 4.2 of [21] does not apply is given. It is also shown by means of an example (example 4.8) that a theorem of the form (4.2) is not in general true when applied directly to systems of the form $(1.1)_v$. In theorem 4.6 we show that theorem 4.2 also holds if $g[P]$ and $g[Q]$ in its statement are replaced by $p[P]$ and $q[Q]$ where p and q are positively homogeneous subadditive and superadditive functionals respectively on the vector space of $n \times n$ symmetric matrices. This theorem is an extension to systems of the form $(1.5)_v$ of theorem 1.1 of [25] (see theorem 2.57, chapter 2) which is applicable only to systems of the form $(1.3)_v$. Our example 4.3 also illustrates an instance where theorem 4.6 is applicable but theorem 1.1 of [25] is not.

Regarding the second type of results we have obtained (theorem 4.15) for systems of the form $(1.1)_v$ with arbitrary $A(x)$ an integral criterion for conjugacy in terms of the co-efficient matrices. This result generalizes a theorem of Došlý (theorem 1 of [14]) (see theorem 2.60, chapter 2) for the system

$$u' = B(x)v$$

$$v' = -C(x)u. \tag{1.6$_v$}$$

Supplementary to this theorem example 4.16 illustrates an instance where theorem 4.15 is applicable but theorem 1 of [14] is not.

To begin with we discuss the conjugacy criteria of $(1.5)_v$ on the interval $-\infty \leq \alpha < \beta \leq \infty$. First we obtain sufficient conditions for the equation $(1.5)_v$ to be conjugate on I by using a generalized Picone type identity (theorem 4.1) in the same way as in theorem 4.2 of [21] for the system $(1.3)_v$.

It may be noted that in the case $R(x) \equiv 0$ this identity reduces to the one given in theorem 4.1 of [21] (see theorem 2.51, chapter 2).

Theorem 4.1. Let g be a positive linear functional on $\mathcal{M}$ and $u : [\alpha, \beta] \to \mathbb{R}$ be piecewise continuously differentiable. If $U : [\alpha, \beta] \to \mathcal{M}$ is L-admissible and nonsingular on $[\alpha, \beta]$ then the following identity holds on $[\alpha, \beta]$.

$$g[(u'E_n - uU'U^{-1})^* \, P\,(u'E_n - uU'U^{-1})] + \{u^2 g[(PU' + RU)U^{-1}]\}'$$
$$= u'^2 g[P] + uu'(g[R] + g[R^*]) - u^2 g[Q] + u^2 g[L[U]U^{-1}] \tag{4.1}$$

Proof. This identity is proved by using the obvious linearity property $\{g[X(x)]\}' = g[X'(x)]$ where $X(x) \in \mathcal{M}$ for each x, $\alpha \leq x \leq \beta$.

$$\begin{aligned}
\text{L.H.S. of (4.1)} &= g\left[\left(u'E_n - uU^{*^{-1}}U^{*'}\right)^* P\left(u'E_n - uU'U^{-1}\right)\right] \\
&\quad +2uu'g\left[(PU' + RU)\,U^{-1}\right] + u^2 g\left[(PU' + RU)'\,U^{-1}\right. \\
&\quad \left. - (PU' + RU)\,U^{-1}U'U^{-1}\right] \\
&= g\left[u'^2 P - uu'(PU'U^{-1} + U^{*^{-1}}U^{*'}P) + u^2 U^{*^{-1}}U^{*'}PU'U^{-1}\right] \\
&\quad +2uu'g\left[PU'U^{-1} + R\right] + u^2 g\left[(L[U] + R^*U' - QU)\,U^{-1}\right] \\
&\quad -u^2 g\left[(PU' + RU)\,U^{-1}U'U^{-1}\right] \\
&= u'^2 g[P] + uu'g\left[-PU'U^{-1} - U^{*^{-1}}U^{*'}P + 2PU'U^{-1} + 2R\right]
\end{aligned}$$

$$
\begin{aligned}
&+u^2 g\left[U^{\star^{-1}} U^{\star'} P U' U^{-1} + L[U]U^{-1} + R^{\star} U' U^{-1} - Q\right. \\
&\left. -(PU' + RU)U^{-1}U'U^{-1}\right] \\
= \;& u'^2 g[P] + uu'g\left[PU'U^{-1} - U^{\star^{-1}} U^{\star'} P + 2R\right] \\
&+u^2 g\left[U^{\star^{-1}} U^{\star'} P U' U^{-1} + R^{\star} U' U^{-1}\right. \\
&\left. -(PU' + RU)U^{-1}U'U^{-1}\right] - u^2 g[Q] + u^2 g\left[L[U]U^{-1}\right] \\
= \;& u'^2 g[P] + uu'g\left[U^{\star^{-1}} U^{\star'} P + R^{\star} - R - U^{\star^{-1}} U^{\star'} P + 2R\right] \\
&-u^2 g[Q] + u^2 g\left[L[U]U^{-1}\right] \\
&(\text{since } U \text{ is L-admissible} \\
&PU'U^{-1} \equiv U^{\star^{-1}} U^{\star'} P + R^{\star} - R \text{ and} \\
&(PU' + RU)\, U^{-1}U'U^{-1} = U^{\star^{-1}} U^{\star'} P U' U^{-1} + R^{\star} U' U^{-1}) \\
= \;& u'^2 g[P] + uu'g\left[R + R^{\star}\right] - u^2 g[Q] + u^2 g\left[L[U]U^{-1}\right] \\
= \;& \text{R.H.S. of (4.1)}
\end{aligned}
$$

The following theorem extends theorem 4.2 of [21] (see theorem 2.52, chapter 2) to the equation $(1.5)_v$.

Theorem 4.2. Assume the hypothesis $\tilde{H}_v$. Let g be a positive linear functional on $\mathcal{M}$ satisfying $g(A) = g(A^{\star})\ \forall A \in \mathcal{M}$. If the scalar equation

$$(g[P]u' + g[R]u)' - (g[R^{\star}]u' - g[Q]u) = 0 \qquad (4.2)$$

is conjugate on $[\alpha, \beta]$ then the equation $(1.5)_v$ is conjugate on $[\alpha, \beta]$.

Proof. Suppose $(1.5)_v$ is disconjugate on $[\alpha, \beta]$. Then there exists a prepared solution $U = U(x)$ of $(1.5)_M$ such that $U(x)$ is nonsingular on $[\alpha, \beta]$ (see theorem 2.38, chapter 2). Further by the conjugacy of equation (4.2) on $[\alpha, \beta]$ there exists a nontrivial solution $u(x)$ of (4.2) and two numbers $x_1, x_2 \in [\alpha, \beta]$, $x_1 < x_2$ such that $u(x_1) = u(x_2)$. Hence the equation (4.1) holds on $[\alpha, \beta]$.

On integrating both the sides of this equation from x_1 to x_2 and using the fact that $u(x)$ vanishes at x_1, x_2 and $L[U](x) \equiv 0$ on $[\alpha, \beta]$, we obtain

$$\begin{aligned}
&\int_{x_1}^{x_2} g[(u'E_n - uU'U^{-1})^* P(u'E_n - uU'U^{-1})]dx \\
&= \int_{x_1}^{x_2} \{u'(g[P]u' + g[R]u) \ + \ u(g[R^*]u' - \ g[Q]u)\}dx \\
&= -\int_{x_1}^{x_2} (g[P]u' + g[R]u)'udx \ + \int_{x_1}^{x_2} (g[R^*]u' - g[Q]u)udx \\
&\quad \text{(on integrating by parts the first integral on the R.H.S.)} \\
&= -\int_{x_1}^{x_2} \{(g[P]u' + g[R]u)' \ - \ (g[R^*]u' - g[Q]u)\}udx \\
&= 0 \text{ (since } u \text{ is a solution of (4.2)).}
\end{aligned}$$

However $u(x)$ is nontrivial and $u(x_1) = 0$ implies $u'(x_1) \neq 0$ and hence $\det(u'E_n - uU'U^{-1})(x_1) = (u'(x_1))^n \neq 0$. Consequently, there exists a non-degenerate subinterval J of $[x_1, x_2]$ with x_1 as left end point such that $(u'E_n - uU'U^{-1})$ is nonsingular and hence $(u'E_n - uU'U^{-1})^* P(u'E_n - uU'U^{-1}) > 0$ on J. Now by the hypothesis that g is positive on $\mathcal{M}$ we have

$$\begin{aligned}
0 &= \int_{x_1}^{x_2} g[(u'E_n - uU'U^{-1})^* P(u'E_n - uU'U^{-1})]dx \\
&\geq \int_J g[(u'E_n - uU'U^{-1})^* P(u'E_n - uU'U^{-1})]dx > 0.
\end{aligned}$$

This contradiction shows that $(1.5)_v$ must be conjugate on $[\alpha, \beta]$.

The following example with $R(x) \not\equiv 0$ illustrates theorem 4.2.

Example 4.3. Consider the 2-dimensional second order equation $(1.5)_v$ with

$$P(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R(x) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \text{ and } Q(x) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \qquad (4.3)$$

Let the linear functional g be such that $g(N) = x_0^* N x_0$ where $x_0 = [1, 0]^*$ so that $g[P] = 1$, $g[Q] = 1$ and $g[R] = 0$. Therefore the scalar differential equation (4.2) becomes

$$u'' + u = 0. \tag{4.4}$$

As the scalar differential equation is conjugate on any interval J such that $[0, \pi] \subset J$ the vector differential equation is also conjugate on J by theorem 4.2. On the other hand the conjugacy of the vector equation can be verified directly by noting that

$$U(x) = \begin{bmatrix} k_2 \cos k_1 x - k_1 \sin k_1 x & e^{k_2 x} \\ \dfrac{-k_2 \sin k_1 x - k_1 \cos k_1 x}{k_1^3} & \dfrac{e^{k_2 x}}{k_2^3} \end{bmatrix}$$

with $k_1 = \sqrt{\dfrac{1+\sqrt{5}}{2}}$ and $k_2 = \sqrt{\dfrac{-1+\sqrt{5}}{2}}$ is a prepared solution of the vector system (4.3) and $\det U(x) = e^{k_2 x}\left[\left(\dfrac{1}{k_2^2} + \dfrac{1}{k_1^2}\right)\cos k_1 x + \left(\dfrac{-k_1}{k_2^3} + \dfrac{k_2}{k_1^3}\right)\sin k_1 x\right]$ has infinitely many zeros on $(-\infty, \infty)$. Hence the vector equation is conjugate on sufficiently large J.

The following example shows that if the hypothesis of theorem 4.2 does not hold, then equation $(1.5)_v$ need not be conjugate.

Example 4.4. Consider the 4-dimensional equation with

$$P(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ Q(x) = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } R(x) = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}.$$

Any positive linear functional g is of the form

$g[A] = x_0^\star A x_0$ with $x_0 = \mathrm{col}[x_1\ x_2]$ where x_1 and x_2 are arbitrary constants.

$$\begin{aligned}
\text{Then } g[P] &= [x_1 x_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2, \\
g[Q] &= [x_1 x_2]\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1 x_2]\begin{bmatrix} -2x_1 \\ 0 \end{bmatrix} = -2x_1^2, \\
g[R] &= [x_1 x_2]\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1 x_2]\begin{bmatrix} 0 \\ -x_1 \end{bmatrix} = -x_1 x_2, \\
\text{and } g[R^\star] &= [x_1 x_2]\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1 x_2]\begin{bmatrix} -x_2 \\ \end{bmatrix} = -x_1 x_2.
\end{aligned}$$

Then (4.2) becomes

$$((x_1^2 + x_2^2)u' - x_1x_2u)' - (-x_1x_2u' + 2x_1^2u) = 0$$

$$(x_1^2 + x_2^2)u'' - x_1x_2u' + x_1x_2u' - 2x_1^2u = 0$$

$$(x_1^2 + x_2^2)u'' - 2x_1^2u = 0.$$

$$\text{Therefore } u = c_1 e^{\frac{\sqrt{2}x_1}{\sqrt{x_1^2+x_2^2}}x} + c_2 e^{-\frac{\sqrt{2}x_1}{\sqrt{x_1^2+x_2^2}}x}$$

is the general solution which implies that the scalar equation is disconjugate on $(-\infty, \infty)$ and it can be verified directly that the given vector equation is also disconjugate on $(-\infty, \infty)$, since $U(x) = \begin{bmatrix} e^{-x} & 0 \\ -e^{-x} & 1 \end{bmatrix}$ is a self conjoined nontrivial solution with $\det U(x) \neq 0$.

Our next result is another conjugacy criterion for the equation $(1.5)_v$ in which $R(x)$ is not necessarily the zero matrix. This result is a modified version of that of Hartman (lemma 2.2 of [25]) (see lemma 2.56, chapter 2) for the equation $(1.3)_v$.

Lemma 4.5. Let $0 \not\equiv f \in H_0'[\alpha, \beta]$ be such that the constant symmetric matrix

$$\mathcal{Z} \equiv \int_\alpha^\beta \{f'^2 P + ff'[R + R^*] - f^2 Q\}\, dx$$

is not positive definite. Then $(1.5)_v$ is conjugate on $[\alpha, \beta]$.

Proof. Let $\mathcal{Z}$ be not positive definite. Then there exists a constant vector $z_0 \neq 0$ such that $\mathcal{Z}z_0.z_0 \leq 0$.

Let $u(x) = f(x)z_0$. Then by the definition 2.36 $u(x) \in D_0[\alpha, \beta]$ and the functional $J[u; \alpha, \beta]$, by the definition 2.37 and equation (2.7) is such that

$$\begin{aligned} J[u; \alpha, \beta] &= \int_\alpha^\beta \{[Pu' + Ru](x).u'(x) + [R^*u' - Qu](x).u(x)\} dx \\ &= \int_\alpha^\beta \{[P(x)f'(x)z_0 + R(x)f(x)z_0].f'(x)z_0 \end{aligned}$$

$$
\begin{aligned}
&+[R^*(x)f'(x)z_0 - Q(x)f(x)z_0].f(x)z_0\}dx \\
&= \int_\alpha^\beta \{[f'^2P + ff'R + ff'R^* - f^2Q](x)z_0.z_0\}dx \\
&= \int_\alpha^\beta \left\{f'^2P + ff'(R+R^*) - f^2Q\right\}(x)dx\ z_0.z_0 \\
&= \mathcal{Z}z_0.z_0 \leq 0.
\end{aligned}
$$

Hence by theorem 2.38, chapter 2, $(1.5)_v$ is conjugate on $[\alpha, \beta]$.

The following theorem also gives a conjugacy criterion of $(1.5)_v$ in terms of the conjugacy of a scalar equation obtained from $(1.5)_v$ by applying to its coefficients suitable linear, concave and convex functionals (see definition 2.54, chapter 2).

Theorem 4.6. Let $g : \mathcal{M} \to \mathbb{R}$ be a positive linear functional and $p\ (q)$: $S \rightsquigarrow \mathbb{R}$ be positively homogeneous sub(super) additive functionals. Further let g, p and q satisfy that (i) $g(A) = g(A^*)\ \forall\ A \in \mathcal{M}$, (ii) $q(A) \leq g(A) \leq p(A)\ \forall A \in S$ and (iii) the scalar equation

$$(p[P]u' + g[R]u)' - (g[R^*]u' - q[Q]u) = 0 \tag{4.5}$$

is conjugate on $[\alpha, \beta]$. Then $(1.5)_v$ is conjugate on $[\alpha, \beta]$.

Proof. Since (4.5) is conjugate on $[\alpha, \beta]$ there exists by the variational principle for scalar equations (theorem 5.1, p 233, [39]) (theorem 2.38 (with $n = 1$), chapter 2) a real valued function $f \in H_0'[\alpha, \beta]$, $f \not\equiv 0$ such that the functional.

$$
\begin{aligned}
J[f; \alpha, \beta] &\equiv \int_\alpha^\beta \{(p[P]f' + g[R]f)f' + (g[R^*]f' - q[Q]f)f\}dx \\
&\leq 0.
\end{aligned}
$$

Consequently,

$$0 \geq \int_\alpha^\beta \{f'^2p[P] + ff'(g[R] + g[R^*]) - f^2q[Q]\}dx$$

$$\begin{aligned}
&\geq \int_\alpha^\beta \{f'^2 g[P] + ff'g[R+R^*]\}dx - \int_\alpha^\beta f^2 q[Q]dx \text{ (by the hypothesis (ii))} \\
&\geq \int_\alpha^\beta \{f'^2 g[P] + ff'g[R+R^*]\}dx - q\left[\int_\alpha^\beta f^2 Q dx\right] \text{ (by the inequality (2.13))} \\
&= g\left[\int_\alpha^\beta \{f'^2 P + ff'[R+R^*]\}dx\right] - q\left[\int_\alpha^\beta f^2 Q dx\right] \\
&\geq g\left[\int_\alpha^\beta \{f'^2 P + ff'[R+R^*]\}dx\right] - g\left[\int_\alpha^\beta f^2 Q dx\right] \text{ (by the hypothesis (ii))} \\
&= g\left[\int_\alpha^\beta \{f'^2 P + ff'[R+R^*] - f^2 Q\}dx\right].
\end{aligned}$$

The above inequality and the positivity of g implies that the matrix

$$\int_\alpha^\beta \{f'^2 P + ff'[R+R^*] - f^2 Q\}dx$$

is not positive definite. Hence by lemma 4.5 we have that $(1.5)_v$ is conjugate on $[\alpha, \beta]$.

Remark 4.7. The above theorem holds if in its hypothesis the convexity of p and concavity of q are replaced by the weaker hypothesis of continuity of p and q.

We remark here that a theorem of the type (4.2) does not hold for the vector system $(1.1)_v$ in the following sense, that is we can find a positive linear functional g on $\mathcal{M}$ such that the scalar differential system

$$\begin{aligned}
u' &= g(A)u + g(B)v \\
v' &= g(C)u - g(A^*)v \qquad (4.6)
\end{aligned}$$

is conjugate on an interval I but the vector system $(1.1)_v$ is not conjugate on I. We illustrate this by the following example.

