

A study of generalized Riemann problems for a system of hyperbolic partial differential equations

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by

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To God
and
my parents

Smt. Sangeeta Devi and Shree Ram Gopal Sharan Arya



CERTIFICATE

This is to certify that the thesis entitled “**A study of generalized Riemann problems for a system of hyperbolic partial differential equations**” by **Mr. Akshay Kumar** bearing Reg. No. 18MMPA02 in partial fulfilment of the requirements for the award of Doctor of Philosophy in **Applied Mathematics** is a bonafide work carried out by him under the supervision and guidance of Prof. R. Radha.

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Further, the student has the following publications:

1. Radha, R., Sharma, V. D. and Akshay Kumar: Riemann problem for rate-type materials with non-constant initial conditions, *Mathematical Methods in the Applied Sciences*, **44** (2021) 13866-13880. <https://doi.org/10.1002/mma.7663>
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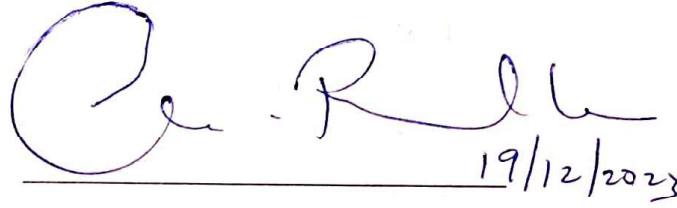
1. International conference on the numerical and analytical techniques in differential equations (ICNATDE-2021) November 16-18, 2021, Ramanujan School of Mathematical Sciences Pondicherry University, Puducherry - 605014, India. **(Online)**
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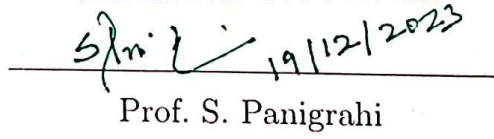
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Declaration

I, Akshay Kumar, Reg. No. 18MMPA02, hereby declare that the thesis entitled “A study of generalized Riemann problems for a system of hyperbolic partial differential equations” submitted by me under the guidance and supervision of Prof. R. Radha is a bonafide research work. I also declare that it has not been submitted previously in part or full to this university or any other university/institute for the award of any degree or diploma.

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AKSHAY

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Chapter 1

Introduction

Exact solutions of partial differential equations have been pivotal in advancing our understanding of the qualitative features of diverse natural phenomena and processes across several scientific disciplines. Notably, the solutions of nonlinear differential equations offer clear illustrations of the intricate mechanisms underlying complex effects, such as spatial localization of transfer processes, the multiplicity or absence of stationary states, the existence of blow-up solutions, and the possible non-smoothness of the unknowns. Even exact solutions of differential equations remain invaluable as test problems that facilitate the assessment of accuracy and the applicability range of various numerical and approximate analytical methods.

The general theory of quasilinear partial differential equations emerged a century ago in the context of mathematical physics. To understand more about quasilinear partial differential equations, in particular nonlinear hyperbolic partial differential equations, the fundamental problem is to study the Riemann problem, which is basically an initial value problem with constant initial data with a discontinuity at a point. Riemann [1] first posed the Riemann problem and solved it in his seminal work on the mathematical theory of shock waves in 1860 for isentropic gas dynamic equations. Riemann's solution is described by elementary waves, i.e., shock waves and rarefaction waves. In the 1940s, Courant and Friedrichs [3] added a new type of elementary wave, characteristic shock waves/contact discontinuities, while studying adiabatic flow. The theory of small solutions to the Riemann problem for strictly hyperbolic systems was established in the seminal work of Lax [4] in 1957. Since then, great progress has been achieved in understanding Riemann's problem for one-dimensional gas dynamic equations, and research interest in the Riemann problem lasted for more than a century. The Riemann problem for gas dynamic equations governing a one-dimensional flow of van der Waals gases was studied by Ambika and Radha [9] and by Hattori [73]. Elemen-

tary waves, whose interactions are determined by perturbed Riemann problems, also reflect the asymptotic behavior of general solutions. Moreover, the Riemann problem and its elementary wave interactions for an isentropic system in magnetogasdynamics and shallow water equations were discussed by Raja Sekhar and Sharma [74–76] in the presence of magnetic field, which make both the shock and the rarefaction stronger as compared to the situation in the absence of a magnetic field. Chun Shen [77] proved that the limiting solutions of the Riemann problem for isentropic magnetogasdynamic equations converge to the corresponding solution of the transport equation in the absence of both pressure and magnetic field. Mathematicians and physicists continue to have a great interest for the generalized Riemann problem which is basically an initial value problem with non-constant initial data with a discontinuity at a point.

It is well known that a large number of physical processes are modeled by systems of quasilinear partial differential equations, but no general methods are available for solving such systems with arbitrary initial or/and boundary conditions [5, 6, 15–20]. In this thesis, we account for the mathematical theory to generalized Riemann problem using the theory of compatibility, which is followed by the basic definition of differential constraints, as explained by Yanenko [30]. In this direction, in 1988, LeFloch and Raviart [56] considered the generalized Riemann problem for nonlinear hyperbolic systems of conservation laws and found the entropy solution in the form of an asymptotic expansion in time to get an explicit method for the construction of the asymptotic expansion, which is explained further through an application to gas dynamic equations by Bourgeade et al. [57]. Using these approximate solutions Ben-Artzi, M. [11] solved the generalized Riemann problem for reactive flows and further, Ben-Artzi, M. and Li, J. [12] derived more precise numerical schemes.

Over the years (see [8]–[29]), a variety of mathematical methods, for example, similarity transformation methods, perturbation methods, etc., leading to the derivation of approximate solutions to quasilinear hyperbolic systems have been proposed. The approach based on the use of differential constraints, proposed by Janenko [30] (see also [31, 32]), has been of considerable interest in recent years (see [33] – [45]). The differential constraints equations play an important role in selecting classes of solutions of the system under interest. Olver and Rosenau [21, 22], Meleshko [26], Kaptsov [27], Levi and Winternitz [28] show that almost all known reduction methods such as partial invariance, separation of variables etc., could be summarized into a most general framework of differential constraints. This general formulation requires that the system under investigation must be appended by additional differential constraints, subject to

compatibility conditions. In 1994, Olver [23] accounted for the relationship between the higher-order direct method of Galaktionov and the method of differential constraints. A new algebraic structure and special classes of determining equations and higher-order differential constraints were introduced by Kaptsov [24]. Within such a theoretical framework, a reduction procedure for the generalized simple wave solution to the generalized Riemann problem with applications to nonlinear transmission lines, constant Astigmatism equation, traffic flow model, ET6 model and ideal chromatography were explored by Curró et al. [39, 41, 54, 55, 59, 60]. Furthermore, the method of differential constraints is also used to determine the exact solution of quasilinear systems depending on various applications like p-systems with relaxation conditions (Curró, and Manganaro [62]), for Chaplygin gas model (Kumar and Radha [61]), for non-homogeneous shallow water equations (Sueet et al. [65]) and for the homogeneous p-system (Manganaro et al. [66]). Recently Meleskho et al. [67] discussed the generalized simple wave solution for magnetic fluid using the differential constraints method and also applied it to systems of equations written in Riemann invariants [68]. Moreover, Chaiyasena et al. [69] discussed generalized Riemann waves and their adjointment through a shock wave.

Recently, Shen and Sun [78] showed that when the pressure tends to zero for isentropic gas dynamic equations, the solution for the Riemann problem, which is composed of two shock waves, tends to a delta shock solution. Whereas the limiting solution for the Riemann problem constituted by two rarefaction waves converges to a solution made up of two contact discontinuities along with the vacuum state encompassed by them when the adiabatic exponent tends to one. Also, they noticed an interesting phenomenon that the internal states in two rarefaction wave fans are transformed gradually into the corresponding vacuum states under this limiting circumstance, which is distinguished from the previously established result that a whole rarefaction wave is concentrated into only one contact discontinuity. Moreover, Guo et al. [79] discussed the limiting behavior of the solution to the Riemann problem to the generalized Chaplygin gas equations with a Coulomb-like friction term along with the formation of a delta shock wave. Also, discussed the formation of delta shock waves and the vacuum states as the pressure vanishes.

