SELBERG–DELANGE METHOD, GODEMENT–JACQUET L-FUNCTION

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by

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CERTIFICATE

This is to certify that the thesis entitled **"Selberg–Delange method, Godement–Jacquet L-function"** submitted by **Amrinder Kaur** bearing **Reg. No. 19MMPP03** in partial fulfillment of the requirements for award of **Doctor of Philosophy** in **Mathematics** is a bonafide work carried out by her under my supervision and guidance.

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The student has the following publications before submission of the thesis for adjudication and has produced evidence for the same.

1. Amrinder Kaur and Ayyadurai Sankaranarayanan, *Godement–Jacquet L-function, some conjectures and some consequences*, Hardy–Ramanujan Journal, **45** (2023), 42–56. `https://doi.org/10.46298/hrj.2023.10747`

2. Amrinder Kaur and Ayyadurai Sankaranarayanan, *On the Rankin–Selberg L-function related to the Godement–Jacquet L-function*, Acta Mathematica Hungarica, **169(1)** (2023), 88–107. `https://doi.org/10.1007/s10474-023-01296-9`

3. Amrinder Kaur and Ayyadurai Sankaranarayanan, *The Selberg–Delange method*

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4. Amrinder Kaur and Ayyadurai Sankaranarayanan, *An analogue of Mertens function for Rankin–Selberg L-function* $L_{f\times f}(s)$, Essays in Analytic Number Theory (in honour of Prof. Helmut Maier's 70th birthday), edited jointly by Michael Th. Rassias, J. Friedlander and C. Pomerance, Springer (accepted on 24 September 2023).

5. Amrinder Kaur and Ayyadurai Sankaranarayanan, *On certain* $GL(6)$ *form and its Rankin–Selberg convolution*, Czechoslovak Mathematical Journal, (accepted on 9 December 2023).

The student has presented parts of this thesis in the following conferences.

1. Gave an online presentation entitled "Godement-Jacquet L-function, some conjectures and some consequences" on 18 November 2022 at the International Conference on Evolution in Pure and Applied Mathematics (ICEPAM-2022), organized by Akal University, Punjab.

2. Gave a presentation entitled "On the Rankin–Selberg L-function related to the Godement–Jacquet L-function" on 27 November 2022 at the International Conference on Special Functions and Applications (ICSFA-2022), organized by the University of Mysore, Mysore.

3. Gave a presentation entitled "On the Rankin–Selberg L-function related to the Godement–Jacquet L-function" on 5 February 2023 in Algebra and Number Theory Symposium (ANT-Hyd-2023) organized by University of Hyderabad, Hyderabad.

Further, the student has passed the following courses towards fulfillment of the coursework requirement for Ph.D.

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DECLARATION

I, **Amrinder Kaur**, hereby declare that this thesis entitled **"Selberg–Delange method, Godement–Jacquet L-function"** submitted by me under the guidance and supervision of **Prof. A. Sankaranarayanan**, School of Mathematics and Statistics, University of Hyderabad, is a bonafide research work. I also declare that it has not been submitted previously in part or in full to this University or any other University or Institution for the award of any degree or diploma. A report on plagiarism statistics from the University Librarian is enclosed.

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ABSTRACT

This thesis is structured into four chapters, each focusing on a distinct aspect of the research. The opening chapter provides an introduction to the primary themes of the thesis. Chapter two delves into the Selberg–Delange method and presents an asymptotic formula for an arithmetic function over short intervals, utilizing the Hooley–Huxley contour. In the third chapter, our exploration centers on the Godement–Jacquet L-function, where we establish an upper bound for the mean square of the logarithmic derivative of this function. Finally, in the fourth chapter, we investigate the Rankin–Selberg L-function associated with the Godement–Jacquet L-function. We provide an asymptotic formula for the k-th Riesz mean of the coefficients of this Rankin–Selberg L-function, subsequently leading to an asymptotic formula for the partial sum of these coefficients.

SYNOPSIS

This thesis comprises four chapters. The gist of the thesis is presented in the following sections.

0.1 Selberg–Delange method

A fundamental problem in analytic number theory is to study the behaviour of the sum $\sum_{n\le x}\mu(n)$. It is well known that a bound of the kind

$$\sum_{n\le x}\mu(n) \ll x^{\frac{1}{2}+\epsilon} \qquad \text{for every } \epsilon>0$$

is equivalent to the unproven Riemann hypothesis.

More generally, given an arithmetical function $f(n)$, studying the behaviour of the sum $\sum_{n\le x} f(n)$ is a classical problem. Perron's formula [42] is a powerful tool when one has a thorough understanding of the analytic properties of

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^s},$$

particularly its growth conditions and the nature of its singularities. However, when dealing with L-functions exhibiting unknown singularities and featuring a natural product representation, a more specialized approach is required.

In this context, we draw upon the insights of Selberg [55] and Delange [10, 11]. Their pioneering work provides a framework that enables us to study the sums in question with exceptional detail, even when facing challenging singularities of the associated L-function.

Towards this direction, in the second chapter, we consider $\mathcal{P}$ type Dirichlet series defined as:

Definition 0.1.1. *Let $\kappa > 0, w \in \mathbb{C}, \alpha > 0, \delta \geq 0, A \geq 0, B > 0, M > 0$ be some constants. A Dirichlet series $\mathcal{F}(s)$ defined as*

$$\mathcal{F}(s) := \sum_{n=1}^{\infty} f(n) n^{-s}$$

is said to be of type $\mathcal{P}(\kappa, w, \alpha, \delta, A, B, M)$ if the following conditions are satisfied:

1. *for any $\epsilon > 0$, we have*

$$\left|f(n)\right| \ll_{\epsilon} n^{\epsilon} \qquad (n \geq 1);$$

2. *we have*

$$\sum_{n=1}^{\infty} \left|f(n)\right| n^{-\sigma} \ll (\sigma - 1)^{-\alpha} \qquad (\sigma > 1);$$

3. *the Dirichlet series*

$$\mathcal{G}(s; \kappa, w) := \mathcal{F}(s) \zeta(s)^{-\kappa} \zeta(2s)^{w}$$

is analytically continued to a holomorphic function in (some open set containing) $\Re(s) \geq \frac{1}{2}$ and, in this region $\mathcal{G}(s; \kappa, w)$ satisfies the bound

$$\left|\mathcal{G}(s; \kappa, w)\right| \leq M \left(|\tau| + 1\right)^{\max\{\delta(1-\sigma), 0\}} \left(\log\left(|\tau| + 1\right)\right)^{A} \qquad (s = \sigma + i\tau)$$

uniformly for $0 < \kappa \leq B$ and $|w| \leq B$.

We prove the following result over short intervals.

Theorem 0.1.1. *Let $\kappa > 0$, $w \in \mathbb{C}$, $\alpha > 0$, $\delta \geq 0$, $A \geq 0$, $B > 0$, $M > 0$ be some constants. Let $\eta_1 > 0$ be such that*

$$\big|\zeta(\sigma + it)\big| \ll \big(|t| + 2\big)^{\eta_1(1-\sigma)} \log\big(|t| + 2\big) \qquad \text{for } \frac{1}{2} \leq \sigma \leq 1 + \frac{1}{\log\big(|t| + 2\big)}.$$

Suppose that

$$\mathcal{F}(s) := \sum_{n=1}^{\infty} f(n) n^{-s}$$

is a Dirichlet series of type $\mathcal{P}(\kappa, w, \alpha, \delta, A, B, M)$. Then for any $\epsilon > 0$ and sufficiently large $x \geq x_0(\epsilon, \kappa, A)$, we have:

$$\sum_{x<n\leq x+y} f(n) = y(\log x)^{\kappa-1} \left\{ \sum_{l=0}^{N} \frac{\lambda_l(\kappa, w)}{(\log x)^l} + O\big(R_N(x, y)\big) \right\}$$

uniformly for

$$x \geq y \geq x^{\theta(\kappa,\delta)+\epsilon},\ N \geq 0,\ 0 < \kappa \leq B,\ |w| \leq B,$$

where

$$\lambda_l(\kappa, w) := \frac{g_l(\kappa, w)}{\Gamma(\kappa - l)},$$

$$R_N(x, y) := \frac{y}{x} \sum_{l=1}^{N+1} \frac{l\big|\lambda_{l-1}(\kappa, w)\big|}{(\log x)^l} + \frac{(a_1 N + 1)^{N+1}}{x^{1/2}} + M \left\{ \left(\frac{a_1 N + 1}{\log x}\right)^{N+1} + e^{-a_2 \frac{\log x}{\log\log x}} \right\}$$

for some constants $a_1, a_2 > 0$ and

$$\theta(\kappa, \delta) := \begin{cases} \frac{5\delta+55\epsilon+7}{5\delta+5\epsilon+12} & \text{if } \kappa \leq \frac{12}{5\eta_1}, \\ \frac{\eta_1\kappa+\delta-1+11\epsilon}{\eta_1\kappa+\delta+\epsilon} & \text{if } \kappa > \frac{12}{5\eta_1}. \end{cases}$$

This improves Theorem 1.1 of [9]. (See also [53].) It is easy to check in either case (whether $\kappa \leq \frac{12}{5\eta_1}$ or $\kappa > \frac{12}{5\eta_1}$) that

$$\theta(\kappa, \delta) < \frac{5\kappa + 15\delta + 21}{5\kappa + 15\delta + 36}$$

of [9] for $\eta_1 = \frac{1}{3}$. Thus the above theorem is an improvement over the short interval length.

$\eta_1 = \frac{1}{3}$ follows from Hardy's estimate

$$\left|\zeta\left(\frac{1}{2} + it\right)\right| \ll \left(|t| + 2\right)^{\frac{1}{6}} \log\left(|t| + 2\right).$$

In fact, one may even take the best–known value $\eta_1 < \frac{1}{3}$ from the work of Bourgain in [4], giving

$$\left|\zeta\left(\frac{1}{2} + it\right)\right| \ll |t|^{\frac{13}{84}+\epsilon}.$$

If one assumes the zero density hypothesis for $\zeta(s)$, then we have

$$N(\sigma, T) \ll T^{2(1-\sigma)}(\log T)^A.$$

Thus the above theorem holds with

$$\theta(\kappa, \delta) := \begin{cases} \frac{1+\delta+11\epsilon}{2+\delta+\epsilon} & \text{if } \kappa \leq \frac{2}{\eta_1}, \\ \frac{\eta_1\kappa+\delta-1+11\epsilon}{\eta_1\kappa+\delta+\epsilon} & \text{if } \kappa > \frac{2}{\eta_1}. \end{cases}$$

0.2 Godement–Jacquet L-function

The L-function attached to a Maass form for $SL(n, \mathbb{Z})$ $(n \geq 2)$ is called the Godement–Jacquet L-function. The analysis of the characteristics and properties of the Godement-Jacquet L-function plays a crucial role in unraveling the intricacies of the generalized Ramanujan conjecture and the broader Langlands program.

In the third chapter, we study the mean square of the logarithmic derivative of the Godement–Jacquet L-function. In particular, we show the following two theorems.

Theorem 0.2.1. *Ramanujan's weak conjecture implies Rudnick–Sarnak conjecture.*

Theorem 0.2.2. *Assume $n \geq 5$ be any arbitrary but fixed integer. Let ϵ be any small positive constant and $T \geq T_0$ where T_0 is sufficiently large. Assume the Rudnick–Sarnak conjecture and Riemann hypothesis for $L_f(s)$. Then the estimate:*

$$\int_T^{2T} \left| \frac{L_f'}{L_f} (\sigma_0 + it) \right|^2 dt \ll_{f,n,\epsilon,\eta} T (\log T)^{2\eta}$$

holds for $\frac{1}{2} + \epsilon \leq \sigma_0 \leq 1 - \epsilon$ with η being some constant satisfying $0 < \eta < \frac{1}{2}$.

Since Rudnick–Sarnak conjecture is true for $2 \leq n \leq 4$, the above theorem holds just with the assumption of Riemann hypothesis for $L_f(s)$ whenever $2 \leq n \leq 4$.

It is not difficult to see from our arguments that only assuming Riemann Hypothesis for $L_f(s)$, the above theorem can be upheld for any σ_0 satisfying $1 - \frac{1}{n^2+1} + \epsilon \leq \sigma_0 \leq 1 - \epsilon$ by using the bound $\theta_n = \frac{1}{2} - \frac{1}{n^2+1}$ of Luo, Rudnick and Sarnak [36, 37].

It is also not difficult to see from our arguments that the generalized Ramanujan conjecture and the Riemann hypothesis for $L_f(s)$ together imply the bound

$$\int_T^{2T} \left| \frac{L_f'}{L_f}(\sigma_0 + it) \right|^2 dt \ll_{f,n,\epsilon} T$$

to hold for any σ_0 satisfying $\frac{1}{2} + \epsilon \leq \sigma_0 \leq 1 - \epsilon$.

Though we expect the bound stated in the above equation to hold unconditionally for σ_0 in the said range, this seems to be very hard to establish.

0.3 Rankin–Selberg L-function

In the fourth chapter, we give k-th Riesz mean for the coefficients of the Rankin–Selberg L-function related to the Godement–Jacquet L-function.

We write

$$L_{f\times f}(s) := \sum_{m=1}^{\infty} \frac{b(m)}{m^s} \qquad \text{for } \Re(s) > 1.$$

We prove the following result.

Theorem 0.3.1. *Let $n \geq 3$ be an arbitrary but fixed integer. For $k \geq k_1(n) = \left[\frac{n^2}{2}\right] + 1$, we have*

$$\sum_{m \leq x} \frac{b(m)}{k!} \left(1 - \frac{m}{x}\right)^k = \frac{Cx}{(k+1)!} + O_n(1).$$

Here C is an effective constant depending only on f.

As a consequence of the above result along with a power-lowering trick for the Riesz mean, we get an asymptotic relation for the partial sum of the coefficients of the Rankin–Selberg L-function.

Theorem 0.3.2. *For sufficiently large x, we have*

$$\sum_{m \leq x} b(m) = \frac{2^{k_1} C}{(k_1+1)} x + O_n\left(x^{1-\frac{1}{2^{k_1}}}\right)$$

where $k_1 = k_1(n) = \left[\frac{n^2}{2}\right] + 1.$

The best conditional bound that we obtain from our method is

$$\sum_{m \leq x} b(m) = Cx + O(x^{\frac{3}{4}+\epsilon}).$$

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NOTATIONS

We use the following standard conventions and notations.

$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of integers
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
$\mathbb{C}^n$	n-dimensional complex vector space
$[x]$	Greatest integer less than or equal to x
p	Prime number
$\epsilon, \epsilon_1, \eta$	Arbitrary small positive constants
a or C	with or without suffixes denote positive constants
x, x_0	Sufficiently large real number
T, T_0	Sufficiently large real number
$s = \sigma + i\tau$ or $s = \sigma + it$	Complex number
$\Re(s)$	Real part of the complex number s
$\Im(s)$	Imaginary part of the complex number s
$\|A\|$	Cardinality of set A
$d(n)$	Number of divisors of n
$d_k(n)$	Number of representations of n as a product of k factors

$\binom{n}{k}$	$= \frac{n!}{k!(n-k)!}$ denotes the binomial coefficient
$f(z) \ll g(z)$	(due to Vinogradov) means that there exists a constant $C > 0$ such that $\left\lvert f(z) \right\rvert \leq Cg(z)$ for all values of z under consideration
$f(z) = O(g(z))$	(due to Bachmann) means $f(z) \ll g(z)$
$N(\sigma, T)$	The number of zeros $\beta + i\gamma$ of the Riemann zeta function such that $\beta > \sigma, 0 < \gamma \leq T$
$\mathfrak{H}$	Hankel contour
$\mathcal{H}^n$	Generalized upper half plane
$SL(n, \mathbb{Z})$	Group of $n \times n$ matrices with integer entries and determinant one
$GL(n, \mathbb{R})$	Group of $n \times n$ invertible matrices of real numbers
$O(n, \mathbb{R})$	Orthogonal group of $GL(n, \mathbb{R})$
Z_n	Center of $GL(n, \mathbb{R})$
$\mathfrak{gl}(n, \mathbb{R})$	Lie Algebra of $GL(n, \mathbb{R})$
$\mathfrak{D}^n$	Center of universal enveloping algebra of $\mathfrak{gl}(n, \mathbb{R})$
$U_n(\mathbb{Z})$	Group of $n \times n$ upper triangular matrices with $1s$ on the diagonal and an integer entry above the diagonal

CHAPTER

ONE

CONTEXT AND OVERVIEW

1.1 Selberg–Delange method

Let

$$\mathcal{F}(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

Studying the analytic properties of Dirichlet series $\mathcal{F}(s)$ helps us understand the average behaviour of an arithmetic function $f(n)$.

When $\mathcal{F}(s)$ is a meromorphic function, the summatory function $\sum_{n \le x} f(n)$ can be obtained by application of the Perron's formula [42]. A limitation of Perron's formula is when $\mathcal{F}(s)$ has singularities which are not poles. Selberg–Delange method (independently developed by Selberg [55] and Delange [10, 11]) deals with such Dirichlet series where the nature of singularity is unknown. Another advantage of Selberg–Delange method is that there is consistency in the asymptotic behaviour of two Dirichlet series whose ratio is a sufficiently regular analytic function.

A useful reference for this theory is Chapter II.5 of Tenenbaum's book [56] from which we will use the following fundamental results.

Definition 1.1.1. Generalized binomial coefficient

For $w \in \mathbb{C}, v \in \mathbb{N}$, the generalized binomial coefficient is defined by

$$\binom{w}{v} := \frac{1}{v!} \prod_{j=0}^{v-1} (w - j).$$

For $|\xi| < 1$, $z \in \mathbb{C}$,

$$(1-\xi)^{-z} = \sum_{v=0}^{\infty} \binom{z+v-1}{v} \xi^v.$$

When z is a negative integer, this formula reduces to the classical binomial formula.

For $\Re(s) > 1$, we have

$$\begin{aligned} \zeta(s)^z &= \prod_p (1-p^{-s})^{-z} \\ &= \prod_p \left(1 + \sum_{v=1}^{\infty} \binom{z+v-1}{v} p^{-vs}\right), \end{aligned}$$

where the infinite product is absolutely convergent.

Thus, $\zeta(s)^z$ is representable in the half-plane $\Re(s) > 1$ as the Dirichlet series of a multiplicative function $d_z(n)$, defined by

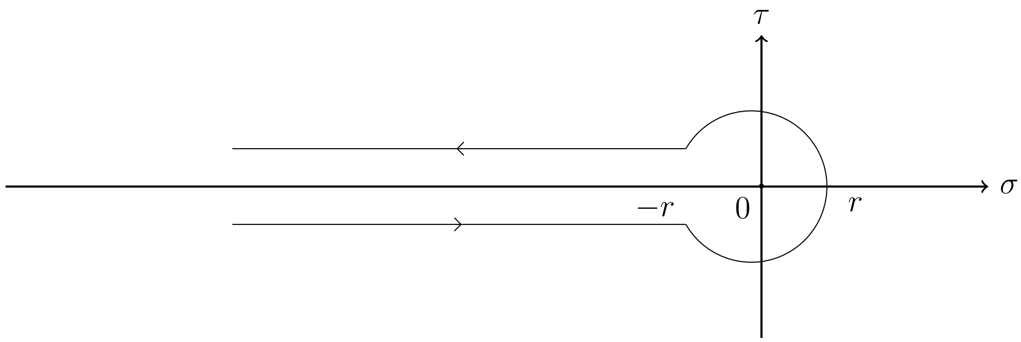
$$d_z(p^v) := \binom{z+v-1}{v}.$$

This definition generalizes that of the function $d_k(n)$ corresponding to the case when $z = k$ is a positive integer.

In order to study Dirichlet series that are close to a complex power of the Riemann zeta function, we need to study Hankel's formula.

Definition 1.1.2. Hankel Contour

Given a positive number r, we designate by Hankel contour $\mathfrak{H}$ the path formed from the circle $|s| = r$ excluding the point $s = -r$, together with the half-line $(-\infty, -r]$ traced *out twice, with respective arguments $+\pi$ and $-\pi$.*



We prove Hankel's formula next.

Theorem 1.1.1. *[56] Let $\mathfrak{H}$ be a Hankel contour. For any complex number z, we have*

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{\mathfrak{H}} s^{-z}e^{s}ds.$$

Proof. The integral is absolutely and uniformly convergent for each z. It thus defines an entire function of z. By the residue theorem, this function is independent of r, since the only singularity of the integrand is at the point $s = 0$. When $\Re(z) < 1$, the integral round the circular part $|s| = r$ of the Hankel contour tends to zero with r. The integral along the doubled half-line tends to

$$\begin{aligned}\frac{1}{2\pi i}\int_0^\infty (e^{i\pi z} - e^{-i\pi z})\sigma^{-z}e^{-\sigma}d\sigma &= \frac{\sin \pi z}{\pi}\int_0^\infty \sigma^{-z}e^{-\sigma}d\sigma\\ &= \frac{\sin \pi z}{\pi}\Gamma(1-z)\\ &= \frac{1}{\Gamma(z)}.\end{aligned}$$

This proves the result when $\Re(z) < 1$. By analytic continuation, the result follows for all z. $\square$

Corollary 1.1.1. *[56] For each $X > 1$, let $\mathfrak{H}(X)$ denote the part of the Hankel contour* *situated in the half-plane* $\sigma > -X$*. Then we have uniformly for* $z \in \mathbb{C}$*,*

$$\frac{1}{2\pi i}\int_{\mathfrak{H}(X)} s^{-z}e^{s}ds = \frac{1}{\Gamma(z)} + O\left(47^{|z|}\Gamma(1+|z|)e^{\frac{-X}{2}}\right).$$

Proof. For $s = \sigma e^{\pm i\pi}, \sigma > 1$, we have

$$\left|s^{-z}e^{s}\right| \leq (e^{\pi}\sigma)^{|z|}e^{-\sigma}.$$

Thus

$$\begin{aligned}\frac{1}{2\pi i}\int_{\mathfrak{H}} s^{-z}e^{s}ds - \frac{1}{2\pi i}\int_{\mathfrak{H}(X)} s^{-z}e^{s}ds &\ll e^{\pi|z|}\int_{X}^{\infty}\sigma^{|z|}e^{-\sigma}d\sigma\\ &\leq e^{\pi|z|-\frac{X}{2}}\int_{0}^{\infty}\sigma^{|z|}e^{\frac{-\sigma}{2}}d\sigma.\end{aligned}$$

Since $2e^{\pi} < 47$, the change of variable $\sigma = 2t$ gives the desired result.

