SETS OF PERIODS OF CONTINUOUS SELF MAPS ON SOME METRIC SPACES

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DECLARATION

I hereby declare that the work presented in this thesis has been carried out by me under the supervision of Prof. V. Kannan in partial fulfillment of my Ph.D. degree. This has not been submitted for any degree or diploma of any other university.

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Certified that the work contained in this thesis entitled 'SETS OF PERIODS OF CONTINUOUS SELF MAPS ON SOME METRIC SPACES', has been carried out by Mr.P.V.S.P. Saradhi, under my supervision and the same has not been submitted for the award of research degree of any university.

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Dedicated

To

LORD VENKATESWARA

ABSTRACT

In this work we generalise Sarkovskii's theorem to some metric spaces . This research was suggested by Alseda et al in [1] . This answers a. problem (Problem 5.1) posed by **Baldwin in** [3] among a large class of spaces. We describe all possible sets of periods of continuous self maps on any zero dimensional metric space, any compact subset of $\mathbb{R}$ and on **any** convex subset of $\mathbb{R}^n$.

CONTENTS

Chapter 0

Notations

$\mathbb{R}$: Set of all real numbers

$\mathbb{Z}$: Set of all integers

$\mathbb{N}$: Set of natural numbers

$\wp(\mathbb{N})$: Set of all subsets of $\mathbb{N}$ (i.e., power set of $\mathbb{N}$)

For $n \in \mathbb{N}$ we denote

$$\mathcal{F}_n = \{A \subset \mathbb{N} \mid A \text{ is nonempty } ; \ \sum a < n\}$$

$$\mathcal{F}_* = \bigcup \{\mathcal{F}_n ; \ n \in \mathbb{N}\} = \{\text{nonempty finite subsets of } \mathbb{N}\}$$

$$\mathcal{G}_1 = \{A \subset \mathbb{N} \mid A \text{ is nonempty and finite or } 1 \in A\}$$

$$\mathcal{G}_n = \{A \subset \mathbb{N} \mid A \text{ is nonempty, there exist two finite subsets } F \text{ and } G \text{ of } A \text{ such that every element of } A \setminus G \text{ is a multiple of some element of } F, \text{ and such that } F \in \mathcal{F}_n\}$$

$$\mathcal{G} = \{A \subset \mathbb{N} \mid \text{ there exists a nonempty finite subset } F \text{ of } A \text{ such that every element of } A \text{ is a multiple of some element of } F\}$$

$$\mathcal{H}_1 = \{A \subset \mathbb{N} \mid 1 \in A\} \cup \mathcal{G}$$

$\mathcal{H}_n$ = $\{A$ C $\mathbb{N}$ | either some element of A is $< n$ or

3 a finite nonempty subset F of A such that

A C { multiples of elements of $F\}\}$

$$\mathcal{U}_1 = \{A \subset \mathbb{N} | 1 \in A\}$$

For any two families $\mathcal{F}$ and $\mathcal{G}$ we denote

$$\mathcal{F} * \mathcal{G} = \{\bigcup_{n \in B} nA_n : B \in \mathcal{F}, A_n \in \mathcal{G}, \ \forall\, n \in B\}$$

$$\mathcal{F} V \mathcal{G} = \{B \cup C : B \in \mathcal{F}, C \in \mathcal{G}\} \cup \mathcal{F} \cup \mathcal{G}$$

For any space X ,we denote :

$|X|$ = cardinality of X

X' = derived set of $X = \{ x \in X | \{x\}$ is not open}

$X'' = (X')'$ and $X''' = (X'')'$

Chapter 1

Introduction

§0

Definitions:

Let IR denote the set of all real numbers, *7L* denote the set of all integers and IN denote the set of all natural numbers. If / is a function from a set *X* to itself, we denote

$$f^2(x) \quad = \quad f(f(x))$$

and in general $f^{n+1}(x) = f(f^n(x))$ for all *n* in IN, and for all *x* in *X*. Then f^n is also a function from *X* to *X*. If *X* is a topological space, and / : $X \to X$ is continuous, we say that / is a continuous self-map of *X*. If for some *x* in X, *n* is the least positive integer such that $f^n(x) = x$, we say that *x* is a periodic point of /, of period *n*. We let *Per(f)* = { periods of periodic points of / }.

We define $PER(X) =$ ***{Per(f)*** : / is a continuous self-map of *X}*.

This is a family of subsets of **IN** .

§1

Sarkovskii's Theorem, $PER(\mathbb{R})$ and $PER(I)$

Let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$. Then one can prove "If f has a point t with period $l > 1$ then f has a fixed point (i.e. point of period 1)". This provokes us to think about the following questions:

Suppose a continuous function f has a point of period $k > 1$.Vill this k imply any other period along with 1?

Can a point of lower period imply the higher one?

Can a point of even period imply odd period > 1 ?

Precisely, if f is a continuous function with points of period m, must it also have points of other periods n, n $\neq m$?

There are some results along these lines. In 1975, the article: "Period three implies chaos ", was published in the American Mathematical Monthly by Li and Yorke [23]. In that article, they proved the following interesting theorem :

Theorem 1.1: If $3 \in Per(f)$ for a continuous function f: $\mathbb{R} \to \mathbb{R}$,thcn $Per(f) = \mathbb{N}$

In otherwords, period 3 $\Rightarrow$ period n for all n. [We say period $m \Rightarrow$ period n, if the existence of a point of period m implies that of another point of period n].

In that paper they also produced a counter example to show that period 5 does not imply period 3. Unknown to Li and Yorke , a Russian mathematician A.N. Sarkovskii [31] had already answered in 1964, the general question of when period m implies period n for continuous selfmaps on real line.

He introduced a new total ordering on the natural numbers as follows:

Sarkovskii ordering $\triangleright$:

$3 \triangleright 5 \triangleright 7 > \bullet\bullet\bullet > 2.3 \triangleright 2.5 \triangleright 2.7 \bullet\bullet\bullet \triangleright 2^2.3 \triangleright 2^2.5 > 2^2.7 \bullet\bullet\bullet \triangleright 2^3 \triangleright 2^2 \triangleright 2 \triangleright 1$

Sarkovskii's Theorem

Theorem 1.2 : Suppose / : $\mathbb{R} \to \mathbb{R}$ is continuous. Suppose / has a periodic point of period *k*. If $k \triangleright$ / in the above ordering, then / has a periodic point of period / also .

From the theorem we make the following observations:

1. If *Per(f)* is finite, then every element of *Per(f)* is a power of 2.
2. period 3 is the greatest period in the Sarkovskii ordering and therefore period 3 implies the existence of all other periods. Hence the main result of [23] becomes a corollary to this theorem.

The converse of Sarkovskii's theorem is also true.

1.3 Converse of Sarkovskii's Theorem:

Theorem 1.3: Let *h* and *k* be two positive integers such that *h* precedes *k* in $\triangleright$ ordering. There always exists a continuous function from $\mathbb{R}$ into IR which has a point of period *k* but no point of period *h*.

Later, people have tried to understand the proof of Sarkovskii theorem (which was given by himself) and found it to be long and complicated. In the words of P.D. Straffin in [36], "He constructs so many sequences of points that eight complex figures and most of the letters of the Greek alphabet are necessary to keep track of them. Sarkovskii's theorem is an example of a common occurence in mathematics of an elegant result whose first proof is extremely inelegant." In fact, the Sarkovskii's theorem can be deduced **from** three basic results:

(i) period $k \Rightarrow$ period 2, for all AT > 1

(ii) any odd period > 1 $\Rightarrow$ all higher odd periods

(iii) any odd period > 1 $\Rightarrow$ all even periods.

Straffin [36] attmpted to give a simpler proof of the sufficiency part of the Sarkovskii's theorem i.e., period m implies period n if m precedes n in the above ordering. Using directed graphs (digraph in short) he proved the following results:

Theorem A: If a k-periodic point digraph associated to / has a non-repetitive cycle of length /, then / $\in$ $Per(f)$.

Theorem B: If a continuous function / : $\mathbb{R} \to \mathbb{R}$ is such that $k \in Per(f)$ where k is an odd integer > 1 then $\{k, k+1, k+2, \cdots\} \subset Per(f)$.

Theorem B, stated above, comes close to provide a more elegant proof of Sarkovskii's theorem for it embodies (i) and most of (iii). The remaining part was left as a gap. In, 1981, Chung-Wu Ho and Charles Morris [15] filled the gap in Strattin's work. They gave a proof of 'necessary' part of the theorem, which was also based on directed graphs, and gave a complete simple proof of Sarkovskii's theorem and its converse. They proved it by a sequence of propositions listed below.

Proposition 1. If a continuous function / : IR $\to$ $\mathbb{R}$ has a point of odd period $2n+1$ $(n > 1)$, then it must have periodic points of all even periods.

Proposition 2. If a continuous function / : $\mathbb{R} \to \mathbb{R}$ is such that $k \in Per(f)$ for some $k >$ 1 then $\{1,2\} \subset Per(f)$

Proposition 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If h precedes k in the Sarkovskii ordering, and if $h \in Per(f)$, then $k \in Per(f)$.

Proposition 4. For each $n > 2$, there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ which has a point of period $2n+1$ but has no point of period $2m+1$ for any m with $1 < m < n$.

Proposition 5. Let h and k be two positive integers such that h precedes k in $\triangleright$ ordering. There always exists a continuous function from $\mathbb{R}$, into $\mathbb{R}$ which has a point of period k but no point of period h.

Proposition 4 is subsumed in Proposition 5, which is indeed the converse of Sarkovskii theorem. There are very few examples in the literature of Dynamical systems, which deal with the maps that have points of period 5 but has no points of period 3. Stefan in [35] gave a general pattern to generate maps that have points of period (2k+ 1) but no points of period (2k-1). He also constructed maps with points of periods $2^k(2n+1)$ but without points of periods $2^k(2n-1)$ for any $n \in \mathbb{N}$ and $k > 0$. In 1996,S.Elayadi [12] gave a new and simple construction of such maps and thus proved the converse of Sarkovskii's theorem again.

An independent proof of Sarkovskii's theorem, also based on digraphs, was discovered jointly by Block, Guckenheirner, Misiurewicz and Young. In [11], one can see the proof of the Sarkovskii's theorem in a simplified form and it is further simplified in [17] .

We must emphasize that Sarkovskii's theorem, together with its converse, can be restated as a theorem describing all sets of periods as stated below:

Theorem 1.4: $PER(\mathbb{R}) = \{A \subset \mathbb{N} \mid m \in A$ and $m \triangleright n$ in Sarkovskii's ordering implies $n \in A\}$.

If X is the unit interval I or any compact interval then by intermediate value theorem every continuous self map on it has a fixed point. Hence empty set cannot be a member of $PER(I)$.

Theorem 1.5 : $PER(I) = PER(\mathbb{R})\setminus\{\phi\}$.

We denote $PER(I)$ by S.

§2

Period sets of Unit circle S^1 in the plane

The theorem of Sarkovskii specifies, for continuous maps of an interval, which sets of positive integers may occur as the sets of periods. Results along these lines have also been obtained for maps of the circle.

2.1 Some results on S^1 about sets of periods:

In 1980, L.S. Block [6] proved the following interesting results on continuous self maps of the unit circle S^1. By $f \in C(S^1,S^1)$ we mean f is a continuous self map from S^1 to S^1

Theorem A: Let $f \in C(S^1,S^1)$ Suppose $1 \in Per(f)$ and $n \in Per(f)$ for some odd integer $n > 1$, then for every integer $m > n$, $m \in Per(f)$.

Theorem B: Let $f \in C(S^1,S^1)$ and suppose that $Per(f)$ is finite. Then there are integers m and n (with $m > 1$ and $n > 0$) such that

$$Per(f)= \{m, 2.m, 2^2.m, \cdots, 2^n.m\}$$

Theorem C: Let $f \in C(S^1,S^1)$. If $\{1,2,3\} \subset Per(f)$ then $Per(f) = \mathbb{N}$. Conversely, if $S \subset \mathbb{N}$ with the property that for any $f \in C(S^1,S^1)$ $S \subset Per(f) \Rightarrow Per(f) = \mathbb{N}$

then {1,2,3} *C S.*

2.2 Descripion of Per(f) when / has a fixed point:

Again in 1981, Block [7] proved the following main result about $Per(f)$for / G $C(S^1S^1)$ when f has a fixed point.

Theorem 1: Let / G $C(S^1, S^1)$. Suppose 1 G *Per(f)* and *n* G $Per(f)$for some integer *n* > 1. Then (atleast) one of the following holds:

(i) For every integer m with *n* < m, *rn* G $Per(f)$

(ii) For every integer *rn* with *n* ⊳ m, m G *Per(f)*

(here < denotes the usual order on $\mathbb{N}$).

He has also proved the converse of the above theorem i.e

Theorem 2: Let *S* C IN with 1 G *S.* Suppose that for each *n* G *S* with *n* $\neq$ 1 atleast one of the following holds.

(a) for every integer m with *n* < m, *rn* G *S.*

(b) for every integer m with *n* ⊳ m, m G *S.*

Then there exists a continuous map / G $C(S^1, S^1)$such that the set of periods of **periodic** points of / is exactly *S.*

These two theorems of Block characterize the sets of periods which can occur for a continuous map of the circle to itself having a fixed point. But not every self map / of the circle has a fixed point. One can see that if the deg(/) $\neq 1$ (where / € $C(S^1, S^1)$) then / has a fixed point. Partly for this reason, degree-one maps of the circle require special attention to study the periodic orbits. See [8] for more details.

2.3 **Per(f)** when degree of f $\neq 1$:

In 1982, M. Misiurewicz [27] proved the result, which describes the possible sets of periods of the periodic points of a continuous degree one map of the circle. For any two real numbers a and 6, let $M(a,6)$ = {n ∈ IN : a < $\frac{t}{n}$ < *b* for some integer t}

If *a* ∈ IR and / ∈ IN U {2°°} , we define a subset

$S(a,l)$C IN as follows:

If *a* is irrational then $S(a,l) = \phi$

If a is rational and if *a* = $\frac{t}{n}, n$ € IN, t ∈ Z, (t,n)= 1 and if / ∈ IN then *S(a,l)* denotes the set of positive integers of the form ns,where / l> *s* (in Sarkovskii ordering).

If / = 2°° then $S(a,/)$ denotes the set of all positive integers of the form ns, where *s* is a power of 2. Now we state Misiurewicz's result about continuous maps of circle.

Theorem 3: Let / be a continuous map of the circle to itself of degree one. Then there exist a, b ∈ IR with a < *b* and l, r ∈ IN U $\{2^\infty\}$ such that *Per(f)* = *M(a, b)* U $S(a,/)$ U $S(b,r)$

Conversely, for every subset A of INT of the form *A* = $M(a$,*b)* U *S(a,* $l)$ U 5(6, r) there is a continuous map of the circle to itself of degree one such that $Per(f)$= *A*

Hence we have a complete answer for describing the sets of periods for continuous self maps on circle,because

$$PER(S^1) = \{Per(f) | /\text{is a continuous self map on} S^1 \text{with deg } 1\}$$

$$\bigcup \{Per(g) | g \text{ is a continuous self map on} S^1 \text{with a fixed point}\}$$

§3

Period sets of Y

In 1989, L. Alseda, J. Llibre and M. Misiurewicz [1] made a generalization of Sarkovaskii's theorem to characterise the possible sets of periods for continuous maps f of the space $Y = \{z \in \mathbb{C} : z^3 \in [0,1]\}$ into itself for which /(0) = 0.

In this section by / ∈ *C(Y)* we mean f is a continuous self map on *Y*. The following was proved by Mumbru in [28] for / € *C(Y)*

Theorem 3.1 :

(a) If / ∈ $C(Y)$ and {2,3,4,5,7} C *Per(f)* then *Per(f)* = IN.

(b) If *W* C IN is a set such that for every $f \in Y$, $VPer(f)$ implies $Per(f)$= IN, then $W \subset$ {2,3,4,5,7}.

3.2 Some notations and definitions:

To describe the result of Alseda et al in [1], we need to introduce the following notations and two new orderings.

S(k) = *{n* ∈ IN : *k* $\rhd n\}$ U *{k}Vk* ∈ IN(here l> is Sarkovskii order)

$S(2^\infty)$= $\{ 2^i \mid i = 0, 1, \cdots\}$

(a) Green ordering is the ordering of IN — {2}, denoted by $<_g$,and defined as:

5 $<_g$ 8 ***Kg*** **4** $<_g$ **11** $<_g$ 1$<_g$ $\cdots <_g$

3.3 $<_g$ 3.5 $<_g$ 3.7 $<_g$ · 3.2.3 $<_g$ 3.2.5 ··• $<_g$ 3.2^2 $<_g$ 3.2 $<_g$ 3.1 $<_g$ 1. The first part of this ordering can be understood as

6 - 1, 6 + 2, 3 + 1, 2.6 - 1, 2.6 + 2, 2.3 + 1, 3.6 - 1, 3.6 + 2, 3.3 + 1, • • •

(b) Red ordering is the ordering of $\mathbb{N}\setminus\{2,4\}$, denoted by $<_r$ and defined as :
$7 <_r 10 <_r 5 <_r 13 <_r 16 <_r 8 <_r 19 <_r 22 <_r 11 <_r 25 <_r 28 <_r 14 <_r \cdots$
$\cdots <_r 3.3 <_r 3.5 <_r 3.7 <_r 3.9 <_r \ldots <_r 3.2.3 <_r 3.2.5 <_r 3.2.5.7 <_r \cdots <_r \cdots$
$\cdots 3.2^3 <_r 3.2^2 <_r 3.2 <_r 3.1 <_r 1$. Here the first part can be viewed as,

$$6 - 1, 6 + 4, 3 + 2, 2.6 + 1, 2.6 + 4, 2.3 + 2, 3.6 + 1, 3.6 + 4, 3.3 + 2 \cdots$$

we denote

$$G(n) = \{n\} \cup \{k : n <_g k\} \text{for } n \in \mathbb{N}\setminus\{2\}$$

$$R(n) = \{n\} \cup \{k : n <_r k\} \text{for } n \in \mathbb{N}\setminus\{2,4\}$$

and

$$G(3.2^\infty) = R(3.2^\infty) = \{1\} \cup \{3.i; i \in 5(2^\infty)\}$$

we also denote

$$N_s = \mathbb{N} \cup \{2^\infty\}$$

$$N_g = (\mathbb{N}\setminus\{2\}) \cup \{3.2^\infty\}$$

$$N_r = (\mathbb{N}\setminus\{2,4\}) \cup \{3.2^\infty\}$$

Now we are ready to state the main result of Alseda, Llubre and Misiurewicz on sets of periods of *Y*.

Theorem 3.3 :

(a) If $f \in C_o(Y)$ (i.e., f is continuous self map on Y with $f(0) = 0$) then $Per(f) = S(n_s) \cup G(n_g) \cup R(n_r)$ for some $n_s \in N_s$, $n_g \in N_g$ and $n_r \in N_r$.

If $n_s \in N_s$, $n_g \in N_g$ and $n_r \in N_r$ then there exists a continuous self map f on $C_o(Y)$ **for** which

$$Per(f) = S(n_s) \cup G(n_g) \cup R(n_r) .$$

In [1] at the end they posed an open question to describe possible sets of periods of continuous self maps f on the space $X_k = \{z \in \mathbb{C} | \ z^k \in [0,1]\}$ for $k > 3$., for which zero is fixed point (i.e., $f(0) = 0$).

This question was taken up by S. Baldwin and he genaralised the theorem of[1] . He described the complete solution for the sets of periods of continuous self maps on n-od with 0 as fixed point.

§4

Sets of periods n-od

The n-od,denoted by X_n is the subspace of the complex plane, which is described as $X_n = \{ \ z \in \mathbb{C} \ | \ z^n \in [0,1]\}$ [Note that 1-od and 2-od are homeomorphic]. This can be viewed as the set obtained by attaching n copies of unit interval to the central point (i.e., at 0.)

4.1 Some definitions and notations:

In order to study the structure of the sets of periods of continuous maps $f : X_n \to X_n$ we need to define partial ordering $\leq_t$ for all positive integers t.

The ordering $\leq_1$ is defined by

$$2^i \leq_1 2^{i+1} \leq_1 2^{j+1}(2m+3) \leq_1 2^{j+1}(2m+1) \leq_1 2^j(2k+3) \leq_1 2^j(2k+1)$$

for all integers $i,j > 0$ and $k,m > 0$.

In other words $\leq_1$ is the usual Sarkovskii ordering.

If $n > 1$ then the ordering $\leq_n$ is defined as follows:

Let m, k be positive integers.

Case 1: k = 1 then m $\leq_n k$ iff m = 1

Case 2: k is divisible by n then $m \leq_n k$ iff

either $m = 1$ or m is divisible by n and $(m/n) \leq_1$ *(k/n)*

Case 3: fc > **1,** k is not divisible by n. Then $m \leq_n$ fc iff

either m = $1, m = k$ or m = $ik + jn$ for some integers i > 0, j > 1.

In [3], some diagrams illuminating these partial orderings are given. From the **definition** we can see that $\leq_1$ and $\leq_2$ coincide with the Sarkovskii ordering. If n > 2, then $\leq_n$ is not a linear ordering. Define a set S C IN to be an initial segment of $\leq_n$ if whenever A: is an element of S and $m \leq_n k$ then m is also an element of 5, i.e., S is closed under $\leq_n$ predecessors. Now we state the theorem of Baldwin in [3].

Theorem 4.2 : Let X_n be the n-od.

1. Let / : $X_n \to X_n$ be a continuous map). Then $Per(f)$is a nonempty finite union of initial segments of $\{\leq_p: 1 < p < \text{n}\}$.

2. Conversely, if S is a nonempty finite union of initial segments of $\{\leq_p: 1 < p \leq n\}$ then there is a continuous map / : $X_n \to X_n$ such that /(0) = 0 and $Per(f)$= *S.* The n-od result is the same, regardless of whether the branching point is required to be fixed or not.

§5

Period sets for Tree maps

The main result of [4] is the extension of n-od theorem to every continous self map on a tree T having all branching points fixed. It is of interest to ask what *Per(f)* can be if all branching points of T are fixed. The result on n-od has been extended to all trees (without assumptions on branching points) by Baldwin, but these results do not specify which sets are possible if the branching points are remaining fixed. For similar results on graphs which characterize sets of periods without specifying which sets of periods correspond to which graphs, see [9]. Now we state the main result of [4]

Given a tree T, let $e(T)$ and $b(T)$ be the number of end points and branching points respectively.

Theorem 5.1 : Let T be a tree.

(a) Let / : $T \to T$ be a continuous map with all branching points fixed. Then *Per(f)* is a non empty finite union of initial segments of $\{\leq_p : 1 < p < e(T)\}$.

(b) Conversely, If S is a nonempty finite union of initial segments of $\{\leq_p : 1 < P < e(T)\}$ then there is a continuous map $f : T \to T$ with all the branching points fixed such that *Per(f)* = S.

This theorem solves a problem which was originally posed by Alseda et al in [1].

§6

Summary of our results

S.Baldwin [3] asked: "Characterize all possible sets Per(f), where / ranges over all continuous functions of a graph G or of a more general topological space if one is really ambitious" (Problem 5.1). Earlier Alseda, Llibre, and Misiurewicz also posed the same problem after stating Sarkovskii's theorem : "Further research **starting** at this point can go in at least six directions : (1) Replace an interval I by another space and another class of maps $\cdots$." in [1].

We take up this problem of characterizing all possible sets of periods of more general topological spaces , and we describe *PER(X)* , when

(i) X = any zero dimensional metric space (Chapter 3)

(ii) X = any compact subset of real line (chapter 4)

(iii) X = any convex subset of $\mathbb{R}^n$ (Chapter 5).

We obtain certain families of subsets of IN (like $\mathcal{F}_n, \mathcal{G}_n, \mathcal{H}_n$ etc) as *PER(X)*. One interesting observation is that these families occuring in Chapter 3 form a chain:

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \mathcal{F}_n \subset \cdots \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots \subset \mathcal{G}_n \subset \cdots \subset \mathcal{H}_1 \subset \cdots \subset \mathcal{H}_n \subset \cdots$$

6.1 Results in Chapter 2:

We say that a space X has the periodic point property (p.p.p.) if $0 \notin PER(X)$. In Chapter 2, we find that the study of p.p.p. takes us to the class of countable compact spaces. The link between the p.p.p. and this class of spaces has been noticed in [33]. In this chapter we prove two main theorems.

Theorem 1: The **following** are equivalent for a metric space *X:*

(a) X is countable and compact.

(b) X is zero-dimensional and has p.p.p.

Theorem 2: Let X be a compact subset of $\mathbb{R}$ such that its boundary is uncountable. Then X does not have p.p.p; and conversely, every compact subset of $\mathbb{R}$ with **countable boundary** has p.p.p.

6.2 Results in Chapter 3:

In chapter3, we describe *PER(X)* when *X* is a zero dimensional metric space. The zero-dimensional spaces deserve this kind of study because:

(a) In the already existing literature in Topological Dynamics, the zero-dimensional spaces are known to have some pleasing recurrence properties for dynamical systems on them.

(b) Unlike the one-dimensional spaces, they yield themselves to our study so coherently that *PER (X)* is now calculated for a big class of spaces simultaneously.

(c) These results will be used in the next chapter.

Now we list the theorems proved in this chapter :

Theorem : Let X be a zero dimensional metric space. Then

$$PER(X) = \begin{cases} \wp(\mathbb{N}) & \text{if } X \text{ is not compact} \\ \wp(\mathbb{N}) & \text{if } X \text{ is not countable} \\ \wp(\mathbb{N})\setminus\{\phi & \text{if } X \text{ is countable, compact} \\ & \text{and } X'' \text{ is infinite} \\ \mathcal{H}_n & \text{if } X \text{ is compact and } |X''| = n \\ \mathcal{G}_n & \text{if } X \text{ is compact and } |X'| = n \\ \mathcal{F}_n & \text{if } X \text{ is finite and } |X| = n \end{cases}$$

(See Chapter 0 for Notations)

This theorem is a consequence of following theorems.Notice that metrizability is not assumed in some theorems.

Theorem A: Let X be a compact HausdorfF space with a unique limit point. Then $PER(X) = \mathcal{G}_1$.

Theorem B: Let n be a positive integer. Let X be a compact Hausdorff space with exactly n limit points. Then *PER(X)* = { A C IN | A is a nonempty set containing two finite subsets F and G satisfying (i). every element of $A \setminus G$ is a multiple of some element of F and (ii). sum of elements of F is $< n$ }.

Theorem C: Let X be a compact Hausdorff space having a unique element x_o every neighbourhood of which contains infinitely many limit points. Then $PER(X) = \mathcal{G}$.

Theorem D: Let X be a compact Hausdorff space such that \X"\ = n.Then *PER(X)* = *{A* C IN | 1 or 2 or • · • or n G *A}* U $\mathcal{G}$.

Theorem E: Let X be any countable compact Hausdorff space such that X'' is infinite. Then *PER(X)* consists of all nonempty subsets **of** $\mathbb{N}$.

Theorem F: Let K be the Cantor set. Then $PER\ (K) = \wp(\mathbb{N})$.

Theorem G: Let X be a zero dimensional metric space.Then the the following are equivalent:

(i)X is countable and compact.

(ii)$PER(X) \neq \wp(\mathbb{N})$.

(iii)$\phi \notin PER(X)$.

6.3 Results in Chapter 4:

We devote Chapter 4 to describe *PER(X)* when *X* is any compact subset of real line. This is achieved by proving the following theorems.

Theorem 1: Let $n \in \mathbb{N}$. Let $X = I_1 \cup I_2 \cup \cdots \cup I_n$, where each I_i is a closed interval in $\mathbb{R}$. Let these I_i's be pairwise disjoint, then

$PER(X) = \{A \subset \mathbb{N} | A$ is of the form $n_1 A_1 \cup n_2 A_2 \cup \cdots \cup n_k A_k$ where each $A_i \in S$ and $n_1 + n_2 + \cdots + n_k < n\}$

Theorem 2: Let X be a topological space. Let $\tilde{X}$ be the space of all connected components of X. Let $q : X \to \tilde{X}$ be the natural map defined as, $q(x) = C_x$, the component of x (viewed as an element of $\tilde{X}$). Let f be a continuous self map on X and $\tilde{f}$ be the induced map of f on $\tilde{X}$. Then

$Per(f) = \bigcup_{n \in per(\tilde{f})} n\, Per(f^n|_{X_n})$ where $X_n = \{x \in X \mid \tilde{f}$-period of q(x) is n$\}$

Theorem 3: Let X be a compact subset of $\mathbb{R}$ such that every component of X is non trivial. Then $PER(X) = \{\bigcup n A_n : E \in PER(\tilde{X}), A_n \in S \;\; \forall n \in E\}$

Theorem 4: Let X be a compact subset of $\mathbb{R}$ such that (i) X has infinitely many non-trivial components. (ii) there is only one component of X that is not open.

