Application of Thermofield Dyanamics to Equilibrium and Non-Equilibrium Problems

a thesis submitted for the degree of
DOCTOR OF PHILOSOPHY

by
P. Shanta

School of Physics
University of Hyderabad
Hyderabad - 500 046, **INDIA**

NOVEMBER 1995

Certificate

This is to certify that the work contained in this thesis entitled '**Application of Thermofield Dynamics to Equilibrium and Non-Equilibrium Problems**' has been carried out by Ms. P. Shanta under my supervision for the full period prescribed under Ph. D. ordinances of the university and the same has not been submitted for award of research degree of any university.

(S. Chaturvedi)
Thesis Supervisor

Place:

Date:

Dean
School of Physics
University of Hyderabad
Hyderabad INDIA 500 046

DECLARATION

I herteby delcare that the matter contained in this thesis is the result of investigations carried out by me in the School of Physics, University of Hyderabad, Hyderabad - 500 046, under the supervision of Professor S. Chatruvedi.

P Shanta

Place: Hyderabad

Date: 16-11-95.

P. Shanta

Aknowledgements

I thank Prof. S. Chaturvedi for his patience, encouragement and concern. I have learnt much from his logical and elegant approach to problems.

I am indebted to Prof. V. Srinivasan for his good will, encouragement and concern for my physics education in general, apart from all the help in tackling the thesis problems. I thank him for the patience and willingness in answering questions and clearing my doubts. I thank both of them for leading me to this interesting area in physics.

I thank the Dean, School of Physics, for the facilities provided by the school.

I thank Prof. Kapoor for help and encouragement. I thank Dr. Panigrahi for his concern and friendliness. I also thank Dr. Bindu Bambah, Dr. Mukku, and Dr. Nagaraja Kumar for their friendliness. I thank Dr. Anantha Lakshmi for her friendship and hospitality.

My long stay on campus has been made pleasant and instructive because of a large number of friends. I thank Biplab, Jyotsna, Meeta, Bidisha, Tara, Bhalla, Saroja, Ajitha, Ramana, Neelima, and others for their cheerful company, for sharing their ideas and views on everything under the sun, for the help, understanding and moral support. I am particularly grateful to my friends in hostel for care and emotional support. I thank Sandhya for her friendship and encouragement. I also thank Bhavani, Jaya Sree, and Kamala.

I am indebted to my father for his affection and constant encouragement. He has been a source of moral strength and courage. I have very special thanks to him for being my best maths teacher during school and for making science and maths fun to learn. I thank my mother for her love, patience and understanding.

I thank Bhayya for his interest and concern in my career over the years. I thank Yamuna for her affection. It is a pleasure to thank Ravi and Raju for being affectionate brothers and true friends. I thank them for all fun we had bird watching, star gazing, reading, gardening, arguing ...

I express my gratitude to the late Prof. S. Swaminathan of IIT Madras for his concern for my career.

I thank Srinivas for his willing help in typing.

I thank the CSIR for financial support.

Curriculum Vitae

Name : P. Shanta

Date of Birth : 30 July 1965

Nationality : Indian

Address For Correspondence :

School of Physics

University of Hyderabad

Hyderabad INDIA 500046

Academic

Ph.D. University of Hyderabad Thesis submitted in November 1995

M. Phil. University of Hyderabad Hyderabad India 1989

M. Sc. Indian Institute of Technology Madras India 1988

B. Sc. Osmania University Hyderabad India 1986

Research Publications:

1. Lamb shift in a squeezed Thermal Vacuum, S. Chaturvedi, P. Shanta, and V. Srinivasan, Physical Review A **43** 521 (1991)

2. The use of quantum mechanical approximation methods in the solution of master equations, S. Chaturvedi, P. Shanta, and V. Srinivasan, Optics Communications, **78** 289 (1990)

3. Para statistical systems at finite temperatures in thermofield dynamics, International Journal of Modern Physics, A **7** 3807 (1992)

4. A model which interpolates between the Jaynes-Cummings model and the Buck-Sukumar models, P. Shanta, S. Chaturvedi, and V. Srinivasan, Journal of Modern Optics, **39** 1301 (1992)

5. On q-deformed para oscillators and para-q oscillators, M. Krisna Kumari, P. Shanta, S. Chaturvedi, and V. Srinivasan, Modern Physics Letters A **7** 2593 (1992)

6. Time-dependent Bogoliubov transformations and the damped harmonic oscillator, P. Shanta, S. Chaturvedi, V. Srinivasan, and F. Mancini, Modern Physics Letters A **8** 1999 (1993)

7. On the number operators of multimode systems of deformed oscillators covariant under quantum groups, R. Jagannathan, R. Sridhar, R. Vasudevan, S. Chaturvedi, M. Krishna Kumari, P. Shanta, and V. Srinivasan, Journal of Physics A **25** 6429 (1992)

8. A unified approach to multiphoton coherent states, P. Shanta, S. Chaturvedi, V. Srinivasan, G.S. Agarwal, and C.L. Mehta, Physical Review Letters, **72** 1447 (1994)

9. Unified approach to the analogues of single-photon and multiphoton coherent states for generalized boson oscillators, P. Shanta, S. Chaturvedi, V. Srinivasan, and R. Jagannathan, Journal of Physics A **27** 6433 (1994)

Papers Presented at Symposia/Conferences:

'Lamb shift in a squeezed thermal vacuum' at the International Conference on Quantum Optics, Hyderabad, India 5-10 Jan 1991

'A q-deformed model that interpolates between the Jaynes-Cummings model and Buck-Sukumar model', at the Conference onRecent Developments in Quantum Optics, University of Hyderabad, India 18-20 March 1993

'A unified approach to multiphoton coherent states' at the conference 'Chorent States - New Developments and Perspectives', University of Hyderabad, India 29-31 Oct 1993

Thermal Generator for Non-Equilibrium Thermofield Dynamics' DAE Symposium on High Energy Physics , Visva-Bharati, 28 Dec-2 Jan 1995

Table of Contents

Chapter 1

The Thermal State and Thermal Operators

1.1 Introduction

The treatment of statistical systems in quantum mechanics is based on the use of density operator, which is the quantum mechanical counterpart of the classical density function.For a quantum mechanical treatment of many body systems it proves convenient to use the method of second quantization. The solutions of single particle Schrodinger equation are used to construct a classical field.The Schrodinger equation is the equation of motion of this classical field. When quantised, this gives a quantum field theory for many body systems. The quantities of interest in these problems are the correlation functions—which are the ground state expectation values of time ordered operators. The techniques of quantum field theory like diagrammatic expansions , which were introduced for relativistic fields, become available for Schrodinger fields. But this applies only to systems at zero temperature, because the field theoretic techniques involved vacuum expectation values. For systems at finite temperature the relevant quantities are the thermal averages. The use of field theoretic methods for many body systems at finite temperature started with the imaginary time formulation [1][2]. This approach was based on the similarity of the statistical factor $e^{-\beta H}$ to the time evolution operator e^{-iHt}.

By treating temperature as an imaginary time argument a perturbation theory is set up. Use is made of the thermodynamic Wick's theorem. The Green's functions

for interacting systems can be calculated perturbatively using diagrammatic techniques. The complex time path formulations of Mills, Schwinger, Keldysh [3][4][10][28] are the generalisation of Matsubara's technique. They have the advantage that the results go over to the ground state results in the zero temperature limit.

An alternate treatment of statistical systems is given by the formalism of thermofield dynamics developed by Umezawa and others [7][8][62]. The aim of this formalism is to treat the mixed states of systems at finite temperature on par with the pure states. The trace operations which occur in thermal averages are replaced by expectation values with respect to specially constructed states. The analogy with the ground state field is complete, as the states so constructed have annihilation operators associated with them.

The thermal average of operator A is given as

$$< A >= \frac{Tr\rho A}{Tr\rho}$$

where ρ is the density operator characterizing the mixed state.In conventional statistical mechanics these thermal averages cannot be written as expectation values $< \psi | A \mid \psi >$. Hence the mixed state averages are on a different footing. But in the formalism of thermofield dyanamics we can construct state vectors $\mid \rho >$ to represent the mixed state. Such a state $\mid \rho >$ can be constructed in a doubled Hilbert space $H \bigotimes \tilde{H}$. The second set of states are refered to as the tildean states. They do not seem to have any physical significance. All quantities of physical interest are expressed purely in terms of the non-tildean degrees of freedom. It is possible to have an unsymmetrical representation of the bra and ket vectors represented by $\mid \rho^{\alpha} >$ and $< \rho^{1-\alpha} \mid$, where $\frac{1}{2} \leq \alpha \leq 1$. Strictly speaking, the states $|\rho^{1-\alpha} >$ are not guaranteed to have a finite norm for all ρ and are to be treated as convenient mathematical constructs. The thermal expectation value of any observable A is given by

$$< A >=< \rho^{1-\alpha} | A \mid \rho^{\alpha} >$$

Thermofield dynamics(TFD) is a real time formulation of finite temperature field theory. It is a canonical formulation in terms of the fields ψ and the tildean counterpart $\tilde{\psi}$. Along with the Lagrangian density $\mathcal{L}(\psi(x), \partial_\mu \psi(x))$ we also have a tildean $\tilde{\mathcal{L}}(\tilde{\psi}(x), \partial_\mu \tilde{\psi}(x))$. Similarly, the Hamiltonian densities $\mathcal{H}(\psi(x), \pi(x))$ and $\tilde{\mathcal{H}}(\tilde{\psi}(x), \tilde{\pi}(x))$ .The quantities $\hat{L} = L - \tilde{L}$ and $\hat{H} = H - \tilde{H}$ determine the field equations in TFD. With vacuum expectation values being replaced with expectation values using the states $\mid \rho^\alpha >$ and $< \rho^{1-\alpha} \mid$ the analogy to ground state field theory is complete. There exist operators $\xi, \tilde{\xi}$ and and $\xi^{\dagger\dagger}, \tilde{\xi}^{\dagger\dagger}$, which annihilate $\mid \rho^\alpha >$ and $< \rho^{1-\alpha} \mid$ respectively. The ξs are related to the operators $a, \tilde{a}, a^\dagger, \tilde{a}^\dagger$ by a Bogoliubov-like transformation; $a, a^\dagger$ etc being the free field annihilation and creation operators. The existence of ξs allows the use of the ordinary Wick's theorem and diagrammatic methods can be used for interacting fields

The other treatments of finite temperature field theories use complex time arguments. The time argument of the evolution operator in interaction picture is continued to complex values. The real part of the argument is assumed to give dynamical information. When only stationary equilibrium quantities are computed, it is set to zero.The imaginary part is taken to be β, the inverse temperature $(1/kT)$, The time ordering which appears in usual field theories is replaced by ordering along a complex contour. The contour can be of arbitrary shapes except for restrictions placed by the analyticity of the quantities to be computed. For Green's functons of a Schrodinger field, for example, analyticity demands that the contour have a non-increasing imaginary part. This general approach due to Mills contains the Matsubara method as a special case, the contour lying entirely on the imaginary axis. The other commonly used contours are the Schwinger and the Keldysh contour.

The exact Green's functions for interacting fields can be calculated for any of these

contours using diagrammatic methods. Use is made of the thermodynamic Wick's theorem. These complex time methods are completely equivalent to TFD for systems in equilibruim [9][13-15]. Identical expressions are obtained when the fields on the two legs of the contour are identified with the tilde and the non-tilde fields of TFD.

The treatment of non-equilibrium systems is a more complex problem. One approach simply extends the linear response theory to finite temperature. In the linear response theory at zero temperature the linear response of interacting systems to weak external influence which drives it off equilibrium is studied using a first order perturbation theory. The pertubation is carried out about the eigenstates of the interacting system. The change in the expectation values of any observable of the interacting system due to the external influence is proportional to the strength of the external perturbation and the expectation values of the observable. By using equilibrium thermal expectation values of the observables this method can be extended to systems slightly out of equilibrium at finite temperature. In the imaginary time (Matsubara) method time dependent correlation functions are obtained as the analytical continuation of the finite temperature correlations. The complex time methods do not need analytical continuation, as the time arguments already have a real part. But as these methods are equivalent to equilibrium TFD, the mere presence of a real time should not be taken to imply non-equilibrium. Such treatments of non-equilibruim processes based on complex time path have the drawback that all the correlation functions are functions only of the time differences, as in equilibrium. The time evolution of the density operator is not taken care of. Only in the presence of external time dependent perturbation do the correlations cease to be functions of time differences and Boltzmann like equations for their evolution can then be derived.

In another approach, non-equilibrium systems are studied using the master equation for the time evolution of the density operator[16-21]. The time dependence of the density

operator is taken of . But this method is not amenable to field theories, though widely used for quantum mechanical systems. The master equation for a quantum mechanical system is derived from the Liouville-Von Neuman equation for the density operator

$$i\frac{\partial \rho_T}{\partial t} = [H, \rho_T]$$

($\hbar = 1$) where H is the Hamiltonian of a system with large number of degrees of freedom and ρ_T is its density operator. The large system contains a few degrees of freedom whose dynamics are of interest (the system). These few degrees of freedom are coupled at some instant to the rest of the variables (the bath) and the subsequent evolution of these few variables is studied.

The coupling is assumed to be linear. The system variables are assumed not to be coupled to each other. From the time evolution of ρ_T, an equation for the evolution of ρ, the density operator of the system is derived by averaging over the bath variables. It is assumed in this process that the correlation times of the bath variables are very short compared with the damping time of the system.The master equation obtained in this manner has the form

$$i\frac{\partial \rho}{\partial t} = [H_o, \rho] + L$$

where H_o is the Hamiltonian for the system alone and L consists of the system variables and ρ. The master equation obtained in this way for a damped harmonic oscillator gives crucial insight in setting up a full-fledged TFD formalism for non-equilibrium.

A master equation of the form given can be converted into an equation for $\mid \rho >$, as will be shown later. The equation for $\mid \rho >$ is a Schrodinger-like equation

$$i\frac{\partial}{\partial t} \mid \rho(t) >= \hat{H} \mid \rho(t) > . \tag{1.1}$$

It has the formal solution

$$\mid \rho(t) = exp - i\hat{H}t|\rho(0) >$$

$\hat{H}$ contains the parameter $\bar{n}$ which characterises the bath, and γ the damping constant which depends on the coupling between the system and the reservoir. Using a disentangling theorem for the operators in $\hat{H}$, which happen to be SU(1,1) generators, the formal solution can be computed algebraically.

For a bosonic oscillator in equilibrium the states $\mid \rho >$ has the form of a 'thermal coherent state':

$$\mid \rho >= (1-f)exp(fa^{\dagger}\tilde{a}^{\dagger})|0,\tilde{0}>$$

where f is related to the mean occupation n as

$$f = \frac{n}{1+n} \quad , \quad n = Tr\{\rho a^{\dagger}a\} \ .$$

Equation (1.1) has the property that it preserves the form of a thermal coherent state as it evolves. Its solution is a thermal coherent state characterised by $n(t)$ where

$$n(t) = n(0)e^{-\gamma t} + \bar{n}(1-e^{-\gamma t}). \tag{1.2}$$

Although this property of (1.1) has been discussed in quantum optics[18] ,it acquires a special significance in the context of TFD. In recent years the formalism of TFD has been extended to non-equilibrium. The aim is to set up a formulation which is similar to the equilibrium formalism and allows a diagrammatic perturbation theory. This is done by making suitable assumptions about the form of the state $\mid \rho(t) >$ as it evolves with time. It is assumed that the structure of $\mid \rho(t) >$ remains the same as that it has in equilibrium, but the parameters which characterize it become time dependent. The bosonic systems, for example, are assumed to have a thermal coherent state for $\mid \rho(t) >$. This form assures the existence of $\xi(t), \tilde{\xi}(t)$ which annihilate the states $\mid \rho^{\alpha}(t) >$. The ξs are related to the a s through a time dependent Bogoliubov transformation.

Equation (1.1) turns out to be a good starting point for extending TFD to non-equilibrium in the way outlined above. An important observation is that the equation

(1.1) preserves the form of a thermal coherent state even when the parameters $\omega, \bar{n}, \gamma$ are made arbitrary functions of time [67]. Using the solution for $n(t)$, the bath parameter $\bar{n}$ can be expressed in terms of $n(t)$ and $\dot{n}$. By doing so we eliminate $\bar{n}$ and hence the reference to the bath. When γ is an unknown function of time it is not possible to solve for the time dependence of $n(t)$. Although the study of a damped harmonic oscillator brings out the central features of the TFD scheme for non-equilibrium, it does not complete it. This happens because the damped oscillator without interactions is unrealistic. It is the interactions which determine the dynamics, and hence $n(t)$. It becomes necessary to express γ in terms of $n(t)$, in order that the Bogoliubov transformation be linear in n. Then, along with the notion of a bath the notion of external damping is also lost. The use of the master equation to derive equation (1.1) becomes irrelevant and it becomes the basic equation consistent with the assumptions made about the nature $\mid \rho(t) >$.

The operator $\hat{H}$ which governs the time evolution of $\mid \rho(t) >$ contains a part $\hat{H}_0 = H_0 - \tilde{H}_0$ corresponding to the free field and a 'thermal' part Q containing bilinear combinations of tildean and non-tildean operators.

When interactions are present we need to have an interaction picture to set up perturbation theory. Let the total Hamiltonian consist of the free part H_0 and a interaction H_I. Since equation (1.1) can be interpreted as the Schrodinger equation in the absence of interactions,it is natural to regard $H_0 - Q$ as the 'free' hamiltonian in the interaction picture. This choice takes care of the assumptions made about the form of $\mid \rho(t) >$. But such an addition of Q would change the original equations of motion, which is not allowable. We do not want to change the dynamics in order to deal with non-equilibrium processes.The Q term is hence added to the interaction as a counter term, as in usual field theories. Such counter terms enable the equations of motion of the field to be written in terms of the physical masses and coupling constants. This is in attempt to retain the particle picture in presence of interactions, and Q too serves a similar purpose, in

non-equilibrium . But at finite temperatures we only have quasi-particle picture as the self energies tend to develelop imaginary parts.

To complete the scheme of non-equilibruim TFD we need to determine the time dependence of $n(t)$. When this is done, the calculations are the same as in equilibruim. An equation for $n(t)$ (the Boltzmann-like equation) is obtained by demanding that the corrected propagator have the form as the free propagator. This requires that the self energy be diagonal (written as a matrix using the doublet notation). The contributions to the self energy due to the counter term and the interaction are added, and the off-diagonal elements are set equal to zero. This gives an equation for $\dot{n}$ analogous to the classical Boltzmann equation. The diagonal elements in general contain both real and imaginary parts.

These concepts are developed in what follows in this thesis. Chapter 1 presents the formalism of TFD briefly, starting from the basics. The concepts of the thermal vacuum, the various α representations , the thermal annihilation operators etc. are developed. The construction of the thermal vacuum for para-statistical systems is also examined.

Chapter 2 contains a brief summary of the complex time formalism and a comparision with the TFD formalism. Their equivalence in equilibrium is demonstrated.

Chapter 3 contains applications of TFD to non-equilibruim processes using the master equation route. The solution of the master equation for a damped harmonic oscillator leads naturally to concepts like time-dependent Bogoliubov transformations, the thermal generator etc. which form the basis of non-equilibruim TFD. This study also justifies the central assumption of non-equilibruim TFD. It also contains some technical results about the parametrization of the Bogoliubov transformation and the evolution equation for the thermal state.

Chapter 4 contains a review of the non-equilibrium TFD and applications to some examples. The diagonalization of the self energy and the derivation of the Boltzmann

non-equilibrium . But at finite temperatures we only have quasi-particle picture as the self energies tend to develelop imaginary parts.

To complete the scheme of non-equilibruim TFD we need to determine the time dependence of $n(t)$. When this is done, the calculations are the same as in equilibruim. An equation for $n(t)$ (the Boltzmann-like equation) is obtained by demanding that the corrected propagator have the form as the free propagator. This requires that the self energy be diagonal (written as a matrix using the doublet notation). The contributions to the self energy due to the counter term and the interaction are added, and the off-diagonal elements are set equal to zero. This gives an equation for $\dot{n}$ analogous to the classical Boltzmann equation. The diagonal elements in general contain both real and imaginary parts.

These concepts are developed in what follows in this thesis. Chapter 1 presents the formalism of TFD briefly, starting from the basics. The concepts of the thermal vacuum, the various α representations , the thermal annihilation operators etc. are developed. The construction of the thermal vacuum for para-statistical systems is also examined.

Chapter 2 contains a brief summary of the complex time formalism and a comparision with the TFD formalism. Their equivalence in equilibrium is demonstrated.

Chapter 3 contains applications of TFD to non-equilibruim processes using the master equation route. The solution of the master equation for a damped harmonic oscillator leads naturally to concepts like time-dependent Bogoliubov transformations, the thermal generator etc. which form the basis of non-equilibruim TFD. This study also justifies the central assumption of non-equilibruim TFD. It also contains some technical results about the parametrization of the Bogoliubov transformation and the evolution equation for the thermal state.

Chapter 4 contains a review of the non-equilibrium TFD and applications to some examples. The diagonalization of the self energy and the derivation of the Boltzmann

equation are given.

Chapter 5 has some results about generalized bosonic oscillators and generalized coherent states. The application of TFD to para and deformed states is based on the fact that these generalized oscillators are related to bosonic oscillators through a similarity transformation. It is possible to construct canonical conjugates for the para and deformed commutation relations. Such constructions also are possible for multiphoton operators — powers of annihilation operators — of both bosonic and generalized oscillators. The canonical conjugates are explicitly constructed. A variety of coherent states can be constructed using the canonical conjugates, and they occur in closely related pairs.

1.2 The Thermal State

The necessity of doubling the Hilbert space in order to construct a state to represent a system in a mixed state was pointed out by Takahashi and Umezawa [7][8][62]. The same conclusions have been drawn from studies in superoperator formalism. The superoperator formalism makes use of the fact that the linear operators acting on a vector space themselves form a linear vector space. This vector space formed by the operators is called the Liouville space. The density operator of a physical system (pure or mixed) can be represented as a vector in Liouville space. Operators that act on the vectors in Liouville space are called superoperators [63].

Takahashi and Umezawa set out to construct a temperature dependent state $|0(\beta)>$ to represent a statistical system in equilibrium at finite temperature with a canonical distribution. The aim was to write the thermal average of observable A which is given as

$$< A >= \frac{Tr\rho A}{Tr\rho} \tag{1.3}$$

as

$$< A >=< 0(\beta)|A|0(\beta) > . \tag{1.4}$$

The density operator ρ has the form $e^{-\beta H}$ where H is the Hamiltonian. Consider the set of eigenstates $|n>$ of H. They form a complete, orthonormal set

$$\begin{aligned} H|n> &= E_n|n> \ , \ \sum\nolimits_n |n><n| = 1 \ , \\ < m|n > &= \delta_{mn} \end{aligned} \tag{1.5}$$

The Hilbert space $\mathcal{H}$ consists of $|n>$. The density operator is written in terms of these states as

$$\rho = \sum \rho_{mn}|m><n| \tag{1.6}$$

Let the state $|0(\beta)>$ which represents the operator ρ be expanded in the same space $\mathcal{H}$ as

$$|0(\beta)>= \sum O_n|n> . \tag{1.7}$$

It is required that

$$\begin{aligned} < A >=< 0(\beta)|A|0(\beta) > &= \sum {O^*}_m O_n < m|A|n > \\ = \frac{Tr\rho A}{Tr\rho} &= \frac{\sum e^{-\beta E_n} < n|A|n >}{\sum e^{-\beta E_n}} \end{aligned} \tag{1.8}$$

which can be satisfied only if

$${O^*}_m O_n = \delta_{mn} \frac{e^{-\beta E_n}}{Tre^{-\beta H}}. \tag{1.9}$$

The δ_{mn} on the right side shows that the coefficients O_n cannot be mere numbers. They need to be vectors in a space similar to $\mathcal{H}$, which has a unique state $|\tilde{n}>$ corresponding to every state $|n>$ in $\mathcal{H}$. Hence, the states $|\tilde{n}>$ are the eigenstates of a Hamiltonian $\tilde{H}$,

$$\tilde{H}|\tilde{n}>= E_n|\tilde{n}> . \tag{1.10}$$

The states $|\tilde{n}>$ are orthonormal. We now consider the direct product space $\mathcal{H}\bigotimes\tilde{\mathcal{H}}$ consisting of states $|n> \bigotimes |\tilde{m}>$. These are denoted as $|m,\tilde{n}>$. Let $|0(\beta)>$ be constructed in this space as

$$|0(\beta)>=[Tr\rho]^{-1/2}\sum_n e^{-\beta E_n/2}|n,\tilde{n}> . \tag{1.11}$$

For any operator A acting only on the physical space $\mathcal{H}$, we have

$$\begin{aligned} <0(\beta)|A|0(\beta> &= [Tr\rho]^{-1}\sum\sum e^{-\beta(E_m+E_n)/2} < m|A|n >< \tilde{m}|\tilde{n}> \\ &= [Tr\rho]^{-1}\sum\sum \delta_{mn} e^{-\beta(E_m+E_n)/2} < m|A|n >< \tilde{m}|\tilde{n}> \\ &= [Tr\rho]^{-1}Tr[\rho A] =< A > . \end{aligned} \tag{1.12}$$

There is another way of showing how the doubling comes about using the Liouville space concept. It is based on the observation that the operators acting on the vector space $\mathcal{H}$ themselves form a vector space $\tilde{\mathcal{H}}$. Any operator A which acts on $\mathcal{H}$ having the form

$$A=\sum A_{mn}|m >< n|$$

can be treated as a vector $|A>$ in the space spanned by the basis $|m >< n|$. A scalar product of two such vectors $|A>$ and $|B>$ in this space is defined as

$$< A|B >= TrA^{\dagger}B = \sum_n < n|A^{\dagger}B|n > . \tag{1.13}$$

With such a scalar product, it is natural now to treat the density operator as a state in the Liouville space. For a density operator with the canonical form

$$\rho=\sum e^{-\beta E_n}|n >< n|$$

the scalar product of states $|A>$ and $|\rho>$ is

$$< \rho|A >= Tr\rho A =< A >$$

. Thus, the scalar product in the Liouville space is suitably defined in order that the thermal averages be represented as a scalar product. The orthonormality of the basis states $|m><n|$ of the Liouville space follows from the definition of the scalar product. When the states $|m><n|$ belonging to $\bar{\mathcal{H}}$ are identified with the states $|m,\tilde{n}>$ of the doubled Hilbert space of TFD, the two approaches — Liouville space formalism and TFD — are essentially the same.

From (1.11) $|\rho>$ can be written as

$$|0(\beta)>=\sum e^{-\beta E_n/2}|n,\tilde{n}>=\rho^{1/2}|I> \tag{1.14}$$

for a normalized density operator. Here, ρ is the diagonal matrix with elements $e^{-\beta E_n}$ representing the density operator in the the basis of the eigenstates of H. The state $|I>$ is

$$|I>=\sum|n,\tilde{n}>\,. \tag{1.15}$$

It is to be noted that it does not have a finite norm. But this vector proves to be useful in defining most of the states $|\rho>$ that are dealt with here. It has the property that it is independent of basis vectors. A change of basis through a unitary transformation U leaves its form unchanged. In terms of

$$|N_i>=\sum_j U_{ij}|n_j> \tag{1.16}$$

we have

$$\begin{aligned} |I> &= \sum\nolimits_i |n_i><n_i| = \sum_i \sum_{jk} U_{ji}U^*_{ki}|N_j><N_k| \\ &= \sum_{jk}\delta_{jk}|N_j><N_k| \\ &= \sum_j |N_j><N_j| \end{aligned} \tag{1.17}$$

where we have used the unitarity condition

$$\sum U_{ji} U^*_{ki} = \sum U_{ji} U^\dagger{}_{ik} = \delta_{jk}. \tag{1.18}$$

Here it is possible to generalize the definition of the thermal state

$$|0(\beta)> = \sum_n e^{-\beta E_n/2} |n, \tilde{n}> = \rho^{1/2} |I> \tag{1.19}$$

to

$$|\rho^\alpha> = \sum_n e^{-\beta E_n \alpha} |n, \tilde{n}> = \rho^\alpha |I> \tag{1.20}$$

where $1/2 \leq \alpha \leq 1$. Correspondingly we have for the bra state

$$< \rho^{1-\alpha}| = \sum_n < n, \tilde{n}| e^{-\beta E_n (1-\alpha)} = < I| \rho^{1-\alpha} \tag{1.21}$$

so that the thermal expectation value is

$$< A > = < \rho^{1-\alpha} | A | \rho^\alpha > . \tag{1.22}$$

This generalization of the thermal state is easily justified by using the cyclic property of trace operation, since

$$< A > = Tr \rho A = Tr \rho^{1-\alpha} A \rho^\alpha. \tag{1.23}$$

The use of $|\rho^\alpha >$ and $< \rho^{1-\alpha}|$ as defined in (1.20) and (1.21) corresponds to use of (1.23).

Using the fact that the vector $|I >$ retains its form under unitary transformations, the thermal state $|\rho^\alpha >$ in any basis can be obtained by applying ρ^α to $|I >$. The nessecity of constructing the asymmetric states can be understood when we apply TFD to non-equilibrium systems, where the $\alpha = 0$ and the $\alpha = 1$ representations are needed. The symmetric state which occur for $\alpha = \frac{1}{2}$ is the 'thermal vacuum' $|0(\beta) >$ mentioned at the beginning of this section, and which was used in the early TFD works.

Let us now consider the simple example of free oscillators, both bosonic and fermionic. We first give the 'tilde conjugation' rules— a set of axioms relating operators $A \otimes 1$ acting

on the first entry in $\mathcal{H} \bigotimes \tilde{\mathcal{H}}$ to the operators $\tilde{A} = 1 \bigotimes A$ which act on the second entry of $\mathcal{H} \bigotimes \tilde{\mathcal{H}}$. These rules are easily justified by looking at the Liouville space operators. After that, the thermal vacuum for the oscillators is constructed. It is shown that for the symmetric $\alpha = 1/2$ case, the thermal vacuum is related to the vacuum at zero temperature , $|0, \tilde{0}>$ by a unitary transformation.