$\square$

1.2 Godement–Jacquet L-function

The Godement–Jacquet L-function is also commonly referred to as the standard L-function. Godement and Jacquet [13] constructed this L-function and showed that it has an analytic continuation to a meromorphic function that satisfies a functional equation. Their method is a generalization from $GL(1)$ of the method of Tate's thesis [6].

According to Langlands' conjectures [34], the most general type of L-function is the one associated with an automorphic representation of $GL(n)$ over a number field. These L-functions are hypothesized to be expressible as products of the "standard" L-functions linked to cuspidal automorphic representations of $GL(n)$ over the rational numbers. These particular L-functions are considered fundamental and are referred to as (principal) primitive L-functions of degree n. Consequently, the behaviour and properties of L-functions for $GL(n)$ are pivotal in understanding the generalized Ramanujan conjecture and the larger Langlands program.

For $n = 1$, these L-functions correspond to the Riemann zeta function and Dirichlet L-functions associated with primitive Dirichlet characters. When $n = 2$, the analytical characteristics and functional equations of such L-functions were explored by Hecke and Maass, and for $n > 3$, this line of investigation was extended by Godement and Jacquet [13].

For an adelic treatment of higher degree L-functions, one can see the self-contained books by Godlfeld and Hundley [15, 16]. Cogdell's lecture notes [8] also give a nice survey for these L-functions.

We will now recollect some basic terminology from [14] that is required to deal with the Godement–Jacquet L-function in Chapter 3.

Definition 1.2.1. Generalized upper half-plane $\mathcal{H}^n$

Let $n \geq 2$. The generalized upper half-plane $\mathcal{H}^n$ associated to $GL(n, \mathbb{R})$ is defined to be the set of all $n \times n$ matrices of the form $z = x \cdot y$ where

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ & 1 & x_{2,3} & \cdots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix}, y = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix},$$

with $x_{i,j} \in \mathbb{R}$ for $1 \leq i < j \leq n$ and $y_i > 0$ for $1 \leq i \leq n-1$.

Let $GL(n, \mathbb{R})$ denote the multiplicative group of all $n \times n$ matrices with coefficients in

$\mathbb{R}$ and non-zero determinant. The orthogonal group $O(n, \mathbb{R})$ is defined as

$$O(n, \mathbb{R}) = \{g \in GL(n, \mathbb{R}) \mid g \cdot g^T = I\}$$

where I is the identity matrix on $GL(n, \mathbb{R})$. The center of $GL(n, \mathbb{R})$ is written as

$$Z_n = \left\{ \begin{pmatrix} d & & 0 \\ & \ddots & \\ 0 & & d \end{pmatrix} \middle| d \in \mathbb{R}, d \neq 0 \right\}.$$

Definition 1.2.2. Iwasawa Decomposition

The fact that every matrix in $GL(n, \mathbb{R})$ can be written as an upper triangular matrix times an orthogonal matrix is called the Iwasawa decomposition [22].

Fix $n \geq 2$. Then we have the Iwasawa decomposition

$$GL(n, \mathbb{R}) = \mathcal{H}^n \cdot O(n, \mathbb{R}) \cdot Z_n,$$

i.e., every $g \in GL(n, \mathbb{R})$ may be expressed in the form

$$g = z \cdot k \cdot d,$$

where $z \in \mathcal{H}^n$ is uniquely determined, $k \in O(n, \mathbb{R})$, and $d \in Z_n$. Further, k and d are also uniquely determined up to multiplication by $\pm I$ where I is the identity matrix on $GL(n, \mathbb{R})$.

For every $n = 1, 2, 3, \ldots,$ we have $Z_n \cong \mathbb{R}^\times$. Thus, we have the isomorphism

$$\mathcal{H}^n \cong GL(n, \mathbb{R}) / (O(n, \mathbb{R}) \cdot \mathbb{R}^\times)$$

Definition 1.2.3. Left invariant measure on the coset space $GL(n,\mathbb{R})/(O(n,\mathbb{R})\cdot \mathbb{R}^\times)$

The left invariant $GL(n,\mathbb{R})$*-measure* d^*z *on* $\mathcal{H}^n$ *can be given explicitly by the formula*

$$d^*z = d^*x \ d^*y$$

where

$$d^*x = \prod_{1\le i<j\le n} dx_{i,j}, \quad d^*y = \prod_{k=1}^{n-1} y_k^{-k(n-k)-1} dy_k.$$

For $n = 2$*, with*

$$z = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix},$$

we have

$$d^*z = \frac{dxdy}{y^2}.$$

For $n = 3$*, with*

$$z = \begin{pmatrix} y_1y_2 & x_{1,2}y_1 & x_{1,3} \\ 0 & y_1 & x_{2,3} \\ 0 & 0 & 1 \end{pmatrix},$$

we have

$$d^*z = dx_{1,2}dx_{1,3}dx_{2,3}\frac{dy_1dy_2}{(y_1y_2)^3}.$$

Definition 1.2.4. Siegel set

Let $a, b \geq 0$ be fixed. A Siegel set $\sum_{a,b} \subset \mathcal{H}^n$ is the set of all

$$\begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ & 1 & x_{2,3} & \cdots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix}$$

with $\left|x_{i,j}\right| \leq b$ for $1 \leq i < j \leq n$ and $y_i > a$ for $1 \leq i \leq n-1$.

Let $v = (v_1, v_2, \ldots, v_{n-1}) \in \mathbb{C}^{n-1}$. Let $\mathfrak{D}^{\mathfrak{n}}$ be the center of the universal enveloping algebra of $\mathfrak{gl}(n, \mathbb{R})$ where $\mathfrak{gl}(n, \mathbb{R})$ is the Lie algebra of $GL(n, \mathbb{R})$. The function

$$J_v(z) = \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} y_i^{b_{i,j} v_j}$$

with

$$b_{i,j} = \begin{cases} ij & \text{if } i + j \leq n, \\ (n-i)(n-j) & \text{if } i + j \geq n, \end{cases}$$

is an eigenfunction of every $D \in \mathfrak{D}^n$. We write

$$DJ_v(z) = \lambda_D \cdot J_v(z) \text{ for every } D \in \mathfrak{D}^n.$$

The function λ_D (viewed as a function of D) is a character of $\mathfrak{D}^n$ because it satisfies

$$\lambda_{D_1 \cdot D_2} = \lambda_{D_1} \cdot \lambda_{D_2} \quad \forall \; D_1, D_2 \in \mathfrak{D}^n.$$

It is sometimes called the Harish–Chandra character.

For $n \geq 2$, a Maass form is defined as a smooth complex valued cuspidal function on

$$\mathcal{H}^n = GL(n,\mathbb{R})/(O(n,\mathbb{R}) \cdot \mathbb{R}^\times)$$

which is invariant under the discrete group $SL(n,\mathbb{Z})$. It is also an eigenfunction of every invariant differential operator in $\mathfrak{D}^n$.

A cuspidal function (or cuspform) is a function whose Fourier expansion has no constant term. This is equivalent to the condition that the function has exponential decay at every cusp. The precise definitions of Maass forms and Godement–Jacquet L-function are given in Chapter 3.

Definition 1.2.5. Character

Fix $n \geq 2$. Let $U_n(\mathbb{R})$ denote the group of upper triangular matrices with 1s on the diagonal and real entries above the diagonal. Fix $\psi : U_n(\mathbb{R}) \to \mathbb{C}^\times$ to be a character of $U_n(\mathbb{R})$ which, by definition, satisfies the identity

$$\psi(u \cdot v) = \psi(u)\psi(v) \quad \forall\ u, v \in U_n(\mathbb{R}).$$

Definition 1.2.6. Whittaker function

Let $n \geq 2$. An $SL(n,\mathbb{Z})$–Whittaker function of type $v = (v_1, v_2, \ldots, v_{n-1}) \in \mathbb{C}^{n-1}$, associated to a character ψ of $U_n(\mathbb{R})$, is a smooth function $W : \mathcal{H}^n \to \mathbb{C}$ which satisfies the following conditions:

1. $W(uz) = \psi(u)W(z) \quad \forall\ u \in U_n(\mathbb{R}), z \in \mathcal{H}^n,$

2. $DW(z) = \lambda_D W(z) \quad \forall\ D \in \mathfrak{D}^n, z \in \mathcal{H}^n,$

3. $\int\limits_{\Sigma_{\frac{\sqrt{3}}{2},\frac{1}{2}}} \left|W(z)\right|^2 d^*z < \infty.$

Definition 1.2.7. Jacquet's Whittaker function

For $n \geq 2$, fix $m = (m_1, \ldots, m_{n-1}) \in \mathbb{Z}^{n-1}$, $v = (v_1, \ldots, v_{n-1}) \in \mathbb{C}^{n-1}$, and let

$$u = \begin{pmatrix} 1 & u_{1,2} & u_{1,3} & \ldots & u_{1,n} \\ & 1 & u_{2,3} & \ldots & u_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & u_{n-1,n} \\ & & & & 1 \end{pmatrix} \in U_n(\mathbb{R}).$$

Denote $u_1 = u_{n-1,n}, u_2 = u_{n-2,n-1}, \ldots, u_{n-1} = u_{1,2}$ Define ψ_m to be the character of $U_n(\mathbb{R})$ defined by

$$\psi_m(u) := e^{2\pi i(m_1u_1+m_2u_2+\cdots+m_{n-1}u_{n-1})}.$$

(All characters of $U_n(\mathbb{R})$ are of this form.)

For $z \in \mathcal{H}^n$ and $m_i \neq 0$ $(1 \leq i \leq n-1)$, define

$$W_J(z; v, \psi_m) := \int_{U_n(\mathbb{R})} J_v(w_n \cdot u \cdot z)\overline{\psi_m(u)}\ d^*(u)$$

to be Jacquet's Whittaker function. Here,

$$w_n = \begin{pmatrix} & & & (-1)^{\left[\frac{n}{2}\right]} \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix} \in SL(n, \mathbb{Z}) \text{ and}$$

$$\int_{U_n(\mathbb{R})} d^*(u) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{1 \leq i < j \leq n} du_{i,j}.$$

1.3 Rankin–Selberg Method

The Rankin–Selberg method was independently developed by Rankin [47, 48] and Selberg [54]. Rankin has remarked that the general idea came from his advisor and mentor, Ingham.

They found out the meromorphic continuation and functional equation of the convolution L-function associated to automorphic forms on $GL(2)$ i.e.,

$$L_{f\times g}(s) = \zeta(2s)\sum_{n=1}^{\infty}\frac{a(n)\overline{b(n)}}{n^s}.$$

They showed that the convolution L-function can be constructed explicitly by taking an inner product of $f \cdot \overline{g}$ with an Eisenstein series. This remarkable development has proven to be of exceptional importance and has generated numerous unforeseen consequences.

Jacquet [23] obtained a broader interpretation of the original Rankin–Selberg convolution within the context of adeles and automorphic representations. Jacquet and Shalika [25] further generalized the theory.

The Rankin–Selberg convolution for the case $GL(n)\times GL(n')$ $(1 \leq n < n')$ necessitates a novel approach. The special case $GL(1)\times GL(n')$ is essentially the Godement–Jacquet L-function. Godement and Jacquet [13] first obtained the holomorphic continuation and functional equation for the Godement–Jacquet L-function. Jacquet, Piatetskii-Shapiro and Shalika [24] further extended the theory for the case of automorphic representations.

The Rankin–Selberg convolution stands as one of the most pivotal constructions within the realm of L-function theory, and it has naturally led to countless generalizations. A comprehensive survey paper by Bump [5] provides an expansive overview of this entire subject, offering valuable insights and perspectives.

Rankin–Selberg method has found numerous applications. The classic application of the Rankin–Selberg method is to obtain strong bounds for the Fourier coefficients of automorphic forms and eigenvalues of a Maass form. Deligne [12] employed this method as the initial step in settling the Ramanujan conjecture for holomorphic modular forms. However, the conjecture remains open for Maass forms. For Maass forms, the bound established by Luo, Rudnick, and Sarnak [36, 37] for the generalized Ramanujan and Selberg conjectures remains as the best-known bound to date. The Rankin–Selberg method played a pivotal role in achieving this result. Another application of the Rankin–Selberg method is the proof for the strong multiplicity one theorem by Jacquet and Shalika [25].

Rankin and Selberg established that

$$\sum_{n\leq x}\lambda_f(n) = C_f x + O_f(x^{\frac{3}{5}}).$$

Here, f is a holomorphic Hecke cusp form or Hecke–Maass cusp form for $SL(2,\mathbb{Z})$. After almost 80 years, the exponent in the error term was slightly improved by B. Huang in [18] to $\frac{3}{5}-\delta$ for any $\delta < 1/560 = 0.001785\ldots$. In recent preprints, S. Pal [41] proved that $\delta < 6/1085 = 0.005529\ldots$ is admissible and B. Huang [19] improved it further to any $\delta < 3/305 = 0.009836\ldots$.

In this direction, we study the asymptotic behaviour of the k-th Riesz mean for the coefficients of the Rankin–Selberg L-function related to the Godement–Jacquet L-function in the fourth chapter. As a result of this relation, we also obtain a relation for the summatory function of the coefficients of the Rankin–Selberg L-function.

CHAPTER

TWO

THE SELBERG–DELANGE METHOD AND MEAN VALUE OF ARITHMETIC FUNCTIONS OVER SHORT INTERVALS

2.1 Introduction

A classical problem in analytic number theory is to study the behaviour of the sum $\sum_{n \le x} \mu(n)$. It is well known that a bound of the kind

$$\sum_{n \le x} \mu(n) \ll x^{\frac{1}{2}+\epsilon} \qquad \text{for every } \ \epsilon > 0$$

is equivalent to the unproven Riemann hypothesis.

More generally, given an arithmetical function $f(n)$, studying the behaviour of the sum $\sum_{n \le x} f(n)$ is a classical problem. If one knows the analytic properties of the L–function attached to $f(n)$, namely

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

(particularly certain growth conditions) and if one knows the nature of the singularity (particularly having only real poles), then Perron's formula [42] is an appropriate tool to obtain the asymptotic nature of the required sum with a possible good error term. However, if the L–function has some singularities whose nature is unknown and has some natural product representation, then Selberg [55] and later Delange [10, 11] developed a method that enables us to study the sum in question in detail.

Throughout the chapter, the constants a with suffixes are positive constants that need not be the same at each occurrence. ϵ, η are small positive constants and x is sufficiently large.

In this chapter, we consider $\mathcal{P}$ type Dirichlet series defined as:

Definition 2.1.1. *Let $\kappa > 0, w \in \mathbb{C}, \alpha > 0, \delta \geq 0, A \geq 0, B > 0, M > 0$ be some constants. A Dirichlet series $\mathcal{F}(s)$ defined as*

$$\mathcal{F}(s) := \sum_{n=1}^{\infty} f(n)n^{-s}$$

is said to be of type $\mathcal{P}(\kappa, w, \alpha, \delta, A, B, M)$ if the following conditions are satisfied:

1. *for any $\epsilon > 0$, we have*

$$\left|f(n)\right| \ll_{\epsilon} n^{\epsilon} \qquad (n \geq 1);$$

2. *we have*

$$\sum_{n=1}^{\infty}\left|f(n)\right| n^{-\sigma} \ll (\sigma - 1)^{-\alpha} \qquad (\sigma > 1);$$

3. *the Dirichlet series*

$$\mathcal{G}(s; \kappa, w) := \mathcal{F}(s)\zeta(s)^{-\kappa}\zeta(2s)^{w}$$

is analytically continued to a holomorphic function in (some open set containing)

$\Re(s) \geq \frac{1}{2}$ *and, in this region* $\mathcal{G}(s;\kappa,w)$ *satisfies the bound*

$$\big|\mathcal{G}(s;\kappa,w)\big| \leq M\left(|\tau|+1\right)^{\max\{\delta(1-\sigma),0\}}\Big(\log\left(|\tau|+1\right)\Big)^{A} \qquad (s=\sigma+i\tau)$$

uniformly for $0<\kappa\leq B$ *and* $|w|\leq B$.

From [56, Theorem II.5.1], the function

$$Z(s;z) := \big\{(s-1)\zeta(s)\big\}^{z} \qquad (z\in\mathbb{C})$$

is holomorphic in the disc $|s-1|<1$, and admits the Taylor series expansion

$$Z(s;z)=\sum_{j=0}^{\infty}\frac{\gamma_j(z)}{j!}(s-1)^j,$$

where the $\gamma_j(z)'s$ are entire functions of z and satisfy: for all $B>0$ and $\epsilon>0$, the estimate

$$\frac{\gamma_j(z)}{j!} \ll_{B,\epsilon} (1+\epsilon)^j \qquad (j\geq 0, |z|\leq B).$$

Under our hypothesis, the function $\mathcal{G}(s;\kappa,w)\zeta(2s)^{-w}Z(s;\kappa)$ is holomorphic in the disc $|s-1|<\frac{1}{2}$ and

$$\big|\mathcal{G}(s;\kappa,w)\zeta(2s)^{-w}Z(s;\kappa)\big| \ll_{A,B,\delta,\epsilon} M$$

for $|s-1|\leq\frac{1}{2}-\epsilon$, $0<\kappa\leq B$ and $|w|\leq B$.

Theorem 2.1.1. *[30] Let* $\kappa>0$, $w\in\mathbb{C}$, $\alpha>0$, $\delta\geq 0$, $A\geq 0$, $B>0$, $M>0$ *be some constants. Let* $\eta_1>0$ *be such that*

$$\big|\zeta(\sigma+it)\big| \ll \left(|t|+2\right)^{\eta_1(1-\sigma)}\log\left(|t|+2\right) \qquad \textit{for } \frac{1}{2}\leq\sigma\leq 1+\frac{1}{\log\left(|t|+2\right)}.$$

Suppose that

$$\mathcal{F}(s) := \sum_{n=1}^{\infty} f(n) n^{-s}$$

is a Dirichlet series of type $\mathcal{P}(\kappa, w, \alpha, \delta, A, B, M)$. *Then for any* $\epsilon > 0$ *and sufficiently large* $x \geq x_0(\epsilon, \kappa, A)$, *we have:*

$$\sum_{x < n \leq x+y} f(n) = y(\log x)^{\kappa-1} \left\{ \sum_{l=0}^{N} \frac{\lambda_l(\kappa, w)}{(\log x)^l} + O\left(R_N(x,y)\right) \right\}$$

uniformly for

$$x \geq y \geq x^{\theta(\kappa,\delta)+\epsilon},\ N \geq 0,\ 0 < \kappa \leq B,\ |w| \leq B,$$

where

$$\lambda_l(\kappa, w) := \frac{g_l(\kappa, w)}{\Gamma(\kappa - l)},$$

$$R_N(x,y) := \frac{y}{x} \sum_{l=1}^{N+1} \frac{l\left|\lambda_{l-1}(\kappa, w)\right|}{(\log x)^l} + \frac{(a_1 N + 1)^{N+1}}{x^{1/2}} + M \left\{ \left(\frac{a_1 N + 1}{\log x} \right)^{N+1} + e^{-a_2 \frac{\log x}{\log\log x}} \right\}$$

for some constants $a_1, a_2 > 0$ *and*

$$\theta(\kappa, \delta) := \begin{cases} \frac{5\delta+55\epsilon+7}{5\delta+5\epsilon+12} & \text{if } \kappa \leq \frac{12}{5\eta_1}, \\ \frac{\eta_1\kappa+\delta-1+11\epsilon}{\eta_1\kappa+\delta+\epsilon} & \text{if } \kappa > \frac{12}{5\eta_1}. \end{cases}$$

Remark 2.1.1. This improves Theorem 1.1 of [9]. (See also [53].) It is easy to check in either case (whether $\kappa \leq \frac{12}{5\eta_1}$ or $\kappa > \frac{12}{5\eta_1}$) that

$$\theta(\kappa, \delta) < \frac{5\kappa + 15\delta + 21}{5\kappa + 15\delta + 36}$$

of [9] for $\eta_1 = \frac{1}{3}$. Thus the above theorem is an improvement over the short interval length. The implied O–constant depends on various parameters like $A, B, \epsilon, \delta, \eta$ etc.

Remark 2.1.2. $\eta_1 = \frac{1}{3}$ follows from Hardy's estimate

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \ll \left(|t|+2\right)^{\frac{1}{6}} \log\left(|t|+2\right).$$

In fact, one may even take the best–known value $\eta_1 < \frac{1}{3}$ from the work of Bourgain in [4], giving

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \ll |t|^{\frac{13}{84}+\epsilon}.$$

Remark 2.1.3. If one assumes the zero density hypothesis for $\zeta(s)$, then we have

$$N(\sigma, T) \ll T^{2(1-\sigma)}(\log T)^A.$$

Thus the above theorem holds with

$$\theta(\kappa,\delta) := \begin{cases} \frac{1+\delta+11\epsilon}{2+\delta+\epsilon} & \text{if } \kappa \le \frac{2}{\eta_1}, \\ \frac{\eta_1\kappa+\delta-1+11\epsilon}{\eta_1\kappa+\delta+\epsilon} & \text{if } \kappa > \frac{2}{\eta_1}. \end{cases}$$

2.2 Construction of the Hooley–Huxley contour of integration

To construct the required Hooley–Huxley contour for our situation, we follow certain descriptions from H. Maier and A. Sankaranarayanan in [38].

Let C^* be a generic absolute constant in the following, which need not be the same at each occurrence.

Definition 2.2.1. A zero $\rho = \beta + i\gamma$ $\left(\text{with } \beta \geq \frac{1}{2}\right)$ of $\zeta(s)$ is said to be *good* if $\beta < 1 - \frac{C^*}{\log\log(|\gamma|+2)}$ and ρ is said to be *exceptional* otherwise.

Let $T \geq T_0$ and $x \geq x_0$ (T_0 and x_0 are sufficiently large). Let $\mathcal{G}$ and $\mathcal{E}$ denote the set of all good and exceptional zeros of $\zeta(s)$ respectively with $|\gamma| \leq T$. We denote by $|\mathcal{G}|$ and $|\mathcal{E}|$ to mean the cardinality of the sets $\mathcal{G}$ and $\mathcal{E}$ respectively.

Let α be any fixed constant satisfying $\frac{1}{2} + \eta \leq \alpha \leq 1 - \eta$ with η being any arbitrarily small fixed positive constant. Since the contour will be symmetric with respect to the real axis, it suffices to describe it in the upper half–plane. We assume that $|\mathcal{E}| = 0$. Hence, $\zeta(s) \neq 0$ in the region $\left\{\sigma > 1 - \frac{C^{**}}{\log\log(U+12)},\ U \leq t \leq 2U\right\}$ where C^{**} is a suitable absolute positive constant and we construct the contour accordingly.