Then $PER(X) = \mathcal{G}_1 * \mathcal{S}$

Theorem 5: Let X be a compact subset of IR with n nontrivial components and with only one non open component. Then $PER(X) = (\mathcal{F}_n * \mathcal{S}) V \mathcal{G}_1$

Theorem 6(a): Let $m, n, p \in$ IN. Let X be a compact subset of IR with n non-open trivial components, m nontrivial open components and p non open non trivial components. Then $PER(X) = \bigcup (\mathcal{G}_r V\ \mathcal{G}_s\ V\ (\mathcal{F}_s * S)\ V\ (\mathcal{F}_t * S))$ where the union is taken over all triples (r,s,t) of positive integers satisfying the inequalities $s < p$, $r + s < n+p$, $s + t < m+p$, and $r + s + t < rn + n+p$

Theorem 6(b): Let X be a compact subset of IR ,such that $|X|' = n$. Let all the open components of X be nontrivial . Then $PER(X) = \mathcal{G}_n * S$. (Note: Some non-open components may be trivial, some others not)

Theorem 6(c): Let X be a compact subset of $\mathbb{R}$ with n non-open trivial components and infinitely many open nontrivial components and p nontrivial non open components, and r nontrivial nonopen components in X such that every open set containing it intersects infinitely many nontrivial components . Then

$$PER(X) = \bigcup_{0<s<r} ((\text{ft} * S)\ \text{V}\ \mathcal{G}_{n+p-s})$$

Theorem 6(d): For any compact subset X C $\mathbb{R}$ such that $|X''|$ = n, $PER(X) = \mathcal{H}_n$

Theorem 6(e): Let X be compact subset of $\mathbb{R}$. Then

$$PER(X) = \begin{cases} \wp(\mathbb{N}) \setminus \{0\} \text{ if boundary of X is countable and } |X''| = \infty \\ \wp(\mathbb{N}) \text{ if boundary of X is uncountable} \end{cases}$$

6.4 **Results** in **Chapter** 5:

In chapter 5, We obtain a higher dimensional analogue of Sarkovskii's theorem . We generalise this theorem, by describing *PER (X)* for all convex subsets X of ET. We prove the following main theorem:

ain Theorem : Let X be a nonempty convex subset of IR". Then

$$PER(X) = \begin{cases} \wp(\mathbb{N}) & \text{if } X \text{ is noncompact} \\ \mathcal{U}_1 & \text{if } X \text{ is compact and not contained in a line} \\ \mathcal{S} & \text{if } X \text{ is infinite, compact and contained in a line} \\ \{1\} & \text{if } X \text{ is singleton} \\ \mathcal{S} \cup \{\emptyset\} & \text{otherwise} \end{cases}$$

To Prove this theorem we have proved the following :

Lemma 1: Every unbounded closed convex subset S of $\mathbb{R}^2$ contains a ray at each of its points.

Lemma 2: Every unbounded convex subset S of $\mathbb{R}^2$ contins a ray.

Lemma 3: Let S be an unbounded closed convex subset of $\mathbb{R}^n$, where $n > 2$. Let $x_o \in S$. Then there exists a hyperspace H such that $S \cap (H + x_o)$ is unbounded.

Theorem-1: **Let** A be any given subset of . Then there is a continuous / : $\mathbb{R}^2 \to \mathbb{R}^2$ such that $Per(f) = A$.

Theorem 2: Let X be a non-compact convex subset of $\mathbb{R}^2$, with nonempty interior. Then $PER(X) = \wp(\mathbb{N})$.

Theorem 3: Let S be a dosed disc. Then $PER(S) = \mathcal{U}_1$.

Theorem 4: Let S be a bounded noncompact convex subset of IR". Then *PER* (S) $= \wp(\mathbb{N})$.

Chapter 2

Periodic Point Property

The results proved in this chapter are required in later chapters. Theoreml will be used in Chapter 3 whereas Proposition 2 and Theorem 2 will be used in Chapter 4.

§1

Introduction and Preliminaries:

The concept of periodic points has been well-studied in the Theory of Functions (See [31], [36]), Ergodic Theory (see [29]), Theory of Fractals (see [24]), Theory of Dynamical systems (see [3],[4] and [11]), Algebraic Topology (see [32]) etc. We say that a topological space *X* has periodic point property (abbreviated as p.p.p) if every continuous self-map of *X* has a periodic point. This is analagous to the fixed point property (see [34]) that has been studied in General Topology, Algebraic Topology and Functional Analysis,Theory of Differential Equations, etc.

In this chapter, we find that the study of p.p.p. takes us to the class of countable compact spaces. This class of topological spaces has attracted the attention of many mathematicians like Sierpinski, Mazurkiewicz, J.de Groot, M. Katetov, Bessaga, Pelczynski, and Rajagopalan. A summary of their contributions is available in [19], It is already known that every countable compact metric space X has p.p.p.[33].It follows that every closed subspace of such X also has p.p.p.It is natural to ask whether the converse is true,i.e.if a metric space X has the property that every closed subspace of X has p.p.p,then should X be countable and compact ? We answer this in the affirmative.This may be compared with the main result of [16] where a similar converse to Banach's contraction mapping theorem has been proved. The only result on countable compact spaces that we need for our discussion, has been proved in [25] and in a different way in [18]. It states that every countable compact space is a well-ordered space.

Now we fix some notations. IN denotes the set of all natural numbers. Let / : $X \longrightarrow X$ be any self-map of X. A point x is called a periodic point of / if $f^n(x) = x$ for some n in $\mathbb{N}$. If this happens with $n = 1$, then it is called a fixed point of /. We repeat that a topological space X is said to have p.p.p. if every continuous self-map of X has a periodic point. We use the word ‘clopen’ as an abbreviated form of 'both open and closed'.

Proposition 1.1: The following spaces satisfy p.p.p.:-

(a) All finite spaces.

(b) All compact convex subsets of $\mathbb{R}^n$; in particular the closed interval [0,1] and the closed unit disc D in $\mathbb{R}^2$,

(c) All strongly rigid spaces.

(d) All well-ordered compact spaces, and in particular [1,0], where ***it*** is the first **uncountable ordinal number.**

Proof: (a) Let X be a finite discrete space having exactly n elements.

Let / : $X \to X$ be any function. Let x_o G X. Among the $n+1$ terms of the sequence x_o, $f(x_o)$, $f^2(x_o)$, • • , $f^n(x_o)$, some two should be equal by pigeon-hole **principle,** say $f^r(x_o)$- $f^s(x_o)$,with r < s. Then $f^r(x_o)$is a periodic point of /.

(b) It is a well-known theorem that every compact convex subset of $\mathbb{R}^n$ has the fixed point property (f.p.p), namely that every continuous self-map has a fixed point (See [34]). It is easily seen that f.p.p implies p.p.p.

(c) A Hausdorff space X is said to be strongly rigid if every continuous self-map of X is either a constant map or the identity map. Obviously, these spaces have f.p.p. and therefore p.p.p.

(d) Every well-ordered compact space is of the form $[1, \alpha]$ for some ordinal number α. When α is a finite ordinal, the result follows from part (a). If at all there is an ordinal α for which the result fails, let α_0 be the least such. Let X = $[1, \alpha_0]$ and let / : $X \mapsto X$ be a continuous function without periodic points. For each $x \in X$ let A_x denote $\{y$ G $X \backslash y = f^n(x)$for some n G $\mathcal{N}\}$. Let A = $\{x$ G $X \backslash x$ is in the closure of $A_x\}$,we first claim that α_0 G A. If not, A_{α_0} C $[1, \beta]$ **for** some β < α_0. Because / is continuous, / takes A_{α_0} to itself. **Noting that A_{α_0} is homeomorphic** to the well-ordered compact space [1,7] for some 7 < *fl* < α_0 it follows from the choice of α_0 that the restriction $f_{|\overline{A}_{\alpha_0}}$has a periodic point in A_{α_0} . The same point is a periodic point of / in X, contradicting the choice of /. This proves that $\alpha_0 \in A$ and thus A is non-empty. Now let α_1 be the least element of A. Then A_{α_1} C A by the continuity of /. And so α_1 < every element of A_{α_1}. Whereas $\alpha_1 \in A_{\alpha_1}$,because α_1 is in A, and this is possible only when α_1 G A_{α_1}

(since otherwise the left ray $[1, \alpha_1]$ will be a neighbourhood of α_1 disjoint from A_{α_1}). This implies that $\alpha_1 = f^n(\alpha_1)$ for some natural number n, and thus α_1 is a periodic point of /. This is a contradiction to the choice of /, hence proves the result.

Proposition 1.2: The following spaces do not possess p.p.p.

(a) An infinite discrete space.

(b) The circle S^1.

(c) The infinite product $\prod_{n=1}^{\infty} X_n$ of finite spaces $X_n = \{1, 2, \cdots, n\}$.

(d) The Cantor set A'.

(e) The real line fll (and more generally any topological group with an element of infinite order)

(f) The Stone-Cěch compactification $\beta\mathbb{N}$ of the discrete space IN of natural numbers.

Proof: (a) If X is an infinite discrete space, let $A = \{x_1, x_2, \cdots, x_n, \cdots\}$ be a countably infinite subset of X. Define / : $X \to X$ by

$$f(x) = \begin{cases} x_1 & \text{if } x \notin A \\ x_{n+1} & \text{if } x = x_n, \quad n \in \mathcal{N} \end{cases} .$$

Then / is a continuous self-map of X without any periodic point.

(b) Let $S^1 = \{(x, y) \in R^2 : x^2 + y^2 = 1\}$. Let *0* be an angle non-commensurate **with** π (that is, $ is not a rational multiple of π). Then the rotation by ***0* radians,** (this is same as multiplication by the complex number $e^{i\theta}$ if S^1 is viewed in the **complex plane)** is a continuous self-map **of** S^1 without **periodic points.**

(c) Let **for** every $n \in$ IN, the set X_n be equal to $\{m \in$ **IN** : m < rc}, **provide** discrete topology to each X_n. Let $P = \prod_{n \in \mathbb{N}} X_n$ with the product topology. Consider / : $P \to P$ defined by the rule: If $x = (x_n) \in P$, then $f(x) = y$ whose n^{th}coordinate y_n is given by

$$\begin{cases} x_{n+1} & \text{if } x \neq n \\ 1 & \text{if } x_n = n \end{cases} .$$

In other words we are taking a cyclic permutation $p_n : X_n \to X_n$ of order n for each n in IN and / = $\prod_{n \in \mathbb{N}} p_n$. The component functions $f_n : P\ X_n$ have the property that $f_n^{-1}(k) = \prod_{m=1}^{\infty} Y_m$

$$\text{where } Y_m = \begin{cases} X_m & \text{if } m \neq n \\ \text{a singleton, namely } p_n^{-1}(k) & \text{if m} = n. \end{cases}$$

So, $f_n^{-1}(k)$ is a basic open set in the product space P, for each k in X_n. Therefore each f_n is continuous. Therefore / is continuous. But / does not have a periodic point. This is because if x 6 P and $n \in$ IN, then x and $f^n(x)$necessarily differ in their $n + 1^t$ coordinate. In fact

$$\begin{cases} x_{n+1} - 1 & \text{if } x_{n+1} \neq 1 \\ \text{n} + 1 & \text{if } x_{n+1} = 1. \end{cases}$$

(d) Let K be the Cantor set consisting of those numbers in [0,1] that admit a ternary representation $0.a_1a_2 \cdots a_n \cdots$ where each a_n is either **0** or 2. Let X_n **have** the meaning as in (c). Let $Q = \prod_{n=1}^{\infty} X_{2^n}$ with product **topology. Define** $h : K \to Q$ by the rule: if x has ternary representation $0.x_1x_2 \cdots x_n \cdots$ where each x_n is either 0 or 2, then the n-th coordinate of $h(x)$ is 1 or 2 or $\cdots$ or 2^n according **as the** block $x_{2^{n-1}}, x_{2^{n-1}+1}, \cdots, x_{2^n-1}$ is $00\cdots0$, or $00\cdots01$, or $00\cdots10$, or ... or $11\cdots1$, respectively. It is easily seen that h is a bijection. Also, if W is a basic open subset of Q, of the form $\prod_{n=1}^{\infty} Y_n$ where

$$Y_n = \begin{cases} X_{2^n} & \text{if } n \neq m \\ \text{a singleton in } X_{2^m} & \text{if } n = m \end{cases}$$

then $h^{-1}(Y_n)$ is a subset of K of the form $\{x \in K$: in the ternary expansion of x, the finite number of terms $x_r, x_{r+1}, \cdots, x_{r+s}$ coincide with the pre-assigned values$\}$ for some r, s in IN and for some pre-assigned block of values from $\{0, 2\}$. It follows that $h^{-1}(Y_n)$ is clopen in K for each basic open set in Q. Therefore h is continuous. Since K and Q are compact, h is a horneomorphism. Lastly by a proof similar to that of (c), one can prove that Q does not have p.p.p. It follows that K does not have p.p.p.

(e) The map $f(x) = x + 1$ for all x in $\mathbb{R}$ is a continuous self-map of $\mathbb{R}$ without any periodic point.

(f) Let $f : \beta\mathbb{N} \to \beta\mathbb{N}$ be the unique continuous extension of the map $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x + 1$ for all x in $\mathbb{N}$. We claim that f has no periodic points. If n is in $\mathbb{N}$, the n_o subsets $A_r = \{kn_o + r; k \in \mathbb{N}\}$ for $r = 0, 1, \cdots, n - 1$ are pairewise disjoint subsets of $\mathbb{N}$. Their closures in $\beta\mathbb{N}$ should be pairwise disjoint. If x is in $\beta\mathbb{N}$, then x belongs to A_r for a unique r. But $f^m(A_r) = A_s$ is disjoint from A_r if m < n. There $f^m(x) \in A_s$, and hence $f^m(x) \neq x$. Since n_o in $\mathbb{N}$ is arbitrary, this means that no x in $\beta\mathbb{N}$ can be a periodic point of f.

§2

Theorem 1: The following are equivalent for a metric space *X:*

(a) *X* is countable and compact.

(b) *X* is zero-dimensional and has p.p.p.

(c) Every closed subspace of *X* has p.p.p.

(d) Every continuous image of *X* has p.p.p.

Proof: **Step 1:** We first prove that every countable compact Hausdorff space *X* has p.p.p. Let $f : X \to X$ be continuous. Call a subset *A* of *X* as /-invariant if *f(A)* C *A*. By Zorn's lemma, there is a minimal non-empty closed /-invariant subset *A* of *X* (such sets are called minimal sets in Topological Dynamics). Because of minimality, *A* has the property that for each *x* in A, $\{f^n(x)$ *n* G *AT}* = *A*. Applying Baire Category theorem to the countable compact Hausdorff space A, we obtain an isolated point $a \in A$. This a is in A — $\{f^n(a) : n \in \mathcal{N}\}$, and therefore $a \in \{f^n(a)$ $n \in \mathcal{N}\}$, because a is isolated in *A*. This implies that a is a periodic point of / (in fact this proves that *A* is a finite set).

Step 2: Whenever *X* is a countable compact space, so is every closed subspace of X, and so is every continuous image of *X*. Therefore it is immediate from step 1 that (1) implies both (3) and (4).

Step 3: It is well-known that every countable $T_{3\frac{1}{2}}$ (i.e., completely regular, Hausdorff) space *X* is zero-dimensional. Indeed, if *x* € *X* and if *F* is a closed subset of X not containing x, then there is a continuous function / : $X \longrightarrow [0,1]$ such that $f(x) = 1$ and $f(y) = 0$ for all y in F. The range of this / has to be countable, since its domain is so. Therefore there is t with $0 < t < 1$, that is not in the range of /. Now the set $f^{-1}([0,t)) = f^{-1}([0,t])$ is both open and closed in X, containing x, and disjoint from F.

Similar proof can be given to show that every countable T_4 space has the property that any two disjoint closed subsets can be separated by disjoint clopen subsets.

In particular every countable compact Hausdorff space is zero-dimensional. This observation together with step 1 proves that (1) implies (2). Thus so far we have proved that (1) implies each of the other three.

Step 4: We now prove that every zero-dimensional metric space X with p.p.p. has to be compact. To prove this, let $A = \{x_1, x_2, \cdots, x_n, \cdots\}$ pe anvcountably infinite subset of X; we shall prove that A has a limit point. Suppose not. Then A is a discrete closed subset of X. For each $n \in$ IN we choose a clopen set V_n such that (i) these are pairwise disjoint (ii) $V_n \cap A = \{x_n\}$and(iii) V_n C $B(x_n \frac{1}{n})$. Let V_o be the complement of$\bigcup_{n=1}^{\infty} V_n$. If $x \in V_o$, let $r > 0$ be such that $B(x,r) \cap A$ is empty. Then $B(x, \frac{r}{2}) \cap V_n$ is empty for all n such that $\frac{1}{n} < \frac{r}{2}$. Therefore x is not in the closure of $\bigcup_{k=n_o}^{\infty} V_k$ for a suitable n_o. Nor is x in the closure of the clopen set $\bigcup_{k=1}^{n_0} V_k$. Therefore the set $\bigcup_{n=1}^{\infty} V_n$ is closed. It is obviously open also. Thus $\{V_o, V_1, V_2, \cdots, V_n, \cdots\}$ is apartition of X into clopen subsets. Define / : $X \longrightarrow X$ by the rule $f(x) = x_{n+1}$ if $x \in V_n$. Then it is easily seen that / is a continuous self-map of X without any periodic point. But X has p.p.p. by assumption. Therefore A should have a limit point. Thus X is compact.

Step 5: Next we prove that every uncountable zero-dimensional compact metric space fails to satisfy p.p.p. Let X be one such space. For the subset S = {x G $X\backslash x$ admits a countable neighbourhood} the family $\{V_x : V_x$ is a countable neighbourhood of x in $S\}$ is an open cover. Because S is second countable, there should be a countable subcover for this open cover. This implies that S is countable. Let Y be the complement of S in X. Then Y is an uncountable, zero-dimensional compact metric space in which every point is a limit point. By a classical theorem (see[14]) Y is homeomorphic to the Cantor set K. Therefore Y admits a base $\{W_1, W_2, \cdots, W_n, \cdots\}$ of nonempty clopen (in Y) subsets such that $Y = W_1 \cup W_2$ and $W_n = W_{2n+1} \cup W_{2n+2}$ for all n G IN. (we have only to note that the standard base for the topology of the Cantor set K has this property.) Moreover every infinite subfamily of this base has at most one element in the intersection. Now the sets $W\backslash$ and W_2 are disjoint compact subsets of the zero-dimensional Hausdorff space Y, and therefore can be separated by some clopen subsets $W\backslash$ and W_2 of Y whose union is Y. Similarly for each n G IN, the disjoint compact subsets W_{2n+1} and W_{2n+2} °f the previously defined zerodimensional Hausdorff space W_n can be separated by disjoint clopen (in W_n and therefore in X) subsets W_{2n+1} and W_{2n+2} whose union is W_n. Thus we recursively arrive finally at a family $\{W_n : n \in \text{IN}\}$ of clopen subsets of X such that $W_n \cap Y = W_n$ for each n. We use this family to define a function r : $X \to Y$ by the rule $r(x)$ is the unique element in Y of $\bigcap\{W_n : n$ G IN$, x$ G $W_n\}$. To show that this intersection set is a singleton, we first note that $\{W_{2^n-1}, W_{2^n}, \cdots, W_{2^{n+1}-2}\}$ is apartition of X into clopen sets, for each $n \in$ IN. Therefore x must belong to one and only one of these sets.

In particular, for every m G IN, $3n > m$ such that $x \in W_n$.

There are therefore infinitely many $n \in$ IN such that $x \in W_n$. Therefore the intersection of the corresponding W_n's cannot have more than one element. On the other hand any two such W_n's being comparable, this family has finite intersection property, and hence by compactness of X, has nonempty intersection. Thus $r(x)$ gets defined uniquely for each $x \in X$. It is also easily seen that

(a) if $x \in Y$, then $x \in W_n \iff x \in W_n$.
Therefore $r(x) = x$ for all x in Y.

(b) $r^{-1}(W_n) = W_n$ for all n. Since $\{W_n : n \in \mathbb{N}\}$ is a base for Y this implies that r is continuous.

We have thus proved that $r : X \to Y$ is a retraction map. Now from proposition 1.2(d), there is a continuous self-map $f : Y \to Y$ without periodic points. Then $f \circ r : X \to Y$, regarded as a map from X to X, has no periodic points at all. Thus X does not have p.p.p.

Step 6: Combining step 4 and step 5, we see that (2) implies (1). Now let X satisfy (3): every closed subspace of X has p.p.p. Then first, X has to be compact, because otherwise, the infinite discrete space IN is homeomorphic to some closed subset, that does not have p.p.p. by proposition 1.2(a). Next, X has to be countable also, since otherwise, by a known classical result (see [14]), X will contain a closed set homeomorphic to the Cantor set K, that does not have p.p.p. by proposition *1.2(d)*. These together prove that (3) implies (1).

Step 7: Lastly, we prove that (4) implies (2). Let X be a metric space every continuous image of which has p.p.p. We consider two cases.

Case 1: Suppose there is some x_o in X and some $\alpha > 0$ in $\mathbb{R}$ such that for every *ft* in $[0, \alpha]$, there is *y* in *X* such that $d(x_0, y)$— *ft*. Then the function $f : X \to S^1$ defined by *f(x)* == $\exp(i.\frac{2\pi}{\alpha}.d(x_o, x))$ is continuous and surjective. Thus S^1 becomes a continuous image of *X*. But by proposition 1.2(6), S^1 does not have p.p.p. Therefore this case cannot arise. The only possibility is the next case.

Case 2: For every *x* in *X* and for every $\alpha > 0$ in IR, there exists *ft* in the open interval $(0, \alpha)$ such that *ft* is not of the form $d(x, y)$ for any *y* in *X*. Then the open ball $B(x$,*ft)* is also the closed ball $B(x$,*ft),* and is therefore a clopen set inside $B(x, \alpha)$.Thus X is zero-dimensional.

Remark 1: We proved in proposition 1.2 that neither the discrete space IN **nor** the Cantor set *K* has p.p.p. In a sense, we can say that these two are the main culprits preventing a strong version of p.p.p. in a metric space. This is because each of the four statements in the main theorem is equivalent to each of the **following:**

(5) Neither IN nor *K* is homeomorphic to a closed subspace of *X.*

(6) *X* is zero-dimensional and neither IN nor *K* is a continuous image of *X.*

Remark 2: We proved in step 1 that every countable compact space has p.p.p. Essentially the same ideas can be used to prove the stronger result that every compact scattered space has p.p.p. A space is said to be scattered if every subset of it contains a point that is isolated in its relative topology. There is no need to **use Baire** category theorem **while imitating** that proof. In fact, the **following result holds: A** compact space *X* is scattered if and only if every continuous image of *X* has p.p.p. For proving the second **half,** we use the **following result of Rudin: If** *X* is **a compact** non-scattered space, then the closed interval $[0, 1]$ is a continuous image of X (see [30]). For proving the first half, we need the result that every continuous image of a compact scattered space is again so (see [21]).

Remark 3: In §1 we stated a theorem of Mazurkiewicz and Sierpinski. This theorem, combined with proposition 1.1(d), gives another proof of the result of step 1. The advantage in this proof is that we do not use Baire category theorem. The disadvantage is that we use the theorem of [25] that is not as popular as Baire category theorem.

Remark 4: After reading §2, one may naturally seek a characterization of all matric spaces that satisfy p.p.p. But no neat characterization of this class is expected, because one can prove that this class is closed neither under continuous images, nor under closed subspaces, nor under arbitrary products. There are haphazard examples provided by Proposition 1.2(c). J.de Groot [10] has proved a surprising result that the Euclidean plane $\mathbb{R}^2$ contains many strongly rigid subspaces. These are neither compact nor countable. These are neither zero-dimensional nor path-connected. For more examples of strongly rigid spaces, see [20]. Every space is a closed subspace of one such space.

Remark 5: We leave this question open: What are all the Hausdorff spaces, every closed subspace of which has p.p.p.? Our main theorem answers this question among metric spaces. One can prove that every compact scattered space has the property that every closed subspace has p.p.p. But the converse is not true.

Remark 6: **(a)** The method of step 5 of §2 can be used to prove the following stronger results: (1) In every zero-dimensional Hausdorff space X, every compact metrizable subspace is a retract. (2) Every retract of every space with p.p.p, again has p.p.p.

(b) The idea of proof of proposition 1.2(c) can be used to prove that for a family $\{Y_n : n \in \mathbb{N}\}$ of topological spaces, if $\prod Y_n$ has p.p.p., then each Y_n has p.p.p. and further there is n_o in IN such that for every $n > n_o$, the period of every periodic point of every continuous self-map of Y_n is $< n_o$. (i.e. there is a uniform bound n_o for these periods, eventually).

(c) In Step 7 of §2 we have actually proved that if a metric space X does not admit [0,1] as a continuous image, then X is zero-dimensional. This result can be supplemented with a companion result. Every compact Hausdorff space X is either zero-dimensional or admits $[0,1]$ as a continuous image. For, if the compact space X is not zero-dimensional, a known result says that there exists a connected subset of X containing two distinct elements x, y. Then there exists a continuous $f : X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$. Since $f(A)$ has to be connected, f is onto.

Remark 7: The main theorem implies that the empty set cannot be the set of all periods of any continuous self-map of any countable compact space. It is natural to ask which subsets A of IN arise as the set of all periods for some such function? This question has been completely answered by us, by describing the family of all possible period sets, for each coutable compact space. The proof involves known results on the structure of such spaces and will appear in next chapter.

Proposition 2: Let $X \subset \mathcal{R}$ be a compact set

Such that $X \setminus (\text{int } X)$ is countable. Then X has p.p.p.

Proof: Let $Y = X \setminus (\text{int } X)$. Let $f : X \to X$ be continuous function. For each y

in Y, let

$$g(y) = \begin{cases} \{f(y)\} & \text{if } f(y) \in Y \\ \{a_y, b_y\} & \text{if } f(y) \text{ belongs to the component interval } [a_y, b_y] \text{ of } X. \end{cases}$$

Then g is a multifunction from Y to Y.

We first prove that g is lower semicontinuous .

(Note : By a multifunction φ from a topological space X to Y we mean that $\varphi(x)$ is a nonempty subset of Y for each x in X. A multifunction $\varphi : X \to Y$ *is* called lower semicontinuous if for every open subset V in Y the set $\{x \in X | \varphi(x) \cap V$ is non empty$\}$ is open in X.)

Let (a, 6) be any open interval in $\mathcal{R}$.

$$\{y \in Y | g(y) \cap (a,b) \text{ is nonempty}\} = \begin{cases} f^{-1}((a,b) \cap Y) \cup S \\ \text{empty if(a,b) is disjoint from Y} \\ \text{Union of } f^{\ 1}(a,6) \text{ and sets of the form } f^{\ 1}(a_y, b_y) \\ \text{where either } a_y \text{ or } b_y \text{ belongs to } (a, b) \end{cases}$$

where S $=\{y \in$ y$|$either $a < a_y < b$ or $a < b_y < b$ where $f(y) \in (a_y, b_y)\}$

In all the cases it is an open set. Now we apply the following selection theorem of Michael in [26] : Let Y be a zero-dimensional complete metric space.

Let $g : Y \to Y$ be a l.s.c. multifunction. Then g admits a continuous selection.

By this, we obtain a continuous function $f : Y \to Y$. We have thus proved:

Let X C $\mathbb{R}$ be a closed set. Let / : $X \to X$ be continuous. Let Y be the boundary of X. Then 3 continuous / $:Y \to Y$ such that

(a) $f(y) = f(y)$ whenever $f(y) \in Y$

(b) $f(y)$ and $f(y)$ always (i.e., $\forall y$ G Y) lie in the same component of X.

We next claim that if y is a periodic point of f, then some element in the X-component of y must be a periodic point of /. Let $(f)^n(y) = y$. Then $f^n(y) \in$ the component of y. [because, / takes components inside components]. Then

$$f^n(\text{component of } y) \subset \text{component of } y.$$

Therefore f^n should have some fixed point. This point is a periodic point of /.