In recent decades, researchers have shown a growing interest in exploring the interactions of elementary waves in hyperbolic systems of partial differential equations [9, 80–82, 84, 86, 87], owing to their extensive practical applications such as stability of the Riemann solution with respect to small perturbations of the Riemann initial data [83, 88].

Luo, T. and Yang, T. [89] discussed the interaction of elementary waves for compressible Euler equations with frictional damping. Raja Sekhar and Sharma, [90], delved into the existence of a vacuum state and briefly discussed wave interactions within the realm of isentropic magnetogasdynamics. Liu and Sun [86] carried out an in-depth analysis of elementary wave interactions in ideal magnetogasdynamics using the characteristic analysis methodology.

Based on the present literature review, we were motivated to investigate the generalized Riemann problem. Consequently, an algorithm was developed to determine a class of non-trivial solutions for a given quasilinear hyperbolic system using differential constraints through which the Riemann problem with non-constant initial data is fully characterized with shock waves/characteristic shocks and/or rarefaction waves.

In the subsequent chapters of this thesis, we explore the generalized Riemann problem for a certain class of physical phenomenon modeled by a system of quasilinear hyperbolic partial differential equations and discuss the interaction of elementary waves of the double Riemann problem.

- The second chapter is connected with the Riemann problem for non-constant initial data for rate-type materials, which is described by the following equations

$$\begin{aligned}\frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left(\frac{1}{v} \right) &= 0, \\ \frac{\partial v}{\partial t} + \frac{\partial u}{\partial x} &= 1 - v,\end{aligned}$$

where u is the Lagrangian velocity of a particle and $1/v$ with $v \neq 0$, denotes the stress in the material that is undergoing loading/unloading processes.

In this chapter, a class of solutions to the partial differential equations describing rate-type material are determined. Further, a family of generalized Riemann problems for the system under consideration are solved completely, and the solutions are characterized through shocks and/or rarefaction waves, the results of which are summarized as follows:

The solution to the generalized Riemann problem for this system subject to the

initial data

$$(u(x, 0), v(x, 0)) = \begin{cases} (x + u_L, v_L), & \text{if } x < 0, \\ (x + u_R, v_R), & \text{if } x \geq 0, \end{cases}$$

consists of

1. a 1-shock wave if and only if one of the following conditions hold:
 - (a.) $u_L - u_R + \frac{v_R - v_L}{\sqrt{v_L v_R}} < 0$, when $v_R > v_L$ and $u_L < u_R$.
 - (b.) $u_L - u_R + \log\left(\frac{v_R}{v_L}\right) < 0$, when $v_R < v_L$ and $u_L > u_R$.
 - (c.) $v_R < v_L$ and $u_L < u_R$.
 2. a 2-shock wave if and only if one of the following conditions hold:
 - (a.) $u_L - u_R - \log\left(\frac{v_R}{v_L}\right) < 0$, when $v_R > v_L$ and $u_L > u_R$.
 - (b.) $u_L - u_R - \frac{v_R - v_L}{\sqrt{v_L v_R}} < 0$, when $v_R < v_L$ and $u_L < u_R$.
 - (c.) $v_R > v_L$ and $u_L < u_R$.
 3. a 1-rarefaction wave if and only if one of the following conditions hold:
 - (a.) $u_L - u_R + \frac{v_R - v_L}{\sqrt{v_L v_R}} > 0$, when $v_R > v_L$ and $u_L < u_R$.
 - (b.) $u_L - u_R + \log\left(\frac{v_R}{v_L}\right) > 0$, when $v_R < v_L$ and $u_L > u_R$.
 - (c.) $v_R > v_L$ and $u_L > u_R$.
 4. a 2-rarefaction wave if and only if one of the following conditions hold:
 - (a.) $u_L - u_R - \log\left(\frac{v_R}{v_L}\right) > 0$, when $v_R > v_L$ and $u_L > u_R$.
 - (b.) $u_L - u_R - \frac{v_R - v_L}{\sqrt{v_L v_R}} > 0$, when $v_R < v_L$ and $u_L < u_R$.
 - (c.) $v_R < v_L$ and $u_L > u_R$.
- The third chapter deals with the generalized Riemann problem for the Chaplygin gas equations given by

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2 + p)_x &= 0, \end{aligned}$$

together with the equation of state given by p as a function of ρ alone such that $p'(\rho) > 0$, where ρ and u are, respectively, the density and velocity of the gas. This system with the equation of state

$$p(\rho) = -\frac{1}{\rho}.$$

describes the dark matter and dark energy in the unified form through an exotic background fluid proposed in [47–50] as a mathematical approximation to calculate the lifting force on a wing of an airplane in aerodynamics. Also, it was a prototype of the unified model [51, 52] where dark energy and dark matter were depicted by a single fluid, and Chaplygin cosmology provides an interesting possibility to account for current observations about the expansion of the universe. It was also predicted by them that the cosmological constant would increase (or that it was less in the past), and this could, in principle, be observed.

Further, the solution to the generalized Riemann problem for the Chaplygin gas equations is characterized subject to the initial data

$$(\rho(x, 0), v(x, 0)) = \begin{cases} (\rho_r(x), v_r(x)), & x \geq 0, \\ (\rho_l(x), v_l(x)), & x \leq 0, \end{cases}$$

by one 1- or/and 2- Characteristic shock(s). Here ρ_l, v_l, ρ_r and v_r are arbitrary functions of x such that

$$\begin{aligned} v_l'(x) - \frac{\alpha}{\rho_l^2(x)} \rho_l'(x) &= C_0 \rho_l(x), \\ v_r'(x) - \frac{\beta}{\rho_r^2(x)} \rho_r'(x) &= C_1 \rho_r(x) \end{aligned}$$

where, $\alpha = \pm 1$, $\beta = \pm 1$ and C_0, C_1 are arbitrary constants.

- ❖ Let the solution to the Riemann problem of these equations subject to the given initial data be connected through only a 1-Characteristic shock. Then $v_L - \frac{1}{\rho_L} = v_R - \frac{1}{\rho_R}$ along with one of the following possibilities:
 1. $\alpha = 1$, $\beta = 1$, and C_0, C_1 are arbitrary constants.
 2. $\alpha = 1$, $\beta = -1$, $C_1 = 0$ and C_0 is an arbitrary constant.
 3. $\alpha = -1$, $\beta = 1$, $C_0 = 0$ and C_1 is an arbitrary constant.
 4. $\alpha = -1$, $\beta = -1$, $C_1 = C_0$ and C_0 is an arbitrary constant.
- ❖ Let the solution of the Riemann problem of these equations subject to the given initial data be connected through only a 2-Characteristic shock. Then

$v_L + \frac{1}{\rho_L} = v_R + \frac{1}{\rho_R}$ along with one of the following cases:

1. $\alpha = 1, \beta = 1, C_1 = C_0$ and C_0 is an arbitrary constant.
2. $\alpha = 1, \beta = -1, C_0 = 0$ and C_1 is an arbitrary constant.
3. $\alpha = -1, \beta = 1, C_1 = 0$ and C_0 is an arbitrary constant.
4. $\alpha = -1, \beta = -1$, and C_0, C_1 are arbitrary constants.

❖ Let the solution of the Riemann problem of these equations subject to the given initial data be connected through both 1- and 2- Characteristic shocks. Then one of the following occurs:

1. $\alpha = 1, \beta = -1$ and C_0, C_1 are arbitrary constants.
2. $\alpha = 1, \beta = 1, C_1 = 0$ and C_0 is an arbitrary constant.
3. $\alpha = -1, \beta = -1, C_0 = 0$ and C_1 is an arbitrary constant.
4. $\alpha = -1, \beta = 1$ and $C_1 = C_0$ where C_0 is an arbitrary constant.

- The fourth chapter deals with the generalized Riemann problem for the system of conservation laws describing gas dynamic equations given by

$$\begin{aligned}\rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2 + p)_x &= \rho f,\end{aligned}$$

where ρ , u , and p denote the density, the velocity, and the pressure of gases, respectively, and the equation of state is given by

$$p(\rho) = C\rho^\gamma, \quad C = \text{constant}.$$

After determining the compatibility conditions using differential constraints equations and the governing system, different cases are considered, and solutions are obtained for the Cauchy problem. In fact, solutions for the generalized Riemann problem by generalized rarefaction waves and/or shock waves are obtained. For $\gamma > -1$, we completely characterize the Riemann problem for constant initial data. Also, for $\gamma = 1$, we completely characterize the Riemann problem for non-constant initial data and the main results are stated as follows for the initial data given by

$$(\rho(x, 0), u(x, 0)) = \begin{cases} (\rho_L, u_L), & \text{if } x < 0, \\ (\rho_R, u_R), & \text{if } x \geq 0, \end{cases}$$

where ρ_L, ρ_R, u_L and u_R are constants.