Let $T = 2^{l_0}$. We choose c with $\frac{1}{2} \leq c \leq 1$ such that $H_0 = c\log\log T = 2^L$ with a positive integer L. For $l \geq L$, write $U = U^{(l)} = 2^l$. We define the contour for $U \leq t \leq 2U$. Let $H = H(U^{(l)}) = c_l \log\log(U^{(l)})$ and choose c_l satisfying $\frac{1}{2} \leq c_l \leq 1$ such that $\frac{U}{2H}$ is a positive integer.

We split the interval $[U, 2U]$ into $\frac{U}{2H}$ disjoint abutting small intervals $I_j = I_j^{(l)}$ of equal length $2H$ for $1 \leq j \leq \frac{U}{2H}$. Let $I_j = [U_j - H, U_j + H]$ and let

$$\beta_j = \sup\left\{\beta \mid \rho = \beta + i\gamma,\ \zeta(\rho) = 0,\ \beta \geq \alpha,\ \gamma \in [U_j - 2H, U_j + 2H]\right\}$$

and

$$\beta_j^* = \beta_j + \frac{C^*}{\log\log 2(U+12)}.$$

We also define $\big(\text{with } H_0' = H_0 + 2(\log H_0)^2\big)$

$$\beta_0 = \sup\big\{\beta \mid \rho = \beta + i\gamma,\ \zeta(\rho) = 0,\ \beta \geq \alpha,\ \gamma \in [0,\, 2H_0']\,\big\}$$

and

$$\beta_0^* = \beta_0 + \frac{C^*}{\log\log 2H_0}.$$

If there is no zero of $\zeta(s)$ in the rectangle $\big\{\sigma \geq \alpha,\ U_j - 2H \leq t \leq U_j + 2H\big\}$, then we define $\beta_j^* = \alpha$. A similar notion applies to β_0^* too.

Then the contour $\mathcal{C}$ consists of

1. Vertical pieces (V_j):

$$V_j = \begin{cases} [\beta_j^* + i(U_j - H + \epsilon),\ \beta_j^* + i(U_j + H - \epsilon)] & \text{if } \beta_j^* < \min(\beta_{j-1}^*, \beta_{j+1}^*) \\ [\beta_j^* + i(U_j - H - \epsilon),\ \beta_j^* + i(U_j + H + \epsilon)] & \text{if } \beta_j^* > \max(\beta_{j-1}^*, \beta_{j+1}^*) \\ [\beta_j^* + i(U_j - H - \epsilon),\ \beta_j^* + i(U_j + H - \epsilon)] & \text{if } \beta_{j-1}^* < \beta_j^* < \beta_{j+1}^* \\ [\beta_j^* + i(U_j - H + \epsilon),\ \beta_j^* + i(U_j + H + \epsilon)] & \text{if } \beta_{j+1}^* < \beta_j^* < \beta_{j-1}^* \end{cases}$$

   and

$$V_0 = \begin{cases} [\beta_0^*,\ \beta_0^* + i(H_0 - \epsilon)] & \text{if } \beta_1^* > \beta_0^* \\ [\beta_0^*,\ \beta_0^* + i(H_0 + \epsilon)] & \text{if } \beta_1^* < \beta_0^* \end{cases}$$

2. Horizontal pieces (h_j):

   (a) If $\beta_j^* < \min(\beta_{j-1}^*, \beta_{j+1}^*)$, then

$$h_j(\text{top}) = [\beta_j^* + i(U_j + H - \epsilon),\ \beta_{j+1}^* + i(U_j + H - \epsilon)] \quad \text{and}$$
$$h_j(\text{bottom}) = [\beta_j^* + i(U_j - H + \epsilon),\ \beta_{j-1}^* + i(U_j - H + \epsilon)]$$

(b) If $\beta_j^* > \max(\beta_{j-1}^*, \beta_{j+1}^*)$, then

$$h_j(\text{top}) = [\beta_{j+1}^* + i(U_j + H + \epsilon),\ \beta_j^* + i(U_j + H + \epsilon)] \quad \text{and}$$
$$h_j(\text{bottom}) = [\beta_{j-1}^* + i(U_j - H - \epsilon),\ \beta_j^* + i(U_j - H - \epsilon)]$$

(c) If $\beta_{j-1}^* < \beta_j^* < \beta_{j+1}^*$, then

$$h_j(\text{top}) = [\beta_j^* + i(U_j + H - \epsilon),\ \beta_{j+1}^* + i(U_j + H - \epsilon)] \quad \text{and}$$
$$h_j(\text{bottom}) = [\beta_{j-1}^* + i(U_j - H - \epsilon),\ \beta_j^* + i(U_j - H - \epsilon)]$$

(d) If $\beta_{j+1}^* < \beta_j^* < \beta_{j-1}^*$, then

$$h_j(\text{top}) = [\beta_{j+1}^* + i(U_j + H + \epsilon),\ \beta_j^* + i(U_j + H + \epsilon)] \quad \text{and}$$
$$h_j(\text{bottom}) = [\beta_j^* + i(U_j - H + \epsilon),\ \beta_{j-1}^* + i(U_j - H + \epsilon)]$$

and similar horizontal pieces $h_{0,l}$ that link the top (respectively the bottom) vertical pieces of the ranges

$$U^{(l-1)} \leq t \leq 2U^{(l-1)} \text{ (respectively) } U^{(l)} \leq t \leq 2U^{(l)}.$$

The vertical piece V_0 and the horizontal piece h_0 pertain to the interval $[2^3, H_0']$ where $H_0' = H_0 + 2(\log H_0)^2$. We also observe that the vertical piece V^* for the interval $[0, 2^3]$ can be taken to be $\alpha = \frac{1}{2} + \eta$ for any small positive constant η. Therefore, the contour $\mathcal{C}$ can be pictorially seen in Figure 2.1 and

$$\mathcal{C} = \Gamma \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_1^R \cup \Gamma_2^R.$$

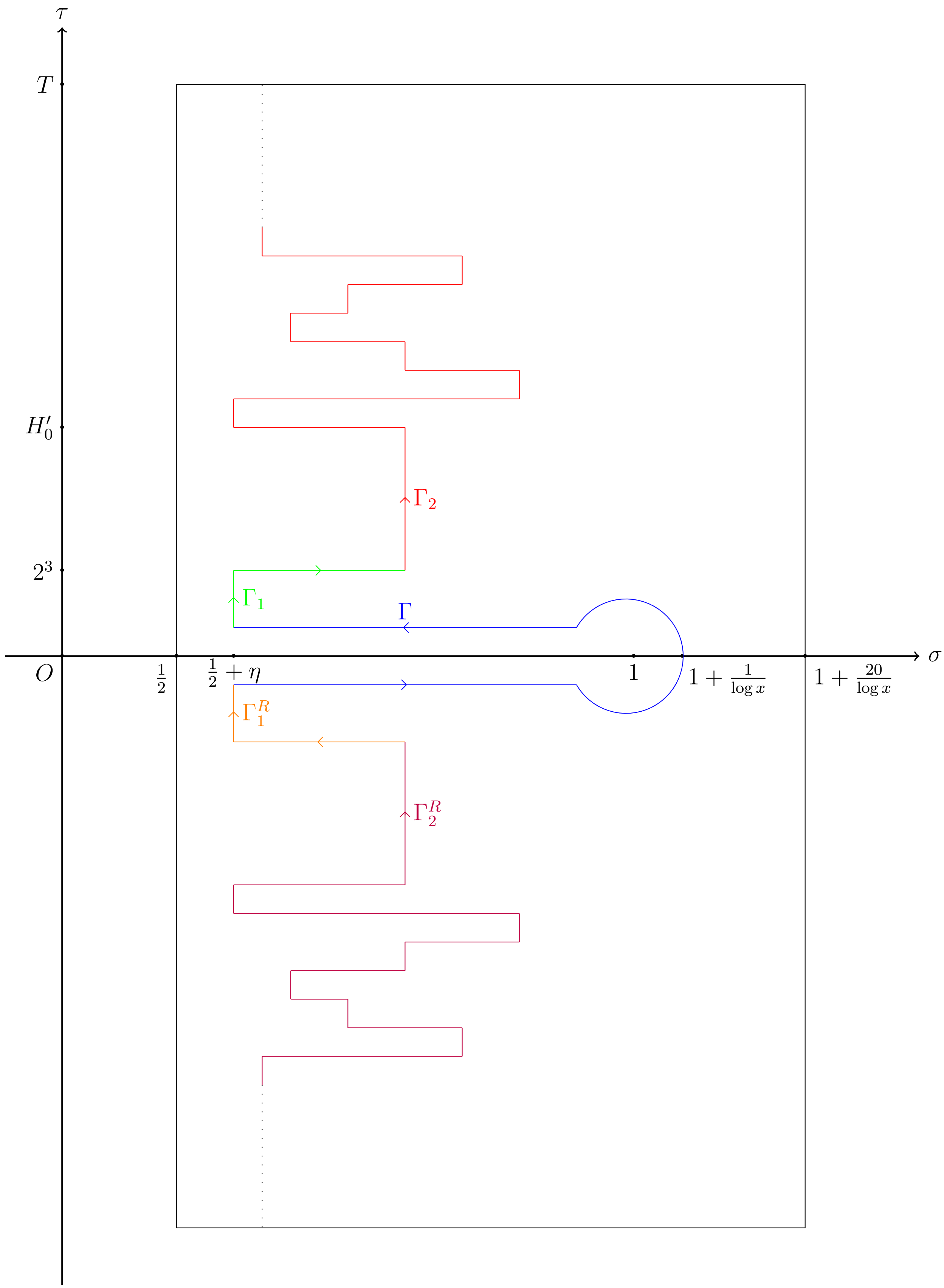


Figure 2.1: Contour $\mathcal{C}$

2.3 Proof of Theorem 2.1.1

2.3.1 Treatment of the sum $\sum_{x<n\leq x+y} f(n)$

Since $\mathcal{F}(s)$ is a Dirichlet series of the type $\mathcal{P}(\kappa, w, \alpha, \delta, A, B, M)$, we apply Corollary II.2.2.1 of [56] with the choice of parameters $\sigma_a = 1, B(n) = n^{\epsilon}, \alpha = \alpha, \sigma = 0$ to obtain

$$\sum_{x<n\leq x+y} f(n) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} \mathcal{F}(s)\frac{(x+y)^s - x^s}{s} ds + O\left(\frac{x^{1+\epsilon}}{T}\right)$$

where $b = 1 + \frac{20}{\log x}, 100 \leq T \leq x$ such that $\zeta(\sigma + iT) \neq 0$ for $0 < \sigma < 1$.

Now we replace the path of integration $[b - iT, b + iT]$ by the contour $\mathcal{C}$ in described above. (See Figure 2.1.)

K. Ramachandra and A. Sankaranarayanan (see Theorems 1 and 2 of [46]) investigated certain upper bound estimates "locally" for the function $\left|\log \mathcal{F}(s)\right|$ (where $\mathcal{F}(s)$ is any Dirichlet series satisfying certain general conditions) under the assumption that $\mathcal{F}(s) \neq 0$ in the rectangle $\{\sigma \geq \frac{1}{2} + \eta\ ,\ T - H \leq t \leq T + H\}$ of t–width $2H$. Here the parameter H can be chosen as small as $H = c \log\log\log T$. We record here a special case of the general theorem as:

Lemma 2.3.1. *Let $\frac{1}{2} \leq \alpha^* \leq 1 - \eta$, $H = a_3 \log\log T$ and suppose that $\zeta(s) \neq 0$ in $\{\sigma > \alpha^*\ ,\ \ T - H \leq t \leq T + H\}$. Then for $\alpha^* < \sigma \leq 1 - \frac{a_4}{\log\log T}$, $\ T - \frac{H}{2} \leq t \leq T + \frac{H}{2}$, we have*

$$\left|\log \zeta(\sigma + it)\right| \leq a_5 \log T (\log\log T)^{-1}$$

where a_3, a_4 and a_5 are certain positive constants.

Therefore by this lemma, for $|t|\,(\geq H_0)$, we have

$$\big|\zeta(\sigma+it)\big| \ll U^{\epsilon}$$

for $\sigma+it \in V_j$ and $\sigma+it \in h_j$. The horizontal slab with $|t| \in \left[\frac{1}{\log x},\, H_0\right]$ is treated as follows. We redefine

$$\beta_0^* = \frac{1}{2}+\eta \quad \text{if } |t| \in \left[\frac{1}{\log x}, 2^7\right].$$

We remark here that by computational results, we know that all the non-trivial zeros of $\zeta(s)$, for instance up to height 2^{10} lie on the line $\Re(s)=\frac{1}{2}$. (See [40], [43].)

For the portion $|t| \in [2^5,\, H_0]$, we first observe that the region

$$\left\{10 \geq \sigma \geq \beta_0 + \frac{C^*}{\log\log H_0}\ ,\ \ 2^3 \leq |t| \leq H_0'\right\}$$

is free from zeros of $\zeta(s)$. Therefore, applying the Borel–Carathéodory theorem, we get $\left(\text{for } 10 \geq \sigma \geq \beta_0 + \frac{C^*}{\log\log H_0}\right)$,

$$\big|\log\zeta(\sigma+it)\big| \ll (\log H_0)^{1-\epsilon} \ll (\log\log\log T)^{1-\epsilon} \ll H_0^{\epsilon}.$$

So this estimate holds when $\beta_0^*+it \in V_0$ with $t \in [2^5, H_0]$ and $T \geq T_0$ where T_0 is sufficiently large.

The portion $|t| \in [0, 2^5]$ is dealt as follows. We observe that the region

$$\left\{10 \geq \sigma \geq \beta_0^*\ ,\ \ |t| \leq 2^7\right\}$$

is zero–free for $\zeta(s)$. This follows from the computational results. Thus,

$$\big|\zeta(s)\big| \ll \log\log(U+2) \ll U^{\epsilon}$$

$$\text{for} \quad \left\{\sigma \geq \beta_0^* \ , \ \ |s-1| \geq \frac{1}{10} \ , \ \ |t| \leq 2^7\right\}.$$

For $|s-1| \leq \frac{1}{10}$,

$$\big|\zeta(s)\big| \ll \frac{1}{|s-1|}.$$

Thus, we need to estimate

$$\begin{aligned}\sum_{x<n\leq x+y} f(n) &= \frac{1}{2\pi i}\left[\int_{\Gamma} + \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_1^R} + \int_{\Gamma_2^R}\right] \mathcal{F}(s)\frac{(x+y)^s - x^s}{s} ds + O\left(\frac{x^{1+\epsilon}}{T}\right) \\ &= I_0 + I_1 + I_2 + I_1^R + I_2^R + O\left(\frac{x^{1+\epsilon}}{T}\right) \ \text{(say)}.\end{aligned}$$

2.3.2 Evaluation of I_0

Let $0 < a_6 < \frac{1}{10}$ be any small constant. Since $\mathcal{G}(s;\kappa,w)\zeta(2s)^{-w}Z(s;\kappa)$ is holomorphic and $O(M)$ in the disc $|s-1| \leq a_6$, the Cauchy's formula implies that

$$g_l(\kappa, w) \ll M{a_6}^{-l} \qquad (l \geq 0,\ 0 < \kappa \leq B,\, |w| \leq B)$$

where $g_l(\kappa, w)$ is defined by

$$\mathcal{G}(s;\kappa,w)\zeta(2s)^{-w}Z(s;\kappa) = \sum_{l=0}^{\infty} g_l(\kappa,w)(s-1)^l$$

with

$$Z(s;\kappa) := \big((s-1)\zeta(s)\big)^{\kappa},$$

$$g_l(\kappa, w) := \frac{1}{l!}\sum_{j=0}^{l}\binom{l}{j}\frac{\partial^{l-j}\left\{\mathcal{G}(s;\kappa,w)\zeta(2s)^{-w}\right\}}{\partial s^{l-j}}\bigg|_{s=1}\gamma_j(\kappa).$$

Hence for any integer $N \geq 0$ and $|s-1| \leq \frac{a_6}{2}$,

$$\mathcal{G}(s;\kappa,w)\zeta(2s)^{-w}Z(s;\kappa) = \sum_{l=0}^{N} g_l(\kappa,w)(s-1)^l + O\left(M\left(\frac{|s-1|}{a_6}\right)^{N+1}\right).$$

We have,

$$\begin{aligned}\mathcal{F}(s) &= \mathcal{G}(s;\kappa,w)\zeta(2s)^{-w}\zeta(s)^{\kappa},\\ &= \mathcal{G}(s;\kappa,w)\zeta(2s)^{-w}Z(s;\kappa)(s-1)^{-\kappa}.\end{aligned}$$

Thus,

$$\begin{aligned}I_0 &:= \frac{1}{2\pi i}\int_{\Gamma}\mathcal{F}(s)\frac{(x+y)^s - x^s}{s}ds\\ &= \sum_{l=0}^{N} g_l(\kappa,w)\frac{1}{2\pi i}\int_{\Gamma}(s-1)^{l-\kappa}\frac{(x+y)^s-x^s}{s}ds\\ &\quad + O\left(M{a_6}^{-N}\int_{\Gamma}(s-1)^{N+1-\kappa}\frac{(x+y)^s-x^s}{s}ds\right)\\ &= \sum_{l=0}^{N} g_l(\kappa,w)M_l(x,y) + O\left(M{a_6}^{-N}E_N(x,y)\right)\ \text{(say)}.\end{aligned}$$

Evaluation of $M_l(x,y)$

$$M_l(x,y) := \frac{1}{2\pi i}\int_\Gamma (s-1)^{l-\kappa}\frac{(x+y)^s - x^s}{s}ds$$

Observe that

$$\frac{(x+y)^s - x^s}{s} = \int_x^{x+y} u^{s-1}\ du.$$

Using Corollary II.5.2.1 of [56], we can write

$$\begin{aligned} M_l(x,y) &= \int_x^{x+y}\left(\frac{1}{2\pi i}\int_\Gamma (s-1)^{l-\kappa}u^{s-1}\ ds\right)du \\ &= \int_x^{x+y}(\log u)^{\kappa-1-l}\left\{\frac{1}{\Gamma(\kappa-l)} + O\left(\frac{(a_7 l+1)^l}{u^{\frac{1}{2}}}\right)\right\}du \end{aligned}$$

where we have used

$$47^{|\kappa-l|}\Gamma\left(1+|\kappa-l|\right) \ll_B (a_7 l+1)^l \qquad (l \geq 0,\ 0<\kappa\leq B).$$

a_7 and the implied constant may depend at most on B. Now for $0<\kappa\leq B$, $0<u<y\leq x$,

$$\begin{aligned} \log(x+u) &= \log x + \log\left(1+\frac{u}{x}\right) \\ &= \log x + O\left(\frac{u}{x}\right). \end{aligned}$$

Therefore,

$$\big(\log(x+u)\big)^{\kappa-1-l} = (\log x)^{\kappa-1-l} + O\left(\frac{(l+1)u(\log x)^{\kappa-2-l}}{x}\right)$$

and

$$\begin{aligned}\int_x^{x+y}(\log u)^{\kappa-1-l}du &= \int_0^y \big(\log(x+u)\big)^{\kappa-1-l}\,du\\ &= y(\log x)^{\kappa-1-l} + O\left(\frac{(l+1)(\log x)^{\kappa-2-l}}{x}\int_0^y u\ du\right)\\ &= y(\log x)^{\kappa-1-l} + O\left(\frac{(l+1)(\log x)^{\kappa-2-l}}{x}y^2\right)\\ &= y(\log x)^{\kappa-1-l}\left\{1+O_B\left(\frac{(l+1)y}{x\log x}\right)\right\}.\end{aligned}$$

Also,

$$\begin{aligned}(a_7l+1)^l\int_x^{x+y}\frac{(\log u)^{\kappa-1-l}}{u^{\frac{1}{2}}}du &\ll \frac{(a_7l+1)^l}{x^{\frac{1}{2}}}\big(\log(2x)\big)^{\kappa-1-l}\,y\\ &\ll_B \frac{(a_7l+1)^l(\log x)^{\kappa-1-l}y}{x^{\frac{1}{2}}}.\end{aligned}$$

Thus, we get

$$M_l(x,y) = y(\log x)^{\kappa-1-l}\left\{\frac{1}{\Gamma(\kappa-l)} + O_B\left(\frac{(l+1)y}{\Gamma(\kappa-l)x\log x}\right) + O_B\left(\frac{(a_7l+1)^l}{x^{\frac{1}{2}}}\right)\right\}$$

for $l\geq 0$, $0<\kappa\leq B$.

Estimation of $E_N(x,y)$

$$E_N(x,y) := \int_\Gamma (s-1)^{N+1-\kappa}\frac{(x+y)^s-x^s}{s}ds$$

We observe that

$$\begin{aligned}\left|\frac{(x+y)^s-x^s}{s}\right| &= \left|\int_x^{x+y} u^{s-1}\ du\right| \le \int_x^{x+y} u^{\sigma-1}\ du\\ &= \frac{u^\sigma}{\sigma}\bigg|_x^{x+y} = \frac{(x+y)^\sigma - x^\sigma}{\sigma}\\ &\ll \frac{x^{\sigma-1}y\sigma}{\sigma} \ll x^{\sigma-1}y.\end{aligned}$$

Therefore, for $r = \frac{1}{\log x}$,

$$\begin{aligned}E_N(x,y) &\ll \int_{\frac{1}{2}+\eta}^{1-\frac{1}{\log x}} (1-\sigma)^{N+1-\kappa} x^{\sigma-1} y\ d\sigma\\ &\quad + \left|\int_{-\pi}^{\pi} (re^{i\theta})^{N+1-\kappa} \frac{(x+y)^{1+re^{i\theta}} - x^{1+re^{i\theta}}}{1+re^{i\theta}} re^{i\theta} i\ d\theta\right|\\ &\ll \frac{y}{(\log x)^{N+1-\kappa}} \int_{\frac{1}{2}}^{1-\frac{1}{\log x}} \left\{(1-\sigma)\log x\right\}^{N+1-\kappa} e^{-(1-\sigma)\log x}\ d\sigma\\ &\quad + \int_{-\pi}^{\pi} |r|^{N+1-\kappa}\, x^{r\cos\theta} yr\ d\theta\\ &\ll \frac{y}{(\log x)^{N+1-\kappa}} \int_1^{\frac{\log x}{2}} u^{N+1-\kappa} e^{-u} \frac{du}{\log x} + yr^{N+2-\kappa}\\ &\ll \frac{y}{(\log x)^{N+2-\kappa}} \Gamma\left(1+|N-\kappa|\right) + \frac{y}{(\log x)^{N+2-\kappa}}\\ &\ll y(\log x)^{\kappa-1} \frac{(a_7 N+1)^{N+1}}{(\log x)^{N+1}}\end{aligned}$$

uniformly for $x \ge y \ge 2$, $N \ge 0$ and $0 < \kappa \le B$ where $a_7 > 0$ and the implied constant depends only on B. Inserting all these estimates, we get

$$I_0 = y(\log x)^{\kappa-1} \left\{ \sum_{l=0}^{N} \frac{\lambda_l(\kappa, w)}{(\log x)^l} + O_B\left(E_N^*(x,y)\right) \right\}$$

where

$$E_N^*(x,y) := \frac{y}{x}\sum_{l=1}^{N+1}\frac{l\left|\lambda_{l-1}(\kappa,w)\right|}{(\log x)^l} + \frac{(a_7N+1)^{N+1}}{x^{\frac{1}{2}}} + M\left(\frac{a_7N+1}{\log x}\right)^{N+1}.$$

(This constant a_7 is denoted as a_1 in the statement of the theorem.)