Thus we have proved that any compact subset of $\mathbb{R}$ with countable boundary has p.p.p . In the next section we deal with compact subsets of IR with uncountable boundary .

§3

Lemma 3: Let X, Y be two closed subsets of $\mathbb{R}$ such that $Y \subset X$. Then the **following** are equivalent:

(1) Y is a retract of X.

(2) $a, h \in Y$ and $[a, b] \subset X$ implies that $[a, b] \subset Y$.

Proof: (1) $\Rightarrow$ (2) : Let $r : X \to Y$ be a retraction. Assume that a and b are elements of Y and let $[a, b] \subset X$. Then $r(a) = a$ and $r(b) = b$. Also $r([a, b])$ **should** be a connected set containing $r(a)$ and $r(b)$. Therefore it contains $[a, b]$. Thus $[a, b] \subset Y$.

(2) $\Rightarrow$ (1) : Let (2) hold. Then no bounded component of Y^c is contained in X. Choose some x_n in each bounded component (a_n, b_n) of Y^c such that $x_n \notin X$.

Now define $r : X \longrightarrow Y$ as $r(x) = \begin{cases} x & \text{if } x \in Y \\ a_n & \text{if } a_n < x < x_n \\ b_n & \text{if } x_n < x < b_n \end{cases}$

This r is continuous . Hence Y is retract of X.

Corollary 3.1: Let X be a closed subset of $\mathbb{R}$ with empty interior . Then every closed subset of X is a retract of X.

Corollary 3.2: Let X be a zero dimensional metric space. Then every closed subset of X is a retract of X. (Since X is homeomorphic to a subspace of $\mathbb{R}$)

Corollary 3.3: A compact metric space X is zero dimensional if and only if every closed subset of X is a retract of X.

Theorem 2: Let X be a compact subset of $\mathbb{R}$ such that its boundary is uncountable. Then X does not have p.p.p.

Proof: The proof is divided into five major steps.

Step 1: First, we fix some notations. dX is the boundary of X. It is equal to $X\backslash$ (interior of X).

Y = {a G $dX\backslash$ every neighbourhood of a contains uncountably many elements of $\partial X\}$.

$Z = Y \cup$ (union of all those components of X whose boundary is contained in Y).

$A = \{a$ G $Y\backslash$ a has an immediate successor a^+ in Y and $[a, a^+]$ C $Z\}$.

B — $\{b \in Y\backslash$ b has an immediate successor b^+ in Y and $[b, b^+] \cap Z$ is empty$\}$.

The following can be proved easily.

Proposition 3.1:

1 X, dX, Y and Z are compact and uncountable.

2 $\partial Z = Y$.

3 A and B are countable.

4 Between any two non-adjacent elements of A lies an element of B.

5 If $y_1 < y_2$ in Y, 3c G A U B such that $y_1 < c < c^+ < y_2$.

Step 2: Now we introduce some more notations. $A = \{a_1, a_2 \cdots, a_n \cdots\}$ is either finite or countable.

$B = \{b_1, b_2 \cdots, b_n \cdots\}$ is countably infinite.

$J_1 = \{0, 1\}$.

$B_o = \{y \in Y | y \leq b_1\}$.

$B_1 = \{y \in Y | y > b_1\}$

α_1 = that element of J_1 such that $a_1 \in B_{\alpha_1}$.

$J_1 = J_1 \cup \{\alpha_1^*\}$ where α_1^* is just a symbol corresponding to α_1. Suppose we have defined for some positive integer n, the index set J_n, the clopen subsets B_α of X for each $\alpha \in J_n$ the special element α_n of J_n determined by a_n and the index set J_n. Then we define

$J_{n+1} = \{\alpha 0 | \alpha \in J_n\}$

$B_{\alpha 0} = \{y \in B_\alpha | y < b_{n(\alpha)}\}$.

$B_{\alpha 1} = \{y \in B_\alpha | y > b_{n(\alpha)}\}$.

$n(\alpha)$ = the least positive integer $\ni b_{n(\alpha)}$ and $b^+_{n(\alpha)}$ are both in B_α.

α_{n+1}=that element of J_{n+1} such that a_{n+1} G $B_{\alpha_{n+1}}$.

$J_{n+1} = J_{n+1} \bigcup \{\alpha^*_{n+1}\}$. We let $J_{n+1} = J_{n+1}$ if \A\ < n.

The **following** can be proved without difficulty

Proposition 3.2:

6 $|J_n| = 2^{n+1} - 1$ for all $n = 1, 2, \cdots$ if |i4| > n

7 **Each** B_α is a clopen subset of Y; it is called a block of ra-th level if $\alpha \in J_n$.

8 **For** every fixed n

$\{B_\alpha | \alpha \in J_n\}$ is a partition of Y.

9 Every block of $n + 1^{\text{th}}$ level is contained in some block of n^{th} level;

10 a_n and a_n^+ belong to different blocks on n^{th} level; similarly, b_n and b_n^+ belong to different blocks of n^{th} level.

11 Each block B_α is of the form $Y \cap 1$ for some closed interval /.

12 Let $J = \bigcup_{n=1}^{\infty} J_n$. Then $\{B_\alpha | \alpha \in J\}$ is a base for the topology of Y.

Step 3: In this step we define for each $n = 1, 2, \cdots$, a function $\sigma_n = J_n \longrightarrow J_n$, which will be used in the next step to define a function / : $Y \rightarrow Y$.

Initially, $\sigma_1 : J_1 \rightarrow J_1$ is defined by letting $\sigma_1(o) = 1$ and $\sigma_1(1) = 0$. Next, it is extended to $\sigma_1 : J\backslash \rightarrow J_1$, by letting $\tilde{\sigma}_1(\alpha_1^*) = \sigma_1(\alpha_1)$.

Suppose we have defined for some positive integer n, the functions $\sigma_n : J_n \rightarrow J_n$ and $\sigma_n : J_n \rightarrow J_n$. Then we define $\sigma_{n+1} : J_{n+1} \rightarrow J_{n+1}$ by defining

$$\sigma_{n+1}(\alpha 0) = \begin{cases} \beta 0 & \text{if } \beta = \sigma_n(\alpha) \neq 00\cdots 0 \\ \beta 1 & \text{if } \beta = \sigma_n(\alpha) = 00\cdots 0 \end{cases}$$

$$\sigma_{n+1}(\alpha 1) = \begin{cases} \beta 1 & \text{if } \beta = \sigma_n(\alpha) \neq 00\cdots 0 \\ \beta 0 & \text{if } \beta = \sigma_n(\alpha) = 00\cdots 0 \end{cases}$$

This defines σ_{n+1} on the first 2^{n+1} elements of J_{n+1}. We define it at the other elements suitably so that the following two conditions are satisfied:

$$\{\sigma_{n+1}(\alpha 0), \sigma_{n+1}(\alpha 1)\} \subset \{\beta 0, \beta 1\} \text{ where } \beta = \sigma_n(\alpha)$$

$$\sigma_{n+1}(\alpha_m^* 1 1 \cdots 1) = \sigma_{n+1}(\alpha_m 0 0 \cdots 0) \text{ for every } m < n$$

Finally we extend σ_{n+1} to $\tilde{\sigma}_{n+1} : \tilde{J}_{n+1} \to \tilde{J}_{n+1}$ by defining

$$\tilde{\sigma}_{n+1}(\alpha^*_{n+1}) = \sigma_{n+1}(\alpha_{n+1})$$

The following can be proved without much difficulty.

Proposition 3.3 :

13 The range of σ_n has exactly 2^n elements. These elements are precisely those words of J_n that do not involve the *-symbol.

14 Every element in the range of σ_n is a periodic point of period 2^n.

15 These σ_n's are mutually compatible in the sense that the following diagram commutes for each $m < n$:

$$\begin{array}{ccc} & \tilde{\sigma}_n & \\ \tilde{J}_n & \longrightarrow & \tilde{J}_n \\ r\downarrow & & \downarrow r \\ J_m & \longrightarrow & J_m \\ & \sigma_m & \end{array}$$

Here r $= J_n \to J_m$ is the restriction map that associates to each word α in J_n, the truncated word in J_m omitting all but the first m letters of α.

Step 4: In this step, we define a function / : $Y \to Y$ using the $\tilde{\sigma}_n$'s constructed in step 3. But this requires some preparatory result:

16 Any strictly decreasing sequence of blocks $\{B_{\alpha_n}\}$has the property that the intersection $\bigcap_{n=1}^{\infty} B_{\alpha_n}$ is a singleton. [To prove this, we use the results (5) and (10)]

We now fix one more notation. For each x in Y and for each positive integer n, let $\alpha(x,n)$denote the unique a in σ_n such that $y \in B_{\alpha(x,n)}$. [Here we use the result (8)].

We define / : $Y \rightarrow Y$ by the rule

$f(x)$ = the unique element of $\bigcap_{n=1}^{\infty} B_{\tilde{\sigma}_n(\alpha(x,n))}$.

Proposition 3.4:

17 $f(B_\alpha)$ C $B_{\tilde{\sigma}(\alpha)}$ for each α in J_n , by definition of f.

18 $f^{-1}(B_{\overline{\alpha}})$ $\cup\{B_\beta|\beta\in J_n,\ \tilde{\sigma}_n(\beta) = \alpha\}$ for each α in J_n.

= a finite union of blocks.

= a clopen set.

19 / i s continuous. [Here, we use the result (12)].

20 If x is a periodic point of / and if n is a positive integer, then $\alpha(x,n)$ is a periodic point of σ_n; moreover the /-period of x divides the $\tilde{\sigma}_n$-period of $\alpha(x,n)$.

21 / has no periodic points. [Here we use the resuilts (20) and (14)].

22 $f(a_n) = f(a^+)$ for $n = 1, 2, \cdots$

Step 5: In this step, we extend the function / defined in step 4. First, we note that Z = Y U $(\bigcup_{n=1}^{\infty}[a_n, a_n^+])$

We define / : $Z \rightarrow Z$ by the rule

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in Y \\ f(a_n) & \text{if } a_n \leq x \leq a_n^+ \end{cases}$$

[We use the result (22) to prove that this definition does give a function]

Proposition 3.5:

23 / is continuous.

24 Range of / is same as the range of / and is contained in *Y*.

25 / has no periodic points. [Here, we use the result (21)].

26 *Z* has the property that

$$\left.\begin{array}{ll} x, y & \in Z \\ [x, y] & \subset X \end{array}\right\} \Longrightarrow [x, y] \subset Z.$$

27 *Z* is a retract of *X*. *[* Here we use the Lemma 3].

[Note however that *Y* is not in general a retract of *Z* or *X]*.

As the last step in the proof, we extend / to a continuous function $f^* : X \to Z$ (using the result (27)). Indeed the range of /* = range of /=range of /.

One can prove:

28 This /* : $X \to X$ has no periodic points at all.

[Here, we use the result (25)].

Chapter 3

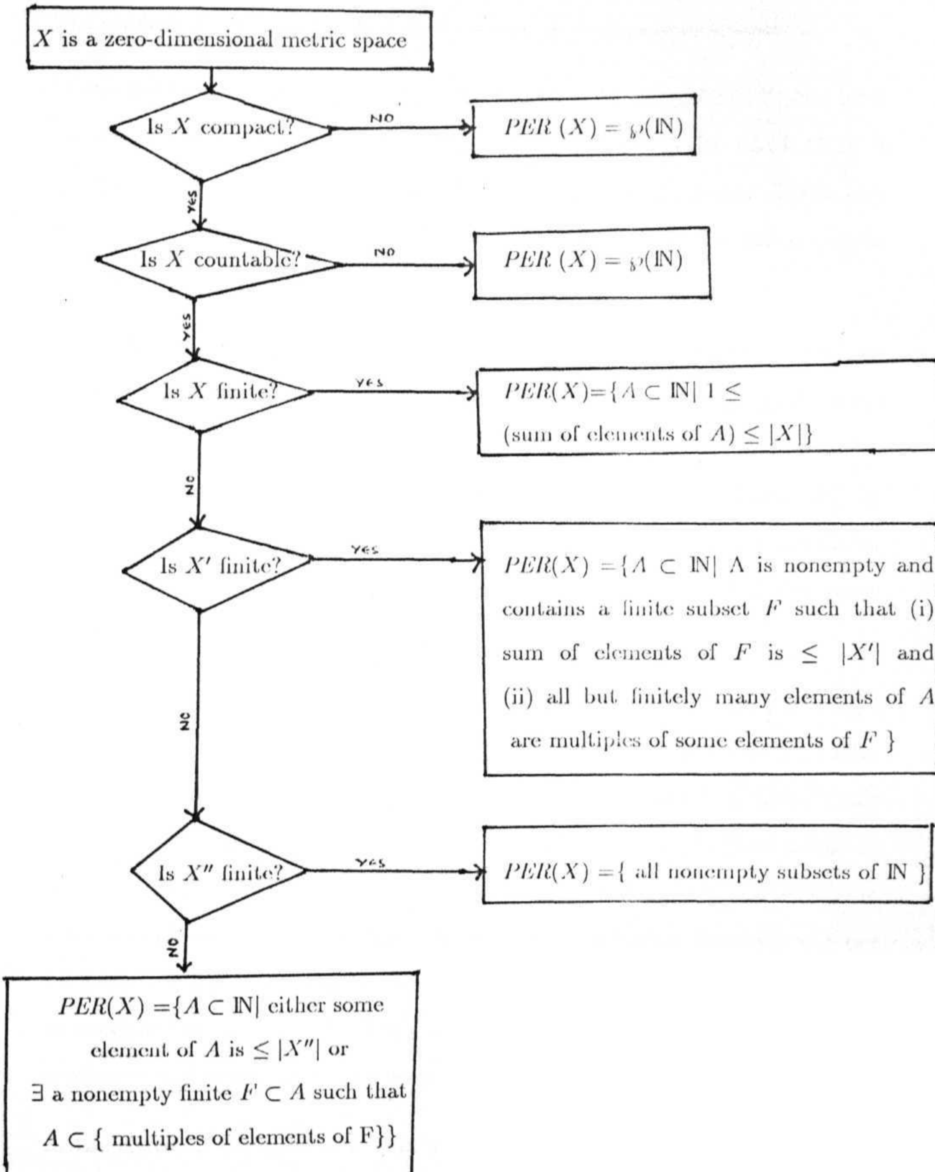
Sets Of Periods Of Zero Dimensional Metric Spaces

§1

Introduction and Preliminaries:

In this Chapter, we take up the problem: Given a topological space X, describe the family *PER (X)* intrinsically. We solve this problem for each X in a class of spaces that includes all zero-dimensional metric spaces, as well as some compact non-metrizable topological spaces. There is only one main result in this chapter, namely Theorem 11. All the other results are intended to provide steps for its proof. In the first seven theorems a little more generality is maintained, than what is needed for the proof of Theorem 11; this generality will be used in our next chapter. It is better to state our main theorem in the form of the flow-chart given below, because that seems more natural in the present context of division of cases.

Table 1.1:


X is a zero-dimensional metric space
Is X compact?
NO
PER (X) = ℘(ℕ)
YES
Is X countable?
NO
PER (X) = ℘(ℕ)
YES
Is X finite?
YES
PER(X)={A ⊂ ℕ| 1 ≤ (sum of elements of A) ≤ |X|}
NO
Is X′ finite?
YES
PER(X) ={A ⊂ ℕ| A is nonempty and contains a finite subset F such that (i) sum of elements of F is ≤ |X′| and (ii) all but finitely many elements of A are multiples of some elements of F }
NO
Is X″ finite?
YES
PER(X) ={ all nonempty subsets of ℕ }
NO
PER(X) ={A ⊂ ℕ| either some element of A is ≤ |X″| or ∃ a nonempty finite F ⊂ A such that A ⊂ { multiples of elements of F}}

The symbols *X', X"* and *\X* used in the above are explained in Chapter 0.

In the history of topological dynamics the problem of finding *PER(X)* has been attacked by many mathematicians. But so far the space *X* for which there is an attempt to describe *PER* (X), mostly has been one-dimensional. But in this chapter, we consider zero-dimensional spaces. As explained in 6.2 of chapter 1, the zero-dimensional spaces deserve this kind of study.

1.2. Terminology: Now we fix the meanings of certain terms from Topology and Dynamics that are often used in this chapter. First, the term'zero-dimensional space' has many mutually non-equivalent meanings depending on whether we use Lebesgue covering dimension, or Urysohn-Menger dimension, or others. We fix that here a space *X* is zero-dimensional if and only if it admits a base consisting of clopen sets (i.e., small inductive dimension is 0). A subset of *X* is said to be clopen, if it is both closed and open.

If / is any continuous self-map of X, then many authors regard the pair (X, f) as a dynamical system. Then / describes the motion where *x* moves to *f(x)* after one instant of time. Now $f^n(x)$ is the position of *x* after *n* instants of time, in this dynamical system. The points that are not moved at all, are the fixed points or the points of period 1. The period of a point is the measure of least time when the point returns to its original position. The orbit of *x* is defined as *{y* ∈ *X\y* = $f^n(x)$ for some *n* in $\mathbb{N}\}$. Some authors call this the semi-orbit of *x.* It is an example of an invariant set. A subset *A* of *X* is said to be invariant under /, if *f(A)* C *A.* **The** continuity of / implies that the closure of an invariant set is again invariant.

In the course of our study, we come across certain topological properties of **the** underlying space X that are expressible as 'common' dynamical properties for all the dynamical systems that X underlies. Examples of such dynamical properties are: (i) admitting a periodic point (ii) the limit points of periodic points, being necessarily periodic, etc. This link between dynamics and topology is the undercurrent of this entire work.

Remark: We conclude this introductory section with a remark on our method of proofs. In §7, we use Baire category theorem; the same is again used in §8 and in §10 indirectly, along with Zorn's lemma, and minimal sets of Topological Dynamics. In §9, we use the theorem that any two totally disconnected compact metric self-dense spaces are homeomorphic [14]. In the first half of this chapter, from §2, to §6, our methods are elementary. However in §5, and §6 we resort to some nice constructions of functions that require geometric imaginations and their translations to the language of analysis.

For a topological space X and for a given T C $\wp(\mathbb{N})$, the statement *PER (X)* — is too brief and misleadingly innocuous, because to prove it we need to proceed as follows: In the first part, we show that for every continuous self-map / of X, the set of all /-periods of all elements of X, satisfies the conditions stipulated for the membership in $\mathcal{F}$; often, this is proved by considering different cases as in Theorem 3,4,5 etc. In the second part, for a given member A of $\mathcal{F}$, first using the elements of A, we construct a self-map / of X; secondly we prove that / is continuous; thirdly we prove that every element of A is a period of /; fourthly we prove that there are no other periods for /. This scheme of proof can be seen to be common in most of our theorems.

§2

In this section, some preliminary results and the first two theorems of the chapter , **are** proved, thereby characterising *PER (X)* for all compact Hausdorff spaces with at most one limit point.

We start with a result whose proof is easy and hence omitted.

Proposition 2.1 : Let *(X,f)* be any dynamical system.

(a) Let *x* and *y* be periodic points of different periods. Then the orbit of *x* and **orbit** of *y* are disjoint.

(b) $Per(f)$— *Per(g)* where *g* is the restriction of / to the range of /.

Remark: These two results will be used more than once later. Regarding (a) note however that for non periodic points, orbit of *x* and orbit of *y* may be different, but non-disjoint. The next result also will be frequently used in later sections:

Proposition 2.2 : Let *Y* be a clopen subset of X, provided with relative topology. Then *PER(X)* D $PER(Y)$.

Proof: If / is a continuous self-map of Y, and if y_o is a fixed element in the range of/, define $f : X \to X$ by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \text{ is in } Y \\ y_o & \text{otherwise.} \end{cases}$$

Then the clopenness of *Y* implies the continuity of /. Moreover / and / have the same range. Therefore by (b) of Proposition 2.1, *Per(f)* $= Per(f)$. Thus every member of $PER(Y)$ is a member of *PER(X)*.

Remark 2.3 : Our first theorem is an elementary exercise of combinatorial nature, relatively easier than the other ten theorems. It is included here not only for the sake of completeness, but also because these ideas are required in the proofs of later theorems.

Theorem 1: Let *X* be a discrete space with *n* elements. Then *PER(X)* = $\{A \subset \mathbb{N} : 1 \leq \left(\sum_{m \in A} m\right) < n\}$ (This family is denoted by$\mathcal{F}_n$).

Proof: *First Part:* Let / : *X* → *X* be a function and let *A* = *Pcr(f)*. Take any *x* in *X*. The $n+1$ terms of the sequence $x, f(x), f^2(x), \cdots, f^n(x)$ cannot be distinct, because \X\ — *n*. Therefore $f^r(x) = f^s(x)$ for some $r < s$. Then $f^r(x)$ is a periodic point. Thus *A* is nonempty, and so is the set *P* of all /-periodic points of *X*. If $a_1, a_2, \cdots, a_r$ are distinct elements of *A* there are elements $x_1, x_2 \cdots, x_r$ in *P* such that the orbit of x_i has a_i elements. By 2.1 (a), these orbits are pairwise disjoint subsets of P, and therefore the sum $a_1 + \cdots + a_r <$ \P\ < \X\.

Second Part: Conversely, let $A = \{a_1, a_2, \cdots, a_r\}$ be a subset of IN such that $a < a_1 + a_2 + \cdots + a_r < n$. Let X — $\{1, 2, \cdots, n\}$. We now construct a function $f : X \to X$ such that *Per(f)* = *A*. We define

$$f(x) = \begin{cases} 1 & \text{if } x > a_1 + \cdots + a_r \text{ or } x = a_1 \\ a_1 + \cdots + a_j + 1 & \text{if } x = a_1 + \cdots + a_{j+1} \text{ for } j = 1, 2, \cdots r-1 \\ x+1 & \text{otherwise.} \end{cases}$$

Our assumption $a_1 + \cdots + a_r < n$ has been used in the above definition of / since otherwise / will not be a self-map of *X*. For this / we have:

all a_1 elements between 1 and a_1 are periodic points with period a_1;

all a_2 elements between $a_1 + 1$ and $a_1 + a_2$ are periodic points with period a_2;

all a_r elements between $a_1 + \cdots + a_{r-1} + 1$ and $a_1 + \cdots + a_r$ are periodic points with period a_r;no other point is a periodic point.

Therefore *Per(f)* = $\{a_1, a_2, \cdots, a_r\} = A$

Example 2.4 : (a) If X is the discrete space with exactly 5 elements, then *PER(X)* = **{{1}, {2}, {3}, {4}, {5},{1,2}, {1,3}, {1,4}, {2,3}},**

(b) If X is the discrete space with exactly 6 elements, then

PER(X)={{1},{2},{3},{4},{5},{6},{1,2},{1,3},{1,4},{1,5}, **{2,3},{2,4},{1,2,3}} .**

Corollary 2.5 : The following are equivalent for a subset A of IN . (i) A is nonempty and finite. (ii) A = *Per(f)* for some self-map / of some finite space X.

Remarks 2.6 : In our next theorem, unlike in the previous one, we see that for many (mutually non-homeomorphic) topological spaces X we have the same *PER(X).*

Theorem 2: Let X be a compact Hausdorff space with a unique limit point. Then *PER(X)* $=\{A \subset$ IN : either $1 \in A$ or A is finite and nonempty$\}$ (This family is same as $\mathcal{F}_* \cup \mathcal{U}_1$ where $\mathcal{U}_1 = \{A$ C IN : l€ $A\})$

Proof : *Let* x_o be the unique limit point of X.

First Part: Let / be a continuous self-map of X. Consider two cases:

Case 1: Let $f(x_o) = x_o$. Then 1 G $Per(f)$.

Case 2: Let $f(x_o) \neq x_o$. Then by the continuity of /, there is a neighbourhood of x_o that is mapped to a singleton-open-set. The complement of this neighbourhood is a finite set, because X is compact. Therefore the range of / is finite. Therefore by Proposition 2.1 (b) and Corollary 2.5, *Per(f)* is a nonempty finite subset of IN

Second Part: Conversely, let A C IN be such that either $1 \in A$ or A is nonempty **and** finite. We now show that $A \in PER(X)$.

Case 1: Let $A = \{a_1, a_2, \ldots, a_r\}$ be a nonempty finite set. Let Y be a subset of X **such** that \Y\ = $a_1 + a_2 + \cdots + a_r$ and such that x_o is not in Y. Then by theorem 1, there is a function $g : Y \to Y$ such that *Pcr(g)* $= A$. Since Y is clopen in X, it **follows** from 2.2 that A € *PER(X).*

Case 2: Let A be infinite and let $1 \in A$. Arrange the elements of A in the increasing order as $1 = a_1 < a_2 < \cdots < a_n < \cdots$

Choose a sequence $x_1, x_2 \cdots, x_n, \cdots$ in X with distinct terms such that no term is a limit point. Define / : $X \to X$ by the rule

/(x) = { the $a_2 + \cdots +$ flfc-th term, if x is the $a_2 + \cdots + a_k + a_{k+1}$-th term
x_{j+1} if $x = x_j$ and if $j \notin \{a_2, a_2 + a_3, \cdots, a_2 + \cdots + a_{r_1} \cdots$
x if x is not in the sequence, or if $x = x_1$.

Then / is one-one and $f(x_o)$ — x_o. One can prove that any function from X to X that fixes x_o and that is one-one is automatically continuous. Therefore / is continuous. Also, the first a_1 terms of the sequence are periodic points with period a_1; the next a_2 terms of the sequence are periodic points with period a_2; and so on; finally, all other elements are fixed points.

Therefore *Per(f)* $= \{1\} \cup \{a_2, a_3 \cdots\} = A$

§ 3

In this section, we consider, for a fixed positive integer n, a compact Hausdorff **space** X with \X'\ = n.We compute *PER(X).* We sketch a proof,but leave out the details,since these are analogous to the details of the previous section.

Theorem 3: Let n be a positive integer. Let X be a compact Hausdorff space with exactly n limit points. Then $PER(X) = \{A \subset \mathbb{N} \mid A$ is a nonempty set containing two finite subsets F and G satisfying (i). every element of $A \setminus G$ is a multiple of some element of F and (ii). sum of elements of F is $< n$ $\}$.

Examples 3.1 : Let $A_1 = \{6n \mid n \in \mathbb{N}\} \cup \{1,2,3,4\}$

Let $A_2 = \{n \in \mathbb{N} | n > 10\}$.

Let $A_3 = \{$ prime numbers $\}$.

Then $A_1 \in PER(X)$ if $n > 2$, because we can take $F = \{2\}$, $G = \{3\}$, [If $n > 6$, we can alternatively take $F = \{6\}$, $G = \{2,3,4\}$] A_2 is never in $PER(X)$, whatever n may be. For if $F = \{a_1, a_2, \cdots, a_n\}$ is any finite subset of A_2,none of whose elements is a multiple of some element of F. Next, A_3 is also never in $PER(X)$. For, if F is any finite set of prime numbers, then no element of A_3 greater than every element of F, is a multiple of some element of F.

Remark 3.2 : Consider n = 1 Then X has unique limit point. In this case $PER(X)$ has been described in one way in Theorem 2 and in another way in Theorem 3. Do these two ways describe the same family? This is not obvious. So, let us look at (i) and (ii) closely.

When $n = 1$, (ii) says that F is either empty or $\{1\}$.

If F is empty, (i) implies that $A \setminus G$ is empty, and so $A = G$, and so A is finite.

If F is $\{1\}$, then (i) poses no more condition on A except that $1 \in A$.

Putting these three pieces of facts together, $PER(X)$ described in Theorem 3 coincides with $\{A \subset \mathbb{N} \mid$ either $1 \in A$ or A is nonempty and finite$\}$.

Proof of Theorem 3: The initial step of Theorem 3 (that it is true when $n = 1$) has been already proved in the previous theorem.

Let $x_1, x_2, \cdots, x_n$ be the limit points of X.

First Part: Let $f : X \to X$ be a continuous function. Let $A = Per(f)$. We want to find two finite subsets F and G of A satisfying (i) and (ii). For this we resort to induction on n. As already noted the result is true for $n = 1$. Assume as induction hypothesis that for a particular n the result is true for all compact spaces having at most $n - 1$ limit points. Consider two cases.

Case 1: The range of f is a space with at most $n - 1$ limit points in the relative topology. Let g = restriction of f to the range of f. Then $Per(g)$, by induction hypothesis, has the property stated in the theorem. Also $Per(f) = Per(g)$. Therefore in this case, we have proved what we want.