Consider a bosonic system with creation and annihilation operators $a^\dagger$ and a which satisfy

$$[a, a^\dagger] = 1. \tag{1.24}$$

The space $\mathcal{H}$ consists of the number states $|n>$ generated by the cyclic action of $a^\dagger$ as

$$|n>= \frac{a^{\dagger n}}{\sqrt{n!}}|0> \tag{1.25}$$

and form a complete orthonormal set. By looking at the action of the operators a and $a^\dagger$ on states $|m><n|$ of the Liouville space from either side we can identify the operators which act on the second entry of the doublet $|m, \tilde{n}>$. We have

$$a|m><n| = \sqrt{m}|m-1><n| \tag{1.26}$$

and the right hand side corresponds to $\sqrt{m}|m-1, \tilde{n}>$. Similarly,

$$a^\dagger|m><n| = \sqrt{m+1}|m+1><n| = \sqrt{m+1}|m+1, \tilde{n}> \tag{1.27}$$

$$|m><n|a = \sqrt{n+1}|m><n+1| = \sqrt{n+1}|m, (\tilde{n}+1)> \tag{1.28}$$

$$|m><n|a^\dagger = \sqrt{n}|m><n-1| = \sqrt{n}|m, (\tilde{n}-1)> . \tag{1.29}$$

Therefore, the operators a and $a^\dagger$ acting on $|m><n|$ from the right correspond to $\tilde{a}^\dagger$ and $\tilde{a}$, the creation and annihilation operators for the tilde mode. We define

$$\begin{aligned} \tilde{a}|m, \tilde{n}> &= \sqrt{n}|m, (\tilde{n}-1)> \\ \tilde{a}^\dagger|m, \tilde{n}> &= \sqrt{n+1}|m, (\tilde{n}+1)> . \end{aligned} \tag{1.30}$$

The tilde operators $\tilde{a}, \tilde{a}^\dagger$ satisfy the bosonic commutation relations

$$[\tilde{a}, \tilde{a}^\dagger] = 1$$

and commute with the non-tilde operators

$$[a, \tilde{a}] = 0 \quad [a, \tilde{a}^\dagger] = 0$$

. The states $|\tilde{n}>$ can be generated from $|\tilde{0}>$ as in (1.25). The state $|I>$ for the bosonic oscillator can be written in terms of the number states as

$$\begin{aligned} |I> &= \sum\nolimits_n |n, \tilde{n}> \\ &= \sum\nolimits_n \frac{a^{\dagger n}\tilde{a}^{\dagger n}}{n!}|0, \tilde{0}> \\ &= \exp(a^\dagger \tilde{a}^\dagger)|0, \tilde{0}> . \end{aligned} \tag{1.31}$$

From this we have

$$\begin{aligned} a|I> &= \sum_{n=1}^{\infty} \sqrt{n}|(n-1), \tilde{n}> \\ \tilde{a}^\dagger|I> &= \sum\nolimits_{n=1} \sqrt{n}|(n-1), \tilde{n}> \end{aligned} \tag{1.32}$$

The identity

$$a|I> = \tilde{a}^\dagger|I> \tag{1.33}$$

and its hermitian conjugate are important and will be made use of frequently. A similar result holds for any operator A consisting of powers of a and $a^\dagger$. Let

$$A(a, a^\dagger) = \sum_{pq} \alpha_{pq} a^p a^{\dagger q} \tag{1.34}$$

Here α_{pq} are complex coeffients. From (1.33) we get

$$A(a, a^\dagger)|I> = \sum_{pq} \alpha_{pq} \tilde{a}^q \tilde{a}^{\dagger p}|I> \tag{1.35}$$

If we define

$$\tilde{A}(\tilde{a},\tilde{a}^\dagger) = \sum_{pq} \alpha^*_{pq} \tilde{a}^p \tilde{a}^{\dagger q}$$

then (1.35) becomes

$$A(a,a^\dagger)|I>= \tilde{A}^\dagger(\tilde{a},\tilde{a}^\dagger)|I> \tag{1.36}$$

This is called the tilde conjugation rule. For arbitrary operators A and B the following axioms are assumed to hold

$$(\tilde{\tilde{A}}) = A \tag{1.37}$$

$$(AB)^\sim = \tilde{A}\tilde{B} \tag{1.38}$$

$$(C_1A + C_2B)^\sim = C^*{}_1\tilde{A} + C^*{}_2\tilde{B} \tag{1.39}$$

where C_1, C_2 are complex numbers.

1.3 The Construction of $|\rho>$ for Bosonic and Fermionic Oscillators

We now construct $|\rho>$ for the bosonic and fermionic oscillators using the rule

$$|\rho^\alpha>= \rho^\alpha|I> .$$

The normalized canonical density operator for a bosonic oscillator with the hamiltonian

$$H = \omega a^\dagger a$$

is

$$\rho = (1 - e^{-\beta\omega}) \exp[-\beta a^\dagger a] \tag{1.40}$$

at temperature T. ($\beta = 1/kT$). The expectation value of the number operator is

$$\begin{aligned} n =< a^\dagger a > &= Tr\rho a^\dagger a \\ &= \frac{e^{-\beta\omega}}{1 - e^{-\beta\omega}}. \end{aligned} \tag{1.41}$$

This can be written in terms of the quantity $f = e^{-\beta\omega}$ which was introduced earlier as

$$n = \frac{f}{1-f}. \tag{1.42}$$

By inverting it we have

$$f = \frac{n}{1+n} \tag{1.43}$$

and we get

$$\rho = (1-f)\exp(-\beta\omega a^\dagger a). \tag{1.44}$$

Now, the state $|\rho^\alpha>$ is given by

$$\begin{aligned} |\rho^\alpha> &= \rho^\alpha |I> \\ &= (1-f)^\alpha (\exp -\beta\omega\alpha a^\dagger a)|I> \\ &= (1-f)^\alpha \sum_n e^{-\beta\omega n\alpha}|n,\tilde{n}> \\ &= (1-f)^\alpha \sum_n e^{-\beta\omega n\alpha}\frac{a^{\dagger n}\tilde{a}^{\dagger n}}{n!}|0,\tilde{0}> \\ &= (1-f)^\alpha \exp(f^\alpha a^\dagger \tilde{a}^\dagger)|0,\tilde{0}>. \end{aligned} \tag{1.45}$$

We can show that the expression (1.45) can be written in the form

$$|\rho^\alpha> = \frac{(1-f)^\alpha}{(1-f^{2\alpha})^{1/2}} \exp -iG_B|0,\tilde{0}> \tag{1.46}$$

where

$$-iG_B = \theta(a^\dagger\tilde{a}^\dagger - a\tilde{a}) \tag{1.47}$$

and

$$\tanh\theta = e^{-\beta} = f. \tag{1.48}$$

The equivalence of (1.45) and (1.46) can be shown using the disentangling theorem for SU(1,1). A realization of SU(1,1) in terms of a, $\tilde{a}$ etc. is given as [19-20]

$$\begin{aligned} K_+ &= a^\dagger\tilde{a}^\dagger \quad K_3 = \frac{1}{2}(a^\dagger a + \tilde{a}^\dagger\tilde{a} + 1) \\ K_- &= a\tilde{a} \quad K_0 = (a^\dagger a - \tilde{a}^\dagger\tilde{a}). \end{aligned} \tag{1.49}$$

The disentangling theorem enables operators like $\exp(\gamma_+ K_+ + \gamma_3 K_3 + \gamma_- K_-)$ to be written as products of exponentials as

$$\exp(\gamma_+ K_+ + \gamma_3 K_3 + \gamma_- K_-) = \exp(\Gamma_+ K_+) \exp(\ln \Gamma_3 K_3) \exp(\Gamma_- K_-) \tag{1.50}$$

where

$$\Gamma_\pm = \frac{2\gamma_\pm \sinh \phi}{2\phi \cosh \phi - \gamma_3 \sinh \phi)} \tag{1.51}$$

$$\Gamma_3 = \frac{1}{(\cosh \phi - (\gamma_3/2\phi) \sinh \phi)^2} \tag{1.52}$$

$$\phi^2 = \frac{1}{4}({\gamma_3}^2 - \gamma_+ \gamma_-). \tag{1.53}$$

Notice that for $\alpha = 1/2$ (1.46) becomes

$$|0(\beta)> = |\rho^{1/2}> = \exp -iG_B |0, \tilde{0}> . \tag{1.54}$$

Since G_B is anti-hermitian, the thermal vacuum $|\rho^{1/2}>$ is related to $|0, \tilde{0}>$ by a unitary transformation $\exp -iG_B$. We also note that the form of $|\rho^\alpha>$ as given in (1.54) is similar to a two mode squeezed state- the Caves-Schumaker state of quantum optics.[59]

A similar construction goes through for a fermionic oscillator with the hamiltonian

$$H = \omega a^\dagger a$$

where the creation and annihilation operators obey the anti-commutation relations

$$\{a, a^\dagger\} = 1, \quad a^2 = 0 \quad {a^\dagger}^2 = 0. \tag{1.55}$$

The Fock space consists of states $|0>$ and $|1>$. The normalized canonical density operator is

$$\rho = (1 + e^{-\beta})^{-1} \sum_{n=0}^{1} e^{-\beta a^\dagger a} |n><n|$$

Writing the number states as $|0>$ and $a^\dagger|0>$ and identifying the states $|n><n|$ with states in the doubled space we have

$$\begin{aligned} |\rho^\alpha> &= \rho^\alpha |I> \\ &= (1+f)^{-\alpha} \sum_n e^{-\beta\omega n\alpha} \frac{a^{\dagger n}\tilde{a}^{\dagger n}}{n!}|0,\tilde{0}> . \end{aligned} \tag{1.56}$$

The tildean operators are also fermionic

$$\{\tilde{a}, \tilde{a}^\dagger\} = 1, \quad \tilde{a}^2 = 0 \quad \tilde{a}^{\dagger 2} = 0$$

and anti-commute with nontildean operators

$$\{a, \tilde{a}\} = \{a, \tilde{a}^\dagger\} = 0 .$$

Equation (56) can be written

$$|\rho^\alpha> = (1+f)^{-\alpha} \exp(f^\alpha a^\dagger \tilde{a}^\dagger)|0,\tilde{0}> . \tag{1.57}$$

As for the bosonic case, we can rewrite (1.57) as

$$|\rho^\alpha> = \frac{(1+f^{2\alpha})^{1/2}}{(1+f)^\alpha} \exp -iG_F|0,\tilde{0}> \tag{1.58}$$

where $-iG_F$ is a hermitian object

$$-iG_F = \theta(\beta)(a\tilde{a} + a^\dagger\tilde{a}^\dagger)$$

where $\tan\theta(\beta) = e^{-\beta\omega} = f$. This follows again from the use of the disentangling theorem for SU(1,1). In terms of the fermionic operators we have the realization

$$\begin{aligned} K_+ &= a^\dagger\tilde{a}^\dagger \quad K_3 = \frac{1}{2}(a^\dagger a + \tilde{a}^\dagger\tilde{a} + 1) \\ K_- &= a\tilde{a} \quad K_0 = (a^\dagger a + \tilde{a}^\dagger\tilde{a}). \end{aligned} \tag{1.59}$$

G_B is of the form $i\theta(\beta)(K_+ - K_-)$ while G_F is of the form $i\theta(\beta)(K_+ + K_-)$. Because of this, the factor ϕ in (1.53) differs by a factor of i for the bosonic and the fermionic cases , the latter being imaginary. The $\alpha = 1/2$ case is special as in the bosonic case,

$$|0(\beta)>= |\rho^{1/2}>= \exp -iG_F|0,\tilde{0}> . \tag{1.60}$$

$|0(\beta)>$ is again related to $|0,\tilde{0}>$ by a unitary transformation.

1.4 The Thermal Operators ξ and $\xi^{\dagger\dagger}$

In both examples above, the thermal state has the general form

$$|\rho^\alpha >= (Tr\rho)^{-1} \exp(f^\alpha a^\dagger \tilde{a}^\dagger)|0,\tilde{0}> \ .$$

This form of $|\rho^\alpha>$ ensures the existence of the 'thermal operators' ξ and $\xi^{\dagger\dagger}$ which annihilate $|\rho^\alpha>$ and $<\rho^{1-\alpha}|$ acting from the left and the right respectively.In the symmetric $\alpha = 1/2$ case these are hermitian adjoints of each other and are denoted by $a(\beta)$ and $a(\beta)^\dagger$ — the thermal creation and annihilation operators. Though not the adjoints of each other, ξ and $\xi^{\dagger\dagger}$ still obey the canonical commutation relations. We find that the ξ's are related to the a's through a Bogoliubov or a Bogoliubov-like transformation (the difference between the two will be pointed out later). The form of ξ and $\xi^{\dagger\dagger}$ are worked out below [7][67] starting from the fact that $a, \tilde{a}$ annihilate the vacuum $|0,\tilde{0}>$.

$$a|0,\tilde{0}>= 0 \quad \tilde{a}|0,\tilde{a}>= 0 \tag{1.61}$$

From this it follows

$$\begin{aligned} \{\exp(f^\alpha a^\dagger \tilde{a}^\dagger) a \exp -(f^\alpha a^\dagger \tilde{a}^\dagger)\} \exp(f^\alpha a^\dagger \tilde{a}^\dagger)|0,\tilde{0}> &= 0 \\ \exp(f^\alpha a^\dagger \tilde{a}^\dagger) a \exp -(f^\alpha a^\dagger \tilde{a}^\dagger)|\rho^\alpha> &= 0 \\ (a - f^\alpha \tilde{a}^\dagger)|\rho^\alpha> &= 0 \end{aligned} \tag{1.62}$$

where we have used the operator identity

$$\exp(A)B\exp -(A) = B + [A, B] + \frac{1}{2}[A, [A, B]] + ... \tag{1.63}$$

Similarly, starting from the second equation of (1.61) we get

$$(\tilde{a} - f^{\alpha}a^{\dagger})|\rho^{\alpha} >= 0. \tag{1.64}$$

Also, starting from the set

$$< 0, \tilde{0}|a^{\dagger} = 0, \quad < 0, \tilde{0}|\tilde{a}^{\dagger} = 0 \tag{1.65}$$

we get

$$\begin{aligned} < \rho^{1-\alpha}|(a^{\dagger} - f^{1-\alpha}\tilde{a}) &= 0 \\ < \rho^{1-\alpha}|(\tilde{a}^{\dagger} - f^{1-\alpha}a) &= 0\ . \end{aligned} \tag{1.66}$$

By identifying the operators in (1.62,64,66) with ξ, $\tilde{\xi}$, $\xi^{\dagger\dagger}$, and $\tilde{\xi}^{\dagger\dagger}$ we write

$$\begin{aligned} \xi|\rho^{\alpha} >= 0 \quad \tilde{\xi}|\rho^{\alpha} > &= 0 \\ < \rho^{1-\alpha}|\xi^{\dagger\dagger} = 0 \quad < \rho^{1-\alpha}|\tilde{\xi}^{\dagger\dagger} &= 0. \end{aligned} \tag{1.67}$$

These are called the 'thermal state conditions'. These conditions completely characterize the thermal state. When we go from the quantum mechanical systems to field theories with infinite degrees of freedom, the transformation (1.46) defining $|\rho^{\alpha} >$ may not exist in a mathematically rigorous sense. The thermal state is then defined only by the existence of the ξs.

The ξs need to be normalized so that the commutation relations are preserved. We require

$$[\xi, \xi^{\dagger\dagger}] = 1, \quad [\tilde{\xi}, \tilde{\xi}^{\dagger\dagger}] = 1$$

Let the normalized ξs be

$$\begin{aligned} \xi &= \frac{1}{N_1}(a - f^{\alpha}\tilde{a}^{\dagger}), \quad \tilde{\xi} = \frac{1}{N_1}(\tilde{a} - f^{\alpha}a^{\dagger}) \\ \xi^{\dagger\dagger} &= \frac{1}{N_2}(a^{\dagger} - f^{1-\alpha}\tilde{a}), \tilde{\xi}^{\dagger\dagger} = \frac{1}{N_2}(\tilde{a}^{\dagger} - f^{1-\alpha}a). \end{aligned} \tag{1.68}$$

We can either set $N_1 = N_2 = (1-f)^{1/2}$ or $N_1 = (1-f)^{-\alpha}$, $N_2 = (1-f)^{-(1-\alpha)}$. In the $\alpha = 1/2$ case the transformations are Bogoliubov transformations, that is, transformations of the type

$$\xi = ua - v\tilde{a}^{\dagger}$$

where $(u^2 - v^2) = 1$. Noting that $\tanh\theta(\beta) = f$ we have

$$\begin{aligned} a(\beta) &= \cosh\theta(\beta)\, a - \sinh\theta(\beta)\tilde{a}^{\dagger} \\ \tilde{a}(\beta) &= \cosh\theta(\beta)\, \tilde{a} - \sinh\theta(\beta)a^{\dagger} \end{aligned} \tag{1.69}$$

In the case of $\alpha \neq 1/2$ for either normalization,the condition $u^2 - v^2 = 1$ does not hold and in these cases by a Bogoliubov transformation we mean a transformation which preserves the commutation relation.

We now introduce what is called the doublet notation for the operators a, $a^{\dagger}$, $\tilde{a}$, and $\tilde{a}^{\dagger}$. It is widely used in TFD literature and allows the Bogoliubov transformations to be written easily. The column matrix a has elements $a^1 = a$, $a^2 = \tilde{a}^{\dagger}$ and the row matrix $\bar{a}$ has elements $\bar{a}^1 = a^{\dagger}$, $\bar{a}^2 = -\tilde{a}$. Similarly for the ξ's and the fields.

1.5 Thermal State for Para and Deformed Oscillators

The formalism developed in the previous section can be applied to para and deformed oscillators too.The spectrum in parabose and deformed cases has the same form as that of a bosonic oscillator, consisting of infinite number of equally spaced states. The parafermi oscillator has a finite number of equally spaced states. These oscillators differ from

the bose and fermi oscillators by not having a simple bilinear commutation or anti-commutation relations between the operators a and $a^\dagger$. The occurrence of a parameter in the commutation relations and the hamiltonian is another feature. The following commutation relations hold for all oscillators:

$$[a, N] = a \qquad [a^\dagger, N] = -a^\dagger.$$

In case of deformed oscillator the number operator turns out to be a power series in $(a^\dagger a)$. Since the spectrum is similar to that of a harmonic oscillator the density operator has the same form. But when transcribed in TFD the thermal state cannot be written in the form of (1.46) or (1.54). This is because the number states are of the form $|n>= \frac{a^{\dagger n}}{[n]!}$ where $[n]$ is some factorial like function of the integer n which contains the parameters p or q. Further, there is no realization of SU(1,1) algebra in terms of para or deformed operators $a, a^\dagger$. There does not seem to be a unitary operator of the form (1.46,54). Neither can we construct thermal operators related to $a, a^\dagger$ as in (1.69). This was the conclusion drawn by the earlier workers [32]. There does not seem to be any point in using TFD for the para and deformed systems since none of its useful features remain.

But it is possible to overcome these limitations and obtain a unitary operator and Bogoliubov transformations for parabose and deformed oscillators. For a parafermi oscillator the unitary operator exists only in the limit $p \to \infty$ It is shown below that the use of a non-linear realization of para and deformed oscillators in terms of bosonic operators makes this possible. The unitary operator and the Bogoliubov transformation apply to the bosonic operators in terms of which the para and deformed algebras are realized.

The parabose oscillator is defined by the relations

$$[a, N] = a \quad [a^\dagger, N] = -a^\dagger \quad N = \frac{1}{2}\{a, a^\dagger\}. \tag{1.70}$$

It follows that the eigenvalues of N are positive and that the successive eigenvalues differ by unity. The application of a and $a^\dagger$ to the eigenstate of N takes the state to a lower

and a higher state respectively. The vacuum state is assumed to satisfy

$$a|0>=0 \qquad aa^{\dagger}=p|0> \tag{1.71}$$

where the real parameter p is the order of the parabose oscillator. The higher states are generated as

$$|n>=(Norm)(a^{\dagger})^{n}|n> \tag{1.72}$$

To find the norm $<0|a^{n}(a^{\dagger})^{n}|0>$, the trilinear commutations

$$a^{2}a^{\dagger}-a^{\dagger}a^{2}=2a, \;\; aa^{\dagger 2}-a^{\dagger 2}a=2a^{\dagger} \tag{1.73}$$

are used repeatedly along with equation (1.71). The normalized even and odd states are

$$|2n> \;=\; \frac{a^{\dagger 2n}}{[2^{n}n!p(p+2)\cdots(p+2n-2)]^{1/2}}|0> \tag{1.74}$$

$$|2n+1> \;=\; \frac{a^{\dagger 2n+1}}{[2^{n}n!p(p+2)\cdots(p+2n)]^{1/2}}|0> \tag{1.75}$$

From this it follows that the action of $a^{\dagger}$ is as

$$a^{\dagger}|2n> \;=\; (p+2n)^{1/2}|2n+1> \tag{1.76}$$

$$a^{\dagger}|2n+1> \;=\; [2(n+1)]^{1/2}|2n+2> \tag{1.77}$$

and that of a is

$$a|2n> \;=\; (2n)^{1/2}|2n-1> \tag{1.78}$$

$$a|2n+1> \;=\; (p+2n)^{1/2}|2n> \tag{1.79}$$

or,

$$a|n>=n+\frac{1}{2}[1-(-1)^{n}](p-1) \tag{1.80}$$

The deformation of the Heisenberg algebra $aa^{\dagger}-a^{\dagger}a=1$ has been drawing a lot of interest currently [37-43]. One such deformation is of the form

$$aa^{\dagger}-qa^{\dagger}a=1 \tag{1.81}$$

The results obtained here also apply to the various other forms of deformation. A number operator N for this algebra can be constructed [68,69] as a power series

$$N = \sum_{n=1}^{\infty} \frac{(1-q)^n}{(1-q^n)} a^{\dagger n} a^n. \tag{1.82}$$

It has positive integer eigenvalues. The quantity $< 0|a^n a^{\dagger n}|0 >$ which is used to normalize the number states is found by the repeated application of (1.81).It is found to be

$$\begin{aligned} < 0|a^n a^{\dagger n}|0 > &= (1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1}) \\ &= \frac{(1-q^n)(1-q^{n-1}\cdots(1-q^2)}{(1-q)^n} \\ &= [n]_q! \end{aligned} \tag{1.83}$$

where we have used the notation

$$[n]_q = \frac{(1-q^n)}{(1-q)} \quad [n]_q! = n_q[n-1]_q \cdots [1]_q. \tag{1.84}$$

The normalized number states and the action of a and $a^\dagger$ on them is

$$\begin{aligned} |n> &= ([n]_q!)^{1/2}|0> \\ a|n> &= \sqrt{[n]_q}|n-1> \qquad a^\dagger|n> = \sqrt{[n+1]_q}|n+1> . \end{aligned} \tag{1.85}$$

We can summarize the results for parabose and deformed oscillators by noting that the action of the creation and annihilation operators on the number states is as

$$a|n> = \sqrt{f(n)}|n-1> \qquad a^\dagger|n> = \sqrt{f(n+1)}|n+1> . \tag{1.86}$$

with $f(n)$ being given by (1.80) or (1.85). From these relations it follows that

$$aa^\dagger|n> = f(n+1)|n>, \quad a^\dagger a|n> = f(n)|n> \tag{1.87}$$

and the following operator identity holds on the Fock space

$$aa^\dagger = f(N+1), \quad a^\dagger a = f(N). \tag{1.88}$$

Note that for any operator $g(N)$ which is a function of N, the operator identities

$$ag(N) = g(N+1)a, \quad g(N)a^\dagger = a^\dagger g(N+1) \tag{1.89}$$

hold on the Fock space. We now define [69][37-39]

$$\begin{aligned} b &= \sqrt{\frac{N+1}{f(N+1)}}a = a\sqrt{\frac{N}{f(N)}} \\ b^\dagger &= a^\dagger\sqrt{\frac{N+1}{f(N+1)}} = \sqrt{\frac{N}{f(N)}}a^\dagger \end{aligned} \tag{1.90}$$

By looking at the action of these operators on the number states we see that

$$bb^\dagger = N+1, \quad b^\dagger b = N, \tag{1.91}$$

and hence

$$[b, b^\dagger] = 1. \tag{1.92}$$

We note that the expression for the number operator in of terms of b and $b^\dagger$ is

$$N = b^\dagger b \tag{1.93}$$

which is the usual expression for bosons. Also, the vacuum $|0>$ is annihilated by b.The relations (1.90) are invertible

$$\begin{aligned} a &= \sqrt{\frac{f(N+1)}{(N+1)}}b = \sqrt{\frac{f(b^\dagger b+1)}{(b^\dagger b+1)}}b \\ a^\dagger &= b^\dagger\sqrt{\frac{f(N+1)}{(N+1)}} = b^\dagger\sqrt{\frac{f(b^\dagger b+1)}{(b^\dagger b+1)}}. \end{aligned} \tag{1.94}$$

This is a realization of para and deformed oscillators as a non-linear function of the bosonic operators. This will now be used in treating the oscillators at finite temperature in TFD. The hamiltonian for these systems is defined as

$$H = \omega N$$

($\hbar = 1$) in analogy with the ordinary oscillator. ω is set equal to unity The normalized density operator is

$$\rho = \frac{e^{-\beta H}}{Tr\rho} = (1 - e^{-\beta}) \exp(-\beta N) \tag{1.95}$$

which becomes, on using (1.93)

$$\rho = (1 - e^{-\beta}) \exp(-\beta b^\dagger b). \tag{1.96}$$

.

It is now straight forward to double the Hilbert space and construct a thermal state which has the form of a thermal coherent state of (1.54). We have

$$|0(\beta)> = \rho^{1/2}|I> = (1 - e^{-\beta})^{1/2} \exp(-\frac{\beta}{2} b^\dagger b)|I> \tag{1.97}$$

$$= (1 - e^{-\beta})^{1/2} \exp(e^{-\frac{\beta}{2}} b^\dagger \tilde{b}^\dagger)|0, \tilde{0}> \tag{1.98}$$

Since the operators b, $b^\dagger$ are bosonic, there exists a realization of SU(1,1) as in (1.49) and (1.98) can be written as

$$|0(\beta)> = \exp -\theta(\beta)[b^\dagger \tilde{b}^\dagger - b\tilde{b}]|0, \tilde{0}> \tag{1.99}$$

as before. This proves the existence of a unitary operator for the para and deformed oscillators too. The unitary operator can be written in terms of $a, a^\dagger, \check{a}, \check{a}^\dagger$, where $\check{a}, \check{a}^\dagger$ are related to $\check{b}, \check{b}^\dagger$ as in (1.94) and the as commute with $\check{a}$s.

The single mode parafermi oscillator is defined by the relations

$$[a, N] = a, \quad [a^\dagger, N] = -a^\dagger \tag{1.100}$$

$$N = \frac{1}{2}[a, a^\dagger] \tag{1.101}$$

along with the assumption

$$(a)^{p+1} = (a^\dagger)^{p+1} = 0. \tag{1.102}$$

p is the order of the parafermi oscillator. (1.100,101) define the SU(2) algebra. $a,a^\dagger$ and N correspond to the familiar J_-, J_+, andJ_z operators. The nature of eigenstates is well known. The eigenstates are of the form

$$-\frac{p}{2}, -\frac{p}{2}+1, \cdots \frac{p}{2}-1, \frac{p}{2},$$

p being an integer. For a given p the representation is $(p+1)$ dimensional. The operator

$$N = N + \frac{p}{2} \tag{1.103}$$

has integer eigenvalues $0, 1, \cdots p$ and is used to define the hamiltonian $H = N$. The normalized eigenstates of N are generated from the vacuum satisfying

$$a|0>=0, \quad aa^\dagger|0>=p|0> \tag{1.104}$$

by repeated application of $a^\dagger$. The norm can be found by the repeated application of the commutation relations (1.93) and the conditions (1.102,104). The eigenstates are

$$|n>=\frac{(a^\dagger)^n}{[n!p(p-1)\cdots(p-n-1)]^{1/2}}|0> . \tag{1.105}$$

The action of $a^\dagger$ and a is as

$$\begin{aligned} a^\dagger|n> &= [(n+1)(p-n)]^{1/2}|n+1>=\sqrt{f(n+1)}|n+1> \\ a|n> &= [n(p-n+1)]^{1/2}|n-1>=\sqrt{f(n)}|n+1> \end{aligned} \tag{1.106}$$

The non-linear realization of parafermi oscillator in terms of boson like operators is refered to as the Holstein-Primakoff transformation [35]. The form of the transformation is the same as in (1.94) with $f(N)$ given by

$$f(N) = N(p-N+1).$$

We have

$$b = a(p-N+1)^{-1/2}, \quad b^\dagger = (p-N+1)^{-1/2}a^\dagger \tag{1.107}$$

The action of $bb^\dagger$ and $b^\dagger b$ on the number states is

$$bb^\dagger|n>=(n+1)|n>, \quad b^\dagger b|n>=n|n>$$

Eventhough the matrix element $< m|[b,b^\dagger]|n >$ is $\delta_{m,n}$ we cannot have the operator identity $[b,b^\dagger]=1$ since the vector space has a limited number of states. Using $b|0>=0$ we get

$$\begin{aligned}(bb^\dagger \mp b^\dagger b) &= \sum_{n=0}^{p-1}(n+1)|n><n| \\ &\mp \sum_{n=1}^{p} n|n><n| \end{aligned} \tag{1.108}$$

In the limit $p \to \infty$, the Hilbert space becomes the entire Fock space and we get

$$[b,b^\dagger]=\lim_{n-\infty}\sum_{n=0}^{p-1}|n><n|=1$$

and in the limit $p=1$ we get

$$\{b,b^\dagger\}=|0><0|+|1><1|=1.$$

We note that eventhough the standard commutation relations do not hold for $b,b^\dagger$ the number operator can still be written as $N=b^\dagger b$ and the density opertor is $e^{-\beta b^\dagger b}$. To construct the thermal state we double the truncated Hilbert space and proceed as usual. The normalized density operator is

$$\rho=\left(\frac{1-e^{-(p+1)\beta}}{1-e^{-\beta}}\right)^{-1}\sum_{n=0}^{p}e^{-\beta n}|n><n|. \tag{1.109}$$

The thermal state for $\alpha=1/2$ is

$$|0(\beta)>=(\frac{1-e^{-(p+1)\beta}}{1-e^{-\beta}})^{-1/2}\sum_{n=0}^{p}\frac{(e^{-\beta/2}b^\dagger\tilde{b}^\dagger)^n}{n!}|0,\tilde{0}>. \tag{1.110}$$

The tilde operators are also assumed to satisfy (1.102). We then have

$$|0(\beta)>=\left(\frac{1-e^{(p+1)\beta}}{1-e^{-\beta}}\right)^{-1/2}\exp(e^{-\beta/2})b^\dagger\tilde{b}^\dagger)|0,\tilde{0}>. \tag{1.111}$$

As b, $b^\dagger$ are not true bosons (or fermions) we cannot have a realization of SU(1,1) and hence the thermal state cannot be written in the form of (1.60). The unitary operator and the Bogoliubov transformation do not exist.