2.3.3 Treatment of I_1 and I_1^R

$$I_1 := \frac{1}{2\pi i}\int_{\Gamma_1}\mathcal{F}(s)\frac{(x+y)^s - x^s}{s}ds$$

Note that

$$\mathcal{F}(s) := \mathcal{G}(s;\kappa,w)\zeta(s)^{\kappa}\zeta(2s)^{-w}$$

so that

$$\left|\mathcal{G}(s;\kappa,w)\right| \leq M\left(|\tau|+1\right)^{\max\{\delta(1-\sigma),0\}}\Big(\log\left(|\tau|+1\right)\Big)^A \qquad (s=\sigma+i\tau),$$

$$\begin{aligned}\left|\mathcal{G}\left(\frac{1}{2}+\eta+i\tau;\kappa,w\right)\right| &\leq M\left(|\tau|+1\right)^{\frac{\delta}{2}}\Big(\log\left(|\tau|+1\right)\Big)^A \\ &\leq M.2^{\delta 2^3}(\log 2^4)^A \\ &\leq M.2^{8\delta}4^A \qquad \text{if } |\tau|\leq 2^3.\end{aligned}$$

In $\sigma > 0$, $\zeta(s)$ admits an analytic continuation as a single–valued function having its only singularity at $s=1$, which is a simple pole and one has the representation (in $\sigma > 0$):

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{(x)}{x^{s+1}} dx \ , \qquad (x) \text{ is the fractional part of } x$$

$$\begin{aligned}\left|\zeta(\sigma+i\tau)\right| &\leq \frac{\sigma+|\tau|}{\sqrt{(1-\sigma)^2+\tau^2}} + \left(\sigma+|\tau|\right)\int_1^\infty \frac{dx}{x^{\sigma+1}} \\ &\leq \frac{\sigma+|\tau|}{\sqrt{(1-\sigma)^2+\tau^2}} + \frac{\sigma+|\tau|}{\sigma} \qquad \text{for } \sigma \geq \eta > 0.\end{aligned}$$

Thus,

$$\left|\zeta\left(\frac{1}{2}+\eta+i\tau\right)\right| \leq 2^6$$

for $|\tau| \leq 2^3$, $\eta > 0$ be any small positive constant. For $\kappa > 0$,

$$\left|\zeta\left(\frac{1}{2}+\eta+i\tau\right)^\kappa\right| \leq 2^{6\kappa}.$$

For $w \in \mathbb{C}$,

$$\left|\zeta(1+2\eta+2i\tau)^{-w}\right| \leq \left|\zeta(1+2\eta+2i\tau)\right|^{a_8|w|} \leq \left(\zeta(1+\eta)\right)^{a_8|w|}$$

where $|\tau| \leq 2^3$ and a_8 is an effective constant.

Hence,

$$\left|\mathcal{F}\left(\frac{1}{2}+\eta+i\tau\right)\right| \leq M2^{8\delta}4^A2^{6\kappa}\left(\zeta(1+\eta)\right)^{a_8|w|}$$

and

$$
\begin{aligned}
|I_1| + \left|I_1^R\right| &\le \frac{1}{2\pi}\left|\int_{-2^3}^{2^3} \mathcal{F}\left(\frac{1}{2}+\eta+i\tau\right) \frac{(x+y)^{\frac{1}{2}+\eta+i\tau} - x^{\frac{1}{2}+\eta+i\tau}}{\frac{1}{2}+\eta+i\tau} i \, d\tau\right| \\
&\le \frac{1}{2\pi} M 2^{8\delta} 4^A 2^{6\kappa} \left(\zeta(1+\eta)\right)^{a_8|w|} \int_{-2^3}^{2^3} \frac{x^{\frac{1}{2}+\eta-1} y}{\left|\frac{1}{2}+\eta+i\tau\right|} d\tau \\
&\le M 2^{8\delta} 4^A 2^{6\kappa} \left(\zeta(1+\eta)\right)^{a_8|w|} \frac{2^4}{\frac{1}{2}} \frac{y}{x^{\frac{1}{2}-\eta}} \\
&\ll_{A,B,\delta,\eta} M \frac{y}{x^{\frac{1}{2}-\eta}}
\end{aligned}
$$

uniformly for $0 < \kappa \le B, |w| \le B$.

2.3.4 Estimation of the integral I_2

$$
I_2 := \frac{1}{2\pi i} \int_{\Gamma_2} \mathcal{F}(s) \frac{(x+y)^s - x^s}{s} ds
$$

Recall that

$$
\mathcal{F}(s) := \mathcal{G}(s;\kappa,w)\zeta(s)^\kappa \zeta(2s)^{-w}
$$

in Γ_2, $\frac{1}{2}+\eta = \alpha \le \sigma \le 1 - \frac{C^*}{\log\log U}$ for $t \in [U,\, 2U]$ and $0 < \kappa, |w| \le B$. Also,

$$
\left|\frac{(x+y)^s - x^s}{s}\right| \ll x^{\sigma-1} y,
$$

$$
\left|\mathcal{G}(s;\kappa,w)\right| \ll M \left(|\tau|+1\right)^{\max\{\delta(1-\sigma),0\}} \left(\log(|\tau|+1)\right)^A \qquad \text{where } \; s = \sigma + i\tau,
$$

$$
\left|\zeta(2s)^{-w}\right| \ll 1.
$$

For $\frac{U}{2} \le |\tau| \le 2U$, we have

$$
\left|\zeta(\beta_j^* + i\tau)\right| \ll \log\log U
$$

when β_j^* is near to the left of the line $\sigma = 1$ and

$$\left|\zeta(\beta_j^* + i\tau)\right| \ll e^{a_9 \frac{\log U}{\log\log U}}$$

when β_j^* is away from the line $\sigma = 1$ and closer to the line $\sigma = \frac{1}{2}$ from its right. The width of V_j is $\ll H$, $\Re(V_j) = \beta_j^*$, $U_j - 2H \leq \Im(V_j) \leq U_j + 2H$, $H \ll a_3 \log\log U$ and

$$\mathcal{F}(s) \ll MU^{\delta(1-\beta_j^*)} \big(\log(U+1)\big)^A e^{a_9\kappa \frac{\log U}{\log\log U}}.$$

The contribution of the vertical path V_j to I_2 is

$$\begin{aligned}
\left|I_2(V_j)\right| := &\left|\frac{1}{2\pi i}\int_{V_j} \mathcal{F}(s)\frac{(x+y)^s - x^s}{s} ds\right| \\
&\ll MU^{\delta(1-\beta_j^*)} \big(\log(U+1)\big)^A e^{a_9\kappa \frac{\log U}{\log\log U}} x^{\beta_j^*-1} yH \\
&\ll My \left(\frac{U^\delta}{x}\right)^{1-\beta_j^*} e^{a_9\kappa \frac{\log U}{\log\log U}} \big(\log(U+1)\big)^A H \\
&\ll My \left(\frac{U^\delta}{x}\right)^{1-\beta_j^*} e^{a_9\kappa \frac{\log U}{\log\log U}} (\log U)^{A+1} \\
&\ll My\, e^{a_{10} \frac{\log U}{\log\log U}} \left(\frac{U^\delta}{x}\right)^{1-\beta_j^*}.
\end{aligned}$$

Using $H_0 \ll U$ and $H_0 \ll \log T$, we get

$$\begin{aligned}
\left|I_2(V_0)\right| :=& \left|\frac{1}{2\pi i}\int_{V_0} \mathcal{F}(s)\frac{(x+y)^s - x^s}{s} ds\right| \\
&\ll M{H_0}^{\delta(1-\beta_0^*)}\left(\log(H_0+1)\right)^A e^{a_9\kappa\frac{\log U}{\log\log U}} x^{\beta_0^*-1} y H_0 \\
&\ll My\left(\frac{U^\delta}{x}\right)^{1-\beta_0^*} e^{a_9\kappa\frac{\log U}{\log\log U}}\left(\log(U+1)\right)^A H_0 \\
&\ll My\left(\frac{U^\delta}{x}\right)^{1-\beta_0^*} e^{a_9\kappa\frac{\log U}{\log\log U}}(\log U)^A(\log T) \\
&\ll My(\log T)e^{a_{10}\frac{\log U}{\log\log U}}\left(\frac{U^\delta}{x}\right)^{1-\beta_0^*}.
\end{aligned}$$

Thus,

$$\left|I_2(V_j)\right| \ll_{A,B,\eta,\epsilon} My(\log T)e^{a_{10}\frac{\log U}{\log\log U}}\left(\frac{U^\delta}{x}\right)^{1-\beta_j^*} \qquad \text{for } j = 0, 1, 2, \ldots, \frac{U}{2H}+1.$$

Let $\beta^{**} = \max\{\beta_j^*, \beta_{j+1}^*\}$. Then in $h_j(\text{top})$, $\alpha \leq \sigma \leq \beta^{**}$ and $|\tau| = U_j + H - \epsilon \leq 2U$. The contribution of the horizontal path h_j to I_2 is

$$\begin{aligned}
\left|I_2(h_j)\right| :=& \left|\frac{1}{2\pi i}\int_{h_j} \mathcal{F}(s)\frac{(x+y)^s - x^s}{s} ds\right| \\
&\ll \int_\alpha^{\beta^{**}} MU^{\delta(1-\sigma)}\left(\log(U+1)\right)^A e^{a_9\kappa\frac{\log U}{\log\log U}} x^{\sigma-1} y \, d\sigma \\
&\ll My\int_\alpha^{\beta^{**}}\left(\frac{U^\delta}{x}\right)^{1-\sigma} e^{a_9\kappa\frac{\log U}{\log\log U}}\left(\log(U+1)\right)^A \, d\sigma \\
&\ll My\, e^{a_{10}\frac{\log U}{\log\log U}}\int_\alpha^{\beta^{**}}\left(\frac{U^\delta}{x}\right)^{1-\sigma} d\sigma.
\end{aligned}$$

Thus,

$$\left|I_2(h_j)\right| \ll_{A,B,\eta,\epsilon} My\ e^{a_{10}\frac{\log U}{\log\log U}} \int_{\alpha}^{\beta^{**}} \left(\frac{U^\delta}{x}\right)^{1-\sigma} d\sigma \qquad \text{for } j = 0, 1, 2, \ldots, \frac{U}{2H} + 1.$$

Analogous estimate also applies for the horizontal pieces $h_{0,l}$.

Let α be any fixed constant satisfying $\frac{1}{2} + \eta \leq \alpha \leq 1 - \eta$ with η being any arbitrarily small fixed positive constant. Assume that $|\mathcal{E}| = 0$. We choose a partition of the interval $[\alpha,\, 1]$ namely

$$\alpha = \alpha_0 < \alpha_1 < \cdots < \alpha_{j-1} < \alpha_j = 1 \qquad \text{with } \alpha_j - \alpha_{j-1} < \epsilon.$$

The number of j–values for which $\beta_j^* \in [\alpha_{l-1},\, \alpha_l]$ is bounded by $N(\alpha_{l-1}, 2U)$. Therefore, on the dyadic t–width $U \leq t \leq 2U$, the vertical bits and the horizontal bits contribute to the integral I_2, a quantity which is

$$\begin{aligned}
\left|c(I_2)\right| &\leq \sum_j \left\{\left|I_2(V_j)\right| + \left|I_2(h_j)\right|\right\} \\
&\ll_{A,B,\eta,\epsilon} My(\log T)e^{a_{10}\frac{\log U}{\log\log U}} \left[\sum_j{}^{*} \left\{\left(\frac{U^\delta}{x}\right)^{1-\beta_j^*} + \int_{\alpha}^{\beta^{**}} \left(\frac{U^\delta}{x}\right)^{1-\sigma} d\sigma\right\}\right] \\
&\ll_{A,B,\eta,\epsilon} My(\log T)e^{a_{10}\frac{\log U}{\log\log U}} \int_{\alpha}^{1-\sigma_0} \left(\frac{U^\delta}{x}\right)^{1-\sigma} dN^*(\sigma, 2U)
\end{aligned}$$

where $N^*(\sigma, U) := \sum\limits^{*}_{\substack{\sigma \leq \beta_j^*,\\ |\gamma_j| \leq U}} 1$. Note that we have presumed here that $x \geq T^\delta$. We also observe that our choice of x made later agrees with this presumption.

Thus,

$$\begin{aligned}
\left|c(I_2)\right| \ll_{A,B,\eta,\epsilon} & \, My(\log T)e^{a_{10}\frac{\log U}{\log\log U}} \left\{ \left(\frac{U^\delta}{x}\right)^{1-\sigma} N^*(\sigma, 2U)\Big|_{\alpha}^{1-\sigma_0} \right. \\
& \left. + \int_{\alpha}^{1-\sigma_0} \left(\frac{U^\delta}{x}\right)^{1-\sigma} \left|\log\left(\frac{U^\delta}{x}\right)\right| N^*(\sigma, 2U)\, d\sigma \right\} \\
\ll_{A,B,\eta,\epsilon,\delta} & \, My(\log T)e^{a_{10}\frac{\log U}{\log\log U}} \left\{ \left(\frac{U^\delta}{x}\right)^{1-\alpha} N(\alpha, 2U) \right. \\
& \left. + \log T \int_{\alpha}^{1-\sigma_0} \left(\frac{U^\delta}{x}\right)^{1-\sigma} N(\sigma, 2U)\, d\sigma \right\}.
\end{aligned}$$

Here, $\sigma_0 = \sigma_0(T) := \frac{C^*}{\log\log T}$. According to our assumption $|\mathcal{E}| = 0$, i.e., for $\sigma \geq 1 - \sigma_0$, $\zeta(s) \neq 0$. Therefore, $N^*(\sigma, 2U) \leq N(\sigma, 2U)$. From [20], it is known that

$$N(\sigma, T) \ll T^{\frac{12}{5}(1-\sigma)}(\log T)^{44}$$

for $\frac{1}{2} \leq \sigma \leq 1$ and $T \geq 2$. Hence,

$$\begin{aligned}
c(I_2) \ll & \, My(\log T)e^{a_{10}\frac{\log U}{\log\log U}} \left\{ \left(\frac{U^\delta}{x}\right)^{1-\alpha} U^{\frac{12}{5}(1-\alpha)}(\log T)^{44} \right. \\
& \left. + \log T \int_{\alpha}^{1-\sigma_0} \left(\frac{U^\delta}{x}\right)^{1-\sigma} U^{\frac{12}{5}(1-\sigma)}(\log T)^{44}\, d\sigma \right\} \\
\ll & \, My(\log T)e^{a_{10}\frac{\log U}{\log\log U}} \left\{ \left(\frac{U^\delta}{x}\right)^{1-\alpha} U^{\frac{12}{5}(1-\alpha)}(\log T)^{44} \right. \\
& \left. + (\log T)^{46} \int_{\alpha}^{1-\sigma_0} \left(\frac{U^{\delta+\frac{12}{5}}}{x}\right)^{1-\sigma} d\sigma \right\}.
\end{aligned}$$

Note that as a function of σ, $\left(\frac{U^{\delta+\frac{12}{5}}}{x}\right)^{1-\sigma}$ is monotonic in $[\alpha,\, 1-\frac{\sigma_0}{2}]$ and hence it attains maximum at its extremities. Therefore,

$$\begin{aligned}
c(I_2) &\ll My(\log T)e^{a_{10}\frac{\log U}{\log\log U}}\left\{\left(\frac{U^{\delta+\frac{12}{5}}}{x}\right)^{1-\alpha}(\log T)^{46}+\left(\frac{U^{\delta+\frac{12}{5}}}{x}\right)^{\sigma_0}(\log T)^{46}\right\}\\
&\ll My\ e^{a_{10}\frac{\log U}{\log\log U}}(\log T)^{47}\left\{\left(\frac{U^{\delta+\frac{12}{5}}}{x}\right)^{1-\alpha}+\left(\frac{U^{\delta+\frac{12}{5}}}{x}\right)^{\sigma_0}\right\}
\end{aligned}$$

and

$$I_2 \ll \sum_{\substack{U=2^l,\\ \frac{\log T}{\log 2}\geq l\geq L}} c(I_2) \ll My\ e^{a_{11}\frac{\log T}{\log\log T}}\left\{\left(\frac{T^{\delta+\epsilon+\frac{12}{5}}}{x}\right)^{1-\alpha}+\left(\frac{T^{\delta+\epsilon+\frac{12}{5}}}{x}\right)^{\sigma_0}\right\}$$

since $H_0 \ll U \ll T$. For the sake of convenience, we have multiplied the first term in the curly bracket by $T^{\epsilon(1-\alpha)}$ and the second term by $T^{\epsilon\sigma_0}$.

A similar estimate holds for I_2^R, of course on the assumption that $|\mathcal{E}| = 0$.

2.3.5 Case $|\mathcal{E}| \geq 1$

One observes that the number of exceptional zeros is

$$\begin{aligned}
|\mathcal{E}| &:= N(1-\sigma_0, T)\\
&\ll T^{\frac{12}{5}\sigma_0}(\log T)^{44}\\
&\ll e^{\frac{C^*\log T}{\log\log T}}\\
&\ll T^{\epsilon}
\end{aligned}$$

for $T \geq T_0$ (T_0 sufficiently large).

Recall that

$$\mathcal{F}(s) := \mathcal{G}(s;\kappa,w)\zeta(s)^{\kappa}\zeta(2s)^{-w},$$

$$\frac{1}{2}+\eta = \alpha \leq \sigma \leq 1 - \frac{C^*}{\log\log U} \leq \beta^*_{j,e},$$

$$\mathcal{G}(s;\kappa,w) \ll M\left(|\tau|+1\right)^{\max\{\delta(1-\sigma),0\}}\Big(\log\left(|\tau|+1\right)\Big)^{A} \qquad \text{where } s=\sigma+i\tau \text{ and}$$

$$\big|\zeta(s)\big|^{\kappa} \ll \left(|\tau|+1\right)^{\eta_1\kappa(1-\sigma)}\log\left(|\tau|+1\right) \qquad \text{with } \eta_1 < \frac{1}{3}.$$

The contribution of the vertical path $V_{j,e}$ pertaining to an exceptional zero $\beta_{j,e}$ is

$$\begin{aligned}\big|I_2(V_{j,e})\big| &\ll MU^{\delta(1-\beta^*_{j,e})}\Big(\log(U+1)\Big)^{A}x^{\beta^*_{j,e}-1}yHU^{\eta_1\kappa(1-\beta^*_{j,e})}\log(U+1)\\ &\ll MU^{(\eta_1\kappa+\delta)(1-\beta^*_{j,e})}\Big(\log(U+1)\Big)^{A+1}yHx^{\beta^*_{j,e}-1}\\ &\ll MU^{(\eta_1\kappa+\delta)(1-\beta^*_{j,e})}\Big(\log(U+1)\Big)^{A+2}yx^{\beta^*_{j,e}-1}.\end{aligned}$$

Similarly,

$$\big|I_2(V_{0,e})\big| \ll MH_0^{(\eta_1\kappa+\delta)(1-\beta^*_{0,e})}(\log T)^{A+2}x^{\beta^*_{0,e}-1}y.$$

Therefore,

$$\big|I_2(V_{j,e})\big| \ll MU^{(\eta_1\kappa+\delta)(1-\beta^*_{j,e})}(\log T)^{A+2}yx^{\beta^*_{j,e}-1}$$

for $j = 0,1,2,\ldots,\frac{U}{2H}+1$.

Similarly, the horizontal path $h_{j,e}$ contributes to I_2,

$$\begin{aligned}\left|I_2(h_{j,e})\right| &\ll My \int_{\alpha}^{\beta^*_{j,e}} U^{(\eta_1\kappa+\delta)(1-\sigma)} (\log T)^{A+2} x^{\sigma-1}\ d\sigma \\ &\ll My \int_{\alpha}^{\beta^*_{j,e}} U^{(\eta_1\kappa+\delta)(1-\sigma)} (\log T)^{A+2} x^{\sigma-1}\ d\sigma\end{aligned}$$

with $\alpha \le \sigma \le \beta^*_{j,e}$ and $1 - \frac{C^*}{\log\log U} \le \beta^*_{j,e}$.

Thus in the case of exceptional set $\mathcal{E}$ being non–empty, we obtain as before

$$\begin{aligned}c_e(I_2) &\ll My(\log T)^{A+2} \left\{ \left(\frac{U^{\eta_1\kappa+\delta}}{x}\right)^{1-\alpha} N^*_e(\alpha, 2U) \right. \\ &\quad \left. + (\log T) \int_{\alpha}^{1} \left(\frac{U^{\eta_1\kappa+\delta}}{x}\right)^{1-\sigma} N^*_e(\sigma, 2U)\ d\sigma \right\} \\ &\ll My(\log T)^{A+3} \left\{ \left(\frac{U^{\eta_1\kappa+\delta+\epsilon}}{x}\right)^{1-\alpha} + \left(\frac{U^{\eta_1\kappa+\delta+\epsilon}}{x}\right) \right\}\end{aligned}$$

where $N^*_e(\sigma, U) := \sum_{\substack{\sigma \le \beta^*_{j,e}, \\ \left|\gamma_{j,e}\right| \le U}} 1$. Thus, $N^*_e(\sigma, U) \le |\mathcal{E}| \ll U^{\epsilon}$. Note that we have made the presumption here that $x \ge T^{\delta+\eta_1\kappa}$. We can observe that our choice of x made later agrees with this presumption.

Therefore,

$$I_{2,e} \ll \sum_{\substack{U=2^l, \\ \frac{\log T}{\log 2} \ge l \ge L}} c_e(I_2) \ll My(\log T)^{A+4} \left\{ \left(\frac{T^{\eta_1\kappa+\delta+\epsilon}}{x}\right)^{1-\alpha} + \left(\frac{T^{\eta_1\kappa+\delta+\epsilon}}{x}\right) \right\}.$$

A similar estimate also holds for $I^R_{2,e}$.