Case 2: Let Case 1 not hold. Then $x_1 x_2, \cdots, x_n$, are in the range of f, and these are limit points of the range of f as well. Let $S = \{x_1, x_2, \cdots, x_n\}$. We first prove that S is f-invariant. If not, let $f(x_i) \notin S$. Then some neighbourhood V of x_i is mapped into the singleton-open-set $\{f(x_i)\}$, and so the range of f is $\{f(x_i)\} \cup f(X \setminus V)$. Here $X \setminus V$ is a compact space with at most $n - 1$ limit points. The continuous image $f(X \setminus V)$ should also be so.More genarally, if $f : X \to Y$ is a continuous surjection between compact spaces, then Y has fewer number of limit points than X. Thus the range of f has atmost $n - 1$ limit points, contradicting the assumption that Case 1 does not hold.

Next we prove that $f(S) = S$. If $x_i \notin f(S)$, there is a clopen neighbourhood V of x_i disjoint from the finite set $f(S)$ and then the clopen set $f^{-1}(V)$ is a compact subset disjoint from 5, and is therefore bereft of any limit point, and is therefore finite. This implies that only finitely many points of V are in the range of /. Then x_i cannot be a limit point of the range of /, a contradiction.

Thus we have proved that $f|_S$ is a permutation on the finite set S.

Let A = *Per(f)*. We want to find two subsets F and G of A satisfying (i) and (ii). We take $F = Per(f|_S)$.That is, F is the set of all lengths of the disjoint cycles in the permutation $f|_S$. Next to find G, we proceed as follows. Let $\{V_o, V_1, V_2, \cdots, V_n\}$ be a partition of X such that

(a) V_o is a finite set disjoint from S

(b) each V_i is clopen

(c) $x_i \in V_i$ for $i = 1, 2, \cdots, n$.

[one can prove easily that such a partition of X exists]

Write $x \sim y$ if for each $j = 0, 1, 2, \cdots$,n, the elements $f^j(x)$ and *ft(y)* belong to the same partition class V_i. This $\sim$ is an equivalence relation on X. It expresses the fact that at each of the first n instants of time, in the dynamical system (X, f), the **elements** x and y behave alike.

For $i = 1, 2, \cdots$, n, let W_i = {x G X\$x \sim x_i\}$. Then we claim: For 1 < i < n.

(a) $W_i \subset V_i$.

(b) W_i is clopen.

(c) $V_i \backslash W_i$ is finite.

(d) $x_i \in W_i$.

Of these, (a) is easy from the definition of $\sim$ taking $j = 0$.

To prove (b), we note that for $1 < \imath < n$, $W_i = \bigcap_{j=o} \{(f^j)^{-1}(V_k) : f^j(x_i) \in V_k\}$.

By the continuity of *p*, we see that each W_i is the intersection of finitely many clopen subsets of *X*. To prove (c), we note that $V_i \backslash W_i$ is a clopen subset of *X* containing no limit points, and hence is a compact discrete set, and therefore finite. Next we let

$$B = X \backslash \cup \{W_i | 1 < i < n\} .$$

Then by a similar argument, *B* is also a finite set. Lastly let *G* = {periods of those *x* in *B* that are /-periodic points}. Then *G* is clearly a finite subset of *A*(= *Per(f))*.

Having thus defined two finite subsets *F* and *G* of *A* we now prove that (i) and (ii) are satisfied.

To prove (i) let ra $\in A \backslash G$.

Then 31 $< i < n$ and *3x* in W_i such that *x* is an /-periodic point with period ra. For this $\imath$, look at x_i. It is also a periodic point. Let its period be raj. then raj G *F*. We claim that m is an integral multiple of m_i. Here are three simple pieces of arguments to prove this:

Suppose among the elements $f(x), f^2(x), \cdots$, one of them is in *B*. Then since that element should have the same period as x, it contradicts the assumption that ra is not in *G*. Therefore the entire orbit of *x* is inside $\cup_{j=1}^{n} W_j$.

This implies that at any future instant r (even when r is greater than n), *x* and x_i behave alike, i.e., the elements $f^r(x)$ and $f^r(x_i)$ are in the same V_k. (This k depends on r).

Now ra is the period of *x* $\Rightarrow f^m(x) = x$

$\Rightarrow f^m(x) \in W_i \subset V_i$

$\Rightarrow f^m(x_i) \in V_i$ (as argued above)

$$\Rightarrow \quad f^m(x_i) = x_i \text{(since } S \text{ is f-invariant)}$$

$$\Rightarrow \quad m \text{ is an integral multiple of period of } x_i$$

Next to prove (ii) we have only to note that the sum of elements of F is $< n_o$ (as already seen in 3.1) since F is in $PER(S)$, as S is a finite set with n_o elements.

Second Part: Conversely let A be a nonempty subset of IN admitting two finite subsets F and G satisfying (i) and (ii). Now we construct a continuous function $f : X \to X$ such that ***Per(f) = A.***

Let $x_1, x_2, \cdots, x_{n_o}, S, V_o, V_1, V_2, \cdots, V_n$ be as in the first part of this proof. Since addition or deletion of a finite number of isolated points does not affect the clopenness of a set, we may, by mutual transfer of finitely many elements between V_o and V_1, assume that the set V_o has exactly $\sum_{m \in G} m$ elements.

Let $F = \{a_1, a_2, \cdots, a_r\}$. Define for $1 < i < r$

$S_i = \cup\{V_j : a_1 + \cdots + a_{i-1} + 1 < j < a_1 + \cdots + a_i\}$

and $S_{r+1} = \cup\{V_j | j > a_1 + \cdots + a_r\}$

Note that $S_1 = \cup\{V_j | 1 < j < a_i\}$ and also note that $\{V_o, S_1, \cdots, S_r, S_{r+1}\}$ is a partition of X.

For $1 < i < n_o$, choose a countably infinite subset B_i of X such that $x_i \in B_i \subset V_i$ and arrange the elements of $B_i \setminus \{x_i\}$ as a sequence $(x_n^i)_{n=1 \text{ to } \infty}$. Next, let $A_i = \{\frac{m}{a_i} | m \in A$ and a_i divides $m\}$ for $1 < i < r$. We note that $1 \in A_i \forall i$ because $F \subset A$. With this much of ground work, we are ready to construct f in pieces.

First, $f : S \to S$ is constructed such that its period set is exactly F. Here we use (ii) and Theorem 1. We may assume that

$$f(x_j) = \begin{cases} x_{a_1+a_2+\cdots+a_{i-1}+1} & \text{if } j = a_1 + \cdots + a_n \\ x_1 & \text{if } j = a_1 \text{ or if } j > a_1 + \cdots + a_r \\ x_{j+1} & \text{otherwise} \end{cases}$$

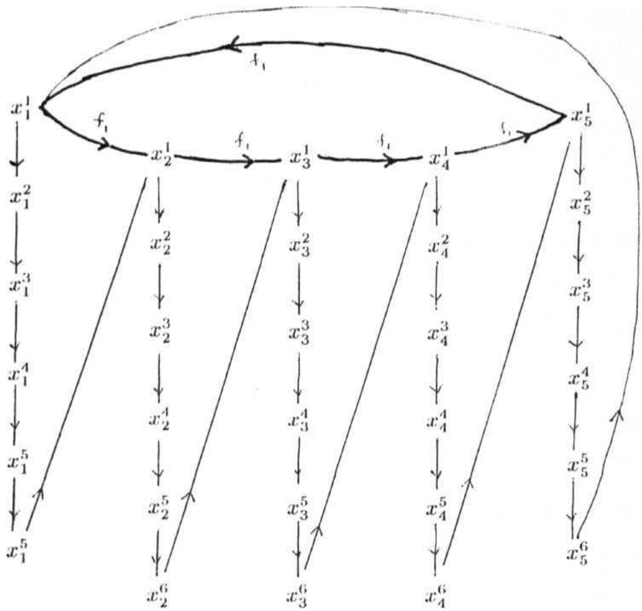
Secondly, $f : V_o \to V_o$ is constructed such that its period-set is G. Here we use Theorem 1 and the fact $|V_o| = \sum m$.

Thirdly, $f_i : B_i \to B_i$ is constructed by applying Theorem 2 to the compact space B_i with unique limit point x_i [It can proved that x_i is the unique limit point of B_i] such that $Per(f_i) = A_i$, after noting that $1 \in A_i$.

Fourthly, we use the above pieces to define $f : S_i \to S_i$ for $1 < i < r$. After fixing one such i, let $k = a_1 + \cdots + a_{i-1} + 1$. We let

$$f(x) = \begin{cases} x_n^{j+1} & \text{if } x = x_n^j \text{ and } a_1 + \cdots + a_{i-1} + 1 < j < a_1 + \cdots + a_i \\ f_k(x_n^k) & \text{if } x = x_n^{a_1+\cdots+a_i} \\ f(x_j) & \text{if } x \in V_j \setminus B_j \text{ (using that } f \text{ has been already defined on } S) \end{cases}$$

In order to explain this part of the definition of f, one of its orbits is depicted diagramatically below, taking $a_1 = 5$ and $6 \in A_1$



Finally, we define $f(x) = x_1$ for all x in S_{r+1}. Thus we have defined f at all points of X. Since V_o and the S_i's are mutually disjoint, the compatibility of the pieces has to be checked only between S and S_i's. This is done by actual verification.

It remains to prove that this f has the required properties:

(a) f is continuous

(b) $Per(f) = A$.

To prove the continuity of f, we first make four simple observations:

(a) Let $1 < i < n_o$. Let j be the unique integer $(1 < j < n_o)$ such that $f(x_i) = Xj$. Then $f(V_i)$ C V_j.

[This is found to be true from the definition of /] .

(b) $f^{-1}(V_j) = \cup\{V_i | f(x_i) = x_j\}$, for $1 \leq j \leq n_o$.

[This follows from (1) after noting that the V_i;s are pairwise disjoint and **that** no element of V_o is mapped outside it]

(c) Since S is /-invariant, no limit point of X is mapped by / to any isolated point. Therefore $f^{-1}(A)$is open whenever A is a finite open set.

(d) Every open subset V of X must be of the form

$$V = (V_{i_1} \cup V_{i_2} \cup \cdots \cup V_{i_s}) \cup (A \text{ finite open set}) \setminus (\text{some finite open set}).$$

It follows from (2), (3) and (4) that $f^{-1}(V)$ is open for each open subset V of X. Thus / is continuous.

Now let us see what the /-periodic points in X are and what their periods are. We make the following observations:

(a) The limit point x_i is a periodic point if and only if $1 < i < a_1 + \cdots + a_r$. Their periods are precisely the elements of the given set F.

(b) Every element of V_o is a periodic point. The set of their periods is precisely the given set G.

(c) Let b_1 be the first element of $A_1 \setminus \{1\}$. Then b_1 is equal to m/a_1 for some m in A. And $x\backslash$ is a /-periodic point whose f_1-period is b_1. Its /-period is however more, and its calculation requires some care.

Its f_1-orbit is $x_1^1, x_2^1, \cdots, x_{b_1}^1 = x_1^1$

Its f-orbit is $x_1^1, x_1^2, \cdots, x_1^{a_1}, x_2^1, x_2^2, \cdots, x_2^{a_1}, x_3^1, \cdots, x_1^1$

It is noted that for 1 < r < b_1, f^{ra_1}(a}) = *fi(x\)* and that if k is not divisible by a_1, then $f^k(x_1^1)$s not in B_1 at all (and so cannot be *x\)*. It follows that the /-period of x_1^1 is a_1 times its f_1-period; that is $a_1 b_1$; that is the element m in *A* that we started with. The same m is the period for *x* etc. also.

(d) The above calculation of the period of x_1^1 is a particular case of this **general** result:

For points in B_1 U • • • U B_{a_1} the periods are precisely those elements of *A* that are divisible by a_1; for points in B_{a_1+1} U • • • U $B_{a_1+a_2}$the periods are precisely those elements of *A* that are divisible by a_2; and so on.

(e) There are no other periodic points. All the periodic points of / are in V_o or *S* or one of the B_i's. This is because the range of / is itself contained in the union of these.

[we can also prove that for /, the periodic points are precisely the **elements** in the range].

Now we argue as follows:

Since every element in *A\G* is divisible by some element of F, it follows from (4) that every element of *A\G* is in *Per(f)*. Together with (2), *A* C ***Per(f).*** Because of (5), **(1),** (2) and (4) ***Per(f)*** C A. Thus we have *A* = *Per(f).*

This completes the proof of the theorem.

Corollary 3: Let $|X'| = n$. Let $B \in \mathcal{G}_n$. Then there exists a continuous self map f on X such that

(i) $Per(f) = B$

(ii) $X \setminus X'$ is f-invariant

(iii) $Per(f|_{X\setminus X'}) = B$

§4

This short section is devoted to the fourth theorem. This is concerning spaces X with $|X''| = 1$.

Notations 4.1 : In the previous three theorems, we have come across a countably infinite number of families that are of the form $PER(X)$. In the rest of this section, we need to refer to them often and hence the need for suitable notations. We recall the **following** notations from Chapter 0.

For $n \in \mathbb{N}$,

$\mathcal{F}_1 = \{\{1\}\}$

$\mathcal{F}_n = \{A \subset \mathbb{N} | A$ is nonempty; $\sum a$ is $< n\}$

$\mathcal{G}_1 = \{A \subset \mathbb{N} \mid A$ is nonempty and finite, or $1 \in A\}$

$\mathcal{G}_n = \{A \subset \mathbb{N} | A$ is nonempty; $\exists$ finite subsets F and G of A such that (i) every element of $A \setminus G$ is a multiple of some element of F and (ii) sum of elements of F is $< n.\}$

$\mathcal{G} = \{A \subset \mathbb{N} | A$ is nonempty; $\exists$ finite subsets F and G of A such that every element of $A \setminus G$ is a multiple of some element of F $\}$

Remark 4.2 : We observe that these families are comparable:

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n \subset \cdots \subset \bigcup_{n\in\mathbb{N}} \mathcal{F}_n \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots \subset \mathcal{G}_n \subset \cdots \subset \bigcup_{n\in\mathbb{N}} \mathcal{G}_n = \mathcal{G}$$

Theorem 4: Let X be the well-ordered space $[1, w^2]$. Or more generally, let X be a compact Hausdorff space having a unique element x_o every neighbourhood of which contains infinitely many limit points. Then $PER(X) = \mathcal{G}$.

Proof:

First Part: Let / : $X \to X$ be a continuous function. Let A — $Per(f)$.We shall prove that $A \in \mathcal{G}$. We consider two cases.

Case 1: Let $f(x_o) = x_o$. Then $1 \in A$. Therefore A € $\mathcal{G}$.

Case 2: Let $f(x_o) \neq y$ x_o. Then there is a clopen neighbourhood V of y such that V has at most one limit point. Then $f^{-1}(V)$is a neighbourhood of x_o. Let $Y = X \setminus f^{-1}(V)$. Then Y has only finitely many limit points, because otherwise the compactness of Y ensures that some point in Y must be the limit point of the set of all limit points of Y, contradicting the uniqueness of x_o. Therefore *f(Y)* has only finitely many limit points. Now *f(X)* C *f(Y)* U *V*. Therefore *f(X)* has only finitely many limit points. Let $g = f|_{f(X)}$.Then $Per(g) = Per(f)$Also applying theorem 3 to the domain of g (that is same as the range of /), we have $Per(g) \in \mathcal{G}_n$ for some n . Therefore A € $\mathcal{G}_n$for some n. Therefore $A \in \mathcal{G}$.

Second Part: Conversely let A 6 $\mathcal{G}$. We construct a function / : $X \to X$ such that $Per(f)$= A. Choose $n \in \mathbb{N}$ such that A 6 $\mathcal{G}_n$. Choose any n limit points $x_1, x_2, \cdots, x_n$of X different from x_o. Find a clopen neighbourhood V of x_o, not containing $x_1 x_2, \cdots, x_n$ but containing all other limit points. Let $W = X \setminus V$. Applying Theorem 3 to the space

W, there is a continuous $g : W \to W$ such that *Per(g)* = *A* (because *A* € $\mathcal{G}_n$).

$$\text{Define/:}X \to X \text{ by } f(x) = \begin{cases} g(x) & \text{if } x \in W \\ x_1 & \text{otherwise} \end{cases}$$

Then the range of / is contained in *W*. Therefore $Per(f)$= *Per(g)* = *A*.

§5

In this long section the ideas of proof are elementary, but lengthy and involved.

Theorem 5: Let *X* be a compact Hausdorff space such that *X"* has exactly 2 elements. (That is, the set X' of all limit points of X, in its relative topology, has precisely two elements. For instance, take *X* as the well-ordered space $[1, w^2.2]$). Then $PER(X) = \mathcal{H}_2$ where $\mathcal{H}_2 = \mathcal{G}$ U {*A* C IN : 2 ∈ *A*}.

Proof:

First Part: Let $X'' = \{x_o, y_o\}$and let / : $X \to X$ be continuous.

Case 1: Let the range of / be a compact Hausdorff space *Y* with at most one point that is the limit point of the set *V* of all limit points of *Y*. Then it follows from the earlier theorems that *Per(f)* £ $\mathcal{G}$.

Case 2: Let either x_o or y_o be a fixed point of /. Then 1 6 *Per(f).* Therefore $Per(f) \in \mathcal{G}$.

Case 3: Let neither case 1 nor case 2 hold. Then $f(x_o)$ has to be y_o. Otherwise, some clopen neighbourhood *V* of x_o will be mapped by / into a clopen set *Y* for which Y'' is empty. Since the complement of *V* has at most one element of that kind (that is in $(X \backslash V)''$),the continuity of / can be used to prove that *f(X\V)* is also so. Thus the range of / is contained in the union of two sets, Y and Z, such that. Y'' is empty and Z'' is atmost a singleten. It can be proved that the range of / therefore has at most one second level-limit point. This lands us in Case 1.

Thus **we proved** that in Case 3, $f(x_o) = y_o$ and similarly $f(y_o) = x_o$. thus $2 \in Per(f)$. Therefore ***Per(f)*** $\in$ $\{A$ C IN : $2 \in A\}$.

Second Part: Let A C IN be such that either A € $\mathcal{G}$ or $2 \in A$. we shall now construct a continuous function / : $X \to X$ such that $Per(f)= A$. For this, we consider two cases.

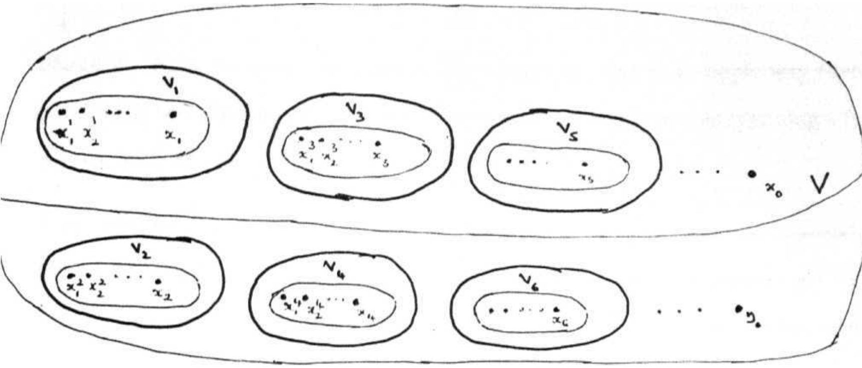
Case 1: Let A € $\mathcal{G}$.

Take a clopen set V containing x_o and not containing y_o.
Then V in its relative topology is such that V'' is a singleton. therefore by Theorem 3, there exists a continuous $g : V \to V$ such that *Per(g)* — A. Now define / : $X \to X$ by

$$f(x) = \begin{cases} g(x) & \text{if } x \in V \\ g(x_o) & \text{if } x \notin V. \end{cases}$$

Then / is continuous, and on the range of / ,it is same as g.
Therefore *Per{f)* = *Per(g)* = A.

Case 2: Let 2€ A. Noting that *if* $1 \in A$ *we* are in case 1, let $A = \{2 — a_o < a_1 < a_2 < \cdots\}$. Let V be a clopen neighbourhood of x_o not containing y_o. Then both $X' \cap V$ and $X'\backslash V$ are infinite sets. Choose a sequence (x_n) in X' with distinct terms such that $x_n \in V$ if n is odd and $x_n \notin V$ if n is even. Choose a clopen neighbourhood V_n of x_n such that for each n, V_n is either contained in V or disjoint from V, and such that $V_n \cap X' = \{x_n\}$. This is possible because $x_n \notin X''$. In each V_m choose a sequence $(x_n^m)_{n=1 to \infty}$ with distinct



We are going to construct a continuous $f : X \to X$ such that $Per(f)$ = the given A. The definition of this f is carried out in four stages.

Stage 1: $X''' = \{x_o, y_o\}$. We define $f : X'' \to X''$ by

$f(x_o) = y_o$ and $f(y_o) = x_o$.

Stage 2: Let $S = \{x_n : n \in \mathbb{N}\}$, where x_n are as chosen above. We define $f : S \to S$ by

$f(x_n) = x_{\sigma(n)}$ where σ is the permutation on $\mathbb{N}$ defined by

$$\sigma(1) = 3, \sigma(2) = 1, \sigma(3) = 4$$

$$\left.\begin{aligned} \sigma(4n+2) &= 4n+1 \\ \sigma(4n+3) &= 4n+4 \\ \sigma(4n) &= 4n+3 \\ \sigma(4n+1) &= 4n-2 \end{aligned}\right\} \text{ for all } n \in \mathbb{N}$$

We pictorially indicate σ as follows:

$$\ldots \to 9 \to 6 \to 5 \to 2 \to 1 \to 3 \to 4 \to 7 \to 8 \to \cdots$$

We note that for any two m, n in $\mathbb{N}$, $\exists k$ in Z such that $\sigma^k(m) = n$

Stage 3: Next we define the values at the points x_n^m, first in a simple way (defining a function *g)* and then modify the same so as to meet our requirements (defining a function /). We let $g(x_n^m) = x_n^{\sigma(m)}$ for all m, n in IN.

Note that the superfix ra is changed to $\sigma(m)$, whereas the suffix *n* is not changed. Thus the sequence $(x_n^m)_{n=1 \text{ to } \infty}$ converging to x_m is mapped by g to the sequence $(x_n^{\sigma(m)})_{n \text{ to } \infty}$ converging to $x_{\sigma(m)}$. Note that *g* is a bijection on the set of points where it is defined.

At this stage (unlike in the other stages of definition of /) we provide a role for the given set *A* — $\{2 = a_o < a_1 < a_2 < \cdots\}$. For each *j* in IN we do the following. We will, by suitably modifying $g(x_n^m)$ at some places, create an orbit of size a_j. We consider two cases.

Sub Case 1: Let a_j be odd, say a_j = 2r + 1. Then we make the element x_{2j+1}^1 as a periodic point with period a_j, by defining

$$f(x) = \begin{cases} g(x) & \text{if } \; x \in \{x_{2j+1}^1 g(x_{2j+1}^1), g^2(x_{2j+1}^1), \cdots, g^{r-1}(x_{2j+1}^1)\} \\ & \text{or if } \; x \in \{g^{-1}(x_{2j+1}^1), g^{-2}(x_{2j+1}^1), \cdots, g^{-r}(x_{2j+1}^1)\} \\ g^{-r}(x_{2j+1}^1) & \text{if } \; x = g^r(x_{2j+1}^1) \end{cases}$$

Now the f-orbit of x_{2j+1}^1 is

$$x_{2j+1}^1, x_{2j+1}^{\sigma(1)}, x_{2j+1}^{\sigma^2(1)}, \cdots x_{2j+1}^{\sigma^r(1)}, x_{2j+1}^{\sigma^{-r}(1)}, x_{2j+1}^{\sigma^{-r+1}(1)}, \cdots, x_{2j+1}^{\sigma^{-2}(1)}, x_{2j+1}^{\sigma^{-1}(1)}, x_{2j+1}^1$$

Subcase 2: Let a_j be even. Then we make $x_2^{a_j}$ *a*/-periodic point with period a_j, by changing the values of *g* at $\frac{a_j}{2}$ points, and creating the /-orbit as

$$x_2^{a_j}, \; x_2^{\sigma(a_j)}, x_4^{a_j}, x_4^{\sigma(a_j)}, \; \cdots, x_{a_j}^{a_j}, x_{a_j}^{\sigma(a_j)}, x_2^{a_j},$$

This stage of definition of / is completed by declaring *f(x)* = *g(x)* for all *x* that do not figure in the finite orbits created in the above two subcases, but that are still of the form x_n^m for some m, n in IN.

Stage 4: Now we define / at all points of X where we have not defined g so far. We let

$$f(x) = x^{\sigma(m)} \quad \text{for all those } x \text{ in } V \text{ for which } f(x) \text{ is undefined so far .}$$

Next we let $f(x) = y_o$ for all those x in V for which $f(x)$ has not been defined, even in the last line.

Lastly we let $f(x) = x_o$, for all those x outside V for which $f(x)$ has not been defined so far.

Thus we have defined a function / : $X \to X$, provided in Stage 3, while prescribing certain deviations from g, we have not unknowingly included an element in more than one orbit. We assure this by proving that the infinite number of finite sets A_j are actually pairwise disjoint, where A_j= the unique orbit of size a_j that we have created in stage 3. We note that

$$\begin{aligned} A_j &= \{x^1_{2j+1}, x^{\sigma(1)}_{2j+1}, \cdots, x^{\sigma^r(1)}_{2j+1}, \cdots, x^{\sigma^{-1}(1)}_{2j+1}\} \text{ if } a_j = 2r+1 \\ \text{and } A_j &= \{x^{a_j}_2, x^{\sigma(a_j)}_2,, \cdots, x^{\sigma(a_j)}_{a_j}\} \text{ if } a_j \text{ is even.} \end{aligned}$$

Let $i \neq j$ in $\mathbb{N}$. We claim that the subsets A_i and A_j of X are disjoint. For this we consider three cases.

Case 1: Let a_i and a_j be both even. Then A_i C 14, U $V_{\sigma(a_i)}$

and A_j C V_{a_j} U $V_{\sigma(a_j)}$

The four numbers a,-, $\sigma(a_i)$, a_j, $\sigma(a_j)$ are distinct, because $i \neq j$, because σ is one-one, and because σ is parity-changing [in the sense that $\sigma(n)$ is odd iff n is even] except when n = 1. Here a_i and a_j are even, and therefore $\sigma(a_i)$ and $\sigma(a_j)$ are odd.

Case 2: Let a_i and a_j be both odd. Then for every element in A_i the suffix is 2i + 1, whereas for every element in A_j, the suffix is $2j + 1$. Therefore A_i and A_j are disjoint.

Case 3: Let a_i be odd a_j be even. Then for every element in A_i the suffix is odd, whereas for every element in A_j, the suffix is even. Therefore A_i and A_j are disjoint.

Case 4: The case where a_i is even and a_j is odd, is similar to Case 3.

Our next job is to prove that this / is continuous. We assert that for each n in IN, there are only finitely many elements of V_n where / and g disagree. This is proved by observing:

(a) In the finite odd-orbit A_j, the only point where / and g disagree is x_{2j+1}, and this element is in $V_{\sigma^r(1)}$ where a_j — 2r + 1.

(b) If n is of the form $\sigma^r(1)$ for some r in IN, (then this r is unique) and if for that r, the number $2r+1$ is in the given subset A of IN, then there is one element of V_n where / and g disagree; If either such r does not exist or if $2r+1$ is not in A, then there is no point of odd period in V_n where / and g disagree.

(c) For every $n \in$ IN, there is at most one j in IN such that $n = a_j$. If this a_j is even, then there are $\frac{a_j}{2}$ points of even period in V_n where / and g disagree.

(d) / and g agree at all points of V_n except possibly at the $\frac{a_j}{2}$ points mentioned in (3) and at one point mentioned in (2).

As the next step in proving the continuity of /, we now make six more observations:

(e) Any point where / and g do not agree is an isolated point of X.

(f) Let $x \in X$. Let m, n in $\mathbb{N}$ be such that $f(x) \in V_m$ and $g(x) \in V_n$. Then $|m-n| \in \{0,2,4\}$.

[Reason: *Case 1:* x has odd f-orbit and $f(x) \neq g(x)$. Then for some r in $\mathbb{N}$, $x \in V_{\sigma^r(1)}$, $m = \sigma^{-r}(1)$ and $n = \sigma^{r+1}(1)$. But, for all r in $\mathbb{N}$, $|\sigma^{r+1}(1) - \sigma^{-r}(1)| = 2$.