- ❖ Let $u_L > u_R$. If $\rho_L > \rho_R$ (respectively; $\rho_L < \rho_R$), then a 1-shock wave (respectively, a 2-shock wave) is a solution to the Riemann problem for this system subject to the given initial conditions if and only if

$$(u_L - u_R) \geq |\rho_L - \rho_R| \sqrt{\frac{C(\rho_L^\gamma - \rho_R^\gamma)}{(\rho_L - \rho_R)\rho_L\rho_R}}.$$

- ❖ Let $u_L < u_R$. If $\rho_L > \rho_R$ (respectively; $\rho_L < \rho_R$), then a 1-rarefaction wave (respectively, a 2-rarefaction wave) is a solution to the Riemann problem for this system subject to the given initial conditions if and only if

$$(u_L - u_R) \leq |\rho_L - \rho_R| \sqrt{\frac{C(\rho_L^\gamma - \rho_R^\gamma)}{(\rho_L - \rho_R)\rho_L\rho_R}}.$$

- ❖ Let $u_L > u_R$. If $\rho_L > \rho_R$ (respectively; $\rho_L < \rho_R$), then a 1-shock wave (respectively, a 2-shock wave) is a solution to the Riemann problem for this system subject to the given initial conditions if and only if

$$(u_L - u_R) \geq -\frac{2\sqrt{C}\gamma}{\gamma - 1} \left| \rho_L^{\frac{\gamma-1}{2}} - \rho_R^{\frac{\gamma-1}{2}} \right|.$$

- ❖ Let $-\frac{2\sqrt{C}\gamma}{\gamma - 1} \left(\rho_L^{\frac{\gamma-1}{2}} + \rho_R^{\frac{\gamma-1}{2}} \right) < (u_L - u_R) < 0$. If $\rho_L > \rho_R$ (respectively; $\rho_L < \rho_R$), then a 1-rarefaction wave (respectively, a 2-rarefaction wave) is a solution to the Riemann problem for this system subject to the given initial conditions if and only if

$$(u_L - u_R) \leq -\frac{2\sqrt{C}\gamma}{\gamma - 1} \left| \rho_L^{\frac{\gamma-1}{2}} - \rho_R^{\frac{\gamma-1}{2}} \right|.$$

- In Chapter 5, the interaction of elementary waves of the double Riemann problem for the system considered in Chapter 2 is investigated subject to the perturbed initial data with two discontinuities as follows

$$(u(x, 0), v(x, 0)) = \begin{cases} (x + u_L, v_L), & \text{if } x \leq 0, \\ (x + u_M, v_M), & \text{if } 0 < x \leq x_0, \\ (x + u_R, v_R), & \text{if } x_0 < x, \end{cases}$$

where u_L, u_R, u_M, v_L and v_R are given constants but arbitrary.

The solution to the double Riemann problem is ultimately given through only a 1-shock wave and only a 2-rarefaction wave, when $u_L < u_M < u_R$ and the constants v_L, v_M and v_R satisfy the following conditions:

$$\diamond \frac{v_M - v_L}{\sqrt{(v_L v_M)}} + \frac{v_R - v_M}{\sqrt{(v_R v_M)}} + \frac{v_L - v_R}{\sqrt{(v_R v_L)}} > 0, \text{ when } v_R < v_M < v_L.$$

Similarly, the solution to the double Riemann problem is ultimately given through a 1-shock wave and a 2-shock wave when $u_L < u_M < u_R$ and when the constants v_L, v_M and v_R satisfy the following conditions:

$$\diamond \frac{v_L - v_M}{\sqrt{(v_L v_M)}} + \frac{v_R - v_M}{\sqrt{(v_R v_M)}} < \frac{v_L - v_R}{\sqrt{(v_R v_L)}} < \frac{v_M - v_L}{\sqrt{(v_L v_M)}} + \frac{v_M - v_R}{\sqrt{(v_R v_M)}},$$

when $v_R < v_M, v_L < v_M$.

Similarly, the solution to the double Riemann problem is ultimately given through a 1-rarefaction wave and a 2-shock wave when $u_L < u_M < u_R$, and the constants v_L, v_M and v_R satisfy the following conditions:

$$\diamond \frac{v_L - v_M}{\sqrt{(v_L v_M)}} + \frac{v_M - v_R}{\sqrt{(v_M v_R)}} + \frac{v_R - v_L}{\sqrt{(v_R v_L)}} > 0, \text{ when } v_L < v_M < v_R.$$

Today, the generalized Riemann problem remains an active area of research, with researchers continuing to explore new ways to generalize classical Riemann solutions. Moreover, for the development of more accurate and efficient numerical methods for solving a quasi-linear hyperbolic system of partial differential equations, it is essential to study the generalized Riemann problem. As we know, quasi-linear hyperbolic systems come into the picture when we try to formulate mathematically natural phenomena such as aerospace engineering, climate modeling, and computational finance.

Overall, the generalized Riemann problem is a fascinating and important problem in the field of quasi-linear hyperbolic partial differential equations, with many practical applications and theoretical challenges. As computational power and numerical methods continue to advance, the generalized Riemann problem will play an increasingly important role in the study of complex systems in physics, engineering, and beyond.

Chapter 2

Riemann problem for rate-type materials with nonconstant initial conditions

2.1 Introduction

It is well known that a large number of physical processes are modeled by systems of quasilinear partial differential equations, but no general methods are available for solving such systems with arbitrary initial or/and boundary conditions ([5–7]). A variety of mathematical methods for finding exact solutions to such systems have been proposed over the years (see [8]–[29]). The approach based on the use of differential constraints, proposed by Janenko [30] (see also [31] – [32]), has been of considerable interest in recent years (see [33] – [45]). Based on Lie symmetry analysis, an approximate rarefaction wave-type solution to the Riemann problem with non-classical discontinuous initial data for a system of balance laws describing rate-type materials was presented in [29]; here, the initial data for the variable u are discontinuous whereas the initial data for the variable v are constants. A class of solutions to the partial differential equations, describing rate-type material, was obtained in [29] to solve a generalized Riemann problem through a rarefaction wave. In this chapter, an attempt is made to solve a family of generalized Riemann problems for the system under consideration and to completely characterize solutions that connect the initial data to regions either through shocks or rarefaction waves.

2.2 Compatibility conditions for Differential Invariants

In this section, compatibility conditions for differential invariants are derived so that the given system of partial differential equations are solved along a family of characteristic curves. For this, we consider the following hyperbolic system

$$\frac{\partial v_j}{\partial t} + a_{jk} \frac{\partial v_k}{\partial x} = b_j; \quad j, k = 1, 2, \dots, n, \quad (2.2.1)$$

where the matrices (a_{jk}) and (b_j) may be functions of x, t , and the unknowns $v_1, v_2, \dots, v_n$. Let $\lambda^{(i)}$ be the real eigenvalues of (a_{jk}) and $\mathbf{R}^{(i)}$ the corresponding eigenvectors; here and through out this section, summation from 1 to n over a repeated subscript is automatic unless stated otherwise. The system (2.2.1) can be written as

$$\frac{\partial v_j}{\partial t} + \lambda^{(i)} \frac{\partial v_j}{\partial x} + q_j^{(i)} = 0, \quad (2.2.2)$$

where

$$q_j^{(i)} = (a_{jk} - \lambda^{(i)} \delta_{kj}) \frac{\partial v_k}{\partial x} - b_j, \quad (2.2.3)$$

with $\delta_{kj} = 0$ for $k \neq j$ and $\delta_{kj} = 1$ for $k = j$. If $\mathbf{q}^{(i)}$ can be determined as functions of x, t and $v_1, v_2, \dots, v_n$ such that the system (2.2.3) is consistent, then the system (2.2.2) can be solved along the characteristic family $\frac{dx}{dt} = \lambda^{(i)}$.