Chapter 2

TFD for Equilibrium Systems and the Complex Time Formalisms

2.1 Introduction

The TFD formalism presented in the previous chapter dealt with a single mode. The extension to a field theory is straight forward. In case of interacting fields which are not exactly solvable a perturbation theory is set up. This allows the use of Feynman diagrams. The process is completely analogous to the ground state field theory. A brief summary of the complex time formulations, which also deal with fields at finite temperature is given in this chapter. A comparision is made with TFD. The equivalence of TFD in equilibrium and the complex time methods is proved for any α. So far the equivalence has been pointed out in literature only for the special case of $\alpha = 1/2$ [9] [13-15]. We consider a Schrödinger field in the rest of this chapter. The results also hold for relativistic fields. As is well known the Schrödinger field is used to describe a system of identical non-relativistic particles. The identical particles could be bosons or fermions. The Schrödinger field is described by the classical Lagrangian density

$$\mathcal{L} = i\psi^*\dot{\psi} - \frac{1}{2m}\nabla\psi^*\nabla\psi - V(\vec{x})\psi^*\psi \tag{2.1}$$

and the corresponding Hamiltonian density

$$\mathcal{H} = -\frac{1}{2m}\psi^*\nabla^2\psi + V(x)\psi^*\psi \quad , \tag{2.2}$$

The field ψ is a function of the variables $\vec{x}$ and t.$V(\vec{x})$ is related to the interaction between the particles as shown later.

The classical equation of motion is

$$i\dot{\psi}(\vec{x},t) = -\frac{1}{2m}\nabla^2\psi(\vec{x},t) + V(x)\psi(\vec{x},t). \tag{2.3}$$

The solutions to this equation are denoted as $u_i(\vec{x})e^{-iE_it}$. The field $\psi(\vec{x},t)$ can be expanded as a linear superposition of these solutions

$$\psi(\vec{x},t) = \sum_i a_i u_i(\vec{x})e^{-iE_it} \quad , \tag{2.4}$$

Here, a_i are complex coefficients. When the field $\psi(\vec{x},t)$ is quantized, it is treated as an operator and obeys the Heisenberg equation of motion. The quantization is done by using the equal time commutation relations

$$\begin{aligned} \left[\psi(\vec{x},t), \psi^\dagger(\vec{x}',t)\right] &= \delta(\vec{x}-\vec{x}') \\ [\psi(\vec{x},t), \psi(\vec{x}',t)] = \left[\psi^\dagger(\vec{x},t), \psi^\dagger(\vec{x}',t)\right] &= 0 \end{aligned} \tag{2.5}$$

for bosons and

$$\begin{aligned} \left\{\psi(\vec{x},t), \psi^\dagger(\vec{x}',t)\right\} &= \delta(\vec{x}-\vec{x}') \\ \{\psi(\vec{x},t), \psi(\vec{x}',t)\} = \left\{\psi^\dagger(\vec{x},t), \psi^\dagger(\vec{x}',t)\right\} &= 0 \end{aligned} \tag{2.6}$$

for fermions.

The equation of motion which results from the Heisenberg equation

$$i\frac{\partial\psi(\vec{x},t)}{\partial t} = [H, \psi(\vec{x},t)] \tag{2.7}$$

($\hbar = 1$) is the quantum mechanical version of equation (2.3), with the c-number fields and gradients replaced by the appropriate operators. The quantized field has an expansion

similar to (2.4) with the complex numbers a_i replaced by operators a_i

$$\begin{aligned} \psi(\vec{x},t) &= \sum_i a_i u_i(\vec{x}) e^{-iE_i t} \\ \psi^\dagger(\vec{x},t) &= \sum_i a_i^\dagger u_i^*(x) e^{iE_i t} \end{aligned} \tag{2.8}$$

for a free Schrödinger field, for example, the modes are plane waves and in the infinite volume limit the modes becomes continuous and we have

$$\psi(\vec{x},t) = \int \frac{d^3\vec{k}}{(2\pi)^3} e^{\vec{k}.\vec{x}-i\omega t} a_{\vec{k}}. \tag{2.9}$$

From the commutation relations (2.5) and (2.6) it follows that

$$[a_i, a_j^\dagger] = \delta_{ij} \qquad [a_i, a_j] = [a^\dagger, a_j^\dagger] = 0 \tag{2.10}$$

for bosons and

$$\{a_i, a_j^\dagger\} = \delta_{ij} \quad , \quad \{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0 \tag{2.11}$$

for fermions. In the continuous limit mentioned above the δ_{ij} are to be replaced by $\delta(i-j)$. Using the expansion (2.8) the Hamiltonian can be written as

$$H = \int d^3\vec{x}\, \mathcal{H}(\vec{x}) = \sum_i E_i a_i^\dagger a_i \tag{2.12}$$

for both bosons and fermions. The number operator $N_i = a_i^\dagger a_i$ gives the number of particles with energy E_i. N_i has integer eigenvalues 0 and 1 for fermions. For bosons the eigenvalues of N_i form the set of positive integers. The total number operator $N = \sum_i N_i$ can be written as

$$N = \int \psi^\dagger(\vec{x})\psi(\vec{x})\, d^3\vec{x} \quad . \tag{2.13}$$

The operators N_i commute with each other. Hence they have simultaneous eigenstates $|n_1, n_2, \ldots >$ which are used as basis of the Hilbert space. In terms of creation operators we have

$$|n_1, n_2, \cdots >= \frac{1}{[n_1! n_2! \cdots]^{1/2}} (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} \cdots |0> \quad . \tag{2.14}$$

The vacuum $|0>$ satisfies $a_i|0>=0$ for all a_i.

To make a connection between this second quantized formulation and the usual n body quantum mechanics we define a configuration space wave function

$$|\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n, t> = \frac{1}{\sqrt{n!}}\psi^\dagger(\vec{x}, t)\psi^\dagger(\vec{x}_2, t)\cdots\psi^\dagger(\vec{x}_n, t)|0> \quad . \tag{2.15}$$

It represents n particles localized at positions $\vec{x}_1, \vec{x}_2, \cdots \vec{x}_n$. The scalar product

$$< \vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n, t|n_1, n_2, \cdots, n_n >= \phi^n_{n_1,n_2,\ldots}(\vec{x}_1, \vec{x}_2, \cdots t) \quad , \tag{2.16}$$

gives the probability amplitude of finding $n = n_1 + n_2 \cdots + n_n$ particles at positions $\vec{x}, \vec{x}, \cdots, \vec{x}_n$ at time t when n_1 of them are in state u_1, n_2 in state u_2 and so on. It is this probability amplitude that obeys the usual n body Schrödinger equation

$$i\frac{\partial}{\partial t}\phi^n_{n_1,n_2}(\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n, t) = \sum_{i=1}^{n}\left[-\frac{1}{2m}\nabla_i^2 + V(x_i)\right]\phi^n_{n_1,n_2\ldots}(\vec{x}_1, \cdots, t) \quad . \tag{2.17}$$

It is symmetric under the exchange of $\vec{x}_i$ and $\vec{x}_j$ for bosons and satisfies the exclusion principle for fermions.

In case of interacting many body problems which are not exactly solvable it is possible to have a perturbation theory using interaction picture. Diagrammatic methods can be then developed. The use of second quantization makes techniques of quantum field theory available for many body systems of condensed matter.

Since temperature comes in naturally in many body systems, the ground state expectation values have to be replaced by thermal averages. The techniques of quantum field theory at zero temperature can be extended to systems at finite temperature in equilibrium. This extension, carried out by Matsubara [2], Umezawa [1], and others [3], is based on the similarity of the statistical factor $e^{-\beta H}$ and the evolution operator e^{-iHt} in quantum mechanics. This suggests that temperature can be treated as an imaginary

time parameter. The possibility of describing quantum statistical systems by continuing the time variable to imaginary times was also made obvious by the path integral approach to quantum mechanics [6].

After a brief survey of TFD applied to free and interacting fields, a review of these complex time methods is given below.

2.2 Free Fields in TFD:

Consider a free Schrödinger field described by the classical Lagrangian density

$$\mathcal{L}_o = \mathcal{L}_o(\psi, \nabla\psi) = i\psi^*\dot{\psi} - \frac{1}{2m}\nabla\psi^*\nabla\psi \ . \tag{2.18}$$

The treatment of a single mode in the previous chapter gives the motivation for doubling the dynamical variables. The thermal vacuum $|0(\beta)>$ is then constructed as before, in the doubled Hilbert space.

Define

$$\tilde{\mathcal{L}}_o = \mathcal{L}_o^*(\tilde{\psi}, \nabla\tilde{\psi}) = -i\tilde{\psi}^*\dot{\tilde{\psi}} - \frac{1}{2m}\nabla\tilde{\psi}\nabla\tilde{\psi}^*. \tag{2.19}$$

The tildian fields $\tilde{\psi}$ are the extra unphysical degrees of freedom required in TFD. $\tilde{\mathcal{L}}_o$ is obtained by replacing the field ψ by $\tilde{\psi}$ in $\mathcal{L}_o$ and taking complex conjugates of all the complex numbers in $\mathcal{L}_o$. This follows from the tilde conjugation rules. The Lagrangian density relevant in TFD is $\hat{\mathcal{L}}_o = \mathcal{L}_o - \tilde{\mathcal{L}}_o$ and it generates the equations of motion for both $\psi(\vec{x}, t)$ and $\tilde{\psi}(\vec{x}, t)$. Similarly we define $\hat{\mathcal{H}}_o = \mathcal{H}_o - \tilde{\mathcal{H}}_o$. The field described by $\hat{H}_o = \int d^3\vec{x}\,\hat{\mathcal{H}}_o$ is quantized by making the fields $\psi(\vec{x}, t)$ and $\tilde{\psi}(\vec{x}, t)$ into operators and assuming the following equal time commutation relations

$$\begin{aligned} [\psi(\vec{x}, t), \psi^\dagger(x', t)] &= \delta(\vec{x} - \vec{x}') \\ [\tilde{\psi}(\vec{x}, t), \tilde{\psi}^\dagger(\vec{x}', t)] &= \delta(\vec{x} - \vec{x}') \end{aligned}$$

$$\begin{aligned}[\psi(x,t),\psi(x',t)] &= [\psi^\dagger(x,t),\psi^\dagger(\vec{x}',t)] \\ &= [\tilde{\psi}^\dagger(\vec{x},t),\tilde{\psi}^\dagger(\vec{x}',t)] = [\tilde{\psi}(\vec{x},t),\tilde{\psi}(\vec{x}',t)] = 0 \quad . \end{aligned} \tag{2.20}$$

All the tilde fields are assumed to commute with all non-tilde fields. For fermions the commutator brackets are replaced by anti-commutator brackets and the tilde and non-tilde fields are assumed to anti-commute. The quantized free fields are expanded using plane wave solutions of the Heisenberg equations

$$\begin{aligned} i\dot{\psi}(\vec{x},t) &= [\psi(\vec{x},t),\hat{H}_o] \\ i\dot{\tilde{\psi}}(\vec{x},t) &= -[\tilde{\psi}(\vec{x},t),\hat{H}_o] \end{aligned} \tag{2.21}$$

as

$$\begin{aligned} \psi(\vec{x},t) &= \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{i\vec{k}.\vec{x}-i\omega t}a_k \\ \tilde{\psi}(\vec{x},t) &= \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{-i\vec{k}.\vec{x}+i\omega t}\tilde{a}_k \quad . \end{aligned} \tag{2.22}$$

The mode label $\vec{k}$ stands as usual for the wave vector and the creation and annihilation operators obey the usual bosonic or fermionic commutation relations. Using (2.22) and the commutation relations the Hamiltonian $\hat{H}$ is written as

$$\hat{H}_o = \sum_{\vec{k}} \omega_k[a_k^\dagger a_k - \tilde{a}_k^\dagger \tilde{a}_k] \quad . \tag{2.23}$$

The equation of the motion for the creation and annihilation operators and their solutions are

$$\begin{aligned} \dot{a}_k = -i\omega_k a_k(t) \quad &, \quad a_k(t) = e^{-i\omega_k t}a_k(0) \\ \dot{\tilde{a}}_k = i\omega_k \tilde{a}_k(t) \quad &, \quad \tilde{a}_k(t) = e^{i\omega_k t}\tilde{a}_k(0) \quad . \end{aligned}$$

It is now easy to construct a thermal vacuum $|0(\beta)>$ and operators $a(\beta),\tilde{a}(\beta)$ that annihilate it.

The thermal state $|\rho^{\alpha}>$ for a field is constructed by generalizing the construction given for the single mode. For $\alpha = \frac{1}{2}$, $|0(\beta)>$ is constructed from the vacuum as

$$|0(\beta)>= e^{-iG_B}|0,\tilde{0}> \tag{2.24}$$

where

$$-iG_B = \sum_{\vec{k}} \theta_k (a_k^{\dagger}\tilde{a}_k^{\dagger} - a_k\tilde{a}_k)$$

and $\tanh\theta_k = e^{-\beta\omega_k}$. The vacuum satisfies

$$a_k|0,\tilde{0}>= \tilde{a}_k|0,\tilde{0}>= 0 \tag{2.25}$$

for all $\vec{k}$. Similarly, the thermal vacuum for fermions is given by

$$|0(\beta)>= e^{-iG_F}|0.\tilde{0}>$$

where

$$-iG_F = \sum_{\vec{k}} \theta_k (\tilde{a}_k a_k - a_k^{\dagger}\tilde{a}_k^{\dagger}) \quad , \quad \tan\theta_k = e^{-\beta\omega_k}.$$

As in case of a single mode (2.25) and (2.24) lead to the thermal operators satisfying

$$a_k(\beta)|0(\beta)>= \quad , \quad \tilde{a}_k(\beta)|0(\beta)>= 0 \ . \tag{2.26}$$

The form of $a_k(\beta)$ and $\tilde{a}_k(\beta)$ is the same as (1.69), except for the mode label.

The annihilation operators for the asymmetric states $|\rho^{\alpha}>$ and $<\rho^{1-\alpha}|$ are respectively ξ_k, $\tilde{\xi}_k$,and $\xi_k^{\dagger\dagger}$, $\tilde{\xi}_k^{\dagger\dagger}$. They are related to the operators $a, \tilde{a}$ etc., through the relation

$$\begin{aligned} \xi_k &= B_k a_k \\ \bar{\xi}_k &= \bar{a}_k B_k^{-1} \end{aligned}$$

using the doublet notation of chapter 1. We notice that $\hat{H}_o$ as given in (2.23) retains its form when written in terms of the thermal operators

$$\begin{aligned} \hat{H}_o &= \sum_{\vec{k}} \omega_k \left\{ a_k^\dagger(\beta) a_k(\beta) - \tilde{a}_k^\dagger(\beta) \tilde{a}_k(\beta) \right\} \\ &= \sum_{\vec{k}} \omega_k \left\{ \xi_k^{\dagger\dagger} \xi_k - \tilde{\xi}_k^{\dagger\dagger} \tilde{\xi}_k \right\} \end{aligned} \tag{2.27}$$

The transformations relating $|0(\beta)$ to $|0,0>$ need further consideration. These transformations are unitary in the case of quantum mechanical systems. In the continuum limit the number of degrees of freedom becomes infinite. It can be shown that under these conditions $|0(\beta)>$ and $|0,\tilde{0}>$ belong to unitarily inequivalent Hilbert spaces [8].

2.3 Interacting Fields:

We now consider the case of interacting fields that cannot be solved exactly. A perturbation theory is set up to calculate quantities like Green's functions. It is necessary as usual, to define an interacting picture in order to set up a perturbation theory.

Let the Hamiltonian of the interacting system be

$$\hat{H} = \hat{H}_o + \hat{H}_I \quad .$$

Here $\hat{H}_o$ is as in (2.23) and $\hat{H}_I = H_I - \tilde{H}_I$.

We define the interaction picture operator $\psi_{ip}(t)$ which evolves according to

$$\begin{aligned} i\dot{\psi}_{ip}(t) &= [\psi_{ip}(t), \hat{H}_o] \\ i\dot{\tilde{\psi}}_{ip}(t) &= -[\tilde{\psi}_{ip}(t), \hat{H}_o] \quad . \end{aligned} \tag{2.28}$$

The interaction picture operators $\psi_{ip}(t)$ and the Heisenberg picture operators $\psi(t)$ are related through unitary transformation

$$\psi_{ip}(t) = e^{iH_o t} e^{-iHt} \psi(t) e^{iHt} e^{-iH_o t}$$

$$= U(t)\psi(t)U^{-1}(t) \quad . \tag{2.29}$$

Similarly

$$\begin{aligned} \tilde{\psi}_{ip}(t) &= e^{-i\tilde{H}_o t} e^{i\tilde{H}t} \tilde{\psi}(t) e^{-i\tilde{H}t} e^{i\tilde{H}_o t} \\ &= \tilde{U}(t)\tilde{\psi}(t)\tilde{U}^{-1}(t) \quad . \end{aligned} \tag{2.30}$$

The explicit form of $U(t)$ given in (2.29) and (2.30) holds only for time independent Hamiltonian. The validity of the perturbation theory on the other hand requires an adiabatic factor in the interaction which gradually switches the interaction on and off.

Corresponding to the definition of the operators in (2.29) and (2.30) we also have interaction picture state $|\phi_{ip}(t)>$ defined in terms of the Schrödinger picture states $|\phi_s(t)>$ and the Heisenberg picture states $|\phi>_H$ as

$$\begin{aligned} |\phi_{ip}(t)> &= e^{iH_o t}|\phi_s(t)> = e^{iH_o t} e^{-iHt}|\phi>_H \\ &= U(t)|\phi>_H \quad . \end{aligned} \tag{2.31}$$

The Schrödinger, Heisenberg and interacting picture coincide at $t = t_o$. which is set to zero here. Using this fact (2.31) can be written as

$$\begin{aligned} |\phi_{ip}> &= U(t)|\phi_H> = U(t)|\phi_{ip}(0)> \\ |\tilde{\phi}_{ip}> &= \tilde{U}(t)|\tilde{\phi}_H> = \tilde{U}(t)|\tilde{\phi}_{ip}(0)> \end{aligned} \tag{2.32}$$

or,

$$\begin{aligned} |\phi_{ip}> &= U(t)U^{-1}(t')|\phi_{ip}(t')> \\ &= U(t)U^{\dagger}(t')|\phi_{ip}(t')> \\ \text{and} \qquad |\tilde{\phi}_{ip}(t)> &= \tilde{U}(t)\tilde{U}^{\dagger}(t')|\tilde{\phi}_{ip}(t')> \end{aligned} \tag{2.33}$$

where t' is some arbitrary time. The operator $U(t)U^{\dagger}(t')$ is denoted by $U(t,t')$. This evolution operator satisfies the equation

$$\begin{aligned} i\frac{\partial U(t,t')}{\partial t} &= H_I^{ip}(t)U(t,t'), \\ i\frac{\partial \tilde{U}(t,t')}{\partial t} &= -\tilde{H}_I^{ip}(t)\tilde{U}(t,t'), \end{aligned}$$

where $H_I^{ip}(t)$ is the interaction Hamiltonian in the interaction picture. $\hat{U}(t,t')$ is defined to be $U(t,t')\tilde{U}(t,t')$ and satisfies

$$i\frac{\partial \hat{U}(t,t')}{\partial t} = \hat{H}_I^{ip}(t)\hat{U}(t,t'). \tag{2.34}$$

The differential equation is converted in to the integral equation

$$\hat{U}(t,t_o) = 1 - i\int_{t_o}^{t} dt_1 \hat{H}_I^{ip}(t_1)\hat{U}(t_1,t_o) \tag{2.35}$$

using the boundary condition $\hat{U}(t_o,t_o) = 1$. Some important properties of the operator $\hat{U}(t,t')$ are listed below

(i) it is unitary,

(ii) it satisfies the group property $\hat{U}(t,t_1)\hat{U}(t_1,t') = \hat{U}(t,t')$

(iii) the inverse of $\hat{U}(t,t')$ is $\hat{U}(t',t) = \hat{U}^{\dagger}(t,t')$.

These general properties hold even when the interaction carries the time dependent adiabatic factor $e^{\epsilon|t|}$, where ϵ is an infinitesimal quantity. For convenience, the interaction picture operators $\psi^{ip}(t)$ will henceforth be written without the superscript as $\psi(t)$. The Heisenberg operators on the other hand will be written with a suffix as $\psi_H(t)$.

Since $\psi(\vec{x},t)$ and $\psi_H(\vec{x},t)$ are related through a unitary operator, the equal time commutation relations hold for $\psi(\vec{x},t)$ and $\psi^{\dagger}(\vec{x}',t)$ too. Also, the equation of motion for the interaction picture operators are the same as those for free fields. This allows an

expansion for $\psi(\vec{x},t)$, $\tilde{\psi}(\vec{x},t)$ similar to (2.22). Only, the operators $a_k, \tilde{a}_k$ etc in (2.22) are to be taken as the interaction picture operators at $t = 0$. They obey the same commutation relations as the free field Heisenberg operators. Hence, for an interacting field the Bogoliubov transformation is applied only to the interaction picture operators

$$\begin{aligned} a_\beta(t) &= \cosh\theta a(t) - \sinh\theta\tilde{a}^\dagger(t) \\ \tilde{a}_\beta(t) &= \cosh\theta\tilde{a}(t) - \sinh\theta a^\dagger(t) \end{aligned} \tag{2.36}$$

where the mode label k has been left out. In terms of these operators the thermal field in interaction picture is defined as

$$\begin{aligned} \psi_\beta(\vec{x},t) &= \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{i\vec{k}.\vec{x}-i\omega t}a_\beta(\vec{k}) \\ \tilde{\psi}_\beta(\vec{x},t) &= \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{-i\vec{k}.\vec{x}+i\omega t}\tilde{a}_\beta(\vec{k}) \end{aligned} \tag{2.37}$$

The $\psi_\beta(\vec{x},t)$ evolve through the equation

$$i\dot{\psi}_\beta(\vec{x},t) = [\psi_\beta(\vec{x},t), \hat{H}_o]$$

The operators $a_\beta(t)$ have the same time dependence as $a(t)$ because $\hat{H}_o$ retains its form when written in terms of $a_\beta(t)$ i.e.,

$$\hat{H}_o = \sum_{\vec{k}} \omega_k \left\{ a^\dagger_{\beta,k}(t)a_{\beta,k}(t) - \tilde{a}^\dagger_{\beta,k}(t)\tilde{a}_{\beta,k}(t) \right\}. \tag{2.38}$$

Hence the total Hamiltonian H in interaction picture when written in terms of the thermal field has the form

$$\hat{H}(t) = \hat{H}_o + \hat{h}(t) \tag{2.39}$$

where

$$H_I(\psi,\psi^\dagger,\tilde{\psi},\tilde{\psi}^\dagger) = h(\psi_\beta,\psi^\dagger_\beta,\tilde{\psi}_\beta,\tilde{\psi}^\dagger_\beta). \tag{2.40}$$

Thus, corresponding to the field theory described by $\hat{H}$ we have a corresponding field theory described by (2.39) whose interaction picture vacuum is $|0(\beta)>$. The Heisenberg fields $\psi_{\beta H}(\vec{x},t)$ are related to the interaction picture fields $\psi_\beta(\vec{x},t)$ as

$$\begin{aligned}\psi_{\beta H}(\vec{x},t) &= \hat{U}^{-1}(t,t_o)\psi_\beta(\vec{x},t)\hat{U}(t,t_o)\\ \tilde{\psi}_{\beta H}(\vec{x},t) &= \hat{U}^{-1}(t,t_o)\tilde{\psi}_\beta(\vec{x},t)\hat{U}(t,t_o)\end{aligned} \tag{2.41}$$

Equation(2.34) can be written as

$$i\frac{\partial\hat{U}(t,t')}{\partial t} = \hat{h}(t)\hat{U}(t,t_o). \tag{2.42}$$

The exact Green's functions $G(t,t')$, which are the quantities of interest for interacting fields are of the general form

$$\frac{Tr\{e^{-\beta H}TA_H(t)B_H(t')\}}{Tre^{-\beta H}} \tag{2.43}$$

where A_H and B_H stand for either ψ_H, $\tilde{\psi}_H$ or their adjoints and T denotes the familiar time ordering operator. Taking the trace with respect to the eigenstates of H is equivalent to taking the expectation value with respect to the exact vacuum of the field described by (2.39). Let $|0(\beta)>_H$ denote the exact Heisenberg vacuum. Using a standard result in field theory due to Gellman and Low [10-12]

$$|0(\beta)>_H = \frac{\hat{U}(0,-\infty)|0(\beta)>}{<0(\beta)|U(0,-\infty)|0(\beta)>}\ , \tag{2.44}$$

the trace in the numerator of (2.43) becomes

$$_H<0(\beta|TA_H(t)B_H(t')|0(\beta)>_H\ .$$

Using (2.44), and the properties of $\hat{U}(t,t')$ listed above, this becomes

$$\begin{aligned}\frac{1}{<0(\beta)|U(\infty,-\infty)|0(\beta)>} &\times\ _H<0(\beta)|\hat{U}(0,\infty)\hat{U}(\infty,0)T\hat{U}(0,t)A(t)\hat{U}(t,t')\\ &\times\ B(t')\hat{U}(t',0)\hat{U}(0,-\infty)\hat{U}(-\infty,0)|0(\beta)>_H\\ &= \frac{<0(\beta)|T\{SA(t)B(t')\}|0(\beta)>}{<0(\beta)|S|0(\beta)>}\end{aligned} \tag{2.45}$$

where $S = U(\infty, -\infty)$ is the S matrix. An iteration of the integral equation which corresponds to (2.42) gives the solution

$$\hat{U}(t,t') = \sum_n \frac{(-i)^n}{n!} \int_t^{t'} dt_1 dt_2 \cdots dt_n T\{\hat{h}(t_1)\hat{h}(t_2)\cdots\hat{h}(t_n)\} \tag{2.46}$$

The expression in (2.45) becomes, leaving aside the denominator

$$\sum_n \frac{(-i)^n}{n!} \int dt_1 \cdots dt_n T\{A(t)B(t')\hat{h}(t_1)\cdots\hat{h}(t_n)\} \tag{2.47}$$

The time ordered product can be decomposed in to a sum of terms containing the normal ordered products. It is tackled using Wick's theorem. It is decomposed into a sum of terms containing normal ordered fields and contractions. The contractions are expectation values of operators with respect to the vacuum $|0(\beta)>$. $A(t)$ and $B(t')$ can written in terms of the $\psi_\beta, \tilde{\psi}_\beta$ fields. Then (2.47) contains time ordered products of various thermal fields. Or, the $\hat{h}(t)$ in (2.47) could be replaced by the equivalent $H_I(t)$. In the first case we have contractions of the form

$$< 0(\beta)|T\psi_\beta^1(t)\psi_\beta^{2\dagger}(t')|0(\beta) > \tag{2.48}$$

where ψ_β^1 and ψ_β^2 stand for either ψ_β or $\tilde{\psi}_\beta^\dagger$. The contractions do not contain any temperature dependent factors. These factors from the Bogoliubov transformation are contained in $h(t)$. When the $h(t)$ are replaced by $H_I(t)$ in (2.47) the contraction are of the form

$$< 0(\beta)|T\psi^1(t)\psi^{2\dagger}(t')|0(\beta) > \quad . \tag{2.49}$$

The temperature dependent factors are contained in these contractions. It is this approach which occurs in all the TFD literature. The tildian interaction hamiltonian H_I is used in perturbation theory and the contractions given in (2.49) are used as propagators. The doublet notation given in chapter 1 is extended to fields

$$\psi = \begin{bmatrix} \psi \\ \tilde{\psi}^\dagger \end{bmatrix} \qquad \bar{\psi} = [\psi^\dagger - \tilde{\psi}]. \tag{2.50}$$

The four basic contractions in (2.49) can be written as a 2×2 propagation matrix

$$G_o(t,t') =< 0(\beta)|T\psi(t)\bar{\psi}(t')|0(\beta) > \tag{2.51}$$

We note that due to the unphysical nature of the tilde fields, the external lines in all diagrams can only be non-tilde fields. The denominator in (2.45) serves to cancel out the disconnected diagrams as in usual field theory.

The perturbation theory could be carried out in any α representation, although the $\alpha = \frac{1}{2}$ case has been used above. The G_o would then be

$$G_o(t,t') =< \rho^{1-\alpha}|T\psi(t)\bar{\psi}(t')|\rho^{\alpha} > \quad . \tag{2.52}$$

Even though some of the propagators depend on α, the results for any computation are independent of α.