Case 2: x has even f-orbit and $f(x) \neq g(x)$. Then for some r in $\mathbb{N}$, $r = V_{a_j}$ and either $n = \sigma(a_j) = m$ or $m = \sigma^{-1}(a_j)$ and $n = \sigma(a_j)$.
But we have $|\sigma^{-1}(k) - \sigma(k)| = 4$ except when $k = 1$.

Case 3: x has infinite f-orbit. Then $f(x) = g(x)$].

(g) The symmetric difference $f^{-1}(V_n)\Delta g^{-1}(V_n)$ is a finite set of isolated points for each n in $\mathbb{N}$.

[Reason: This is the set $\{x \in X |$ exactly one of $f(x)$ and $g(x)$ belongs to $V_n\}$. This is contained in the set

$$\{x \in X | f(x) \neq g(x); f(x) \in V_m \text{ for some } m \text{ with } |m - \sigma(n)| < 4\}$$

$$\subset \{x \in X | f(x) \neq g(x)\} \cap (\cup\{V_m \cap f^{-1}(V_n) | m \in \{\sigma(n), \sigma(n) \pm 1, \sigma(n) \pm 2, \sigma(n) \pm 4\}$$

This is because of (6). For each m, the set

$$\{x \in V_m | f(x) \in V_n, f(x) \neq g(x)\} \text{ is finite, because of (4)]}$$

(h) It follows from (7) that $f^{-1}(V_n)$ is clopen for each n in $\mathbb{N}$.

(i) For each x in $X' \backslash X''$, let V_x be a clopen neighbourhood, chosen **arbitrarily,** and let V_{x_n} be same as the already chosen V_n. Then the family

$$\{\{y\}, V_x \backslash \{y\}, V \backslash V_x \backslash \{y\}, X \backslash V \backslash V_x \backslash \{y\} : x \in X' \backslash X'', y \text{ is isolated in } X\}$$

is a subbase for the topology of X.

(j) For each member W of this subbase, the preimage $f^{-1}(W)$ is clopen in X. [Reason: (a) If x in $X'\backslash X''$ is different from the x_n's, then only finitely many elements of V_x are in the range of f and these elements are isolated. Therefore $f^{-1}(V_x)$ is an open set in X contained in $X\backslash X'$. (b) Because of (6), if $f(x) \in V_m$ and $g(x) \in V_n$, then m and n are of the same parity. Therefore $f(x) \in V$ if and only if $g(x) \in V$. Thus $f^{-1}(V) = g^{-1}(V)$. (c) Now combine (a), (b), (7) and (8) to prove (10)]

Lastly, we prove that $Per(f) = A$. We note that the points in X are of three types with regard to their behaiour in the dynamical system (X, f).

First kind: The points explicitly mentioned in the finite f-orbits in the two subcases in Stage 3 of the definition of f. These are f-periodic points. By construction, for any such point, the f-period is one of the a_i's in A. Conversely for every a_j in A there is one such point.

Second kind: On all points that are not of the first kind, f and g agree. Among them, there are some points x such that $g^m(x)$ is of the first kind for some m in $\mathbb{N}$. These points have finite f-orbits, but are not f-periodic. So, they do not contribute any element to the set $Per(f)$.

Third kind: Lastly, we have the remaining points of X. For them, the f-orbits are infinite.

These observations prove that all f-periodic points are of the first kind and $Per(f) = A$.

§6

In this section,we consider ,for a fixed positive integer n,a compact HausdorfF space *X* with $|X''| = n$.We compute *PER(X)* We sketch a proof,but omit the details,since these are analogous to the details of the previous section.

Theorem 6: Let *X* be a compact Hausdorff space such that \X"\ $= n$. Then ***PER(X)*** = *{A* C IN|1 or 2 or $\cdots$ or *n* G *A]* U $\mathcal{G}$

Proof: Denote by $\mathcal{H}_n$ the family {*A* C IN|1 or 2 or • • • or *n* G *A*} U $\mathcal{G}$.Note that when n = 1, $\mathcal{H}_n$ is same as $\mathcal{G}$, because all subsets containing 1 are already in $\mathcal{G}$. Therefore this result has been already proved, when *n* = 1, as Theorem 4. When *n* = 2,this is same as Theorem 5.

First Part: Let / : *X* —> *X* be continuous. Let *Per(f)* = *A.* We shall prove that *A* G $\mathcal{H}_n$ • We prove this by induction on n. As already noted, it is true for *n* = 1. Assume as induction hypothesis that $PER(Y) \subseteq$ {A C IN| 1 or2 or • • • or n-1 $\in$ *A]* U $\mathcal{G}$ = $\mathcal{H}_{n-1}$ for any compact Hausdorff space *Y* with $|Y''|$ < *n* ~] . Given $|X''|$ = *n* . Suppose *X"* = $\{x_1, x_2,$ • • • $x_n\}$. Let / : *X* → *X* be a continuous function such that *Per(f)* — *A* . If *X"* is /-invariant then *f(X")* C X''.

hence by Theorem 1, 1 or 2 or • • • $n \in A$. Hence $A \in \mathcal{H}_n$ .If X'' is not /-invariant, then there exists a x_i in it such that $f(x_i) \notin X''$, then there exists some clopen neighbourhood *V* of x_i such that it is mapped into a clopen subset *Y* for which Y'' is empty. Since V^c has at most *n* — 1 elements, the continuity of / implies that the image of V^c has at most *n* — 1 elements. Then the range of / (also compact T_2-space) is contained in the union of *Y* and *Z* such that *Y"* is empty and *Z"* has exactly *n* — 1 elements. Now let *g* —$f|_Z$ then g is continuous and *Per{g)* = $Per(f|_Z)$ = $Per(f)$ = A. As \Z"\ = *n* - 1, by our induction hypothesis *A* € $\mathcal{H}_{n-1}$ .But ft$_n$_i C $\mathcal{H}_n$, hence, $A \in \mathcal{H}_n$.

Second Part: Conversly, let X be as above let A C IN such that either 1 or 2 or • • • $n \in A$, or $A \in \mathcal{G}$. We now construct a continuous self map / of X such that *Perf* = *A*.

Case (i): Let n be the least element of A. Let $z_1, z_2, \cdots, z_n X''$ and let $W_i's$ be pairwise disjoint clopen sets such that $z_i \in W_i$, i € $\{1,2,\cdots,n\}$. For each i in $\{1, 2$ • • • n$\}$, choose a sequence $x_{i1}, x_{i2}, \cdots, x_{ik}$ • • • of distinct elements in X'' converging to z_i ;let V_i, Vn+i, V_{2n+i} • • • be pairwise disjoint clopen subsets of W_i such that V_{kn+i} $\cap$ X" = $\{x_{ik}\}$;let ft $= \{x_{ik} \mid k= 1,2,\cdots\}$.

Next step is to define a permutation σ on IN.

Firstly let $\sigma(rn$ + /) = (r + l)n + l; $0 \leq r \leq l-1$; $0 < l \leq n-1$.

Secondly $\sigma(k)$ = min $\{\{sn+ / + l \backslash s \in$ IN$\} \backslash S_k\}$.

where k = rn +/; / < n

Thirdly, if / = n, then k = (r + l)n; and $\sigma(k)$ = min $\{\{sn+ 1|s \in$ IN$\}\backslash S_k\}$

where $S_k = \{\sigma(1), \sigma^{-1}(1), \sigma(2)\sigma^{-1}(2) \cdots \sigma^{-1}(k$- 1)$\}$

and $\sigma^{-1}(k)$ = *min{{sn* + / - *l\s* $\in$ IN$\} \backslash S_k$ } where k = rn + /; / > 1;

Fourthly ,if / = 1; then fc = rn + 1 and $\sigma(k)$ = min $\{\{sn|s \in$ IN$\}\backslash S_k$ } .

Next we use this σ to define a function g : $X \to$ X such that : *g{Vi)* — $V_{\sigma(i)}$ if i= *rn* + /; and $\sigma(i)$ — mn + p.

$g(x_{lr})$ = x_{pm} since x_{lr} and x_{pm} € X', 3 sequences in V_i and $V_{\sigma(i)}$ X converging to x_{lr} and x_{pm} respectively, g maps the n-th element of the sequence in V_i to the n-th element of the sequence in $V_{\sigma(i)}$ and all the other elements of V_i are mapped to x_{pm}.

and all the other points outside $\bigcup^{\infty} V_i$ are mapped to z_n.

and *g(zi)* = z_{i+1} ;$g(z_n) = z_1$ V $i \in \{1,2,$• • •, n $- 1\}$. Clearly from the definition of g there are no finite orbits other than the orbit of length n. Next we perturb g at some points, to define a new self-map / of X. Look at A = *{n* < n_2, < n_3, • • •$\}$ $\subset$ IN

Let $\mathcal{W} = \{2,3,4,5,\cdots\}$

Now suppose $n_i = nl + k$ $k < n$;pick the n_i-th element of the chosen sequence in V_k call it x; and look at the path

$x, g(x), g^2(x), \cdots g^k(x), \cdots g^{(n-1)k}(x)$[the perturbation is $g^{(n-1)k}(x) \longmapsto x$] and get an orbit of length $n + k$. if $n_i = nl + k$, $l \in \mathcal{W}$, and l is odd, $k \in \mathbb{N}$, $k < n$, g is perturbed at one point to achieve a cycle of length n_i ; the peturbation is as follows. Pick the n_i th element of the sequence in V_k; call it z, follow the path

$z, g(z), g^2(z), \cdots g^k(z), \cdots g^{qn+k}(z), g^{-[qn+k]}(z), g^{-[(q-1)n+k]}(z), \cdots g^{-2}(z), g^{-1}(z), z$

to get a n_i cycle when l is odd and $l = 2q + 1$

and follow the path $z, g(z), g^2(z), \cdots g^k(z), \cdots g^{qn+k}(z)$,

$g^{-[(q-1)n+k]}(z), g^{-[(q-2)n+k]}(z), \cdots g^{-2}(z), g^{-1}(z), z$, if l is even and $l = 2q$.

Note that the above paths are along the q-orbit of z except at one point; $g^{-[qn+k]}(z)$ is not mapped to $g^{-[(q-1)n+k]}(z)$ under g when l is odd and $g^{qn+k}(z)$ is not mapped to $g^{-[(q-1)n+k]}(z)$ under g when l is even.

If $k = n$; i.e. $n_i = rn$, for some r; Pick n_i-th element of the sequence s_{rn} in V_{rn} call it θ ,map it to the n_i-th element of the corresponding sequence $S_{(r-1)n+1}$ in $V_{r-1)n+n+1}$. and continue doing so till the $n-1$-th row is reached and the n_i-th element of the corresponding sequence in $V_{(r-1)n+n-1}$ is mapped to the $(n_i + 1)$-th element of the corresponding sequence in V_{rn}; and this element in V_{rn} is mapped to the $(n_i + 1)$-th element of $V_{(r-1)n+1}$ and continued till the $(rn + n_i)$-th element of the corresponding sequence in V_{rn-1} is reached. Call it r; map r to θ to complete a n_i orbit.

Let f be the perturbed form of g .Clearly $Per(f) = A$

Note the following

- (i) / and g differ at only finitely many points in each of the V_i's.

- (ii) For each i in IN there are only finitely many j in **IN such that** $f^{-1}(V_i) \cap V_j$ is nonempty.

- (iii) For every x in X, both $f(x)$ and $g(x)$ belong to the same row ; in fact if x is in the i-th row, and $f(x)$ is in the j-th row, then $j — i + 1$ (mod n).

Further details of the proof are omitted. Case (ii): Let case (i) not hold. Then either $n \notin A$ or $m \in A$ for some $m < n$. Then $A \in \mathcal{H}_{n-1}$. Therefore by induction hypothesis, there is a continuous $g : X \backslash W \longrightarrow X \backslash W_1$ such that $Per(g) = A$. This is because $X \backslash W_1$ is a compact Hausdorff space Y such that $|Y''|$ — n — 1. This g can be extended to a continuous / : $X \rightarrow X$ as in proposition 1.2 so that $Per(f) = A$.

§7

In this section we describe a class of spaces X such that $PER(X)$={ all nonempty subsets of IN}.

Lemma 7.1 : Let X be a compact Hausdorff space.Let $x \in X$. Let V be an open neighbourhood of x. Let for each y in $V \backslash \{x\}$ there exist a clopen neighbourhood V_y not containing x. Then there is a clopen neighbourhood of x contained in V. [This means in an imprecise sense: If all other points around x have sufficiently small clopen neighbourhoods, so does x\.

Proof: Choose a neighbourhoodd V_x of x whose closure is contained in V. This is possible because X is regular. Now $\{V_x\}$ U $\{V_y : y \in V_x \backslash \{x\}\}$is an open cover for the compact space V_x. Let V_x, V_{y_1}, $\cdots$, V_{y_n} be a finite subcover. Then $V_x \setminus V_{y_1} \setminus \cdots \setminus \overline{V_{y_n}}$ is equal to $V_x \setminus V_{y_1} \setminus \cdots \setminus V_{y_n}$ and is therefore a clopen neighbourhood of x.

Lemma 7.2 : Let X be a compact Hausdorff space. Then (i) every element x of $X'\setminus X''$ admits a clopen neighbourhood V_x such that $V_x \cap X' = \{x\}$.

(ii) every element x of $X''\setminus X'''$ admits a clopen neighbourhood V_x in X such that $V_x \cap X'' = \{x\}$

Proof: (i) Since x is in $X'\setminus X''$, there exists an open neighbourhood V of x such **that** $V \cap X' = \{x\}$ (since otherwise x will be a limit point of X'). Now every y in V other then x is isolated and hence has $\{y\}$ as a clopen neighbourhood. Therefore by the lemma above, there is a clopen neighbourhood V_x of x contained in V. This has the required property.

(ii) Since x is not a limit point of X'', there is a neighbourhood V of x such that $V \cap X'' = \{x\}$. If y is any element in this V, other then x, then y is not in X''. Therefore y is either in $X'\setminus X''$ or y is isolated. In either case, y admits a clopen neighbourhood contained in $V\setminus\{x\}$. Here we use the part (i) of this lemma. Next, using the previous lemma, we obtain a clopen V_x contained in V, and containing x. This proves (ii).

Remark 7.3 : The above lemmas are not new. More general results are already known. For instance it is known that all comapct Hausdorff scattered spaces are zero-dimensional. Still we have included the proofs of these lemmas here, because, when we do not require the more general results, these proofs are simpler.

Example 7.4 :

Let $X = \{(a_1, a_2, a_3) \in R^3 |$ each a_i is either 0 or the reciprocal of a positive integer} . Let X be provided with relative topology from 1R^3. Then

$X' = \{(a_0, a_1, a_3) \in X \mid a_1 a_2 a_3 = 0\}$

$X'' — \{(a_0, a_1, a_3) \in X \mid a_1 a_2 + a_2 a_3 + a_1 a_3 = 0\}$

$X''' = \{(0,0,0)\}$ .This is an example of a countable compact metric space X such that X''' is a singleton.

Theorem 7: Let X be a compact Hausdorff space such that X''' is a nonempty finite set. Then $PER(X) = \wp(\mathbb{N}) \setminus \{\phi\}$.

Proof:

First part: Let / : $X \to X$ be continuous. We want to prove that $Per(f)$is not empty. Case 1: Suppose there is some x in X''' such that $f^n(x)$is in X''' for infinitely many values of n. Since X''' is finite, it follows that $f^m(x) = f^n(x)$for some $m < n$ and therefore $f^m(x)$is a periodic point. Let X''' be finite.Let $f : X \to X$ *he* continuous. Then either some element of X''' is a periodic point of / or there exists $n_1 \in$ IN and a clopen W_1 D X''' such that if $Y - f^{n_1}(W_1)$,then Y'' is finite. Arguing similarly ,either some element of Y'' is a periodic pioint of f^{n_1} (and therefore of f) or there exists $n_2 \in$ IN and a clopen W_2 D Y'' C W_2 such that if $Z = f^{n_1 n_2}(W_2)$,then Z' is finite. Once again arguing similarly, either some element of Z' is a periodic point of $f^{n_1 n_2}$ (and therefore of f) or there exists $n_3 \in$ IN and clopen W_3 containing Z' such that $f^{n_1 n_2 n_3}(W_3)$is a finite set.

Case 2: If Case 1 does not hold, there is n_1 in IN such that $f^{n_1}(X''')$is disjoint from X'''. This finite set admits a clopen neighbourhood disjoint from the finite X''' .

Let W_1 =the preimage of this clopen neighbourhood under f^{n_1} .Then $f^{n_1}(W_1)$ is a compact subset of X disjoint from X''' .On the otherhand $X\setminus W_1$ is also a compact subset of X disjoint from X''' .Therefore $(f^{n_1}(X\setminus W_1))''$ is finite. Let $Y_1 = f^{n_1}(X)$.Then $Y_1 = f^{n_1}(W_1) \cup (f^{n_1}(X\setminus W_1)$ is a compact subset of X such that $(Y_1)''$ is finite.[We are not claiming that $Y_1 \cap X'''$ is empty]. Thus so far we have proved in these two cases the following : Either there exists a periodic point in X''' (in case 1) or there exists $n_1 \in \mathbb{N}$ such that Y_1'' is finite , where $Y_1 = f^{n_1}(X)$.

Arguing similarly with $f^{n_1} : Y_1 \to Y_1$ in place of $f : X \to X$ we conclude:

Either there exists a periodic point in Y_1 for f^{n_1} (and therefore for f) or there exists $n_2 \in \mathbb{N}$ such that Y_2' is finite where $Y_2 = f^{n_1 n_2}(Y_1)$ .Arguing once again with $f^{n_1 n_2} : Y_2 \to Y_2$, we conclude that either there is a periodic point in Y_2 for $f^{n_1 n_2}$ (and therefore for f) or $f^{n_1 n_2 n_3}(Y_2)$ is finite. In the later case ,obviously there is a periodic point in Y_2 in $f^{n_1 n_2 n_3}(Y_2)$ and therefore for f.Thus in all cases , f has a periodic point .

Note: By this method we can prove the more general result that all compact scattered spaces X have the property that Per(f) is non empty for all continuous functions $f : X \to X$.

Second Part: Conversely let A be a nonempty subset of $\mathbb{N}$. We construct a continuous function $f : X \to X$ such that *Per(f)* = *A*. Let a_o be the least element of *A*. Choose from the infinite set $X''\setminus X'''$,any a_o elements, say $x_1 x_2, \cdots, x_{a_o}$. By the above lemma, choose clopen neighbourhoods $W_1, W_2, \cdots, W_{a_o}$ of thesee points respectively such that $W_i \cap X''$ is a singleton for each *i*.

Let $Y = W_1 \cup W_2 \cup \cdots \cup W_{a_o}$. Then Y is a clopen subset of X, and is a compact Haudorff space on its own right, satisfying $|Y''| = a_o$. Therefore by Theorem 6, there exists a continuous $g : Y \to Y$ such that *Per(g)* = A, because $a_o \in A$. Now defining $f : X \to X$

by the rule,

$$f(x) = \begin{cases} g(x) & \text{if } x \in Y \\ x_1 & \text{if } x \notin Y \end{cases}$$

Then / is continuous (bcause Y is clopen in X), and *Per(g)* = *Per(f)* [because *g* is nothing but the restriction of / to its range]. Thus *Per(f)* = *A.*

§8

In this last section, we restrict our attention to metrizable spaces. We prove four theorems describing spaces *X* for which $PER(X)$ is either $\wp(\mathbb{N})$ or $\wp(\mathbb{N}) \setminus \{\phi\}$.

Theorem 8: Let *X* be any countable compact Hausdorff space such that *X''* is infinite. Then *PER(X)* consists of all nonempty subsets of IN.

Proof: Let / : *X* → *X* be any continuous function. It is known that *Per(f)* has to be nonempty. This result has been first stated in [33] and see Theorem 1 of Chapter 2 for a proof.

Conversely let *A* be any nonempty subset of IN. We want to find a continuous function / : *X* → *X* such that *Per(f)* = *A.* We proceed as follows.

Since *X* is compact, the infinite set *X''* must have a limit point in *X.* That is *X'''* is nonempty. this *X'''* is a countable compact Hausdorff space on its own right; when written as the union of its singletons, Baire category theorem implies that at least one of these singletons must be open in X'''. This means that there exists an open set *V* such that $V \cap X'''$ is a singleton, say $\{x_o\}$. By parts (i) and (ii) of Lemma 7.2, every element of $V \setminus \{x_o\}$ has a clopen neighbourhood not containing x_o. Therefore by Lemma 7.1, there is a clopen neighbourhood *W* of x_o contained in V. This *W* is a compact Haudorff space on its own right, such that $W''' = \{x_o\}$. Therefore by Theorem 7, there exists a continuous

$g : W \rightarrow W$ such that $Per(g) = A$ (where A is the given nonempty subset of $\mathbb{N}$). Now define / : $X \rightarrow X$ by the rule

$$f(x) = \begin{cases} g(x) & \text{if } x \in W \\ x_o & \text{if } x \notin W \end{cases}$$

Then / is continuous and $Per(f)$is also equal to A.

Theorem 9: Let K be the Cantor set. Then $PER\ (K)=\wp(\mathbb{N})$.

8.1. **Definition:** A closed subset Y of a topological space X is called a retract of X if there is a continuous function $r : X \rightarrow Y$ such that the restriction $f|_Y$is the identity function on Y. This r is called a retraction map.

Example 8.2 : Every clopen subset Y of X is a retract of X. The closed interval $[0,1]$ is a retract of $\mathbb{R}$.

Lemma 8.3 : Let Y be a retract of X. Then $PER(Y)$ is contained in $PER(X)$.

Proof: Let $A \in PERY$. then there is a continuous / : $Y \rightarrow Y$ such that $Per(f)= A$. Let r : $X \rightarrow Y$ be a retraction map. Let i : $Y \rightarrow X$ be the inclusion map. Then $i\ o\ f\ o\ r : X \longrightarrow X$ is a continuous function. Its range is Y. Its restriction to the range coincides with /. Therefore $Per(i\ o\ f\ o\ r) = Per\{f) = A$. Thus A € $PER(X)$.

Proof of Theorem 9: Let X be a fixed compact metric space with X''' a singleton, described in Example 7.4. Consider the product space X x $K = Y$. Then Y is a zero dimensional compact metric space in which every point is a limit point. Therefore by a classical theorem of Hausdorff, Y is homeomorphic to the Cantor set. See [14].

let $p : Y \rightarrow X$ x $\{0\}$ be the projection map defined by $p(x, y) = x$ for all (x,y) in X x A'. Then p is a retraction map, exhibiting X x $\{0\}$ as a retract of Y. Therefore by Lemma

8.3, $PER(Y) \supset PER(X \times \{0\})$.

Now $X \times \{0\}$ is homeomorphic to X. By Theorem 7, $PER(X)=\{$ nonempty subsets of IN$\}$. This proves that every nonempty subset of IN belongs to $PER(K)$.

It has been proved in proposition 1.2 of Chapter 2, that the empty set belongs to $PER(K)$.

Combining these two results, we obtain that $PER(K)= \wp(\mathbb{N})$.

Theorem 10: Let X be a zero dimensional metric space.Then the the following are **equivalent:**

(i)X is countable and compact.

(n)$PER(X) \neq \wp(\mathbb{N})$.

(iii)$\phi \notin PER(X)$.

Proof: (i)$\Rightarrow$(iii), has been proved in chapter 2. It also follows from Theorems 1 to 9 above. (iii)$\Rightarrow$(ii), is obvious. To prove (ii)$\Rightarrow$(i), we take an arbitrary zero- **dimensional** metric space X not satisfying(i)and we prove that $PER(X)=\wp(\mathbb{N})$.

We consider two cases:

Case1: X is not compact.Then there exists a countably infinite closed subset Y without any limit points in X.It is possible to partition X into clopen subsets $W_1, W_2, \cdots W_n \cdots$ such that $W_n \cap Y$ is a singleton for each n in IN (see [14]). Define a map r: $X \to Y$ by the rule r(x)= the unique point of $W_n \cap Y$ where n is the unique element of IN such that x is in W_n.Then r is continuous(because each W_n is clopen),and r is a retraction. Therefore ***PER(Y)*** C *PER(X)* by lemma 9.3 We claim that $PER(Y) = \mathbb{N}$.(and so is *PER(X)*).To prove this we note that Y is homeomorphic to IN with discrete topology.Let $A = \{n_1, n_2, ...\}$,be any subset of IN.

Define f :IN $\to$ IN by

$$
f(m) = \begin{cases} m+1 & \textbf{if } \mathbf{1} < m < n_1 \\ & \text{or if } n_1 + n_2 + \dots + \text{ra*} + 1 < n_1 + \text{ra}_2 + \dots + n_k \textbf{for} \text{ some k} \\ 1 & \text{if } rn = n_1 \\ n_1 + n_2 + \dots + n_{k-1} + 1 & \text{if } m = n_1 + n_{2+} + \dots + n_k \\ 1 & \text{if } A \text{ is finite and if m > sum of all elements of A} \end{cases}
$$

then it is easily verified that Per(f)=A.Thus we have proved that if X is not compact,then *PER(X)* $= \wp(\mathbb{N})$.

Case2: Let X be com pact. Si nce X does not satisfy (i), X must be uncountable. By a known classical theorem, X should contain a homeo- morphic copy of the Cantor set K. By a known property of the cantor set, K is a retract of every zero-dimensional metric space containing it. [See 14] for a proof.Therefore *PER(X)* $\supset$ *PER(K)* by lemma 8.3. But $PER(K) = \wp(\mathbb{N})$ by Theorem 9. Therefore *PER{X)* $= \wp(\mathbb{N})$. These two cases together prove (ii)$\Rightarrow$ (i).

Remark 10.1 : The equivalence of (i) and (iii) has been proved in the previous chapter. The equivalence of (ii) and (iii) implies the following **surprising** result : For zero dimensional metric spaces X ,if some continuous self-map of X exists without periodic points,then for every A C $\mathbb{N}$, there is a continuous self-map of X whose set of periods is exactly A .

Remark 10.2 : (ii)and(iii) are not eqvivalent for a general metric space (in the absence of **zero-dimensionality**).The simplest counter example is the real line $\mathbb{R}$.**By** a known theorem of Sarkovskii ,*PER* (IR) $\neq \wp(\mathbb{N})$;**for** example {1,3} $\notin PER(\mathbb{R})$.**But** ϕ G $PER(\mathbb{R})$; this can be proved by considering the function *f(x)* = $x + 1$ for all x in IR .

Now combining all the results we arrive at the following

THEOREM 11: Let X be a zero dimensional metric space. Then

$$PER(X) = \begin{cases} \wp(\mathbb{N}) & \text{if } X \text{ is not compact} \\ \wp(\mathbb{N}) & \text{if } X \text{ is not countable} \\ \wp(\mathbb{N})\backslash\{\phi\} & \text{if } X \text{ is countable and compact} \\ & \text{and if } X'' \text{ is infinite} \\ \mathcal{H}_n & \text{if } X \text{ is compact and } |X''| = n \\ \mathcal{G}_n & \text{if } X \text{ is compact and } |X'| = n \\ \mathcal{F}_n & \text{if } X \text{ is finite and } |X| = n \end{cases}$$

where for each n in $\mathbb{N}$,

$\mathcal{H}_n = \{A \subset \mathbb{N} | A$ is nonempty; if A is infinite,

some element of A is $< n\} \cup \mathcal{G}$ where

$\mathcal{G} = \{A \subset \mathbb{N} | \exists$ a nonempty finite subset F of A

such that every element of A is a multiple of some element of F $\}$

$\mathcal{G}_n = \{A \subset \mathbb{N} | A$ is nonempty; there exist two finite subsets F and G

of A such that every element of $A \backslash G$ is a multiple of some

element of F, and such that the sum of all elements of F is $< n\}$

and $\mathcal{F}_n = \{A \subset \mathbb{N} | 1 <$ (sum of elements of A) $< n\}$.