Example 4.8. Consider the 4 dimensional vector system $(1.1)_v$ with

$$A(x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B(x) = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{bmatrix} \text{ and } C(x) = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}. \qquad (4.7)$$

Let g be defined by $g(K) = \mathrm{Tr}(K)$, $K \in \mathcal{M}$, so that $g(A) = 1$, $g(B) = 1$, $g(C) = -2$ and the scalar system (4.6) becomes

$$\begin{aligned} u' &= u + v \\ v' &= -2u - v. \end{aligned}$$

This system is conjugate on $[-\pi/2, \pi/2]$ since

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos x \\ -\cos x - \sin x \end{pmatrix}$$

is a nontrivial solution of (4.6) with $u(x)$ having two zeros on $[-\pi/2, \pi/2]$. However the vector system (4.7) is disconjugate on $(-\infty, \infty)$ and hence on any finite interval since

$$U(x) = \begin{pmatrix} e^{\frac{x}{2}} & 0 \\ 0 & e^{\frac{\sqrt{3}}{2}x} \end{pmatrix}, \quad V(x) = \begin{pmatrix} -2e^{\frac{x}{2}} & 0 \\ 0 & \frac{2}{\sqrt{3}}e^{\frac{\sqrt{3}}{2}x} \end{pmatrix}$$

is a self conjoined solution of (4.7) such that $U(x)$ is nonsingular on $(-\infty, \infty)$.

Further note that the first order 4-dimensional system (4.7) can be expressed in the form $(1.5)_v$ as a second order two dimensional system with

$$P(x) = B^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & \frac{4}{3} \end{bmatrix}, R(x) = -B^{-1}A = \begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix} \text{ and}$$

$$Q(x) = -C - R^{*}P^{-1}R = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

However theorem 4.2 does not apply to this example since with $g(K) = \mathrm{Tr}(K)$ we have $g[P] = \frac{16}{3}$, $g[Q] = -2$ and $g[R] = -4 = g[R^{*}]$, and consequently the second order scalar differential equation (4.2) becomes

$$(\frac{16}{3}u' - 4u)' - (-4u' + 2u) = 0$$

that is, $\frac{16}{3}u'' - 2u = 0$.
This equation is disconjugate on $(-\infty, \infty)$ since

$$u = e^{\sqrt{3/8}x}$$

is a solution with out any zeros on $(-\infty, \infty)$.

Now we discuss the conjugacy criteria for the system $(1.1)_v$ on the interval $-\infty \leq \alpha < \beta \leq +\infty$.

In the following discussion we will be also using the system $(1.1)_v$ with $C(x) \equiv 0$, that is

$$u' = A(x)u + B(x)v$$

$$v' = -A^*(x)v \qquad (1.2)_{a,b}$$

which will be referred to as the 'unperturbed system associated with $(1.1)_v$' under the hypothesis H_v.
Moreover, we need in our proofs the form of the general solution (U, V) of $(1.2)_v$ in terms of a fundamental matrix $U_0(x)$ of the equation

$$u' = A(x)u. \qquad (4.8)$$

We state this and other related results in the following lemma.
Lemma 4.9. Let $U(x), V(x)$ be $n \times r$ matrix valued functions defined on I.
(i) $(U, V) = (U(x), V(x))$ is a solution of $(1.2)_v$ on I if and only if

$$U = U_0(x) \left[M + \int_\gamma^x U_0^{-1}(s)B(s)V(s)ds \right]$$

$$\text{and } V = U_0^{*^{-1}}(x)N$$

where M and N are arbitrary constant $n \times r$ matrices and $\gamma \in I$ is arbitrary but fixed.

(ii) (U, V) is a self conjoined solution of $(1.2)_v$ on I if and only if $M^\star N - N^\star M = 0$.

(iii) Under the hypothesis H_v the system $(1.2)_v$ is identically normal and disconjugate on I.

Proof. (i) This follows from the fact that $U_0^{\star^{-1}}(x)$ is a fundamental matrix of the equation $(1.2)_{v_b}$ and by an application of the variation of constants formula for $(1.2)_{v_a}$.

(ii) This is a consequence of the fact $(U^\star V - V^\star U)(x) \equiv (U^\star V - V^\star U)(\gamma) = M^\star N - N^\star M$.

(iii) The identical normality is a consequence of the hypothesis H_v (see remark 2.22, chapter 2).

To prove disconjugacy, note that if $u(x), v(x)$ are $n \times 1$ vector valued functions then by part (i) of this lemma $(u(x), v(x))$ is a solution of the system $(1.2)_v$ if and only if

$$u(x) = U_0(x)\left[k_2 + \int_\gamma^x U_0^{-1}(s)B(s)v(s)ds\right]$$

$$v(x) = U_0^{\star^{-1}}(x)k_1 \tag{4.9}$$

where k_1 and k_2 are constant $n \times 1$ vectors (see note 2.30, chapter 2). Now if $u(x_2) = 0 = u(x_1)$, $x_1, x_2 \in I$ $(x_1 < x_2)$, we have from equation 4.9 by the nonsingularity of $U_0(x)$

$$\begin{aligned} k_2 + \left(\int_\gamma^{x_1} U_0^{-1}(s)B(s)U_0^{\star^{-1}}(s)ds\right)k_1 &= 0 \\ k_2 + \left(\int_\gamma^{x_2} U_0^{-1}(s)B(s)U_0^{\star^{-1}}(s)ds\right)k_1 &= 0 \end{aligned}$$

The above equations imply $k_1 = 0 = k_2$ by the hypothesis H_v and the assumption $x_1 < x_2$. Thus the system $(1.2)_v$ is disconjugate on I.

In the rest of the chapter for the sake of convenience we shall denote

$$G(x) = U_0^{-1}(x)B(x)U_0^{\star^{-1}}(x) \tag{4.10}$$

Remark 4.10. The matrix $\int_{x_1}^{x_2} G(s)ds$ is nonsingular for every $x_1, x_2 \in (\alpha, \beta)$ with $x_1 < x_2$, by the nonsingularity of $U_0(x)$ and positive definiteness of $B(x)$.

The following lemma gives under hypothesis H_v necessary and sufficient conditions for $(1.2)_v$ to be 0-general (see definition 2.43, chapter 2).

Proposition 4.11. Assume hypothesis H_v and let

$$L_1(x) = \int_x^\gamma G(s)ds, \quad \alpha < x < \gamma$$

$$\text{and} \quad L_2(x) = \int_\gamma^x G(s)ds, \quad \gamma < x < \beta$$

where $\gamma \in I$ is arbitrary but fixed. The system $(1.2)_v$ is 0-general on I if and only if

$$L_1^{-1}(x) \to 0 \quad \text{as} \quad x \to \alpha+$$

and

$$L_2^{-1}(x) \to 0 \quad \text{as} \quad x \to \beta- \tag{4.11$_{a,b}$}$$

Proof. If (4.11) holds it follows from the definition of principal solutions (see theorem 2.39, chapter 2) that

$$(U(x), V(x)) = (U_0(x), 0)$$

is the principal solution at both α and β.

Conversely, assume that the system $(1.2)_v$ is 0-general on I and (4.11) does not hold.

Now consider the following three possible cases

Case (i). $(4.11)_a$ does not hold but $(4.11)_b$ holds.

In this case it can be directly verified that

$$(U_\alpha(x), V_\alpha(x)) = \left(U_0(x)\int_\alpha^x G(s)ds,\ \ U_0^{\star^{-1}}(x)\right)$$

$$\text{and}\ \ (U_\beta(x), V_\beta(x)) = (U_0(x), 0)$$

are the principal solutions at α and β respectively.

Case (ii). $(4.11)_a$ holds but not $(4.11)_b$.

In this case we can verify that

$$(U_\alpha(x), V_\alpha(x)) = (U_0(x), 0)$$

$$\text{and}\ \ (U_\beta(x), V_\beta(x)) = \left(-U_0(x)\int_x^\beta G(s)ds,\ \ U_0^{\star^{-1}}(x)\right)$$

are the principal solutions at α and β respectively.

Case (iii). $(4.11)_a$ and $(4.11)_b$ both do not hold.

In this case $(U_\alpha(x), V_\alpha(x))$ as in case (i) and $(U_\beta(x), V_\beta(x))$ as in case (ii) are the principal solutions at α and β respectively.

Thus in each of the three possible cases we have a contradiction to $(1.2)_v$ being 0-general. Hence the proposition.

Recall that $\lambda_n(M)$ and $\lambda_1(M)$ stand for the smallest and largest eigenvalues respectively of an $n \times n$ Hermitian matrix M; E_r and O_r stand for the $r \times r$ identity and zero matrices respectively. Also we follow the convention that if an $n \times n$ matrix M is written in the form

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

it is meant that the partition is done after the first k rows and k columns.

Lemma 4.12. Assume the hypothesis H_v. Let the system $(1.2)_v$ be k-general on I, $\gamma \in I$ be arbitrary but fixed and (i),(ii) and (iii) be the following statements:

(i) $$\lim_{x\to\beta-} \lambda_n\left(\int_\gamma^x G(s)ds\right) = \infty \tag{4.12}$$

(ii) $$U_\beta(x) = U_0(x), \quad V_\beta(x) = 0_n$$

and

(iii) $$U_\alpha(x) = U_0(x)\left[K + \int_\gamma^x G(s)K_0 ds\right], \quad V_\alpha(x) = U_0^{\star^{-1}}(x)K_0$$

where

$$K_0 = \text{diag }(E_k, 0_{n-k}), \quad K = \begin{pmatrix} K_1 & 0 \\ K_3 & K_4 \end{pmatrix} \tag{4.13}$$

and K_1, K_3 and K_4 are arbitrary constant matrices with K_1 Hermitian and K_1, K_4 nonsingular.

Then (i) and (ii) are equivalent and (ii) implies (iii).

Proof. $(i) \Longleftrightarrow (ii)$. This is a consequence of the definition of the principal solution at β and equation 4.10.

$(ii) \Longrightarrow (iii)$. This follows from (i) and (ii) of lemma 4.9, and the fact that the system $(1.1)_v$ is k-general if and only if the rank of the constant matrix $U_\beta^\star V_\alpha - V_\beta^\star U_\alpha$ is equal to k. and the assumption (ii).

The following Lemma gives some necessary conditions for the system $(1.2)_v$ to be k-general $(0 < k \leq n)$ on I under the hypothesis H_v. It remains unknown whether these conditions are sufficient also.

Lemma 4.13. Assume the hypothesis H_v, the system $(1.2)_v$ is k-general and the assertion (i) of lemma 4.12 holds. Let U_α, V_α be as in lemma 4.12, $\{e_1, ..., e_n\}$ be the standard basis of $\mathbb{R}^n$ and V_{n-k} = linear span $\{e_{k+1}, ..., e_n\}$. Then

(i) $$\lim_{x\to\alpha+} \int_x^\gamma u^\star G(s)u d_s = \infty \ \forall \ (0 \neq) \ u \in V_{n-k} \tag{4.14}$$

and

(ii) $$\lim_{x\to\alpha+} \lambda_{k+j}\left(\int_x^\gamma G(s)ds\right) = \infty \ \text{ for } j = 1, ..., n-k$$

Proof. (i) Let
$$U_0(x) = \begin{bmatrix} U_1(x) & U_2(x) \\ U_3(x) & U_4(x) \end{bmatrix}$$

$$U_0^{-1}(x) = \begin{bmatrix} D_1(x) & D_2(x) \\ D_3(x) & D_4(x) \end{bmatrix}$$

and
$$B(x) = \begin{bmatrix} B_1(x) & B_2(x) \\ B_2^\star(x) & B_3(x) \end{bmatrix}. \qquad (4.15)_{a,b,c}$$

First we obtain a self conjoined solution $(U(x), V(x))$ of the system $(1.2)_v$ satisfying $U_\alpha^\star V - V_\alpha^\star U$ is nonsingular. For this choose

$$N = \begin{bmatrix} 0 & 0 \\ 0 & E_{n-k} \end{bmatrix} \text{ and } M = \begin{bmatrix} E_k & 0 \\ 0 & 0 \end{bmatrix}$$

in lemma 4.9 so that

$$U(x) = U_0(x)\left[M + \int_\gamma^x U_0^{-1}(s)B(s)V(s)ds\right]$$

$$\text{and } V(x) = U_0^{\star^{-1}}(x)N. \qquad (4.16)_{a,b}$$

Since $M^\star N - N^\star M = 0$ it follows from (ii) of lemma 4.9 that $(U(x), V(x))$ is self conjoined.

Further equations (4.13) and (4.16) imply

$$(U_\alpha^\star V - V_\alpha^\star U)(\gamma) = K^\star N - K_0^\star M = \begin{bmatrix} -E_k & K_3^\star \\ 0 & K_4^\star \end{bmatrix} = \text{a nonsingular matrix.}$$

Consequently, the solution $(U(x), V(x))$ satisfies (by theorem 10.5, p 393 of [23]) (see theorem 2.40, chapter 2)

$$\lim_{x \to \alpha} U^{-1}(x)U_\alpha(x) = 0.$$

In particular the (2,2)th block entry of $U^{-1}(x)U_\alpha(x)$ denoted by $S_{2,2}(x) \longrightarrow 0$ as $x \longrightarrow \alpha$.

Now to determine $S_{2,2}(x)$ note that by virtue of equations $(4.15)_b$, and $(4.16)_{a,b}$ the second block row of $U^{-1}(x)$ is given by

$$\left(\left(\int_\gamma^x w_2\right)^{-1} D_3, \left(\int_\gamma^x w_2\right)^{-1} D_4\right)$$

$$\begin{aligned}\text{where } w_2(x) &= (2,2)\text{th entry of } G(x)\\ &= (D_4B_3D_4^\star + D_4B_2^\star D_3^\star + D_3B_2D_4^\star + D_3B_1D_3^\star)(x).\end{aligned}$$

In view of $(4.15)_a$ and (iii) of lemma 4.12 we have

$$\text{2nd column block of } U_\alpha(x) = \text{col}(U_2K_4, U_4K_4)$$

Therefore a direct computation using the relation $D_3U_2 + D_4U_4 \approx E_{n-k}$ gives that

$$S_{2,2}(x) = \left(\int_\gamma^x w_2(s)ds\right)^{-1} K_4$$

Now taking the limit as $x \longrightarrow \alpha$ the nonsingularity of K_4 implies

$$\lambda_1 \left(\int_x^\gamma w_2(s)ds\right)^{-1} \longrightarrow 0 \text{ as } x \longrightarrow \alpha + .$$

$$\text{and hence } \lambda_n \left(\int_x^\gamma w_2(s)ds\right) \longrightarrow +\infty \text{ as } x \longrightarrow \alpha + .$$

Now let $(0 \neq)\ u \in V_{n-k}$ be arbitrary and $u = \text{col}\ (0, c)$ in the partitioned form. Using equations $(4.15)_{b,c}$ we can show by direct computation that $u^\star G(s)u = c^\star w_2(s)c$.

$$\begin{aligned}\text{Hence L.H.S.of (4.14)} &= \lim_{x\to\alpha+} \int_x^\gamma c^\star w_2(s)c\ ds\\ &\geq \lim_{x\to\alpha+} \lambda_n \left(\int_x^\gamma w_2(s)ds\right) = \infty.\end{aligned}$$

(ii) This is a direct consequence of (i) and Courant-Fischer min-max principle (p 115, [6]) (see theorem 2.58, chapter 2).

The following example illustrates lemma 4.13 with $n = 2$ and $k = 1$.

Example 4.14. Consider the system

$$u' = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} u + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} v$$

$$v' = -\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} v \qquad (4.17)_{a,b}$$

on $I = (-\infty, \infty)$ where u, v are 2-dimensional column vectors. It can be easily shown by solving the system that this system is disconjugate on $(-\infty, \infty)$ with

$$\left(\begin{bmatrix} \frac{e^x}{2} & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} e^x & 0 \\ 0 & 0 \end{bmatrix}\right) \text{ and } \left(\begin{bmatrix} e^{-x} & 0 \\ 0 & 1 \end{bmatrix}, 0_2\right)$$

as its principal solutions at $-\infty$ and $+\infty$ respectively and hence is 1-general.

Note that $U_0(x) = \begin{bmatrix} e^{-x} & 0 \\ 0 & 1 \end{bmatrix}$, $U_0^{-1}(x) = \begin{bmatrix} e^x & 0 \\ 0 & 1 \end{bmatrix}$ and $G(x) = \begin{bmatrix} e^{2x} & 0 \\ 0 & 1 \end{bmatrix}$.

Choosing $\gamma = 0$ we have

$$\int_0^x G(s)ds = \begin{bmatrix} \frac{e^{2x}-1}{2} & 0 \\ 0 & x \end{bmatrix}$$

and hence

$$\lim_{x\to\infty} \lambda_2\left(\int_0^x G(s)ds\right) = \lim_{x\to\infty} x = \infty.$$

Thus the system satisfies all the hypotheses of the lemma.

Now $V_1 = \{ce_2 : c \in \mathbb{R}\}$ and

$$\int_x^0 G(s)ds = \begin{bmatrix} \frac{1-e^{2x}}{2} & 0 \\ 0 & -x \end{bmatrix}$$

and hence $\lambda_1(\int_x^0 G(s)ds) = -x \longrightarrow +\infty$ as $x \longrightarrow -\infty$.

Theorem 4.15. Assume the hypothesis H_v, the system $(1.2)_v$ is k-general and the assertion (i) of lemma 4.12 holds. Let $\alpha_1, \beta_1 \in (\alpha, \beta)$ be such that the matrix $C(x)$ is nonpositive definite for $x \in (\alpha, \alpha_1) \cup (\beta_1, \beta)$. If there exists a $(k+1)$-dimensional vector space $V_{k+1} \subset \mathbb{R}^n$ such that

$$\limsup_{\substack{t \to \alpha+ \\ z \to \beta-}} \int_t^z w^\star U_0^\star(x) C(x) U_0(x) w \, dx = l_w < 0 \tag{4.18}$$

$\forall\ (0 \neq)\ w \in V_{k+1}$, then equation $(1.1)_v$ is conjugate on (α, β).