Since the matrix $(a_{jk} - \lambda^{(i)} \delta_{kj})$ is of rank $n - 1$, the system (2.2.3) is said to be consistent if the corresponding augmented matrix is also of rank $n - 1$. Thus, if the system (2.2.3) is consistent then the derivatives $\frac{\partial v_k}{\partial x}$ can be expressed in the form

$$\frac{\partial v_k}{\partial x} = R_k^{(i)} \frac{\partial v_\alpha}{\partial x} + Q_k^{(i)}, \quad (2.2.4)$$

for some $\alpha \in \{1, 2, \dots, n\}$ with $R_\alpha^{(i)} = 1$, where $R_k^{(i)}$ is the k^{th} component of eigenvector $\mathbf{R}^{(i)}$, v_α is the α^{th} component of $\mathbf{v}$, and $\mathbf{Q}^{(i)}$ satisfy

$$Q_\alpha^{(i)} = 0 \text{ and } q_j^{(i)} = (a_{jk} - \lambda^{(i)} \delta_{kj}) Q_k^{(i)} - b_j. \quad (2.2.5)$$

The equations (2.2.1) and (2.2.3) are said to be compatible if $\frac{\partial}{\partial x} \left(\frac{\partial v_i}{\partial t} \right)$, obtained by differentiating the equation (2.2.1) with respect to x , and $\frac{\partial}{\partial t} \left(\frac{\partial v_i}{\partial x} \right)$, obtained by differentiating the equation (2.2.4) with respect to t , are equal, i.e., $\frac{\partial}{\partial x} \left(\frac{\partial v_i}{\partial t} \right) = \frac{\partial}{\partial t} \left(\frac{\partial v_i}{\partial x} \right)$, which leads to the conditions for determining $\mathbf{Q}^{(i)}$; these conditions are known as compatibility conditions.

In view of (2.2.4), equation (2.2.1) can be written as

$$\frac{\partial v_j}{\partial t} + a_{jk} \left(R_k^{(i)} \frac{\partial v_\alpha}{\partial x} + Q_k^{(i)} \right) = b_j. \quad (2.2.6)$$

Thus, equation (2.2.2) can be solved along a family of characteristics, which in turn gives a class of solutions to the equation (2.2.1). In order to achieve this objective, we differentiate (2.2.6) with respect to x to obtain

$$\begin{aligned} & \frac{\partial^2 v_j}{\partial x \partial t} + \frac{\partial a_{jk}}{\partial v_\ell} \left(R_\ell^{(i)} \frac{\partial v_\alpha}{\partial x} + Q_\ell^{(i)} \right) \left(R_k^{(i)} \frac{\partial v_\alpha}{\partial x} + Q_k^{(i)} \right) \\ & + a_{jk} R_k^{(i)} \frac{\partial^2 v_\alpha}{\partial x^2} + \frac{\partial a_{jk}}{\partial x} \left(R_k^{(i)} \frac{\partial v_\alpha}{\partial x} + Q_k^{(i)} \right) \\ & + a_{jk} \left(\frac{\partial R_k^{(i)}}{\partial x} \frac{\partial v_\alpha}{\partial x} + \frac{\partial Q_k^{(i)}}{\partial x} \right) \\ & + a_{jk} \left(\frac{\partial R_k^{(i)}}{\partial v_\ell} \frac{\partial v_\alpha}{\partial x} + \frac{\partial Q_k^{(i)}}{\partial v_\ell} \right) \left(R_\ell^{(i)} \frac{\partial v_\alpha}{\partial x} + Q_\ell^{(i)} \right) \\ & = \frac{\partial b_j}{\partial x} + \frac{\partial b_j}{\partial v_k} \left(R_k^{(i)} \frac{\partial v_\alpha}{\partial x} + Q_k^{(i)} \right). \end{aligned} \quad (2.2.7)$$

Here and throughout this section, α and i are fixed and these indices are not to be summed. Similarly, differentiating the equation (2.2.4) with respect to t and then substituting $\frac{\partial^2 v_\alpha}{\partial x \partial t}$ from (2.2.7) for $j = \alpha$, we obtain the following system on using (2.2.6):

$$\begin{aligned}
\frac{\partial^2 v_j}{\partial x \partial t} = & \frac{\partial R_j^{(i)}}{\partial t} \frac{\partial v_\alpha}{\partial x} + \left(\frac{\partial R_j^{(i)}}{\partial v_k} \frac{\partial v_\alpha}{\partial x} + \frac{\partial Q_j^{(i)}}{\partial v_k} \right) \left(b_k - a_{k\ell} R_\ell^{(i)} \frac{\partial v_\alpha}{\partial x} - a_{k\ell} Q_\ell^{(i)} \right) \\
& + \frac{\partial Q_j^{(i)}}{\partial t} + R_j^{(i)} \frac{\partial b_\alpha}{\partial x} + R_j^{(i)} \frac{\partial b_\alpha}{\partial v_k} \left(R_k^{(i)} \frac{\partial v_\alpha}{\partial x} + Q_k^{(i)} \right) \\
& - R_j^{(i)} \frac{\partial a_{\alpha k}}{\partial v_\ell} \left(R_\ell^{(i)} \frac{\partial v_\alpha}{\partial x} + Q_\ell^{(i)} \right) \left(R_k^{(i)} \frac{\partial v_\alpha}{\partial x} + Q_k^{(i)} \right) \\
& - R_j^{(i)} \frac{\partial a_{\alpha k}}{\partial x} \left(R_k^{(i)} \frac{\partial v_\alpha}{\partial x} + Q_k^{(i)} \right) - R_j^{(i)} a_{\alpha k} \left(\frac{\partial R_k^{(i)}}{\partial x} \frac{\partial v_\alpha}{\partial x} + \frac{\partial Q_k^{(i)}}{\partial x} \right) \\
& - R_j^{(i)} a_{\alpha k} \left(\frac{\partial R_k^{(i)}}{\partial v_\ell} \frac{\partial v_\alpha}{\partial x} + \frac{\partial Q_k^{(i)}}{\partial v_\ell} \right) \left(R_\ell^{(i)} \frac{\partial v_\alpha}{\partial x} + Q_\ell^{(i)} \right) \\
& - R_j^{(i)} a_{\alpha k} R_k^{(i)} \frac{\partial^2 v_\alpha}{\partial x^2}.
\end{aligned} \tag{2.2.8}$$

Equations (2.2.7) and (2.2.8) imply that

$$T_j^{(i)} + S_j^{(i)} \frac{\partial v_\alpha}{\partial x} + \tilde{T}_j^{(i)} \frac{\partial^2 v_\alpha}{\partial x^2} + \tilde{S}_j^{(i)} \left(\frac{\partial v_\alpha}{\partial x} \right)^2 = 0, \tag{2.2.9}$$

where

$$\begin{aligned}
T_j^{(i)} = & \left(\frac{\partial Q_j^{(i)}}{\partial v_k} \right) \left(b_k - a_{k\ell} Q_\ell^{(i)} \right) + \frac{\partial Q_j^{(i)}}{\partial t} + R_j^{(i)} \frac{\partial b_\alpha}{\partial x} + R_j^{(i)} \frac{\partial b_\alpha}{\partial v_k} Q_k^{(i)} \\
& - R_j^{(i)} \frac{\partial a_{\alpha k}}{\partial v_\ell} Q_\ell^{(i)} Q_k^{(i)} - R_j^{(i)} \frac{\partial a_{\alpha k}}{\partial x} Q_k^{(i)} - R_j^{(i)} a_{\alpha k} \frac{\partial Q_k^{(i)}}{\partial x} \\
& - R_j^{(i)} a_{\alpha k} \frac{\partial Q_k^{(i)}}{\partial v_\ell} Q_\ell^{(i)} + \frac{\partial a_{jk}}{\partial v_\ell} Q_\ell^{(i)} Q_k^{(i)} + \frac{\partial a_{jk}}{\partial x} Q_k^{(i)} + a_{jk} \frac{\partial Q_k^{(i)}}{\partial x} \\
& + a_{jk} \frac{\partial Q_k^{(i)}}{\partial v_\ell} Q_\ell^{(i)} - \frac{\partial b_j}{\partial x} - \frac{\partial b_j}{\partial v_k} Q_k^{(i)}, \\
\tilde{T}_j^{(i)} = & a_{jk} R_k^{(i)} - a_{\alpha k} R_k^{(i)} R_j^{(i)}, \\
\tilde{S}_j^{(i)} = & R_\ell^{(i)} \frac{\partial (a_{jk} R_k^{(i)})}{\partial v_\ell} - R_j^{(i)} R_\ell^{(i)} \frac{\partial (a_{\alpha k} R_k^{(i)})}{\partial v_\ell} - a_{k\ell} R_\ell^{(i)} \frac{\partial R_j^{(i)}}{\partial v_k},
\end{aligned}$$