Proof: Combine the results of the earlier ten theorems. More specifically: The first assertion that $PER(X)=\wp(\mathbb{N})$ if the zero-dimensional metric space X is not compact, has been proved in case 1 of the proof of (2)⇒(1) of Theorem 10, by exhibiting the discrete space $\mathbb{N}$ as a retract of X. The second assertion that $PER(X)=\wp(\mathbb{N})$ when the zero-dimensional metric space X is not countable, has been proved in case 2 of the same, by exhibiting the Cantor set K as a retract of X and by applying Theorem 9. The third assertion that $PER(X)=\wp(\mathbb{N})\setminus\{\phi\}$ if X is countable and compact and if X'' is infinite, is exactly the statement of Theorem 8. The fourth assertion that if X is compact and if $|X''| = n$, then $PER(X)=\mathcal{H}_n$, is proved in Theorem 6 (even without assuming X to be a metric space). Similarly, the fifth and sixth assertions are proved respectively as Theorem 4 and Theorem 2.

Remark 11.1 : In the first seven theorems, we have worked among general topological spaces that need not be metrizable. It is only in the last four theorems, that we confined our discussion to metric spaces. One can therefore expect to harvest a lot more from them. For example we can deduce from them complete answers to the following questions: which families arise as $PER(X)$ for compact scattered spaces X ? Which families arise as $PER(X)$ for well-ordered compact spaces X? But their thorough descriptions require the knowledge of ordinal arithmatic and derived length (that are not required when we work among metric spaces) and therefore we defer these answers, even though the complete machinery for their proofs are already available in this chapter.

Remark 11.2 : Three facts, are worth-noting:

We have described countably many families. The class of zero-dimensional metric spaces is a proper class, not restricted to any cardinality. Even the compact ones among them are of uncountably many homeomorphic types. But to describe their period-sets-family, countably many families suffice.

These countably many families form a totally ordered set: $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n \subset \cdots \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots \subset \mathcal{G}_n \subset \cdots \subset \mathcal{H}_1 \subset \mathcal{H}_2 \subset \cdots \subset \mathcal{H}_n \subset \cdots \subset \wp(\mathbb{N})\setminus\{\phi\} \subset \wp(\mathbb{N})$.

This is not the case, when we consider one-dimensional spaces also. For instance **Sarkovskii's** period-set-family for $\mathbb{R}$ is not comparable with $\mathcal{G}_1$.

Open Problem: Find *PER (X)* for all zero-dimensional spaces X, without assuming metrizability.

Chapter 4

Sets of Periods of Compact Subsets of *JR*

In this Chapter, we calculate $PER(X)$ where X is a general compact subset of $\mathbb{R}$. We note that this is already accomplished in the previous chapter, when X does not contain a nontrivial interval. Our answer is that $PER(X)$, in general, has to be a certain kind of combination of the Sarkovskian family and the families obtained in Chapter 3. The answer depends not only on the cardinalities of X' and X'', but also on the number of nontrivial components of X, of X' and of X''. The new families no more form a totally ordered collection.

§1.

Theorem 1:

Let $n \in$ **IN** . Let $X = I_1 \cup I_2 \cup \ldots\ldots I_n$, where each I_i is a closed interval in $\mathbb{R}$. Let these I_i's be pairwise disjoint. Then $PER(X) = \{A$ C IN : A is of the **form** $n_1 A_1$ U $n_2 A_2$U • • • U $n_k A_k$ where each $A_i \in$ $ and $1 < n_1 + n_2 + \cdots n_k < n\}$

Proof:

First Part: Let $f : X \to X$ be continuous. Let $A = Per(f)$. It is obvious that for each $i \in \{1, 2, \cdots n\}$ — J, $f(I_i)$ is a connected subset of X and $f(I_i)$ C I_j for some $j \in$ J. Thus we get a function from $J \to J$ induced by /, i.e., ϕ: $J \to$ J defined by $\phi(i)$ = **the**

unique j such that $f(I_i)$ C lj

(Note: $x \in I_i \Rightarrow f(x) \in I_{(i)}$).

Let $Per(\phi) = \{n_1 n_2, \cdots, n_k\}$. Then this set belongs to $\mathcal{F}_n$. Therefore $1 \leq n_1 + n_2 + \cdots + n_k < n$. Now consider $f^{n_j} : X \to X$.

As we have a cycle of length n_j for , say $i_1, i_2, \cdots, i_{n_j}, i_1$, $x \in I_{i_1} \Rightarrow f^{n_j}$ (a?) € I_{i_1}

So I_{i_1} is invariant under f^{n_j}.

Now restrict f^{n_j} to I_{i_1} and call it as g_{j,i_1} i.e., $f^{n_j}|_{I_{i_1}} = g_{j,i_1}$

since this g_{j,i_1} is self map on I_{i_1} its period set is described in S , call $Per(g_{j,i}) = A_{j,i_1}$.

Let $A_j = \bigcup\{A_{j,i} \mid i \in J;$ -period of % is $n_j\}$. Thus / gives rise to $n_1, n_2, \cdots, n_k$ and also to $A_1, A_2, \cdots, A_k$.

Claim: $Per(f) = n_1 A_1 \cup n_2 A_2 \cup \cdots \cup n_k A_k$

Firstly, let $m \in Per(f)$.

Let x be a point of period m in X.

Then there is a unique i 6 J such that $x \in I_i$. Then $f^m(x) \in I_{m(i)}$. Therefore $\phi^m(i) = i$.

Therefore m is divisble by the -period of i in $Per(\phi)$, say $i = n_j$.

Therefore $m = kn_j$ for some k . Now see the orbit sequence of x,

$$x, f(x), f^2(x), \cdots, f^m(x)(= x)$$

and corresponding intervals in which it is moving,

$$\underbrace{I_i, I_{\phi(i)} \cdots}_{n_j\text{steps}} \quad \underbrace{I_i, I_{\phi(i)} \cdots I_i}_{n_j\text{steps}} \cdots \underbrace{I_i, I_{\phi(i)} \cdots I_i}_{n_j\text{steps}}$$

Thus we observe that I_i takes n_j steps to come back under ϕ and x in I_i , takes k steps to come back under f^{n_j} hence $k \in Per(f^{n_j}) = A_{j,i}$. So fc G A_j hence m $\in n_j A_j$.

On the other hand,

let m G $n_1 A_1 \cup n_2 A_2 \cup \cdots n_k A_k$

Then $m = kn_j$ for some $k \in A_j$,for some $j \in J$

but $A_j = \bigcup A_{j,i}$

whereA = $\{i \in J$: -period of i is rc,}

Since $k \in A_j$

choose one such i in A and consider the function on $g_{j,i} = f^{n_j}$|/. on I_i. Then $(f^{n_j})^k(x) = x$

for some x in I_i.

Thus we have a periodic point with period k for f^{n_j} when it is restricted to I_i.

The same x (chosen in I_i) is /-periodic point with period $n_j k(= m)$.

Hence / has periodic point in I_i with period *rn.*

Hence $m \in Per(f)$.

Second Part: Let $n_1, n_2, \cdots, n_k$ be given positive integers whose sum is < *n.* Let $A_1, A_2, \cdots, A_k$ be given members of *S.* We now construct a continuous / : $X \to X$ such that *Per(f)* = $n_1 A_1$U $n_2 A_2$U • • · U $n_k A_k$.

Step(a): Let J = {1,2,• • · ,n}. Construct ϕ : $J \to J$ with the following properties:

(i) $Per(\phi) = \{n_1, n_2, \cdots, n_k\}$

(ii) If 1 < i < k, then the ϕ-period of i is rc$_t$.

Such ϕ exists by Theorem1 of Chapter-3.

Step (b): For each 1 < i < k, choose f_i : $[0,1] \to [0,1]$ be a continuous function such that $Per(f_i) = A_i$. Here we are using the known converse of Sarkovskii's theorem . See [12 1.

Step (c): Let for $1 < i < k$, $h_i : [0,1] \to I_i$ be the canonical homeomorphism, having a formula of the form $h_i(t) = \lambda t + \mu$ for all t in [0,1]. We define f from X to X by the formula

$$f(x) = \begin{cases} h_{\phi(i)} f_i h_i^{-1}(x) & \text{if } 1 < i < k \text{ and if } x \in I_i \\ h_{\phi(i)} h_i^{-1}(x) & \text{otherwise} \end{cases}$$

Clearly f is continuous, since its restriction to each I_j is so.

Step (d): Let us now calculate $Per(f)$.

The range of f is contained in $\cup\{I_j : j$ is in the range of $\phi\}$.

If $1 < i < k$ and $x \in I_i$, then the f-orbit sequence of x is

$$x,\ h_{\phi(i)} f_i h_i^{-1}(x h_{\phi^2(i)} f_i h_i^{-1}(x), \cdots \quad ;$$

the n_i-th term is $h_i f_i h_i^{-1}(x)$; the next term is $h_{\phi(i)} f_i^2 h_i^{-1}(x)$; $\cdots$

The f-period of x is equal to n_i(f_i-period of x).

If y is any other in f-orbit of some element x of $\bigcup^{k} I_i$, then we have period of y equal to period of x. If y is any other element(i.e if y is not in the orbit of any element in $\bigcup_{i=1}^{k} I_i$) then y is not a periodic point. Thus

$$Per(f) = \bigcup_{i=1}^{k} n_i A_i$$

§2

We now describe a functor from the category of all compact subspaces of $\mathbb{R}$ to the category of all zero dimensional compact metric spaces. This is suggested by the methods of §1. We use this functor to derive the main theorem of Chapter 4 from that of Chapter 3.

Let X be any topological space.

Let X be the set of all connected components of X

Let $q : X \to X$ be the natural map. For all x in X, $q(x)$ = the component of x, viewed as an element of X. Provide X with the quotient topology through q. One can prove:

(a) If W is a clopen subset, of X then

$W = q^{-1}q(W)$= union of components in X of elements in W

(b) q takes clopen subsets of X to clopen subsets of X

(c) If X is connected then X is connected
(hence, X is connected $\Leftrightarrow$ X is connected)

(d) If $y \in X$ and if C_y is its connected component in X then the **restriction**

$$q\Big|_{q^{-1}(C_y)} : q^{-1}(C_y) \to C_y$$

is also a quotient map. Indeed it is the "q-map" for the space $q^{-1}(C_y)$.

Proposition 2.1

A is totally disconnected, i.e., the component of each element is singleton.

Proof: Let $y \in$ A. Let C_y be the connected component of y in A. To prove: $C_y = \{y\}$; it is enough to prove that $q^{-1}(C_y)$ is a connected subset of A.

If possible, let $q^{-1}(C_y) = F_1$ U F_2 where F_1 and F_2 are two disjoint closed subsets of X (noting that therefore $q^{-1}(C_y)$ be closed in A). If x € F_1 and C_x is component of x then C_x C F_1 (otherwise, $F_1 \cap C_x$ and $F_2 \cap C_x$ form a separation for C_x ; neither of them is empty, their intersection is empty, and their union is

$$(F_1 \cap C_x) \cup (F_2 \cap C_x) = q^{-1}(C_y) \cap C_x = C_x \qquad \textbf{(because } C_x \textbf{ C } q^{-1}(C_y))$$

Therefore F_1 is the union of components of its dements. $F_1 = q^{-1}(q(F_1))$ similarly, $F_2 =$

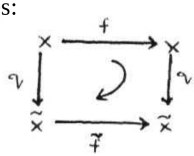
Now $q(F_1)$ and $q(F_2)$ are closed in X and $q(F_1) \cup q(F_2) = C_y$ and $q(F_1) \cap q(F_2) = \phi$. Thus contradicts the fact that C_y is connected. Therefore $q^{-1}(C_y)$ is connected and therefore $C_y = \{y\}$.

Therefore X is totally disconnected.

Definition 2.1 : Let $f: X \to$ A' be continuous map.

Define / : $X \to X$ by $f(C_x) = C_{f(x)}$ for every C_x in A'.

Note that this diagram commutes:



This / **will** be called the self map of A' induced by f.

Theorem 2: Let X be a topological space. Let / be a continuous self map on A' , / be the continuous self map on A', (the space of all components in A) induced by j ,and q is the map defined earlier. Then

$$Per(f) = \bigcup_{n \in Per(\tilde{f})} n.Per(f^n|_{X_n})$$

where X_n = {x G X| /-period of $q(x)$ is n}

Proof: **For** / : A $\to X$, f: $X \to X$, and $q : X \to X$ we have qof = *foq*. Consider $x \in X$.

Let x be a /-periodic point with period m. The f-orbit of x in X is

$$x, f(x), f^2(x), \cdots, f^m(x) = x$$

The /-orbit sequence of $q(x)$ in *x* is $q(x), q(f(x)) q(f^2(x)), \cdots, q(f^m(x)) = q(x)$. This is same as

$$q(x), \tilde{f} o q(x), \tilde{f}^2 o q(x) \cdots \tilde{f}^m o q(x) = q(x)$$

This shows that *q{x)* is periodic point of /.

Let /-period of *q(x)* be *n*. Then we have that *n* divides m (say m = *nk).*

Now $f(X_n)$ C X_n

(Because

$$\begin{aligned} y \in X_n \;&\Rightarrow\; \tilde{f} - \text{period of } q(y) \text{ is } n \\ &\Rightarrow\; \tilde{f} - \text{period of } \tilde{f}(q(y)) \text{ is } n \\ &\Rightarrow\; \tilde{f} - \text{period of } q(f(y)) \text{ is } n \\ &\Rightarrow\; f(y) \in X_n) \end{aligned}$$

Therefore $f^n(X_n)$C X_n.

Let $g_n = f^n|_{X_n} : X_n \to X_n$. Then the g_n-orbit of *x* is

$$x, f^n(x), f^{2n}(x), \cdots, f^{nk}(x) = x$$

Also note that these *nk* elements are distinct.

Therefore g_n-period of *x* is *k* and m = rc&

Therefore $m \in n.Per(g_n)$ therefore *Per(f)* C $\bigcup_{n \in Per(\tilde{f})}$ rc.Per(/U$_n$)Conversely, let m$\in \bigcup_{n \in Per(\tilde{f})} n.Per(f^n|_{X_n})$ then $\exists$ n in *Per(f)* such that m = *nk* for some k is $Per(g_n)$

Therefore $g_n^k(x)$ — *x* for some *x* € X_n

But $g_n^k(x) = (f^n)^k(x) = (f^{nk})(x) = f^m(x) = x$

Thus $m \in Per(f)$.(Because, if / is any positive integer such that $f^l(x) = x$ then n divides 1. This is because we have $f(q(x)) = q(f(x))$ which implies $f^l(q(x)) = q(x)$ and /-period of $q(x)$ is n.

Therefore/ $= nk_1$ for some $k_1 \in \mathbb{N}$. Hence $f^{nk_1}(x) = (f^n)^{k_1}(x) = x$ Therefore k_1 divides k . So m divides / . Hence / > m.)

Hence $Per(f) = \bigcup_{n \in Per(\bar{f})} n.Per\left(f^n\Big|_{X_n}\right)$

Note:

(i) Each X_n is a union of components.

If x_1 and x_2 belong to the same component of ar, and if x_1 G X_n then x_2 also in X_n. This is because,

$$x_1 \in X_n \Rightarrow f\text{— period of } q(x_1) \text{ is } n$$

$$\Rightarrow f\text{— period of } q(x_2) \text{ is also } n \qquad (\text{because } q(x_1) = q(x_2))$$

$$\Rightarrow x_2 \in X_n$$

(ii) If x G X_n then $f^n(C_x)$ C C_x

$x_1 \in C_x \Rightarrow q(x_1) = q(x)$

$f^n(q(x_1)) = f^n(q(x))$

$qof^n(x_1) = qof^n(x) = q(x)$

$\Rightarrow f^n(x_1)$ and $f^n(x)$ are in the same component namely C_x

(iii) $Per\left(f^n\right) = \bigsqcup Per(f^n|_{C_x})$. If $x \in X_n$ then $f^n(q(x)) = g(x)$

so, $f^n(x_1) \in C_x$ hence $f^n(C_x) C \quad C_x$

Proposition 2.2:

Let X be a topological space. / be a continuous selp map on X and / be the induced map on X.

Then $Per(f)$ is of the form, $\bigcup_{n \in Per(f)} n.A_n$ where each A_n € S .

Proof: Let

$$A_n = Per(f^n|_{X_n})$$

$$= \bigcup Per(f^n|_{C_x}) \text{ (by (ii))}$$

But C_x is either a singleton or a closed interval.

hence $Per(f^n|_{C_x}) \in \mathcal{S}$ Since S is closed under unions , M $Per(f^n|_{C_x})$ is also in $\mathcal{S}$

Now by theorem 2, it follows that

Per(f) is of the form (I rc.j4$_n$, where each A_n is in $\mathcal{S}$
$n \in Per(f$

§3

Hereafter we assume that X is a compact subset of ℝ.

The symbols q, X, C_x have the same meanings as in §2.

Proposition 3.1:

X is a zero dimensional metric space.

Proof:

We know that X^c is countable union of open intervals. Choosing one element from each of these open intervals, form a countable set A.

If a, $b \in A, a < b$ then (a, *b)* 0 X is clopen.

((a, *b)* $\cap X$ is open,

[a, 6] $\cap X$ is closed and these are equal)

Hence $q((a,b) \cap X)$ is a clopen subset of $X(by$§2, **6)**

Let = $\{q((a, b) \cap X) : a, b \in A, a < b\}$

Claim: is a base for X.

let $y_o \in X$ and let F be a closed subset in X such that $y_o \notin F$. Then $q^{-1}(y_o)$ is a closed interval and is disjoint from $q^{-1}(F)$,say, .

$$q^{-1}(y_o) = [a_o, b_o] \quad \text{(may be } a_o = b_o)$$

$\exists\, a, b \in A$ such that $a < a_o < b_o < b$ and such that $q^{-1}(F) \cap (a, b)$ is empty.

then

$$q((a,b) \cap X) \; 6 \; B$$

$$F \cap q([a,b] \cap X) = \phi$$

Therefore is a base.

is a clopen base for the space X. X is zerodimensional.

Note: Since is a countable clopen base and since we can prove that it separates points of X it follows that X is metrizable.

Let (5, <) be totally ordered set.

We say $x, y \in S$ are adjacent if there exists no element strictly between *x* and *y*.

Proposition 3.2:

Let (5, <) be countable and complete (every nonempty subset which is bounded above has supremum). Then between any two distinct elements there exists a pair of adjacent elements.

Proof: Let $x < y$ be distinct is *S*

Let $S \cap [x, b]$ be written as a a sequence of

$$x = x_1, x_2 = y, x_3, x_4, \cdots$$

Now define two sequences, with

$s_1 = x$ $\quad$ $t_1 = y$

After defining $s_1, s_2, \cdots, s_n, t_1, t_2, \cdots, t_n$, see whether s_{n+1} and t_n are adjacent. If yes, we are through. Otherwise, take the least *k* such that $s_{n+1} < x_k < t_n$, and let $t_{n+1} = x_k$. proceeding in this way, either we have a pair of adjacent elements between x and y or we obtain two infinite sequences (s_n) and (t_n) in $S \cap [x, y]$ with the following properties:

(s_n) is increasing

(t_n) is decreasing

Each s_m is less than each t_n.

Every element of S is either less than some s_m or greater than some t_n

[To prove the last of these four properties, let z € S . If z is not in [x,y] then the result is obvious because z is either less than s_1 or greater than t_1. If z is in $S \cap [x, y]$, then there is a j such that $z = x_j$. Not every s_n can belong to the finite set $\{x_1, x_2, \cdots, x_{j-1}\}$.

Take least $k > j$ such that $x_k = s_n$ for some n. If k=j, we have $x_j = s_n < s_{n+1}$ and we are through . If $k \neq j$, this implies, by our **definition** of s_n that x_j does not lie between s_{n+1} and t_{n-1}. Therefore either $x_j < s_{n-1}$ or $x_j > t_{n-1}$ and again we are through .]
Now the increasing sequence is bounded above by y. Because of completeness, it should have a supremum, say x_0. This is in *S*. Therefore either x_0 is less than some s_n or greater than some t_n . The former is not possible because x_0 is the supremum of all $s_n' s$. Therefore $x_0 > t_n$ for some n. This implies that some s_m is greater than t_n contradicting the third property. This contradiction proves that between x and y **there should** be a pair of adjacent elements. Let $(S, <)$ be a totally ordered set. Define,

for all $x \in S$ $\quad L_x = \{y \in s \backslash y < x\}$ (called as left ray)

$R_x = \{y \in S \backslash y < x\}$ (called as right ray)

Now, $\{L_x | x \in S\} \cup \{R_x | x \in S\}$ is a subbase for some topology on *S*. This is known as **the order** topology.

It is easily seen that the following are equivalent for y_1, y_2 in *X*

(a) every element of $q^{-1}(y_1) <$ every element of $q^{-1}(y_2)$

(b) some element a of $q^{-1}(y_1) <$ some element b of $q^{-1}(y_2)$

(c) largest element of $q^{-1}(y_1) <$ smallest element of $q^{-1}(y_2)$

Definition: If (a) or (b) holds we say $y_1 < y_2$ in *X*

It is easy to see that *(X, <)* is a totally ordered set.

Proposition 3.3:

Let X be a compact subset of $\mathbb{R}$. Let q and X be as before. Then q-quotient topology agrees with order topology on X.

Proof: Let $y \in X$. Then $q^{-1}(L_y) = \{x \in X | q(x) < y\}$
$= \{x \in X \backslash x <$ the least element of $q^{-1}(y)\}$
= a left ray in X. therefore L_y is open in the quotient topology on X.

Similarly right ray R_y ,is open in quotient topology in X . Hence., order topology on $X \subset q$-quotient topology on X, This order topology on X is T_2. Since X is compact in the quotient topology, quotient topology agrees with order topology.
[If two compact T_2 topologies are comparable, then they are equal].

Lemma 3 :

Let X be a countable compact ordered space. Then there exists a closed subset S of X such that

(i) S is homeomarphic to X and

(ii) every x in S (except possibly supremum and infimum of S) has an adjacent element in S.

Proof: By transfinite induction on the derived length of X

(For a space X, we define

$X^1 =$ the set of all limit points of X .

$X^{\alpha+1} = (X^{\alpha})^1$ for all limit ordinals α

$X^{\alpha} = \bigcap_{\beta < \alpha} X^{\beta}$ forall limit ordinals α

Derived length 6(X) as the infimum of the set $\{\alpha : X^\alpha = \phi\})$

As an initial step of transfinite inducion, take S(X) = 1 Then $X' = \phi$.
Thus X is finite. (since X is compact)

Take S — X. X clearly has property (ii) by Proposition 3.2

As induction hypothesis, assume the theorem to be true whenever S(X) < a. Now to prove the result, when $\delta(X)$= a + 1.

For S(X) = α+ 1 we have $X^\alpha \neq \phi$, $X^{\alpha+1} = \phi$

$X^\alpha$is finite

Let $X^\alpha = \{y_1, y_2, \bullet\bullet\bullet, y_n\}$. Assume $y_1 < y_2 < \cdot \bullet\bullet < y_n$. Take y_1. Find an interval around y_1 disjoint from an interval around y_2. Call it I_1.

Let $I_q^r = \{y \in I_1 | y > y_1\}$ and $I_1^l = \{y \in I_1 | y < y_1\}$ so that / = I_1^l U I_1^r we prove
(i) $\delta(I_1) = \alpha + 1$ **(ii)** $\delta(I_1) = \max(\delta(I_1^n), \delta(I_1^l))$

let J_1=either I_1^r or /{such that $\delta(J_1) = \alpha + 1$
In I_1, y_1 is the only point of X^α.
Take a monotone sequence (x_n) with distinct terms in I_1, converging to y, such that no x_n is in X^α.
In case α is limit ordinal, X^α — $\bigcap_{\beta \leq \alpha} X^\beta$

We can choose a sequence (α_n) such that $x_n \in X^{\alpha_n}$ and (α_n) is strictly increasing *to a*
If *a* = *fi* + 1, then each x_n is chosen in X^β, Choose V_1 any compact neighbourhood axound x_1

such that $\delta(V_1) = \alpha_1 < \alpha$. Then by induction hypothesis *3* a closed subset S_1 in V_1 such that S_1 homeomorphic to V_1 and every element in S_1 has an adjacent element.

Similarly for $x_2, x_3, \cdots$ we get neighbourhoods $V_2, V_3, \cdots$ and subspac.es $S_2, S_3, \cdots$ of them.

Let $T_1 = \{S_1 \cup S_2 \cdot \cup S_n \cdots\} \cup \{y_1\}$.

Then $\delta(T_1) = \alpha + 1\}$and $\delta(T_1 \setminus \{y_1\}) < \alpha$

For each n, define T_n similarly and define $S = T_1 \cup T_2 \cup \cdots \cup T_n$, Clearly, $S \subset X$, S is closed, and $\delta(S) = \alpha + 1$ and satisfies (i) and (ii) Hence the theorem is proved, by transfinite induction, after noting that *6(X)* cannot be a limit ordinal, because A is compact.

Now we state a Proposition whose proof follows from corollary '.\.*2* of Chapter 2.

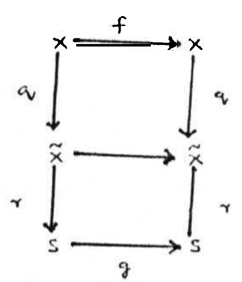
Proposition 3.4:

In every countable compact space every closed subset is a retract.

Let *X* be compact subset of JR. X is as defined as earlier.

This *X* is countable compact ordered space. Then by Theorem '•'> ∃ a closed subset *S* of *X*. with the properties that (i) *S* is homeomorphic to *X* (ii) every element of *S* has an adjacent element. (except possibly the smallest and the largest element)

Let g be a map from *S* to *S*. Let *r* be a retraction from S to A, then q, r, g will give a lift f from X to *X* such that following diagram commutes. Hence we have $q \circ f — g \circ r \circ q$



Proposition 3.5:

Let *X,* $\tilde{X}$, S be as above. Then every continuous selfmap g of S can be "lifted" to a continuous selfmap f of *X.*

Proof: Define ψ : *S*

$$\psi(y) = \begin{cases} \text{left end point of } q^{-1}(y) & \text{if y is a left limit point of } S \\ \text{right end point of } q^{-1}(y) & \text{otherwise} \end{cases}$$

Now define / : $X \to X$ by $f(x) = \psi(g(r(q(x))))$ for each x in$X$. We claim that ψ is continuous.

Let $y \in S$. Let $\psi(y) = x$ • Then x is an end point of C_x. Let (a, b) be an interval in IR containing x. Then $\exists$ csuch that a < c < b and $c \notin X$ Let $W = q((c, x] \cap X)$. This is open in S (because, $(q(c), y^+) \cap S$ is open.)

Let $w \in W$. Then $q^{-1}(w)$ C $(c, x]$ (because, $W = q((c, x] \cap X)$) Therefore $\psi(w)$ G $q^{-1}(W)$which implies $\psi(W)$ C (a, b). Hence ψ is continuous, hence f is continuous as others are continuous.

Theorem 3:

For any compact subset X in IR *PER{X) D PER(X)*

Proof: Let B € $PER(X)$Let S C X be as in Lemma 3. Then $B \in PER(S)$ (because S is homeomorphic to $\tilde{X}$). Therefore there exists a a continuous function $g : S \to S$ such that $Per(g) = B$. Then by above proposition there is a continuous lift $f : X \to X$ such that $q \circ f = g \circ r \circ q$ and such that f is constant on each component of ***X,*** . , . „ . ... ,

Claim: *Per(f)* = $Per(g)$

By Theorem 2, we have *Per(f)* - $\bigcup_{x£P} n_x Per(f_x^n|_{C_x})$ Since f^n is constant on each component of X we have $Per(f^{n_x}|_{C_x})$= {1}, Therefore *Per(f)* = $\{n_x \setminus x \in P\}$

But, for $x \in P$ n_x = /period *oiq{x)* = g- period of$r(q(x))$

$\{n_x|x\in P\}$ = {g- period of all g-periodic points of S = $Per(g)$

Hence Per(f) = Per(g)

Thus $PER(X)$ c *PER(X)*

Corollary 3.6 :

For any *X* C IR $PER(X)$C (X) C *PER(X)* $* \mathcal{S}$

Proof: The first part is nothing but Theorem 3. To prove the second part, let / : $X \to X$ be continuous. Then by Proposition 2 of §2,

Per(f) is of the form $\bigcup_{n\in Per(\tilde{f})}$ rc,4n where each $A_n \in \mathcal{S}$. Thus $Per(f) \in$ *PER(X)* $*$ 5.