Proof. To prove $(1.1)_v$ is conjugate it is enough by theorem 5.1 (p 337,[39]) (see theorem 2.38, chapter 2) if we show that there exists a pair of vector functions $(u(x), v(x))$ such that

(i) $u(x)$ has compact support in I,

(ii) $u' = A(x)u + B(x)v$ a.e. on I and

$$I(u, v, \alpha, \beta) = \int_\alpha^\beta (v^\star Bv + u^\star Cu)dx < 0.$$

First suppose $-\infty < l_w < 0$ for some $(0 \neq)\ w \in V_{k+1}$.

$$\text{Let } \ l_0 = \max\{(l_w) : w \in V_{k+1},\ w^\star w = 1\}.$$

Note that $l_0 < 0$ and given $\epsilon > 0$ there exist $x_2, x_3 \in I$ such that

(a)
$$\int_{t_1}^{t_2} w^\star U_0^\star(x) C(x) U_0(x) w \, dx \ < \ l_0 + \epsilon \tag{4.19}$$

for every $w \in V_{k+1}$, $w^\star w = 1$ whenever $t_1 \leq x_2$, $t_2 \geq x_3$ and

(b) $C(x)$ is nonpositive definite for $x \in (\alpha, x_2) \cup (x_3, \beta)$ (In case $l_w = -\infty$ for every $w \in V_{k+1}$ then (a) and (b) hold with arbitrary l_0.).

Claim 1. $x_1 \in (\alpha, x_2)$ can be chosen such that

$$\left(\epsilon \int_{x_1}^{x_2} G(s)ds\right) w_m = d_m w_m \tag{4.20}$$

where $w_m (1 \leq m \leq n-k)$ is the unit vector belonging to both $(n-k)$ dimensional vector space V_{n-k} and $(k+1)$-dimensional vector space V_{k+1}. From lemma 4.13 with $\gamma = x_2$ we have

$$\lim_{x \to \alpha+} \int_x^{x_2} w^* G(s) w \, ds = \infty$$

for every $0 \neq w \in V_{n-k}$ where V_{n-k} is as in lemma 4.13 and

$$\lim_{x \to \alpha+} \lambda_{k+j} \left(\int_x^{x_2} G(s) ds \right) = \infty \quad j = 1, \cdots, n-k.$$

Choose $x_1 \in (\alpha, x_2)$ such that

$$\lambda_{k+1} \left(\int_{x_1}^{x_2} G(s) \, ds \right) = 1/\epsilon$$

and let $w_j, \ j = 1, \cdots, (n-k)$ be the unit eigen vector corresponding to the eigenvalue $\lambda_{k+j} \ (\epsilon \int_{x_1}^{x_2} G(s) \, ds)$. Note that at least one of the vectors w_j, say w_m must belong to V_{k+1}, where $1 \leq m \leq n-k$ and let d_m denote the corresponding eigen value. Then w_m and d_m satisfy equation (4.20).

Claim 2. There exist $x_4 \in (x_3, \beta)$ such that

$$w_m^* \left(\int_{x_3}^{x_4} G(s) \, ds \right)^{-1} w_m = \epsilon \tag{4.21}$$

From the equation (4.12) with $\gamma = x_3$ we have

$$\lim_{x \to \beta-} \lambda_n \left(\int_{x_3}^{x} G(s) \, ds \right) = \infty$$

which implies

$$\lim_{x \to \beta-} \lambda_1 \left(\int_{x_3}^{x} G(s) \, ds \right)^{-1} = 0.$$

Consequently

$$\sup_{\|w\|=1} w^* \left(\int_{x_3}^{x} G(s) ds \right) w \longrightarrow 0 \text{ as } x \longrightarrow \beta$$

and inparticular

$$w_m^\star \left(\int_{x_3}^{x} G(s)ds \right)^{-1} w_m \longrightarrow 0 \text{ as } x \longrightarrow \beta - .$$

This implies that equation (4.21) holds for some x_4, $x_3 < x_4 < \beta$ and hence the claim.

Now define a pair of functions $(u(x), v(x))$ as follows.

$$(u(x), v(x)) = \begin{cases} (0,0) & x \in (\alpha, x_1] \\ (\epsilon d_m^{-1} U_0(x) \int_{x_1}^{x} G(s)ds\ w_m, \epsilon d_m^{-1} U_0^{\star^{-1}}(x) w_m) & x \in (x_1, x_2] \\ (U_0(x) w_m, 0) & x \in (x_2, x_3] \\ (U_0(x) \int_x^{x_4} G(s)ds (\int_{x_3}^{x_4} G(s)ds)^{-1} w_m, & \\ -U_0^{\star^{-1}}(x) (\int_{x_3}^{x_4} G(s)ds)^{-1} w_m) & x \in (x_3, x_4] \\ (0,0). & x \in (x_4, \beta) \end{cases}$$

Now it can be easily verified that

(i) $u' = A(x)u + B(x)v$ a.e. on (α, β),

(ii) Support of $u(x) \subset (\alpha, \beta)$ and

$$\begin{aligned} I(u, v, \alpha, \beta) &= I(u, v, x_1, x_4) \\ &= \int_{x_1}^{x_4} (v^\star(x) B(x) v(x) + u^\star(x) C(x) u(x)) dx \\ &< \epsilon d_m^{-1} + \epsilon + \epsilon + l_0 \text{ (by the nonpositivity of } C(x) \\ &\qquad \text{together with the equations (4.19), (4.20) and (4.21).)} \\ &< 3\epsilon + l_0 \text{ (since } d_m \gtrsim 1). \end{aligned}$$

Now we can choose $0 < \epsilon < -l_0/3$ and hence $I(u, v, \alpha, \beta) < 0$ as required.

This completes the proof of the theorem.

The following example illustrates theorem 4.15.

Example 4.16. Let $A(x)$, $B(x)$ be as in example 4.14 and

$$C(x) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

so that the system $(1.2)_v$ is 1-general and $C(x)$ is negative definite on $I = (-\infty, \infty)$.

$V_2 = \text{Span}\,\{e_1, e_2\}$ so that for $t < a < b < z$ we have

$$\int_t^z U_0^* C(x) U_0(x) = \int_t^z \begin{bmatrix} -e^{-2x} & 0 \\ 0 & -1 \end{bmatrix} dx \quad \left(\text{since } U_0(x) = \begin{bmatrix} e^{-x} & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$\begin{aligned} \int_t^z ce_1^* U_0^* C(x) U_0(x) ce_1 \, dx &= \int_t^z c^2(-e^{-2x}) dx \\ &= \frac{c^2}{2}(e^{-2z} - e^{-2t}) \end{aligned}$$

$$\Longrightarrow \limsup_{\substack{t \to -\infty \\ z \to +\infty}} \int_t^z (ce_1^* U_0^* C(x) U_0(x) ce_1) \, dx = -\infty < 0$$

and

$$\int_t^z (ce_2^* U_0^* C(x) U_0(x) ce_2) \, dx = \int_t^z -c^2 \, dx = c^2(t - z)$$

$$\Longrightarrow \limsup_{\substack{t \to -\infty \\ z \to +\infty}} \int_t^z (ce_2^* U_0^* C(x) U_0(x) ce_2) \, dx = -\infty < 0$$

Hence the system is conjugate on $(-\infty, \infty)$ by the theorem. A direct verification also shows that this system is conjugate on $(-\infty, \infty)$ (since

$$(u(x), v(x)) = \left(\begin{pmatrix} 0 \\ \sin x \end{pmatrix}, \begin{pmatrix} 0 \\ \cos x \end{pmatrix} \right)$$

is a vector solution with $u(x) \not\equiv 0$ but has more than two zeros.

Chapter 5

Oscillation Criteria for Linear Hamiltonian Matrix Systems

In this chapter we discuss oscillation criteria for the linear Hamiltonian matrix system

$$\begin{aligned} U' &= A(x)U + B(x)V \\ V' &= C(x)U - A^*(x)V \end{aligned} \qquad (1.1)_M$$

under the hypothesis

H_v : $A(x)$, $B(x) = B^*(x) > 0$ and $C(x) = C^*(x)$ are $n \times n$ matrices of real valued continuous functions on the interval $I = [\alpha, \infty)$, $(-\infty < \alpha)$.

Oscillation criteria given here are of two types. One type of results are motivated by the works of Etgen and Pawlowski [22] and some related references contained therein. More specifically these are modelled on the oscillation criteria due to Wintner [47] for the second order self-adjoint scalar equation

$$(p(x)u')' + q(x)u = 0 \qquad (1.3)$$

($p(x) > 0$ and $q(x)$ continuous on $[\alpha, \infty)$) which states that (1.3) is oscillatory on $[\alpha, \infty)$ if

$$\int^{\infty} p^{-1}(t)dt = \infty \text{ and } \int^{\infty} q(t)dt = \infty.$$

Second type of criteria are based on the results of Erbe, Qingkai Kong and Shigui Ruan [19] and Fanwei Meng, Jizhong Wang and Zhaowen Zheng [34]. These criteria in turn are modelled on the result due to Kamenev [29] which states that the scalar equation

$$u'' + q(x)u = 0 \tag{1.3)$_1$}$$

is oscillatory if for some positive integer $m > 2$ and $t_0 > \alpha$

$$\limsup_{t\to\infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} (t-s)^{m-1} q(s)ds = \infty.$$

In this chapter we first obtain oscillation criteria of Wintner type (theorems 5.1 and 5.2) for the system $(1.1)_M$ which include as special cases, results of Etgen and Pawlowski (theorems 1 and 2, [22]) (see theorems 2.74 and 2.75, chapter 2) for the system

$$U'' + Q(x)U = 0. \qquad (1.3)_{1_M}$$

Next we obtain oscillation criteria of Kamenev type (theorem 5.3 and corollories 5.5-5.8) for the system $(1.1)_M$ which are generalizations of the criteria given by Erbe, Qingkai Kong and Shigui Ruan [19] (see theorem 2.76, chapter 2) for the system

$$(P(x)U')' + Q(x)U = 0. \qquad (1.3)_M$$

Further in theorem 5.9 we generalize a more recent Kamenev type oscillation criterion due to Fanwei Meng, Jizhong Wang and Zhaowen Zheng (theorem 1, [34]) (see theorem 2.77, chapter 2) for the system $(1.3)_{1_M}$ to the more general system $(1.1)_M$.

Lastly, we present a set of six examples, more specifically a pair of examples for each theorem, one to illustrate the theorem and the other to illustrate the significance of the hypotheses of the theorem in the sense that if any one of the hypotheses is not satisfied, the conclusion may not hold. Further example 5.12 demonstrates that theorem 5.2 is a strict generalization of theorem 5.1 as well as the fact that theorem 2 of [22] is a strict generalization of theorem 1 of [22]. Moreover example 5.14 illustrates an instance where theorem 1 of [34] is not applicable but theorem 5.9 is applicable.

The following theorem extends to the system $(1.1)_M$ the theorem 1 of Etgen and Pawlowski [22] (theorem 2.74, chapter 2) for the system $(1.3)_{1_M}$.

Theorem 5.1. If there exists a positive linear functional g on M such that

$$\lim_{x\to\infty}\int^{x}\frac{1}{g[B^{-1}(s)]}ds=\infty \tag{5.1}$$

and

$$\lim_{x\to\infty} g\left[-\int^{x}(C+A^{\star}B^{-1}A)(s)ds-B^{-1}(x)A(x)\right]=\infty \tag{5.2}$$

then the system $(1.1)_M$ is oscillatory on $[\alpha,\infty)$.

Proof. Suppose the hypotheses (5.1) and (5.2) hold and $(1.1)_M$ is not oscillatory on $[\alpha,\infty)$. Then we arrive at a contradiction.

By our assumption and definition 2.61 there exists a nontrivial, prepared solution $(U(x),\ V(x))$ of $(1.1)_M$ such that $U(x)$ is nonsingular on $[\beta,\infty)$ for some $\beta>\alpha$.

Let $W(x)=-V(x)U^{-1}(x)$. Then $W(x)$ is well defined, Hermitian and satisfies the Riccati equation

$$W'(x)+A^{\star}(x)W(x)+W(x)A(x)-W(x)B(x)W(x)+C(x)=0.$$

on $[\beta,\infty)$ by virtue of (theorem 5.1, p 337 [39]) (see theorem 2.38, chapter 2).

On integrating both the sides of the above equation from β to x and rearranging the sides we obtain

$$W(x)-W(\beta)=\int_{\beta}^{x}(WBW-A^{\star}W-WA)(t)dt+\int_{\beta}^{x}(-C(t))dt.$$

Now the substitution

$$P(x)=W(x)-B^{-1}(x)A(x)$$

in the above equation gives us

$$P(x) = W(\beta) + \int_\beta^x (P^\star BP)(t)dt + \int_\beta^x -(C + A^\star B^{-1}A)(t)dt - B^{-1}(x)A(x)$$

by virtue of the relation

$$(P^\star BP)(t) = (WBW - A^\star W - WA + A^\star B^{-1}A)(t).$$

Consequently on applying the linear functional g to both sides of the integral equation given above we obtain

$$\begin{aligned} g[P(x)] &= g[W(\beta)] + g\left[\int_\beta^x (P^\star BP)(t)dt\right] + g\left[\int_\beta^x -(C + A^\star B^{-1}A)(t)dt\right] \\ &\quad -g[B^{-1}(x)A(x)]. \end{aligned} \tag{5.3}$$

Now the hypothesis (5.2) implies that there exists $\gamma > \beta$ such that

$$g[W(\beta)] + g\left[\int_\beta^x -(C + A^\star B^{-1}A)(t)dt\right] - g[(B^{-1}A)(x)] > 0 \quad \text{on} \quad [\gamma, \infty)$$

and hence by (5.3) and the hypothesis H_v we have on the interval $[\gamma, \infty)$ the inequalities

$$g[P(x)] > g\left[\int_\beta^x (P^\star BP)(t)dt\right] \tag{5.4}$$

$$\text{and} \quad g[P(x)] > 0. \tag{5.5}$$

We now claim that for $x \in [\gamma, \infty)$

$$g[(P^\star BP)(x)] \geq \{g[B^{-1}(x)]\}^{-1} \{g[P(x)]\}^2 > 0. \tag{5.6}$$

For this it suffices to show by virtue of (5.5) that on $[\gamma, \infty)$

$$g[B^{-1}(x)]\, g[(P^\star BP)(x)] \geq \{g[P(x)]\}^2 .$$

This is however true since

$$\begin{aligned} g[B^{-1}(x)]g[(P^*BP)(x)] &= g[(B^{-\frac{1}{2}*}B^{-\frac{1}{2}})(x)]g[(B^{\frac{1}{2}}P)^*(B^{\frac{1}{2}}P)(x)] \\ &\geq \{g[(B^{-\frac{1}{2}}B^{\frac{1}{2}}P)(x)]\}^2 \text{ (by lemma 2.73)} \\ &\geq \{g[P(x)]\}^2 > 0 \text{ for all } x \text{ in } [\gamma, \infty). \end{aligned}$$

Hence the claim is true.

Now defining

$$Q(x) = \int_\beta^x (P^*BP)(t)dt, \quad x \in [\gamma, \infty)$$

we have by (5.4)

$$\begin{aligned} g[P(x)] &> g[Q(x)] \qquad (5.7) \\ \text{and } g[Q(x)] &= g\left[\int_\beta^x (P^*BP)(t)dt\right] = \int_\beta^x g[(P^*BP)(t)]dt \\ &\geq \int_\beta^x \{g[B^{-1}(t)]\}^{-1}\{g[P(t)]\}^2 dt \text{ (by (5.6))} \\ &> 0 \text{ on } [\gamma, \infty) \text{ (by (5.5))}. \\ \text{Further } \{g[Q(x)]\}' &= g[Q'(x)] \text{ (by the linearity of } g) \\ &= g[(P^*BP)(x)] \text{ (by the definition of } Q(x)) \\ &\geq \frac{\{g[P(x)]\}^2}{g[B^{-1}(x)]} \text{ (by (5.6))} \\ &> \frac{\{g[Q(x)]\}^2}{g[B^{-1}(x)]} \text{ on } [\gamma, \infty) \text{ (by (5.7)). Hence} \\ \frac{1}{g[B^{-1}(x)]} &< \frac{\{g[Q(x)]\}'}{\{g[Q(x)]\}^2} \text{ on } [\gamma, \infty). \qquad (5.8) \end{aligned}$$

Now integrating both the sides of the above inequality from γ to x we obtain

$$\begin{aligned} \int_\gamma^x \frac{1}{g[B^{-1}(t)]}dt &< \frac{1}{g[Q(\gamma)]} - \frac{1}{g[Q(x)]} \\ &< \frac{1}{g[Q(\gamma)]} \text{ (since } g[Q(x)] > 0 \text{ on } [\gamma, \infty)). \end{aligned}$$

The above inequality holds for all $x > \gamma$ and thus we have a contradiction to the hypothesis (5.1). This completes the proof of the theorem.

The next theorem is an extension to systems of the form $(1.1)_M$ of the theorem 2 of Etgen and Pawlowski [22] for the special case of the form $(1.3)_{1_M}$. In this theorem we let $I = [0, \infty)$ in the hypothesis H_v.

Theorem 5.2. Suppose there exists a positive function $a(x) \in C^1[0,\infty)$ and a positive linear functional g on M such that

$$\lim_{x\to\infty} \int_0^x \frac{1}{a(t)g[B^{-1}(t)]} dt = \infty \tag{5.9}$$

Further suppose the operator J defined by

$$\begin{aligned} J(x) &= \int_0^x \left\{ \frac{a'}{2}(A^*B^{-1} + B^{-1}A) - \frac{a'^2}{4a}B^{-1} - a(A^*B^{-1}A + C) \right\}(t)dt \\ &\quad -a(x)B^{-1}(x)A(x) + \frac{a'(x)}{2}B^{-1}(x) \end{aligned} \tag{5.10}$$

has the property that

$$\lim_{x\to\infty} g[J(x)] = \infty. \tag{5.11}$$

Then the system $(1.1)_M$ is oscillatory on $[0, \infty)$.