We observe that

$$N^*(\sigma, 2U) = N^*_{\mathcal{G}}(\sigma, 2U) + N^*_{\mathcal{E}}(\sigma, 2U).$$

Thus we get in any case whether $|\mathcal{E}| = 0$ or $|\mathcal{E}| \geq 1$,

$$I_2 \ll My\ e^{a_{11}\frac{\log T}{\log\log T}} \left\{ \left(\frac{T^{\delta+\epsilon+\frac{12}{5}}}{x} \right)^{1-\alpha} + \left(\frac{T^{\delta+\epsilon+\frac{12}{5}}}{x} \right)^{\sigma_0} \right\}$$
$$+ My(\log T)^{A+4} \left\{ \left(\frac{T^{\eta_1\kappa+\delta+\epsilon}}{x} \right)^{1-\alpha} + \left(\frac{T^{\eta_1\kappa+\delta+\epsilon}}{x} \right) \right\}.$$

A similar estimate holds for I_2^R. Thus,

$$I_1 + I_1^R + I_2 + I_2^R \ll_{A,B,\delta,\eta} M\frac{y}{x^{\frac{1}{2}-\eta}} + My\ e^{a_{11}\frac{\log T}{\log\log T}} \left\{ \left(\frac{T^{\delta+\epsilon+\frac{12}{5}}}{x} \right)^{1-\alpha} \right.$$
$$\left. + \left(\frac{T^{\eta_1\kappa+\delta+\epsilon}}{x} \right)^{1-\alpha} + \left(\frac{T^{\delta+\epsilon+\frac{12}{5}}}{x} \right)^{\sigma_0} + \left(\frac{T^{\eta_1\kappa+\delta+\epsilon}}{x} \right) \right\}.$$

2.3.6 Case 1: If $\kappa \leq \frac{12}{5\eta_1}$

$$I_1 + I_1^R + I_2 + I_2^R \ll M\frac{y}{x^{\frac{1}{2}-\eta}} + My\ e^{a_{11}\frac{\log T}{\log\log T}} \left\{ \left(\frac{T^{\delta+\epsilon+\frac{12}{5}}}{x} \right)^{1-\alpha} \right.$$
$$\left. + \left(\frac{T^{\delta+\epsilon+\frac{12}{5}}}{x} \right)^{\sigma_0} + \left(\frac{T^{\delta+\epsilon+\frac{12}{5}}}{x} \right) \right\}$$

We choose T such that $T^{\delta+\epsilon+\frac{12}{5}} \sim x^{1-10\epsilon}$ so that

$$\begin{aligned}
I_1 + I_1^R + I_2 + I_2^R &\ll M\frac{y}{x^{\frac{1}{2}-\eta}} + My\ e^{a_{12}\frac{\log x}{\log\log x}}\left\{x^{-10\epsilon(1-\alpha)} + x^{-10\epsilon\sigma_0} + x^{-10\epsilon}\right\} \\
&\ll My\ e^{-a_{13}(\alpha,\epsilon)\frac{\log x}{\log\log x}}
\end{aligned}$$

for sufficiently large $x \geq x_0(\epsilon, \kappa, A)$.

From the error term in the Perron's formula,

$$\begin{aligned}
\frac{x^{1+\epsilon}}{T} &\ll x^{1+\epsilon-\frac{1-10\epsilon}{\delta+\epsilon+\frac{12}{5}}} \ll x^{\theta+\epsilon} \\
x^{\theta} &\gg x^{1-\frac{1-10\epsilon}{\delta+\epsilon+\frac{12}{5}}} \\
\theta &\geq 1 - \frac{5-50\epsilon}{5\delta+5\epsilon+12} \\
\theta &\geq \frac{5\delta+55\epsilon+7}{5\delta+5\epsilon+12}.
\end{aligned}$$

2.3.7 Case 2: If $\kappa > \frac{12}{5\eta_1}$

$$\begin{aligned}
I_1 + I_1^R + I_2 + I_2^R \ll M\frac{y}{x^{\frac{1}{2}-\eta}} + My\ e^{a_{11}\frac{\log T}{\log\log T}} &\left\{\left(\frac{T^{\eta_1\kappa+\delta+\epsilon}}{x}\right)^{1-\alpha}\right. \\
&\left. + \left(\frac{T^{\eta_1\kappa+\delta+\epsilon}}{x}\right)^{\sigma_0} + \left(\frac{T^{\eta_1\kappa+\delta+\epsilon}}{x}\right)\right\}
\end{aligned}$$

We choose T such that $T^{\eta_1\kappa+\delta+\epsilon} \sim x^{1-10\epsilon}$ so that

$$\begin{aligned} I_1 + I_1^R + I_2 + I_2^R &\ll M\frac{y}{x^{\frac{1}{2}-\eta}} + My\ e^{a_{14}\frac{\log x}{\log\log x}} \left\{x^{-5\epsilon} + x^{-10\epsilon\sigma_0} + x^{-10\epsilon}\right\} \\ &\ll My\ e^{-a_{15}(\epsilon)\frac{\log x}{\log\log x}} \end{aligned}$$

for sufficiently large $x \geq x_0(\epsilon, \kappa, A)$.

From the error term in the Perron's formula,

$$\begin{aligned} \frac{x^{1+\epsilon}}{T} &\ll x^{1+\epsilon-\frac{1-10\epsilon}{\eta_1\kappa+\delta+\epsilon}} \ll x^{\theta+\epsilon} \\ x^{\theta} &\gg x^{1-\frac{1-10\epsilon}{\eta_1\kappa+\delta+\epsilon}} \\ \theta &\geq \frac{\eta_1\kappa+\delta-1+11\epsilon}{\eta_1\kappa+\delta+\epsilon}. \end{aligned}$$

This completes the proof of Theorem 2.1.1.

2.4 Consequences of Halász–Turán theorem

Theorem 2.4.1. *[17] Assume the Lindelöf hypothesis for $\zeta(s)$ in the form*

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \leq t^{\eta_2^2} \qquad \text{for } t > t_0$$

for all sufficiently small positive numbers η_2. Then the inequality

$$N\left(\frac{3}{4}+2\eta_2^{\frac{1}{2}}, T\right) < T^{3\eta_2}$$

holds for $T > t_0$.

For a more general theorem along the same flavour, see Theorem 1.3 and Theorem 1.4 of [51].

Therefore, if one assumes Lindelöf hypothesis in the form given in the above theorem, then by taking $\alpha = \frac{3}{4} + 2\eta_2^{\frac{1}{2}}$ in the earlier arguments, we find that

$$\begin{aligned} I_1 + I_1^R + I_2 + I_2^R &\ll_{A,B,\delta,\eta} M \frac{y}{x^{1-\alpha}} \\ &+ My\,(\log T)^{A+10} \left\{ \left(\frac{T^{2\eta_2^2\kappa+\delta+\frac{3\eta_2}{(1-\beta^*)}}}{x} \right)^{1-\alpha} + \left(\frac{T^{2\eta_2^2\kappa+\delta+\frac{3\eta_2}{(1-\beta^*)}}}{x} \right) \right\}. \end{aligned}$$

Since for $\sigma \geq \frac{3}{4} + 2\eta_2^{\frac{1}{2}}$, under Lindelöf hypothesis,

$$\begin{aligned} N(\sigma, T) &\leq N\left(\frac{3}{4} + 2\eta_2^{\frac{1}{2}}, T\right) \\ &< T^{3\eta_2} \\ &= T^{\frac{3\eta_2(1-\sigma)}{(1-\sigma)}} \\ &< T^{\frac{3\eta_2(1-\sigma)}{(1-\beta^*)}}. \end{aligned}$$

Here,

$$\begin{aligned} \beta^* &= \max_j \beta_j^* + \frac{c^*}{2(\log T)^{\frac{2}{3}}(\log\log T)^{\frac{1}{3}}} \\ &< 1 - \frac{c^*}{2(\log T)^{\frac{2}{3}}(\log\log T)^{\frac{1}{3}}} \end{aligned}$$

follows from Korobov–Vinogradov's zero-free region for $\zeta(s)$. (See [33], [58].)

From convexity principle for $\frac{1}{2} \leq \sigma \leq 1$, we find that

$$\begin{aligned}\zeta(\sigma+it) &\ll t^{\eta_2^2} x^{\frac{1}{2}-\sigma} + (\log t) x^{1-\sigma} \\ &\ll t^{2\eta_2^2(1-\sigma)} (\log t)\end{aligned}$$

by choosing $x = t^{2\eta_2^2}$.

Now we choose

$$\eta_2 = \frac{c^*}{2(\log T)^{\frac{2}{3}} (\log\log T)^{\frac{1}{3}}} (1-\beta^*)$$

so that $\eta_2^2 < \epsilon$ for large T and

$$\begin{aligned}I_1 + I_1^R + I_2 + I_2^R &\ll M \frac{y}{x^{\frac{1}{4}-2\eta_2^{\frac{1}{2}}}} \\ &+ My\, (\log T)^{A+10} \left\{ \left(\frac{T^{2\kappa\epsilon+\delta+3\epsilon}}{x} \right)^{1-\alpha} + \left(\frac{T^{2\kappa\epsilon+\delta+3\epsilon}}{x} \right) \right\}.\end{aligned}$$

By choosing $T^{2\kappa\epsilon+\delta+3\epsilon} \sim x^{1-10\epsilon}$, we observe that

$$I_1 + I_1^R + I_2 + I_2^R \ll M \frac{y}{x^{\frac{1}{4}-2\eta_2^{\frac{1}{2}}}} + My\, (\log x)^{A+10} \left\{ x^{-10\epsilon(1-\alpha)} + x^{-10\epsilon} \right\}.$$

Hence under the assumption of Lindelöf hypothesis, the above theorem holds with

$$\theta(\kappa, \delta) := \frac{\delta - 1 + 2\kappa\epsilon + 13\epsilon}{\delta + 2\kappa\epsilon + 3\epsilon}.$$

One observes that $\frac{\delta-1}{\delta} < \frac{1}{2}$ when $1 < \delta < 2$ and $\frac{\delta-1}{\delta} < 1$ for any positive δ. One needs to assume that $\delta > 1$ so that the numerator is positive.

Applying the above contour with $\alpha = \frac{3}{4} + 2\eta_2^{\frac{1}{2}}$ (assuming Lindelöf hypothesis in the stated form), we obtain

$$\begin{aligned} I_2 &\ll \sum_{\frac{\log T}{\log 2} \geq l \geq L} c(I_2) \\ &\ll My\ (\log T)^{A+3} \left\{ \left(\frac{T^{2\eta_2^2\kappa+\delta}}{x} \right)^{1-\alpha} N^{**}(\alpha, T) \right. \\ &\quad \left. + (\log T) \int_{\alpha}^{1} \left(\frac{T^{2\eta_2^2\kappa+\delta}}{x} \right)^{1-\sigma} N^{**}(\sigma, T)\ d\sigma \right\} \\ &\ll My\ (\log T)^{A+10} T^{3\eta_2} \left\{ \left(\frac{T^{2\eta_2^2\kappa+\delta}}{x} \right)^{1-\alpha} + \left(\frac{T^{2\eta_2^2\kappa+\delta}}{x} \right) \right\} \\ &\ll My\ T^{3\eta_2+\epsilon} \left\{ \left(\frac{T^{2\eta_2^2\kappa+\delta}}{x} \right)^{1-\alpha} + \left(\frac{T^{2\eta_2^2\kappa+\delta}}{x} \right) \right\}. \end{aligned}$$

Here $N^{**}(\sigma, T)$ has its relevant meaning with the current context of α. Choose $\eta_2 = \epsilon$ and T such that $T^{2\eta_2^2\kappa+\delta} = x^{1-20\epsilon}$. Then,

$$\begin{aligned} I_2 &\ll My\ T^{4\epsilon} \{ x^{-20\epsilon(1-\alpha)} + x^{-20\epsilon} \} \\ &\ll My\ x^{4\epsilon} \{ x^{-5\epsilon+40\epsilon^{\frac{3}{2}}} + x^{-20\epsilon} \}. \end{aligned}$$

Hence, we can also take

$$\begin{aligned}\theta(\kappa,\delta) &:= 1 - \frac{1-20\epsilon}{2\eta_2^2\kappa+\delta}\\ &= \frac{2\eta_2^2\kappa+\delta-1+20\epsilon}{2\eta_2^2\kappa+\delta}\\ &= \frac{2\epsilon^2\kappa+\delta-1+20\epsilon}{2\epsilon^2\kappa+\delta}.\end{aligned}$$

Again of course, one needs to assume that $\delta > 1$. We observe that the earlier unconditional estimate for $\theta(\kappa,\delta)$ $\left(\text{relevant when } \kappa < \frac{12}{5\eta_1}\right)$ is $\frac{5\delta+7}{5\delta+12} + \epsilon_1$ which may be compared with the Lindelöf hypothesis conditional estimate $\frac{\delta-1}{\delta} + \epsilon_2$. Clearly,

$$\frac{5\delta+7}{5\delta+12} > \frac{\delta-1}{\delta}$$

for any $\delta > 0$. (However, our relevance here is $\delta > 1$.)

CHAPTER

THREE

GODEMENT–JACQUET L-FUNCTION

3.1 Introduction

Definition 3.1.1. Maass form

Let $n \geq 2$, and let $v = (v_1, v_2, \ldots, v_{n-1}) \in \mathbb{C}^{n-1}$. A Maass form [14] for $SL(n, \mathbb{Z})$ of type v is a smooth function $f \in \mathcal{L}^2(SL(n, \mathbb{Z})\backslash\mathcal{H}^n)$ which satisfies

1. *$f(\gamma z) = f(z)$, for all $\gamma \in SL(n, \mathbb{Z}), z \in \mathcal{H}^n$,*

2. *$Df(z) = \lambda_D f(z)$, for all $D \in \mathfrak{D}^n$ where $\mathfrak{D}^{\mathfrak{n}}$ is the center of the universal enveloping algebra of $\mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{gl}(n, \mathbb{R})$ is the Lie algebra of $GL(n, \mathbb{R})$,*

3. *$\displaystyle\int_{(SL(n,\mathbb{Z})\cap U)\backslash U} f(uz)\, du = 0,$*
   for all upper triangular groups U of the form

$$U = \left\{ \begin{pmatrix} I_{r_1} & & & \\ & I_{r_2} & & * \\ & & \ddots & \\ & & & I_{r_b} \end{pmatrix} \right\},$$

with $r_1 + r_2 + \cdots + r_b = n$. Here, I_r denotes the $r \times r$ identity matrix, and $*$ denotes arbitrary real entries.

A Hecke–Maass form is a Maass form which is an eigenvector for the Hecke operators algebra.

Let $f(z)$ be a Hecke–Maass form of type $v = (v_1, v_2, \ldots, v_{n-1}) \in \mathbb{C}^{n-1}$ for $SL(n, \mathbb{Z})$. Then it has the Fourier expansion

$$f(z) = \sum_{\gamma \in U_{n-1}(\mathbb{Z}) \backslash SL(n-1,\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1} \neq 0} \frac{A(m_1, \ldots, m_{n-1})}{\prod_{j=1}^{n-1} \left|m_j\right|^{\frac{j(n-j)}{2}}}$$
$$\times W_J \left(M \cdot \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, v, \psi_{1,\ldots,1,\frac{m_{n-1}}{\left|m_{n-1}\right|}} \right),$$

where

$$M = \begin{pmatrix} m_1 \ldots m_{n-2} \cdot \left|m_{n-1}\right| & & & & \\ & \ddots & & & \\ & & m_1 m_2 & & \\ & & & m_1 & \\ & & & & 1 \end{pmatrix},$$

$$A(m_1, \ldots, m_{n-1}) \in \mathbb{C}, \qquad A(1, \ldots, 1) = 1,$$

$$\psi_{1,\ldots,1,\epsilon} \left(\begin{pmatrix} 1 & u_{n-1} & & & \\ & 1 & u_{n-2} & & * \\ & & \ddots & \ddots & \\ & & & 1 & u_1 \\ & & & & 1 \end{pmatrix} \right) = e^{2\pi i (u_1 + \cdots + u_{n-2} + \epsilon u_{n-1})},$$

$U_{n-1}(\mathbb{Z})$ denotes the group of $(n-1) \times (n-1)$ upper triangular matrices with $1s$ on the diagonal and an integer entry above the diagonal and W_J is the Jacquet Whittaker function.

Definition 3.1.2. Dual Maass form

If $f(z)$ is a Maass form of type $(v_1, \ldots, v_{n-1}) \in \mathbb{C}^{n-1}$, then

$$\tilde{f}(z) := f(w \cdot (z^{-1})^T \cdot w),$$

$$w = \begin{pmatrix} & & & (-1)^{\left[\frac{n}{2}\right]} \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}$$

is a Maass form of type $(v_{n-1}, \ldots, v_1)$ for $SL(n, \mathbb{Z})$ called the dual Maass form. *If $A(m_1, \ldots, m_{n-1})$ is the $(m_1, \ldots, m_{n-1})$–Fourier coefficient of f, then $A(m_{n-1}, \ldots, m_1)$ is the corresponding Fourier coefficient of $\tilde{f}$.*

We note that the Fourier coefficients $A(m_1, \ldots, m_{n-1})$ satisfy the multiplicative relations

$$A(m_1 m_1', \ldots, m_{n-1} m_{n-1}') = A(m_1, \ldots, m_{n-1}) \cdot A(m_1', \ldots, m_{n-1}'),$$

if

$$(m_1 \ldots m_{n-1}, m_1' \ldots m_{n-1}') = 1,$$

$$A(m, 1, \ldots, 1) A(m_1, \ldots, m_{n-1}) = \sum_{\substack{\prod_{l=1}^{n} c_l = m \\ c_1 | m_1, c_2 | m_2, \ldots, c_{n-1} | m_{n-1}}} A\left(\frac{m_1 c_n}{c_1}, \frac{m_2 c_1}{c_2}, \ldots, \frac{m_{n-1} c_{n-2}}{c_{n-1}}\right),$$

and

$$A(m_{n-1}, \ldots, m_1) = \overline{A(m_1, \ldots, m_{n-1})}.$$

Definition 3.1.3. Godement–Jacquet L-function

The Godement–Jacquet L-function $L_f(s)$ [26] attached to f is defined for $\Re(s) > 1$ by

$$L_f(s) = \sum_{m=1}^{\infty} \frac{A(m, 1, \ldots, 1)}{m^s} = \prod_p \prod_{i=1}^{n} (1 - \alpha_{p,i} p^{-s})^{-1},$$

where $\{\alpha_{p,i}\}, 1 \leq i \leq n$ are the complex roots of the monic polynomial

$$X^n + \sum_{j=1}^{n-1} (-1)^j A(\overbrace{1, \ldots, 1}^{j-1 \; terms}, p, 1, \ldots, 1) X^{n-j} + (-1)^n \in \mathbb{C}[X], \quad and$$

$$A(\overbrace{1, \ldots, 1}^{j-1}, p, 1, \ldots, 1) = \sum_{1 \leq i_1 < \cdots < i_j \leq n} \alpha_{p,i_1} \ldots \alpha_{p,i_j}, \qquad for \;\; 1 \leq j \leq n-1.$$

$L_f(s)$ satisfies the functional equation:

$$\begin{aligned}\Lambda_f(s) &:= \prod_{i=1}^{n} \pi^{\frac{-s+\lambda_i(v_f)}{2}} \Gamma\left(\frac{s - \lambda_i(v_f)}{2}\right) L_f(s) \\ &= \Lambda_{\tilde{f}}(1-s),\end{aligned}$$

where $\tilde{f}$ is the Dual Maass form.

In the case of Godement–Jacquet L-function, Yujiao Jiang and Guangshi Lü [26] have studied cancellation on the exponential sum $\sum_{m \leq N} \mu(m) A(m, 1) e^{2\pi i m\theta}$ related to

$SL(3, \mathbb{Z})$ where $\theta \in \mathbb{R}$.

Throughout the chapter, we assume that f is self dual i.e., $\widetilde{f} = f$.

ϵ, ϵ_1 and η always denote any small positive constants.

If $N_f(T)$ denotes the number of zeros of $L_f(s)$ in the rectangle mentioned below, then from the functional equation and the argument principle of complex function theory we have,

$$N_f(T) \sim c(n) T \log T,$$

where $c(n)$ is a non zero constant depending only on the degree n of $L_f(s)$.

(i) The generalized Ramanujan conjecture:

It asserts that

$$\left| A(m, 1, \ldots, 1) \right| \leq d_n(m)$$

where $d_n(m)$ is the number of representations of m as the product of n natural numbers. The current best estimates are due to Kim and Sarnak [31] for $2 \leq n \leq 4$ and Luo, Rudnick and Sarnak [36, 37] for $n \geq 5$

$$\begin{aligned} \left|A(m)\right| &\leq m^{\frac{7}{64}} d(m), \\ \left|A(m,1)\right| &\leq m^{\frac{5}{14}} d_3(m), \\ \left|A(m,1,1)\right| &\leq m^{\frac{9}{22}} d_4(m), \\ \left|A(m,1,\ldots,1)\right| &\leq m^{\frac{1}{2}-\frac{1}{n^2+1}} d_n(m). \end{aligned}$$

We note that the generalized Ramanujan conjecture is equivalent to

$$\left|\alpha_{p,i}\right| = 1 \qquad \forall \text{ primes } p \text{ and } i = 1, 2, \ldots, n.$$

Other estimates are equivalent to

$$\left|\alpha_{p,i}\right| \leq p^{\theta_n} \qquad \forall \text{ primes } p \text{ and } i = 1, 2, \ldots, n \text{ where}$$

$$\theta_2 := \frac{7}{64}, \qquad \theta_3 := \frac{5}{14}, \qquad \theta_4 := \frac{9}{22}, \qquad \theta_n := \frac{1}{2} - \frac{1}{n^2+1} (n \geq 5).$$

(ii) Ramanujan's generalized weak conjecture:

We formulate this conjecture as:

For $n \geq 2$, the inequality

$$\left|\alpha_{p,i}\right| \leq p^{\frac{1}{4}-\epsilon_1}$$

holds for some small $\epsilon_1 > 0$, for every prime p and for $i = 1, 2, \ldots, n$. Of course, this weak conjecture holds good for $n = 2$. For $n \geq 3$, this conjecture is still open.