2.4 Complex Time Methods

The complex time methods are procedures for calculating thermal averages for interacting systems which are not exactly solvable. The interactions are treated as perturbations of a free, solvable Hamiltonian. Consider the partition function

$$z = Tre^{-\beta H} \quad . \tag{2.53}$$

It can be written as

$$z = Tr\{e^{-\beta H_o}e^{\beta H_o}e^{-\beta H}\} \tag{2.54}$$

The thermal average of $e^{-\beta H}$ using the statistical weight $e^{-\beta H_o}$ is

$$< e^{-\beta H} >_o = \frac{Tr\{e^{-\beta H_o}e^{-\beta H}\}}{Tre^{-\beta H_o}} = \frac{Tr\{e^{-\beta H_o}e^{-\beta H}\}}{z_o}.$$

From (2.54) we have

$$z = z_o < e^{\beta H_o} e^{-\beta H} >_o \quad . \tag{2.55}$$

The quantity inside the angular brackets is the evolution operator of (2.29) with an imaginery time agreement $-i\beta$,

$$z = z_o < U(-i\beta) >_o . \tag{2.56}$$

The expression in (2.56) can be generalized to complex arguments of U as follows

$$z = Tr \left\{ e^{-\beta H_o} e^{(\beta + i\tau) H_o} e^{-(\beta + i\tau) H} e^{i\tau H} e^{-i\tau H_o} \right\} \tag{2.57}$$

$$= z_o < U(\tau - i\beta) U^\dagger(\tau) >_o \quad . \tag{2.58}$$

Here τ is an arbitrary complex number. $U^\dagger(\tau)$ is not the adjoint of $U(\tau)$. It stands for the operator $e^{i\tau H} e^{-i\tau H_o}$, it satisfies

$$U(\tau) U^\dagger(\tau) = 1 \quad , \tag{2.59}$$

and is related to $U(\tau)$ as

$$U^\dagger(\tau) = [U(\tau^*)]^\dagger \quad . \tag{2.60}$$

$U^\dagger(\tau)$ defined this way is used to define an operator $U(\tau, \tau')$ as the analytic continuation of $U(t, t')$ as follows

$$U(\tau, \tau') = U(\tau) U^\dagger(\tau') \quad , \tag{2.61}$$

we then have

$$z = z_o < U(\tau - i\beta, \tau) >_o \quad . \tag{2.62}$$

The expression for thermal averages of the kind given in (2.43) is , leaving aside the time ordering for a moment, as follows

$$< A_H(t) B_H(t') > \; = \; z^{-1} Tr \{ e^{-\beta H} A_H(t) B_H(t') \}$$

$$
\begin{aligned}
&= z^{-1}Tr\{e^{-\beta H_o}e^{(\beta+i\tau)H_o}e^{-(\beta+i\tau)H}A_H(t)B_H(t')\cdot e^{i\tau H}e^{-i\tau H_o}\} \\
&= z^{-1}Tr\{e^{-\beta H_o}U(\tau-i\beta)A_H(t)B_H(t')U^\dagger(\tau)\} \\
&= \frac{z_o}{z} < U(\tau-i\beta)U^\dagger(t)A(t)U(t)U^\dagger(t')B(t')U(t')U^\dagger(\tau) >_o \\
&= \frac{z_o}{z} < U(\tau-i\beta,t)A(t)U(t,t')B(t')U(t',\tau) >_o \quad . \qquad (2.63)
\end{aligned}
$$

Here, as in (2.43) $A(t)$ and $B(t')$ stand for field operators ψ or $\psi^\dagger$. The analytical continuation of $U(t,t')$ defined in (2.61) satisfies a differential equation like (2.34) when the interaction picture operator $H_I(\tau)$ is defined to be $e^{iH_o\tau}H_Ie^{-iH_o\tau}$, where H_I is the Heisenberg picture operator at $\tau = 0$. The group properties of $U(t,t')$ (ii) and (iii) listed earlier also hold for $U(\tau,\tau')$. Hence the iterative solution (2.35) also holds for the analytically continued operator. But the time integrals are replaced by integrals over a complex contour. A single complex contour passing through t and t' could be used for all the time integrals. Time ordering is replaced by ordering along the complex contour. The external times t and t' could be complex.

Using a single continuous contour to join the points $\tau, (t,t' : interchangeable), \tau - i\beta$ in the given order the expression for $< TA_H(t)B_H(t') >$ becomes

$$
\begin{aligned}
&< TA_H(t)B_H(t') >=< T_cA_H(t)B_H(t') > \\
&= \frac{z_o}{z}\left\langle \sum_{n=o}^{\infty}\frac{(-i)^n}{n!}\int_c dt_1\cdots dt_nT_c\left\{A(t)B(t')H_I(t_1)\cdots H_I(t_n)\right\}\right\rangle_o \qquad (2.64)
\end{aligned}
$$

T_c here stands for ordering along the contour C. A similar expression can be written for the partition function z. The time arguments t and t' can in general be complex. In the literature computations done for purely imaginary time arguments $t = t' = -i\beta$ are associated with stationary equilibrium averages, while those done for complex time arguments $t = t_1 - i\beta$, $t' = t_2 - i\beta$ are assumed to give dynamical information for systems out of equilibrium.

A wide range of contours can be chosen in the above procedure. The analyticity of

the Green's functions places some restrictions on the shape and domain of the contour.

Consider $< T_c A_H(t) B_H(t') >$ when t occurs later on the contour than t'. Then the Green's function becomes

$$G(t,t') = G^+(t,t') =< A_H(t) B_H(t') > \tag{2.65}$$

To find its domain of analyticity we write the operators in terms of time independent Schrödinger operators A_s and B_s. Extending the relation between Heisenberg and Schrödinger pictures to complex times we have

$$G^+(t,t') = \frac{1}{z} Tr \left\{ e^{\beta H} e^{iHt} A_s e^{-iHt} e^{iHt'} B_s e^{-iHt'} \right\} \ . \tag{2.66}$$

Inserting a complete set of eigenstates of H,

$$G^+(t,t') = \frac{1}{z} \sum_{m,n} e^{iE_m(t-t'-i\beta)} e^{-iE_m(t-t')} < m|A_s|n >< n|B_s|m > \tag{2.67}$$

Assuming that the convergence of the summations are dominated by the exponentials, we need to have

$$-\beta < Im(t-t') < 0 \tag{2.68}$$

for the convergence of $G^+(t-t')$. Similarly when t' lies farther on the contour than t we have

$$G(t,t') = G^-(t,t') =< B(t') A(t) > \tag{2.69}$$

and it is analytical when

$$0 < Im(t-t') < \beta \ . \tag{2.70}$$

Here Im stands for the imaginary part of the argument.

From the domain of analiticy of $G^+(t,t')$ and $G^-(t,t')$ and from the definition

$$T_c\{ A(t)B(t') = \theta_c(t-t') A(t) B(t') + \theta_c(t'-t) B(t') A(t) \tag{2.71}$$

it can be seen that $G(t,t')$ is analytic in the strip $-\beta < Im(t-t') < \beta$ provided θ_c has the following form. If $\theta_c(t-t')$ is set equal to $\theta(-Im(t-t'))$, then $G(t,t')$ is equal to $G^+(t,t')$ when $Im(t-t')$ is negetive and $G^-(t,t')$ when $Im(t-t')$ is positive. In either case the functions are in their domains of analyticity.

If we fix a contour and define the Green's functions for all points t,t', lying on it, the contour should be such that $Im(t-t')$ does not change sign along it. For any point t which precedes t' on the contour $Im(t-t')$ should positive.The negative imaginary part increases monotonically along the contour.The T_c ordering can be expressed as

$$T_c\{A(t)B(t')\} = \theta(Im(t'-t)G^+ + \theta(Im(t-t'))G^- \tag{2.72}$$

The allowed and the forbidden contours are shown in Figures 1-2. An important periodicity condition for the propagator can be obtained from the definitions (2.65) and (2.69). From (2.65) we have

$$\begin{aligned} G^+(t.t') &= z^{-1}Tr\{e^{-\beta H}A_H(t)B(t')\} \\ &= z^{-1}Tr\{e^{-\beta H}e^{\beta H}B_H(t')e^{-\beta H}A(t)\} \\ &= z^{-1}Tr\{e^{-\beta H}B_H(t'-i\beta)A(t)\} \\ &= G^-(t,t'-i\beta) \end{aligned} \tag{2.73}$$

Similarly it can be proved that

$$G^-(t,t') = G^+(t-i\beta,t'). \tag{2.74}$$

These results are used to prove what are known as the KMS conditions. In the range $-\beta < Im(t-t') < 0$, we have

$$G(t.t') = G^+(t,t') = G^-(t,t'-i\beta) \tag{2.75}$$

from (2.73).In this range of $Im(t-t')$, $G(t,t'-i\beta)$ is also equal to $G^{-}(t,t'-i\beta)$, as $Im(t-t'+i\beta)$ is positive. Hence

$$G(t,t') = G(t,t'-i\beta). \tag{2.76}$$

Similarly,

$$G(t,t') = G(t-i\beta,t')$$

in the range $0 < Im(t-t') < \beta$. The KMS condition for $G(t,t')$ can be proved in TFD by using asymptotic fields [7]. For $G_o(t,t')$ it follows diectly from the tilde conjugation rules for the interaction picture operators.

The general contour used by Mills (Fig. 3) has a limb which starts from τ and goes parallel to the real axis to some point τ'. It then goes down parallel to the imaginery axis and returns to $\tau - i\sigma$ parallel to the real axis. Finally it goes down to $\tau - i\beta$ In the Matsubara contour it goes from the origin straight down to $-i\beta$. In the Keldysh contour τ and τ' are set to ∞ and $-\infty$ respectively.

The finite length of the Matsubara contour, together with the KMS periodicity leads to the Matsubara frequencies. When the propagator $G(t,t')$ is Fourier analysed in terms of imaginery frequencies, only discrete values of the frequencies are used.

The integral in (2.64) can be split in to the four parts of the contour C_1, C_2, C_3, C_4. The T_c ordered product are dealt with using a thermodynamic version of Wicks theorem [2][10][11]. The integrand in (2.64) boils to T_c ordered product of fields ψ and $\psi^\dagger$ of the form

$$\int \frac{d\vec{k}_1}{(2\pi)^3}\cdots\frac{d\vec{k}_n}{(2\pi)^3}\int_{T_c} dt_1 dt_2\cdots dt_n e^{\pm(\vec{k}.\vec{x})}\cdots e^{\pm\vec{k}_n.\vec{x}_n} T_c\{a^i(t_1)a^j(t_2)\cdots a^n(t_n)\} \tag{2.77}$$

Here $a^i(t)$ etc. stand for either $a(t)$ or $a^\dagger(t)$. The exponential time dependence of the $a^i(t)$ can be taken out and we are left to evaluate

$$_o < T_c a^i a^j \cdots a^n >_o \tag{2.78}$$

The thermodynamic Wick's theorem states that (2.78) is given by

$$\sum \pi < T_c a_i a_j >_o \tag{2.79}$$

The sum is over all possible pairings of the operators among $a_i \cdots a_n$. We then get a sum of terms each consisting of products of simple contractions $G_o(t,t')$. From the exponential time dependence in (2.77) it follows that G_o depends only on $(t-t')$. The factor z_o/z serves to cancel the disconnected diagrams as usual.

Let us now consider the integration of the product in (2.64) over the parts C_2 and C_4 of the contour as τ and τ' tend to ∞ and $-\infty$. Since these regions have infinite real parts, the exponential gives a rapidly oscillating contribution. The contributions from C_2 and C_4 can therefore be dropped. Then the contribution over C_1 and C_3 becomes

$$\int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n T_c\{H_I(t_1) - H_I(t_1 - i\sigma)\} \cdots \{H_I(t_n) - H_I(t_n - i\sigma)\} \tag{2.80}$$

The use of Wick's theorem now gives the four basic contractions which can be written as a 2×2 matrix

$$G_o(t-t') = \left\langle \begin{bmatrix} \psi(x,t) \\ \psi^\dagger(x,t-i\sigma) \end{bmatrix} [\psi^\dagger(x',t')\psi(x',t'-i\sigma)] \right\rangle_o \tag{2.81}$$

2.5 Equivalence of TFD and Mills Contour Method

Comparing the expression for

$$\langle A_H(t) B_H(t') \rangle = \tag{2.82}$$

$$\frac{\left\langle \sum_n \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \cdots T_c\{A(t)B(t')[H_I(t) - H_I(t-i\sigma)] \cdots [H_I(t_n) - H_I(t_n - i\sigma)] \right\rangle_o}{\left\langle \sum_n \frac{(-i)^n}{n!} \int dt_1 dt_n \cdots T_c\{H_I(t_1) - H_I(t_1 - i\sigma)\} \cdots \{H_I(t_n) - H_I(t_n - i\sigma)\} \right\rangle_o}$$

with that from TFD

$$\frac{< 0(\beta)|\sum_n \frac{(-i)^n}{n!}\int dt_1\cdots dt_n T A(t)B(t)[H_I(t)-\tilde{H}_I(t)]\cdots[H_I(t_n)-\tilde{H}_I(t_n)]|0(\beta)>}{< 0(\beta)|\sum_n \frac{(-i)^n}{n!}\int dt_1\cdots dt_n T\{H_I(t_1)-\tilde{H}_I(t_1)]\cdots|0(\beta)>}$$

we find that the fields on the return path of C_3 of the contour can be identified with the tilde fields. Note that TFD deals with a pair of commuting fields ψ and $\tilde{\psi}$. In the complex time method even though there are no independent commuting operators, the contour ordering in effect gives us two distinct fields, the T_c ordered form of $\psi(t_1)\psi(t_2-i\sigma)$ has $\psi(t_2-i\sigma)$ to the left irrespective of the values of t_1 and t_2.

A comparison of the basic propagators also shows the equivalence of the two approaches. G_o^{11} from (2.81), for example is

$$\begin{aligned} < T\psi(\vec{x},t)\psi^\dagger(\vec{x},t') >_o &= \theta(t-t')\int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}.\vec{x}-i\omega t}(1+< n >_o) \\ &+ \theta(t'-t)\int \frac{d^3\vec{k}}{(2\pi)^3} e^{-i\vec{k}.\vec{x}+i\omega t} < n >_o \end{aligned} \tag{2.83}$$

where $< n >_o = \frac{1}{e^{\beta\omega_k}-1}$. Using the Fourier representation for the θ function the propagator matrix $G(t,t')$ can be written as

$$\begin{aligned} &\frac{i}{(2\pi)^4}\int d^3\vec{k}d\omega e^{-i\vec{k}.(\vec{x}-\vec{x}')+i\omega(t-t')} \\ &\left\{ \frac{(e^{\beta\omega_k}-1)^{-1}}{k_o-\omega_k+i\epsilon}\begin{bmatrix} e^{\beta\omega_k} & e^{\sigma\omega_k} \\ e^{(\beta-\sigma)\omega_k} & 1 \end{bmatrix} \right. \\ &\left. - \frac{(e^{\beta\omega_k}-1)^{-1}}{k_o-\omega_k-i\epsilon}\begin{bmatrix} 1 & e^{\sigma\omega_k} \\ e^{(\beta-\sigma)\omega_k} & e^{\beta\omega_k} \end{bmatrix} \right\} \end{aligned} \tag{2.84}$$

A comparison shows that the expression for $G_o(t-t')$ from (2.52) and (2.84) are identical when we identify the quantity σ with $\beta(1-\alpha)$. Hence the use of α freedom in TFD corresponds to the freedom in choice of the controur. The equivalence of the two approaching has been pointed out in literature only for the special case of $\alpha = \frac{1}{2}$ [9][13-15]. Eventhough G_o^{12} and G_o^{21} contain α (or σ) explicitly, no result for physical quantities depend on it. The arguments for this conclusion, based on conservation of energy at each vertex [13] hold for both Mills method and TFD.

2.6 Conclusion

The main features of the two formalisms available for finite temperature field theories have been compared. The equivalence of equilibrium TFD and complex time formalism have been shown for a general choice of α. The correspondence between α and the choice of the contour has been given.

Chapter 3

Time-Dependent Bogoliubov Transformation and Non-equilibrium TFD

3.1 The Damped Harmonic Oscillator and Non-Equilibrium TFD

In this chapter we start with the study of a damped harmonic oscillator. The damping mechanism in quantum mechanics is conventionally modelled by the interaction of the oscillator with a large number of modes of the bath or the reservoir. It is assumed that the oscillator

$$H_o = \omega a^\dagger a \tag{3.1}$$

($\hbar = 1$) is coupled to a reservoir with a large number of degrees of freedom

$$R = \sum_i \omega_i b_i^\dagger b_i \tag{3.2}$$

through the interaction

$$V = \sum_i \left(K_i b_i a^\dagger + K_i^* b_i^\dagger a \right) . \tag{3.3}$$

K is a coupling constant. The density operator of the system and reservoir together

evolves according to the equation

$$i\frac{\partial}{\partial t}\rho_T(t) = [H_o + R + V, \rho_T(t)] = [H, \rho_T(t)] \tag{3.4}$$

with the formal solution

$$\rho_T(t) = e^{-iHt}\rho(0)e^{iHt} \quad . \tag{3.5}$$

The expectation value of any variable A consisting of the oscillator (system) operators is

$$< A >= Tr_{rs}\rho_T(t)A \quad . \tag{3.6}$$

where the suffix rs indicates that the trace is over both the reservoir and system variables. The trace over reservoir states is taken first and we deal with the reduced density operator $\rho(t)$

$$< A >= Tr_s A Tr_r \rho_T(t) = Tr_s A\rho(t) \quad . \tag{3.7}$$

As we need to deal only with $\rho(t)$, we seek an equation giving the time evolution of $\rho(t)$ from (3.4). This is done by going over to the interaction picture and obtaining a perturbation expansion for $\rho_T(t)$. If the interaction is switched on at $t = 0$, it is assumed that we are dealing with times t much smaller than the damping time and much larger than the correlation time for the reservoir variables. Physically it means that the fluctuations of the system are smoothened out over time periods over which the reservoir variables are correlated. The system loses its memory over these time scales.

The equation derived for $\rho_T(t)$ under this Markovian approximation is referred to as the master equation [16][17].

For the linear coupling that is assumed in (3.3) the correlation functions for the reservoir modes are the thermal averages

$$\begin{aligned} < b_i^\dagger b_j > &= \frac{Tr_r e^{-\beta R} b_j b_j^\dagger}{Tr_R e^{-\beta R}} \\ &= \delta_{ij}\bar{n}_i = \delta_{ij}\frac{1}{e^{\beta\omega_i}-1} \end{aligned} \tag{3.8}$$

The parameter $\bar{n}_i$ gives the mean occupation of the modes in the reservoir, and is determined by the temperature of the reservoir. The equation for $\rho(t)$, which is in Schrödinger picture is

$$\begin{aligned} \frac{\partial\rho(t)}{\partial t} &= -i\omega[a^\dagger a, \rho(t)] + \frac{\gamma}{2}[2a\rho a^\dagger - a^\dagger a\rho(t) - \rho(t)a^\dagger a] \\ &+ \gamma\bar{n}[a^\dagger\rho(t)a + a\rho(t)a^\dagger - a^\dagger a\rho(t) - \rho(t)aa^\dagger] \end{aligned} \tag{3.9}$$

where the damping constant γ depends on the coupling constant and the density of the reservoir modes. $\bar{n}$ is $< b_i^\dagger b >$ for the mode with $\omega_i = \omega$. The ω in (3.9) should actually read $(\omega + \Delta\omega)$ where $\Delta\omega$ is the small shift in the oscillator frequency because of the interaction with the reservoir. Equation (3.9) is the master equation for a damped oscillator [16][17]. The use of the coherent state representation [16] for the density operator, $\rho = \int P(\alpha)|\alpha >< \alpha|d\alpha$ converts the operators a and $a^\dagger$ in to differentiation with respect

to α and factors of α respectively. Equation (3.9) is converted in to a partial differential equation for the distribution function $P(\alpha)$. The master equation has been studied extensively in this manner [18].

An important feature of (3.9) is that the density operator for the harmonic oscillator in equilibrium at the inverse temperature β,

$$\rho = \frac{1}{e^{\beta\omega} - 1} \exp(-\beta\omega a^\dagger a) \tag{3.10}$$

retains its form as it evolves. It goes over to

$$\rho(t) = \frac{1}{e^{\omega\beta(t)} - 1} \exp(-\beta(t)\omega a^\dagger a) \tag{3.11}$$

where

$$\begin{aligned} \frac{e^{-\beta(t)\omega}}{1 - e^{-\beta(t)\omega}} &= \frac{f(t)}{1 + f(t)} = n(t) \\ &= n(0)e^{-\gamma t} + \bar{n}(1 + e^{-\gamma t}) \ . \end{aligned} \tag{3.12}$$

$n(0)$ is the initial distribution (corresponding to β) and $\bar{n}$ the final distribution determined by the temperature of the bath. [18][20]

The equation (3.9) for $\rho(t)$ can be easily converted into an equation for the state $|\rho(t)>$ of TFD. The prescription given in Chapter 1,

$$|\rho^\alpha> = \rho^\alpha |I>$$

is used for this purpose. For dissipative systems, however it is possible to work only in $\alpha = 1$ representation. This can be seen as follows. For non-dissipative systems the equation for $\rho(t)$ is of the form

$$i\frac{\partial}{\partial t}\rho(t) = [H, \rho(t)] \quad . \tag{3.13}$$

A formal solution

$$\rho(t) = e^{-iHt}\rho(0)e^{iHt}$$

can be written. It follows from it that $\rho^{\alpha}(t) = e^{-iHt}\rho^{\alpha}(0)e^{-iHt}$ and $|\rho^{\alpha}(t) >= \rho^{\alpha}(t)|I >$. For dissipative systems, however, the equation for $\rho(t)$ has the general form

$$i\frac{\partial}{\partial t}\rho(t) = [H, \rho(t)] + L \quad . \tag{3.14}$$

as in (3.9). A formal solution cannot be written. We cannot write an expression for $\rho^{\alpha}(t)$, and hence, for $|\rho^{\alpha}(t) >$. Even though the total density operator $\rho_T(t)$ satisfies (3.13), and a formal solution can be written for it, the expression for $\rho^{\alpha}(t) = \{Tr_r\rho_T^{\alpha}(t)\}$ does not follow from $\rho_T^{\alpha}(t)$ by taking a trace as

$$Tr_r\{\rho_T^{\alpha}(t)\} \neq \{Tr_r\rho_T(t)\}^{\alpha} \quad .$$

Hence we use the $\alpha = 1$ representation. To convert (3.9) for $\rho(t)$ into an equation for $|\rho(t) >$ we apply $|I >$ on the right side of equation (3.9). $\rho(t)|I >$ is replaced by $|\rho(t) >$.

For terms like $a^\dagger \rho a|I>$ we use the rule $a|I>=\grave{a}^\dagger|I>$ given in Chapter 1. We then get $a^\dagger\rho(t)\tilde{a}^\dagger|I>$. The tildian operator is commuted past $\rho(t)$ giving $a^\dagger\grave{a}^\dagger|\rho(t)>$. In this manner we get

$$\begin{aligned}\frac{\partial}{\partial t}|\rho(t)> &= -i\omega(a^\dagger a-\tilde{a}^\dagger\tilde{a})+\frac{1}{2}\gamma(\bar{n}+1)(2a\tilde{a}-a^\dagger a-\tilde{a}^\dagger\tilde{a}) \\ &+ \frac{1}{2}\gamma\bar{n}(2a^\dagger\tilde{a}^\dagger-aa^\dagger-\tilde{a}\tilde{a}^\dagger) \\ &= -i\hat{H}|\rho(t)> \end{aligned} \tag{3.15}$$

The formal solution is $\exp -i\hat{H}t|\rho(0)>$ and the explicit form of the solution can be algebraically computed. Using the realization of $SU(1,1)$ given in (1.49) of Chapter 1, the solution can be written as

$$\exp(-i\hat{H}t)|\rho(0)>=\exp(-i\omega tK_o+\gamma_o)\exp(\gamma_+K_++\gamma_-K_-+\gamma_3K_3)|\rho(0)> \tag{3.16}$$

where

$$\begin{aligned}\gamma_+ &= \gamma\bar{n}t \\ \gamma_- &= \gamma(\bar{n}+1)t \\ \gamma_3 &= -\gamma(2\bar{n}+1)t\end{aligned}$$

$$\gamma_o = \frac{1}{2}(\gamma_- - \gamma_+) \tag{3.17}$$

Using the disentangling theorem of (1.50)[19][20], the exponent $\exp(\gamma_+ K_+ + \gamma_- K_- + \gamma_3 K_3)$ can be written as the product

$$\exp(\Gamma_+ K_+)\exp(\ln \Gamma_3 K_3)\exp(\Gamma_- K_-) \ .$$

The Γ's are calculated from γ's using (1.51-53). The initial state $|\rho(0)>$ which is a 'thermal coherent state' has the form

$$|\rho(0)>= \exp\{f(0)a^\dagger \tilde{a}^\dagger\}|0,\tilde{0}> .$$

It is written as $\exp[f(0)K_+]|0,\tilde{0}>$. Then we need to compute

$$\exp\{-i\omega t K_o + \gamma_o\}\exp(\Gamma_+ K_+)\exp(\ln \Gamma_3 K_3)$$

$$\exp(\Gamma_- K_-)\exp(f(0)K_+)|0,\tilde{0}> \tag{3.18}$$

To evaluate it a factor $\exp(-\Gamma_- K_-)$ is inserted to the left of the state $|0,\tilde{0}>= 0$. The operator identity

$$\exp\{\xi A\} B \exp -\{\xi A\} = B + \xi[A,B] + \frac{\xi^2}{2}[A,[A,B]]$$

and the disentangling theorem are used repeatedly along with the $SU(1,1)$ algebra. The result of the computation is

$$|\rho(t)>=\{1-f(t)\}\exp[f(t)a^{\dagger}\tilde{a}^{\dagger}]|0,\tilde{0}> \tag{3.19}$$

where

$$f(t)=\frac{n(t)}{1+n(t)}$$

and

$$n(t)=n(0)e^{-\gamma t}+\bar{n}(1-e^{-\gamma t}) \tag{3.20}$$

Hence evolution according to (3.9) preserves the form of a thermal coherent state.

We next prove that this property of (3.9) holds even when ω and the parameters γ and $\dot{n}$ are made arbitrary functions of time [67]. With the time dependent parameters the solution of

$$\frac{\partial}{\partial t}|\rho(t)>=-i\hat{H}|\rho(t)>$$

is no longer $\exp\{-i\hat{H}t|\rho(0)>$, but is given by

$$|\rho(t)>=S(t)|\rho(0)> \tag{3.21}$$

In order to prove that $|\rho(t)>$ has the form of a thermal coherent state we prove that there exist operators which annihilate $|\rho(t)>$ as follows

$$\{a - f(t)\tilde{a}^\dagger\}|\rho(t)> = 0$$

$$\{\tilde{a} - f(t)a^\dagger\}|\rho(t)> = 0. \tag{3.22}$$

Assuming that (22) is true it follows that

$$S^{-1}(t)\{a - f(t)\tilde{a}^\dagger\}S(t)|\rho(0)>= 0$$

i.e.,

$$S^{-1}(t)\{a - f(t)\tilde{a}^\dagger\}S(t)|\rho(0)>= \{a(t) - f(t)\tilde{a}^\dagger(t)\}|\rho(0)>= 0$$

and a corresponding equation for $\{\tilde{a} - f(t)a^\dagger\}$. Here $a(t) = S^{-1}(t)aS(t)$ etc. When we look at the evolution of the operator $\{a(t) - f(t)\tilde{a}^\dagger(t)\}$ we find that it is proportional to $\{a(0) - f(0)\tilde{a}^\dagger\}$. Hence (22) holds true and $|\rho(t)>$ does have the form of a thermal coherent state.

The evolution of $\{a(t) - f(t)\tilde{a}^\dagger(t)\}$ is given by

$$\frac{d}{dt}\{a(t) - f(t)\tilde{a}^\dagger(t)\} = -i[\hat{H}, a(t) - f(t)\tilde{a}^\dagger(t)] - \frac{d}{dt}f(t)\tilde{a}^\dagger(t) \tag{3.23}$$

Using $\hat{H}$ from (3.15) and computing commutators like

$$[\hat{H}, a(t)] = S^{-1}(t)[\hat{H}, a]S(t)$$

we get

$$\begin{aligned} \frac{d}{dt}\{a(t) - f(t)\tilde{a}^{\dagger}(t)\} &= -\{i\omega(t) + \frac{1}{1-f(t)}\frac{d}{dt}f(t) + \frac{\gamma(t)}{2}\} \\ &\times \{a(t) - f(t)\tilde{a}^{\dagger}(t)\} \ . \end{aligned} \tag{3.24}$$

It follows from this that

$$\{a(t) - f(t)\tilde{a}^{\dagger}(t)\} \propto \{a - f(0)\tilde{a}^{\dagger}\} \ .$$

The proportionality constant is an exponential of some time dependent function. Hence, equation (22) holds true.

3.2 Non-Equilibrium TFD using Time-Dependent Bogoliubov Transformations

The two results given above make contact with the non-equilibrium TFD formalism which uses time dependent Bogoliubov transformations. The result that the form of the state $|\rho(0)>$ remains unchanged as it evolves according to (3.15) (and its generalized version with time dependent parameters) corresponds to the basic assumption about

the form of $\rho(t)$ in non-equilibrium TFD [22-27]. The existence of a time dependent Bogoliubov transformation is based on this assumption about the form of $|\rho(t)>$. The result about the operator $\{a(t) - f(t)\tilde{a}^\dagger\}$ evolving according to a simple exponential time dependence anticipates the addition of the thermal generator to the equation of motion in non-equilibrium TFD. We now review the time dependent Bogoliubov transformations which are the basis of non-equilibrium TFD. The derivation of the equations of motion for the state $|\rho(t)>$ and the operators $\xi(t)$ is studied closely. The equation of motion for $|\rho(t)>$ is compared with (3.15) and the parameter of the Bogoliubov transformation which corresponds to the damping constant γ is identified.