$$\begin{aligned}
S_j^{(i)} = & \frac{\partial R_j^{(i)}}{\partial t} + \frac{\partial R_j^{(i)}}{\partial v_k} \left(b_k - a_{k\ell} Q_\ell^{(i)} \right) - \frac{\partial Q_j^{(i)}}{\partial v_k} a_{k\ell} R_\ell^{(i)} + R_j^{(i)} \frac{\partial b_\alpha}{\partial v_k} R_k^{(i)} \\
& - R_j^{(i)} \frac{\partial a_{\alpha k}}{\partial v_\ell} \left(R_\ell^{(i)} Q_k^{(i)} + Q_\ell^{(i)} R_k^{(i)} \right) - R_j^{(i)} \frac{\partial a_{\alpha k}}{\partial x} R_k^{(i)} - R_j^{(i)} a_{\alpha k} \frac{\partial R_k^{(i)}}{\partial x} \\
& - R_j^{(i)} a_{\alpha k} \left(\frac{\partial R_k^{(i)}}{\partial v_\ell} Q_\ell^{(i)} + \frac{\partial Q_k^{(i)}}{\partial v_\ell} R_\ell^{(i)} \right) + \frac{\partial a_{jk}}{\partial x} R_k^{(i)} + a_{jk} \frac{\partial R_k^{(i)}}{\partial x} \\
& + \frac{\partial a_{jk}}{\partial v_\ell} \left(R_\ell^{(i)} Q_k^{(i)} + Q_\ell^{(i)} R_k^{(i)} \right) + a_{jk} \left(\frac{\partial R_k^{(i)}}{\partial v_\ell} Q_\ell^{(i)} + \frac{\partial Q_k^{(i)}}{\partial v_\ell} R_\ell^{(i)} \right) \\
& - \frac{\partial b_j}{\partial v_k} R_k^{(i)},
\end{aligned}$$

for $j = 1$ to n for each $i \in \{1, 2, \dots, n\}$. It may be noticed that $a_{\alpha k} R_k^{(i)} = \lambda^{(i)} R_\alpha^{(i)} = \lambda^{(i)}$ and $a_{jk} R_k^{(i)} = \lambda^{(i)} R_j^{(i)}$, which lead to $\tilde{T}_j^{(i)} = 0$ and $\tilde{S}_j^{(i)} = 0$. Thus, the system (2.2.9) becomes

$$T_j^{(i)} + S_j^{(i)} \frac{\partial v_\alpha}{\partial x} = 0, \quad (2.2.10)$$

where $j = 1$ to n for each $i \in \{1, 2, \dots, n\}$. Observe that $T_\alpha^{(i)} \equiv 0$ and $S_\alpha^{(i)} \equiv 0$. In the following section, we use this methodology to a system of conservation laws describing rate-type materials.

2.3 Solutions to the Cauchy problem

We consider the following system of balance laws describing rate-type materials ([30]-[32])

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left(\frac{1}{v} \right) = 0, \quad (2.3.1)$$

$$\frac{\partial v}{\partial t} + \frac{\partial u}{\partial x} = 1 - v, \quad (2.3.2)$$

where u is the Lagrangian velocity and $1/v$ with $v \neq 0$ denotes the stress in the material that is undergoing loading/unloading processes.

The eigenvalues $\lambda^{(i)}$, $i = 1, 2$ represent the characteristic speeds of the system (2.3.1)

and the corresponding right eigenvectors $\mathbf{R}^{(i)}$, are given by

$$\lambda^{(1)} = -\frac{1}{v}, \quad \lambda^{(2)} = \frac{1}{v},$$

$$\mathbf{R}^{(1)} = \begin{bmatrix} -v^{-1} \\ 1 \end{bmatrix}, \quad \mathbf{R}^{(2)} = \begin{bmatrix} v^{-1} \\ 1 \end{bmatrix}.$$

As the system (2.3.1) is strictly hyperbolic and genuinely nonlinear (since $\mathbf{R}^{(i)} \cdot \nabla \lambda^{(i)} \neq 0$ for $i = 1, 2$) for any smooth initial data:

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad (2.3.3)$$

there exists a unique solution of the Cauchy problem (2.3.1), (2.3.3) involving either a rarefaction wave or a shock wave depending on whether $\lambda^{(i)}$ is monotonically increasing or decreasing as (u, v) vary along an integral curve of the vector field $\mathbf{R}^{(i)}$.

In view of (2.2.4), we have $\frac{\partial u}{\partial x} = Q_1^{(1)} - \frac{1}{v} \frac{\partial v}{\partial x}$, and so, equations (2.3.1) can be written as

$$\frac{du}{dt} = -\frac{Q_1^{(1)}}{v}, \quad \frac{dv}{dt} = 1 - v - Q_1^{(1)}, \quad (2.3.4)$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \lambda^{(1)} \frac{\partial}{\partial x}$; and $Q_1^{(1)}$ is a function of x , t , u and v , which is to be determined from the equation (2.2.10), i.e.,

$$T_1^{(1)} + S_1^{(1)} \frac{\partial v}{\partial x} = 0; \quad (2.3.5)$$

here $T_1^{(1)}$ and $S_1^{(1)}$ are given by

$$T_1^{(1)} = \frac{1}{v} \left(\frac{\partial Q_1^{(1)}}{\partial x} + v \frac{\partial Q_1^{(1)}}{\partial t} + Q_1^{(1)} \frac{\partial Q_1^{(1)}}{\partial u} + v \left(1 - v - Q_1^{(1)} \right) \frac{\partial Q_1^{(1)}}{\partial v} \right),$$

$$S_1^{(1)} = \frac{1}{v^2} \left(1 - Q_1^{(1)} - 2 \frac{\partial Q_1^{(1)}}{\partial u} + 2v \frac{\partial Q_1^{(1)}}{\partial v} \right).$$

Similarly, when $\frac{\partial u}{\partial x} = Q_1^{(2)} + \frac{1}{v} \frac{\partial v}{\partial x}$, equations (2.3.1) can be written as

$$\frac{du}{dt} = \frac{Q_1^{(2)}}{v}, \quad \frac{dv}{dt} = 1 - v - Q_1^{(2)}, \quad (2.3.6)$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \lambda^{(2)} \frac{\partial}{\partial x}$ and $Q_1^{(2)}$ is a function of x, t, u and v , which is to be determined from the equation (2.2.10), i.e.,

$$T_1^{(2)} + S_1^{(2)} \frac{\partial v}{\partial x} = 0; \quad (2.3.7)$$

here $T_1^{(2)}$ and $S_1^{(2)}$ are given by

$$\begin{aligned} T_1^{(2)} &= -\frac{1}{v} \left(\frac{\partial Q_1^{(2)}}{\partial x} - v \frac{\partial Q_1^{(2)}}{\partial t} + Q_1^{(2)} \frac{\partial Q_1^{(2)}}{\partial u} - v \left(1 - v - Q_1^{(2)} \right) \frac{\partial Q_1^{(2)}}{\partial v} \right), \\ S_1^{(2)} &= \frac{1}{v^2} \left(-1 + Q_1^{(2)} - 2 \frac{\partial Q_1^{(2)}}{\partial u} - 2v \frac{\partial Q_1^{(2)}}{\partial v} \right). \end{aligned}$$

It may be noticed that (2.3.5) and (2.3.7) admit the cases $T_1^{(i)} = 0$ and $S_1^{(i)} = 0$ for $i = 1, 2$. For the case $i = 1$, the equation $S_1^{(1)} = 0$ implies that

$$Q_1^{(1)} = 1 + e^{-u/2} \phi(x, t, \xi), \quad (2.3.8)$$

where $\xi = u + \log v$ and ϕ is an arbitrary function of x, t and ξ . In view of (2.3.8), the equation $T_1^{(1)} = 0$ implies that