Proof. Suppose the hypotheses (5.9), (5.10) and (5.11) hold and $(1.1)_M$ is not oscillatory. Then we arrive at a contradiction.

Let $(U(x), V(x))$ be any nontrivial prepared solution of $(1.1)_M$. Then $U(x)$ is nonsingular on the interval $[\alpha, \infty)$ for some $\alpha > 0$ and the operator

$$S(x) = -(aVU^{-1})(x)$$

exists on $[\alpha, \infty)$.

Note that $S(x)$ is Hermitian (since $(U(x), V(x))$ is prepared) and satisfies on $[\alpha, \infty)$ the equation

$$S'(x) = (-a'VU^{-1} - aV'U^{-1} + aVU^{-1}U'U^{-1})(x)$$

$$
\begin{aligned}
&= (-a'VU^{-1} - a[CU - A^*V]U^{-1} + aVU^{-1}[AU + BV]U^{-1})(x) \\
&= (-a'VU^{-1} - aC + aA^*VU^{-1} + aVU^{-1}A + aVU^{-1}BVU^{-1})(x) \\
&= \left(\frac{a'S}{a} - aC - A^*S - SA + \frac{SBS}{a}\right)(x). \text{ That is,}
\end{aligned}
$$

$$
S'(x) = \left(\frac{1}{a}\left[a'S - a\{A^*S + SA\} + SBS\right] - aC\right)(x) \text{ on } [\alpha, \infty). \quad (5.12)
$$

Define

$$
R(x) = \left(S - aB^{-1}A + \frac{a'}{2}B^{-1}\right)(x), \quad \text{for } x \geq \alpha. \quad (5.13)
$$

We claim

$$
\begin{aligned}
S'(x) &= \frac{1}{a}\left[R^*BR - a^2A^*B^{-1}A + \frac{aa'}{2}\{A^*B^{-1} + B^{-1}A\} - \frac{a'^2}{4}B^{-1}\right](x) \\
&\quad -a(x)C(x), \quad \text{for } x \geq \alpha \quad (5.14)
\end{aligned}
$$

Note that

$$
\begin{aligned}
(R^*BR)(x) &= \left[S - aA^*B^{-1} + \frac{a'}{2}B^{-1}\right]B\left[S - aB^{-1}A + \frac{a'}{2}B^{-1}\right](x) \\
&= (SBS - a[SA + A^*S] + a'S + a^2A^*B^{-1}A \\
&\quad -\frac{aa'}{2}[A^*B^{-1} + B^{-1}A] + \frac{a'^2}{4}B^{-1})(x).
\end{aligned}
$$

Consequently

$$
\begin{aligned}
&\left(R^*BR - a^2A^*B^{-1}A + \frac{aa'}{2}[A^*B^{-1} + B^{-1}A] - \frac{a'^2}{4}B^{-1}\right)(x) \\
&= (SBS - a[SA + A^*S] + a'S)(x) \\
&= a(x)[S' + aC](x) \text{ by } (5.12).
\end{aligned}
$$

Hence the claim.

Now integrating from α to x both sides of (5.14) we obtain

$$
S(x) = S(\alpha) + \int_\alpha^x \left(\frac{1}{a}R^*BR\right)(t)dt
$$

$$+\int_{\alpha}^{x}\left\{\frac{a'}{2}(A^*B^{-1}+B^{-1}A)-a(A^*B^{-1}A+C)-\frac{a'^2}{4a}B^{-1}\right\}(t)dt.$$

Thus by (5.13) for $x \geq \alpha$ we have

$$\begin{aligned}R(x) &= S(\alpha)+\int_{\alpha}^{x}\left(\frac{1}{a}R^*BR\right)(t)dt\\&+\int_{\alpha}^{x}\left\{\frac{a'}{2}(A^*B^{-1}+B^{-1}A)-a(A^*B^{-1}A+C)-\frac{a'^2}{4a}B^{-1}\right\}(t)dt\\&-a(x)B^{-1}(x)A(x)+\frac{a'(x)}{2}B^{-1}(x).\end{aligned}$$

Consequently by the definition 5.10 of $J(x)$ and on denoting

$$P(\alpha)=\int_{0}^{\alpha}\left\{\frac{a'}{2}(A^*B^{-1}+B^{-1}A)-a(A^*B^{-1}A+C)-\frac{a'^2}{4a}B^{-1}\right\}(t)dt$$

we obtain

$$R(x)=S(\alpha)+\int_{\alpha}^{x}\left(\frac{1}{a}R^*BR\right)(t)dt+J(x)-P(\alpha) \qquad (5.15)$$

and hence

$$g[R(x)]=g[S(\alpha)-P(\alpha)]+g\left[\int_{\alpha}^{x}\left(\frac{1}{a}R^*BR\right)(t)dt\right]+g[J(x)].$$

This implies

$$g[R(x)]\geq g[S(\alpha)-P(\alpha)]+g[J(x)] \text{ (since } g \text{ is positive on } M).$$

Further by the property (5.11) of g it follows that there exists $\beta \geq \alpha$ such that

$$\begin{aligned}g[S(\alpha)-P(\alpha)]+g[J(x)] &> 0 \text{ on } [\beta,\infty).\\ \text{Hence } g[R(x)] &> 0 \text{ on } [\beta,\infty)\\ \text{and } g[R(x)] &> g\left[\int_{\alpha}^{x}\left(\frac{1}{a}R^*BR\right)(t)dt\right].\end{aligned}$$

Define an operator $Z(x)$ for $x \geq \beta$ by

$$Z(x) = \int_{\alpha}^{x} \left(\frac{1}{a} R^{*} B R\right)(t) dt$$

so that for $x \geq \beta$ we have

$$\begin{aligned}
(i) \qquad g[R(x)] &> g[Z(x)] \qquad (5.16) \\
(ii) \qquad g[Z(x)] &= \int_{\alpha}^{x} \left(\frac{1}{a} g[R^{*} B R]\right)(t) dt \text{ (since } g \text{ is linear)} \\
&\geq \int_{\alpha}^{x} \frac{\{g[R(t)]\}^{2} dt}{a(t) g[B^{-1}(t)]} > 0 \text{ (by lemma 2.73)} \\
\text{and } (iii) \qquad \{g[Z(x)]\}' &= g[Z'(x)] = g\left[\left(\frac{1}{a} R^{*} B R\right)(t)\right] \\
&\geq \frac{\{g[R(x)]\}^{2}}{a(x) g[B^{-1}(x)]} \text{ (by lemma 2.73)} \\
&> \frac{\{g[Z(x)]\}^{2}}{a(x) g[B^{-1}(x)]} \text{ (by (5.16)).}
\end{aligned}$$

$$\text{Hence } \frac{1}{a(x) g[B^{-1}(x)]} < \frac{\{g[Z(x)]\}'}{g[Z(x)]^{2}}.$$

On integrating from β to x both sides of the above inequality we obtain

$$\begin{aligned}
\int_{\beta}^{x} \frac{1}{a(t) g[B^{-1}(t)]} dt &< \frac{1}{g[Z(\beta)]} - \frac{1}{g[Z(x)]} \\
&< \frac{1}{g[Z(\beta)]} \quad (\text{since } g[Z(x)] > 0 \text{ on } [\beta, \infty))
\end{aligned}$$

which is a contradiction to (5.9). Hence $(1.1)_M$ is oscillatory on $[\alpha, \infty)$.

The following theorem is an extension of theorem 1 of [19] (see theorem 2.76, chapter 2) to the system $(1.1)_M$.

Theorem 5.3. Let $g(x, s)$ be a real valued function such that it is continuous on $D = \{(x, s) : x \geq s \geq \alpha \geq 0\}$, $g(x, s) > 0$ for $x > s \geq \alpha$, $g(x, x) = 0$ for $x \geq \alpha$ and the partial derivative $g_s(x, s)$ is nonpositive and continuous for $x \geq s \geq \alpha$ (for instance $(x - s)^r$, $r > 1$, $\ln(x/s)$, $\rho(x - s)$ where

$\rho \in C^1[0,\infty] > 0$, $\rho'(u) > 0$ for $u > 0$ and $\rho(0) = 0$ are possible choices for $g(x,s)$). Let $h(x,s)$ and $H(x)$ be real valued and matrix valued functions respectively defined by

$$\begin{aligned} g_s(x,s) &= -h(x,s)g^{\frac{1}{2}}(x,s),\ x > s \geq \alpha & (5.17) \\ \text{and } H(x) &= -\int_\alpha^x \{g(x,s)[C + A^*B^{-1}A](s) \\ &\quad + \frac{1}{2}h(x,s)g^{\frac{1}{2}}(x,s)[A^*B^{-1} + B^{-1}A] \\ &\quad + \frac{1}{4}h^2(x,s)B^{-1}\Big\}(s)ds,\ x \geq \alpha. & (5.18) \end{aligned}$$

If

$$\limsup_{x\to\infty} \frac{1}{g(x,\alpha)}\lambda_1[H(x)] = \infty \tag{5.19}$$

holds then the system $(1.1)_M$ is oscillatory on $[\alpha,\infty)$.

It will now be convenient to state as a lemma, two identities (which can be easily verified) needed in the proof of the theorem.

Lemma 5.4. Let g, h, H, A, B and C be as in theorem 5.3 with $R(x) = B^{\frac{1}{2}}(x)$ and $W(x)$ an arbitrary Hermitian $n \times n$ matrix-valued function for $x \geq \alpha$. Also let

$$\begin{aligned} G(x,s) &= h(x,s)g^{\frac{1}{2}}(x,s)(RWR)(s) + g(x,s)(R[WA + A^*W]R)(s) \\ &\quad + g(x,s)[RWR][RWR](s) \\ \text{and } Q(x,s) &= g^{\frac{1}{2}}(x,s)[RW + R^{-1}A](s)R(s) + \frac{1}{2}h(x,s)E_n \text{ for } x > s \geq \alpha. \end{aligned}$$

Then the following identities hold for $x > s \geq \alpha$.

$$\begin{aligned} (i)\ & h(x,s)g^{\frac{1}{2}}(x,s)W(s) + g(x,s)[WA + A^*W + WBW](s) \\ &= R^{-1}(s)G(x,s)R^{-1}(s) \end{aligned}$$

$$= R^{-1}(s)(Q^*Q)(x,s)R^{-1}(s) - \{g(x,s)(A^*B^{-1}A)(s)$$
$$+\frac{1}{2}h(x,s)g^{\frac{1}{2}}(x,s)[A^*B^{-1}+B^{-1}A](s)+\frac{1}{4}h^2(x,s)B^{-1}(s)\}$$

and

$$(ii) \quad \int_\alpha^x R^{-1}(s)G(x,s)R^{-1}(s)ds = \int_\alpha^x R^{-1}(s)(Q^*Q)(x,s)R^{-1}(s)ds + H(x) + \int_\alpha^x g(x,s)C(s)ds.$$

Proof of theorem 5.3. Suppose that the hypothesis (5.19) holds and the system $(1{\cdot}1)_M$ is not oscillatory on $[\alpha,\infty)$. Then there exists a prepared solution $(U(x),V(x))$ of $(1.1)_M$ which we may suppose without loss of generality, satisfies that $\det U(x)\neq 0$ for $x\geq\alpha$.

Define for $x\geq\alpha$

$$W(x)=V(x)U^{-1}(x).$$

Then on $[\alpha,\infty)$ $W(x)$ satisfies the Riccati equation

$$W'(x)+W(x)A(x)+A^*(x)W(x)+W(x)B(x)W(x)-C(x)=0.$$

On multiplying the Riccati equation (with x replaced by s) by $g(x,s)$, integrating with respect to s from α to x and rearranging the terms we obtain for $x>\alpha$

$$\begin{aligned}\int_\alpha^x g(x,s)C(s)ds &= \int_\alpha^x g(x,s)W'(s)ds \\ &\quad + \int_\alpha^x g(x,s)[WA+A^*W+WBW](s)ds \\ &= -g(x,\alpha)W(\alpha)+\int_\alpha^x h(x,s)g^{\frac{1}{2}}(x,s)W(s)ds \\ &\quad + \int_\alpha^x g(x,s)[WA+A^*W+WBW](s)ds\end{aligned}$$

(on integration by parts and using (5.17))

$$
\begin{aligned}
&= -g(x,\alpha)W(\alpha) + \int_\alpha^x R^{-1}(s)(Q^*Q)(x,s)R^{-1}(s)ds \\
&\quad + H(x) + \int_\alpha^x g(x,s)C(s)ds \\
&\quad \text{(by } (i) \text{ and } (ii) \text{ of lemma 5.4)}
\end{aligned}
$$

yielding

$$
\begin{aligned}
H(x) &= g(x,\alpha)W(\alpha) - \int_\alpha^x R^{-1}(s)(Q^*Q)(x,s)R^{-1}(s)ds \\
&\leq g(x,\alpha)W(\alpha).
\end{aligned}
$$

Therefore

$$\lambda_1[H(x)] \leq \lambda_1[g(x,\alpha)W(\alpha)]$$

implying

$$\frac{1}{g(x,\alpha)}\lambda_1[H(x)] \leq \lambda_1[W(\alpha)] \text{ for } x > \alpha, \tag{5.20}$$

a contradiction to (5.19). This completes the proof of theorem 5.3.

Corollory 5.5. Let $g(x,s)$, $h(x,s)$ and $H(x)$ be as in theorem 5.3 and define the matrix valued function

$$G_1(x) = H(x) + \int_\alpha^x \frac{1}{4}h^2(x,s)B^{-1}(s)ds.$$

(i) If $$\limsup_{x\to\infty}\frac{1}{g(x,\alpha)}\lambda_1[G_1(x)] = \infty \tag{5.21}$$

and

$$\limsup_{x\to\infty}\frac{1}{g(x,\alpha)}\lambda_1\left[\int_\alpha^x h^2(x,s)B^{-1}(s)ds\right] < \infty \tag{5.22}$$

then the system $(1.1)_M$ is oscillatory on $[\alpha,\infty)$.

(ii). In particular if $B(x) = \text{diag}(b_1(x), b_2(x), ..., b_n(x))$ where $b_i(x)$ are continuous and positive for $x \geq \alpha$ and $b(x) = \min_{1<i<n}\{b_i(x)\}$ then the condition (5.22) can be replaced by the condition

$$\limsup_{x\to\infty}\frac{1}{g(x,\alpha)}\int_\alpha^x \frac{h^2(x,s)}{b(s)}ds < \infty. \tag{5.23}$$

(iii). Further if $B(x) = E_n$ and the conditions (5.21) and (5.23) are respectively replaced by

$$\limsup_{x\to\infty} \frac{1}{g(x,\alpha)}\lambda_1 \left[-\int_\alpha^x \{g(x,s)(C+A^*A)(s) + \frac{1}{2}h(x,s)g^{\frac{1}{2}}(x,s)[A^*+A](s)\}ds\right] = \infty \tag{5.24}$$

and

$$\limsup_{x\to\infty} \frac{1}{g(x,\alpha)} \int_\alpha^x h^2(x,s)ds < \infty \tag{5.25}$$

then the system $(1.1)_{1_M}$ is oscillatory on $[\alpha, \infty)$.

Proof. (i) By the definition of $G_1(x)$ and lemma 2.72, we have

$$\begin{aligned}\lambda_1[H(x)] &\geq \lambda_1[G_1(x)] + \lambda_n\left[-\int_\alpha^x \frac{1}{4}h^2(x,s)B^{-1}(s)ds\right] \\ &= \lambda_1[G_1(x)] - \lambda_1\left[\int_\alpha^x \frac{1}{4}h^2(x,s)B^{-1}(s)ds\right] \\ &\geq \lambda_1[G_1(x)] - \limsup \lambda_1\left[\int_\alpha^x \frac{1}{4}h^2(x,s)B^{-1}(s)ds\right] - 1 \\ &\quad \text{for sufficiently large } x.\end{aligned}$$

Therefore

$$\limsup \frac{1}{g(x,\alpha)}\lambda_1[H(x)] = \infty \quad \text{(by (5.21) and (5.22))}$$

and hence $(1.1)_M$ is oscillatory on $[\alpha, \infty)$.

The second and the third parts are consequences of the first.

Corollory 5.6. Let $m > 2$ be an integer and assume that

$$\limsup_{x\to\infty} \frac{1}{x^{m-1}}\lambda_1\left[-\int_\alpha^x \{(x-s)^{m-1}(C+A^*A)(s) + \frac{1}{2}(m-1)(x-s)^{m-2}[A^*+A](s)\}ds\right] = \infty. \tag{5.26}$$

Then the system $(1.1)_{1_M}$ is oscillatory on $[\alpha, \infty)$.

Proof. With $g(x,s) = (x-s)^{m-1}$ we have $h(x,s) = (m-1)(x-s)^{\frac{m-3}{2}}$ for $x \geq s \geq \alpha$ and

$$\frac{1}{g(x,\alpha)} \int_\alpha^x h^2(x,s)ds = \frac{(m-1)^2}{(m-2)(x-a)}.$$

Hence

$$\begin{aligned}
\text{l.h.s of } (5.25) &= \lim_{x\to\infty} \frac{(m-1)^2}{(m-2)(x-a)} = 0 \\
\text{and l.h.s of } (5.24) &= \limsup_{x\to\infty} \frac{1}{(x-a)^{m-1}} \lambda_1 \left[-\int_\alpha^x \{(x-s)^{m-1}(C + A^\star A)(s) \right. \\
&\quad \left. + \frac{1}{2}(m-1)(x-s)^{m-2}[A^\star + A](s)\} ds \right] \\
&= \infty \ (\text{by } (5.26)).
\end{aligned}$$

Hence by corollory 5.5 the system $(1.1)_{1_M}$ is oscillatory on $[\alpha, \infty)$.