Taking the logarithmic derivative of $L_f(s)$, we have

$$-\frac{L_f'}{L_f}(s) := \sum_{m=1}^{\infty} \frac{\Lambda_f(m)}{m^s} = \sum_{m=1}^{\infty} \frac{\Lambda(m)a_f(m)}{m^s}$$

where $a_f(m)$ is multiplicative and

$$a_f(p^r) = \sum_{i=1}^{n} \alpha_{p,i}^r$$

for any integer $r \geq 1$.

In particular,

$$a_f(p) = \sum_{i=1}^{n} \alpha_{p,i} = A(p, 1, \ldots, 1).$$

(iii) Rudnick–Sarnak conjecture:

For any fixed integer $r \geq 2$,

$$\sum_p \frac{\left|a_f(p^r)\right|^2 (\log p)^2}{p^r} < \infty.$$

We know that this conjecture is true for $n \leq 4$. (See [32, 49].)

(iv) Riemann hypothesis for $L_f(s)$:

It asserts that $L_f(s) \neq 0$ in $\Re(s) > \frac{1}{2}$.

This chapter aims to establish:

Theorem 3.1.1. *[27] Ramanujan's weak conjecture implies Rudnick–Sarnak conjecture.*

Remark 3.1.1. Theorem 3.1.1 is indicated in [32].

Theorem 3.1.2. *[27] Assume $n \geq 5$ be any arbitrary but fixed integer. Let ϵ be any small positive constant and $T \geq T_0$ where T_0 is sufficiently large. Assume the Rudnick–Sarnak conjecture and Riemann hypothesis for $L_f(s)$. Then the estimate:*

$$\int_T^{2T} \left| \frac{L_f'}{L_f}(\sigma_0 + it) \right|^2 dt \ll_{f,n,\epsilon,\eta} T(\log T)^{2\eta}$$

holds for $\frac{1}{2} + \epsilon \leq \sigma_0 \leq 1 - \epsilon$ with η being some constant satisfying $0 < \eta < \frac{1}{2}$.

Remark 3.1.2. Since Rudnick–Sarnak conjecture is true for $2 \leq n \leq 4$, Theorem 3.1.2 holds just with the assumption of Riemann hypothesis for $L_f(s)$ whenever $2 \leq n \leq 4$.

Remark 3.1.3. It is not difficult to see from our arguments that only assuming Riemann Hypothesis for $L_f(s)$, Theorem 3.1.2 can be upheld for any σ_0 satisfying $1 - \frac{1}{n^2+1} + \epsilon \leq \sigma_0 \leq 1 - \epsilon$ by using the bound $\theta_n = \frac{1}{2} - \frac{1}{n^2+1}$ of Luo, Rudnick and Sarnak.

It is also not difficult to see from our arguments that the generalized Ramanujan conjecture and the Riemann hypothesis for $L_f(s)$ together imply the bound

$$\int_T^{2T} \left| \frac{L_f'}{L_f}(\sigma_0 + it) \right|^2 dt \ll_{f,n,\epsilon} T \tag{3.1}$$

to hold for any σ_0 satisfying $\frac{1}{2} + \epsilon \leq \sigma_0 \leq 1 - \epsilon$.

Though we expect the bound stated in Equation 3.1 to hold unconditionally for σ_0 in the said range, this seems to be very hard to establish.

3.2 Some Lemmas

Lemma 3.2.1. *If $f(s)$ is regular and*

$$\left|\frac{f(s)}{f(s_0)}\right| < e^M \qquad (M > 1)$$

in $|s - s_0| \leq r_1$, *then for any constant b with* $0 < b < \frac{1}{2}$,

$$\left|\frac{f'}{f}(s) - \sum_{\rho} \frac{1}{s - \rho}\right| \ll_b \frac{M}{r_1}$$

in $|s - s_0| \leq \left(\frac{1}{2} - b\right) r_1$, *where ρ runs over all zeros of $f(s)$ such that* $|\rho - s_0| \leq \frac{r_1}{2}$.

Proof. See Lemma α in Section 3.9 of [57] or see [45].

□

Lemma 3.2.2. *Let $N_f^*(T)$ denote the number of zeros of $L_f(s)$ in the region* $0 \leq \sigma \leq 1$, $0 \leq t \leq T$. *Then,*

$$N_f^*(T+1) - N_f^*(T) \ll_n \log T.$$

Proof. Let $n(r_1)$ denote the number of zeros of $L_f(s)$ in the circle with centre $2 + iT$ and radius r_1. By Jensen's theorem,

$$\int_0^3 \frac{n(r_1)}{r_1} dr_1 = \frac{1}{2\pi} \int_0^{2\pi} \log\left|L_f\left(2 + iT + 3e^{i\theta}\right)\right| d\theta - \log\left|L_f(2 + iT)\right|.$$

From the functional equation, we observe that

$$\left|L_f(s)\right| \ll_f t^A \qquad \text{for } -1 \leq \sigma \leq 5 \ \text{ where } A \text{ is some fixed positive constant,}$$

and hence we have,

$$\log\left|L_f\left(2+iT+3e^{i\theta}\right)\right| \ll A\log T.$$

Note that

$$\begin{aligned}\left|1-\frac{\alpha_{p,i}}{p^{2+it}}\right| &\geq 1-\frac{\left|\alpha_{p,i}\right|}{p^2}\\ &\geq 1-\frac{p^{\frac{1}{2}}}{p^2}\\ &= 1-\frac{1}{p^{\frac{3}{2}}}.\end{aligned}$$

Thus we have,

$$\begin{aligned}\left|L_f(2+it)\right| &= \prod_p\prod_{i=1}^{n}\left|\left(1-\frac{\alpha_{p,i}}{p^{2+it}}\right)\right|^{-1}\\ &\leq \prod_p\prod_{i=1}^{n}\left(1-\frac{1}{p^{\frac{3}{2}}}\right)^{-1}\\ &\leq \left(\zeta\left(\frac{3}{2}\right)\right)^{n}\\ &\ll_n 1.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_0^3\frac{n(r_1)}{r_1}dr_1 \ll A\log T + A \ll \log T,\\ &\int_0^3\frac{n(r_1)}{r_1}dr_1 \geq \int_{\sqrt{5}}^3\frac{n(r_1)}{r_1}dr_1 \geq n(\sqrt{5})\int_{\sqrt{5}}^3\frac{dr_1}{r_1} \geq c.n(\sqrt{5}).\end{aligned}$$

Hence,

$$N_f^*(T+1) - N_f^*(T) \ll_n \log T.$$

$\square$

Lemma 3.2.3. *Let a_m(m=1,2,...,N) be any set of complex numbers. Then*

$$\int_T^{2T} \left| \sum_{m=1}^{N} a_m m^{-it} \right|^2 dt = \sum_{m=1}^{N} |a_m|^2 \left(T + O(m)\right).$$

Lemma 3.2.4. *Let b_m be any set of complex numbers such that $\sum m \left(|b_m|\right)^2$ is convergent. Then*

$$\int_T^{2T} \left| \sum_{m=1}^{\infty} b_m m^{-it} \right|^2 dt = \sum_{m=1}^{\infty} |b_m|^2 \left(T + O(m)\right).$$

Proof. See [39] or [44] for Montgomery and Vaughan theorem.

$\square$

Hereafter, $Y \geq 10$ is an arbitrary parameter depending on T which will be chosen suitably later. Also, σ_0 satisfies the inequality $\frac{1}{2} + \epsilon \leq \sigma_0 \leq 1 - \epsilon$ for any small positive constant ϵ.

Lemma 3.2.5. *For $\frac{1}{2} + \epsilon \leq \sigma_0 \leq 1 - \epsilon$, we have*

$$\sum_{m > \frac{Y}{2}(\log Y)^2} \frac{m\left|\Lambda_f(m)\right|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}} \ll 1.$$

Proof. We have,

$$\sum_{m>\frac{Y}{2}(\log Y)^2} \frac{m\left|\Lambda_f(m)\right|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}} \ll \sum_{m>\frac{Y}{2}(\log Y)^2} \frac{m\left|\Lambda_f(m)\right|^2 e^{-\frac{m}{Y}} \frac{Y^2}{m^2}}{m^{2\sigma_0}}$$

$$\ll Y^2 \sum_{m>\frac{Y}{2}(\log Y)^2} \frac{\left|\Lambda_f(m)\right|^2 e^{-\frac{m}{Y}}}{m^{1+2\sigma_0}}.$$

Since $\frac{m}{Y} \geq \frac{1}{2}(\log Y)^2$ for $m \geq \frac{Y}{2}(\log Y)^2$, we have $e^{\frac{m}{Y}} \gg Y^B$ for any large positive constant B. Therefore,

$$\sum_{m>\frac{Y}{2}(\log Y)^2} \frac{m\left|\Lambda_f(m)\right|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}} \ll \frac{Y^2}{Y^B} \sum_{m>\frac{Y}{2}(\log Y)^2} \frac{\left|\Lambda_f(m)\right|^2}{m^{1+2\sigma_0}}$$

$$\ll 1.$$

□

Lemma 3.2.6. *Assuming Rudnick–Sarnak conjecture and taking Y sufficiently large, we have*

$$\sum_{m\leq\frac{Y}{2}(\log Y)^2} \frac{\left|\Lambda_f(m)\right|^2}{m^{2\sigma_0}} e^{-\frac{2m}{Y}} \ll (\log Y)^2.$$

Proof. Note that

$$\sum_{m\leq\frac{Y}{2}(\log Y)^2} \frac{\left|\Lambda_f(m)\right|^2}{m^{2\sigma_0}} e^{-\frac{2m}{Y}} \leq \sum_{p\leq\frac{Y}{2}(\log Y)^2} \frac{(\log p)^2\left|a_f(p)\right|^2}{p^{2\sigma_0}} + \sum_{r=2}^{\left[\frac{\log\frac{Y}{2}}{\log 2}\right]+1} \sum_{p} \frac{(\log p)^2\left|a_f(p^r)\right|^2}{(p^r)^{2\sigma_0}},$$

and

$$\left|a_f(p)\right| = \left|\sum_{i=1}^{n} \alpha_{p,i}\right| = \left|A(p,1,\ldots,1)\right|.$$

We have,

$$\begin{aligned}\sum_{m\le Y} \frac{c_m}{m^l} &= \int_1^Y \frac{d\left(\sum_{m\le u} c_m\right)}{u^l} \\ &= \left.\frac{\sum_{m\le u} c_m}{u^l}\right|_1^Y - \int_1^Y (-l)\frac{\sum_{m\le u} c_m}{u^{l+1}} du.\end{aligned}$$

From Remark 12.1.8 of [14], we have

$$\sum_{m_1^{n-1} m_2^{n-2} \ldots m_{n-1} \le Y} \left|A(m_1, m_2, \ldots, m_{n-1})\right|^2 \ll_f Y.$$

Therefore,

$$\sum_{m\le Y} \left|A(m,1,\ldots,1)\right|^2 \le \sum_{m_1^{n-1} m_2^{n-2} \ldots m_{n-1} \le Y} \left|A(m_1, m_2, \ldots, m_{n-1})\right|^2 \ll_f Y.$$

Taking $l = 2\sigma_0$ and $c_m = \left|A(m,1,\ldots,1)\right|^2$,

$$\sum_{m\le \frac{Y}{2}(\log Y)^2} \frac{\left|A(m,1,\ldots,1)\right|^2}{m^{2\sigma_0}} \ll 1.$$

Hence,

$$\sum_{p \leq \frac{Y}{2}(\log Y)^2} \frac{(\log p)^2 \left|a_f(p)\right|^2}{p^{2\sigma_0}} \ll (\log Y)^2 \sum_{m \leq \frac{Y}{2}(\log Y)^2} \frac{\left|A(m,1,\ldots,1)\right|^2}{m^{2\sigma_0}} \ll (\log Y)^2.$$

By Rudnick–Sarnak conjecture and the bound $\left|\alpha_{p,i}\right| \leq p^{\theta_n}$ with $\theta_n = \frac{1}{2} - \frac{1}{n^2+1}$,

$$\sum_{r \geq 2} \sum_{p} \frac{(\log p)^2 \left|a_f(p^r)\right|^2}{p^r}$$

converges (as in proof of Theorem 3.1.1) and in particular,

$$\sum_{r=2}^{\left[\frac{\log \frac{Y}{2}}{\log 2}\right]+1} \sum_{p} \frac{(\log p)^2 \left|a_f(p^r)\right|^2}{p^r} \ll 1.$$

Therefore,

$$\sum_{m \leq \frac{Y}{2}(\log Y)^2} \frac{\left|\Lambda_f(m)\right|^2}{m^{2\sigma_0}} \ll (\log Y)^2.$$

□

Lemma 3.2.7. *Assume Rudnick–Sarnak conjecture. Then we have,*

$$\sum_{m \leq \frac{Y}{2}(\log Y)^2} \frac{\left|\Lambda_f(m)\right|^2}{m^{2\sigma_0-1}} e^{-\frac{2m}{Y}} \ll Y(\log Y)^4$$

where Y is sufficiently large.

Proof. We have,

$$\begin{aligned}\sum_{m\le \frac{Y}{2}(\log Y)^2} \frac{\left|\Lambda_f(m)\right|^2}{m^{2\sigma_0-1}} e^{-\frac{2m}{Y}} &\le \sum_{m\le \frac{Y}{2}(\log Y)^2} \frac{\left|\Lambda_f(m)\right|^2}{m^{2\sigma_0}} m \\ &\ll Y(\log Y)^2 \sum_{m\le \frac{Y}{2}(\log Y)^2} \frac{\left|\Lambda_f(m)\right|^2}{m^{2\sigma_0}} \\ &\ll Y(\log Y)^4\end{aligned}$$

by using Lemma 3.2.6.

□

Lemma 3.2.8. *For sufficiently large Y, we have*

$$\sum_{m> \frac{Y}{2}(\log Y)^2} \frac{\left|\Lambda_f(m)\right|^2}{m^{1+2\sigma_0}} e^{-\frac{2m}{Y}} \ll_{n,\epsilon} 1.$$

Proof. If $m = p_1^{l_1} p_2^{l_2} \dots p_k^{l_k}$, then

$$\begin{aligned}\left|a_f(m)\right| &= \left|a_f(p_1^{l_1}) \dots a_f(p_k^{l_k})\right| \\ &\le np_1^{l_1\theta_n} \dots np_k^{l_k\theta_n} \\ &\le n^k m^{\theta_n}\end{aligned}$$

where $k = \omega(m) \le \frac{2\log m}{\log\log m}$ and m is sufficiently large.

We have,

$$\begin{aligned}n^{2\omega(m)} &\le n^{\frac{4\log m}{\log\log m}} \\ &= m^{\frac{4\log n}{\log\log m}} \\ &\ll_{n,\epsilon} m^{\epsilon}\end{aligned}$$

and hence,

$$\begin{aligned}
\sum_{m>\frac{Y}{2}(\log Y)^2} \frac{\left|\Lambda_f(m)\right|^2}{m^{1+2\sigma_0}} e^{-\frac{2m}{Y}} &= \sum_{m>\frac{Y}{2}(\log Y)^2} \frac{\left|\Lambda(m)a_f(m)\right|^2}{m^{1+2\sigma_0}} e^{-\frac{2m}{Y}} \\
&\ll \sum_{m>\frac{Y}{2}(\log Y)^2} \frac{(\log m)^2 n^{2\omega(m)} m^{2\theta_n}}{m^{1+2\sigma_0}} \\
&\ll_{n,\epsilon} \sum_{m>\frac{Y}{2}(\log Y)^2} \frac{m^{2\epsilon} m^{2\theta_n}}{m^{1+2\sigma_0}} \\
&\ll_{n,\epsilon} \sum_{m>\frac{Y}{2}(\log Y)^2} \frac{1}{m^{1+\frac{2}{n^2+1}}} \\
&\ll_{n,\epsilon} 1.
\end{aligned}$$

□

Lemma 3.2.9. *For sufficiently large Y, we get*

$$\sum_{m>\frac{Y}{2}(\log Y)^2} \frac{\left|\Lambda_f(m)\right|^2}{m^{2\sigma_0}} e^{-\frac{2m}{Y}} \ll 1.$$

Proof. From Lemma 3.2.5,

$$\sum_{m>\frac{Y}{2}(\log Y)^2} \frac{\left|\Lambda_f(m)\right|^2}{m^{2\sigma_0}} e^{-\frac{2m}{Y}} \leq \sum_{m>\frac{Y}{2}(\log Y)^2} m\frac{\left|\Lambda_f(m)\right|^2}{m^{2\sigma_0}} e^{-\frac{2m}{Y}} \ll 1.$$

□

3.3 Proof of Theorem 3.1.1

Assuming $\left|\alpha_{p,i}\right| \leq p^{\theta_n}$ with $\theta_n \leq \frac{1}{4} - \epsilon_1$, we need to prove that for every integer $n \geq 5$ and for every integer $r \geq 2$,

$$\sum_{p} \frac{(\log p)^2 \left|a_f(p^r)\right|^2}{p^r} < \infty.$$

It is enough to show that

$$\sum_{r=2}^{\infty} \sum_{p} \frac{(\log p)^2 \left|a_f(p^r)\right|^2}{p^r} < \infty.$$

Using

$$a_f(p^r) := \sum_{i=1}^{n} \alpha_{p,i}^r \qquad \text{and} \qquad \left|\alpha_{p,i}\right| \leq p^{\theta_n}$$

we get,

$$\begin{aligned}
\sum_{r=2}^{\infty} \sum_{p} \frac{(\log p)^2 \left|a_f(p^r)\right|^2}{p^r} &\leq \sum_{r=2}^{\infty} \sum_{p} \frac{(\log p)^2 \left(\sum_{i=1}^{n} p^{r\theta_n}\right)^2}{p^r} \\
&= \sum_{r=2}^{\infty} \sum_{p} \frac{(\log p)^2 n^2 p^{2r\theta_n}}{p^r} \\
&\leq n^2 \sum_{p} (\log p)^2 \sum_{r=2}^{\infty} \frac{p^{2r\left(\frac{1}{4}-\epsilon_1\right)}}{p^r} \\
&= n^2 \sum_{p} (\log p)^2 \sum_{r=2}^{\infty} \frac{1}{p^{\frac{r}{2}+2r\epsilon_1}} \\
&= n^2 \sum_{p} (\log p)^2 \frac{p^{-(1+4\epsilon_1)}}{1 - p^{-(\frac{1}{2}+2\epsilon_1)}} \\
&= n^2 \sum_{p} (\log p)^2 \frac{1}{p^{\frac{1}{2}+2\epsilon_1}\left(p^{\frac{1}{2}+2\epsilon_1} - 1\right)} \\
&\ll_{n,\epsilon_1} 1.
\end{aligned}$$

This proves Theorem 3.1.1.

3.4 Proof of Theorem 3.1.2

First, we wish to approximate $\frac{L_f'}{L_f}(s)$ uniformly for $\frac{1}{2} < \sigma_0 \leq \sigma \leq \sigma_1 < 1$ when $T \leq t \leq 2T$. We assume throughout below the Riemann hypothesis for $L_f(s)$.

From the work of Godement–Jacquet [13], it is known that the function $L_f(s)$ is of finite order in any bounded vertical strip. Hence, we can very well assume that

$$L_f(s) \ll T^A = e^{A \log T}$$

for $-1 \leq \sigma \leq 2$, $T \leq t \leq 2T$ and A some fixed positive constant.

Taking $s_0 = 2 + it$ with $t \in \mathbb{R}$, we have

$$L_f(2+it) = \prod_p \prod_{i=1}^{n} \left(1 - \frac{\alpha_{p,i}}{p^{2+it}}\right)^{-1}.$$

Observe that

$$\begin{aligned} \left|1 - \frac{\alpha_{p,i}}{p^{2+it}}\right| &\leq 1 + \frac{\left|\alpha_{p,i}\right|}{p^2} \\ &\leq 1 + \frac{p^{\theta_n}}{p^2} \\ &= 1 + \frac{1}{p^{2-\theta_n}} \\ &\leq 1 + \frac{1}{p^{\frac{3}{2}}} \end{aligned}$$

because $\theta_n \leq \frac{1}{2}$ for $n \geq 2$.

Therefore,

$$\begin{aligned}
\left|L_f(2+it)\right| &\geq \prod_p \prod_{i=1}^{n} \left(1+\frac{1}{p^{\frac{3}{2}}}\right)^{-1} \\
&= \prod_p \left(1+\frac{1}{p^{\frac{3}{2}}}\right)^{-n} \\
&= \prod_p \left(\frac{1-\frac{1}{p^{\frac{3}{2}}}}{1-\frac{1}{p^3}}\right)^{n} \\
&= \left(\frac{\zeta(3)}{\zeta\left(\frac{3}{2}\right)}\right)^{n}
\end{aligned}$$

which is a constant depending only on n. Therefore, $L_f(2+it) \neq 0 \;\; \forall \;\; t \in \mathbb{R}$.