Theorem-4:

Let *X* be a compact subset of IR such that every component of *X* is nontrivial. Then

$$PER(X) = \left\{ \bigcup_{n\in B} nA_n \middle| B \in PER(\tilde{X}),\ \text{each} A_n \in \mathcal{S} \right\}$$

Proof: By Theorem 2 for any continuous self map f on *X* we have

$$Per(f) = \{ \bigcup_{n\in Per(\tilde{f})} nA_n \middle| \text{each} A_n \in \mathcal{S}\}$$

Hence

$$PER(X) \subset \left\{ \bigcup_{n\in B} nA_n \middle| B \in PER(\tilde{X})\ \ \text{each} A_n \in \mathcal{S} \right\}$$

We prove the other way as follows:

Step-1: For a given $A \in S$ there is a continuous function / : $I \to I$ such that /(0) = 0 and $f(1) = 1$ and *Per(f)* = *A*

Step-2: Here X is a countable compact ordered space. Therfore by Lemma-3, there exists a closed subset S of X such that

(i) S is homeomorphic to X and

(ii) every x in S (except possibly supremum and infimum of *S)* has an adjacent element in *S.*

Step-3 Given $X, X, q, \mathcal{S}$ in §3. (All components in X are non trivial)

$B \in PER(S)$

$A_n \in \mathcal{S} \quad \forall n \in B$

We want to construct a continuous function / : $X \to X$ such that *Per(f)* = $\bigcup nA_n$..

Take $g : S \to S$ such that *Per(g)* = B. Using Step-1, take for each n in B a continuous function $g_n : I \to$ / such that $Per(g_n) = A_n, g_n(0) = 0$and $g_n(1) = 1$. Then we define a continuous function / : $X \to X$ as follows:

Let $x \in X$. Then f is so defined as to take C_x to $q^{-1}(r^{-1}(g(r(q(x)))$so that / agrees with *g or* on X

For each y in X, let $h_y : I \to C_y$ be the homeomorphism of the form $t \to \lambda t + \mu$, such that $h_y(0) = \psi(r(q(y))$.

Consider two cases.

Case1: If q(x) is a g-periodic point of g-period n, and is the least element (in the order **of *X*) in its g-orbit**, then we define

$$/ = h_{-}^{-1} \circ g_n \circ h_y \text{ on } C_x, \text{ where } y = g(rq(x)).$$

Case2: If casel does not hold, we define

$$f = h_{-}^{-1} \circ id \circ h_y \text{ on } C_x, \text{ where } y = g(rq(x))$$

where *id* denotes the identity function on I .

Thus we have defined f on C_x for each x in *X*. We shall prove in the remaining steps that this f has the required properties .

Step-4 f is continuous.

Let $(x_n) \rightarrow x$ in *X*.

case(i): All $x_n's$ lie in one component.

Since the restriction of f to any component is continuous ,we have $f(x_n) \rightarrow f(x)$. case(ii): Disctinct x_i' s lie in distinct components .

Then x has to be an end point of its component . From the definition of f, we see that f(x) is also an end point of its component. Take an open interval *V* around f(x). Then there exists an open interval *W* C *V* with $f(x) \in W$, such that one end point of W is in $C_{f(x)}$ and other end point of W is not in X.

Now $rq(f(x_n)) = g(rq(x_n))$butg ,q are continuous $g(rq(x_n)) \rightarrow g(rq(x))$.

gq(x) lies entirely in rq(W$\cap X$).Therefore $f(x_n)$ should lie eventually in ***W*. Therefore** $f(x_n) \rightarrow f(x)$.

Any $(x_n) \rightarrow x$ contains a subsequence of type either in case(i) or in case(ii).

Therfore $f(x_n) \rightarrow f(x)$ whenever $(x_n) \rightarrow x$.

Thus f is continuous.

Step 5: We claim that *Per(f)* = (J nA_n

First we note that $f^n|_{C_x} = h_x^{-1}$ *o* g_n *o* h_x. It follows that $\text{Per}(f^n|_{C_x}) = Per(g_n) = A_n$

Therefore $Per(f^n|X_n)$(which is (J $Per(f^n|_{C_x}))$ is also equal to A_n.
$x \in x_n$

Now from Theorem 2, We obtain $Per(f) = \bigcup_{n \in Per(f)} nA_n$.

But $Per(f)$= *Per(g o r)* = *Per(g)* = *b*

Thus Per(f)= **(J** nA_n.

§5

Theorem 5A:

Let X be a compact subset of $\mathbb{R}$ such that (i) X has infinitely many nontrivial components and (ii) there is only one component of X that is not open. Then

$$PER(X) = \mathcal{G}_1 * S$$

where $\mathcal{G}_1$ = {*A C* 1N|1 € *A* or *A* is finite}

Remark:

There are many examples of X satisfying (i) and (ii).

Some of them are:

(a) $X_1 = \{0\} \cup \bigcup_{n=1}^{\infty} \left[\frac{1}{2n}, \frac{1}{2n-1}\right]$

(b) $X_2 = X_1 \cup \{2, 3, \cdots, n\}$

(c) $X_3 = X_1 \cup \left\{\frac{4n-3}{2(2n-2)(2n-1)} : n \in \mathbb{N}\right\}$

(d) $X_4 = (X_1 + 1) \cup (-X_1) \cup [0, 1]$

(e) $X_5 = (X_3 + 1) \cup (-X_3) \cup [0, 1]$

These are mutually nonhomeomorphic.

Proof of theorem 5A:

We note that X has exactly one limit point.

$PER(X) = \mathcal{G}_1$ (by Theorem 2 of Chapter 3)

Now by corollary to Theorem 2 in §2 , we have $PER(X)$ C $PER(X)^{*}$ S

$PER(X) \subset \mathcal{G}_1 * \mathcal{S}$. Conversely, to prove $PER(X)$ C $\mathcal{G}_1 * \mathcal{S}$.

Let B in $\mathcal{G}_1$ be given and let for each n *in* B a member A_n of S be given.

We have to find a continuous function / : $X \to X$ such that $Pcr(f) = \text{M}_{n \in B}\, nA_n$.

Write $X = X_1$ U X_2, where $X\backslash$ — $\{x \in X \backslash x <$ some element of $C\}$ $X_2 =$ $\{x \in X\backslash x >$ some element of $C\}$ where C is the non open component.

Atleast one of these two has infinitely many non-trivial components. Without loss of generality let it be X_1. Define

$$X_o = C \cup \text{ union of nontrivial components of } X_1$$

then $q(X_o)$ is a compact space with unique limit point.

Choose / : $q(X_o) \to q(X_o)$ continuous such that $Per(f) = B$.

Define as in §4, / : $X_o \to X_o$ using this / and the f_n's from / to / for which $Per(f_n)= A_n$. Then / is **continuous** (by theorem 4).

Define

$$f^* : X \to X \text{by}$$

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in X_o \\ \text{left end point of } C & \text{if } x \in X_1 \backslash X_o \\ \text{right end point of } C & \text{if } x \in X_2 \backslash X_o \end{cases}$$

then range of /* = range of /

$$\text{period set of/*} = \text{period set of/}$$

$$= \bigcup nA_n \text{ (as in §4)}$$

Theorem 5B:

Let X have n nontrivial components and only one non open component. Then

$$PER(X) = (\mathcal{F}_n * \mathcal{S}) \text{ V } \mathcal{G}_1$$

Proof: Let / : $X \to X$ be continuous, then by theorem 2,

$$Per(f) = \bigcup_{n \in Per(f)} n.A_n \text{ where } A_n = Per(f^n|_{X_n})$$

$$\text{where} X_n = \{x \text{ € } X \backslash \text{/-period of } q(x) \text{ is n}\}$$

We have to prove

$$Per(f) \text{€ } (\mathcal{F}_n * S) \text{ V ft}$$

let $A = \{x \in X | f^m(x) \in \text{nontrivial component } \forall m = 0, 1, 2, \cdots\}$ •

$Per(f|_A)$ 6 $\mathcal{F}_n * \mathcal{S}$,except when A is empty.

Next let $B = \{x \in X | f^m(x \notin A^c \quad \forall m = 0, 1, 2 \cdots\}$

B C X. In case the limit component is not invariant under f, then the range of / is contained in the union of finitely many components of X. Then

$Per(f|_B)$ a finite set, hence either belongs to $\mathcal{G}_1$ or is empty

Also, Per(f) can be proved to be equal to $Per(f|_A)$U $Per(f|_B)$. Therefore Per(f) € $(\mathcal{F}_n *$ $S)$ V $\mathcal{G}_1$ In the other case, the limit component is invariant then $1 \in Per(f)$.

$$Per(f) \in \mathcal{G}_1$$

$$PER(X) \subset (\mathcal{F}_n * \mathcal{S}) \vee \mathcal{G}_1$$

§6

In this section we use the following Proposition which is easy to prove.

Proposition-6: Let X be a compact subset of $\mathbb{R}$. Let X_1 and X_2 be clopen subsets of X such that $X = X_1 \cup X_2$ then $PER(X_1) \vee PER(X_2) \subset PER(X)$

Theorem 6(a):

Let $m, n, p \in \mathbb{N}$. Let X be a compact subset of IR with n non-open trivial components, m nontrivial open components and p non open non trivial components. Then $PER(X)$ $= \bigcup(\mathcal{G}_r \vee \mathcal{G}_s \vee (\mathcal{F}_s * \mathcal{S}) \vee (\mathcal{F}_t * \mathcal{S}))$

where the union is taken over all triples (r,s,t) of non-negative integers satisfying the inequalities $s < p$, $r + s < n + p$, $s + t < m + p$, and $r + s + t < m + n + p$

Proof: Let / : $X \to X$ be a continuous function. To show $Per(f) = A \cup B \cup C \cup D$ where A $\in \mathcal{G}_r \cup \{\emptyset\}$, B€ $\mathcal{G}_s \cup$ {0}, C € $(\mathcal{F}_s * 5) \cup \{\emptyset\}$, $D \in (\mathcal{F}_t * 5)$ U {0}

We have $n + p$ non-open components all together, we claim that in X 3 mutually disjoint clopen sets $\{V_i\}_{i=o}^{n+p}$ such that each non-open component is contained inside only one of the V_i's $1 < i < n + p$ which form a base for X.

Let X be the quotient space of X. Let $\tilde{x}_1, x_2, \cdots, \tilde{x}_p$ be the q images of the non-open non trivial components. and $\tilde{x_{p+1}}, \tilde{x_{p+2}}, \cdots, \tilde{x_{p+n}}$ be the q images of the trivial components. See the map $f : X \to X$. Suppose there is a subset, without loss of generality, say $\{ \tilde{x}_1, \tilde{x}_2, \cdots \tilde{x}_s\}$ which is invariant under f, where $0 < s < p$. When $s = 0$, this set is empty. Then by a previous theorem $\exists$ mutually disjoint $W_1, W_2 \cdots, W_s$ open in X such that $V_i \supset W_i$ and $\tilde{W}_i^c$ is finite in V_i and $x_i \in W_i$, W_i clopen in V.

Let $H = \bigcup^s \tilde{V}_i \cup \{W_i \mid 1 < i < s\} \cup V_o$

H is finite. Let G be the set of all periods of f-periods of f-periodic points in H. (Note: $H = \bigcup \tilde{V}_i \setminus \bigcup \tilde{W}_i$

Let F be the set of all periods of f-periodic points in $\{\tilde{x}_1 x_2, \cdots, \tilde{x}_s\}$. If $m \in Per(f)$ $\exists$ x in $\bigcup^s V_i \setminus \bigcup^s q^{-1}(\tilde{x}_i)$ such that $f^m(x) = x$

Further if $x \notin H$ then m is a multiple of some member in F. If $x \in B$ then $m \in G$

Thus Per(f) contains a subset A of IN such that $A \in \mathcal{G}_s$ (when $s = 0$, A is empty),where A is the set of all f-periods of f-periodic points in $\bigcup^s (W_i \setminus q^{-1}\ (\tilde{x}_i)) \cup G \cup F$

Let $\{ x_{i_1}, x_{i_2}, \cdots \tilde{x}_{i_r}\}$ be invarint under f, $i_j \in \{ s+1, s+2, \cdots, p, p+1, \cdots, n+p\}, 0 < j < r$.

Note that there is atleast one $q^{-1}(\tilde{x}_{i_j})$ which is a trivial non-open component in X. Again by the same previous arguement $\exists W_{i_1}, W_{i_2}, \cdots, W_{i_r}$ mutually disjoint sets, each clopen in V and $(W_{i_j})^c$ is finite in $(V_{i_j})^c$ and $x_{i_j} \in W_{i_j}$.

Let $Z = \bigcup^r V_{i_j} \setminus \bigcup^r W_{i_j}$ This Z is finite.

$$Z = \bigcup_{j=1}^{r} V_{i_j} \setminus \bigcup_{j=1}^{r} W_{i_j} Z \text{ is also finite.}$$

Let G' be the set of all periods of f-periodic points in Z. and F' be the set of all periods of f-periodic points in $\{x_{i_1}, x_{i_2}, \cdots, x_{i_r}\}$.If $m \in Per(f)$,then there is $x \in \bigcup^{0} V_{i_j}$ and further,if $x \notin Z$ then m is a multiple of some number in F' . Thus $Per(f)$contain a subset B of $\mathbb{N}$ such that $B \in \mathcal{G}_r$ (when $r = 0$, B is empty) where B is the set of all periods of f-periodic points in $\bigcup^{r}(W_{i_j} \setminus q^{-1}(\tilde{x_{i_j}}) \cup G' \cup F'$

Since $\{\tilde{x_1}, \tilde{x_2}, , \tilde{x_s}\}$is f-invariant , let $g = f|_{\{q^{-1}(\tilde{x_1}), q^{-1}(\tilde{x_2}), \cdots, q^{-1}(\tilde{x_s})\}}$. Notice that if $m \in Per(f)$ then $m \in Pcr(g)$. Now applying Theorem 1 of Chapter 3, we have : if $C \subset \mathbb{N}$and $C \subset Per(f)$ (when $s = 0$, C is empty) where C is the set of periods of all periodic points in $\bigcup^{s} q^{-1}(\hat{x}_{i})$ then $C \in \mathcal{F}_s * S$

There can be t non trivial components that are f-invarint where, $s + t < m + p$, and $r + s + t < m + n + p$

If $D \subset \mathbb{N}$and $D \subset Per(f)$ (when $t = 0$, D is empty) where D is the set of periods of all f-periodic points in the t nontrivial components, then $D \in \mathcal{F}_t * S$ by Theorem 1 of Chapter 3.

Conversely, let $A \cup B \cup C \cup D \in \bigcup(\mathcal{G}_r \vee \mathcal{G}_s \vee (\mathcal{F}_s * 5) \vee (\mathcal{F}_t * \mathcal{S}))$

where the union is taken over all triples (r,s,t) of non-negative integers satisfying the inequalities $s \le p$, $r + s \le n + p$, $s + t < m + p$, and $r + s + t < m + n + p$ where $A \in \mathcal{G}_r$, $B \in \mathcal{G}_s$, $C \in \mathcal{F}_s * \mathcal{S}, D \in \mathcal{F}_t * \mathcal{S}$

Assume A £ $\mathcal{G}_r, A \notin \mathcal{G}_k$ for $k < r$, and $B \notin \mathcal{G}_k$ for fc $< s$. Let C = $\{aS \setminus a \in A \in \mathcal{F}_s, S \in 5\}$ and D = $\{bS \mid 6 \in B \in \mathcal{F}_t, S \in \mathcal{S}\}$

Let $y_1, y_2, \cdots y_m$ be the open components of X and $\tilde{x}_1, \tilde{x}_2, \cdots x_{p+n}$be the nonopen components of X, where as $q^{-1}(\tilde{x}_1), q^{-1}(\tilde{x}_2), \cdots q^{-1}(\tilde{x}_p)$ are nontrivial components of X, and $q^{-1}(\tilde{x}_{p+1}), q^{-1}(\tilde{x}_{p+2}), \cdots q^{-1}(\tilde{x}_{n+p})$ are trivial components of X.

Let $\{W_i\}_{i=1}^{n}$ be disjoint clopen cover of the X such that $_i \in W_i$ and $_{1,2}, \cdots _m \in W_1$.

Let $V_1 = q^{-1}(\bigcup_{i=1} \tilde{W}_i)$, $V_2 = q^{-1}(\bigcup_{i=t} \tilde{W}_i)$, $V_3 = q^{-1}(\bigcup_{i=t+s+1} W_i)$, $V_4 = q^{-1}(\bigcup_{i=t+r+2s+1} W_i)$

Clearly $X = V_1 \cup V_2 \cup V_3 \cup V_4$ and

V_1 contains a retract homeomorphic to / x $\{1, 2, \cdots t\}$.

V_2 contains a retract homeomorphic to / x $\{1, 2, \cdots s\}$.

V_3 contains a retract homeomorphic to a compact zero dimensional space having s limit points

V_4 contains a retract homeomorphic to a compact zero dimensional space having r limit points

Now we can define continuous selp maps g_1, g_2, g_3, g_4 on V_1, V_2, V_3, V_4 respectively such that $Per(g_1) = D, Per(g_2) = C, Per(g_3) = B, Per(g_4) = A$.

Now define / : $X \to X$ such that $f|_{V_i} = g_i$ for i — 1,2,3 and 4 Then f is continuous and $Per(f) = A \cup B \cup C \cup D$.

(Note: If some of A, B, C and D are empty then by making suitable modifications in the above proof we can construct function / with required conditions.)

It is also clear that

$PER(V_1) \supseteq \mathcal{F}_t * \mathcal{S}$

$PER(V_2) \supseteq \mathcal{F}_s * \mathcal{S}$

$PER(V_3) \supset \mathcal{G}_s$

$PER(V_4) \supseteq \mathcal{G}_r$

Therefore by Proposition-6 , $PER(X) \supset PER(V_1) \vee PER(V_2) \vee PER(V_3) \vee PER(V_4)$

Hence the converse.

Theorem 6(b):

Let X be a compact subset of $\mathbb{R}$,such that $|X|' = n$. Let all the open components of X

be **nontrivial** . Then *PER(X)* = $\mathcal{G}_n * \mathcal{S}$.

(Note: Some non-open components may be trivial, some others not)

Proof: We know from Corollary 3.6 *PER(X)* C $\mathcal{G}_n$ * *S* as $\mathrm{PER}(X) = \mathcal{G}_n$.

To prove the reverse inequality, let *B* £ $\mathcal{G}_n$ and let for each m in B , A_m£ *S*.

Consider $A = \bigcup_{m \in B} mA_m$. Then by corollary 3 in Chapter 3, there is a continuous

$g : X \to X$ such that $(i) Per(g) = B$ (ii) the set *Y* of isolated points of *X* is g-invariant and (iii) $Per(g|_Y) = B$

With these data (viz. $B, A'_m s, g$), we can construct a / : $X \to X$ as in the proof of step 3 of Theorem 4. The proofs of steps 4 and 5 of that theorem yield that f is continuous and $Per(f)$— *A*,(using that for every m in B, we have a nontrivial component in X, whose q-image in X is having m as g-period).

Thus *PER(X)* $= \mathcal{G}_n * \mathcal{S}$.

Theorem 6(c):

Let *X* be a compact subset of $\mathbb{R}$ with *n* non-open trivial components and infinitely many open nontrivial components and *p* nontrivial non open components, and *r* nontrivial nonopen components in *X* such that every open set containing it intersects infinitely many nontrivial components. Then *PER(X)* = $\bigcup_{0 < s < r} ((\mathcal{G}_s * \$) \ V \ \mathcal{G}_{n+p-s})$

Proof: Let / : $X \to X$ be continuous function. Without loss of generality, let $Z = \{\tilde{x}_1, \tilde{x}_2, \cdots \tilde{x}_r\}$ be all non-open components of *X* of the $\{\tilde{x}_1, \tilde{x}_2, \bullet \bullet \bullet, \tilde{x}_{p+n}\}$ total non open components such that every open set containing $q^{-1}(_i)$ intersects infinitely many open trivial components of *X*.

Again with out loss of generality let $\{\tilde{x}_1, \tilde{x}_2, \bullet \bullet \bullet, \tilde{x}_s\}$ be the maximal / invariant subset of Z . Let $q^{-1}(_1), q^{-1}(_2), \cdot \cdot q^{-1}(_r)$ be disjoint clopen subsets of *X* such that $_i \in_i$.

Denote $\bigcup_{i=1}^{8} q^{-1}(W_i) = Y_1$.

Then by Theorem 6(b) $Per(f|_{Y_1}) \in \mathcal{G}_s * S$.

Similarly denote $\bigcup^{n} q^{-1}(W_i) = Y_2$ Then $Per(f|_{Y_2}) \in (\mathcal{F}_m * S) \vee \mathcal{G}_a$

where $m + a + s = n + p, 0 \leq m \leq p, 0 < a < n$

But $\mathcal{F}_m * \mathcal{S} \subset \mathcal{G}_s * S$.

Hence $PER(X) \subset \bigcup ((\mathcal{G}_s * \mathcal{S}) \vee \mathcal{G}_{n+p-s})$

Conversely,

For given $0 < s < r$ define X_1 as a clopen set containing exactly s nonopen components , such that every open set containing it intersects infinitely many nontrivial components. Take X_2 as complement of X_1 in X.

Then one can notice that $PER(X_1) = \mathcal{G}_s * S$ (by Theorem 6(b))

and $PER(X_2) \supset \mathcal{G}_{n+p-s}$

Hence $(\mathcal{G}_s * S) \vee \mathcal{G}_{n+p-s} \subset PER(X_1) \vee PER(X_2) \subset PER(X)$

(since $X = X_1 \cup X_2$ and by Proposition-6) This is true for s in $0 < s < r$, therefore

$$\bigcup_{0<s<r} ((\mathcal{G}_s * \mathcal{S}) \vee \mathcal{G}_{n+p-s}) \subset PER(X)$$

Theorem 6(d):

For any compact subset $X \subset \mathbb{R}$ such that $|X''| = n$, then $PER(X) = \mathcal{H}_n$

Proof: For this space X , we know PER $(X) = \mathcal{H}_n$. and $\mathcal{H}_n * S = \mathcal{H}_n$. Hence by Corollary 3.6, $PER(X) = \mathcal{H}_n$.

We close this chapter by stating one more result whose proof follows from the theorems of Chapter-2 :

Theorem 6(e):

Let X be compact subset of IR . Then

$$PER(X) = \begin{cases} \wp(\mathbb{N}) \setminus \{0\} \text{if boundary of X is countable and } |X''| = \infty \\ \wp(\mathbb{N}) \text{ if boundary of X is uncountable} \end{cases}$$

Chapter 5

Sets Of Periods Of Convex Subsets Of $\mathbb{R}^n$

§0

Introduction and Preliminaries

According to [31], there is a countable family S of subsets of IN such that for every continuous self map f of I (the closed interval) , $Per(f) \in S$. Conversely it has been proved [12] that every member of S is of the form Per(f) for some continuous f from I to I. Combining these results, we write $PER(I) = S$. It is also known that $PER(\mathbb{R}) = \mathcal{S} \cup \{\emptyset\}$. It is not difficult to deduce a stronger version of Sarkovskii's theorem which is as follows : If X is a nonempty convex subset of $\mathbb{R}$ then $PER(X)$ has to be one of the three families $\{1\}$, $\mathcal{S}$ or $\mathcal{S} \cup \{\emptyset\}$.

In this chapter we generalise the above result to higher dimensions. We prove that if X is a nonempty convex subset of $\mathbb{R}^n$, then $PER(X)$ should be one of the five families: $\{1\}$, $\mathcal{S}$, $\mathcal{S} \cup \{\emptyset\}$, $\mathcal{U}_1$, or $\wp(\mathbb{N})$.

Summary: First, we calculate PER ($\mathbb{R}^2$) (Theorem 1). Then we use it to prove **that** $PER(X) = \wp(\mathbb{N})$ for all noncompact convex subsets X of $\mathbb{R}^2$ (Theorem 2). This is **done** by proving that every noncompact convex subset of $\mathbb{R}^2$ with nonempty interior contains a retract which is homeomorphic to the quadrant C_1. Next we prove that $PER(D) = \mathcal{U}_1$,where D is the closed unit disc (Theorem 3). We use all these togther with Sarkovskii's Theorem to prove the main theorem of this chapter (Theorem 5):

Theorem : Let X be a nonempty convex subset of $\mathbb{R}^n$. Then

$$PER(X) = \begin{cases} \wp(\mathbb{N}) & \text{if } X \text{ is noncompact} \\ \mathcal{U}_1 & \text{if } X \text{ is compact and not contained in a line} \\ \mathcal{S} & \text{if } X \text{ is infinite, compact and contained in a line} \\ \{1\} & \text{if } X \text{ is singleton} \\ \mathcal{S} \cup \{\emptyset\} & \text{otherwise} \end{cases}$$

§1

Period Sets For The Plane

Now we take up the question: Which subsets of $\mathbb{N}$ arise as $Per(f)$ for some continuous self-map / of the plane $\mathbb{R}^2$? Unlike the case of $\mathbb{R}$, here it turns out that there are no such conditions of the type: "If 2 is an element, then 1 is also an element"; indeed there are no conditions at all; we prove that every subset of $\mathbb{N}$ is of the form $Per(f)$. More concisely, $PER(\mathbb{R}^2) = \wp(\mathbb{N})$.

Theorem 1: Let A be any given subset of IN. Then there is a continuous / : $\mathbb{R}^2 \to \mathbb{R}^2$ such that $Per(f) = A$.

Proof: If A is empty, we take / to be a 'translation' by a non-zero vector; for instance,

$$f(x,y) = (x+1,y) \text{ for all } x,y \text{ in } \mathbb{R}^2$$

It is easily seen that / admits no periodic point and therefore $Per(f)$ is empty.

If A is a nonempty subset of $\mathbb{N}$, arrange its elements in the increasing **order** $n_1 < n_2 < \cdots$

Step 1: For each positive integer n we first construct an unbounded convex strip X_n in $\mathbb{R}^2$ and a continuous self-map of X_n, whose period set is $\{n\}$. We let

$$X_n = \{(x, y) \in \mathbb{R}^2 \quad | \, 1 \leq x \leq n\}$$

We define

$$f(x,y) = \begin{cases} (x+1, y+t-t^2) & \text{if } 1 < x < n - 1 \\ (n^2 - n + 1 + x - nx, y + t - t^2) & \text{if } n - 1 < x < n \end{cases}$$

where $t = x - [x]$ = fractional part of x.

To see that / is a self-map of X_n we observe:

(i) $1 < x + 1 < n$ holds whenever $1 < x < n - 1$

(ii) $1 < n^2 - n + 1 + x - nx < n$ holds whenever $n - 1 < x < n$.

Of these (i) is obvious. (ii) is easy on noting that

$$n^2 - n + 1 + x - nx = (n - x)(n - 1) + 1$$

$$\text{and the same is also} = n - (n - l)(x - (n - 1)$$

To see that / is continuous, we have only to note that the two pieces of definition agree on the 'boundary line' $x = n - 1$.

Now we calculate the periods of the periodic points of /. We claim that

(A) Every point of the form (m, y) where m is an integer between 1 and n, is a periodic point of period exactly n.

(B) No other point is a periodic point.

To prove (A), we have only to note that

$$f^k(m,y) = \begin{cases} (m+k, y) & \text{if } m + k < n \\ (n - m - k, y) & \text{if } m + k > n \end{cases}$$

and therefore $f^n(m,y) \neq (m,y)$ for all $1 < m < n$ and n is the least such **positive** integer. To prove (B), we first prove that

(C) for each (x,y) in X_n where x is not a positive integer, the second coordinate of $f(x,y)$ is strictly greater than y.

To prove (C), we note that the y-coordinate of $f(x,y)$ is $y + t - t^2$ where $t = x - [x]$. When x is not an integer, i is strictly between 0 and 1; so $t - t^2$ is positive; so the y-coordinate of $f(x,y)$ is greater than y .

Now to prove (B) from (C) we consider two cases.

In the first case, the first coordinate of $f^k(x,y)$ is an integer for some k in IN. Then for **all** higher values of k, the first coordinate of $f^k(x,y)$ remains an integer (as can be seen from the definition of /) and therefore (x,y) cannot be a periodic point, since x is not an integer. In the second case, the abscissa of $f^k(x,y)$ is an integer for no value of k in IN. Therefore by (C) above, the ordinates of the points $y, f(x,y), f^2(x,y), \cdots$ form a strictly increasing sequence of numbers and therefore it is not possible for $f^k(x,y)$ to be same as (x,y) for any k in IN.

Now that we have proved both (A) and (B), we conclude that $Per(f) = \{n\}$.

[**Remark:** When $n = 1$, the strip X_n reduces to a line

$$X_n = \{(x,y) | x = 1\}$$

and the function / reduces to the identity function on X_n.]