3.3 Time-Dependent Bogoliubov Transformation and the Thermal Generator:

Consider a single mode described by $\hat{H}_o = \omega\{a^\dagger a - \tilde{a}^\dagger\tilde{a}\}$. In non-equilibrium TFD it is assumed that the state $|\rho(t)>$ has same form as it does for the mode in equilibrium, but with a time dependent parameter $f(t)$

$$|\rho^\alpha(t)>= [1 - f(t)]^\alpha \exp(\alpha \ln f(t) a^\dagger a)|I>$$

and that the thermal state conditions

$$\xi(t)|\rho^\alpha(t)>= 0 \qquad \tilde{\xi}(t)|\rho^\alpha(t)>= 0$$

$$<\rho^{1-\alpha}(t)|\xi^{\dagger\dagger}(t) = 0 \qquad <\rho^{1-\alpha}(t)|\tilde{\xi}^{\dagger\dagger}(t) = 0. \tag{3.25}$$

The ξ's are related to the operators $a, a^\dagger$ etc., as follows.

$$\xi(t) = \frac{e^{s(t)}[a - f^\alpha(t)\tilde{a}^\dagger]}{\sqrt{1-f(t)}} \qquad \tilde{\xi}(t) = \frac{e^{s(t)}[\tilde{a} - f^\alpha(t)a^\dagger]}{\sqrt{1-f(t)}}$$

$$\xi^{\dagger\dagger}(t) = \frac{e^{-s(t)}[a^\dagger - f^{1-\alpha}(t)\tilde{a}]}{\sqrt{1-f(t)}} \qquad \tilde{\xi}^{\dagger\dagger}(t) = \frac{e^{-s(t)}[\tilde{a}^\dagger - f^{1-\alpha}(t)a]}{\sqrt{1-f(t)}} \tag{3.26}$$

Note that the definition of the thermal operators given in (3.26) is made more general by the presence of the exponential factors. Using the doublet notation $\xi^1 = \xi$, $\xi^2 = \tilde{\xi}^{\dagger\dagger}$, $\bar{\xi}^1 = \xi^{\dagger\dagger}$, $\bar{\xi}^2 = -\tilde{\xi}$, the relations (3.25) can be written as

$$\xi(t) = B(t)a \qquad , \bar{\xi}(t) = \bar{a}B^{-1}(t)$$

where the Bogoliubov matrix is

$$B(t) = \frac{1}{\sqrt{1-f(t)}} \begin{bmatrix} e^{s(t)} & -e^{s(t)}f^\alpha(t) \\ -e^{-s(t)}f^{1-\alpha}(t) & e^{-s(t)} \end{bmatrix} \tag{3.27}$$

The $\xi(t)'s$ obey the same commutation relations,

$$[\xi^\mu(t), \bar{\xi}^\nu(t)] = \delta_{\mu\nu}$$

as

$$[a^\mu, \bar{a}^\nu] = \delta_{\mu\nu} \quad .$$

Here $\mu, \nu = 1, 2$. Also, $\hat{H}_o$ is invariant under the transformation, as in equilibrium. $B(t)$ is characterized by three parameters $n(t)$, $s(t)$ and α. The Heisenberg equations of motion for the operators $\xi(t)$ and $\bar{\xi}(t)$ are ($\hbar = 1$)

$$i\frac{d\xi^{\mu}(t)}{dt} = [\xi^{\mu}(t), \hat{H}_o] + i\frac{\partial\xi^{\mu}(t)}{\partial t} \tag{3.28}$$

where

$$\frac{\partial\xi^{\mu}(t)}{\partial t} = \dot{B}(t)^{\mu\nu}a^{\nu} = [\dot{B}(t)\bar{B}(t)]^{\mu\nu}\xi^{\nu}(t) \tag{3.29}$$

and $\dot{B}(t) = \frac{\partial}{\partial t}B(t)$.

Defining the quantity

$$\begin{aligned} Q(\alpha) &= i\bar{\xi}^{\mu}(t)[\dot{B}(t)B^{-1}(t)]^{\mu\nu}\xi^{\nu}(t) \\ &= i\bar{a}^{\mu}(t)[B^{-1}(t)\dot{B}(t)]^{\mu\nu}a^{\nu}(t) \end{aligned} \tag{3.30}$$

we can write (3.29) as

$$\begin{aligned} i\frac{\partial\xi^{\mu}(t)}{\partial t} &= [\xi^{\mu}(t), Q(\alpha)] \\ i\frac{\partial\bar{\xi}^{\mu}(t)}{\partial t} &= [\xi^{-\mu}(t), Q(\alpha)] \end{aligned} \tag{3.31}$$

The Heisenberg equation now reads as

$$i\frac{d}{dt}\xi^{\mu}(t) = [\xi^{\mu}(t), \hat{H}_o] + [\xi^{\mu}(t), Q(\alpha)] \quad . \tag{3.32}$$

Note that the second term on the right arises due to the time dependence of the Bogoliubov transformation. The quantity $Q(\alpha)$ which occurs in addition to $\hat{H}_o$ in the evolution of the system away from equilibrium is called the thermal generator.

3.4 The Evolution Equation for $|\rho(t)>$:

The state $|\rho^{\alpha}>$ in equilibrium in the Schrödinger picture is time-independent since

$$|\rho^{\alpha}>=\sum_{n} e^{-\beta\omega n\alpha}|n,\tilde{n}>$$

and the time dependence of the tilde and non-tilde states have positive and negative exponents, respectively. But the $|\rho^{\alpha}(t)>$ of non-equilibrium TFD has an explicit time dependence. We now derive an evolution equation for $|\rho^{\alpha}(t)>$ in Schrödinger picture in terms of $Q(\alpha)$.

By differentiating

$$\begin{aligned} \xi(t)|\rho^{\alpha}(t)> &= 0 \\ \\ <\rho^{1-\alpha}(t)|\xi^{\dagger\dagger}(t) &= 0 \end{aligned} \tag{3.33}$$

with respect to t we have

$$\left\{\frac{d\xi(t)}{dt}+\xi(t)\frac{d}{dt}\right\}|\rho^{\alpha}(t)>=0$$

$$= \left\{-i[\xi(t),Q(\alpha)]+\xi(t)\frac{d}{dt}\right\}|\rho^{\alpha}(t)>$$

$$= \xi(t)\left\{-iQ(\alpha)+\frac{d}{dt}\right\}|\rho^{\alpha}(t)>=0 \qquad (3.34)$$

and similarly

$$<\rho^{1-\alpha}(t)\left\{Q(\alpha)+i\frac{d}{dt}\right\}\bar{\xi}=0 \qquad (3.35)$$

where we used (3.30) and (3.31). The quantity in curly brackets in (3.34) acting on $|\rho^{\alpha}(t)>$ should be proportional to $|\rho^{\alpha}(t)>$ since the action of $\xi(t)$ annihilates it. We set it equal to $i\mu(\alpha)|\rho^{\alpha}(t)>$ where $\mu(\alpha)$ is a constant to be determined below [22]. Earlier works in TFD [24-26] set the action of the expression in the curly brackets on $|\rho^{\alpha}(t)>$ directly equal to zero giving the evolution equation

$$i\frac{d}{dt}|\rho^{\alpha}(t)=-Q(\alpha)|\rho^{\alpha}(t)> \quad . \qquad (3.36)$$

As we shall see below this reduces the number of independent parameters and leads to a trivial solution for $|\rho^{\alpha}(t)>$ [22]. With the introduction of the constant $\mu(\alpha)$, the equations of motion which follows from (3.33) become

$$i\frac{d}{dt}|\rho^{\alpha}(t)> = -\{Q(\alpha)-i\mu(\alpha)\}|\rho^{\alpha}(t)>$$

$$< \rho^{1-\alpha} | i\frac{d}{dt} = < \rho^{1-\alpha}(t) | \{Q(\alpha) - i\lambda(\alpha)\} \tag{3.37}$$

By requiring that

$$\frac{d}{dt}\{Tr\rho(t)\} = \frac{d}{dt}\left\langle \rho^{1-\alpha} \middle| \rho^{\alpha}(t) \right\rangle = 0$$

we have $\mu(\alpha) = \lambda(\alpha)$. Also, $\mu(\alpha)$ must be real in order that the equations (3.35) remain invariant under tilde conjugation.

The equations of motion then become

$$i\frac{d}{dt}|\rho^{\alpha}(t) > = -[Q(\alpha) - i\mu(\alpha)]\,|\rho^{\alpha}(t) >$$

$$< \rho^{1-\alpha}(t)|i\frac{d}{dt} = < \rho^{1-\alpha}(t)|\,[Q(\alpha) - i\mu(\alpha)] \tag{3.38}$$

To evaluate $\mu(\alpha)$ we compare equation (3.37) with those obtained by differentiating the defining expressions for $|\rho^{\alpha}(t) >$ and $< \rho^{1-\alpha}(t)|$. The following expressions which can be derived from the defining expressions for the $\xi(t)$ and $\xi^{\dagger\dagger}(t)$ are used in the process

$$a^{\dagger}\tilde{a}^{\dagger}|\rho^{\alpha}(t) > = f^{-\alpha}(t)a^{\dagger}a|\rho^{\alpha}(t) >$$

$$\tilde{a}a|\rho^{\alpha}(t) > = f^{\alpha}(t)[1 + a^{\dagger}a]|\rho^{\alpha}(t) >$$

$$\begin{aligned}
\check{a}\check{a}^{\dagger}|\rho^{\alpha}(t)> &= [1+a^{\dagger}a]|\rho^{\alpha}(T)> \\
<\rho^{1-\alpha}(t)|a^{\dagger}\tilde{a}^{\dagger} &= <\rho^{1-\alpha}(t)|[1+a^{\dagger}a]f^{-1\alpha}(t) \\
<\rho^{1-\alpha}|\tilde{a}a &= <\rho^{1-\alpha}(t)|a^{\dagger}af^{\alpha-1}(t) \\
<\rho^{1-\alpha}|\check{a}\check{a}^{\dagger} &= <\rho^{1-\alpha}(t)|[1+a^{\dagger}a] \qquad (3.39)
\end{aligned}$$

Using the defining expression for $Q(\alpha)$ we have

$$\begin{aligned}
Q(\alpha) &= ia^{-\mu}(B^{-1}(t)\dot{B}(t))^{\mu\nu}a^{\nu} \\
&= i[A_{11}(\alpha)a^{\dagger}a + A_{12}(\alpha)a^{\dagger}\tilde{a}^{\dagger} - A_{21}(\alpha)\tilde{a}a - A_{22}\check{a}\check{a}^{\dagger}]
\end{aligned}$$

where the matrix $A(\alpha)$ is given by

$$\begin{aligned}
A(\alpha) &= \frac{\dot{n}}{2(1+n)}\begin{pmatrix} 2\alpha-1 & -2\alpha(1+\frac{1}{n})^{1-\alpha} \\ -2(1-\alpha)(1+\frac{1}{n})^{\alpha} & -(2\alpha-1) \end{pmatrix} \\
&+ \dot{s}\begin{pmatrix} 1+2n & -2n^{\alpha}(1+n)^{1-\alpha} \\ 2n^{1-\alpha}(1+n)^{\alpha} & -(1+2n) \end{pmatrix}
\end{aligned}$$

n is a function of time, though , for convenience it is not written as $n(t)$. Using the

conditions (3.39), the equations (3.38) can be written as

$$i\frac{d}{dt}|\rho^{\alpha}(t)> = i\left[\frac{\alpha\dot{n}}{n(1+n)}a^{\dagger}a - \dot{s} - \frac{\dot{n}}{2(1+n)} + \mu(\alpha)\right]|\rho^{\alpha}(t)>$$

$$<\rho^{1-\alpha}(t)|i\frac{d}{dt} = <\rho^{1-\alpha}(t)|i\left[\frac{\dot{n}(1-\alpha)}{n(1+n)}a^{\dagger}a + \dot{s} - \frac{\dot{n}}{2(1+n)} - \mu(\alpha)\right] \quad (3.40)$$

By differentiating the defining expression for $|\rho^{\alpha}(t)>$ and $<\rho^{1-\alpha}(t)|$ given above (3.25), on the other hand, we get

$$i\frac{d}{dt}|\rho^{\alpha}(t)> = i\frac{\alpha\dot{n}}{n(1+n)}[a^{\dagger}a - n]|\rho^{\alpha}(t)> \quad (3.41)$$

$$<\rho^{1-\alpha}|i\frac{d}{dt} = <\rho^{1-\alpha}(t)|\frac{i(1-\alpha)\dot{n}}{n(1+n)}[a^{\dagger}a - n] \quad (3.42)$$

A comparision gives

$$\mu(\alpha) = \dot{s} + \frac{\dot{n}(1-2\alpha)}{2(1+n)} \quad (3.43)$$

with this value for $\mu(\alpha)$, we obtain consistency between the second of equations (3.38) and the Hermitian adjoint of the first, with α replaced by $(1-\alpha)$, i.e., the second should be equivalent to

$$<\rho^{1-\alpha}(t)|i\frac{d}{dt} =<\rho^{1-\alpha}(t)|\left[Q^{\dagger}(1-\alpha) + i\mu(1-\alpha)\right] \quad . \quad (3.44)$$

This consistency is necessary for the following reason [22]. Let α_1 and α_2 be two choices

of α such that $\alpha_1 = 1 - \alpha_2$.Then the equation for $|\rho^{\alpha_1}>$ is given by the first equation in (3.38) and that for $<\rho^{1-\alpha_2}|$ is given by the second. Since $<\rho^{1-\alpha_2}|$ is the usual ket state corresponding to $|\rho^{\alpha_1}>$, the equations for these states should be related by hermitian conjugation. This happens when the consistency condition given above is satisfied.

By using the relations (3.39), the right hand side of (3.44) is written as

$$<\rho^{1-\alpha}(t)|i\left[\frac{\dot{n}(1-\alpha)}{n(1+n)}a^{\dagger}a - \dot{s} - \frac{\dot{n}}{2(1+n)} + \mu(1-\alpha)\right].$$

Comparing it with the second of equations (3.38), or equivalently (3.40), we find (3.44) is identical to them as

$$\mu(1-\alpha) - \dot{s} = \dot{s} - \mu(\alpha)$$

when $\mu(\alpha)$ is given as in (3.43).

The necessity of the constant $\mu(\alpha)$ can be seen as follows. In the absence of $\mu(\alpha)$ the equations (3.38) read

$$i\frac{d}{dt}|\rho^{\alpha}(t)> = -Q(\alpha)|\rho^{\alpha}(t)> \tag{3.45}$$

$$<\rho^{1-\alpha}(t)|i\frac{d}{dt} = <\rho^{1-\alpha}(t)|Q(\alpha) \ . \tag{3.46}$$

Comparing (3.45) with (3.41) we have

$$\dot{s} = \frac{\dot{n}(2n-1)}{2(1+n)}$$

Further, demanding the consistency of (3.46) with the Hermitian conjugate of (3.45) with α replaced by $(1-\alpha)$ gives $\dot{s}=0$ implying $\dot{n}=0$ and hence a trivial solution.

The correct evolution equations for the states $|\rho^{\alpha}(t)>$ and $<\rho^{1-\alpha}(t)|$ which follow from the time dependent Bogoliubov transformation are of the form

$$\begin{aligned} i\frac{d}{dt}\rho^{\alpha}(t)> &= -G(\alpha)|\rho^{\alpha}(t)> \\ <\rho^{1-\alpha}(t)|i\frac{d}{dt} &= <\rho^{1-\alpha}(t)|G^{\dagger}(1-\alpha) \end{aligned} \tag{3.47}$$

where

$$G(\alpha)=Q(\alpha)-i\left[\dot{s}+\frac{\dot{n}(1-2\alpha)}{2(1+n)}\right]$$

The generator $G(\alpha)$ consists of a part $Q(\alpha)$ which follows directly from the Bogoliubov transformation and the additional part $\mu(t)$. We have retained all the three independent parameters $\alpha, s(t)$ and $n(t)$ which characterise $B(t)$. Let us now consider some special choices for the parameters. The choice $\alpha=1$ gives us the master equation (3.15), when we make the identification

$$\frac{\gamma}{2}=\dot{s}-\frac{\dot{n}}{2(1+n)} \quad . \tag{3.48}$$

Further, by choosing $\dot{s}$ to be equal to $\frac{\dot{n}}{2(1+n)}$ we get $\gamma=0$. We shall see in the next chapter that this is the choice of α and $\dot{s}$ used for non-equilibrium computations. Thus,

eventhough the non-equilibrium TFD appears to be equivalent to using a bath, it actually uses zero coupling to bath. Hence it can be said that TFD generates dissipation without external damping.

For the $\alpha = \frac{1}{2}$ case we have the equation

$$i\frac{d}{dt}|\rho^{1/2}(t)>= i\frac{\dot{n}}{2n(1+n)}[a^\dagger a - n]|\rho^{1/2}(t)> \tag{3.49}$$

Since

$$\begin{aligned} a^\dagger a|\rho^{1/2}> &= (1+n)(1-f)a^\dagger a|\rho^{1/2}> \\ &= (1+n)f^{1/2}(a^\dagger \check{a}^\dagger - a\check{a})|\rho^{1/2}> \end{aligned}$$

using relations (3.39), we have

$$i\frac{d}{dt}|\rho^{1/2}(t)>= i\frac{\dot{n}}{2\sqrt{n(1+n)}}(a^\dagger \check{a}^\dagger - a\check{a})|\rho^{1/2}(t)> \quad . \tag{3.50}$$

This is identical to the equation derived by Celeghini et al [21].

$$i\frac{d}{dt}|\rho^{1/2}(t)>= i\Gamma(\check{a}^\dagger a^\dagger - a\check{a})|\rho^{1/2}(t)>$$

where $\sinh^2 \Gamma t = n$. In their work on quantum dissipation they use two coupled modes, one a system of interest and the other an effective single mode bath.

It is to be noted that a special choice of $s(t)$, in terms of $n(t)$ and α makes $\mu(\alpha)$ zero:

$$\dot{s}(t) = \frac{\dot{n}}{2(1+n)}(2\alpha - 1) \quad .$$

Then the evolution is governed purely by $Q(\alpha)$ for all α.

3.5 Reparametrization of $B(t)$:

It is seen that the following reparametrization of $B(t)$ has some desirable features. The reparametrization follows from a slightly modified definition

$$\begin{aligned} \xi(t) &= \frac{e^{\chi(t)}(a - f^{\alpha}(t)\tilde{a}^{\dagger})}{[1-f(t)]^{\alpha}} \\ \xi^{\dagger\dagger}(t) &= \frac{e^{-\chi(t)}(a^{\dagger} - f^{1-\alpha}(t))}{[1-f(t)]^{1-\alpha}} \end{aligned} \tag{3.51}$$

and similarly for $\tilde{\xi}(t)$ and $\tilde{\xi}^{\dagger\dagger}(t)$.

The expression for $Q(\alpha)$ is now $Q(\alpha) = \bar{a}A(\alpha)a$ with

$$\begin{aligned} A(\alpha) &= \dot{n} \begin{pmatrix} 2\alpha - 1 & -\frac{(2\alpha-1)n+\alpha}{n^{1-\alpha}(1+n)^{\alpha}} \\ \frac{(2\alpha-1)n+(\alpha-1)}{n^{\alpha}(1+n)^{1-\alpha}} & -(2\alpha-1) \end{pmatrix} \\ &+ \dot{\chi} \begin{pmatrix} 1+2n & -2n^{\alpha}(1+n)^{1-\alpha} \\ 2n^{1-\alpha}(1+n)^{\alpha} & -(1+2n) \end{pmatrix} . \end{aligned} \tag{3.52}$$

It is useful to examine the properties of this matrix under 'α-conjugation'. An α-Hermitian matrix $0(\alpha)$ satisfies

$$0(\alpha) = 0^{\dagger}(1-\alpha) \tag{3.53}$$

For $\alpha = 1/2$, α-Hermiticity coincides with usual Hermiticity. The matrix $A(\alpha)$ above can be rewritten as

$$A(\alpha) = \dot{n}N(\alpha) + \dot{\chi}X(\alpha)$$

where $N(\alpha)$ and $X(\alpha)$ are the matrices in (3.52) which go with $\dot{n}$ and $\dot{\chi}$. We can check that $N(\alpha)$ is anti-α-Hermitian while $X(\alpha)$ is α-Hermitian,

$$N^{\dagger}(1-\alpha) = -N(\alpha)$$

$$X^{\dagger}(1-\alpha) = X(\alpha) \ .$$

The equations of motion are now obtained as follows. We have from (3.52) and (3.39)

$$\begin{aligned} Q(\alpha)|\rho^{\alpha}(t)> &= \{\bar{a}A(\alpha)a\}|\rho^{\alpha}(t)> \\ &= i\left[\frac{\alpha\dot{n}}{n(1+n)}[a^{\dagger}a - n) - \dot{\chi}\right]|\rho^{\alpha}(t)> \end{aligned} \tag{3.54}$$

and

$$< \rho^{1-\alpha}(t)|Q^{\dagger}(1-\alpha) = i < \rho^{1-\alpha}(t)| \left[\frac{(1-\alpha)\dot{n}}{n(1+n)}(a^{\dagger}a - n) - \dot{\chi} \right] \quad . \tag{3.55}$$

Looking at (3.41) and (3.42) we can rewrite (3.54) and (3.55) as

$$\begin{aligned} i\frac{d}{dt}|\rho^{\alpha}(t) > &= -[Q(\alpha) - i\lambda]|\rho^{\alpha}(t) > \\ < \rho^{1-\alpha}(t)|i\frac{d}{dt} &= < \rho^{1-\alpha}(t)|[Q^{\dagger}(1-\alpha) + i\dot{\chi}] \quad . \end{aligned} \tag{3.56}$$

As $\dot{\chi}$ is real we can go from one equation to the other by α-conjugation. thus the reparametrization gives a pair of α-conjugate equations. There is no need to add the extra term $\mu(\alpha)$.

The above decomposition of $Q(\alpha)$ into α-Hermitian and α-anti-Hermitian parts shows that for $Q(\alpha)$ to be α-Hermitian we need to set the coefficient $\dot{\chi}$ of the α-anti-Hermitian part zero. Then, according to (3.54) and (3.55) the evolution is governed only by the parameter $\dot{n}(t)$.

We shall see in the next chapter that we do not deal with the constant $\mu(\alpha)$, even-though we use $B(t)$ as in (3.27). This comes about from the choice of α and $s(t)$. In order to have $B(t)$ linear in $n(t)$ we set

$$\dot{s} = \frac{\dot{n}}{2(1+n)} \qquad \text{and} \qquad \alpha = 1$$

This corresponds to using $\gamma = 0$ in the master equation. This choice of $\dot{s}$ and α makes $\mu(\alpha)$ zero.

3.6 Conclusion

The master equation for the damped harmonic oscillator can be converted into a Schrodinger like equation for $|\rho^{\alpha}>$ when transcribed in TFD. This equation retains the form of a thermal coherent state as it evolves. Non-equilibrium TFD is based on the use of time dependent Bogoliubov transformations and also assumes that the state $|\rho^{\alpha}>$ retains its form as the system evolves. The two approaches have been compared in this chapter. It is shown that the evolution equation for $|\rho>$ must contain the constant $\mu(\alpha)$ apart from the thermal generator. The constant is necessary to ensure consistence between the equations for $|\rho^{\alpha}>$ and $<\rho^{1-\alpha}|$. In its absence, the parameters $s(t)$ and $n(t)$ would have to be set to zero in order to have consistency.

The equation obtained this way is shown to yield the master equation. A relation between the damping constant and the parameters $s(t)$ and $n(t)$ of the Bogoliubov transformation is found. The transformation is reparametrized in a way that the evolution equations for $|\rho^{\alpha}>$ and $<\rho^{1-\alpha}|$ are consistent without having to add the constant $\mu(\alpha)$.

Chapter 4

Non-Equlibrium TFD and Boltzmann Equation

4.1 The Thermal Generator in Non-Equilibrium TFD

The time dependent Bogoliubnov transformations considered in the previous chapter are between the interaction picture operators $a(t)$ and $\xi(t)$ of a particular mode when interacting fields are considered. The evolution equation of $\xi(t)$ contains a term that arises from the time dependence of the Bogoliubov transformation.

$$\xi(t) = B(t)a(t) \tag{4.1}$$

leads to

$$\begin{aligned} i\frac{d\xi(t)}{dt} &= [\xi(t), \hat{H}_o] + i\frac{\partial\xi(t)}{\partial t} \\ &= \omega\xi(t) + [\xi(t), Q(\alpha)] \quad . \end{aligned} \tag{4.2}$$

$Q(\alpha)$ is defined as

$$Q(\alpha) = i\int d^3\vec{k}\tilde{a}_k(t)P_k(t)a(t)$$

for a field. $P_k = B^{-1}(t)\dot{B}(t)$. The momentum subscript $\vec{k}$ of ξ, B and Q has been dropped for convenience. The thermal operators $\xi(t)$ are the relevant ones for the physical particle (quasi particle) picture at finite temperature. Hence it is desirable that $\xi(t)$ have the simple time dependence $\xi(0)e^{-i\omega t}$. In that case $\xi(t)$ also annihilates the thermal state annihilated by $\xi(0)$. In analogy with the usual ground state field theory, the vacuum of the interacting field determines the physical particles. In order that $\xi(t)$ evolve as a

free particle operator, we have to use $[\hat{H}_o - Q(\alpha)]$ as the free Hamiltonian . Then the Heisenberg equation is

$$\begin{aligned} i\frac{d}{dt}\xi(t) &= \Big[\xi(t), \hat{H}_o - Q(\alpha)\Big] + [\xi, Q(\alpha)] \\ &= -i\omega\xi^{\mu}(t) - [\xi(t), Q(\alpha)] + [\xi(t), Q(\alpha)] \ . \end{aligned} \tag{4.3}$$

The extra term which arises on using $[\hat{H}_o - Q(\alpha)]$ for free evolution cancels the term arising out of the explicit time dependence of $B(t)$. We have $\xi(t) = \xi(0)e^{-i\omega t}$. The equations of motion for the $a(t)$, $\bar{a}(t)$ operators now becomes

$$\frac{d}{dt}a(t) = [-i\omega - P]a(t) \frac{d}{dt}\bar{a}(t) = \bar{a}[i\omega + P] \ .$$

It can be pointed out here that the same result was seen in Chapter 3, when the evolution of $\xi(t)$ with the evolution operator $S(t)$ was studied.

Such an arbitrary subtraction of $Q(\alpha)$ from the free Hamiltonian would change the dynamics. For an interacting field $Q(\alpha)$ is added to the interaction H_I and so the total Hamiltonian remains unchanged. Hence $Q(\alpha)$ acts as a counter term of the usual field theories when we use $[\hat{H}_o - Q(\alpha)]$ as the free Hamiltonian [23-27]. And this choice is necessary to retain a quasiparticle picture of the interacting field, where the thermal quasiparticles are annihilated by the thermal operators $\xi(t), \tilde{\xi}(t)$.

In the last chapter we saw that when evolution is governed by the thermal generator the basic assumption about the form of the thermal vacuum $|\rho(t) >$ is satisfied. But when the bath parameter $\bar{n}$ in the master equation is eliminated in favour of $\dot{n}(t)$, the functional dependence of $n(t)$ is not determined. It is not surprizing that when the concept of external damping is removed the evolution of $n(t)$ cannot be determined for a free field. When interactions are introduced it is expected that they determine the evolution of $n(t)$.

A systematic procedure for determining $n(t)$ is outlined here. The procedure has been

developed by Umezawa, Hardman, Evans and others over a period of few years [23-27]. A few calculations are given as examples. A perturbation theory is given for computing Green's functions in non-equilibrium. The total self energy including the contribution from the counter term is required to be diagonal. This ensures that the corrected Green's functions have the same form as the free ones. This diagonalization gives an equation for $n(t)$. In equilibrium when there is no $Q(\alpha)$ counter term to enforce the diagonalization, the Greens functions can be shown to be naturally diagonal [23].

4.2 Perturbation Theory

The interaction picture operators $A(t)$ evolve as

$$i\frac{dA(t)}{dt} = [A(t), H_o - Q(\alpha)] + i\frac{\partial A}{\partial t} \quad . \tag{4.4}$$

The operator $U(t)$ connects the interaction picture operators and the Heisenberg picture operators as usual

$$A_H(t) = U^{-1}(t)A(t)U(t) \tag{4.5}$$

where

$$U(t) = \exp i\int (\hat{H}_o - Q)\,dt\, \exp -i\hat{H}t \tag{4.6}$$

and $U(t,t')$ is defined to be $U(t)U^{-1}(t')$ as usual. $U(t,t')$ obeys the differential equation

$$i\frac{dU(t,t')}{dt} = H_I(t)U(t,t') \tag{4.7}$$

where $H_I(t)$ is the interaction Hamiltonian in the interaction picture. We note here that $U(t)$ as defined in (4.7) does not satisfy $U^\dagger(t) = U^{-1}(t)$, as $Q(\alpha) = i\bar{a}B^{-1}(t)\dot{B}(t)a$ is not hermitian. But $U(t,t')$ does satisfy the group properties:

$$U(t,t_1)U(t_1,t') = U(t,t') \tag{4.8}$$

$$U(t,t')U(t',t) = 1 \tag{4.9}$$

The interaction and Heisenberg pictures are assumed to coincide at $t = t_o$. From the differential equation and the properties listed above, it follows that $U(t,t')$ satisfies the integral equation (2.35) and its iteration gives the usual solution

$$U(t,t') = \sum \frac{(-i)^n}{n!} \int dt_1, \cdots dt_n T\left\{H_I(t_1)\cdots H_I(t_n)\right\} \quad . \tag{4.10}$$

It is to be noted that the operators $H_I(t)$ given here are in the interaction picture as defined in (4.6).