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial \xi} - \frac{1}{2} \phi + e^{(\xi-u)} \left(\frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial \xi} \right) - \frac{1}{2} e^{-u/2} \phi^2 = 0,$$

which leads to $\phi \equiv 0$, i.e., $Q_1^{(1)} \equiv 1$. Similarly, with the assumption that $T_1^{(2)} = S_1^{(2)} = 0$, we have $Q_1^{(2)} \equiv 1$. Thus, on solving (2.3.4) and (2.3.6), equations (2.3.1) admit the following solutions, which recovers the solution obtained in [45] that follows a different line of approach:

$$\begin{aligned}
v(x, t) &= v_0(\xi)e^{-t}, \\
u(x, t) &= u_0(\xi) + \delta \left(\frac{e^t - 1}{v_0(\xi)} \right), \\
x &= \xi + \delta \left(\frac{e^t - 1}{v_0(\xi)} \right), \quad \delta = \pm 1.
\end{aligned} \tag{2.3.9}$$

Here, $\xi(x, t)$ denotes the unique point on the x -axis, which lies on the characteristic through (x, t) and is given by (2.3.9)₃. For $\delta = \mp 1$, the above equations (2.3.9) give two solutions of the system (2.3.1) and (2.3.3), one for each characteristic family; indeed, the above solutions are characterized by the differential constraints:

$$\frac{du_0(x)}{dx} = \frac{\delta}{v_0(x)} \frac{dv_0}{dx} + 1. \tag{2.3.10}$$

Observe that, for a given x and t , the equations (2.3.9)₁ and (2.3.9)₂ admit unique values for v and u provided there exists a unique ξ satisfying (2.3.9)₃; in other words, the existence of a unique solution is guaranteed for every x in $(-\infty, \infty)$ and for every $t > 0$ provided that

$$\delta \left(\frac{e^t - 1}{(v_0(\xi))^2} \right) \frac{dv_0}{d\xi} \neq 1. \tag{2.3.11}$$

2.4 Shocks and rarefaction waves

There are two distinct families of discontinuous solutions of (2.3.1), (2.3.3), referred to as 1-shocks (or back shocks) and 2-shocks (or front shocks). Similarly, there are two families of continuous solutions of (2.3.1), (2.3.3), referred to as rarefaction waves corresponding to either characteristic family $\lambda^{(1)}$ or $\lambda^{(2)}$.

Let $x = X(t)$ be a curve representing a discontinuity across which the flow variables u and v are discontinuous and let $\sigma = \frac{dX}{dt}$ be the speed of propagation of the discontinuity. Then R-H conditions for the system (2.3.1) are

$$\sigma(u_\ell(t) - u_r(t)) = \frac{1}{v_r(t)} - \frac{1}{v_\ell(t)}, \quad \sigma(v_\ell(t) - v_r(t)) = (u_\ell(t) - u_r(t)), \tag{2.4.1}$$

where $u_\ell(t) = \lim_{x \rightarrow X(t)^-} u(x, t)$, $u_r(t) = \lim_{x \rightarrow X(t)^+} u(x, t)$, $v_\ell(t) = \lim_{x \rightarrow X(t)^-} v(x, t)$, and $v_r(t) = \lim_{x \rightarrow X(t)^+} v(x, t)$. Equations (2.4.1) imply that

$$\sigma = \pm \frac{1}{(v_r v_\ell)^{1/2}}, \quad u_\ell = u_r + \sigma (v_\ell - v_r). \quad (2.4.2)$$

If the admitted discontinuity $x = S_1(t)$ is a consequence of the intersection of characteristics belonging to the family $\frac{dx}{dt} = -\frac{1}{v}$, satisfying

$$-\frac{1}{v_\ell} > \sigma > -\frac{1}{v_r}, \quad (2.4.3)$$

then the discontinuity $x = S_1(t)$ is called a 1-shock or a back shock; the inequality (2.4.3) shows that $\sigma < 0$ and therefore, for a 1-shock, we have

$$\sigma = \frac{dS_1}{dt} = -\frac{1}{(v_r v_\ell)^{1/2}}, \quad u_\ell = u_r - \frac{(v_\ell - v_r)}{(v_r v_\ell)^{1/2}}, \quad (2.4.4)$$

with $v_\ell(t) > v_r(t)$ and $u_\ell(t) < u_r(t)$.

Similarly, if the admitted discontinuity $x = S_2(t)$ is a consequence of the intersection of characteristics belonging to the family $\frac{dx}{dt} = \frac{1}{v}$, satisfying

$$\frac{1}{v_\ell} > \sigma > \frac{1}{v_r}, \quad (2.4.5)$$

then the discontinuity $x = S_2(t)$ is called a 2-shock or a front shock satisfying $\sigma > 0$ with

$$\sigma = \frac{dS_2}{dt} = \frac{1}{(v_r v_\ell)^{1/2}}, \quad u_\ell = u_r + \frac{(v_\ell - v_r)}{(v_r v_\ell)^{1/2}}. \quad (2.4.6)$$

with $v_\ell < v_r$ and $u_\ell < u_r$.

We now turn to the rarefaction wave solutions of (2.3.1), (2.3.3) which are continuous solutions corresponding to the eigen modes $\lambda^{(1)}$ and $\lambda^{(2)}$, referred to as 1-rarefaction wave and 2-rarefaction waves, respectively. Let $u_L = \lim_{x \rightarrow 0^-} u_0(x)$, $v_L =$

$\lim_{x \rightarrow 0^-} v_0(x)$, $u_R = \lim_{x \rightarrow 0^+} u_0(x)$, $v_R = \lim_{x \rightarrow 0^+} v_0(x)$ such that the initial step function is expansive with $v_L < v_R$. Let $x = R_1(t)$ and $x = R_2(t)$ be the curves that pass through $(0, 0)$ such that $R_1(t) < R_2(t)$ for all $t > 0$ with $R_1(t)$ and $R_2(t)$ satisfying

$$\begin{aligned} \frac{dR_1}{dt} &= \frac{-1}{v(R_1(t), t)}, \\ \frac{dR_2}{dt} &= \frac{-1}{v(R_2(t), t)}. \end{aligned} \quad (2.4.7)$$

In view of (2.3.9), equations (2.4.7) lead to $R_1(t) = \frac{(1 - e^t)}{v_L}$, $R_2(t) = \frac{(1 - e^t)}{v_R}$. Since, $v_L < v_R$, we have $v(R_1(t), t) < v(R_2(t), t)$; a continuously varying solution in the region $R_1(t) < x < R_2(t)$, which is continuous across the curves $x = R_1(t)$ and $x = R_2(t)$, referred to as 1-rarefaction wave, can be obtained from (2.3.9) as follows. Since all the values of u (respectively, v) between u_L and u_R (respectively, v_L and v_R) are taken on characteristics in a fan through origin, where $\xi = 0$, the solution in the fan, bounded by the characteristics $x = R_1(t) = \frac{(1 - e^t)}{v_L}$ and $x = R_2(t) = \frac{(1 - e^t)}{v_R}$ is given by

$$\begin{aligned} v(x, t) &= ze^{-t}, \quad \text{if } R_1(t) < x < R_2(t), \\ u(x, t) &= \zeta - \left(\frac{e^t - 1}{z} \right), \quad \text{if } R_1(t) < x < R_2(t), \\ x &= - \left(\frac{e^t - 1}{z} \right), \end{aligned} \quad (2.4.8)$$

where $R_1(t) < x < R_2(t)$, $v_L < z < v_R$, and $u_L < \zeta < u_R$. Here, the characteristics are emanating from the origin and given by $\frac{dx}{dt} = -\frac{1}{v} = -\frac{e^t}{z}$ whose speeds are varying from $\frac{-1}{v_L}$ to $\frac{-1}{v_R}$. Differentiating the equations (2.4.8) with respect to x and t and substituting in (2.3.1) we get

$$\begin{aligned} \frac{\partial \zeta}{\partial t} + \frac{e^t}{e^t - 1} &= 0, \\ \frac{\partial \zeta}{\partial x} - \frac{1}{x} &= 0. \end{aligned} \quad (2.4.9)$$