Hence from Lemma 3.2.1, with $r = 12$, $s_0 = 2 + iT$, $f(s) = L_f(s)$, $M = A \log T$, we obtain

$$-\frac{L_f'}{L_f}(s) = \sum_{|s-s_0|\leq 6} \frac{1}{s-\rho} + O(\log T).$$

For $|s - s_0| \leq 3$ and so in particular for $-1 \leq \sigma \leq 2, t = T$, replacing T by t in the particular case, we obtain

$$-\frac{L_f'}{L_f}(s) = \sum_{|\rho-s_0|\leq 6} \frac{1}{s-\rho} + O(\log t).$$

Any term occurring in $\sum\limits_{|t-\gamma|\leq 1} \frac{1}{s-\rho}$ but not in $\sum\limits_{|s-s_0|\leq 6} \frac{1}{s-\rho}$ is bounded and the number of such terms does not exceed

$$N_f^*(t+6) - N_f^*(t-6) \ll \log t,$$

where $N_f^*(t)$ is the number of zeros of $L_f(s)$ in the region $0 \le \sigma \le 1$ and $0 \le t \le T$. Thus, we get

$$-\frac{L_f'}{L_f}(s) = \sum_{|t-\gamma|\le 1} \frac{1}{s-\rho} + O(\log t).$$

Assume $\frac{1}{2} < \sigma < 1$ and $T \le t \le 2T$, then

$$\sum_{m=1}^{\infty} \frac{\Lambda_f(m)}{m^s} e^{-\frac{m}{Y}} = -\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{L_f'}{L_f}(s+w)\Gamma(w)Y^w dw.$$

Note also that from the above reasoning

$$\frac{L_f'}{L_f}(s) \ll \log t \qquad \text{on any line } \sigma \ne \frac{1}{2}.$$

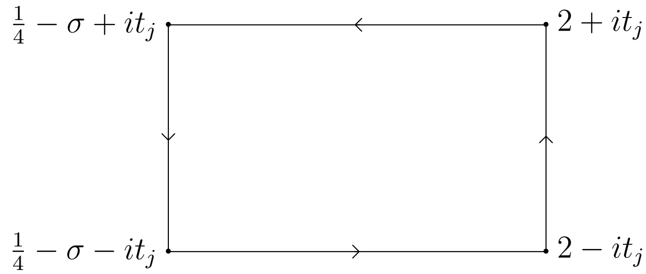
Also,

$$\frac{L_f'}{L_f}(s) \ll \frac{\log t}{\min(|t-\gamma|)} + \log t \qquad \text{uniformly for } -1 \le \sigma \le 2.$$

From Lemma 3.2.2, we observe that each interval $(j, j+1)$ contains values of t whose distance from the ordinate of any zero exceeds $\frac{A}{\log j}$, there is a t_j in any such interval for which

$$\frac{L_f'}{L_f}(s) \ll (\log t)^2 \qquad \text{where } -1 \le \sigma \le 2 \text{ and } t = t_j.$$

Applying Cauchy's residue theorem to the rectangle, we get



$$\frac{1}{2\pi i}\left(\int_{2-it_j}^{2+it_j} + \int_{2+it_j}^{\frac{1}{4}-\sigma+it_j} + \int_{\frac{1}{4}-\sigma+it_j}^{\frac{1}{4}-\sigma-it_j} + \int_{\frac{1}{4}-\sigma-it_j}^{2-it_j}\right)\frac{L_f'}{L_f}(s+w)\Gamma(w)Y^w dw$$
$$= \frac{L_f'}{L_f}(s) + \sum_{-t_j<\gamma<t_j}\Gamma(\rho - s)Y^{\rho-s}.$$

In the sum appearing on the right-hand side above, zeros ρ are counted with its multiplicity if there are any multiple zeros. The integrals along the horizontal lines tend to zero as $j \to \infty$ since the gamma function decays exponentially and Y is going to be at most a power of T only, so that

$$\sum_{m=1}^{\infty}\frac{\Lambda_f(m)}{m^s}e^{-\frac{m}{Y}} = \frac{1}{2\pi i}\int_{\frac{1}{4}-\sigma-i\infty}^{\frac{1}{4}-\sigma+i\infty}\frac{L_f'}{L_f}(s+w)\Gamma(w)Y^w dw - \frac{L_f'}{L_f}(s) - \sum_{\rho}\Gamma(\rho - s)Y^{\rho-s}.$$

Note that $\Gamma(w) \ll e^{-A|v|}$ so that the integral on $\Re(w) = \frac{1}{4} - \sigma$ is

$$\begin{aligned}
&\ll \int_{-\infty}^{\infty} e^{-A|v|} \log(|t+v|+2) Y^{\frac{1}{4}-\sigma} dv \\
&\ll \int_{0}^{2t} e^{-A|v|} \log(10|t|+2) Y^{\frac{1}{4}-\sigma} dv + \left(\int_{-\infty}^{0} + \int_{2t}^{\infty} \right) e^{-A|v|} \log(|v|+10) Y^{\frac{1}{4}-\sigma} dv \\
&\ll Y^{\frac{1}{4}-\sigma} \log T + Y^{\frac{1}{4}-\sigma} \\
&\ll Y^{\frac{1}{4}-\sigma} \log T.
\end{aligned}$$

Note that for $\frac{1}{2} < \sigma_0 \le \sigma \le \sigma_1 < 1$,

$$\big|\Gamma(\rho - s)\big| < A_1 e^{-A_2|\gamma - t|}$$

uniformly for σ in the said range. Therefore,

$$\sum_{\rho} \big|\Gamma(\rho - s)\big| < A_1 \sum_{\rho} e^{-A_2|\gamma - t|} = A_1 \sum_{m=1}^{\infty} \sum_{m-1 \le \gamma \le m} e^{-A_2|t-\gamma|}.$$

The number of terms in the inner sum is

$$\ll \log(|t|+m) \ll \log|t| + \log(m+1)$$

and hence

$$\sum_{\rho} \big|\Gamma(\rho - s)\big| \ll \sum_{m=1}^{\infty} e^{-A_2 m} (\log|t| + \log(m+1)) \ll \log T,$$

$$\left|\sum_{\rho} \Gamma(\rho - s) Y^{\rho - s}\right| \ll Y^{\frac{1}{2}-\sigma} \log T.$$

Thus for $\frac{1}{2} < \sigma_0 \le \sigma \le \sigma_1 < 1$, we have

$$-\frac{L_f'}{L_f}(s) = \sum_{m=1}^{\infty} \frac{\Lambda_f(m)}{m^s} e^{-\frac{m}{Y}} + O_f(Y^{\frac{1}{2}-\sigma} \log T).$$

Thus for $\frac{1}{2} + \epsilon \le \sigma_0 \le 1 - \epsilon$ and $T \le t \le 2T$, we obtain

$$\left|\frac{L_f'}{L_f}(\sigma_0 + it)\right|^2 \ll \left|\sum_{m=1}^{\infty} \frac{\Lambda_f(m) e^{-\frac{m}{Y}}}{m^{\sigma_0 + it}}\right|^2 + \left(Y^{\frac{1}{2}-\sigma_0} \log T\right)^2.$$

Thus,

$$\int_T^{2T} \left|\frac{L_f'}{L_f}(\sigma_0 + it)\right|^2 dt \ll_f \int_T^{2T} \left|\sum_{m=1}^{\infty} \frac{\Lambda_f(m) e^{-\frac{m}{Y}}}{m^{\sigma_0 + it}}\right|^2 dt + Y^{1-2\sigma_0} T (\log T)^2.$$

We note that

$$\left|\sum_{m=1}^{\infty} \frac{\Lambda_f(m) e^{-\frac{m}{Y}}}{m^{\sigma_0 + it}}\right|^2 \ll \left|\sum_{m \le \frac{Y}{2}(\log Y)^2} \frac{\Lambda_f(m) e^{-\frac{m}{Y}}}{m^{\sigma_0 + it}}\right|^2 + \left|\sum_{m > \frac{Y}{2}(\log Y)^2} \frac{\Lambda_f(m) e^{-\frac{m}{Y}}}{m^{\sigma_0 + it}}\right|^2,$$

and hence

$$\int_T^{2T}\left|\frac{L_f'}{L_f}(\sigma_0+it)\right|^2 dt \ll_f \int_T^{2T}\left|\sum_{m\le \frac{Y}{2}(\log Y)^2}\frac{\Lambda_f(m)e^{-\frac{m}{Y}}}{m^{\sigma_0+it}}\right|^2+\int_T^{2T}\left|\sum_{m> \frac{Y}{2}(\log Y)^2}\frac{\Lambda_f(m)e^{-\frac{m}{Y}}}{m^{\sigma_0+it}}\right|^2$$
$$+Y^{1-2\sigma_0}T(\log T)^2.$$

By Montgomery–Vaughan theorem (Lemmas 3.2.3 and 3.2.4) and Lemma 3.2.5, we get

$$\begin{aligned}\int_T^{2T}\left|\frac{L_f'}{L_f}(\sigma_0+it)\right|^2 dt \ll_f & \sum_{m\le \frac{Y}{2}(\log Y)^2}\frac{\left|\Lambda_f(m)\right|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}}\big(T+O(m)\big)\\ &+\sum_{m> \frac{Y}{2}(\log Y)^2}\frac{\left|\Lambda_f(m)\right|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}}\big(T+O(m)\big)+Y^{1-2\sigma_0}T(\log T)^2\\ &\ll_f T\sum_{m\le \frac{Y}{2}(\log Y)^2}\frac{\left|\Lambda_f(m)\right|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}}+\sum_{m\le \frac{Y}{2}(\log Y)^2}m\frac{\left|\Lambda_f(m)\right|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}}\\ &+T\sum_{m> \frac{Y}{2}(\log Y)^2}\frac{\left|\Lambda_f(m)\right|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}}+\sum_{m> \frac{Y}{2}(\log Y)^2}m\frac{\left|\Lambda_f(m)\right|^2 e^{-\frac{2m}{Y}}}{m^{2\sigma_0}}\\ &+Y^{1-2\sigma_0}T(\log T)^2.\end{aligned}$$

By Lemmas 3.2.6, 3.2.7, 3.2.8 and 3.2.9, we obtain

$$\int_T^{2T}\left|\frac{L_f'}{L_f}\left(\frac{1}{2}+\epsilon+it\right)\right|^2 dt \ll_{f,n,\epsilon} T(\log Y)^2+Y(\log Y)^4+Y^{1-2\sigma_0}T(\log T)^2.$$

We choose $Y = \exp\{(\log T)^\eta\}$ with any η satisfying $0 < \eta < \frac{1}{2}$ so that we obtain

$$\int_T^{2T} \left| \frac{L_f'}{L_f}(\sigma_0 + it) \right|^2 dt \ll_{f,n,\epsilon,\eta} T(\log T)^{2\eta}.$$

This proves Theorem 3.1.2.

CHAPTER

FOUR

RANKIN–SELBERG L-FUNCTION RELATED TO THE GODEMENT–JACQUET L-FUNCTION

4.1 Introduction

Definition 4.1.1. *[14] For $n \geq 2$, let f, g be two Maass forms for $SL(n,\mathbb{Z})$ of type $v_f, v_g \in \mathbb{C}^{n-1}$, respectively, with Fourier expansions:*

$$f(z) = \sum_{\gamma \in U_{n-1}(\mathbb{Z}) \backslash SL(n-1,\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1} \neq 0} \frac{A(m_1, \ldots, m_{n-1})}{\prod_{j=1}^{n-1} \left| m_j \right|^{\frac{j(n-j)}{2}}} \\ \times W_J \left(M \cdot \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, v_f, \psi_{1,\ldots,1,\frac{m_{n-1}}{\left|m_{n-1}\right|}} \right),$$

$$g(z) = \sum_{\gamma \in U_{n-1}(\mathbb{Z})\backslash SL(n-1,\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1}\neq 0} \frac{B(m_1, \ldots, m_{n-1})}{\prod_{j=1}^{n-1} \left|m_j\right|^{\frac{j(n-j)}{2}}}$$
$$\times W_J \left(M \cdot \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, v_g, \psi_{1,\ldots,1,\frac{m_{n-1}}{\left|m_{n-1}\right|}} \right).$$

Let $s \in \mathbb{C}$. Then the Rankin–Selberg L-function, denoted as $L_{f\times g}(s)$, is defined by

$$L_{f\times g}(s) = \zeta(ns) \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-1}=1}^{\infty} \frac{A(m_1, \ldots, m_{n-1}) \cdot \overline{B(m_1, \ldots, m_{n-1})}}{(m_1^{n-1} m_2^{n-2} \ldots m_{n-1})^s},$$

which converges absolutely provided $\Re(s)$ is sufficiently large.

In the special case $g = f$, we have

$$L_{f\times f}(s) = \zeta(ns) \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-1}=1}^{\infty} \frac{\left|A(m_1, \ldots, m_{n-1})\right|^2}{(m_1^{n-1} m_2^{n-2} \ldots m_{n-1})^s}$$

for $\Re(s) > 1$.

Let $E_v(z)$ denote the minimal parabolic Eisenstein series. The L-function associated to E_v (see [14, Equation (10.8.5)]) is computed as

$$L_{E_v}(z) = \sum_{c_1=1}^{\infty} \cdots \sum_{c_{n-1}=1}^{\infty} \sum_{m=1}^{\infty} (mc_1 \cdots c_{n-1})^{-s} J_{v-\frac{1}{n}} \left(\begin{pmatrix} \frac{c_1}{m} & & & \\ & \ddots & & \\ & & \frac{c_{n-1}}{m} & \\ & & & 1 \end{pmatrix} \right).$$

From [14, Theorem 10.8.6], there exist functions $\lambda_i : \mathbb{C}^{n-1} \to \mathbb{C}$ satisfying $\Re\left(\lambda_i(v)\right) = 0$

if $\Re(v_i) = \frac{1}{n}(i = 1, \ldots, n-1)$ such that the L-function associated to E_v is just a product of shifted Riemann zeta functions of the form

$$L_{E_v}(z) = \prod_{i=1}^{n} \zeta\left(s - \lambda_i(v)\right).$$

We write

$$L_{f\times f}(s) := \sum_{m=1}^{\infty} \frac{b(m)}{m^s} \qquad \text{for } \Re(s) > 1.$$

Also, $s = \sigma + it$ and t is sufficiently large.

In [28], we proved the following two theorems. Theorem 4.1.1 is an unconditional result while Theorem 4.1.2 is a conditional result.

Theorem 4.1.1. *[28] Let $n \geq 3$ be an arbitrary but fixed integer. For $k \geq k_0(n) = \frac{n^2(n+1)}{2} + n$, we have*

$$\sum_{m\leq x} \frac{b(m)}{k!}\left(1 - \frac{m}{x}\right)^k = \frac{Cx}{(k+1)!} + O_n(\log x).$$

Here C is an effective constant depending only on f.

Hypothesis 4.1.1. (*Coefficient Growth Hypothesis*)

For every $\epsilon > 0$,

$$A(m_1, \ldots, m_{n-1}) \ll_\epsilon m^\epsilon$$

where $m = m_1^{n-1} m_2^{n-2} \ldots m_{n-1}$.

Remark 4.1.1. In some sense, this is slightly weaker than the generalized Ramanujan conjecture namely,

$$\left|\alpha_{p,i}\right| = 1,$$

for every prime p and $i = 1, 2, \ldots, n$.

Hypothesis 4.1.2. (*Lindelöf Hypothesis for $L_{f\times f}(s)$*)
For every $\epsilon > 0$ and for every $\sigma \geq \frac{1}{2}$, the inequality

$$L_{f\times f}(\sigma + it) \ll_\epsilon \left(|t| + 10\right)^\epsilon$$

holds for sufficiently large t.

Theorem 4.1.2. *[28] Assume Hypotheses 4.1.1 and 4.1.2. Let $n \geq 3$ be any arbitrary but fixed integer, then the asymptotic formula*

$$\sum_{m\leq x} \frac{b(m)}{k!}\left(1 - \frac{m}{x}\right)^k = \frac{Cx}{(k+1)!} + O_{n,\epsilon}(x^{\frac{1}{2}+\epsilon})$$

holds for every positive integer $k \geq 1$.

The aim of this chapter is twofold. First, we want to improve the range of k in Theorem 4.1.1 with a better error term. Then, by a reduction argument, we will obtain an unconditional result, namely an asymptotic formula for the sum $\sum\limits_{m\leq x} b(m)$. Thus, we prove:

Theorem 4.1.3. *[29] Let $n \geq 3$ be an arbitrary but fixed integer. For $k \geq k_1(n) = \left[\frac{n^2}{2}\right] + 1$, we have*

$$\sum_{m\leq x} \frac{b(m)}{k!}\left(1 - \frac{m}{x}\right)^k = \frac{Cx}{(k+1)!} + O_n(1).$$

Here C is an effective constant depending only on f.

Theorem 4.1.4. *[29] For sufficiently large x, we have*

$$\sum_{m\leq x} b(m) = \frac{2^{k_1}C}{(k_1+1)}x + O_n\left(x^{1-\frac{1}{2^{k_1}}}\right)$$

where $k_1 = k_1(n) = \left[\frac{n^2}{2}\right] + 1$.

Remark 4.1.2. When proving Theorem 4.1.2 in [28], we assumed hypotheses 4.1.1 and 4.1.2. From Lemma 4.3.1, we can see that Hypothesis 4.1.1 is redundant. Just with the assumption of Hypothesis 4.1.2 for $k = 1$, we have

$$\sum_{m\leq x} b(m)\left(1-\frac{m}{x}\right) = \frac{Cx}{2} + O(x^{\frac{1}{2}+\epsilon}).$$

Using Lemma 4.3.5, we observe that conditionally we get

$$\sum_{m\leq x} b(m) = Cx + O(x^{\frac{3}{4}+\epsilon}).$$

Though the error term obtained in Theorem 4.1.4 is weaker than what is expected, it is an unconditional result.

Remark 4.1.3. We note that the reduction process in Lemma 4.3.5 originated in [21] due to Ingham. This idea has been successfully exploited under various circumstances by several researchers. For instance, see [1, 2, 3].

Remark 4.1.4. Although the error term derived in Theorem 4.1.4 may not match the level of precision found in the results of H. Lao [35], it's important to note that our approach to addressing this problem differs significantly in the sense that we could get as a byproduct an unconditional asymptotic formula using a reduction argument from the k-th Riesz mean.

Throughout the chapter, we assume that f is a self-dual Hecke–Maass form for $SL(n,\mathbb{Z})$ and ϵ is any small positive constant.

4.2 Preliminaries

In this section, we present some necessary properties of the Rankin–Selberg L-function which are used later.

4.2.1 Euler Product

Fix $n \geq 2$. Let f, g be two Maass forms for $SL(n,\mathbb{Z})$ with Euler products

$$L_f(s) = \sum_{m=1}^{\infty} \frac{A(m,1,\ldots,1)}{m^s} = \prod_p \prod_{i=1}^{n} (1-\alpha_{p,i}p^{-s})^{-1},$$

$$L_g(s) = \sum_{m=1}^{\infty} \frac{B(m,1,\ldots,1)}{m^s} = \prod_p \prod_{i=1}^{n} (1-\beta_{p,i}p^{-s})^{-1},$$

then $L_{f\times g}(s)$ will have an Euler product of the form:

$$L_{f\times g}(s) = \prod_p \prod_{i=1}^{n} \prod_{j=1}^{n} (1-\alpha_{p,i}\overline{\beta_{p,j}}p^{-s})^{-1}.$$

4.2.2 Functional Equation

For $n \geq 2$, let f, g be two Maass forms of types v_f, v_g for $SL(n, \mathbb{Z})$ whose associated L-functions L_f, L_g satisfy the functional equations:

$$\begin{aligned}
\Lambda_f(s) &:= \prod_{i=1}^{n} \pi^{\frac{-s+\lambda_i(v_f)}{2}} \Gamma\left(\frac{s-\lambda_i(v_f)}{2}\right) L_f(s) \\
&= \Lambda_{\tilde{f}}(1-s), \\
\Lambda_g(s) &:= \prod_{j=1}^{n} \pi^{\frac{-s+\lambda_j(v_g)}{2}} \Gamma\left(\frac{s-\lambda_j(v_g)}{2}\right) L_g(s) \\
&= \Lambda_{\tilde{g}}(1-s),
\end{aligned}$$

where $\tilde{f}, \tilde{g}$ are the Dual Maass forms.

Then the Rankin–Selberg L-function $L_{f\times g}(s)$ has a meromorphic continuation to all $s \in \mathbb{C}$ with at most a simple pole at $s = 1$ with residue proportional to $\langle f, g\rangle$, the Petersson inner product of f with g. $L_{f\times g}(s)$ satisfies the functional equation:

$$\begin{aligned}
\Lambda_{f\times g}(s) &:= \prod_{i=1}^{n} \prod_{j=1}^{n} \pi^{\frac{-s+\lambda_i(v_f)+\overline{\lambda_j(v_g)}}{2}} \Gamma\left(\frac{s-\lambda_i(v_f)-\overline{\lambda_j(v_g)}}{2}\right) L_{f\times g}(s) \\
&= \Lambda_{\tilde{f}\times\tilde{g}}(1-s).
\end{aligned}$$

From Equation (10.8.5) and Remark 10.8.7 of [14], the powers of π take the much simpler form:

$$\prod_{i=1}^{n} \pi^{\frac{-s+\lambda_i(v)}{2}} = \pi^{\frac{-ns}{2}}, \qquad \prod_{i=1}^{n} \prod_{j=1}^{n} \pi^{\frac{-s+\lambda_i(v_f)+\overline{\lambda_j(v_g)}}{2}} = \pi^{\frac{-n^2 s}{2}}.$$

Hence, we get

$$\Lambda_{f\times g}(s) := \pi^{\frac{-n^2 s}{2}} \prod_{i=1}^{n}\prod_{j=1}^{n}\Gamma\left(\frac{s-\lambda_i(v_f)-\overline{\lambda_j(v_g)}}{2}\right) L_{f\times g}(s)$$
$$= \Lambda_{\tilde{f}\times\tilde{g}}(1-s).$$

We take $g = f$ and f to be a self-dual Maass form of type v so that

$$\Lambda_{f\times f}(s) := \pi^{\frac{-n^2 s}{2}} \prod_{i=1}^{n}\prod_{j=1}^{n}\Gamma\left(\frac{s-\lambda_i(v)-\overline{\lambda_j(v)}}{2}\right) L_{f\times f}(s)$$
$$= \Lambda_{f\times f}(1-s).$$

4.2.3 Bound for the conversion factor

Let f be a self-dual Hecke–Maass form. Then we have the functional equation

$$\Lambda_{f\times f}(s) = \Lambda_{f\times f}(1-s).$$

If we write $L_{f\times f}(s) = \chi_{f\times f}(s)L_{f\times f}(1-s)$, then the conversion factor $\chi_{f\times f}(s)$ can be written as

$$\chi_{f\times f}(s) := \frac{\pi^{n^2 s-\frac{n^2}{2}} \prod_{i=1}^{n}\prod_{j=1}^{n}\Gamma\left(\frac{1-s-\lambda_i(v)-\overline{\lambda_j(v)}}{2}\right)}{\prod_{i=1}^{n}\prod_{j=1}^{n}\Gamma\left(\frac{s-\lambda_i(v)-\overline{\lambda_j(v)}}{2}\right)},$$
$$= \pi^{n^2 s-\frac{n^2}{2}} \exp\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\log\Gamma\left(\frac{1-s-\lambda_i(v)-\overline{\lambda_j(v)}}{2}\right)\right]$$
$$\times \exp\left[-\sum_{i=1}^{n}\sum_{j=1}^{n}\log\Gamma\left(\frac{s-\lambda_i(v)-\overline{\lambda_j(v)}}{2}\right)\right].$$

From [21], we know that the following asymptotic formula for the Gamma function,

$$\log \Gamma(z+\alpha) = \left(z+\alpha-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi + O\left(\frac{1}{|z|}\right)$$

holds uniformly in any fixed angle $\left|\arg(z)\right| \leq \pi - \delta < \pi$ and any bounded range of α as $|z| \to \infty$.