4.3 Exact Green's Functions

The two point functions to be evaluated are of the form

$$G(t,t') = \langle \psi_H(t)\bar{\psi}_H(t')\rangle \tag{4.11}$$

where the doublet notation for the field is used. $G(t,t')$ is a 2×2 matrix. The diagrammatic expansion for this exact propagator uses the following free propagators.

$$\begin{aligned} G_o^{ij}(t,t') &= \langle T\psi^i(x,t)\bar{\psi}^j(x',t')\rangle \\ &= \int d^3k e^{i\vec{k}.(x-x')}\langle Ta^i(t)\bar{a}^j(t')\rangle \qquad (4.12)\\ &= \int d^3k e^{i\vec{k}.(x-x')}G_{o_k}^{ij}(t-t') \qquad (4.13)\end{aligned}$$

where $G_{o_k}(t,t')$ is the Fourier component

$$G_{o_k}(t,t') = \langle Ta_k(t)\bar{a}_k(t')\rangle \quad .$$

In terms of the ξ operators this can be written as

$$\begin{aligned} & B^{-1}(t)\langle T\xi(t)\bar{\xi}(t')\rangle B(t') \\ = \; & B^{-1}(t)\begin{bmatrix} e^{-i\omega(t-t')}\theta(t-t') & 0 \\ 0 & -e^{-i\omega(t-t')}\theta(t'-t) \end{bmatrix} B(t') \qquad (4.14)\\ = \; & B^{-1}(t)\;\Theta\;(t-t')B(t') \qquad (4.15)\end{aligned}$$

where $\Theta\ (t-t')$ is the matrix $< T\xi(t)\bar{\xi}(t') >$. Note that it depends only on $(t-t')$ unlike $G_{o_k}(t,t')$, which depends on both t and t'. It is convenient to define the quantity

$$\Delta_k(t,t') = \langle T a_k(t)\bar{a}_k(t')\tau_3 \rangle \tag{4.16}$$

where τ_3 is the Pauli matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. All the elements of Δ_k are positive, unlike G_{o_k} which has negetive (12) and (22) elements. This is useful in writing the expression for self energy. From their definitions and from the equations of motion of $a(t)$ and $\xi(t)$, we find that $G_{o_k}(t,t')$ and $\Theta\ (t-t')$ satisfy the equations

$$\left[\frac{d}{dt} + i\omega_k + P_k(t)\right] G_{o_k}(t,t') = \delta(t-t') \tag{4.17}$$

$$\left[\frac{d}{dt} + i\omega_k\right]\ \Theta\ (t-t') = \delta(t-t') \tag{4.18}$$

The expression for the exact propagator $G(t,t') =< T\upsilon_H(t)\bar{\psi}_H(t') >$ is now obtained. The derivation is for the special case of $\alpha = 1$ and $\dot{s}(t) = \frac{\dot{n}}{2(1+n(t))}$ is dealt with. It follows that $s(t) = \frac{1}{2}\ln(1+n(t)$. Only this choice of $s(t)$ and the cases $\alpha = 0, 1$ are found in the TFD literature. From expression (3.27) for $B[n(t)]$, we find that for this choice of $s(t)$ and $\alpha = 1$ we have

$$B[n(t)] = \begin{bmatrix} 1+n(t) & -n(t) \\ -1 & 1 \end{bmatrix}, \quad B^{-1}[n(t)] = \begin{bmatrix} 1 & n(t) \\ 1 & 1+n(t) \end{bmatrix} \tag{4.19}$$

and

$$Q(t) = i\bar{a}(t)P(t)a(t) \tag{4.20}$$

with

$$P(t) = \dot{n}(t) \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

where the momentum $\vec{k}$ has been dropped from $a(t)$, $\bar{a}(t)$, $B(t)$ etc. The form of the thermal state conditions for this choice of α is given below. It is to be noted that

for interacting fields they conditions are to be written in terms of interaction picture operators .

$$\begin{aligned} \left[a(t) - f(t)\tilde{a}^{\dagger}(t)\right] |\rho> &= 0 \\ < I| \left[a^{\dagger}(t) - \tilde{a}(t)\right] &= 0 \end{aligned} \tag{4.21}$$

and similarly for the tilde operators. From the second condition it follows , for any H_I such that $H_I^{\dagger} = H_I$, that

$$< I|H_I(t) =< I|\tilde{H}_I(t) \tag{4.22}$$

and hence

$$< I|(H_I(t) - \tilde{H}_I(t)) =< I|\hat{H}(t) = 0 \tag{4.23}$$

$$< I|(\hat{U}(t,t') =< I|T \exp \hat{H}_I(t) = < I| \ . \tag{4.24}$$

In the expression for $< T\psi_H(t)\bar{\psi}_H(t') >$

$$= \frac{< I|TU^{-1}(t,t_o)\psi(t)U(t,t')\psi(t')U(t',t_o)|\rho >}{< I|\rho >} \tag{4.25}$$

the denominator

$$_H < I|\rho >_H =< I|U(\infty,-\infty)|\rho >=< I|\rho >= 1 \ . \tag{4.26}$$

The numerator

$$\begin{aligned} &< I|U(t_o,\infty)U(\infty,t_o)\left\{TU(t_o,t)\psi(t)U(t,t')\bar{\psi}(t')U(t',t_o)\right\} \times \\ &U(t_o,-\infty)U(-\infty,t_o)|\rho >= \\ &< I|U(t_o,\infty)\left\{TU(\infty,-\infty)\psi(t)\bar{\psi}(t')\right\}U(t_o,-\infty)|\rho > \end{aligned} \tag{4.27}$$

where the sequence of time variables allows the factors $U(\infty,t_o)$ and $U(t_o,-\infty)$ to be absorbed in to the time ordering. Using the thermal state conditions, the factor $U(t_o,\infty)$ on the left disappears. The choice $t_o = -\infty$ gets rid of the factor on the right, giving

$$\langle T\psi_H(t)\bar{\psi}_H(t')\rangle = \langle I|T\left\{U(\infty,-\infty)\psi(t)\bar{\psi}(t')\right\}|\rho > \tag{4.28}$$

This is the derivation given in all the TFD works. The choice $\alpha = 0$ is also considered [25]. Even there the thermal state conditions reduce the denominator in (4.26) to unity. In the $\alpha = 0$ and 1 cases, the disconnected diagrams do not arise. And an expression like (4.29) can be obtained for time ordered and antitime ordered products. In the case of any other α, the denominator does not reduce to unity. We have to deal with a time dependent norm in such a case. The presence of two time arguments t and t' would cause an ambiguity about evaluating this time dependent norm. Even if the denominator is taken care of by ignoring the disconnected diagrams in the numerator, the factors $U(\infty, t_o)$ and $U(t_o, -\infty)$ outside the time ordered product in (4.28) have to be dealt with. The choice of t_o alone,along with the thermal state conditions does not get rid of both of them. It would require the use of a Gell-Mann-Low like result. It is desirable to avoid the use of such a result in the time-dependent formalism. Also, the choice of $s(t)$ for $\alpha \neq 0, 1$ and the procedure for diagonalizing the self energy are not clear. There is no way of relating the parameter $s(t)$ to any physical variable, like the external damping in case of $\alpha = 1$. The use of $\alpha = 0$ has also been ruled out [25] on grounds that the Boltzmann equation so derived has an unphysical exponentially growing solution.

4.4 The Self Energy

In what follows we set $\alpha = 1$ and use the Schrödinger field with $|\psi|^4$ interaction as an example. We also study the Friedrich model which models a bath interacting with a single oscillator, but without the usual seperation in to the bath and system variables coupled to each other.

Consider the second order correction to $G(t, t')$ due to

$$\hat{H}_I(t) = \lambda \int d^3x \left\{ \psi^{\dagger 2}(x)\psi^2(x) - \tilde{\psi}^{\dagger 2}(x)\tilde{\psi}^2(x)) \right\} . \tag{4.29}$$

The correction is

$$-\lambda^2 \int d^3x_1 d^3x_2 dt_1 dt_2 \left\{ T\psi(t)\bar{\psi}(t')\hat{H}_I(t_1)\hat{H}_I(t_2) \right\} \quad . \tag{4.30}$$

where the negative sign comes from $(-i)^2$ of U. The corrected propagator can be written as

$$G(t,t') = G_o(t,t') + i\lambda^2 \int dt_1 dt_2 G_o(t,t_1)\Sigma(t_1,t_2)G_o(t_2,t') \tag{4.31}$$

When the fields in (4.31) are written in terms of the Fourier components and the integration with respect to x_1 and x_2 is carried out, we have the correction to $G^{11}(t,t')$ given by

$$\begin{aligned} -\lambda^2 \int dt_1 dt_2 \quad & \int d^3k d^3k' d^3p d^3q d^3r \delta(\vec{p}+\vec{q}-\vec{r}-\vec{k})\delta(\vec{k}'-\vec{p}-\vec{q}+\vec{r}) \\ \times \quad & G^{11}_{o_k}(t,t_1)G^{11}_{o_p}(t_1,t_2)G^{11}_{o_q}(t_1,t_2)G^{11}_{o_r}(t_2,t_1)G^{11}_{o'_k}(t_2,t') \\ \times \quad & \exp\{i(\vec{k}.x - \vec{k}'.x')\} \qquad (4.32) \\ = \quad & -\lambda^2 \int d^3k d^3p d^3q d^3r \, \delta(\vec{k}-\vec{p}-\vec{q}+\vec{r}) \exp\{i\vec{k}.(x-x')\} \\ \times \quad & G^{11}_{o_k}(t,t_1)G^{11}_{o_p}(t_1,t_2)G^{11}_{o_q}(t_1,t_2)G^{11}_{o_r}(t_2,t_1)G^{11}_{o_k}(t_2,t') \qquad (4.33) \end{aligned}$$

Similarly, the corrections to the other elements of $G(t,t')$ can be computed. Taking the Fourier transform of (4.34), and of the other corrections we get the matrix $\Sigma_k(t_1,t_2)$ as

$$\begin{aligned} \Sigma_k(t_1,t_2) = 2i \int d^3p d^3q d^3r \delta(\vec{k}-\vec{p}-\vec{q}+\vec{r}) \quad . \\ \begin{bmatrix} \Delta_p^{11}(t_1,t_2)\Delta_q^{11}(t_1,t_2)\Delta_r^{11}(t_2,t_1) & -\Delta_p^{12}(t_1,t_2)\Delta_q^{12}(t_1,t_2)\Delta_r^{21}(t_2,t_1) \\ \Delta_p^{21}(t_1,t_2)\Delta_q^{21}(t_1,t_2)\Delta_r^{12}(t_2,t_1) & -\Delta_p^{22}(t_1,t_2)\Delta_q^{22}(t_1,t_2)\Delta_r^{22}(t_2,t_1) \end{bmatrix} \end{aligned} \tag{4.34}$$

where

$$\Delta(t_1,t_2) = \langle Ta(t_1)\bar{a}(t_2)\rangle = G_o(t_1,t_2)\tau_3$$

has already been introduced in (4.17). The Fourier transform of (4.32) can be written as

$$G_k(t,t') = G_{o_k}(t,t') + iG_{o_k}(t,t_1)\Sigma_k(t_1,t_2)G_{o_k}(t_2,t') \tag{4.35}$$

using $\Sigma_k(t_1, t_2)$. We require that $G_k(t, t')$ have the same form as $G_{o_k}(t, t')$, as given in (4.16) i.e., $B[n_k(t)]G_k(t, t')B^{-1}[n_k(t')]$ should be diagonal. It follows that $\Sigma_k(t_1, t_2)$ is required to diagonal under the same matrix transformation.

For a general interaction, $\Sigma_k(t_1, t_2)$ has the general form

$$2i \begin{bmatrix} \Delta^{11\ m}(t_1,t_2)\Delta^{11\ n}(t_2,t_1) & \Delta^{12\ m}(t_1,t_2)\Delta^{21\ n}(t_2,t_1) \\ \Delta^{21\ m}(t_1,t_2)\Delta^{21\ n}(t_2,t_1) & \Delta^{22\ m}(t_1,t_2)\Delta^{22\ m}(t_2,t_1) \end{bmatrix} \times \tau_3 \tag{4.36}$$

For the $\upsilon|^4$ interaction we have $m = 2$ and $n = 1$. Consider now the explicit form of $\Delta_k(t_1, t_2)$. From (4.16)

$$\begin{aligned} \Delta_k(t_1,t_2) &= B^{-1}[n(t_1)]\ \Theta\ (t_1 - t_2)B[n(t_2)]\tau_3 \\ &= e^{-i\omega_k(t_1-t_2)}\theta(t_1 - t_2) \begin{bmatrix} \{1+n(t_2)\} & n(t_2) \\ \{1+n(t_2)\} & n(t_2) \end{bmatrix} \\ &+ e^{-i\omega_k(t_1-t_2)}\theta(t_2 - t_1) \begin{bmatrix} n(t_1) & n(t_1) \\ \{1+n(t_1)\} & \{1+n(t_1)\} \end{bmatrix} \end{aligned} \tag{4.37}$$

which can be rewritten as

$$e^{-i\omega_k(t_1-t_2)} \left\{ \theta(t_1 - t_2) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} f^{-1}(t_2) & 0 \\ 0 & 0 \end{bmatrix} + \theta(t_2 - t_1) \begin{bmatrix} f(t_1) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\} \tag{4.38}$$

Then the expression for $\Sigma_k^{11}(t_1, t_2)$ from (4.35) is as follows, apart from the integration and the delta function

$$\begin{aligned} \Sigma^{11}(t_1,t_2) &= \Delta_p^{11}(t_1,t_2)\Delta_q^{11}(t_1,t_2)\Delta_r^{11}(t_2,t_1) \\ &= e^{i(\omega_p+\omega_q-\omega_r)(t_1-t_2)} \Big\{ \theta(t_1 - t_2)\{1+n_p(t_2)\}\{1+n_q(t_2)\}n_r(t_2) \\ &+ \theta(t_2 - t_1)\{n_p(t_1)n_q(t_1)\{1+n_r(t_1)\}\Big\} \end{aligned} \tag{4.39}$$

where terms containing $\theta(t_1 - t_2)\theta(t_2 - t_1)$ are dropped

$$\Sigma(t_1,t_2)\tau_3 = 2ie^{iW(t_1-t_2)}\left\{\theta(t_1-t_2)\begin{bmatrix}\{1+n_p\}\{1+n_q\}n_r & n_pn_q\{1+n_r\}\\ \{1+n_p\}\{1+n_q\}n_r & n_pn_q\{1+n_r\}\end{bmatrix}_{t_2}\right.$$
$$\left. + \theta(t_2-t_1)\begin{bmatrix} n_pn_q\{1+n_r\} & n_pn_q\{1+n_r\}\\ \{1+n_p\}\{1+n_q\}n_r & \{1+n_p\}\{1+n_q\}n_r\end{bmatrix}_{t_1}\right\} \quad (4.40)$$

where the suffixes t_1 and t_2 for the matrices indicate that all the functions $n(t)$ in the matrix are to be evaluated for $t = t_1$ and t_2 respectively. Defining the quantities

$$F(t) = f_p(t)f_q(t)f_r^{-1}(t) \quad \text{and} \quad N(t) = \frac{F(t)}{1-F(t)}$$
$$f_p(t) = \frac{n_p(t)}{1+n_p(t)} \qquad F(t) = \frac{N(t)}{1+N(t)} \quad (4.41)$$

the matrix in (4.41) becomes

$$\theta(t_1-t_2)\quad n_p(t_2)n_q(t_2)\{1+n_r(t_2)\}\begin{bmatrix}F^{-1}(t_2) & 1\\ F^{-1}(t_2) & 1\end{bmatrix}$$
$$+ \theta(t_2-t_1)\{1+n_p(t_1)\}\{1+n_q(t_1)\}n_r(t_1)\begin{bmatrix}F(t_1) & F(t_1)\\ 1 & 1\end{bmatrix} \quad (4.42)$$

where $F^{-1}(t) = \frac{1}{F(t)}$.

The coefficient of the first matrix can be written as

$$n_p(t_2)n_q(t_2)\{1+n_r(t_2)\} = S(t_2)N(t_2) \quad (4.43)$$

where

$$S(t) = \frac{1}{N(t)}n_p(t)n_q(t)\{1+n_r(t)\}\ . \quad (4.44)$$

And the coefficient of second matrix is

$$\begin{aligned}\{1+n_p(t)\}\{1+n_q(t)\}n_r(t) &= \frac{1}{F(t)}n_p(t)n_q(t)\{1+n_r(t)\}\\ &= \frac{\{1+N(t)\}}{N(t)}n_p(t)n_q(t)\{1+n_r(t)\}\\ &= S(t)\{1+N(t)\}\ . \end{aligned} \quad (4.45)$$

Hence the matrix in (4.43) becomes

$$S(t_2)N(t_2)\theta(t_1-t_2)\begin{bmatrix} F^{-1}(t_2) & 1 \\ F^{-1}(t_2) & 1 \end{bmatrix} + \theta(t_2-t_1)S(t_1)\{1+N(t)\}\begin{bmatrix} F(t_1) & F(t_1) \\ 1 & 1 \end{bmatrix} \quad (4.46)$$

$$\begin{bmatrix} F^{-1}(t_2) & 1 \\ F^{-1}(t_2) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} F^{-1}(t_2) & 0 \\ 0 & 0 \end{bmatrix}$$

and the second matrix is also similar to that in (4.39). Hence $\Sigma_k(t_1,t_2)\tau_3$ has the same matrix form as $\Delta_k(t_1,t_2)$, i.e.,

$$B^{-1}[N(t_2)]\begin{bmatrix} S(t_2) & 0 \\ 0 & S(t_1) \end{bmatrix}\Theta\,(t_1-t_2)B[N(t_1)]\tau_3 \quad (4.47)$$

where the exponential factor of $\Theta(t_1-t_2)$ is $e^{-iW(t_1-t_2)}$. $\Sigma_k(t_1,t_2)$ is obtained by multiplying on the right with τ_3. Expression (4.48) with the τ_3 dropped, is the expression for $\Sigma_k(t_1,t_2)$.

4.5 The On-Mass-Shell Condition

$$\Theta\,(t_1-t_2) = e^{-iW(t_1-t_2)}\begin{bmatrix} \theta(t_1-t_2) & 0 \\ 0 & -\,\theta\,(t_2-t_1) \end{bmatrix}$$

is written as

$$\frac{1}{2\pi i}\int d\alpha e^{-i\alpha(t_1-t_2)}\begin{bmatrix} \frac{1}{W-\alpha-i\epsilon} & 0 \\ 0 & -\frac{1}{\alpha-W-i\epsilon} \end{bmatrix} \quad (4.48)$$

using the standard representation for theta function. ϵ is a infinitesimal quantity. The Fourier transforms $1/(\alpha-W-i\epsilon)$ and $1/(W-\alpha-i\epsilon)$ are expanded in a Taylor series around $\alpha=\omega_k$ and only the first term is retained. This constitutes the on-shell condition [23]. When this is done we get

$$\frac{1}{2\pi i}\qquad\int d\alpha e^{-i\alpha(t_1-t_2)}\begin{bmatrix} \frac{1}{W-\omega_k-i\epsilon} & 0 \\ 0 & \frac{1}{W-\omega_k+i\epsilon} \end{bmatrix}$$

$$= \frac{1}{2\pi i}\delta(t_1 - t_2)\left\{\frac{1}{(W-\omega_k)} + i\pi\delta(W-\omega_k)\tau_3\right\} \tag{4.49}$$

4.6 Diagonalization and Boltzmann Equation

Putting this expression for $\Theta\ (t_1 - t_2)$ into (4.48) we get

$$\begin{aligned}\Sigma_k(t_1,t_2) &= \frac{2i}{2\pi i}\delta(t_1-t_2)\int d^3p d^3q d^3r \delta(\vec{k}-\vec{p}-\vec{q}+\vec{r}) \\ &\times\ S(t_1)B^{-1}[N(t_1)]\left\{\frac{1}{W-\omega_k} + i\pi\delta(W-\omega_k)\tau_3\right\}B[N(t_1)] \end{aligned} \tag{4.50}$$

The total self energy $\Sigma'_k(t_1,t_2)$ is $\Sigma_k(t_1,t_2)$ plus the contribution from the counter term Q. $\Sigma'_k(t_1,t_2)$ is required to diagonalizable i.e., $B[n_k(t_1)]\Sigma'_k(t_1,t_2(B^{-1}[n_k(t_1)]$ is required to be diagonal. Considering only Σ_k for the time being, we see that the first term in (4.51) is diagonal, as the B's and B^{-1}'s cancel each other across the diagonal matrix . We consider the second term and compute

$$\begin{aligned} & S(t)B[n_k(t)]B^{-1}[N(t)]\tau_3 B[N(t)]B^{-1}[n_k(t)] \\ & B[n_k(t)]\begin{bmatrix} 1+2N & -2N \\ 2+2N & -1-2N \end{bmatrix}B^{-1}[n_k(t)] \\ = \ & B[n_k(t)]\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}B^{-1}[n_k(t)] + 2N(t)B[n_k(t)]\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}B^{-1}[n_k(t)] \\ = \ & \begin{bmatrix} 1 & 2n_k(t) \\ 0 & -1 \end{bmatrix} + 2N(t)\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{4.51}$$

We now compute the contribution Σ'' from

$$Q(t) = i\int d^3k \bar{a}_k(t)P_k(t)a_k(t)\ .$$

The correction to $G_{o_k}(t,t')$ in first order due to this interaction is

$$-i.i\int dt_1 d^3p\{Ta_k(t)\bar{a}_k(t')\bar{a}_p(t_1)P_p(t_1)a_p(t_1)\}\ .$$

The factor $-i$ comes from the expression for $U(\infty, -\infty)$. When this correction is written in form of equation(4.32), we find the self energy

$$i\Sigma''_{\kappa}(t_1, t_2) = \delta(t_1 - t_2)P_k(t_1) \ , \tag{4.52}$$

$$\Sigma''_{\kappa}(t_1, t_2) = -i\dot{n}\delta(t_1 - t_2)\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \tag{4.53}$$

Computing $B[n_k(t_1)]\Sigma''_k(t_1, t_2)B^{-1}[n_k(t)]$

We get

$$-i\dot{n}_k(t)\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \ . \tag{4.54}$$

Finally requiring $B[n_k(t)]\Sigma_k\prime(t)B^{-1}[n_k(t)]$ to be diagonal. we set the (12) component to zero,

$$i\pi \int d^3p d^3q d^3r \delta(\vec{k} - \vec{p} - \vec{q} + \vec{r})\delta(W - \omega_k)$$

$$\times \ \{2n_k(t)S(t) - 2N(t)S(t)\} + i\dot{n}_k(t) = 0 \tag{4.55}$$

$$\dot{n}_k(t) = 2\pi \int [dp]\delta(W - \omega_k)\{N(t)S(t) - n_k(t)S(t)\} \tag{4.56}$$

where the notation $[dp]$ is self explanatory.

This equation is the analogue of the classical Boltzmann equation and is referred to as the Boltzmann equation in non-equilibrium TFD literature. Its solution completes the non-equilibrium TFD scheme.

Going back to the expression for $B[n_k(t)]\Sigma'_k(t)B^{-1}[n_k(t)]$ (this will be referred to as Σ_{ξ_k}) we find that its diagonal elements are complex. The imaginery parts are

$$\int [dq]\delta(W - \omega_k)S(t)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ . \tag{4.57}$$

The factor outside the matrix above in denoted $\kappa(t_1)$, the damping constant. Note that

in terms of this damping constant the Boltzmann equantion can be written as

$$\dot{n}_k(t) = -2\pi\kappa(t)n_k(t) + 2\pi \int [dq]\delta(W - \omega_k)S(t)N(t) \tag{4.58}$$

The diagonal Σ_{ξ_k} gives the correction to ω_k due to the interaction, in non-equilibrium. The imaginary part of the renormalized ω_k i.e., $\kappa(t)$ gives the life time of the quasi-particles. The hamiltonian of the interacting system can be diagonalized in terms of the time dependent operators

$$\sum_k (\omega_k + \delta\omega_k)\bar{\xi}_k(t)\xi_k(t) \ ,$$

where $\delta\omega_k$ is the complex correction.

The procedure worked out above applies to other interactions too. The general form of $\Sigma(t_1, t_2)$ shows that the correction in general can be diagrammatically given as in Fig. 4 where there are m lines directed from t_1 to t_2 and n from t_2 to t_1. The definition of $F(t)$, $N(t)$ and $S(t)$ can be generalized to

$$\begin{aligned} F(t) &= f_{p_1}(t)f_{p_2}(t)\cdots f_{p_m}(t)f_{q_1}^{-1}(t)\cdots f_{q_n}^{-1}(t) \\ S(t) &= \frac{1}{N(t)} n_{p_1}(t)\cdots n_{p_m}(t)\{1 + n_{q_1}(t)\}\cdots\{1 + n_{q_n}(t)\} \end{aligned} \tag{4.59}$$

4.7 The Thermal Reservoir Model in TFD

As another example we consider the thermal reservoir model, which led to the master equation in Chapter 3. The reservoir model led us to the master equation and made contact with non-equilibrium TFD. Here we shall reverse the process, starting with a TFD model and show it reproduces the reservoir model [70].

Consider a single mode with hamiltonian $\omega_o b^\dagger b$ interacting with a field with Hamiltonian $\int d^3x\psi^\dagger\nabla\psi$ where the total H is

$$H = H_o + H_I \ = \ \omega_o b^\dagger b - \frac{1}{2}\int d^3x\psi^\dagger(x)\nabla^2\psi(x)$$

$$+ \quad \lambda \int d^3x(\psi(x)b^\dagger + \psi^\dagger(x)b) \tag{4.60}$$

The corrections to $G_b(t,t') =< Tb(t)\bar{b}(t') >$ and $G_k(t,t') =< Ta(t)\bar{a}(t') >$ are determined by the self energy matrices Σ_a and Σ_b. Note that there are two parameters $n_a(t)$ and $n_b(t)$. We set

$$\Sigma_b(t_1,t_2) = 2 \begin{bmatrix} \Delta_{a_p}^{11}(t_1,t_2) & \Delta_{a_p}^{12}(t_1,t_2) \\ \Delta_{a_p}^{21}(t_1,t_2) & \Delta_{a_p}^{22}(t_1,t_2) \end{bmatrix} \tau_3 \tag{4.61}$$

and a similar expression for $\Sigma_a(t_1,t_2)$, where the Δ_a's are replaced by Δ_b's.

. The delta function $\delta(W-\omega_k)$ which occurs in the earlier example is replaced by $\delta(\omega_p - \omega_o)$. Requiring that $B[n_b(t)]\Sigma_b(t)B^{-1}[n_b(t)]$ be diagonal gives the equation for $\dot{n}_b(t)$

$$\begin{aligned} \dot{n}_b(t) &= -2\pi n_b(t)\lambda^2 \int d^3p\delta(\omega_p-\omega_o)\delta(\vec{p}-\vec{k}_o)S(t) \\ &+ 2\lambda^2\pi \int d^3p\delta(\omega_p-\omega_o)\delta(\vec{p}-\vec{k}_o)S(t)N(t) \end{aligned} \tag{4.62}$$

where k_o is the momentum corresponding to ω_o. The diagonalization of $\Sigma_a(t)$ similarly leads to the equation for $\dot{n}_a(t)$

$$\dot{n}_{a_k}(t) = -2\pi\lambda^2 \frac{n_{a_k}(t)}{1+n_b(t)} + 2\pi\lambda^2 \frac{n_b(t)}{1+n_b(t)} \tag{4.63}$$

It is reasonable to assume that the parameter $n_a(t)$ characterizing the field does not change with time. In this case the equation for $\dot{n}(t)$ becomes

$$\begin{aligned} \dot{n}_b(t) &= -2\pi\lambda^2 \int d^3p\delta(\omega_p-\omega_o)\delta(\vec{p}-\vec{k}_o)\left\{\frac{n_b(t)-n_{a_p}}{1+n_{a_p}}\right\} \\ &= \frac{-2\lambda^2}{1+n_{a_{k_o}}}\{n_b(t)-n_{k_o}\} \end{aligned} \tag{4.64}$$

with the solution

$$n_b(t) = n_{k_o}\{1-e^{-\gamma t}n_{k_o}(0)\}$$

Where γ is the factor outside the brackets in (4.65). The oscillator attains a time independent n_a as $t \to \infty$, the equilibrium value being given by the field distribution n_{k_o}. The result mimicks the solution of the master equation. This example has been treated in [70], where the field is treated as a collection of large number of modes, but not continuous.

Equations (63) and (64) need a very careful study in order to show that the assumption about the n_a being stationary actually comes out from their solutions.

4.8 The Diagonal Nature of $\Sigma_{\xi k}$ in Equilibrium

In section 6 we saw that the diagonal self energy $\Sigma_{\xi k}$, which gives the correction to the energies ω_k of the quasi-particles can be obtained by diagonalizing Σ'_k through the similarity transformation $B[n_k(t)]$. In equilibrium we do not have the thermal generator Q, and hence there is no contribution Σ''_k to Σ'_k. There is no way of enforcing the diagonalization condition. But it turns out that the off-diagonal terms are automatically zero in equilibrium [23]. The proof of this is based on the fact that in equilibrium the two time correlation functions are functions of time differences $(t - t')$. The Bogoliubov transformation $B[n_k]$ is independent of time. They do not depend on t and t' explicitly, as in non-equilibrium.

That the 21 component of $B[n_k(t)]\Sigma'_k B^{-1}[n_k(t)]$ is zero, for the form of $B[n_k(t)]$ we have chosen, can be shown by using only one of the thermal state conditions:

$$< I|a_k =< I|\tilde{a}_k^\dagger \tag{4.65}$$

and its tildean

$$< I|\tilde{a}_k =< I|\tilde{a}_k^\dagger \ .$$

More generally,

$$< I|A =< I|\tilde{A}^\dagger.$$

for any operator A consisting of $a_k, \tilde{a}_k, a_k^\dagger$ and $\tilde{a}_k^\dagger$. The general form of the correlation function is $< TA(t)\bar{B}(t') >$, where the doublet notation is used for the operators A and B i.e., $A^1(t) = A(t), A^2(t) = \tilde{A}^\dagger(t)$ is the column matrix, and $\bar{B}'(t') = B^\dagger(t')$, $\bar{B}^2(t') = -\tilde{B}(t')$ is a row matrix. The self energy in (4.62) for example, can be written in this form

$$\left\langle T \quad \begin{bmatrix} \psi^\dagger(t)\psi^2(t) \\ \tilde{\psi}^{\dagger 2}(t)\tilde{\psi}(t) \end{bmatrix} \left[\tilde{\psi}^{\dagger 2}(t')\psi(t') - \tilde{\psi}^\dagger(t')\tilde{\psi}^2(t')\right] \right\rangle$$

$$= \int d^3p d^3q d^3r \delta(\vec{k} - \vec{p} - \vec{q} + \vec{r}) \begin{bmatrix} \Delta^{11^2}(t,t')\Delta^{11}(t',t) & -\Delta^{12^2}(t,t')\Delta^{21}(t',t) \\ \Delta^{21^2}(t,t')\Delta^{12}(t',t) & -\Delta^{22^2}(t,t')\Delta^{22}(t',t) \end{bmatrix}$$

We have for $\Sigma_{\xi,k}$

$$\begin{aligned} B[n_k(t)] \quad & < TA(t)\bar{B}(t') > [B^{-1}[n_k(t)] \\ = \; & B[n_k(t)] \begin{bmatrix} < T\{A(t)B^\dagger(t')\} > & - < TA(t)\tilde{B}(t') > \\ < T\{\tilde{A}^\dagger(t)B^\dagger(t')\} > & - < T\tilde{A}^\dagger(t)\tilde{B}(t') > \end{bmatrix} B^{-1}[n_k(t)] \\ = \; & B[n_k(t)] \begin{bmatrix} < TA(t)B^\dagger(t') > & - < B^\dagger(t')A(t) > \\ < TA(t)B^\dagger(t') > & - < \bar{T}B^\dagger(t')A(t) > \end{bmatrix} B^{-1}[n_k(t)] \quad (4.66) \end{aligned}$$

where $\bar{T}$ denotes the antitime ordering operator. The time ordering in 12 and 21 elements has been removed as the tilde and non-tilde operators commute. And condition (4.66) has also been used here. Writing the time ordered products using theta functions and carrying out the matrix multiplication we get

$$\begin{bmatrix} -\theta(t-t') < A(t)B^\dagger(t') - B^\dagger(t')A(t) > & \{1+n_k\} < B^\dagger(t')A(t) > -n_k < A(t)B^\dagger(t') > \\ 0 & \theta(t'-t) < A(t)B^\dagger(t') - B^\dagger(t')A(t) > \end{bmatrix}$$

It can be shown now that for equilibrium Greens functions, which are functions only of $(t-t')$, the 12 element vanishes. The other thermal state condition is used for this

purpose

$$\begin{aligned} a_k|\rho> &= f_k\tilde{a}^\dagger|\rho> \\ \tilde{a}_k|\rho> &= f_k a^\dagger|\rho> \end{aligned}$$

The 12 element of the self energy $\Sigma_{\xi k}$, using expression (4.35) for Σ_k and the general expression in (4.67) is

$$\begin{aligned} \int d^3pd^3qd^3r \qquad & \delta(\vec{k}-\vec{p}-\vec{q}+\vec{r})e^{-i(\omega_p+\omega_q-\omega_r)(t-t')}\Big\{f_k^{-1} < a_{t'}^\dagger a_t >_p < a_{t'}^\dagger a_t)_q < a_{t'}a_t^\dagger >_r \\ - & < a_t a_{t'}^\dagger >_p < a_t a_{t'}^\dagger >_q < a_t^\dagger a_{t'} >_r \Big\} n_k \end{aligned}$$

Here $< a_t^\dagger a_{t'} >_p$ etc., denote $< a_{tp}^\dagger a_{t'p} >$. Taking the Fourier transform of $\Sigma_{\xi,k}(t-t')$,

$$\begin{aligned} \Sigma_{\xi,k}^{21}(\omega_k) &= \int e^{i\omega_k(t-t')}\Sigma_{\xi,k}^{21}(t-t')d(t-t') \\ &= \delta(\omega_k-\omega_p-\omega_q+\omega_r)\int d^3pd^3qd^3r\delta(\vec{k}-\vec{p}-\vec{q}+\vec{r}) \\ &\quad \Big\{f_k^{-1}f_pf_qf_r^{-1} < a_t a_{t'}^\dagger >_p < a - ta_{t'}^\dagger >_r \\ &- < a_t a_{t'}^\dagger >_p < a_t a_{t'}^\dagger >_q < a_t^\dagger a_{t'} >_r \Big\} n_k \end{aligned}$$

Since

$$\begin{aligned} & \delta(\omega_k-\omega_p-\omega_q+\omega_r)f_k^{-1}f_pf_qf_r^{-1} \\ = \; & \delta(\omega_k-\omega_p-\omega_q+\omega_r)e^{-(\omega_k-\omega_p-\omega_q+\omega_r)} \\ = \; & 1 \;, \end{aligned}$$

the Fourier transform of $\Sigma_{\xi,k}^{21}$ and hence $\Sigma_{\xi,k}^{21}$ itself vanishes.