Corollory 5.7. Assume that there exists a C^1 function $\rho(u)$ on $[0, \infty)$, with $\rho(0) = 0$, $\rho(u) \geq 0$ for $u > 0$ such that

$$\begin{aligned}
&\limsup_{x\to\infty} \frac{1}{\rho(x)} \lambda_1 \left[-\int_\alpha^x \{\rho(x-s)(C + A^\star A)(s) \right. \\
&\qquad \left. + \frac{1}{2}\rho'(x-s)[A^\star + A](s)\} ds \right] = \infty \qquad (5.27)
\end{aligned}$$

and

$$\limsup_{x\to\infty} \frac{1}{\rho(x)} \int_0^x \frac{[\rho'(x-s)]^2}{\rho(x-s)} ds < \infty. \qquad (5.28)$$

Then the system $(1.1)_{1_M}$ is oscillatory on $[0, \infty)$.

Proof. With $g(x,s) = \rho(x-s)$ as stated in theorem 5.3 the conditions (5.24) and (5.25) respectively reduce to (5.27) and (5.28).

Corollory 5.8. Assume that $m > 2$ is an integer with

$$\limsup_{x\to\infty} \frac{1}{(\ln x)^{m-1}} \lambda_1 \left[-\int_\alpha^x \left\{ \left[\ln \frac{x}{s}\right]^{m-1} (C + A^\star A)(s) + \frac{1}{2}\left(\frac{m-1}{s}\right) \left[\ln \frac{x}{s}\right]^{m-1} (A^\star + A)(s) \right\} ds \right] = \infty \quad (5.29)$$

Then the system $(1.1)_{1_M}$ is oscillatory on $[0, \infty)$.

Proof. Choose $g(x,s) = \left[\ln \frac{x}{s}\right]^{m-1}$ so that $h(x,s) = \frac{m-1}{s}\left[\ln \frac{x}{s}\right]^{(m-3)/2}$ for $x \geq s \geq \alpha > 1$. Then

$$\begin{aligned}
\text{l.h.s of } (5.25) &= \limsup_{x\to\infty} \frac{1}{(\ln(x/\alpha))^{m-1}} \int_\alpha^x \frac{(m-1)^2}{s^2} \left[\ln \frac{x}{s}\right]^{\frac{m-3}{2}} ds \\
&< \infty
\end{aligned}$$

$$\text{and l.h.s. of } (5.24) = \infty \text{ (by (5.29))}.$$

Therefore by corollory 5.5 the result follows.

The following theorem is an extension to systems of the form $(1.1)_M$ of the theorem 1 of Fanwei Meng, Jizhong Wang and Zhaowen Zheng [34] (see theorem 2.77, chapter 2) for the special case of the form $(1.3)_{1_M}$.

Theorem 5.9. Let $g(x,s)$ and $h(x,s)$ be as in theorem 5.3. Suppose there exists a function $f \in \mathcal{C}^1[\alpha, \infty]$ such that

$$\limsup_{x\to\infty} \frac{1}{g(x,\alpha)} \lambda_1 \left[\int_\alpha^x \{g(x,s)T(s) + H_1(x,s)\} ds \right] = \infty \quad (5.30)$$

where

$$\begin{aligned}
H_1(x,s) &= g(x,s)[bf(A + A^\star) - bA^\star B^{-1} A](s) \\
&\quad - b(s)\left[\frac{1}{2} h(x,s) g^{\frac{1}{2}}(x,s) + f(s) g(x,s)\right] [A^\star B^{-1} + B^{-1} A](s)
\end{aligned}$$

$$-b(s)\left[\left\{\frac{1}{2}h(x,s)+f(s)g^{\frac{1}{2}}(x,s)\right\}B^{\frac{-1}{2}}-f(s)g^{\frac{1}{2}}(x,s)B^{\frac{1}{2}}(s)\right]^2,$$

$$\begin{aligned} b(x) &= \exp\{-2\int_\alpha^x f(s)ds\} \text{ and} \\ T(x) &= b(x)[-C-f(A+A^\star)+f^2B-f'E_n](x). \end{aligned} \tag{5.31}$$

Then the system $(1.1)_M$ is oscillatory.

Proof. Suppose that hypothesis (5.30) holds and the system $(1.1)_M$ is not oscillatory. Then there exists a prepared solution $(U(x),V(x))$ of $(1.1)_M$ concerning which, without loss of generality we may assume that $\det U(x)\neq 0$ for $x\geq\alpha$.

Define for $x\geq\alpha$

$$W(x)=b(x)[V(x)U^{-1}(x)+f(x)E_n].$$

Then $W(x)$ satisfies on $[\alpha,\infty)$ the Riccati equation

$$\left(W'+WA+A^\star W+\frac{WBW}{b}-f[WB+BW-2W]+T\right)(x)=0.$$

On multiplying the Riccati equation (with x replaced by s) by $g(x,s)$, integrating by parts with respect to s from α to x and rearranging the terms as in the proof of theorem 5.3 we obtain for $x>\alpha$

$$\begin{aligned} \int_\alpha^x g(x,s)T(s)ds &= -\int_\alpha^x g(x,s)W'(s)ds-\int_\alpha^x\left\{\frac{g(x,s)}{b(s)}[WBW](s)\right. \\ &\quad \left.+g(x,s)[A^\star W+WA-f(WB+BW-2W)](s)\right\}ds \\ &= g(x,\alpha)W(\alpha)-\int_\alpha^x\left\{\frac{g(x,s)}{b(s)}[WBW](s)\right. \\ &\quad +g(x,s)[A^\star W+WA-f(WB+BW)](s) \\ &\quad \left.+[h(x,s)g^{\frac{1}{2}}(x,s)+2f(s)g(x,s)]W(s)\right\}ds \\ &= g(x,\alpha)W(\alpha)-\int_\alpha^x\{(Q_1^\star Q_1)(x,s)+H_1(x,s)\}ds \end{aligned}$$

on defining $Q_1(x,s)$ by

$$\begin{aligned} Q_1(x,s) &= \left\{\frac{g(x,s)}{b(s)}\right\}^{\frac{1}{2}}(RW)(s) - (b(s)g(x,s))^{\frac{1}{2}}\{fR - R^{-1}A\}(s) \\ &\quad + \left(\frac{1}{2}b^{\frac{1}{2}}(s)h(x,s) + [b(s)g(x,s)]^{\frac{1}{2}}f(s)\right)R^{-1}(s) \end{aligned}$$

where $R(x) = B^{1/2}(x)$ and noting that

$$\begin{aligned} (Q_1^*Q_1)(x,s) &= \frac{g(x,s)}{b(s)}[WBW](s) + [h(x,s)g^{\frac{1}{2}}(x,s) + 2f(s)g(x,s)]W(s) \\ &\quad + g(x,s)[A^*W + WA - f(WB + BW)](s) - H_1(x,s). \end{aligned}$$

Hence

$$\begin{aligned} \int_\alpha^x \{g(x,s)T(s) + H_1(x,s)\}ds &= g(x,\alpha)W(\alpha) - \int_\alpha^x \{(Q_1^*Q_1)(x,s)\}ds \\ &\leq g(x,\alpha)W(\alpha) \end{aligned}$$

$$\begin{aligned} \text{implying } \lambda_1\left[\int_\alpha^x \{g(x,s)T(s) + H_1(x,s)\}ds\right] &\leq \lambda_1[g(x,\alpha)W(\alpha)] \\ \text{and } \frac{1}{g(x,\alpha)}\lambda_1\left[\int_\alpha^x \{g(x,s)T(s) + H_1(x,s)\}ds\right] &\leq \lambda_1[W(\alpha)], \end{aligned}$$

a contradiction to (5.30). This completes the proof of theorem 5.9.

Now we give an example to illustrate theorem 5.1 and the nonapplicability of theorem 1 of [22].

Example 5.10. Consider the 4-dimensional system $(1.1)_M$ where

$$A(x) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ B(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ C(x) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.32}$$

and U, V are 2×2 matrix functions of x on $[0,\infty)$.

Define $g[P] = p_{11}$ where $P = (p_{ij})$ so that $g[B^{-1}(x)] = 1$ and

$$\lim_{x\to\infty}\int_0^x \frac{1}{g[B^{-1}(s)]}ds = \lim_{x\to\infty} x = \infty,$$

Further a simple computation shows that

$$\lim_{x\to\infty} g\left[-\int_0^x \{C + A^\star B^{-1}A\}(s)ds - B^{-1}(x)A(x)\right] = \lim_{x\to\infty} g\begin{bmatrix} x & -1 \\ 0 & -x \end{bmatrix}$$
$$= \lim_{x\to\infty} x = \infty.$$

Therefore by theorem 5.1 the system (5.32) is oscillatory.

This fact is directly verified by noting that if k_1 is any one of the real roots of $\lambda^4 - \lambda^2 - 1 = 0$ and k_2 is any one of the real roots of $\lambda^4 + \lambda^2 - 1 = 0$ then

$$\begin{pmatrix} U(x) \\ \\ V(x) \end{pmatrix} = \begin{pmatrix} \begin{bmatrix} k_2\cos k_1x - k_1\sin k_1x & e^{k_2x} \\ (-k_2\sin k_1x - k_1\cos k_1x)/k_1^3 & (e^{k_2x})/k_2^3 \end{bmatrix} \\ \\ \begin{bmatrix} (-k_2\sin k_1x - k_1\cos k_1x)/k_1 & (-e^{k_2x})/k_2 \\ (-k_2\cos k_1x + k_1\sin k_1x)/k_1^2 & (e^{k_2x})/k_2^2 \end{bmatrix} \end{pmatrix}$$

is a nontrival prepared solution of the system (5.32)

$$\text{with} \quad \det U(x) = e^{k_2x}\left[\left(\frac{1}{k_2^2} + \frac{1}{k_1^2}\right)\cos k_1x + \left(\frac{-k_1}{k_2^3} + \frac{k_2}{k_1^3}\right)\sin k_1x\right]$$

having infinitely many zeros on $[0, \infty)$.

The next example is to show that if the criteria of theorem 5.1 are not satisfied then system $(1.1)_M$ need not be oscillatory.

Example 5.11. Consider the 4-dimensional system $(1.1)_M$ where

$$A(x) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ B(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ C(x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.33}$$

and U, V are 2×2 matrix functions of x on $[0,\infty)$.

Note that any positive linear functional g on the space of 2×2 matrix functions is of the form $g[P(x)] = x_0^\star P(x)x_0$ where $x_0 = \mathrm{col}(x_1,\ x_2)$ is an arbitrary but fixed vector in $\mathbb{R}^2$.

Moreover if $\alpha \geq 0$ is arbitrary and $x > \alpha$ we have

$$\begin{aligned} -\int_\alpha^x \{C + A^* B^{-1} A\}(s) ds - B^{-1}(x) A(x) &= \begin{bmatrix} 2(\alpha - x) & 0 \\ -1 & 0 \end{bmatrix} \\ \text{and } \lim_{x\to\infty} g[-\int_\alpha^x \{C + A^* B^{-1} A\}(s) ds - B^{-1}(x) A(x)] &= 2x_1^2(\alpha - x) - x_1 x_2 \\ &= -\infty. \end{aligned}$$

Hence the hypothesis (5.2) of theorem 5.1 is violated whereas the system (5.33) is not oscillatory since $(U(x), V(x))$ where

$$U(x) = \begin{bmatrix} e^{-x} & 0 \\ -e^{-x} & 1 \end{bmatrix}, V(x) = \begin{bmatrix} -e^{-x} & 0 \\ 0 & 0 \end{bmatrix}$$

is a prepared solution of $(1.1)_M$ with $\det U(x) \neq 0$ on $(0, \infty)$.

In the following example, theorem 5.2 is applicable whereas theorem 5.1 is not, thereby making theorem 5.2 a strict generalization of theorem 5.1.

Example 5.12. Consider the 4-dimensional system $(1.1)_M$ where

$$A(x) \equiv 0,\ B(x) = \frac{1}{(x+1)^2} E_2 \text{ and } C(x) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.34}$$

and U, V are 2×2 matrix functions of x on $[0, \infty)$.

Let g be an arbitrary functional as in example 5.11. If $\alpha \geq 0$ is arbitrary and $x > \alpha$ we have

$$\lim_{x\to\infty} \int_\alpha^x \frac{1}{g[B^{-1}(s)]} ds = \lim_{x\to\infty} \frac{1}{x_1^2 + x_2^2} \int_\alpha^x \frac{1}{(s+1)^2} ds = \frac{1}{(x_1^2 + x_2^2)(1+a)} \neq \infty.$$

Hence theorem 5.1 is not applicable.

Now let $a(x) = \dfrac{1}{(x+1)}$ so that $0 < a(x) \in C^1[0, \infty)$.

Define $g[P] = p_{11}$ where $P = (p_{ij})$ so that $g[B^{-1}(x)] = (x+1)^2$ and

$$\lim_{x\to\infty} \int_0^x \frac{1}{a(s) g[B^{-1}(s)]} ds = \lim_{x\to\infty} \log(x+1) = \infty.$$

Further we have

$$\begin{aligned} g[J(x)] &= \int_0^x \left\{ \frac{-a'^2}{4a} g[B^{-1}] - ag[C] \right\}(s)ds + \frac{a' g[B^{-1}](x)}{2} \\ &= \int_0^x \left\{ \frac{-1}{4(s+1)} + \frac{1}{s+1} \right\} ds - \frac{1}{2} = \frac{3}{4}\log(x+1) - \frac{1}{2} \end{aligned}$$

so that $\lim_{x\to\infty} g[J(x)] = \infty$

Therefore by theorem 5.2 the system (5.34) is oscillatory on $[0,\infty)$.

It can be verified directly that the system (5.34) is oscillatory by noting that $(U(x), V(x))$ where

$$U(x) = \begin{bmatrix} \frac{1}{(x+1)^{1/2}} \sin\left(\frac{\sqrt{3}}{2}\log(x+1)\right) & 0 \\ 0 & \frac{-1}{(x+1)} \end{bmatrix},$$

$$V(x) = \begin{bmatrix} \frac{(x+1)^{1/2}}{2}\left[-\sin\left(\frac{\sqrt{3}}{2}\log(x+1)\right) + \sqrt{3}\cos\left(\frac{\sqrt{3}}{2}\log(x+1)\right)\right] & 0 \\ 0 & 1 \end{bmatrix}$$

is a nontrivial prepared solution of the system (5.34) with

$$\det U(x) = \frac{-1}{(x+1)^{3/2}} \sin\left[\frac{\sqrt{3}}{2}\log(x+1)\right]$$

having infinitely many zeros on $[0,\infty)$.

We show by the following example that if the criteria of theorem 5.2 are not satisfied then system $(1.1)_M$ need not be oscillatory.

Example 5.13. Consider the 4-dimensional system $(1.1)_M$ with

$$A(x) = E_2 = B(x) \text{ and } C(x) = 0 \tag{5.35}$$

and U, V are 2×2 matrix functions of x on $[0,\infty)$.

The system (5.35) is nonoscillatory since

$$U(x) = \frac{-e^{-x}}{2}E_2 \text{ and } V(x) = e^{-x}E_2$$

is a prepared solution of $(1.1)_M$ with $\det U(x) \neq 0$ on $[0, \infty)$.

Let $a(x) > 0$ on $[0, \infty)$ and the positive linear functional g be as in example 5.11 so that $g[E_2] = x_1^2 + x_2^2$ and

$$\lim_{x\to\infty} \int_0^x \frac{1}{a(t)g[B^{-1}(t)]}dt = \lim_{x\to\infty} \frac{1}{x_1^2 + x_2^2}\int_0^x \frac{1}{a(t)}dt$$

Now we consider the two possible cases of violation of the hypotheses of theorem 5.2 according as the integral in the equation (5.9) $< \infty$ or $= +\infty$.

Case (i). Suppose $\lim_{x\to\infty} \frac{1}{x_1^2 + x_2^2}\int_0^x \frac{1}{a(t)}dt < \infty.$

Then the condition (5.9) of theorem 5.2 is violated and we are done.

Case (ii). Suppose $\lim_{x\to\infty} \frac{1}{x_1^2 + x_2^2}\int_0^x \frac{1}{a(t)}dt = \infty.$ (5.36)

From equations (5.35) and (5.10) we have $A^*B^{-1} + B^{-1}A = 2E_2$, $A^*B^{-1}A + C = E_2$,

$$\begin{aligned} J(x) &= \left\{\int_0^x \left(a' - a - \frac{a'^2}{4a}\right)(t)dt - a(x) + \frac{a'(x)}{2}\right\}E_2 \\ &= \left\{-\int_0^x \left(\sqrt{a} - \frac{a'}{2\sqrt{a}}\right)^2 dt - a(x) + \frac{a'(x)}{2}\right\}E_2 \\ \text{and } g[J(x)] &= (x_1^2 + x_2^2)\left\{-\int_0^x \left(\sqrt{a} - \frac{a'}{2\sqrt{a}}\right)^2 dt - a(x) + \frac{a'(x)}{2}\right\} \end{aligned}$$

for all $x > 0$. Hence

$$g[J(x)] \leq (x_1^2 + x_2^2)\left[\frac{a'(x)}{2} - a(x)\right] \text{ for all } x > 0. \tag{5.37}$$

Claim. By the assumption (5.36) there exists a sequence $s_n \longrightarrow \infty$ such that $s_{n+1} > s_n$ and $(a'(s_n)/2) - a(s_n) \leq 0$ for all n.
If otherwise $\exists$ X such that $\frac{a'(x)}{2} - a(x) > 0$ for all $x \geq X$.

$$\begin{aligned}
\Longrightarrow \quad a(x) &> a(X)e^{2(x-X)} \text{ for all } x \geq X \\
\Longrightarrow \quad \frac{1}{a(x)} &< \frac{e^{2(X-x)}}{a(X)} \text{ for all } x \geq X \\
\Longrightarrow \quad \int_X^\infty \frac{1}{a(x)}dx &< \frac{e^{2X}}{a(X)} \int_X^\infty e^{-2x}dx \\
&< \frac{1}{2a(X)} \text{ for all } x \geq X,
\end{aligned}$$

a contradiction to (5.36). Hence the claim is true. Therefore by (5.37)

$$g[J(s_n)] \leq (x_1^2 + x_2^2)\left[\frac{a'(s_n)}{2} - a(s_n)\right] \leq 0 \text{ for all } n$$

implying $\lim_{x\to\infty} g[J(x)] = \infty$ cannot hold.