The system (2.4.9), subject to the condition $\zeta = u_L$ when $x = R_1(t)$, yields the unique solution given by $\zeta = u_L + \log \left(\frac{xv_L}{1 - e^t} \right)$. Thus, the solution for 1-rarefaction wave is given by

$$v(x, t) = \begin{cases} v_0(\xi)e^{-t}, & \text{if } x \leq R_1(t), \\ \left(\frac{e^{-t} - 1}{x}\right), & \text{if } R_1(t) < x < R_2(t), \\ v_0(\xi)e^{-t}, & \text{if } x \geq R_2(t), \end{cases} \quad (2.4.10)$$

$$u(x, t) = \begin{cases} u_0(\xi) - \left(\frac{e^t - 1}{v_0(\xi)}\right), & \text{if } x \leq R_1(t), \\ u_L + \log\left(\frac{xv_L}{1 - e^t}\right) + x, & \text{if } R_1(t) < x < R_2(t), \\ u_0(\xi) - \left(\frac{e^t - 1}{v_0(\xi)}\right), & \text{if } x \geq R_2(t), \end{cases} \quad (2.4.11)$$

$$x = \xi - \left(\frac{e^t - 1}{v_0(\xi)}\right), \quad \frac{du_0}{dx} = 1 - \frac{1}{v_0} \frac{dv_0}{dx},$$

with $u_R = u_L + \log\left(\frac{v_L}{v_R}\right)$, $R_1(t) = \frac{(1 - e^t)}{v_L}$, $R_2(t) = \frac{(1 - e^t)}{v_R}$ and $v(R_1(t), t) < v(R_2(t), t)$.

Similarly, let $x = R_3(t)$ and $x = R_4(t)$ be the curves that pass through $(0, 0)$ such that $R_3(t) < R_4(t)$ for all $t > 0$ then

$$\begin{aligned} \frac{dR_3}{dt} &= \frac{1}{v(R_3(t), t)}, \\ \frac{dR_4}{dt} &= \frac{1}{v(R_4(t), t)}, \end{aligned} \quad (2.4.12)$$

which implies that $v(R_3(t), t) > v(R_4(t), t)$ since $v_L > v_R$ and $R_3(t) < R_4(t)$ for all $t > 0$. A continuously varying solution in the region $R_3(t) < x < R_4(t)$, which is continuous across the curves $x = R_3(t)$ and $x = R_4(t)$, referred to as a 2-rarefaction wave, can be obtained in a similar manner, and is given by

$$v(x, t) = \begin{cases} v_0(\xi)e^{-t}, & \text{if } x \leq R_3(t), \\ \left(\frac{1-e^{-t}}{x}\right), & \text{if } R_3(t) < x < R_4(t), \\ v_0(\xi)e^{-t}, & \text{if } x \geq R_4(t), \end{cases} \quad (2.4.13)$$

$$u(x, t) = \begin{cases} u_0(\xi) + \left(\frac{e^t - 1}{v_0(\xi)}\right), & \text{if } x \leq R_3(t), \\ u_R - \log\left(\frac{xv_R}{1-e^t}\right) + x, & \text{if } R_3(t) < x < R_4(t), \\ u_0(\xi) + \left(\frac{e^t - 1}{v_0(\xi)}\right), & \text{if } x \geq R_4(t), \end{cases} \quad (2.4.14)$$

$$x = \xi + \left(\frac{e^t - 1}{v_0(\xi)}\right), \quad \frac{du_0}{dx} = 1 + \frac{1}{v_0} \frac{dv_0}{dx}$$

with $u_L = u_R - \log\left(\frac{v_R}{v_L}\right)$, $R_3(t) = \frac{(e^t - 1)}{v_L}$, $R_4(t) = \frac{(e^t - 1)}{v_R}$ and $v(R_3(t), t) > v(R_4(t), t)$.

The above results can be summarized as:

- Across a 1-shock wave, we have $v_\ell(t) > v_r(t)$ and $u_\ell(t) < u_r(t)$, where $(u_\ell(t), v_\ell(t))$ and $(u_r(t), v_r(t))$ are the limiting values of (u, v) as the discontinuity $x = S_1(t)$ is approached from left and right, respectively.
- Across a 2-shock wave, we have $v_\ell(t) < v_r(t)$ and $u_\ell(t) < u_r(t)$, where $(u_\ell(t), v_\ell(t))$ and $(u_r(t), v_r(t))$ are the limiting values of (u, v) as the discontinuity $x = S_2(t)$ is approached from left and right, respectively.
- Across a 1-rarefaction wave, we have $v_\ell(t) < v_r(t)$ and $u_\ell(t) > u_r(t)$, where $v_\ell(t) = v(R_1(t), t)$, $u_\ell(t) = u(R_1(t), t)$, $v_r(t) = v(R_2(t), t)$ and $u_r(t) = u(R_2(t), t)$.
- Across a 2-rarefaction wave, we have $v_\ell(t) > v_r(t)$ and $u_\ell(t) > u_r(t)$, where $v_\ell(t) = v(R_3(t), t)$, $u_\ell(t) = u(R_3(t), t)$, $v_r(t) = v(R_4(t), t)$ and $u_r(t) = u(R_4(t), t)$.

Based on solutions (2.4.10), (2.4.11), (2.4.13) and (2.4.14), we solve a Riemann problem with non-constant and smooth initial data, in the next section.

2.5 Riemann problem with non-constant initial state

Consider the initial profile

$$(u(x, 0), v(x, 0)) = (u_0(x), v_0(x)) = \begin{cases} (x + u_L, v_L), & \text{if } x < 0, \\ (x + u_R, v_R), & \text{if } x \geq 0, \end{cases} \quad (2.5.1)$$

where u_L , u_R , v_L and v_R are constants.

If 1-wave is a shock wave then

$$v(x, t) = \begin{cases} v_L e^{-t}, & \text{if } x \leq S_1(t), \\ \tilde{v} e^{-t}, & \text{if } x > S_1(t), \end{cases} \quad (2.5.2)$$

$$u(x, t) = \begin{cases} u_L + x, & \text{if } x \leq S_1(t), \\ \tilde{u} + x, & \text{if } x > S_1(t), \end{cases} \quad (2.5.3)$$

where $\frac{dS_1}{dt} = -\frac{e^t}{\sqrt{(v_L \tilde{v})}}$, which yields on integration that $S_1(t) = \frac{(1 - e^t)}{\sqrt{(v_L \tilde{v})}}$. In view of (2.4.4)₂ we have $\tilde{u} = u_L - \frac{\tilde{v} - v_L}{\sqrt{(v_L \tilde{v})}}$, $\tilde{v} < v_L$ and $\tilde{u} > u_L$.

Similarly, if 2-wave is a shock wave then

$$v(x, t) = \begin{cases} v_R e^{-t}, & \text{if } x \geq S_2(t), \\ \hat{v} e^{-t}, & \text{if } x < S_2(t), \end{cases} \quad (2.5.4)$$

$$u(x, t) = \begin{cases} u_R + x, & \text{if } x \geq S_2(t), \\ \hat{u} + x, & \text{if } x < S_2(t), \end{cases} \quad (2.5.5)$$

where $\frac{dS_2}{dt} = \frac{e^t}{\sqrt{(v_R \hat{v})}}$, which yields on integration that $S_2(t) = \frac{(e^t - 1)}{\sqrt{v_R \hat{v}}}$. In view of (2.4.6)₂ we have $\hat{u} = u_R + \frac{\hat{v} - v_R}{\sqrt{v_R \hat{v}}}$ with $\hat{v} < v_R$ and $\hat{u} > u_R$.

If 1-wave is a rarefaction wave then

$$v(x, t) = \begin{cases} v_L e^{-t}, & \text{if } x \leq R_1(t), \\ \left(\frac{e^{-t} - 1}{x} \right), & \text{if } R_1(t) < x < R_2(t), \\ \tilde{v} e^{-t}, & \text{if } x \geq R_2(t), \end{cases} \quad (2.5.6)$$

$$u(x, t) = \begin{cases} u_L + x, & \text{if } x \leq R_1(t), \\ u_L + \log \left(\frac{x v_L}{1 - e^t} \right) + x, & \text{if } R_1(t) < x < R_2(t), \\ \tilde{u} + x, & \text{if } x \geq R_2(t), \end{cases} \quad (2.5.7)$$

where $\tilde{u} = u_L + \log \left(\frac{v_L}{\tilde{v}} \right)$, $R_1(t) = \frac{(1 - e^t)}{v_L}$, $R_2(t) = \frac{(1 - e^t)}{\tilde{v}}$ and $\tilde{v} > v_L$.