4.9 Complex Time Formalism and Non-equilibrium

Various articles in the complex time literature use the formalism for non-equilibrium processes [28-29]. The matrix propagators are defined, as in Chapter 2, to be the thermal

averages of the contour ordered fields lying on either leg of the contour. As already seen such propagators are functions only of $(t - t')$ and do not depend on both t and t'. We have already seen the complete equivalence of the complex time method and equilibrium TFD. When the correlations are evaluated for complex time arguments, it is not right to associate the real part of the time argument with a non-equilibrium evolution.The lack of time translation symmetry and hence the lack of the energy conserving δ function at each vertex is a necessary feature of non-equilibrium. These features are not incorporated in the complex time methods at a microscopic level. But in systems where external fields are present these features occur because of the explicit time dependence of these fields. The propagators become functions of $\vec{x}$, $\vec{x}'$ and t, t'. In the presence of an external field a Boltzmann like equation is derived in the following way. The Dyson equation

$$G(t,t') = G_o(t,t') + i\lambda^2 \int G_o(t,t_1)\Sigma(t_1,t_2)G_o(t_2,t')dt_1dt_2 \qquad (4.67)$$

is written. The self energy can be perturbatively evaluated using a diagrammatic expansion and the Keldysh propagators. The Keldysh propagators are constructed from the solutions of the free Schrodinger fields in presence of an external field.This equation is converted in to an integro-differential equation by applying the operator $G_o^{-1}(t)$ to (4.68) from the left, where

$$G_o^{-1}(t)G_o(t-t') = \delta(t-t')$$

and $G_o^{-1}(t) = [\frac{d}{dt} - \epsilon(-i\nabla - \frac{e}{c}\vec{A})]$ where $\vec{A}$ is the external vector potential and $\epsilon(\vec{k})$ is the dispersion law for the Schrodinger field in presence of the external field. The reason for replacing equation (4.68) this way is that it contains G_o. And the free Greens function in this formalism is associated with thermal averaging with respect to the initial, equilibrium, canonical distribution. The new equation is

$$G_o^{-1}(t)G(t-t') = i\lambda^2 \int \sum(t,t_1)G(t_1,t')dt_1 + \delta(t-t')\,. \qquad (4.68)$$

It leads to a Boltzmann equation in the complex time method in the following way. A linear transformation is applied to the matrix

$$G(t,t') = \left\langle T_c \begin{bmatrix} \psi(t) \\ \psi^\dagger(t-i\sigma) \end{bmatrix} [\psi(t')\psi^\dagger(t'-i\sigma)] \right\rangle ,$$

$$G' = \frac{(1-i\tau_2)}{\sqrt{2}} G \frac{(1+i\tau_2)}{\sqrt{2}}$$

where $i\tau_2$ is the Pauli matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The field operators ψ consist of the solutions in the presence of the external field, they are not the plane wave solutions of free Schrodinger fields. The transformed Greens function is

$$G'(t,t') = \begin{pmatrix} 0 & G^a \\ G^r & F \end{pmatrix}$$

where G^a is the advanced Greens function $\theta(t-t') < \psi(t)\psi^\dagger(t') + \psi^\dagger(t')\psi(t) >$, G^r the retarded function $\theta(t'-t) < \psi(t)\psi^\dagger(t') + \psi^\dagger(t')\psi(t) >$ and F has the form $< \psi(t)\psi^\dagger(t') - \psi^\dagger(t')\psi(t) >$. Equation (4.69) is a matrix equation. After the transformation is applied, the 22 element of this equation is used to derive a Boltzmann equation. This involves a change of variables from $\vec{x}$, $\vec{x}'$, t, t' to $(\vec{x}+\vec{x}')$, $(\vec{x}-\vec{x}')$, $(t+t')$, and $(t-t')$. A fourier transform is taken with respect to the variables $(\vec{x}-\vec{x}'$ and $(t-t')$.The transforms are functions of the variables $(\vec{x}+\vec{x}'$, $t+t'$ and also the momentum $\vec{p}$ and energy ω. Equation when applied to such a transform of $F(t,t')$ gives the Boltzmann like equation for the transform when averaged over ω.The transform of F is considered the distribution function and its evolution with $(t+t')$ is given.

In some other works [30-31], the possibility that the initial distribution differs from the canonical distribution is addressed. This is referred to as the problem of initial correlations. The complex time method is used to the extent of getting the T_c ordered product of operators. But the T_c ordered product is not written in terms of contractions

$< T_c a_k(t) a_k^\dagger(t')$ etc., making use of the thermodynamic Wicks theorem, which holds only for canonical distributions. Instead, these higher order correlation functions are related to the lower order correlations using physical considerations. Some cluster approximation are used at this stage. The lowest order correlation $< T_c a_k(t) a_k^\dagger(t') >$ etc., are undetermined, as no assumptions are made about the ensemble. But a diagrammatic expansion is possible.

As was already shown in Chapter 3, the complex time method is equivalent to equilibrium TFD. In order to have the non-equilibrium feature of lack of time translation symmetry the Q term is necessary. A Q term could be added to the complex time formalism too, by rewritting Q in terms of the fields on the two legs of contour. The correspondence between these fields and the doublet fields has already been pointed out. But such an addition of Q does not come about naturally in complex time methods. It follows naturally in TFD from the time dependent Bogoliubov transformation.

4.10 Conclusion

In conclusion, it can be said that thermofield dynamics treats systems at zero temprature, those at finite temperature and at equilibrium, and those at finite temperature and out of equilibrium in essentially the same way. Once the Hilbert space is doubled, the difference between the three is only in the choice of the vacuum. The zero temperature system has a vacuum which is annihilated by the annihilation operator a. The physical particles are generated by the corresponding creation operator. The finite temperature system is characterized by the thermal vacuum , which is not unitarily equivalent to the previous vacuum. It is annihilated by the thermal annihilation operators ξ. The ξs are related to the as by a Bogoliubov transformation. The Hamiltonian of the system can be diagonalized in terms of the physical particles generated by the corresponding creation

operators.

It is necessary to have a time dependent vacuum to describe non-equilibrium. In TFD it is assumed that this time dependent vacuum is characterized by the time dependent annihilation operators $\xi(t)$ which are related to the a by a time dependent Bogoliubov transformation. But in order that these time dependent operators be the physical particle operators, it is necessary to add the thermal generator Q to the Hamiltonian. Due to this, a counter term arises. The counter term, together with the self energy due to the interaction , determines the evolution of the time dependent vacuum.

Chapter 5

Generalized Bosonic Oscillators and their Coherent States

5.1 A Deformed Jaynes-Cummings Model

In Chapter 1 a non-linear realization of the deformed and para oscillators in terms of the bosonic operator was used. That construction made it possible to get the thermal vacuum for these systems through a unitary transformation.

A pair of bosonic operators [69][37-39]

$$b = \sqrt{\frac{N+1}{f(+1)}}a \qquad b^\dagger = a^\dagger\sqrt{\frac{N+1}{f(N+1)}} \tag{5.1}$$

can be constructed from the generalized bosonic operator

$$\begin{aligned} [a, N] = a \qquad & [a^\dagger, N] = -a^\dagger \\ [a, a^\dagger] \;\; = \;\; & g(N) \end{aligned} \tag{5.2}$$

where $g(N) = f(N+1) - f(N)$ and

$$a|n>= \sqrt{f(N)}|n-1> \qquad a^\dagger|n>= \sqrt{f(N+1)}|n+1> \tag{5.3}$$

The operators b and a are related by a similarity transformation

$$b = SaS^\dagger \;\; ,$$

as the relation (1) can be written in this form with

$$S = \sqrt{\frac{f(N)!}{N!}} \quad , \quad S^{-1} = \sqrt{\frac{N!}{f(N)!}} \;\; .$$

The form of $f(N)$ for the q-deformed and the para oscillator have already been given. And $f(N!)|n>= f(n)\cdots f(1)|n>$.

Such a realization is used to study the deformed version of the Jaynes-Cummings model [43]. The Jaynes-Cummings (JC) model [44-45] describes the interaction of a two level atom with a single cavity mode. The Hamiltonian of the model is

$$H = \omega_c N + \omega_a S_z + g(S_+ b + S_- b^\dagger)$$

where ω_c is the energy of the mode and ω_a the energy gap of the two level atom. g is a coupling constant, S_z, S_+ and S_- are the operators of the atom. S_z gives the atomic inversion and S_+ and S_- are the raising and lowering operators. They obey the algebra

$$\begin{aligned} [S_+, S_-] &= 2S_z \\ [S_\pm, S_z] &= \pm S_\pm \end{aligned} \tag{5.4}$$

The model is exactly solvable and the solution shows interesting quantum mechanical effects like the collapse and revival of atomic inversion, the inhibition of decay of excited states in a cavity and Rabi oscillations [46-48]. Another model for the interaction of a two level atom with cavity mode is due to Buck and Sukumar (BS) [49-51]. It is also exactly solvable and its solutions can be given in closed form and are more tractable analytically. The model has the Hamiltonian

$$H = \omega_c N + \omega_a S_z + g\left[S_+(1+N)^{1/2} b + S_- b^\dagger (1+N)^{1/2}\right] \tag{5.5}$$

It is an effective model for a non-linearly coupled system.

It can now be shown [43] that the deformed version of this model,

$$H = \omega_c N + \omega_a S_z + g\left[S_+(1+N)^{1/2} a + S_- a^\dagger (1+N)^{1/2}\right] \tag{5.6}$$

interpolates between the two models given above as the deformation parameter goes from 0 to 1. Here, N is the number operator for the deformed oscillator. It has the form of

an infinite series [68] in terms of $a, a^\dagger$. In terms of the corresponding bosons it is simply $b^\dagger b$. Earlier studies of deformed JC model simply replaced N by $a^\dagger a$ [52-54].

The interpolation is shown below. For the deformation [37-42]

$$aa^\dagger - qa^\dagger a = 1$$

that is considered $f(N) = \frac{1-q^N}{1-q}$, when $q = 0$, $f(N) = 1$ and $b = (N+1)^{1/2}a$. Then (5.6) is just the JC model in terms of b and $b^\dagger$ when $q = 1$, $f(N) = N$ and $b = a$. Hence (5.6) in this case is the BS model in terms of b and $b^\dagger$. For intermediate values of q it is a deformed model.

The expressions for the evolution of the atomic inversion $< S_z(t) >$ for this deformed model have been obtained for various initial states of the atom and the radiation mode [43]. When the radiation is in the usual coherent state, the evolution of $< S_z(t) >$ has the same features as in JC or BS models. The evolution has also been studied when the radiation is in the q-coherent state, the eigenstate of operator a. The collapse and revival features are blurred in this case.

The eigenstates of the generalized annihilation operators can be constructed easily when a canonical conjugate $A^\dagger$ is available, such that $[a, A^\dagger] = 1$ [37-39][69].

The operator $A^\dagger$ conjugate to a is

$$A^\dagger = \frac{a^\dagger(N+1)}{f(N+1)} \quad .$$

It satisfies the canonical commutation relation on the Fock space. We also have $[A, a^\dagger] = 1$. In terms of $A^\dagger$, the expression for the number operator is $A^\dagger a$. The eigenstate of a is $\exp \alpha A^\dagger |0>$ and that of A is $\exp \alpha a^\dagger |0>$.

5.2 A Unified Approach to Multiphoton Coherent States

Such a construction of a canonical conjugate can be also carried out for the powers of the bosonic annihilation operator [61]. These operators denoted by F have the general form $F = \phi(N)b^m$ and are called multiphoton annihilation operators in quantum optics literature. $\phi(x)$ is some function of x which does not have zeros for positive integral values of x including zero. F could also be a multimode annihilation operator.

As in the case of deformed and para oscillators it is possible to construct $G^\dagger$ satisfying

$$[F, G^\dagger] = 1 \quad . \tag{5.7}$$

Note that $G^\dagger \neq F^\dagger$ in general. The operator $G^\dagger$ can be used to construct the eigenstates of F starting from the states annihilated by F. Since F contains b^m, there are $(m-1)$ states that are annihilated by it. Denoting them by $|v_i>$, $i = 1, \cdots m-1$,

$$F|v_i> = 0$$

from which it follows that

$$\begin{aligned} \exp(fG^\dagger)F \qquad & \exp(-fG^\dagger)|f_i> \\ &= (F-f)|f_i> = 0 \\ F|f_i> &= f|f_i> \end{aligned} \tag{5.8}$$

where $|f_i> = \exp(fG^\dagger)|v_i>$. The use of canonical conjugate allows the construction of the eigenstates of F in a manner analogous to the construction of the eigenstate of b , the coherent state. Since $G^\dagger \neq F^\dagger$, we have another distinct relation

$$[G, F^\dagger] = 1 \quad .$$

From the explicit construction of $G^\dagger$ given below it can be seen that $|v_i>$ are the vacuua of G also.

Hence, the commutation relation (5.7) is two faced and it generates two sets of eigenstates.

When the operator $G^\dagger$ is constructed below we find it has a different form in each sector. A sector is the set of number states $|mn+i>$, $n=0,1,\cdots$ generated from the vacuum $|v_i>$ by the application of $F^\dagger$.

The operator

$$N_c = \frac{1}{m}[a^\dagger a + \text{a constant}] \tag{5.9}$$

which satisfies

$$[F, N_c] = F \tag{5.10}$$

is used in the construction of $G^\dagger$. The constant, c, is any real number. Now, let $G_c^\dagger$ be an operator consisting of $a^{\dagger m}$ and some function of N such that

$$FG_c^\dagger = N_c \quad .$$

Then, from $[F, N_c] = [F, FG_c^\dagger] = F$ it follows that $[F, G_c^\dagger] = 1 + X$. The operator X satisfies $FX = 0$. Such an operator can be constructed from the states annihilated by F as

$$X = \sum_{i=0}^{m-1} \mu_i |i><i| \quad .$$

Only diagonal operators are chosen because $[F, G_c^\dagger]$ is diagonal for the general form of F and $G_c^\dagger$. Using the orthogonality of the states $|i>$, we get the coefficients μ_i.

$$[F, G_c^\dagger]|i> = FG_c^\dagger|i> = (1+\mu_i)|i>$$

But since $FG_c^\dagger = N_c$,

$$\begin{aligned} <i|N_c|i> &= \frac{1}{m}(i+m-c) = (1+\mu_i) \\ \mu_i &= \frac{1}{m}(i-c). \end{aligned} \tag{5.11}$$

We have

$$[F, G_c^\dagger] = 1 + \sum_{i=0}^{m-1} \mu_i |i><i|$$

and the action of the commutator on all number states is a unit operator except for $|i>$. The freedom to choose c is now made use of to ensure that $[F, G_c^\dagger] = 1$ hold on all states of a sector. If we set $c = i$, then the operator obtained from

$$FG_i^\dagger = N_i$$

is the canonical conjugate of F in the whole of the sector generated from $|i>$. Explicity,

$$G_i^\dagger = \frac{1}{m} F^\dagger \frac{1}{FF^\dagger}(b^\dagger b + m - i) \tag{5.12}$$

when $F = b^2$, for example,

$$G_o^\dagger = \frac{1}{2} b^{\dagger 2} \frac{1}{1+N}$$

in the even number sector generated from $|0>$ and

$$G_1^\dagger = \frac{1}{2} b^{\dagger 2} \frac{1}{N+2}$$

in the odd number sector generated from $|1>$.

The eigenstates of a^2 generated from the vacuua $|0>$ and $|1>$ are $\exp fG_o^\dagger|0>$ and $\exp fG_1^\dagger|1>$. They are known as the cat states in literature. They can be expressed as the linear combinations of coherent states which are macroscopic states.

The second set of coherent states are the eigenstates of G_o and G_1 — $\exp ga^{\dagger 2}|0>$ and $\exp ga^{\dagger 2}|1>$. The former when normalized is the well known squeezed state [60] and the latter have also been studied recently [65]. The operator $G_o^\dagger$ of which the squeezed state is an eigenstate was also constructed earlier.

The same procedure holds for multimode operators. The operator N_c is of the form $\{a^\dagger a + b^\dagger b + c\}$, where $a, b, ..$ are the modes.a here stands for one of the bosonic modes,

not the generalized oscillator of the previous section. For $F = ab$, for example, there are three vacua of F. They are $|0,p>$, $|p,0>$ and $|0,0>$ where p is a positive integer. The canonical conjugates are

$$G_1^\dagger = a^\dagger b^\dagger \frac{1}{1+a^\dagger a}, G_2^\dagger = a^\dagger b^\dagger \frac{1}{1+b^\dagger b}$$
$$G_3^\dagger = \frac{1}{2}(G_1^\dagger + G_2^\dagger). \tag{5.13}$$

The eigenstates of ab, $\exp(fG_1^\dagger)|p,0>$, $\exp(fG_2^\dagger)|0,p>$, and $\exp(fG_3^\dagger)|00>$ are the pair coherent states [55-57]. These states are also the eigenstates of the operator$(a^\dagger a - b^\dagger b)$, as can be easily seen. The eigenstates of G_1, G_2, and G_3 are the second set of special states associated with the operator ab. The eigenstate of G_3 , $\exp fa^\dagger b^\dagger|0,0>$ is the well known Caves-Schumaker state [57-59].

Pairs of canonical conjugate operators were earlier constructed by Brandt and Greenberg [64]. Both operators of the kind $G^\dagger = F^\dagger$ and $G^\dagger \neq F^\dagger$ were constructed. They have the form

$$F = a^m \phi(N)^{(1-\eta)},\ G^\dagger = \phi(N)^\eta a^{\dagger n}$$

where $\phi(N)$ has the form $([N/m](N-m)!/N!)^{1/2}$ with $[x]$ denoting the largest integer smaller than or equal to x. They can be made symmetric by choosing η appropriately. But these constructions do not include pairs like

$$F = \frac{1}{N+1}a^3,\ \ G_o^\dagger = a^{\dagger 3}\frac{(N+3)}{(N+2)(N+4)}.$$

The Brandt-Greenberg states were subsequently studied for non-classical properties like squeezing [64]. The canonical conjugates of $F = a^m$ in the sector generated from $|0>$ were constructed by Buzek, Jex and Quang [66].

This construction of the canonical conjugates can also be extended to the generalized operators a^m [71]. The operator $A^\dagger$ constructed in section is used for this purpose.

The pairs of operators $(a, A^\dagger)$ and $(A, a^\dagger)$ satisfy the same commutation relations as the bosonic operators. As shown earlier, a set of bosonic operators $(b, b^\dagger)$ can be constructed corresponding to every pair $(A, a^\dagger)$.

The action of any expression consisting of the operators $(b, b^\dagger, N)$ on any number state is the same when the set is replaced by $(a, A^\dagger, N)$ or by $(A, a^\dagger, N)$. The canonical conjugates of $F = \phi(N)a^m$ are obtained from the corresponding bosonic operator by replacing $(b, b^\dagger, N)$ by $(a, A^\dagger, N)$. For $F = a^2$, for example,

$$G_o^\dagger = \frac{1}{2} A^{\dagger 2} \frac{1}{N+1}$$

,

$$G_1^\dagger = \frac{1}{2} A^{\dagger 2} \frac{1}{N+2}$$

The conjugate of G_o and G_1 is $a^{\dagger 2}$.

When the bosonic operators are replaced by $(A, a^\dagger, N)$, we have $F' = \phi(N)A^m$ and its conjugates $G_o'^\dagger$, $G_1'^\dagger$ are

$$\frac{1}{2} a^{\dagger 2} \frac{1}{N+1}$$

and

$$\frac{1}{2} a^{\dagger 2} \frac{1}{N+2}$$

respectively. The eigenstates of F, G, F', G' can be constructed as before

In conclusion, we have put to use the fact that the various generalized bosonic oscillators are related to the bosonic oscillator and a canonical conjugate and be constructed for any of them. This was applied to study a deformed version of the Jaynes-Cummings model.

Canonical conjugates have been constructed for operators of the form $\phi(N)b^m$. A systematic procedure has been given for the construction of these operators in the various sectors. A variety of eigenstates which can be called generalized coherent states, as they

are the eigenstates of annihilation operators, are constructed easily. The relation between the closely related states occuring in pairs like the squeezed states and the cat states is explained.

These results have been easily extended to the generalized bosonic oscillators using the operator $A^{\dagger}$.

References

1. H. Ezawa, Y. Tomozawa,and H. Umezawa, Nuovo Cimento **5**, 810 (1957)

2. T. Matsubara, Progress in Theoretical Physics **14**, 351, (1955)

3. P.C. Martin and J. Schwinger, Physical Review, **115**, 1342 (1959)

4. N.P. Landsman and Ch.G. Van Weert, Physics Reports, **145**, 141 (1987)

5. L. Dolan and R. Jackiw, Physical Review D **93320** (1974)

6. R.P. Feynman and A.R. Hibbs, Quantum Mechanics and Path Integrals, McGraw-Hill, 1965

7. Y. Takahashi and H. Umezawa, Collective Phenomena **2**, 55 (1975)

8. H. Umezawa, H Matsumoto, and M. Tachiki, Thermo Field Dynamics (North Holland, 1982)

9. G.W. Semenoff and H. Umezawa, Nuclear Physics B **220**, 196, (1983)

10. Propagators for Many Particle Systems, R. Mills, Gordon and Breach Science Publishers, New York, 1969

11. Quantum Theory of Many Particle Systems, A.L Fetter and J.D. Walecka, McGraw-Hill, 1971

12. M. Gell-Mann, and F. Low, Physical Review, **84**, 350,(1951)

13. H. Matsumoto, Y. Nakano, H. Umezawa, F. Mancini, and M. Marinaro Prog. in Theor. Physics, **70**, 599, (1983)

14. M. Marinaro, Physics Reports, **137**, 81, (1986)

15. H. Matsumoto, Y Nakano, and H. Umezawa, Journal of Physics, **25**, 3076 , (1984)

16. Handbook of Stochastic Methods, Springer, Berlin 1983

17. Quantum Statistical Properties of Radiation, W.H. Louisell, Wiley,1973

18. G.S Agarwal, Progress in Optics, Edited by E. Wolf, **11**,1

19. K. Wodkiewicz and J.H. Eberly, Journal of Optical society of America,B **2**458,(1985)

20. S. Chaturvedi, and V. Srinivasan, Physical Review A **43**, 4054, (1991)

21. E. Celighini, M. Rasetti, and G. Vitiello, Annals of Physics, **215**, 156, (1992)

22. P. Shanta, S. Chaturvedi, V. Srinivasan, and F. Mancini, Modern Physics Letters A **8**, 1999. (1993)

23. H. Chu and H. Umezawa, Int. Jour. Mod. Phys. A, **9**, 2363, (1994); ' Equilibrium and Non-Equilibrium Thermal Physics', H. Umezawa, in the proceedings of the Banff Workshop on Finite Temperature Field Theories, 1994

24. T. Arimitsu, M. Guida, H. Umezawa, Physica A **148**, 1, (1987)

25. J.S. Evans, I. Hardman, H.Umezawa, Y. Yamanaka, Fortschr. Phys., **41**,151, (1993)

26. I. Hardman, H. Umezawa, Y. Yamanaka, Journal of Math. Physics, **28**, 2925, (1987)

27. Y. Yamanaka, H. Umezawa, K. Nakamura, T. Arimitsu, Int. Jour. of Math. Physics, **9**, 1153, (1994)

28. L.V. Keldysh, JETP, **47**, 1515, (1964); Many Particle Physics, G.D. Mahan, Plenum Press, N.Y. 1990

29. J. Rammer and H. Smith, Reviews of Modern Physics, **58**, 323, (1986)

30. A.G. Hall, Journal of Physics, **8**, 214 (1975)

31. Yu.A. Kukharenko, and S.G. Tikhodeev,JETP, **56**, 831, (1982)

32. S.N. Biswas and A. Das, Modern Physics Letters, **A3**, 549, (1988)

33. H.S. Green Physical Review, **90**, 270(1953)

34. O.W. Greenberg and A.M.L. Massiah, Journal of Mathematical Physics **6**,500 (1965); Physical Review B **138**, 1155 (1965)

35. T. Holstein and R. Primakoff, Physical Review, **58**, 1098, (1940)

36. P. Shanta, S. Chaturvedi, A.K. Kapoor, and V. Srinivasan International Journal of Physics A **7** 3807, (1992)

37. P.P. Kulish, and E.V. Damaskinsky,Journal of Physics A **23**, L415,(1990)

38. J. Katriel and A.I. Solomon, Journal of Physics A **24** , 20093, (1991)

39. S. Chaturvedi,and V. Srinivasan, Physical Review A **44**, 8020 (1991)

40. A.J. McFarlane, Journal of Physics A **22**, 4581, (1989)

41. L.C. Beidenharn, Journal of Physics A **22**, L873 (1989)

42. C.Zachos, Symmetries in Science, edited by V.B. Gruber, Plenum, New York,1991

43. P. Shanta, S. Chaturvedi, and V. Srinivasan, Journal of Modern Optics, **39**, 1301, (1992)

44. E.T. Jaynes and F.W. Cummings, Proceedings of IEEE, **51**, 89, (1963)

45. F.W. Cummings, Physical Review A, **140**, 1051, (1965)

46. L. Allen and J.H. Eberly,Optical Resonance and Two Level Atoms (New York, Wiley, 1975)

47. J.H. Eberly, N.B. Narozhny, and J.J. Sanchez-Mondragon, Physical Review Letters, **44**, 1323, (1980)

48. H.I. Yoo,J.J. Sanchez-Mondragon and J.H. Eberly, Journal of Physics,A **14**,1383,(1981)

49. B. Buck and C.V. Sukumar, Physics Letters A, **81**, 132 (1981)

50. B.Buck and C.V. Sukumar, Journal of Physics, **17**, 885, (1984)

51. C.V. Sukumar, Journal of Modern Optics, **36**, 1591 , (1989)

52. M. Chaichian, D. Ellinas, and P. Kulish,Physical Review Letters **65**, 980,(1990)

53. V. Buzek, 1991

54. J.I. Ciriac and L.L. Sanchez-Soto Physical Review A, **42**, 2185, (1990)

55. B. Bhaumik, K. Bhaumik, and B. Dutta-Roy, Journal of Physics A **9**, 1507 (1976)

56. A.O. Barut, and C. Girardello, Communications in Math. Physics **21**, 41 (1971)

57. G.S Agarwal, Journal of Opt. Soc. of Am. B **5**, 1940 , (1988)

58. C.M. Caves and B.L. Schumaker, Physical Review A **31** , 3068, 1985)

59. C.M. Caves, Physical Review D **23**, 1683, (1981)

60. D.F. Walls, Nature **280**, 451, (1979)

61. P. Shanta, S. Chaturvedi, V. Srinivasan, G.S. Agarwal, and C.L. Mehta, Physical Review Letters,**72** 1447, (1994)

62. F. Mancini, L. Leplae, and H. Umezawa, Physics Reports, **10C** 151, (1974)

63. J.A. Crawford, Nuovo Cimento **10**, 698,(1958); B.J. Dalton, Journal of Physics, **A15**, 2157, (1982)

64. R.A. Brandt and O.W. Greenberg, Journal of Math. Physics, **10** 1168 , (1969); G.D' Ariano, M. Rasseti, J. Katriel, A. Solomon, Squeezed and Non-Classical Light, Edited by P. Tombesi and E.R. Pike, Plenum, N.Y. 1989

65. C.L. Mehta, A.K. Roy, and G.M. Saxena, Physical Review A **46**, 1565, (1992); G.S. Agarwal, Optics Communications, **100**, 479,(1993)

66. V. Buzek, I. Jex, and Tran Quang, Journal of Modern Optics, **37**, 159, (1990)

67. Opertor Methods for Master Equations in Quantum Optics, Univ. of Hyderabad preprint, HUTP-89-3

68. S. Chaturvedi, A.K. Kapoor, R. Sandhya, V. Srinivasan and R. Simon, Physical Review A, **43**, 4555, (1991)

69. S. Chaturvedi and V. Srinivasan, Physical Review A **44** 8024, (1988)

70. H. Umezawa and Y. Yamanaka, Advances in Physics, **37** 531,(1988)

71. P. Shanta, S. Chaturvedi, V. Srinivasan, and R. Jagannathan, Journal of Physics A **27**6433, (1994)

Fig.1a

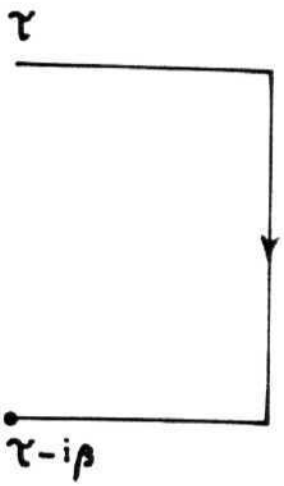

Fig.1b

Fig. 2

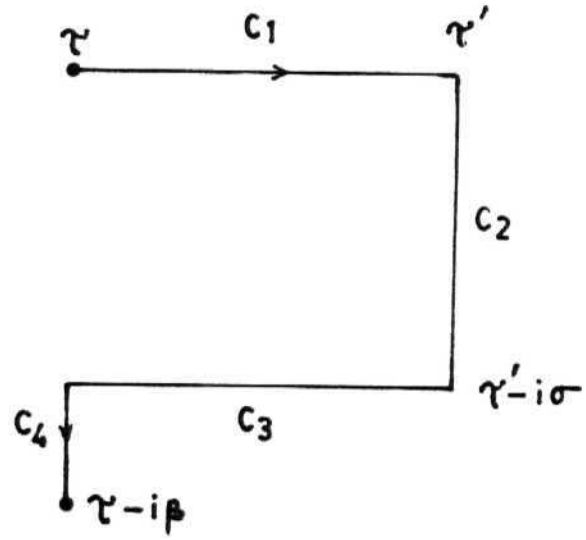

Fig.3

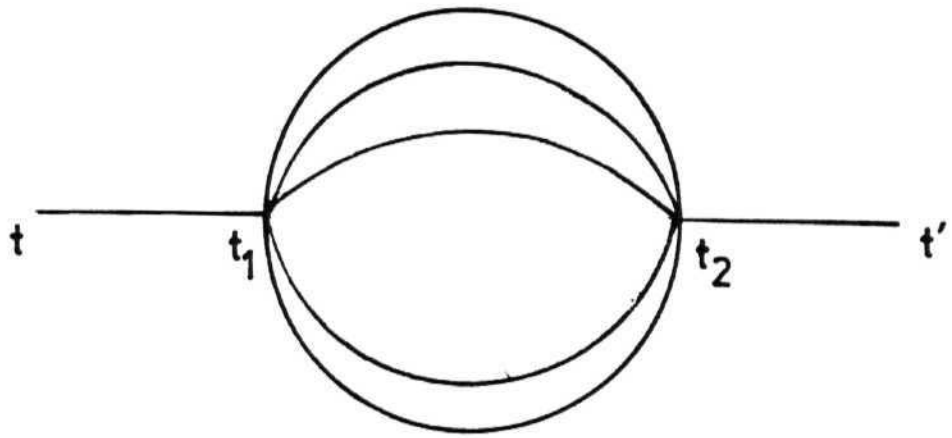

Fig. 4

JOURNAL OF MODERN OPTICS, 1992, VOL. 39, NO. 6, 1301–1308

A model which interpolates between the Jaynes–Cummings model and the Buck–Sukumar model

P. SHANTA, S. CHATURVEDI and V. SRINIVASAN
School of Physics, University of Hyderabad,
Hyderabad, 500134 India

(*Received 11 November 1991; revision received 26 February 1992*)

Abstract. A model is proposed which continuously interpolates between the Jaynes–Cummings model and the Buck–Sukumar model. This is achieved by replacing the Bosonic algebra by a q-deformed algebra. The deformation parameter q is shown to provide the desired interpolation. The atomic inversion in this model is investigated for various deformed coherent states as the initial states for the field.