Thus one or the other of the hypotheses of theorem 5.2 is not satisfied whereas the system (5.35) is nonoscillatory on $[0, \infty)$.

The following example illustrates theorem 5.3 (and theorem 5.9 with $f = 0$). Clearly theorem 1 of [34] is not applicable to this example since $A \neq$ zero matrix.

Example 5.14. Consider the 4-dimensional oscillatory system on $[0, \infty)$ given in example 5.10.

$$\begin{aligned}
\text{Let } g(x,s) &= (x-s)^2 \\
\text{so that } h(x,s) &= 2(x-s)(x-s)^{-1} = 2, \\
\int_0^x g(x,s)C(s)ds &= \int_0^x (x-s)^2 ds \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \frac{x^3}{3}\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\int_0^x g(x,s)(A^*B^{-1}A)(s)ds &= \int_0^x (x-s)^2 ds \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \frac{x^3}{3}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \\
\frac{1}{2}\int_0^x h(x,s)g^{\frac{1}{2}}(x,s)(A^*B^{-1})(s)ds &= \int_0^x (x-s)ds \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\
&= \frac{x^2}{2}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \\
\frac{1}{2}\int_0^x h(x,s)g^{\frac{1}{2}}(x,s)(B^{-1}A)(s)ds &= \int_0^x (x-s)ds \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
&= \frac{x^2}{2}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\
\frac{1}{4}\int_0^x h^2(x,s)B^{-1}(s)ds &= \frac{1}{4}\int_0^x 4E_2 ds = x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
\text{and } H(x) &= \begin{bmatrix} -x+\frac{x^3}{3} & -\frac{x^2}{2} \\ -\frac{x^2}{2} & -x-\frac{x^3}{3} \end{bmatrix}.
\end{aligned}$$

The characteristic polynomial of $H(x)$ is

$$\lambda^2 + 2x\lambda + (x^2 - (x^4/4) - (x^6/9)) = 0$$

and the eigen values of $H(x)$ are

$$\lambda_1[H(x)] = -x + x^3\sqrt{\frac{1}{9}+\frac{1}{4x^2}}, \quad \lambda_2[H(x)] = -x - x^3\sqrt{\frac{1}{9}+\frac{1}{4x^2}}.$$

$$\text{Hence } \limsup_{x\to\infty}\left[\frac{\lambda_1 H(x)}{g(x,0)}\right] = \limsup_{x\to\infty}\left[x\sqrt{\frac{1}{9}+\frac{1}{4x^2}} - \frac{1}{x}\right] = \infty.$$

Thus the hypotheses of theorem 5.3 are satisfied.

Hence theorem 5.3 is a strict generalization of theorem 1 of [34]. The following example shows that if the criteria of theorem 5.3 are not satisfied then system $(1.1)_M$ need not be oscillatory.

Example 5.15 Consider the system $(1.1)_M$ with $A(x) \equiv 0 \equiv C(x)$ and

$B(x) = E_2$.

Let $g(x,s)$ be an arbitrary function having the properties stated in theorem 5.3 and $\alpha > 0$ be arbitrary. Then

$$\begin{aligned} H(x) &= \frac{-1}{4}\int_\alpha^x h^2(x,s)E_2\,ds, \\ \lambda_1[H(x)] &= \frac{-1}{4}\int_\alpha^x h^2(x,s)ds \\ \text{and hence l.h.s. of (5.19)} &= \limsup_{x\to\infty} -\frac{\int_\alpha^x h^2(x,s)ds}{4g(x,\alpha)} \neq \infty. \end{aligned}$$

Thus the hypothesis of theorem 5.3 is violated. However it follows that the given system is nonoscillatory on $[\alpha,\infty)$ since $(U(x), V(x))$ where

$$U(x) = (x+1)E_2 \text{ and } V(x) = E_2$$

is a prepared solution of $(1.1)_M$ with $\det U(x) > 1$ on $[\alpha,\infty)$ for $\alpha > 0$.

Chapter 6

Conjugacy Criteria for a Linear Second Order Difference Equation

Conjugacy, disconjugacy and oscillation criteria for discrete linear Hamiltonian systems have been discussed in the recent past by several authors as for instance in [1] to [5], [17], [20] and [30] and further for singular systems in [3], [4], [7] and [8].

In this chapter we obtain conjugacy criteria for the linear second order scalar difference equation

$$\Delta(c_{n-1}\Delta x_{n-1}) + a_n x_n = 0 \tag{1.7}$$

or the equivalent three term equation

$$c_n x_{n+1} + (a_n - c_n - c_{n-1})x_n + c_{n-1}x_{n-1} = 0 \tag{1.8}$$

on an integer interval I which may be bounded or unbounded under one or the other of the following hypotheses:

H^d : $c_n > 0$ on $[M, N+1]$ (where $-\infty < M < N < \infty$) and a_n real on $[M+1, N+1]$,

H_+^d : $c_n > 0$ on $[M, \infty)$ (where $-\infty < M < \infty$) and a_n real on $[M+1, \infty)$,

H_-^d : $c_n > 0$ on $(-\infty, N+1]$ (where $-\infty < N < \infty$) and a_n real on $(-\infty, N+1]$

and

H_0^d : $c_n > 0$ on $(-\infty, \infty)$ and a_n real on $(-\infty, \infty)$.

Hereafter whenever we refer to equation (1.7) it will be understood that the reference applies equally well to equation (1.8). Along with equation (1.7) we also consider the 'unperturbed equation'

$$\Delta(c_{n-1}\Delta x_{n-1}) = 0 \tag{1.9}$$

or the equivalent three term equation

$$c_n x_{n+1} - (c_n + c_{n-1})x_n + c_{n-1}x_{n-1} = 0$$

associated with (1.7) under the above mentioned hypotheses.

Here we give sufficient conditions for the conjugacy of the equation (1.7) explicitly in terms of the coefficient functions. These criteria are analogous to the ones given by Došlý [16] for the continuous case. We also consider for the special case of a scalar disconjugate difference equation on $(-\infty, \infty)$ the question raised by Ahlbrandt and Patula in [5] concerning the linear dependence or independence of the recessive solutions at $\pm\infty$. More specifically we give necessary and sufficient conditions for the linear independence of such solutions. This result is also analogous to that of the continuous case given by Došlý [16].

Lastly, we give two examples, one to illustrate the main theorem (theorem 6.5) and the other to show that the criterion given in that theorem is significant in the sense that if that criterion is not satisfied the equation may not be conjugate.

Remark 6.1. In the paper [18] (yet to appear in print) (see theorem 2.103, chapter 2) conjugacy criteria for the equation (1.7) on $(-\infty, \infty)$ with $c_n \equiv 1$, that is,

$$\Delta^2 x_{n-1} + a_n x_n = 0 \tag{1.7$_1$}$$

are given. Conjugacy criteria for finite intervals are not given explicitly in [18] whereas in our paper such criteria are also given without the assumption

$c_n \equiv 1$. In particular example 6.11 of this chapter illustrates an instance in which the criteria of [18] are not applicable but the criterion of theorem 6.5 is.

Further even in the case the equation $(1.7)_1$ is conjugate on $(-\infty, \infty)$ the criterion of theorem 6.5 unlike those given in [18] (theorem 2.103, chapter 2) provides the inherent advantage of yielding a subinterval $[M, N+2]$ on which the equation is possibly conjugate.

Now we shall prove some preliminary results which will be useful in our further discussion. In the following lemma we express the general solution x_n of (1.9) explicitly in terms of the coefficient function c_n. Hereafter we follow the convention $\sum_{m=j}^{k} p_m = 0$ and $\prod_{m=j}^{k} p_m = 1$ for arbitrary p_m if $j > k$.

Lemma 6.2. x_n is solution of (1.9) on $[M, N+2]$ if and only if it is of the form

$$x_n = \begin{cases} B + A \displaystyle\sum_{m=k+1}^{n} \frac{1}{c_{m-1}}, & n \geq k \\ B - A \displaystyle\sum_{m=n+1}^{k} \frac{1}{c_{m-1}}, & n \leq k \end{cases} \tag{6.1}$$

where $k \in [M, N+2]$ is arbitrary but fixed and A, B are arbitrary constants.

Proof. The 'if' part follows by direct verification using equation (6.1) and the 'only if' part follows by an iteration using equation (1.9).

In the next lemma we show that the hypothesis H^d guarantees that the unperturbed equation (1.9) is disconjugate on I. This lemma can be proved either by using theorem 2.1 of [1] (see theorem 2.102, chapter 2) or independently as given here below.

Lemma 6.3. Under the hypothesis H^d equation (1.9) is disconjugate on

$I = [M, N+2]$.

Proof. Suppose if possible (1.9) is conjugate on I. Then there exists a nontrivial solution x_n with a pair of generalized zeros say p and q $(p < q)$ belonging to $[M, N+2]$. If so, x_n satisfies exactly one of the following conditions (i) to (iv)

(i) $\quad x_{p-1}x_p < 0, \quad x_{q-1}x_q < 0, \; M+1 \le p < q \le N+2$

(ii) $\quad x_p = 0, \quad x_{q-1}x_q < 0, \; M \le p < q-1 \le N+1$

(iii) $\quad x_{p-1}x_p < 0, \quad x_q = 0, \; M+1 \le p < q \le N+2$

(iv) $\quad x_p = 0, \quad x_q = 0, \; M \le p < q \le N+2.$

Now observe that (6.1) implies $\Delta x_n = A/c_n$ for all $n \in [M, N+1]$ and hence $\text{sgn}(x_n - x_m) = \text{sgn}A$ whenever $M \le m < n \le N+2$. In case (i) there exists $m_1,\ m_2 \in \{p-1, p\}$ and $n_1,\ n_2 \in \{q-1, q\}$ with $x_{m_1} < 0,\ x_{m_2} > 0,\ x_{n_1} < 0$ and $x_{n_2} > 0$. Hence

$$-1 = \text{sgn}(x_{n_1} - x_{m_2}) = \text{sgn}A = \text{sgn}(x_{n_2} - x_{m_1}) = 1,$$

a contradiction. For the case (ii) (and similarly for case (iii)) we have

$$1 = (\text{sgn}A)^2 = \text{sgn}(x_{q-1} - x_p)\text{sgn}(x_q - x_p) = (\text{sgn}x_{q-1})(\text{sgn}x_q) = -1,$$

a contradiction as well. Finally, case (iv) implies $0 = \text{sgn}(x_q - x_p) = \text{sgn}A$, so $A = 0$, hence $B = 0$ and hence x_n is trivial, which is our final contradiction.

The following theorem is a discrete analogue of Došlý's result (remark 4 of [16]) (see lemma 2.15, chapter 2) for the continuous case. In this theorem we give necessary and sufficient conditions for the linear independence of recessive solutions of (1.9) at $\pm\infty$.

Theorem 6.4. Equation (1.9) is 1-special on $(-\infty, \infty)$ if and only if

$$\sum_{m=-\infty}^{k} \frac{1}{c_{m-1}} = \infty = \sum_{m=k+1}^{\infty} \frac{1}{c_{m-1}} \tag{6.2}$$

where $k \in (-\infty, \infty)$.

Proof. Let (6.2) hold. Then the solution $x_n \equiv 1$ obtained by letting $A = 0,\ B = 1$ in the formula (6.1) is the recessive solution of (1.9) at $-\infty$ and $+\infty$. This is so since $x_n > 0\ \ \forall\ \ n \geq k$ and

$$\sum_{i=k}^{n}(x_i c_i x_{i+1})^{-1} = \sum_{i=k}^{n}\frac{1}{c_i} = \sum_{i=k+1}^{n+1}\frac{1}{c_{i-1}} \longrightarrow \infty \text{ as } n \longrightarrow \infty$$

by (6.2).

The recessiveness at $-\infty$ follows from the other equation in (6.2).

Therefore (1.9) is 1-special.

Conversely, let (1.9) be 1-special and assume that (6.2) does not hold. Now we consider the three possible cases and arrive at a contradiction in each case by obtaining linearly independent recessive solutions at $+\infty$ and $-\infty$.

Case (i). $\displaystyle\sum_{m=-\infty}^{k}\frac{1}{c_{m-1}} < \infty$ and $\displaystyle\sum_{m=k+1}^{\infty}\frac{1}{c_{m-1}} = \infty$

We claim that

$$x_{-\infty}(n) = \sum_{m=-\infty}^{n}\frac{1}{c_{m-1}} \quad \text{and} \quad x_{+\infty}(n) = 1$$

are the recessive solutions at $-\infty$ and $+\infty$ respectively.

That $x_{+\infty}(n)$ is the recessive solution at $+\infty$ follows as in the proof of the 'if part'.

On the other hand $x_{-\infty}(n)$ is a solution of (1.9) since it can be obtained from (6.1) by choosing

$$A = 1 \quad \text{and} \quad B = \sum_{m=-\infty}^{k}\frac{1}{c_{m-1}}.$$

Moreover it is a recessive solution at $-\infty$ since $x_{-\infty}(n)$ is positive for every

$n \geq k$ and

$$\begin{aligned}\sum_{i=n}^{k}(x_i c_i x_{i+1})^{-1} &= \sum_{i=n}^{k}\left[\left(\sum_{m=-\infty}^{i}\frac{1}{c_{m-1}}\right)^{-1}\frac{1}{c_i}\left(\sum_{m=-\infty}^{i+1}\frac{1}{c_{m-1}}\right)^{-1}\right] \\ &= \sum_{i=n}^{k}\left[\left(\frac{1}{d_i}\right)\Delta d_i\left(\frac{1}{d_{i+1}}\right)\right] \quad \text{where } d_i = \sum_{m=-\infty}^{i}\frac{1}{c_{m-1}} \\ &= \sum_{i=n}^{k}\left(\frac{d_{i+1}-d_i}{d_i \times d_{i+1}}\right) \\ &= \sum_{i=n}^{k}\left(\frac{1}{d_i}-\frac{1}{d_{i+1}}\right) = \frac{1}{d_n}-\frac{1}{d_{k+1}} \longrightarrow \infty \text{ as } n \longrightarrow -\infty, \\ &\quad \text{since } d_n \longrightarrow 0 \text{ as } n \longrightarrow -\infty.\end{aligned}$$

Thus, $x_{-\infty}(n)$ is a recessive solution at $-\infty$.

Case (ii). $\displaystyle\sum_{m=-\infty}^{k}\frac{1}{c_{m-1}} = \infty$ and $\displaystyle\sum_{m=k+1}^{\infty}\frac{1}{c_{m-1}} < \infty$.

Define

$$x_{-\infty}(n) = 1 \quad \text{and} \quad x_{+\infty}(n) = \sum_{m=n+1}^{+\infty}\frac{1}{c_{m-1}}.$$

Now we can show as in case (i) that $x_{-\infty}(n)$ and $x_{+\infty}(n)$ defined as above are recessive solutions at $-\infty$ and $+\infty$ respectively.

Case (iii). $\displaystyle\sum_{m=-\infty}^{k}\frac{1}{c_{m-1}} < \infty$ and $\displaystyle\sum_{m=k+1}^{\infty}\frac{1}{c_{m-1}} < \infty$.

In this case $x_{-\infty}(n)$ as in case (i) and $x_{+\infty}(n)$ as in case (ii) are the recessive solutions at $-\infty$ and $+\infty$ respectively.

Thus in all the three possible cases, the recessive solutions at $-\infty$ and $+\infty$ are linearly independent which is a contradiction to (1.9) being 1-special on $(-\infty, +\infty)$. This completes the proof of the theorem.

We now give a conjugacy criterion for equation (1.7) on the interval $I = [M, N+2]$.

Theorem 6.5. Let hypothesis H^d hold with $-\infty < M < N < \infty$. Suppose that there exist $\epsilon_1, \epsilon_2 > 0$ and $k \in (M, N+1)$ such that

$$\sum_{p=k+1}^{N+2} \tan^{-1}\left[\frac{\epsilon_1 c_{p-1}^{-1}}{2}\left(\prod_{s=k+1}^{p} q_s\right)^{-1}\left(\prod_{s=k+1}^{p-1} q_s\right)^{-1}\right] > \frac{\pi}{4}$$

and $(6.3)_{a,b}$

$$\sum_{p=M}^{k} \tan^{-1}\left[\frac{\epsilon_2 c_p^{-1}}{2}\left(\prod_{s=p+1}^{k} r_s\right)\left(\prod_{s=p}^{k} r_s\right)\right] > \frac{\pi}{4}$$

with

$$q_s = 1 + c_{s-1}^{-1}\left(\epsilon_1 - \sum_{m=k+1}^{s-1} a_m\right) \neq 0, \quad k+1 \leq s \leq N+2$$

and $(6.4)_{a,b}$

$$r_s = 1 + c_s^{-1}\left(\frac{-\epsilon_2 c_k}{\epsilon_2 + c_k} + \sum_{m=s+1}^{k} a_m\right), \quad M \leq s \leq k.$$

Then (1.7) is conjugate on $I = [M, N+2]$.

Proof. Suppose the hypotheses (6.3) and (6.4) hold and (1.7) is disconjugate on I. Then we arrive at a contradiction. Let x_n be the solution of (1.7) with $x_k = 1$, $\Delta x_k = 0$.

Since (1.7) is disconjugate on I, we have

$$x_n x_{n+1} \leq 0 \tag{6.5}$$

for at most one n on I. So we first assume that (6.5) holds for exactly one n on I and consider the two possible cases.