Evaluating appropriately, we get

$$\begin{aligned}
\log \Gamma\left(\frac{1-s-\lambda_i(v)-\overline{\lambda_j(v)}}{2}\right) &= \left(\frac{-s-\lambda_i(v)-\overline{\lambda_j(v)}}{2}\right)\log\left(\frac{-s}{2}\right) \\
&\quad + \frac{s}{2} + \frac{1}{2}\log 2\pi + O\left(\frac{1}{|s|}\right), \\
\log \Gamma\left(\frac{s-\lambda_i(v)-\overline{\lambda_j(v)}}{2}\right) &= \left(\frac{s-1-\lambda_i(v)-\overline{\lambda_j(v)}}{2}\right)\log\left(\frac{s}{2}\right) \\
&\quad - \frac{s}{2} + \frac{1}{2}\log 2\pi + O\left(\frac{1}{|s|}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n}\log \Gamma\left(\frac{1-s-\lambda_i(v)-\overline{\lambda_j(v)}}{2}\right) \\
&\quad = n^2\left\{\left(\frac{-s}{2}\right)\log\left(\frac{-s}{2}\right) + \frac{s}{2} + \frac{1}{2}\log 2\pi + O\left(\frac{1}{|s|}\right)\right\}, \\
&\sum_{i=1}^{n}\sum_{j=1}^{n}\log \Gamma\left(\frac{s-\lambda_i(v)-\overline{\lambda_j(v)}}{2}\right) \\
&\quad = n^2\left\{\left(\frac{s-1}{2}\right)\log\left(\frac{s}{2}\right) - \frac{s}{2} + \frac{1}{2}\log 2\pi + O\left(\frac{1}{|s|}\right)\right\}.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\chi_{f\times f}(s) \\
&= \pi^{n^2 s-\frac{n^2}{2}} \exp\left[n^2\left\{\left(\frac{-s}{2}\right)\log\left(\frac{-s}{2}\right)+\left(\frac{1-s}{2}\right)\log\left(\frac{s}{2}\right)+s+O\left(\frac{1}{|s|}\right)\right\}\right]
\end{aligned}
$$

and hence,

$$
\begin{aligned}
\left|\chi_{f\times f}(s)\right| &\ll \left|\left(\frac{-s}{2}\right)^{\frac{-n^2 s}{2}}\left(\frac{s}{2}\right)^{\frac{n^2(1-s)}{2}}\right| \\
&\ll |s|^{\frac{-n^2\sigma}{2}}|s|^{\frac{n^2-n^2\sigma}{2}} \\
&\ll |t|^{n^2\left(\frac{1}{2}-\sigma\right)}.
\end{aligned}
$$

This bound is true in any fixed vertical strip $a \le \sigma \le b$ and sufficiently large t. *Hereafter, throughout the chapter, we assume* $n \ge 3$.

4.3 Some Lemmas

Lemma 4.3.1. *For* $\Re(s) \ge 1+\epsilon$*,* $L_{f\times f}(s)$ *is absolutely convergent.*

Proof. The Rankin–Selberg L-function $L_{f\times f}(s)$ has a meromorphic continuation to all $s \in \mathbb{C}$ with a simple pole at $s=1$. It is easy to see that

$$
L_{f\times f}(s) = \zeta(ns)\sum_{m_1=1}^{\infty}\cdots\sum_{m_{n-1}=1}^{\infty}\frac{\left|A(m_1,\ldots,m_{n-1})\right|^2}{(m_1^{n-1}m_2^{n-2}\ldots m_{n-1})^s}
$$

implies that the coefficients $b(m)$ are non-negative. Landau's lemma [7, Page 115] asserts that a Dirichlet series with non-negative coefficients must be absolutely convergent up to its first pole. Hence, $L_{f\times f}(s)$ is absolutely convergent in the half-plane $\Re(s) \geq 1+\epsilon$.

$\square$

Lemma 4.3.2. *For sufficiently large t, we have*

$$L_{f\times f}(s) \ll \big(|t|+10\big)^{\frac{n^2}{2}(1+\epsilon-\sigma)}$$

uniformly for $-\epsilon \leq \sigma \leq 1+\epsilon$.

Proof. We prove along the same lines as in [50, Lemma 3.5]. From Lemma 4.3.1, we have

$$\big|L_{f\times f}(1+\epsilon+it)\big| \ll 1,$$

and by the functional equation

$$\begin{aligned}\big|L_{f\times f}(-\epsilon+it)\big| &= \big|\chi_{f\times f}(-\epsilon+it)L_{f\times f}(1+\epsilon-it)\big| \\ &\ll \big(|t|+10\big)^{n^2\left(\frac{1}{2}+\epsilon\right)}.\end{aligned}$$

Now we apply the maximum modulus principle to the function

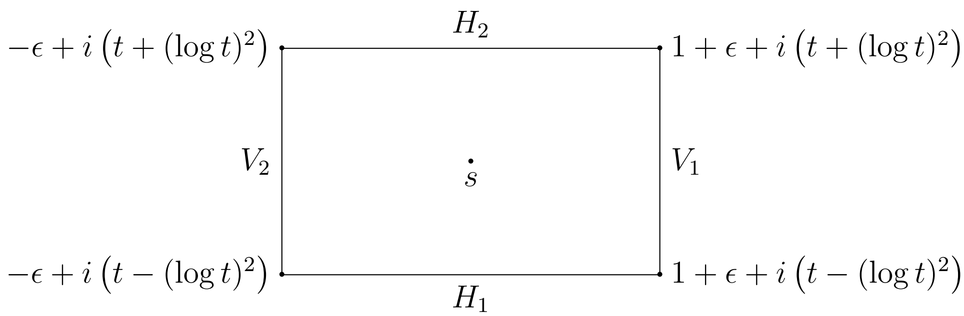
$$F(w) = L_{f\times f}(w)e^{(w-s)^2}X^{w-s}$$

in the rectangle

so that

$$\big|L_{f\times f}(s)\big| \ll V_1+V_2+H_1+H_2.$$

Here V_1, V_2 are the contributions from the vertical lines and H_1, H_2 are the contributions from the horizontal lines.



Let $w = u + iv$ and $s = \sigma + it$. As

$$\begin{aligned}
\exp\{(w-s)^2\} &= \exp\{(u-\sigma)^2 - (v-t)^2 + 2i(u-\sigma)(v-t)\} \\
\left|\exp\{(w-s)^2\}\right| &= \exp\{(u-\sigma)^2 - (v-t)^2\} \\
&\ll \exp\{-(\log t)^2\},
\end{aligned}$$

we see that $\exp\{(w-s)^2\}$ decays exponentially for large t on horizontal lines. Thus,

$$\begin{gathered}
H_1 \ll 1,\ H_2 \ll 1,\ V_1 \ll X^{1+\epsilon-\sigma} \\
V_2 \ll \left(|t| + 10\right)^{n^2\left(\frac{1}{2}+\epsilon\right)} X^{-\epsilon-\sigma}.
\end{gathered}$$

Therefore,

$$\left|L_{f\times f}(s)\right| \ll X^{1+\epsilon-\sigma} + \left(|t| + 10\right)^{n^2\left(\frac{1}{2}+\epsilon\right)} X^{-\epsilon-\sigma} + 1.$$

We choose X such that

$$\begin{gathered}
X^{1+\epsilon-\sigma} \sim \left(|t| + 10\right)^{n^2\left(\frac{1}{2}+\epsilon\right)} X^{-\epsilon-\sigma} \\
\text{i.e., } X \sim \left(|t| + 10\right)^{\frac{n^2}{2}}
\end{gathered}$$

so that

$$\left|L_{f\times f}(s)\right| \ll \left(|t|+10\right)^{\frac{n^2}{2}(1+\epsilon-\sigma)}.$$

This completes the proof of this lemma.

$\square$

Lemma 4.3.3. *For $0 \leq \Re(s) \leq 1+\epsilon$, we have uniformly*

$$L_{f\times f}(s) \ll \left(|t|+10\right)^{\frac{n^2}{2}+\epsilon}.$$

Proof. Follows from Lemma 4.3.2.

$\square$

Lemma 4.3.4. *Let c and y be any positive real numbers and T is sufficiently large. Then we have,*

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^s}{s(s+1)\dots(s+k)}ds = \begin{cases} \frac{1}{k!}\left(1-\frac{1}{y}\right)^k + O\left(\frac{4^k y^c}{T^k}\right) & ,y \geq 1 \\ O\left(\frac{1}{T^k}\right) & ,0 < y \leq 1. \end{cases}$$

Proof. See [52, Lemma 3.2].

$\square$

Remark 4.3.1. Let

$$B(x) = \frac{1}{x}\int_1^x A(t)\ dt.$$

If we know the asymptotic formula for $A(x)$, we can find the asymptotic relation for $B(x)$. But the converse is not true. However, if $A(x)$ is monotonic, then using the asymptotic formula for $B(x)$, we can deduce the asymptotic relation for $A(x)$.

Lemma 4.3.5. *Let $A(x)$ be a monotonically increasing function such that*

$$B(x) = \frac{1}{x}\int_1^x A(t)\ dt.$$

If
$$B(x) = cx + O\left(\frac{x}{E(x)}\right),$$
then
$$A(x) = 2cx + O\left(\frac{x}{\sqrt{E(x)}}\right).$$

Proof. Since

$$B(x) = \frac{1}{x}\int_1^x A(t)\ dt,$$

we have

$$(x+\delta)B(x+\delta) - xB(x) = \int_x^{x+\delta} A(t)\ dt > A(x)\delta$$

where $\delta = o(x)$ is chosen later. Thus

$$\begin{aligned} A(x) &< \left(1+\frac{x}{\delta}\right)\left(cx + c\delta + O\left(\frac{x}{E(x)}\right)\right) - \frac{x}{\delta}\left(cx + O\left(\frac{x}{E(x)}\right)\right) \\ &= cx + c\delta + O\left(\frac{x}{E(x)}\right) + \frac{cx^2}{\delta} + cx + O\left(\frac{x^2}{\delta E(x)}\right) - \frac{cx^2}{\delta} + O\left(\frac{x^2}{\delta E(x)}\right) \\ &= 2cx + c\delta + O\left(\frac{x^2}{\delta E(x)}\right). \end{aligned}$$

The parameter δ is chosen such that

$$\frac{x^2}{\delta E(x)} < \delta$$

i.e.,

$$\delta > \frac{x}{\sqrt{E(x)}}.$$

Thus, we get

$$A(x) < 2cx + \left(\frac{x}{\sqrt{E(x)}}\right).$$

Also,

$$xB(x) - (x-\delta)B(x-\delta) = \int_{x-\delta}^{x} A(t)\ dt < A(x)\delta$$

gives

$$\begin{aligned} A(x) &> \frac{x}{\delta}\left(cx + O\left(\frac{x}{E(x)}\right)\right) + \left(1 - \frac{x}{\delta}\right)\left(cx - c\delta + O\left(\frac{x}{E(x)}\right)\right) \\ &= \frac{cx^2}{\delta} + O\left(\frac{x^2}{\delta E(x)}\right) + cx - c\delta + O\left(\frac{x}{E(x)}\right) - \frac{cx^2}{\delta} + cx + O\left(\frac{x^2}{\delta E(x)}\right) \end{aligned}$$

We choose δ so that

$$\frac{x^2}{\delta E(x)} < \delta$$

i.e.,

$$\delta > \frac{x}{\sqrt{E(x)}}.$$

Thus, we get

$$A(x) = 2cx + \left(\frac{x}{\sqrt{E(x)}}\right).$$

□

4.4 Proof of Theorem 4.1.3

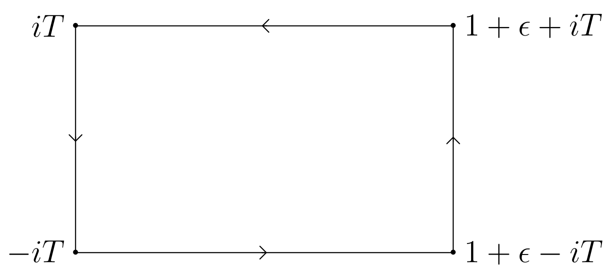
Let $y = \frac{x}{m} \geq 1$ and $c = 1 + \epsilon$ in Lemma 4.3.4 so that

$$\frac{1}{k!}\left(1-\frac{m}{x}\right)^k = \frac{1}{2\pi i}\int_{1+\epsilon-iT}^{1+\epsilon+iT} \frac{\left(\frac{x}{m}\right)^s}{s(s+1)\dots(s+k)}ds + O\left(\frac{4^k x^{1+\epsilon}}{T^k m^{1+\epsilon}}\right).$$

Hence,

$$\begin{aligned}\sum_{m\le x}\frac{b(m)}{k!}\left(1-\frac{m}{x}\right)^k &= \sum_{m\le x}\frac{b(m)}{2\pi i}\int_{1+\epsilon-iT}^{1+\epsilon+iT} \frac{\left(\frac{x}{m}\right)^s}{s(s+1)\dots(s+k)}ds \\ &\quad + O\left(\frac{4^k x^{1+\epsilon}}{T^k}\sum_{m\le x}\frac{b(m)}{m^{1+\epsilon}}\right) \\ &= \frac{1}{2\pi i}\int_{1+\epsilon-iT}^{1+\epsilon+iT} \frac{L_{f\times f}(s)x^s}{s(s+1)\dots(s+k)}ds + O\left(\frac{4^k x^{1+\epsilon}}{T^k}\right).\end{aligned}$$

Summation and integral can be interchanged because of absolute convergence. Now we move the line of integration to $\Re(s) = 0$.



By Cauchy's residue theorem,

$$\begin{aligned}&\frac{1}{2\pi i}\left[\int_{1+\epsilon-iT}^{1+\epsilon+iT} + \int_{1+\epsilon+iT}^{iT} + \int_{iT}^{-iT} + \int_{-iT}^{1+\epsilon-iT}\right]\frac{L_{f\times f}(s)x^s}{s(s+1)\dots(s+k)}ds \\ &= \mathrm{Res}_{s=1}\frac{L_{f\times f}(s)x^s}{s(s+1)\dots(s+k)} \\ &= \lim_{s\to 1}\frac{(s-1)L_{f\times f}(s)x^s}{s(s+1)\dots(s+k)} \\ &= \frac{Cx}{(k+1)!}\end{aligned}$$

where $C = \lim_{s\to 1}(s-1)L_{f\times f}(s)$, depends on f.

Hence,

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{1+\epsilon-iT}^{1+\epsilon+iT}\frac{L_{f\times f}(s)x^s}{s(s+1)\dots(s+k)}ds\\
&=\frac{Cx}{(k+1)!}+\frac{1}{2\pi i}\left[\int_{iT}^{1+\epsilon+iT}+\int_{-iT}^{iT}+\int_{1+\epsilon-iT}^{-iT}\right]\frac{L_{f\times f}(s)x^s}{s(s+1)\dots(s+k)}ds.
\end{aligned}$$

Horizontal line contributions are in absolute value:

$$\begin{aligned}
&\left|\frac{1}{2\pi i}\int_{iT}^{1+\epsilon+iT}\frac{L_{f\times f}(s)x^s}{s(s+1)\dots(s+k)}ds\right|\\
&\quad=\left|\frac{1}{2\pi i}\int_{0}^{1+\epsilon}\frac{L_{f\times f}(\sigma+iT)x^{\sigma+iT}}{(\sigma+iT)(\sigma+iT+1)\dots(\sigma+iT+k)}d\sigma\right|\\
&\quad\leq\frac{1}{2\pi}\int_{0}^{1+\epsilon}\frac{\left|L_{f\times f}(\sigma+iT)\right|x^{\sigma}}{T^{k+1}}d\sigma\\
&\quad\ll T^{\frac{n^2}{2}-k-1+\epsilon}x^{1+\epsilon}.
\end{aligned}$$

The left vertical line contribution is:

$$\begin{aligned}
\frac{1}{2\pi i}\int_{-iT}^{iT}\frac{L_{f\times f}(s)x^s}{s(s+1)\dots(s+k)}ds&=\frac{1}{2\pi i}\int_{\substack{|t|\leq t_0,\\ \sigma=0}}\frac{L_{f\times f}(s)x^s}{s(s+1)\dots(s+k)}ds\\
&\quad+\frac{1}{2\pi i}\int_{\substack{t_0\leq|t|\leq T,\\ \sigma=0}}\frac{L_{f\times f}(s)x^s}{s(s+1)\dots(s+k)}ds.
\end{aligned}$$

We note that

$$\begin{aligned}
&\left|\frac{1}{2\pi i}\int_{\substack{|t|\le t_0,\\ \sigma=0}}\frac{L_{f\times f}(s)x^s}{s(s+1)\dots(s+k)}ds\right| \\
&=\left|\frac{1}{2\pi i}\int_{|t|\le t_0}\frac{L_{f\times f}(it)x^{it}}{(it)(it+1)\dots(it+k)}idt\right| \\
&\le \frac{1}{2\pi}\int_{|t|\le t_0}\frac{t^{\frac{n^2}{2}-1+\epsilon}}{k!}dt \\
&\ll_n 1
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{1}{2\pi i}\int_{\substack{t_0\le|t|\le T,\\ \sigma=0}}\frac{L_{f\times f}(s)x^s}{s(s+1)\dots(s+k)}ds\right| \\
&=\left|\frac{1}{2\pi i}\int_{t_0\le|t|\le T}\frac{L_{f\times f}(it)x^{it}}{(it)(it+1)\dots(it+k)}idt\right| \\
&\le \frac{1}{2\pi}\int_{t_0\le|t|\le T}\frac{t^{\frac{n^2}{2}+\epsilon}}{t^{k+1}}dt \\
&\ll T^{\frac{n^2}{2}-k+\epsilon}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{1}{2\pi i}\int_{1+\epsilon-iT}^{1+\epsilon+iT}\frac{L_{f\times f}(s)x^s}{s(s+1)\dots(s+k)}ds = \frac{Cx}{(k+1)!} &+ O(T^{\frac{n^2}{2}-k-1+\epsilon}x^{1+\epsilon}) \\
&+ O(T^{\frac{n^2}{2}-k+\epsilon}) + O_n(1).
\end{aligned}$$

This implies that

$$\sum_{m\leq x}\frac{b(m)}{k!}\left(1-\frac{m}{x}\right)^{k}=\frac{Cx}{(k+1)!}+O(T^{\frac{n^2}{2}-k-1+\epsilon}x^{1+\epsilon})+O(T^{\frac{n^2}{2}-k+\epsilon})$$
$$+O(T^{-k}x^{1+\epsilon})+O_n(1).$$

First we choose $T=\frac{x}{10}$ so that

$$\sum_{m\leq x}\frac{b(m)}{k!}\left(1-\frac{m}{x}\right)^{k}=\frac{Cx}{(k+1)!}+O(x^{\frac{n^2}{2}-k+\epsilon})+O(x^{\frac{n^2}{2}-k+\epsilon})+O(x^{1-k+\epsilon})+O_n(1).$$

Thus for $k\geq k_1(n)=\left[\frac{n^2}{2}\right]+1$, we finally arrive at

$$\sum_{m\leq x}\frac{b(m)}{k!}\left(1-\frac{m}{x}\right)^{k}=\frac{Cx}{(k+1)!}+O_n(1)$$

which holds good for all integers $k\geq k_1(n)$.

4.5 Proof of Theorem 4.1.4

From Theorem 4.1.3 with $k=k_1$ we have,

$$\sum_{m\leq x}\frac{b(m)}{k_1!}\left(1-\frac{m}{x}\right)^{k_1}=\frac{Cx}{(k_1+1)!}+O_n(1).$$

Note that

$$\begin{aligned}
\sum_{m\le x}\frac{b(m)}{k_1!}\left(1-\frac{m}{x}\right)^{k_1} &= \sum_{m\le x}\frac{b(m)}{k_1!}\left(1-\frac{m}{x}\right)^{k_1-1}\left(1-\frac{m}{x}\right)\\
&= \frac{1}{x}\sum_{m\le x}\frac{b(m)}{k_1!}\left(1-\frac{m}{x}\right)^{k_1-1}(x-m)\\
&= \frac{1}{x}\sum_{m\le x}\frac{b(m)}{k_1!}\left(1-\frac{m}{x}\right)^{k_1-1}\int_m^x dt\\
&= \frac{1}{x}\int_1^x\left(\sum_{m\le t}\frac{b(m)}{k_1!}\left(1-\frac{m}{t}\right)^{k_1-1}\right)dt.
\end{aligned}$$

Using Lemma 4.3.5 with $E(x)=10x$, we can find the (k_1-1)-th Riesz mean. In particular, we get

$$\sum_{m\le x}\frac{b(m)}{k_1!}\left(1-\frac{m}{x}\right)^{k_1-1} = \frac{2Cx}{(k_1+1)!}+O_n(x^{1-\frac{1}{2}}).$$

Once again using Lemma 4.3.5, we get

$$\sum_{m\le x}\frac{b(m)}{k_1!}\left(1-\frac{m}{x}\right)^{k_1-2} = \frac{2^2Cx}{(k_1+1)!}+O_n(x^{1-\frac{1}{2^2}}).$$

Repeatedly using the result in Lemma 4.3.5 k_1 times, we get

$$\sum_{m\le x}\frac{b(m)}{k_1!} = \frac{2^{k_1}Cx}{(k_1+1)!}+O_n\left(x^{1-\frac{1}{2^{k_1}}}\right).$$

This proves the theorem.

CONCLUDING REMARKS

We presented three primary outcomes in this thesis.

In Chapter 2, we improved an earlier result of Z. Cui and J. Wu [9] and gave an asymptotic formula for the mean value of arithmetic functions over shorter intervals. This improvement is achieved by utilizing the Hooley–Huxley contour as the contour of integration.

In Chapter 3, we formulated Ramanujan's weak conjecture and showed that it implies the Rudnick–Sarnak conjecture. We also studied the mean square of the logarithmic derivative of the Godement–Jacquet L-function on the line σ_0 with $\frac{1}{2}+\epsilon \leq \sigma_0 \leq 1-\epsilon$. Under the assumption of Rudnick–Sarnak conjecture and Riemann hypothesis for Godement–Jacquet L-function, we gave $T(\log T)^{2\eta}$ as the upper bound for this mean square where η is some constant such that $0 < \eta < \frac{1}{2}$. It's worth noting that the anticipated upper bound for this mean square is T.

In Chapter 4, we studied the k-th Riesz mean for the coefficients of the Rankin–Selberg convolution of f with itself. We mention a binary improvement of our earlier results. One is the improvement in the range of k for which the asymptotic formula holds. Another improvement is in the error term. As a by-product, we obtained an asymptotic formula for the partial sum of coefficients of this Rankin–Selberg L-function. The conditional error term that we obtain for this partial sum is $x^{\frac{3}{4}+\epsilon}$, although the best error that is expected here is $x^{\frac{1}{2}+\epsilon}$.

The findings that are presented in this thesis not only contribute to the current body of knowledge but also pave the way for future investigations in these intriguing areas of study.

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