Step 2: As in step 1, let n be a fixed positive integer. In this step, we construct a bigger strip Y_n strictly **containing X_n**, and a **continuous extension / of / (as a self-map of Y_n)**

such that

(i) The property that the period-set is $\{n\}$ is not lost while extending, and

(ii) a new property is gained, namely that each of the two boundary lines of Y_n is invariant under /; indeed $f(x,y) = (x, y+1)$ holds for all points on these lines.

For this purpose, we proceed as follows:

Let Y_n — $\{(x,y)|0< x < n+1\}$. Thus Y_n is a strip whose width is greater than that of X_n, by one unit on each side. Next we define, if $n > 2$,

$$\tilde{f}(x,y) = \begin{cases} (2x, y-x+1) & \text{if } 0 < x < 1 \\ (nx-n^2+1, x+y-n) & \text{if } n < x < n+1 \\ f(x,y) & \text{if } 1 < x < n \end{cases}$$

when $n = 1$, we define

$$f(x,y) = \begin{cases} f(x,y) & \text{if } x = 1 \\ (x, y-x+1) & \text{if } 0 \leq x < 1 \\ (x, y+x-1) & \text{if } 1 < x \leq 2 \end{cases}$$

We first note that $f(x,y)$always belongs to Y_n. [the numbers $2x$ and $nx — n^2 + 1$ are between 0 and $n + 1$, respectively when $0 < x < 1$ and $n < x < n+1$]

Next, we note that when x — l,the point $(2x, y — x + 1)$ is same as the point $f(x,y)$= $(x+1, y)$. Similarly, when x = n, the point $(nx — rc^2 + 1, x + y — n)$ is same as the point $f(x,y) = (n^2 — n + 1 + x — nx, y + s — ,s^2)$ with $s = x — n + 1$, because both simplify to $(1, y)$.

These observations prove that / is continuous.

Next, we note that if $(x,y) \in Y_n \backslash X_n$ (that is if $0 < x < 1$ or $n — 1 < x < n$), then the y-coordinate of $f(x,y)$is strictly greater than y. Therefore, similar to the argument in step 1 in two cases, we argue that none of these points can be periodic points of f.

Step 3: Now let $n_1 < n_2 < \cdots$ be the elements of the given set A in the increasing order. A may be finite or infinite. Let for each k in $\mathbb{N}$, the strip Z_k be $\{(x, y) \in \mathbb{R}^2 | n_1 + n_2 + \cdots + n_{k-1} + k - 1 < x \leq n_1 + \cdots + n_k + k\}$, if $k > 1$; and $Z_1 = \{ (x, y) \in \mathbb{R}^2 \mid 0 < x < n_1 + 1 \}$. Define $g : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$g(x,y) = \begin{cases} (x, y+1) & \text{if } x < 0 \\ \tilde{f}_{n_1}(x,y) & \text{if } (x,y) \in Z_1 \\ \vdots & \\ \tilde{f}_{n_k}(x - n_1 - n_2 - \cdots - n_{k-1} - k + 1, y) + (n_1 + \cdots + n_{k-1} + k, 0) & \text{if } (x,y) \in Z_k \\ \vdots & \\ (x, y+1) \text{ if } x > \text{ every element of } A & \text{(if } A \text{ is finite)} \end{cases}$$

To prove that g is continuous, we note:

(i) On the line $x = 0$, the two pieces in the definition of $g(x,y)$ namely $(x, y+1)$ and $f_{n_1}(x,y)$ coincide (see step 2).

(ii) On the common boundary of Z_1 and Z_2, that is, on the line $x = n_1 + 1$, the two pieces in the definition of $g(x, y)$, namely $f_{n_1}(x,y)$ and $f_{n_2}(x - n_1 - 1, y) + (n_1 + 1, 0)$ coincide because both are equal to $(n_1 + 1, y + 1)$

And so on. In fact, on each of the boundary lines of each of the Z_k's, $g(x,y)$ is nothing but $(x, y+1)$. Therefore the several pieces in the definition of $g(x,y)$ agree on all the common portions and therefore g is continuous.

Next, we calculate the periods of the periodic points of g. For this purpose, we note the following:

(1) Each Z_k is invariant under g. That is $g(Z_k) \subset Z_k$.

(2) If ϕ_k - $Y_{n_k} \rightarrow Z_k$ is the translation map

$$\phi_k(x,y) = (x + n_1 + n_2 + \cdots + n_{k-1} + k - 1, 2/)$$

then $\phi_k . \tilde{f}_{n_k} . \phi_k^{-1} = g|_{Z_k}$.

It follows from (2) that

$$Per(g|_{Z_k}) = Per(f_{n_k}) = \{n_k\}.$$

Now because of (1),

$$\textbf{Per(g)} = \bigcup_k Per(g|_{Z_k}) = \bigcup_{n_k \in A} \{n_k\} = \textbf{A}.$$

This proves the theorem, g being the function / required in the statement.

Remark 1: The result can be extended to higher dimensions without difficulty, as follows:

Let n > 2. Let A C IN. Then there is a continuous / : $\mathbb{R}^n \rightarrow \mathbb{R}^n$ such that *Per(f)* = *A*. For the proof, we either define

$$f(x_1, x_2, \cdots, x_n) = (g(x_1, x_2), x_3, \cdots, x_n) \text{ for all } (x_1, x_2, \cdots, x_n) \text{ in } \mathbb{R}^n$$

where $g(x_1, x_2)$ is defined as in the proof of the above theorem, or use the following result:

Proposition-1 : Let X be a topological space, Y be a subset of X and let <j> : $X \rightarrow Y$ be a continuous map such that $\phi(y) = y$ for all y in Y. (i.e. ϕ is a retraction). Then the family *PER(X)* contains the family *PER(Y).* In particular, if *PER(Y)* = $\wp(\mathbb{N})$, **then** *PER(X)* = $\wp(\mathbb{N})$.

The same proposition can be used to prove that every normed linear space X of dimension > 2 has the property $PER(X) = \wp(\mathbb{N})$.

Remark 2: In order to facilitate a better understanding of the function g constructed in Theorem 1, we provide the geometric ideas behind its definition.

First, the elements $n_1 < n_2 < \cdots$ in A are used to consider the partition of $\mathbb{N}$ consisting of {first n_1 elements of $\mathbb{N}$}, {next one element of $\mathbb{N}$}, {next n_2 elements of $\mathbb{N}$}, {next one element of $\mathbb{N}$}, {next n_3 elements of $\mathbb{N}$}, {next one element of $\mathbb{N}$}, and so on. We take a function $\phi : \mathbb{N} \longrightarrow \mathbb{N}$ that is a permutation, keeping each of these 'blocks' (partition classes) invariant, and acting as a cyclic permutation on each block. In other words

$$\begin{aligned}
\phi(1) &= 2\,;\ \phi(2) = 3\,;\cdots;\ \phi(n_1 - 1) = n_1\,;\ \phi(n_1) = 1;\\
\phi(n_1 + 1) &= n_1 + 1;\\
\phi(n_1 + 2) &= n_1 + 3\,;\ \phi(n_1 + 3) = n_1 + 4\,;\cdots;\ \phi(n_1 + n_2) = n_1 + n_2 + 1,;\\
\phi(n_1 + n_2 + 1) &= n_1 + 2\,;\ \phi(n_1 + n_2 + 2) = n_1 + n_2 + 3 \text{ and so on}
\end{aligned}$$

We take $\phi(n) = n$ for all integers not covered by the above.

As one can guess, this is meant for defining g at integer points on the x-axis by the rule

$$g(n,0) = \begin{cases} (\phi(n), 0) & \text{if } n \text{ belongs to the blocks of } n_1, n_2 \text{ etc..} \\ (\phi(n), 1) & \text{otherwise} \end{cases}$$

We observe that thus we have provided periodic points of periods $n_1, n_2, \cdots$ etc., for g.

In the second stage, we define the first component g_1 of g. To make matters simpler, we extend the function $\phi : \mathbb{Z} \to \mathbb{Z}$ to a function $\tilde{\phi} : \mathbb{R} \to \mathbb{R}$ by mere linear interpolation. This means: For each $0 < t < n$ and for each $n \in \mathbb{Z}$, $\tilde{\phi}(n + t) = \phi(n) + t[\phi(n + 1) - \phi(n)]$. We also make our job easy by stipulating to ourselves that every line parallel to y-axis, will be mapped by f (bijectively) to some line of the same kind. This essentially means that the abscissa (x-coordinate) of $g(x, y)$ is determined by x (without any role for y). In other words $g(x, y) = (\phi(x), \psi(x, y))$, for some functions ϕ and ψ of one and two variables respectively. The function ϕ here, is same as what we defined above.

In the third stage, we fix our attention on the second component of g. That is, we want to describe the ordinate of $g(x, y)$. The simplest method is to take it as a function of only x or of only y. But unfortunately, this kind of g admits extraneous periodic points whose periods may not be in the given set A. Indeed, if $g(x, y)$ is independent of y, then g behaves like a function from $\mathbb{R}$ to $\mathbb{R}$, and therefore its set of periods should obey Sarkovski's theorem. Also, if $g(x, y)$ is of the form $(\phi(x), \psi(y))$, the situation is no better. [eg. if (x_o, y_o) is a point of period 3 for g, then $\phi^3(x_o) = x_o, \psi^3(y_o) = y_o$, and one of them fails to be a fixed point, say $\phi(x_o) \neq x_o$. Then by Sarkovski's theorem, there is a point x_1 of period 5 for ϕ. Then the point (x_1, y_o) is a periodic point of g with period 5 or 15 (depending on whether y_o is a fixed point of ψ or not), but it is possible that neither 5 nor 15 is in A]. So far we have explained why we are compelled to complicate the definition of the second component of g. We have the following constraints on $\psi(x, y)$:

(a) It has to depend on both x and y genuinely,

(b) Whenever x is an integer and y — 0, we have already specified $\psi(x, y)$. It is either 1 or 0.

(c) In order to avoid extra periodic points, we now willingly impose the condition that $\psi(x, y) > y$ at all other points

(d) To make matters simple, we take $\psi(x, y)$ to be of the form $y + p(x)$ where $p(x) > 0 \ \forall x$.

This leads us to the search of a continuous function h from $[0,1]$ to itself with the properties $h(0) = 0 = h(1)$ and $h(t) > 0$ for all $0 < t < 1$. We have chosen the function $h(t) = t - t^2$.

[If some one else thinks of such other functions as $h(t) = 1 - |\frac{1}{2} - t|$, there is no objection. The same purpose will be served by any such function]. What all remains to be done is to choose $p(x)$ as a suitable amendment of this h, through suitable magnifying and translating by amounts varying in different intervals, so that the pre-specified values of $\psi(x, y)$ are not changed. This leads us to the function g that we have constructed.

We observe that on each strip of the form $\{(x, y) | n < x < n + 1\}$, both the components of g are polynomials of low degrees in one or two variables.

Remark 3: Let C_1 be the first quadrant, i.e. $\{(x, y) \in \mathbb{R}^2 | x > 0, y > 0\}$. The function f constructed in the proof of the theorem 1 has the property that $f(C_1) \subset C_1$ and that all the periodic points of f are in C_1. Therefore we deduce that $PER(C_1) = \wp(\mathbb{N})$. Similarly, $PER(H_u) = \wp(\mathbb{N})$, where $H_u = \{(x, y) \in \mathbb{R}^2 | y > 0\}$.

§2

Period Sets For The Convex Subsets Of The Plane

In this section, first we use the result of the previous section §1 to improve the same by showing that if S is any noncompact convex subset of $\mathbb{R}^2$, then $PER(S) = \wp(\mathbb{N})$. We also find $PER(S)$ where S is a closed disc.

First we recall some well-known definitions mainly for the purpose of fixing our notations and terminology. $e_1 = (1,0)$ and $e_2 = (0,1)$. If x and y are two vectors in $\mathbb{R}^2$, then $[x, y] = \{tx + (1 - t)y : 0 < t < 1\}$. A subset S is convex if $[x, y] \subset S$ whenever

x and y are in S. If x and y are two fixed vectors in $\mathbb{R}^2$, and if $y \neq 0$, then $\{x + ty \mid t \in \mathbb{R}, t > 0\}$ is the closed ray at x in the direction of y. Such sets will be called simply as rays. A strip is a closed region in $\mathbb{R}^2$ whose boundary consists of two parallel lines. It is of the form $\{(x_1, x_2) \in \mathbb{R}^2 \mid c_1 < ax_1 + bx_2 < c_2\}$ where a, b, c_1 and c_2 are fixed real numbers such that $c_1 < c_2$. A half plane is a closed region in $\mathbb{R}^2$ whose boundary consists of one line. It is a set of the form $\{(x_1, x_2) \in \mathbb{R}^2 \mid ax_1 + bx_2 + c > 0\}$ where a, b, c are three fixed real numbers. The intersection of a half plane and a strip, except when it happens to be a line or a strip, is called a half-strip. It is a set of the form $\{ax + by + cz \mid a, b, c > 0, a + b = 1\}$ where x,y,z are three distinct vectors and $z \neq 0$. It is the closed convex hull of a ray and a point not collinear with it. A cone is a closed convex region whose boundary consists of two rays at a point. The cone at x determined by the two independent vectors y and z is the set $C = \{x + ay + bz \mid a, b > 0\}$. The bisector ray of this cone is the set of all points in C that are equidistant from the two boundary rays of C. Equivalently, it is the ray at x in the direction of $\left(\frac{y}{\|y\|} + \frac{z}{\|z\|}\right)$. This ray divides the cone into two closed regions. We call them the two half-subcones of C. The cone at the origin determined by e_1 and e_2 is called the first quadrant. Three other quadrants are defined similarly, in terms of the four vectors $e_1, e_2, -e_1$ and $-e_2$. Lastly if $S \subset \mathbb{R}^2$ and if $x \in \mathbb{R}^2$, then $S + x = \{y + x \mid y \in S\}$ and $S - x = S + (-x)$.

Lemma 1: Every unbounded closed convex subset S of $\mathbb{R}^2$ contains a ray at each of its points.

Proof: Let $x \in S$ be fixed. There is at least one quadrant C such that $(C + x) \cap S$ is unbounded. Without loss of generality, let C be the first quadrant C_1. There is at least one half-subcone C_2 of C_1 such that $(C_2 + x) \cap S$ is unbounded. Thus proceeding, we get a decreasing sequence $C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n \cdots$ of cones such that for every n, $(i) C_{n+1}$

is a **half-subcone** of C_n and $(ii)(C_n + x) \cap S$ is unbounded. By applying the nested intervals theorem to the sequence of sets $(C_n \cap [e_1, e_2])$, we can prove that $\bigcap_{n=1}^{\infty} C_n$ is a ray. Call it A. Then $A + x$ is a ray at x and we claim that $A + x$ C S . To prove this, let $y(\neq 0)$ G A and let $\epsilon > 0$. Then there is a positive integer n_0 such that C_{n_o} is contained in the smallest cone at 0 that contains the ball $B(y,t)$ with center y and radius €. Because $(C_{n_o} + x) \cap S$ is unbounded, there is some z in it such that \\z — x\\ > \\y\\ . Then the segment [0,2 — x], on one hand meets the ball $B(y, \epsilon)$, and on the other hand, is contained in S — x by the convexity of S. Thus $B(y + x, \epsilon) \cap S$ is nonempty. Since this is true for every $c > 0$ and since S is closed, we conclude that $y + x$ £ S . Since this is true for every y G A, we conclude that S contains the ray $A + x$.

Lemma 2: Every unbounded convex subset S of $\mathbb{R}^2$ contins a ray.

Proof: The closure S of S is also a convex set, by a known result. Now by Lemma1 , 5 contains a ray A. Consider two cases:

Case-1: If S is contained in a line, then S is an unbounded convex subset of a line. It is easy to prove that such a set contains a ray (at each of its points).

Case-2: If case-1 does not hold, take a vector x in S outside A not collinear with it. Let r be the distance between x and A. Let a $\in A$. Then the ball $B(a, \frac{r}{4})$ contains an element y of S, because a G S . Then $[x, y]$ C 5 because S is convex. Take an element z in this segment such that $\frac{r}{4} < ||z — x\\ < \frac{r}{2}$. Consider the ray B at z parallel to A. We claim that $B \subseteq S$. For every S > \\z\\ take a_δ in A such that $||a_\delta|| > 6$. Since a_δ is in S , there is some x_δ in SO B(a_δ, $\frac{r}{4}$). If y_δ is the unique element of $[x, x_\delta] \cap B$, then \\ys\\ > $_4$ $||x - a_\delta|| \geq_4 (||a_\delta|| - ||x||)$ • Here we use the elementary theorem that any line parallel to the base of a triangle divides the other two sides in the same ratio. Thus we have proved that $B \cap S$ is unbounded. Therefore S contains B, because of convexity, and because $z \in S$

Theorem 2: Let X be a non-compact convex subset of $\mathbb{R}^2$, with nonempty interior. Then *PER(X)* =

Proof: First we prove that X contains a retract horneomorphic to C_1.

Case (i) Let X be unbounded.

By lemma 2, X contains a ray A. Because $X^\circ \neq \emptyset, \exists\, x \in X$ such that x is not collinear with elements of A. Therefore the convex hull of *A* U *{x}* contains a half strip H .This H is retract of X; if the map r : $X \longrightarrow H$ is such that r(y) = the unique element of H nearest to y, then r is a retraction map.

But it can be proved that H is horneomorphic to the quardrant C_1.

Case(ii): Let X be bounded. Then X is contained in some ball $B(x_o, r)$;$x_o \in X^o$,the interior of X. Consider *X.* Then there exists a **homeomorphism** *h* : *X* $\rightarrow$ $B(x_o, r)$ satisfying

(i) $h(x_o) = x_o$

(ii) h "blows up"all the other points of Xin such a way that h takes boundary to boundary. Let *h(X)* = *Y.* Then $B(x_o, r)$C *Y* C $B(x_o, r)$(because *Y* is not compact).

It is known from complex analysis that there is a fractional linear transformation from $\overline{B}(x_o$, r) \ {a point on the boundary } onto the **upper half-plane** H_u = { (x,y) \ y > 0} . This yields a homeomorphism from *Y* to H_1 where H_1 = H_u \ some subset of the x-axis. **Now** $C = \{(x, y)|\ x > 0, y > 1\}$ is a closed convex subset of H_1, and is **therefore a retract** of H_2. Also it is easily seen that *C* is homeomorphic to the first quadrant C_1 .

Thus in both cases we have proved that X contains a retract homeomorphic to C_1 and **therefore** $PER(X) \supset PER(C_1)$. But $PER(C_1) = \wp(\mathbb{N})$ by Remark 3 of §1. Therefore $PER(X) = \wp(\mathbb{N})$.

Theorem 3: Let S be a closed disc. Then $PER(S) = \mathcal{U}_1$.

Proof: Without loss of generality, let S be the closed unit disc with centre as the origin 0. Let $S\backslash$ be the boundary of S. Let S_n be the circle with origin 0 as centre and radius $\frac{1}{n}$*.

Let $A = \{1 < n_1 < n_2 < n_3 < ...\} \subset \mathbb{N}$. Now to exhibit a continuous function $f : S \to S$ with $Per(f) = A$, we proceed in the following way :

First let $g : [0,1] \to [0,1]$ be a strictly increasing continuous function such that $\{t \in [0.1] \mid g(t) = i\} = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ and such that $t < g(t)$ for all $t \in [0,1]$.

Secondly $h : [0,1] \to [0,2\pi]$ is the unique function satisfying

(i) $h(\frac{1}{k}) = \frac{2\pi}{n_k}$ for all k in $\mathbb{N}$.

(ii) on each subinterval $[\frac{1}{(k+1)}, \frac{1}{k}]$, h is linear.

Lastly, we use the above g and h to define $f(\rho e^{i\theta}) = g(\rho)e^{i(\theta+h(\rho))}$ for all $\rho e^{i\theta}$ in S.

We observe the following:

(a) f takes origin to origin.

(b) For each k in $\mathbb{N}$, f takes S_k to itself.

(c) On S_k, f is nothing but rotation by the angle of —.

(d) f is continuous.

(e) $Per(f|_{S_k})$= $\{n_k\}$,this follows from (c).

(f) {periodic points of f} = {0} U $(\cup_{n=1}^{\infty} S_n)$.

(g) A C $Per(f)$follows from (a) and (e) and *Per(f)* C *A* follows from (a),(e), and (f).

These together prove that $\mathcal{U}_1$ C $PER(S)$. The reverse inclusion follows from **Brouwer's** fixed point theorem .

§3

Period Sets For Convex Subsets Of ET

Lemma 3: Let *S* be an unbounded closed convex subset of $\mathbb{R}^n$, where *n* > *2.* Let $x_o \in S$. Then there exists a hyperspace *H* such that $S \cap (H + x_o)$ is unbounded.

Proof: Let $\pi : \mathbb{R}^n \to \mathbb{R}^2$ be the projection map defined by $\pi(x_1, x_2, \cdots, .x_n) = (x_1, x_2)$. Let Q_1, Q_2, Q_3, Q_4,be the four quadrants of $\mathbb{R}^2$. Then because

$$S = \bigcup_{n=1}^{4} S \cap \pi^{-1}(Q_n).$$

It follows that S $\cap$ $\pi^{-1}(Q_n)$ is unbounded for some n in {1,2,3,4} . Without loss of generality assume that it is the first quadrant *C* for which we have S $\cap$ $\pi^{-1}(C_1)$ is **unbounded.Similarly** ,let C_2 be a half- subcone of C_1 such that S $\cap$ $\pi^{-1}(C_2)$ is unbounded. Proceeding in the same way,we get a decreasing sequence $C_1 \supseteq$ ft $\supseteq$ ft $\supseteq \cdots \supseteq C_n \supset \ldots$ of cones such that each C_{n+1} is a half- subcone of C_n and such that S

$\cap\ \pi^{-1}(C_n)$ is unbounded for all n.As seen in the proof of lemmal, $\bigcap_{n=1}^{\infty} S \cap\ \pi^{-1}(C_n)$ is same as S $\cap\ \pi^{-1}(\cap_{n=1}^{\infty} \in_n$ $ $\cap\ \pi^{-1}$(a ray at 0). Now π^{-1} (a ray at 0) is contained in π^{-1} (a line through 0) which is a hyperspace H of $\mathbb{R}^n$. If 0 € 5,then we can imitate the arguments in the proof of lemmal to conclude that S $\cap$ H is unbounded : however some modifications are required in the proof.

Given r > 0 and a positive integer k choose n large enough such that every element of C_n whose norm is less than 2r is at a distance less than 1/k from $\pi(H)$.Choosex_k in S $\cap\ \pi^{-1}(C_n)$such that $\| x_k \|$ > 2r . Since the segment *[0,* x_k*]* has to be a subset of S , there is some y_k in S such that r < *yk*\\ < 2r and therefore y_k is at a distance less than 1/k from $\pi(H)$. The sequence (y_k)in the compact ball *B* (0 , 2r) has to admit a subsequence converging to some y_0 . This y_0 has to be in H . Also $\| y_0 \|$ > r since for each k , $\| y_k \|$ > r. Also $y_0 \in$ S because S is closed . Thus there are elements in S $\cap$ H with arbitararily large norm . Therefore S $\cap$ H is unbounded .

Thus we have proved the result if x_0 = 0. For a general x_0 in S we apply this to the convex set S - x_0 and get a hyperspace H such that (S - x_0) $\cap$ H is unbounded . Now it follows that S $\cap$ (H + x_0) is also unbounded.

Theorem 4: Let S be a bounded noncompact convex subset of $\mathbb{R}^n$. Then *PER(S)* = $\wp(\mathbb{N})$.

Proof: It is known that every convex subset of $\mathbb{R}^n$ is homeomorphic to some *p* = $B^m\backslash$ { A subset of the boundary}, where B^m is the closed m-dimensional unit ball for some *m* < *n.* Without loss of genarality let (0 , 0 , 0 , • • • , 1) not be in P. Consider the sphere S_1 with centre(0 , 0 , 0 , • • • , 1/2) and radius 1/2 . Then S_1 \ *{ y }* is a retract of P by the nearest-point map . This is because , for each x in S_1 , the segment [x, y] meets the smaller sphere S_1 at a point other than y , and therefore y cannot be the nearest point to x among the elements of S_1 .

Next let D ={ all $x \in S_1$ such that all but the last two co-ordinates of x are 0 } . Then D is homeomorphic to the 2-dimensional unit disc $\{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$

The projection map P : $S_1 \setminus \{ y \} \to D \setminus \{ y \}$ given by $P(x_1, x_2, x_3, \cdots, x_n) = (0, 0, 0, \cdots, x_{n-1}, x_n)$ is easily seen to be a retraction. We have already proved that (in the proof of theorem 2) $D \setminus \{ y \}$ is homeomorphic to the half plane H_u of $\mathbb{R}^2$. Therefore $PER(S) = PER(P) \supset PER(S_1 \setminus \{ y \}) \supset PER(D \setminus \{ y \}) = PER(H_u) = \wp(\mathbb{N})$, by remark 3 .

Theorem 5: Let *X* be a nonempty convex subset of $\mathbb{R}^n$. Then *PER(X)* has to be one of the following:

(a) $\{\}$, (6) $\mathcal{S}$, (c) $\mathcal{S} \cup \{\emptyset\}$, (*d*) $\mathcal{U}_1$, or (e) $\wp(\mathbb{N})$ according as

(a) X is a singleton,

(b) X is an infinite compact set in which all points are collinear,

(c) X is a noncompact set in which all points are collinear,

(d) X is a compact set in which there exist 3 points that are not collinear, **or**

(e) X is a noncompact set having 3 noncollinear points.

Proof:

(a) obvious

(b) Sarkovskii's theorem, together with its converse, proves (b) because X is homeomorphic to I.

(c) There are only two subsets of $\mathbb{R}^n$ upto homeomorphism, that satisfy the condition in (c). They are a line L and a ray A (A is homeomorphic to [0, oo)) I = [0,1] is a retract of [0, oo) which is a retract of 1R

Hence *PER(I)* C *PER(A)* C $PER(\mathbb{R})$. Therefore *PER(A)* is either ***S* or *S*** U $\{\emptyset\}$.

The map $x \to x+1$ is a continuous self map of [0, oo) to [0, oo) without periodic points. Therefore *PER(A)* = $S \cup \{0\}$. (in both the cases, whether A is a line or a ray)

(d) In this case the set X is homeomorphic to the m-dimensional closed ball B^m for some m < n with 2 < m < n . It contains a retract homeomorphic to the closed unit disc. Therefore by theorem 3 and Brouwer's fixed point theorem $PER(X) = \mathcal{U}_1$.

(e) X is noncompact. Consider two cases . Case(l): If X is bounded then by theorem 4, *PER(X)* = $\wp(\mathbb{N})$.

Case(2): If X is unbounded then by applying lemma-3 to X and using induction along with lemma-1 we can prove that X contains a ray. Since X has atleast 3 non-collinear points , there is a point y in X outside the line of this ray . Then $X \cap$ M = $X \cap M$. Applying lemma-2 to the unbounded convex set $X \cap M$, and noting that we have case 2 of its proof , we conclude that $X \cap M$ contains a half strip . This half strip , being a closed convex set , is a retract of X . It follows that $PER(X)$ = p (IN) .

§4

Concluding Remarks

Some **further results:**

Some examples of non-convex subsets of $\mathbb{R}^2$ are

(a). The punctured plane $\mathbb{R}^2 \setminus \{0\}$.

(b). The compact annul us $\{ (x, y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2 \}$.

(c). $\{ (x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq, x + y < 1, (x - 1)^2 + (y - 1)^2 > 1\}$

If X is any one of the these, we can find *PER(X)* ,by using the methods of proofs of this chapter. We note that there are compact subsets S of $\mathbb{R}^2$ such that $PER(S) = p(\mathbb{N})$.

Some **limitations:**

If we consider the larger class of connected sets , instead of convex sets then the answer cannot be elegant . There are infinite number of families that arise in the form of *PER(X)* . The n-ods studied in [3] are not only connected but also star-convex .

An open question :

We know that every convex subset S of $\mathbb{R}^n$ has the property that there is an open connected subset V of $\mathbb{R}^m$ with $0 < m < n$ and a subset **T with** $V \subset T \subset \bar{V}$ such that S and T are homeomorphic . Therefore our main theorem naturally leads us to the question : If T is as above , should *PER(T)* be one of the 5 families listed in theorem 5 ? The answer seems to be affirmative.

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