Case (i). Suppose (6.5) holds for some n, $M \leq n \leq k$. In this case by the assumption of disconjugacy of (1.7) on I, we have $x_n x_{n+1} > 0$, $k < n \leq N+2$ and since $x_{k+1} = 1$ we must have $x_n > 0$ for all $k < n \leq N+2$.

Let y_n be the solution of (1.7) given by the initial conditions $y_k = 1$ and $c_k \Delta y_k = \epsilon_1$. We now claim

$$y_n > x_n > 0, \quad k < n \le N+2. \tag{6.6}$$

Note that $y_n - x_n$ is a solution of (1.7) satisfying $y_k - x_k = 0$ and hence the disconjugacy of (1.7) implies $(y_n - x_n)c_n(y_{n+1} - x_{n+1}) > 0$ for all $k < n \le N+2$.

Since $y_{k+1} - x_{k+1} = \dfrac{\epsilon_1}{c_k} > 0$, by hypothesis we must have $y_n - x_n > 0$ for all $k < n \le N+2$ and hence the claim (6.6).

Define

$$w_n = \frac{c_{n-1}\Delta y_{n-1}}{y_{n-1}}, \quad k+1 \le n \le N+2. \tag{6.7}$$

Then w_n is well defined on $[k+1, N+2]$ by (6.6) and it satisfies (by theorem 2.1 of [1]) (see theorem 2.102, chapter 2) the Riccati difference equation

$$\Delta w_n = -a_n - \frac{w_n^2}{w_n + c_{n-1}}, \quad k+1 \le n \le N+1$$

where

$$w_n + c_{n-1} > 0, \quad k+1 \le n \le N+2. \tag{6.8}$$

$$\text{Thus } \ w_{n+1} = w_n - a_n - \frac{w_n^2}{w_n + c_{n-1}}, \quad k+1 \le n \le N+1.$$

Now on iteration and using the initial condition $w_{k+1} = \epsilon_1$ we obtain

$$\begin{aligned} w_n &= \epsilon_1 - \sum_{m=k+1}^{n-1} \left(a_m + \frac{w_m^2}{w_m + c_{m-1}} \right), \quad k+1 \le n \le N+2 \\ \Longrightarrow \frac{\Delta y_{n-1}}{y_{n-1}} &= c_{n-1}^{-1} \left[\epsilon_1 - \sum_{m=k+1}^{n-1} \left(a_m + \frac{w_m^2}{w_m + c_{m-1}} \right) \right] \text{ (by equation (6.7))}. \end{aligned}$$

On adding 1 to both sides of the equation and multiplying the resulting equation by y_{n-1} which is positive for $k+1 \le n \le N+2$ by (6.6) we obtain

$$0 < y_n = y_{n-1} \left\{ 1 + c_{n-1}^{-1} \left[\epsilon_1 - \sum_{m=k+1}^{n-1} \left(a_m + \frac{w_m^2}{w_m + c_{m-1}} \right) \right] \right\}.$$

Hence it follows from (6.6) that the factor multiplying y_{n-1} in the above inequality is positive. Moreover another iteration for $k+1 \leq n \leq N+2$ along with the initial condition $y_k = 1$ yields

$$\begin{aligned} 0 < y_n &= \prod_{s=k+1}^{n} \left\{ 1 + c_{s-1}^{-1} \left[\epsilon_1 - \sum_{m=k+1}^{s-1} \left(a_m + \frac{w_m^2}{w_m + c_{m-1}} \right) \right] \right\} \\ &< \prod_{s=k+1}^{n} \left\{ 1 + c_{s-1}^{-1} \left[\epsilon_1 - \sum_{m=k+1}^{s-1} a_m \right] \right\} \quad \text{(by (6.8))} \\ \implies 0 < y_n &< \prod_{s=k+1}^{n} q_s, \quad k+1 \leq n \leq N+2. \end{aligned} \tag{6.9}$$

Now define

$$\alpha_n = \tan^{-1}(y_n/x_n), \quad k \leq n \leq N+2.$$

Note $(\pi/4) \leq \alpha_n < (\pi/2)$ since $1 \leq (y_n/x_n), \quad k \leq n \leq N+2$.
Further $k \leq n \leq N+1 \implies \Delta\alpha_n = \Delta \tan^{-1}(y_n/x_n)$

$$\begin{aligned} \implies \alpha_{n+1} - \alpha_n &= \tan^{-1}\frac{y_{n+1}}{x_{n+1}} - \tan^{-1}\frac{y_n}{x_n} \\ &= \tan^{-1}\left(\frac{y_{n+1}x_n - x_{n+1}y_n}{x_{n+1}x_n + y_{n+1}y_n}\right). \\ &= \tan^{-1}\left(\frac{\epsilon_1 c_n^{-1}}{x_{n+1}x_n + y_{n+1}y_n}\right) \\ &\quad \text{(by note 2.83)} \\ \implies \alpha_{n+1} &= \alpha_n + \tan^{-1}\left(\frac{\epsilon_1 c_n^{-1}}{x_{n+1}x_n + y_{n+1}y_n}\right), \quad k \leq n \leq N+1. \end{aligned}$$

Again by iteration for $k \leq n \leq N+2$ and using the initial condition $\alpha_k = \pi/4$ we obtain

$$\alpha_n = \frac{\pi}{4} + \sum_{p=k+1}^{n} \tan^{-1}\left(\frac{\epsilon_1 c_{p-1}^{-1}}{x_p x_{p-1} + y_p y_{p-1}}\right)$$

$$> \frac{\pi}{4} + \sum_{p=k+1}^{n} \tan^{-1}\left(\frac{\epsilon_1 c_{p-1}^{-1}}{2y_p y_{p-1}}\right), \; k+1 \le n \le N+2 \;\; (\text{by } (6.6)).$$

Thus for $k+1 \le n \le N+2$ we have by (6.9)

$$\alpha_n > \frac{\pi}{4} + \sum_{p=k+1}^{n} \tan^{-1}\left[\frac{\epsilon_1 c_{p-1}^{-1}}{2}\left(\prod_{s=k+1}^{p} q_s\right)^{-1}\left(\prod_{s=k+1}^{p-1} q_s\right)^{-1}\right]. \tag{6.10}$$

In particular for $n = N+2$ we have

$$\begin{aligned}\frac{\pi}{2} \ge \alpha_{N+2} \;&\ge\; \frac{\pi}{4} + \sum_{p=k+1}^{N+2} \tan^{-1}\left[\frac{\epsilon_1 c_{p-1}^{-1}}{2}\left(\prod_{s=k+1}^{p} q_s\right)^{-1}\left(\prod_{s=k+1}^{p-1} q_s\right)^{-1}\right]\\ &>\; \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2} \;\; (\text{by inequality } (6.3)_a \text{ of the hypothesis})\end{aligned}$$

which is a contradiction.

Case (ii). Suppose (6.5) holds for some n, $k < n \le N+1$. Then we arrive at a contradiction to the inequality $(6.3)_b$ of the hypothesis.

In this case $x_n x_{n+1} > 0$ for all n, $M \le n \le k$ and since $x_k = 1$ it follows that $x_n > 0$ for all n, $M \le n \le k$.

Let z_n be the solution of (1.7) given by the initial conditions $z_{k+1} = 1$ and $c_k \Delta z_k = -\epsilon_2$ so that $z_k = 1 + \epsilon_2/c_k$. We now claim

$$z_n > x_n > 0, \quad M \le n \le k. \tag{$6.6)_1$}$$

Note that $z_n - x_n$ is a solution of (1.7) satisfying $z_{k+1} - x_{k+1} = 0$ and hence the disconjugacy of (1.7) implies $(z_{n-1} - x_{n-1})c_n(z_n - x_n) > 0$ for all $M+1 \le n \le k$.

Since $z_k - x_k = 1 + \dfrac{\epsilon_1}{c_k} - 1 = \dfrac{\epsilon_2}{c_k} > 0$, by hypothesis we must have $z_n - x_n > 0$ for all $M \le n \le k$ and hence the claim.

Define

$$w_n = \frac{c_{n-1}\Delta z_{n-1}}{z_{n-1}} \;\; \text{for all } \; M+1 \le n \le k+1. \tag{$6.7)_1$}$$

Then w_n is well defined on $[M+1, k+1]$ by $(6.6)_1$ and it satisfies (by theorem 2.1 of [1]) (see theorem 2.102, chapter 2) the Riccati difference equation

$$\Delta w_n = -a_n - \frac{w_n^2}{w_n + c_{n-1}}, \quad M+1 \le n \le k$$

where

$$w_n + c_{n-1} > 0, \quad M+1 \le n \le k. \tag{6.8$_1$}$$

$$\text{Thus } w_n = w_{n+1} + a_n + \frac{w_n^2}{w_n + c_{n-1}}, \quad M+1 \le n \le k.$$

Now on iteration and using the initial condition $w_{k+1} = -\dfrac{\epsilon_2 c_k}{\epsilon_2 + c_k}$ we obtain

$$\begin{aligned} w_n &= \frac{-\epsilon_2 c_k}{\epsilon_2 + c_k} + \sum_{m=n}^{k} \left(a_m + \frac{w_m^2}{w_m + c_{m-1}} \right), \quad M+1 \le n \le k+1 \\ \Longrightarrow \quad \frac{\Delta z_{n-1}}{z_{n-1}} &= c_{n-1}^{-1} \left[\frac{-\epsilon_2 c_k}{\epsilon_2 + c_k} + \sum_{m=n}^{k} \left(a_m + \frac{w_m^2}{w_m + c_{m-1}} \right) \right], \\ & \quad M+1 \le n \le k+1 \ \text{(by equation } (6.7)_1). \end{aligned}$$

On adding 1 to both sides of the equation and multiplying the resulting equation by z_{n-1} which is positive for $M+1 \le n \le k+1$ by $(6.6)_1$ we obtain

$$0 < z_n = z_{n-1} \left\{ 1 + c_{n-1}^{-1} \left[\frac{-\epsilon_2 c_k}{\epsilon_2 + c_k} + \sum_{m=n}^{k} \left(a_m + \frac{w_m^2}{w_m + c_{m-1}} \right) \right] \right\}.$$

Hence it follows from $(6.6)_1$ that the factor multiplying z_{n-1} in the above inequality is positive and hence

$$0 < z_{n-1} = z_n \left\{ 1 + c_{n-1}^{-1} \left[\frac{-\epsilon_2 c_k}{\epsilon_2 + c_k} + \sum_{m=n}^{k} \left(a_m + \frac{w_m^2}{w_m + c_{m-1}} \right) \right] \right\}^{-1},$$

$M+1 \le n \le k+1$.

Equivalently,

$$0 < z_n = z_{n+1} \left\{ 1 + c_n^{-1} \left[\frac{-\epsilon_2 c_k}{\epsilon_2 + c_k} + \sum_{m=n+1}^{k} \left(a_m + \frac{w_m^2}{w_m + c_{m-1}} \right) \right] \right\}^{-1},$$

$M \leq n \leq k$.

Moreover another iteration for $M \leq n \leq k$ along with the initial condition $z_{k+1} = 1$ yields

$$\begin{aligned} 0 < z_n &= \prod_{s=n}^{k} \left\{ 1 + c_s^{-1} \left[\frac{-\epsilon_2 c_k}{\epsilon_2 + c_k} + \sum_{m=s+1}^{k} \left(a_m + \frac{w_m^2}{w_m + c_{m-1}} \right) \right] \right\}^{-1}, \\ & \quad M \leq n \leq k+1 \\ &< \prod_{s=n}^{k} \left\{ 1 + c_s^{-1} \left[\frac{-\epsilon_2 c_k}{\epsilon_2 + c_k} + \sum_{m=s+1}^{k} a_m \right] \right\}^{-1} \quad \text{(by } (6.8)_1) \\ \implies (1/z_n) &> \prod_{s=n}^{k} r_s > 0, \quad M \leq n \leq k. \end{aligned} \qquad (6.9)_1$$

Now define

$$\beta_n = \tan^{-1}(z_n/x_n), \quad M \leq n \leq k+1.$$

Note $(\pi/4) \leq \beta_n < (\pi/2)$ since $1 \leq (z_n/x_n)$, $\quad M \leq n \leq k+1$.

Further $M \leq n \leq k \implies \Delta\beta_n = \Delta \tan^{-1}(z_n/x_n)$

$$\begin{aligned} \implies \beta_{n+1} - \beta_n &= \tan^{-1}\frac{z_{n+1}}{x_{n+1}} - \tan^{-1}\frac{z_n}{x_n} \\ &= \tan^{-1}\left(\frac{z_{n+1}x_n - x_{n+1}z_n}{x_{n+1}x_n + z_{n+1}z_n}\right). \\ &= \tan^{-1}\left(\frac{-\epsilon_2 c_n^{-1}}{x_{n+1}x_n + z_{n+1}z_n}\right) \\ & \quad \text{(by note 2.83)} \\ \implies \beta_n &= \beta_{n+1} + \tan^{-1}\left(\frac{\epsilon_2 c_n^{-1}}{x_{n+1}x_n + z_{n+1}z_n}\right), \quad M \leq n \leq k. \end{aligned}$$

Again by iteration for $M \leq n \leq k$ and using the initial condition $\beta_{k+1} = \pi/4$ we obtain

$$\beta_n = \frac{\pi}{4} + \sum_{p=n}^{k} \tan^{-1}\left(\frac{\epsilon_2 c_p^{-1}}{x_{p+1}x_p + z_{p+1}z_p}\right), \ M \leq n \leq k+1$$

$$> \frac{\pi}{4} + \sum_{p=n}^{k} \tan^{-1}\left(\frac{\epsilon_2 c_p^{-1}}{2z_{p+1}z_p}\right), \ M \le n \le k \ \text{(by } (6.6)_1).$$

Thus for $M \le n \le k$ we have by $(6.9)_1$

$$\beta_n > \frac{\pi}{4} + \sum_{p=n}^{k} \tan^{-1}\left[\frac{\epsilon_2 c_p^{-1}}{2} \prod_{s=p+1}^{k} r_s \prod_{s=p}^{k} r_s\right]. \qquad (6.10)_1$$

In particular for $n = M$ we have

$$\begin{aligned} \frac{\pi}{2} \ge \beta_M \ &\ge \ \frac{\pi}{4} + \sum_{p=M}^{k} \tan^{-1}\left[\frac{\epsilon_2 c_p^{-1}}{2} \prod_{s=p+1}^{k} r_s \prod_{s=p}^{k} r_s\right] \\ &> \ \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2} \ \text{(by inequality } (6.3)_b \text{ of the hypothesis)} \end{aligned}$$

which is a contradiction.

The contradictions in cases (i) and (ii) imply that (6.5) cannot hold for any n in I, that is, $x_n x_{n+1} > 0$ for all n in I.

This final contradiction completes the proof of the theorem.

Corollary 6.6. Assume hypothesis H_+^d holds. Suppose that there exist $\epsilon_1, \epsilon_2 > 0$ and $k \in (M, \infty)$, such that

$$\sum_{p=k+1}^{\infty} \tan^{-1}\left[\frac{\epsilon_1 c_{p-1}^{-1}}{2} \left(\prod_{s=k+1}^{p} q_s\right)^{-1} \left(\prod_{s=k+1}^{p-1} q_s\right)^{-1}\right] > \frac{\pi}{4} \qquad (6.14)$$

and inequality $(6.3)_b$ holds where q_s and r_s are as given in theorem 6.5. Then (1.7) is conjugate on $[M, \infty)$.

Proof. Proceeding as in the proof of the theorem, we can show that if case (i) holds as in the proof of the theorem then the inequality (6.10) holds for all $n \ge k$. Now taking the limit as $n \longrightarrow \infty$ in (6.10) and using the inequality (6.14) we obtain the contradiction,

$$\frac{\pi}{2} \ge \lim_{n\to\infty} \alpha_n \ge \frac{\pi}{4} + \sum_{p=k+1}^{\infty} \tan^{-1}\left[\frac{\epsilon_1 c_{p-1}^{-1}}{2} \left(\prod_{s=k+1}^{p} q_s\right)^{-1} \left(\prod_{s=k+1}^{p-1} q_s\right)^{-1}\right] > \frac{\pi}{4} + \frac{\pi}{4}.$$

On the other hand if case (ii) holds as in the proof of the theorem then the same argument as in the proof of the theorem gives the final contradiction. Hence the corollory.

Corollary 6.7. Assume hypothesis H_-^d holds. Suppose there exist $\epsilon_1, \epsilon_2 > 0$ and $k \in (-\infty, N+1)$, such that inequality $(6.3)_a$ and

$$\sum_{p=-\infty}^{k} \tan^{-1}\left[\frac{\epsilon_2 c_p^{-1}}{2}\left(\prod_{s=p+1}^{k} r_s\right)\left(\prod_{s=p}^{k} r_s\right)\right] > \frac{\pi}{4} \tag{6.15}$$

hold where q_s and r_s are as given in theorem 6.5. Then (1.7) is conjugate on $(-\infty, N+2]$.

Proof. This is similar to that of corollary 6.6 and hence is omitted.

Corollary 6.8. Assume hypothesis H_0^d holds. Suppose that there exist $\epsilon_1, \epsilon_2 > 0$, $k \in (-\infty, \infty)$, such that (6.14) and (6.15) hold where q_s and r_s are as in theorem 6.5. Then (1.7) is conjugate on $(-\infty, \infty)$.

Proof. This is contained in the proofs of corollary 6.6 and corollary 6.7 and hence is omitted.