1. Introduction

The Jaynes–Cummings model (JCM) [1–6]

$$H=\omega_c N+\omega_a S_3+g(S_+b+S_-b^\dagger), \tag{1}$$

describes the interaction of a two-level atom with a single cavity mode. Here b and $b^\dagger$ are the Bosonic operators of the field, $N=b^\dagger b$ is the number operator for the Bosonic mode and S_+, S_- and S_3 are the operators of the two level atom. These operators obey the algebra

$$[S_+, S_-]=2S_3, \qquad [S_\pm, S_3]=\pm S_\pm, \qquad [b, b^\dagger]=1. \tag{2}$$

This remarkably simple and exactly soluble model shows several interesting quantum-mechanical effects, such as the collapse and revivals of atomic inversion, the inhibition of decay of excited states in a cavity and Rabi oscillations and has been the subject of numerous investigations [7–12]. Recent technological advances have made it possible to experimentally realize this rather idealized model [13–16] and to verify some of the theoretical predictions.

An alternative model which retains all the gross features of the JCM and has the added virtue of yielding expressions for the physical quantities of interest which are analytically more tractable than those in the JCM was proposed by Buck and Sukumar [17]. The Hamiltonian for the Buck–Sukumar model (BSM) is

$$H=\omega_c N+\omega_a S_3+g[S_+(N+1)^{1/2}b+S_-b^\dagger(N+1)^{1/2}]. \tag{3}$$

Buck and Sukumar [18] and Sukumar [19] have also considered a class of revival Hamiltonians of which (1) and (3) are particular cases. Of course, these Hamiltonians in which the field–atom interaction involves m-photon transitions and/or intensity dependent couplings should be regarded as 'effective' Hamiltonians in contrast to the JCM which follows from the fundamental principles.

The purpose of this work is to propose a model which provides a continuous interpolation between the JCM and the BSM. This is achieved by deforming the

algebra obeyed by b and $b^\dagger$ [19]. The deformation parameter q then provides the desired interpolation between (1) and (3). For the model thus obtained, we calculate the atomic inversion for various deformed coherent states.

2. The model

We consider the following Hamiltonian

$$H=\omega_c\mathcal{N}+\omega_a S_3+g[S_+(\mathcal{N}+1)^{1/2}a+S_-a^\dagger(\mathcal{N}+1)^{1/2}], \tag{4}$$

having the same structure as the BSM and assume that the operators a and $a^\dagger$ satisfy the q-commutation relations [20–26]

$$aa^\dagger-qa^\dagger a=1. \tag{5}$$

The operator $\mathcal{N}$ in (4) is the number operator for the algebra (5) satisfying

$$[a,\mathcal{N}]=a, \qquad [a^\dagger,\mathcal{N}]=-a^\dagger. \tag{6}$$

and is explicitly given as an infinite series in a and $a^\dagger$ as

$$\mathcal{N}=\sum_{n=1}^{\infty}\frac{(1-q)^n}{(1-q^n)}(a^\dagger)^n(a)^n. \tag{7}$$

We now show that (4) does indeed interpolate between JCM and BSM with q as the interpolation parameter. This is most easily seen by using the nonlinear realization of the q algebra, in terms of a single Boson, given by [24–27]

$$a=\left[\frac{(N+1)_q}{(N+1)}\right]^{1/2}b, \qquad a^\dagger=b^\dagger\left[\frac{(N+1)_q}{(N+1)}\right]^{1/2}, \tag{8}$$

where b and $b^\dagger$ obey the Bosonic commutation relations and

$$N\equiv b^\dagger b, \tag{9}$$

is the corresponding number operator and $(x)_q$ denotes

$$(x)_q\equiv\frac{(1-q^x)}{(1-q)}. \tag{10}$$

Further, it can be shown that

$$N=\mathcal{N}, \tag{11}$$

and we may invert (8) to obtain

$$b=\left[\frac{(\mathcal{N}+1)}{(\mathcal{N}+1)_q}\right]^{1/2}a, \qquad b^\dagger=a^\dagger\left[\frac{(\mathcal{N}+1)}{(\mathcal{N}+1)_q}\right]^{1/2}. \tag{12}$$

Now, for $q=1$, (4) is clearly the Buck–Sukumar model (3), For $q=0$, since $(x)_q=1$, it follows from (12) that

$$b=(\mathcal{N}+1)^{1/2}a, \qquad b^\dagger=a^\dagger(\mathcal{N}+1)^{1/2}. \tag{13}$$

Using (13) and (11) the Hamiltonian (4) can be written in terms of b and $b^\dagger$ as

$$H=\omega_c N+\omega_a S_3+g(S_+b+S_-b^\dagger). \tag{14}$$

which is evidently the Jaynes–Commings model.

Effects of q-deformed algebras on the BSM and the JCM have also been discussed by Chaichian *et al.* [28] and by Bužek [29] respectively. The Hamiltonian considered by Chaichian *et al.* is

$$H=\omega_c(c^\dagger c+1/2)+\omega_a S_3+g[S_+c(c^\dagger c)^{1/2}+S_-(c^\dagger c)^{1/2}c^\dagger], \tag{15}$$

and that considered by Bužek [29] is

$$H=\omega_c(c^\dagger c+cc^\dagger)+\omega_a S_3+g(S_+c+S_-c^\dagger). \tag{16}$$

The operators c and $c^\dagger$ are taken to obey the following algebra

$$cc^\dagger-Qc^\dagger c=Q^{-\mathcal{N}}. \tag{17}$$

For $Q=1$, the two models reduce to the BSM and to the JCM respectively.

These models evidently differ from the one considered here in both the structure of the free part as well as the interaction part of the Hamiltonian. They are obtained respectively by writing the BSM (JCM) in terms of b and $b^\dagger$ alone and then replacing b and $b^\dagger$ by c and $c^\dagger$ which are assumed to obey (17). Our model, on the other hand is obtained by writing the BSM in terms of b, $b^\dagger$ and N and then replacing them by a, $a^\dagger$ and $\mathcal{N}$ obeying (5) and (6). It may be noted here that, for $q\neq 0$, one can go from (5) to (17) by defining

$$c=q^{-\mathcal{N}/4}a, \qquad c^\dagger=a^\dagger q^{-\mathcal{N}/4}, \tag{18}$$

if $Q=q^{1/2}$.

Further, if the algebra (17) is taken as the starting point, instead of $(x)_q$ defined in (10), one has

$$[x]_Q\equiv\frac{Q^x-Q^{-x}}{Q-Q^{-1}}$$

occurring naturally [23–25].

3. Expressions for inversion with deformed coherent states as the initial states

For simplicity we consider the case $\omega_c=\omega_a$. We assume that the initial state of the atom-field system is given by

$$|\psi(0)\rangle=|\psi_a\rangle\otimes|\psi_f\rangle, \tag{19}$$

where

$$|\psi_a\rangle=\frac{1}{(|\alpha|^2+|\beta|^2)^{1/2}}(\alpha|+\rangle+\beta|-\rangle), \tag{20}$$

and

$$|\psi_f\rangle=\sum_{n=0}^{\infty}C_n|n\rangle, \qquad \sum_{n=0}^{\infty}|C_n|^2=1. \tag{21}$$

With this initial condition the expression for the inversion at time t is found to be

$$\langle S_3(t)\rangle=\Big\{-\tfrac{1}{2}|\beta|^2+\tfrac{1}{2}\sum_{n=0}^{\infty}\{(|\alpha|^2|C_n|^2-|\beta|^2|C_{n+1}|^2)\cos\{[(n+1)(n+1)_q]^{1/2}\tau\}$$
$$+\operatorname{Im}(\alpha^*\beta c_n^* c_{n+1})\sin\{[(n+1)(n+1)_q]^{1/2}\tau\}\}\Big\}(|\alpha|^2+|\beta|^2)^{1/2}, \tag{22}$$

where $\tau=2gt$.

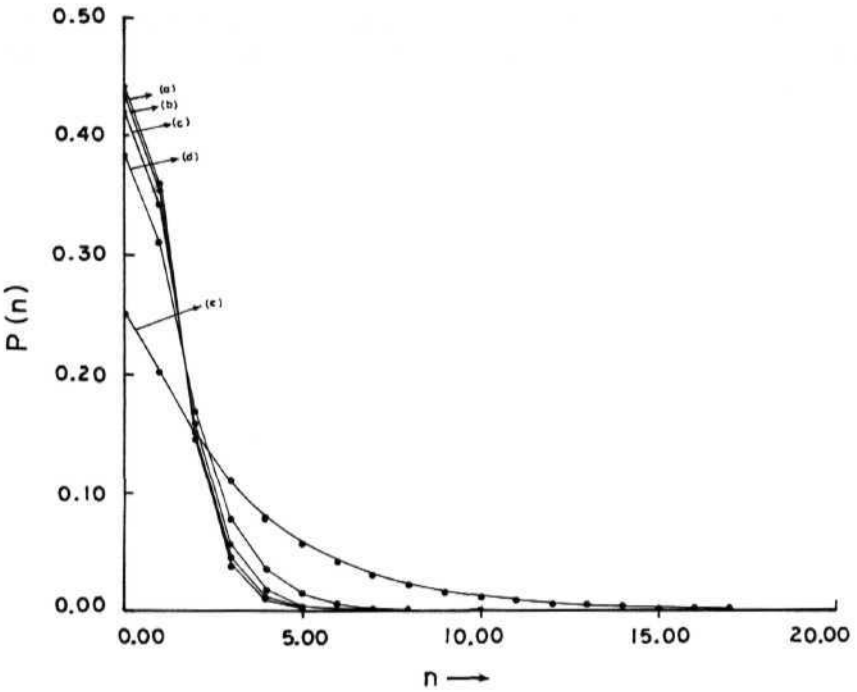

Figure 1. The number distribution for the deformed coherent state for $\lambda = 0{\cdot}9$ and $q' = 1{\cdot}0$ (*a*), 0·9 (*b*), 0·75 (*c*), 0·5 (*d*) and 0·1 (*e*).

Hereafter we restrict ourselves to the case in which the initial state of the field is a deformed coherent state [23–27]

$$|\psi_f\rangle = |\lambda, q'\rangle. \tag{23}$$

The deformed coherent states $|\lambda, q'\rangle$ are defined to be the eigenstates (corresponding to the eigenvalue λ) of the annihilation operator a obeying the algebra $aa^\dagger - q'a^\dagger a = 1$ and are explicitly given by

$$|\lambda, q'\rangle = [\exp_{q'}(|\lambda|^2)]^{-1/2} \sum_{n=0}^{\infty} \frac{\lambda^n}{[(n)_{q'}!]^{1/2}} |n\rangle. \tag{24}$$

where the q exponential $\exp_q(x)$ is defined as

$$\exp_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n)_q!}, \tag{25}$$

with

$$(n)_q! \equiv (n)_q (n-1)_q \ldots (1)_q. \tag{26}$$

For $q' = 1$ these states reduce to the well known Bosonic coherent states. The number distribution for these states—the q-Poisson distribution— is shown in figure 1.

We have numerically computed $\langle S_3(t)\rangle$ for some representative values of α, β, λ, q and q'. We consider three situations.

(1) We first consider the case $\alpha = 0$ and $q' = 1$ corresponding to the atom being initially in the ground state and the field in a Bosonic coherent state. The atomic inversion as a function of τ for $\lambda = 3$ is shown in figure 2 (*a*, *b*, *c*) which respectively correspond to $q = 0$ (JCM), $q = 1$ (BSM) and $q = 0{\cdot}5$.

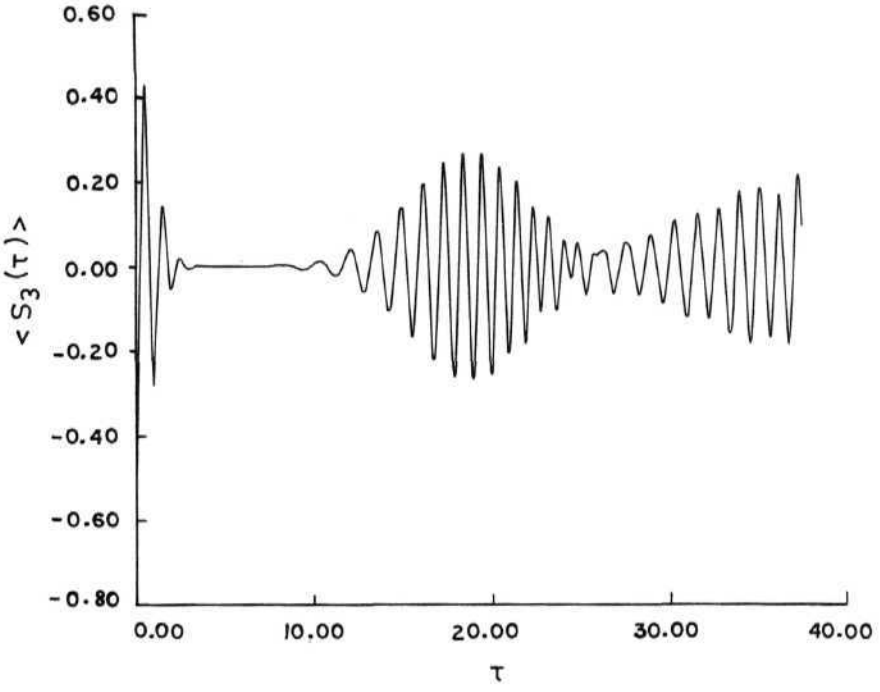

Figure 2. (*a*) $\langle S_3(\tau)\rangle$ versus τ for $q'=1$ (Bosonic coherent state), $\lambda=3$ and $q=0$ (Jaynes–Cummings model).

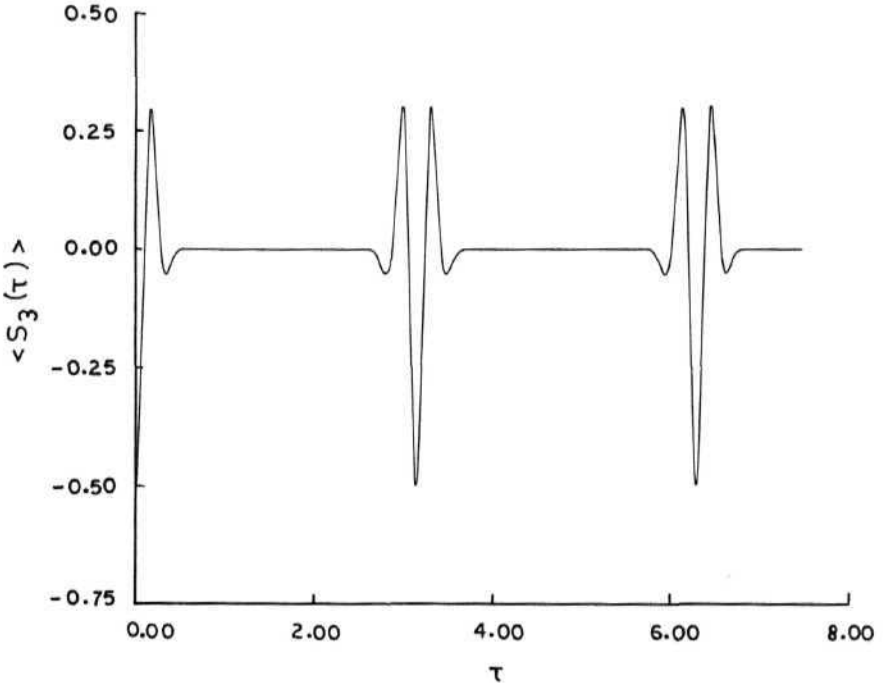

Figure 2. (*b*) Same as figure 2 (*a*) but for $q=1$ (Buck–Sukumar model).

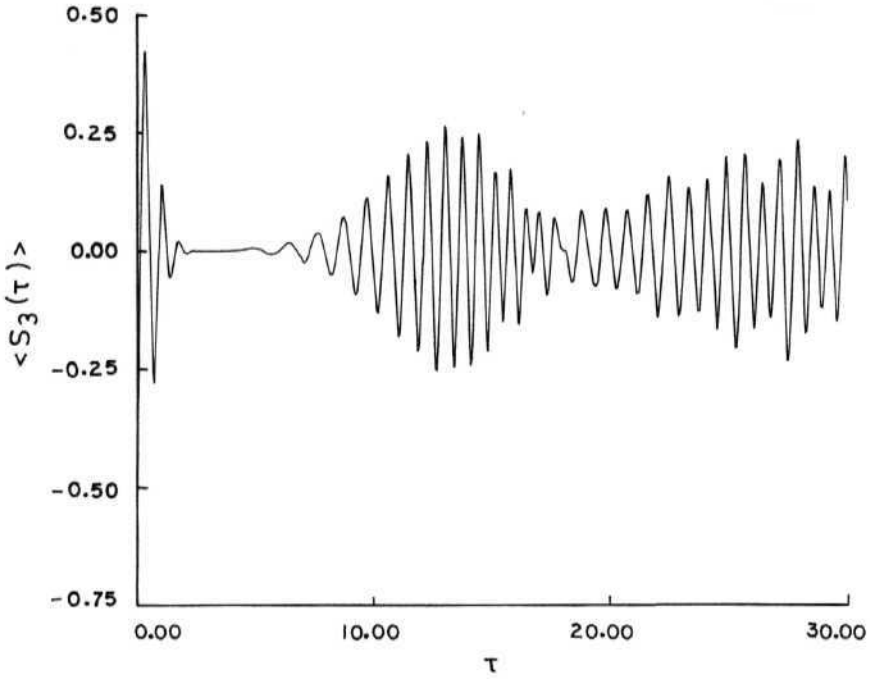

Figure 2. (*c*) Same as figure 2 (*a*) but for $q=0{\cdot}5$.

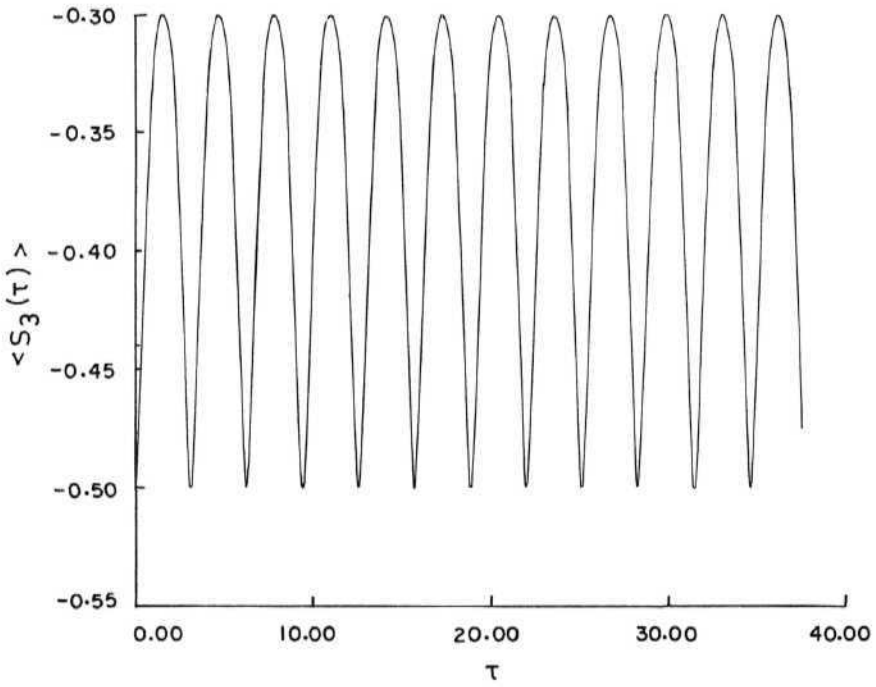

Figure 3. (*a*) $\langle S_3(\tau)\rangle$ versus τ for $q'=0$ (coherent phase state), $\lambda=0{\cdot}5$ and $q=0{\cdot}5$.

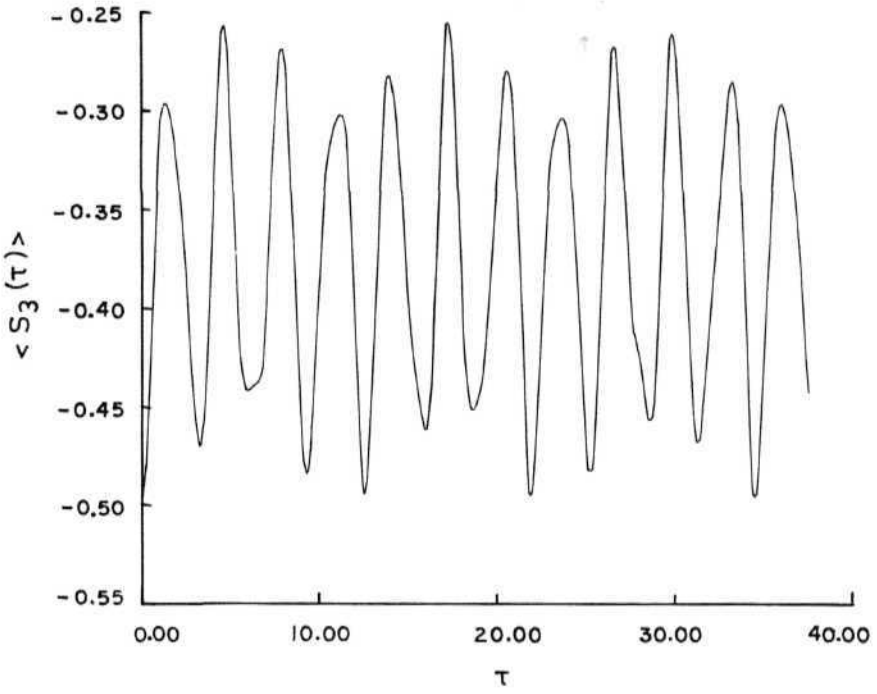

Figure 3. (*b*) Same as figure 3 (*a*) but for $q=1{\cdot}0$.

(2) Next we consider the case $\alpha=0$ and $q'=0$ corresponding to the atom being in the ground state and the field in a 'coherent phase state' [23, 25]. For $\lambda=0{\cdot}5$, $\langle S_3(t)\rangle$ is displayed in figure 3 (*a*, *b*) for $q=0{\cdot}5$ and $q=1{\cdot}0$ (BSM) respectively. Note that for $q=1$ and $q'=0$ the infinite series in (22) can be easily summed to obtain an analytic expression for $\langle S_3(t)\rangle$ for all values of α, β and λ ($|\lambda|<1$).

(3) Finally, we consider the case α, $\beta\neq 0$ and $q'=0$. This corresponds to the situation in which the initial state of the atom is a coherent mixture of the ground and excited states and the field is in a 'coherent phase state' characterized by the parameter λ. If one further chooses $\alpha=\lambda^*$ and $\beta=1$ then it follows from (22) that the atomic inversion, for all q, does not change with time as was first noted by Ciriac and Sanchez-Soto [30] in the context of the JCM.

Acknowledgments

We are grateful to Professor G. S. Agarwal for discussions. P.S. thanks the Council of Scientific and Industrial Research for financial support.

References

[1] Jaynes, E. T., and Cummings, F. W., 1963, *Proc. IEEE*, **51,** 89.
[2] Cummings, F. W., 1965, *Phys. Rev.* A, **140,** 1051.
[3] Stenholm, S., 1973, *Phys. Rep.* C, **6,** 1.
[4] Allen, L., and Eberly, J. H., 1975, *Optical Resonance and Two Level Atoms* (New York: Wiley).
[5] von Foerster, T., 1975, *J. Phys.* A, **8,** 95.
[6] Knight, P. L., and Milonni, P. W., 1980, *Phys. Rep.* C, **66,** 21.

[7] EBERLY, J. H., NAROZHNY, N. B., and SANCHEZ-MONDRAGON, J., J., 1980, *Phys. Rev. Lett.*, **44,** 1323.
[8] YOO, H.-I., SANCHEZ-MONDRAGON, J. J., and EBERLY, J. H., 1981, *J. Phys.* A, **14,** 1383.
[9] YOO, H.-I., and EBERLY, J. H., 1985, *Phys. Rep.*, **118,** 239.
[10] AGARWAL, G. S., 1984, *Phys. Rev. Lett.*, **53,** 1732.
[11] KNIGHT, P. L., 1986, *Physica scripta* T, **12,** 51.
[12] BARNETT, S. M., FILIPOWICZ, P., JAVANAINEN, J., KNIGHT, P. L., and MEYSTRE, P., 1986, *Frontiers in Quantum Optics* edited by E. R. Pike and S. Sarkar (Bristol: Adam-Hilger), p. 485.
[13] GOY, P., RAIMOND, J. M., GROSS, M.. and HAROCHE, S., 1983, *Phys. Rev. Lett.*, **50,** 1903.
[14] GABRIELSE, G., and DEHMELT, H., 1985, *Phys. Rev. Lett.*, **55,** 67.
[15] MESCHEDE, D., WALTHER, H., and MÜLLER, G., 1985, *Phys. Rev. Lett.*, **54,** 551.
[16] HAROCHE, S., and RAIMOND, J. M., 1985, *Advances in Atomic and Molecular Physics* edited by D. R. Bates and B. Bederson (New York: Academic) vol. 20, p. 347.
[17] BUCK, B., and SUKUMAR, C. V., 1981, *Phys. Lett.* A, **81,** 132.
[18] BUCK, B., and SUKUMAR, C. V., 1984, *J. Phys.* **17,** 885.
[19] SUKUMAR, C. V., 1989, *J. mod. Optics*, **36,** 1591.
[20] MCFARLANE, A. J., 1989, *J. Phys.* A, **22,** 4581.
[21] BIEDENHARN, L. C., 1989, *J. Phys.* A, **22,** L873.
[22] ZACHOS, C., 1991, *Symmetries in Science*, edited by V. B. Gruber (New York: Plenum). This work contains an exhaustive list of references on q algebras.
[23] SOLOMON, A. I., and KATRIEL, J., 1990, *J. Phys.* A, **23,** L1209.
[24] KULISH, P. P., and DAMASKINSKY, E. V., 1990, *J. Phys.* A, **23,** L415.
[25] KATRIEL, J., and SOLOMON, A. I., 1991, *J. Phys.* A, **24,** 2093.
[26] CHATURVEDI, S., KAPOOR, A. K., SANDHYA, R., SRINIVASAN, V., and SIMON, R., 1991, *Phys. Rev.* A, **43,** 4555.
[27] CHATURVEDI, S., and SRINIVASAN, V., 1991, *Phys. Rev.* A, **44,** 8020.
[28] CHAICHIAN, M., ELLINAS, D., and KULISH, P., 1990, *Phys. Rev. Lett.*, **65,** 980.
[29] BUŽEK, V., 1991, Imperial college Preprint.
[30] CIRIAC, J. I., and SÁNCHEZ-SOTO, L. L., 1990, *Phys. Rev.* A, **42,** 2185.