Note 6.9. In the following corollary we use the result

$$\sum_{p=1}^{n} \tan^{-1}(x_p) \geq \tan^{-1}\left[\sum_{p=1}^{n} x_p\right], \quad \text{where } x_p \geq 0 \text{ for } 1 \leq p \leq n.$$

Corollary 6.10. Let hypothesis H^d hold. Suppose that there exist $\epsilon_1, \epsilon_2 > 0$, $k \in (M, N+1)$ such that

$$\sum_{p=k+1}^{N+2} A_p > 1 \tag{6.16}$$

and

$$\sum_{j=M}^{k} B_j > 1 \tag{6.17}$$

where

$$A_p = \left(\frac{\epsilon_1 c_{p-1}^{-1}}{2} \left(\prod_{s=k+1}^{p} q_s \right)^{-1} \left(\prod_{s=k+1}^{p-1} q_s \right)^{-1} \right),$$

$$B_j = \left(\frac{\epsilon_2 c_j^{-1}}{2} \left(\prod_{s=j+1}^{k} r_s \right) \left(\prod_{s=j}^{k} r_s \right) \right)$$

and q_s and r_s are as given in theorem 6.5. Further suppose $A_p > 0$, $B_j > 0$ for all p, $k+1 \le p \le N+2$ and for all j, $M \le j \le k$. Then (1.7) is conjugate on $[M, N+2]$.

Proof.
$$\sum_{p=k+1}^{N+2} \tan^{-1} A_p \ge \tan^{-1} \left[\sum_{p=k+1}^{N+2} A_p \right] > \frac{\pi}{4} \quad \text{(by (6.16))}$$

so that inequality $(6.3)_a$ is satisfied.
Similarly using (6.17) we can prove that inequality $(6.3)_b$ is also satisfied and hence the corollary.

We now give two examples, one to illustrate theorem 6.5 and the other to show that if the hypotheses of theorem 6.5 are not satisfied then equation (1.7) need not be conjugate.

Example 6.11. Consider equation (1.7) with $M = 0$, $N = 2$,

$$c_n = \begin{cases} (3/2) & n = 0 \\ 2 & n = 1 \\ 1 & n = 2 \\ (1/4) & n = 3 \end{cases}$$

and

$$a_n = \begin{cases} 3 & n = 1 \\ (3/2) & n = 2 \\ (1/2) & n = 3. \end{cases}$$

Let $k = 2$ in corollary 6.10. Then from equation (6.4)

$$q_3 = 1 + c_2^{-1} \left(\epsilon_1 - \sum_{m=3}^{2} a_m \right) = 1 + \epsilon_1$$

$$\begin{aligned}
\text{and } q_4 &= 1 + c_3^{-1}\left(\epsilon_1 - \sum_{m=3}^{3} a_m\right) = 4\epsilon_1 - 1 \\
\text{so that } A_3 &= \frac{\epsilon_1 c_2^{-1}}{2}\left(\prod_{s=3}^{3} q_s\right)^{-1}\left(\prod_{s=3}^{2} q_s\right)^{-1} = \frac{\epsilon_1}{2q_3} = \frac{\epsilon_1}{2(1+\epsilon_1)} \\
\text{and } A_4 &= \frac{\epsilon_1 c_3^{-1}}{2}\left(\prod_{s=3}^{4} q_s\right)^{-1}\left(\prod_{s=3}^{3} q_s\right)^{-1} = \frac{2\epsilon_1}{q_3^2 \times q_4} \\
&= \frac{2\epsilon_1}{(1+\epsilon_1)^2(4\epsilon_1 - 1)}\,. \\
\text{Now } \sum_{p=3}^{4} A_p &= A_3 + A_4 = \frac{\epsilon_1}{2(1+\epsilon_1)} + \frac{2\epsilon_1}{(1+\epsilon_1)^2(4\epsilon_1-1)} = \frac{392985}{137842} > 1 \\
&\quad \left(\text{choosing } \epsilon_1 = \frac{9}{32}\right).
\end{aligned}$$

Moreover from equation (6.4)

$$\begin{aligned}
r_0 &= 1 + c_0^{-1}\left(\frac{-\epsilon_2 c_2}{\epsilon_2 + c_2} + \sum_{m=1}^{2} a_m\right) = \frac{10\epsilon_2 + 12}{3(1+\epsilon_2)}, \\
r_1 &= 1 + c_1^{-1}\left(\frac{-\epsilon_2 c_2}{\epsilon_2 + c_2} + \sum_{m=2}^{2} a_m\right) = \frac{5\epsilon_2 + 7}{4(1+\epsilon_2)} \\
\text{and } r_2 &= 1 + c_2^{-1}\left(\frac{-\epsilon_2 c_2}{\epsilon_2 + c_2} + \sum_{m=3}^{2} a_m\right) = \frac{1}{1+\epsilon_2} \\
\text{so that } B_0 &= \frac{\epsilon_2 c_0^{-1}}{2}\left(\prod_{s=1}^{2} r_s\right)\left(\prod_{s=0}^{2} r_s\right) = \frac{\epsilon_2}{3} \times r_0 \times r_1^2 \times r_2^2 \\
&= \frac{\epsilon_2}{3} \times \frac{10\epsilon_2 + 12}{3(1+\epsilon_2)} \times \left(\frac{5\epsilon_2 + 7}{4(1+\epsilon_2)}\right)^2 \times \left(\frac{1}{1+\epsilon_2}\right)^2, \\
B_1 &= \frac{\epsilon_2 c_1^{-1}}{2}\left(\prod_{s=2}^{2} r_s\right)\left(\prod_{s=1}^{2} r_s\right) = \frac{\epsilon_2}{4} \times r_1 \times r_2^2 \\
&= \frac{\epsilon_2}{4} \times \frac{5\epsilon_2 + 7}{4(1+\epsilon_2)} \times \left(\frac{1}{1+\epsilon_2}\right)^2
\end{aligned}$$

$$\begin{aligned}
\text{and } B_2 &= \frac{\epsilon_2 c_2^{-1}}{2}\left(\prod_{s=3}^{2} r_s\right)\left(\prod_{s=2}^{2} r_s\right) = \frac{\epsilon_2}{2}\times r_2 = \frac{\epsilon_2}{2}\times\frac{1}{1+\epsilon_2}.\\
\text{Now } \sum_{j=0}^{2} B_j &= B_0+B_1+B_2 = \frac{99}{144}+\frac{3}{32}+\frac{1}{4} = \frac{1188}{1152} > 1\\
&\qquad (\text{choosing } \epsilon_2 = 1).
\end{aligned}$$

As inequalities (6.16) and (6.17) of corollary 6.10 are satisfied, the given equation is conjugate on $[0,4]$.

To verify independently that equation (1.7) is conjugate on $[0,4]$ note that the three term equation (1.8) yields

$$x_{n+1} = \frac{-(a_n - c_n - c_{n-1})}{c_n}x_n - \frac{c_{n-1}}{c_n}x_{n-1}.$$

If x_n is the solution of the given system with $x_0 = 0$ and $x_1 = 1$ then

$$\begin{aligned}
x_2 &= \frac{-(a_1 - c_1 - c_0)}{c_1}x_1 - \frac{c_0}{c_1}x_0 = \frac{1}{4} > 0,\\
x_3 &= \frac{-(a_2 - c_2 - c_1)}{c_2}x_2 - \frac{c_1}{c_2}x_1 = \frac{-13}{8} < 0.
\end{aligned}$$

Hence x_n has a zero at $n = 0$ and a generalized zero at $n = 3$.

The following example shows that if the hypotheses of theorem 6.5 are not satisfied then there is no guarantee that equation (1.7) is conjugate.

Example 6.12. Consider equation (1.7) with $M = 0$, $N = 1$, $c_n = 1$ on $[0,2]$ and $a_n = 0$ on $[1,2]$ and $a_n \neq 0$ for every $n < 1$ or > 2.

For $k \in (0,2)$, that is, for $k = 1$ and $2 \le s \le 3$ we have from equation (6.4)

$$\begin{aligned}
q_s &= 1 + c_{s-1}^{-1}\left(\epsilon_1 - \sum_{m=2}^{s-1} a_m\right)\\
&= 1 + \epsilon_1.
\end{aligned}$$

$$\text{Therefore,} \quad \sum_{p=2}^{3} \tan^{-1}\left[\frac{\epsilon_1}{2}\left(\prod_{s=2}^{p} q_s\right)^{-1}\left(\prod_{s=2}^{p-1} q_s\right)^{-1}\right]$$

$$\begin{aligned}
&= \tan^{-1}\left[\frac{\epsilon_1}{2} \times \frac{1}{q_2}\right] + \tan^{-1}\left[\frac{\epsilon_1}{2} \times \frac{1}{q_2^2} \times \frac{1}{q_3}\right] \\
&= \tan^{-1}\left[\frac{\epsilon_1}{2(1+\epsilon_1)}\right] + \tan^{-1}\left[\frac{\epsilon_1}{2(1+\epsilon_1)^3}\right]. \\
&= \tan^{-1}\left[\{\frac{\epsilon_1}{2(1+\epsilon_1)} + \frac{\epsilon_1}{2(1+\epsilon_1)^3}\}/\{1 - \frac{\epsilon_1}{2(1+\epsilon_1)} \times \frac{\epsilon_1}{2(1+\epsilon_1)^3}\}\right] \\
&\qquad \left(\text{since } 0 < \frac{\epsilon_1}{2(1+\epsilon_1)} < 1 \text{ and}\right. \\
0 &< \left.\frac{\epsilon_1}{2(1+\epsilon_1)^3} < 1, \text{ for arbitrary } \epsilon_1 > 0\right) \\
&= \tan^{-1}\left[\frac{2(1+\epsilon_1)\epsilon_1(\epsilon_1^2 + 2\epsilon_1 + 2)}{4(1+\epsilon_1)^4 - \epsilon_1^2}\right] \\
&< \tan^{-1}\left[\frac{2(1+\epsilon_1)\epsilon_1(\epsilon_1^2 + 2\epsilon_1 + 2)}{2(1+\epsilon_1)^4}\right] \quad (\text{since } 4(1+\epsilon_1)^4 - \epsilon_1^2 > 2(1+\epsilon_1)^4) \\
&= \tan^{-1}\left[\frac{\epsilon_1(\epsilon_1^2 + 2\epsilon_1 + 2)}{(1+\epsilon_1)^3}\right] \\
&= \tan^{-1}\left[\frac{\epsilon_1^3 + 2\epsilon_1^2 + 2\epsilon_1}{1 + 3\epsilon_1 + 3\epsilon_1^2 + \epsilon_1^3}\right] \\
&< \frac{\pi}{4}.
\end{aligned}$$

Thus inequality $(6.3)_a$ of the hypothesis of theorem 6.5 is violated for any choice of $\epsilon_1 > 0$. However it follows from lemma 6.2 that the given equation is disconjugate on $[0, 3]$.

References

[1] CD. Ahlbrandt and J.W. Hooker, Disconjugacy criteria for second order linear difference equations, Edmonton-summer 1984 Conference Proceedings (1984), 1-9.

[2] C.D. Ahlbrandt and J.W. Hooker, Recessive solutions of symmetric three term recurrence relations, Canadian Mathematical Society Conference Proceedings 8 (1987), 3-42.

[3] CD. Ahlbrandt and M. Heifetz, Discrete Riccati equations of filtering and control, Proceedings of the first international conference on difference equations, Gordon and Breach (1994), 1-16.

[4] CD. Ahlbrandt and A. C. Peterson, The (n,n)-disconjugacy of a $2n^{th}$ order linear difference equation, Computers Math. Appl. 28 (1994), 1-9.

[5] CD. Ahlbrandt and W.T. Patula, Recessive solutions of block tridiagonal nonhomogeneous systems, J. Difference Equations, 1 (1995), 1-15.

[6] R. Bellman, "Introduction to matrix analysis", 2nd ed, McGraw-Hill , New York 1970.

[7] M. Bohner, Contrallability and disconjugacy for linear Hamiltonian difference systems, Proceedings of the first international conference on difference equations, Gordon and Breach (1994), 65-77.

[8] M. Bohner, Linear Hamiltonian difference systems: disconjugacy and Jacobi-type conditions, J. Math. Anal. Appl. 199 (1996), 804-826.

[9] **G.J.** Butler, **L.H.** Erbe and A.B. Mingarelli, Riccati techniques and **variational** principles in oscillation theory **for linear systems, Trans.** Amer. **Math. Soc.** 303 (1987), 263-282.

[10] R. Byers, B.J. Harris and M.K. Kwong, Weighted means **and** oscillation conditions for second order matrix differential equations, J. Differential Equations 61 (1986), 164-177.

[11] W.J. Coles and Micheal K. Kinyon, Summability methods **for** oscillation of linear second-order matrix differential equations, Rocky Mountain J. Math. 24 (1994), 19-36.

[12] W.A. Coppel, "Disconjugacy", Lecture Notes in Mathematics, 220 Springer-Verlag, New York/Berlin, 1971.

[13] 0. Došlý, Existence of conjugate points for self-adjoint linear differential equations, Proc. Roy. Soc. Edinburgh 113A (1989), 73-85.

[14] 0. Došlý, On the existence of conjugate points for **linear** differential systems, Math. Slovaca 40 (1990), 87-99.

[15] 0. Došlý, The multiplicity criteria for zero points of second order differential equation, Math. Slovaca 42 (1992), 181-193.

[16] 0. Došlý, Conjugacy criteria for second order differential equations, Rocky Mountain J. Mathematics 23 (1993), 849-861.

[17] 0. Došlý, Transformation of linear Hamiltonian difference systems and some of their applications, J. Math. Anal. Appl. 191 (1995), 250-265.

[18] 0. Došlý and Pavel Řehák, Conjugacy criteria for second order linear difference equations, Arch. Math. (Brno), to appear.

[19] L.H. Erbe, Qingkai Kong and Shigui Ruan, Kamenev type theorems for second order matrix differential systems, Proc. Amer. Math. Soc. 117 (1993), 957-962.

[20] L. Erbe and P. Yan, Disconjugacy for linear Hamiltonian difference systems, J. Math. Anal. Appl. 167 (1992), 355-367.

[21] G.J. Etgen and R.T. Lewis, Positive functionals and oscillation criteria for differential systems, in "Optimal control and differential equations" (A.B.Schwarzkopf, W.G. Kelley and S.B.Eliason Eds) pp. 245-275, Academic Press, New York.

[22] G.J. Etgen and J.F. Pawlowski, Oscillation criteria for second order self adjoint differential systems, Pacific J. Math. 66 (1976), 99-110.

[23] P. Hartman, "Ordinary Differentia] Equations", S.M. Hartman, Baltimore 1973.

[24] P. Hartman, Difference equations: disconjugacy, principal solutions, Green's functions, complete monotonicity, Trans. Amer. Math. Soc. 246 (1978), 1-30.

[25] P. Hartman, Oscillation criteria for self-adjoint second- order differential systems and "Principal sectional curvatures", J. Differential Equations 34 (1979), 326-338.

[26] Hartman and Wintner, On an oscillation criterion of Liapunoff, Amer. J. Mathematics, 73 (1951), 885-900.

[27] D.B. Hinton and R. T. Lewis, Oscillation theory for generalized second-order differential equations, Rocky Mountain J. Mathematics 10 (1980), **751-766.**

[28] R.A. Horn and C.R. Johnson, "Matrix Analysis", Cambridge University Press, New York, 1985.

[29] I.V. Kamenev, Integral criterion for oscillations of linear differential equations of second order , Mat. Zametki 23 (1978), 249-251.

[30] W.G. Kelley and A.C. Peterson, "Difference equations: an introduction with applications", Academic Press, 1991.

[31] M.K. Kwong, H. G. Kaper, K. Akiyama and A.B. Mingarelli, Oscillation of linear second order differential systems, Proc. Arner. Math. Soc. 91 (1984), 85-91.

[32] R.T. Lewis and L.C. Wright, Comparison and oscillation criteria for self-adjoint vector-matrix differentia] equations, Pacific J. Math. 90 (1980), 125-134.

[33] A.B. Mingarelli, On a conjecture for oscillation of second order ordinary differential systems, Proc. Amer. Math. Soc. 2 (1981), 593-598.

[34] Fanwei Meng, Jizhong Wang and Zhaowen Zheng, A note on Kamenev type theorems for second order matrix differentia] systems, Proc. Amer. Math. Soc. 126 (1998), 391-395.

[35] M. Morse, "The calculus of variations in the large", Amer. Math. Soc. Colloq. Publ. 18 (1960).

[36] E. Müller-Pfeiffer, Existence of conjugate points for second and fourth order differential equations, Proc. Roy. Soc. Edinburgh 89A (1981) , 281-291.

[37] E.S. Noussair and CA. Swanson, Oscillation criteria for differential systems, J. Math. Anal. Appl. 36 (1971), 575-580.

[38] A. Peterson and J. Ridenhour, Oscillation of second order linear matrix difference equations, J. Differential Equations 89 (1991), 69-88.

[39] W.T. Reid, Ordinary differential equations, John Wiley, New York, 1971.

[40] W.T. Reid, "Applied Mathematical Sciences vol.31", Springer-Verlag, New York, 1980.

[41] C.E. Rickart, Banach algebras, Van Nostrand, New york, 1960.

[42] Terry Walters, A charecterization of positive linear functionals and oscillation criteria for matrix differential equations, Proc. of the Amer. Math. Soc. 78 (1980), 198-202.

[43] F.J. Tipler, General relativity and conjugate ordinary differential equations, J. Differential equations 30 (1978), 165-174.

[44] E.G. Tomastik, Singular quadratic functionals of n dependent variables, Trans. Amer. Math. Soc. 124 (1966), 60-76.

[45] E.C. Tomastik, Oscillation of systems of second order differential equations, J. Differential Equations 9 (1971), 436-442.

[46] William Simons, Disconjugacy criteria for systems of self-adjoint differential equations, J. London Math. Soc, 6 (1973), 373-381.

[47] A. Wintner, A criterion of oscillatory stability, Quart. Appl. Math. 7 (1949), 115-117.

The following papers contain the main results of this thesis:

[1] Conjugacy Criteria for a Two Dimensional Linear Hamiltonian System, J. Differential Equations and Dynamical Systems, 4(1996), 441-452.

[2] Conjugacy Criteria for a 2n- dimensional Linear Hamiltonian System, J. Differential Equations and Dynamical Systems, 6 (1998), 449-469.

[3] Conjugacy Criteria for a Linear Second Order Difference Equation, J. Dynamical Systems and Applications (to appear).

[4] Oscillation Criteria for Linear Hamiltonian Matrix Systems (to be communicated).