Similarly, if 2-wave is a rarefaction wave then

$$v(x, t) = \begin{cases} \hat{v} e^{-t}, & \text{if } x \leq R_3(t), \\ \left(\frac{1 - e^{-t}}{x} \right), & \text{if } R_3(t) < x \leq R_4(t), \\ v_R e^{-t}, & \text{if } x \geq R_4(t), \end{cases} \quad (2.5.8)$$

$$u(x, t) = \begin{cases} \hat{u} + x, & \text{if } x \leq R_3(t), \\ u_R - \log \left(\frac{x v_R}{e^t - 1} \right) + x, & \text{if } R_3(t) < x < R_4(t), \\ u_R + x, & \text{if } x \geq R_4(t), \end{cases} \quad (2.5.9)$$

where $\hat{u} = u_R - \log \left(\frac{v_R}{\hat{v}} \right)$, $R_3(t) = \frac{(e^t - 1)}{\hat{v}}$, $R_4(t) = \frac{(e^t - 1)}{v_R}$ and $\hat{v} > v_R$. Here, $\tilde{v}$ and $\hat{v}$ are arbitrary constants.

Let A and C be the quantities defined by

$$A = u_L - u_R, \quad C = \log(v_R/v_L). \quad (2.5.10)$$

Then, to continue our development, it is useful to state the following Lemmas:

Lemma 2.5.1. *Let A and C be defined as in (2.5.10). If the solution to the Riemann problem for the system (2.3.1), with initial conditions (2.5.1), consists of 1-shock wave and 2-shock wave then $A + 2 \sinh(|C|/2) < 0$ and $A < 0$.*

Proof. Given that 1-wave is a shock wave, $x = S_1(t)$, implies that $\tilde{u} > u_L$ and $\tilde{v} < v_L$; similarly, if 2-wave is a shock wave, $x = S_2(t)$, then $\hat{u} < u_R$ and $\hat{v} < v_R$. In the region,

$S_1(t) < x < S_2(t)$ the solution given in the equations (2.5.2) and (2.5.3), through 1-shock, and the solutions (2.5.4) and (2.5.5), through 2-shock, should coincide; that is $\tilde{v} = \hat{v} = z(\text{say})$, $\tilde{u} = \hat{u}$, i.e., $A = u_L - u_R < 0$ and $f_1(z) = 0$ where

$$f_1(z) = u_L - u_R - \frac{z - v_L}{(v_L z)^{(1/2)}} - \frac{z - v_R}{(v_R z)^{(1/2)}},$$

for $0 < z < \min\{v_L, v_R\}$. Observe that $\lim_{z \rightarrow 0} f_1(z) = \infty$ and

$$\frac{df_1}{dz} = -\frac{z + v_L}{2z\sqrt{(v_L z)}} - \frac{z + v_R}{2z\sqrt{(v_R z)}} < 0,$$

implying thereby that f_1 is decreasing.

- Let $v_L < v_R$, i.e., $C > 0$. Since $0 < z < \min\{v_L, v_R\}$, i.e., $0 < z < v_L$, $f_1 = 0$ has a solution if $f_1(v_L) < 0$, where

$$f_1(v_L) = u_L - u_R - \frac{v_L - v_R}{(v_R v_L)^{(1/2)}} = A + 2 \sinh(C/2). \quad (2.5.11)$$

- Similarly, let $v_R < v_L$, i.e., $C < 0$ then $f_1 = 0$ has a solution if $f_1(v_R) < 0$, where

$$f_1(v_R) = u_L - u_R - \frac{v_R - v_L}{(v_R v_L)^{(1/2)}} = A - 2 \sinh(C/2). \quad (2.5.12)$$

Thus, in view of (2.5.11) and (2.5.12), if $A < 0$ then the solution exists for $f_1(z) = 0$ only when $A + 2 \sinh(|C|/2) < 0$.

This situation is depicted in Figure 2.1. Further, when $v_R = v_L$, i.e., $C = 0$, it follows from $f_1(z) = 0$ that $z = v_L \left(\sqrt{\frac{(u_R - u_L)^2}{4} + 1} - \frac{u_R - u_L}{4} \right)^2$ which recovers the result obtained in [29] for $v_L = v_R = v_0$, $u_R = u_r$, $u_L = u_\ell$ and $z = v_m$. $\square$

Lemma 2.5.2. *Let A and C be defined as in (2.5.10). If the solution to the Riemann problem for the system (2.3.1), with initial conditions (2.5.1), consists of 1-shock wave and 2-rarefaction wave, then either of the following inequalities holds*

- (i.) $A < 0$, $C < 0$ and $A - 2 \sinh(C/2) > 0$.

(ii.) $A > 0$, $C < 0$ and $A + C < 0$.

Proof. Let 1-wave be a shock wave and 2-wave be a rarefaction wave. This implies that $\tilde{u} > u_L$, $\tilde{v} < v_L$ and $\hat{v} > v_R$. In view of (2.5.2), (2.5.3), (2.5.8) and (2.5.9), it follows that the solutions given by (2.5.2)-(2.5.3) and (2.5.8)-(2.5.9) should coincide in the region $S_1(t) < x < R_3(t)$; this means that $\tilde{v} = \hat{v} = z$ (say), i.e., $v_R < z < v_L$ and $f_2(z) = 0$ where

$$f_2(z) = u_L - u_R - \frac{z - v_L}{\sqrt{(v_L z)}} + \log\left(\frac{v_R}{z}\right),$$

for $v_R < z < v_L$. Observe from the equation

$$\frac{df_2}{dz} = -\frac{z + v_L}{2z\sqrt{(v_L z)}} - \frac{1}{z},$$

that f_2 is decreasing. Since, $v_R < v_L$, i.e., $C < 0$, the equation $f_2 = 0$ has a solution only when $f_2(v_R) > 0$ and $f_2(v_L) < 0$, i.e.,

$$f_2(v_L) < 0 \Rightarrow u_L - u_R + \log\left(\frac{v_R}{v_L}\right) < 0, \quad (2.5.13)$$

$$f_2(v_R) > 0 \Rightarrow u_L - u_R - \frac{v_R - v_L}{\sqrt{(v_L v_R)}} > 0. \quad (2.5.14)$$

Since $v_R < v_L$, and if $u_L > u_R$ then (2.5.14) always holds. Thus, a solution for $f_2(z) = 0$ over $[v_R, v_L]$ is possible if (2.5.13) holds, i.e.,

$$A + C = u_L - u_R + \log\left(\frac{v_R}{v_L}\right) < 0.$$

Hence, $A > 0$, $C < 0$ and $A + C < 0$.

Further, since $v_R < v_L$, and if $u_L < u_R$ then the equation (2.5.13) always holds. Thus, the number of solutions for $f_2(z) = 0$ over $[v_R, v_L]$ is possible only if (2.5.14) holds, i.e.,

$$u_L - u_R - \frac{v_R - v_L}{\sqrt{(v_L v_R)}} = A - 2 \sin(C/2) > 0.$$

Hence, $A < 0$, $C < 0$ and $A - 2 \sinh(C/2) > 0$. This situation is depicted in Figure 2.2. $\square$

Lemma 2.5.3. *Let A and C be defined as in (2.5.10). If the solution to the Riemann problem for the system (2.3.1), with initial conditions (2.5.1), consists of 1-rarefaction wave and 2-shock wave then either of the following inequalities holds*

$$(i.) \quad A < 0, \quad C > 0 \text{ and } A + 2 \sinh(C/2) > 0.$$

$$(ii.) \quad A > 0, \quad C > 0 \text{ and } A - C < 0.$$

Proof. Given that 1-wave is a rarefaction wave implies that $\tilde{v} > v_L$; similarly if the 2-wave is a shock wave then $\hat{u} < u_R$ and $\hat{v} < v_R$. In view of the equations (2.5.4)-(2.5.7), the solutions given by the equations (2.5.4)-(2.5.5) and (2.5.6)-(2.5.7) should coincide in the region $R_2(t) < x < S_2(t)$, i.e., $\tilde{v} = \hat{v} = z$ (say), $\hat{u} = \tilde{u}$, i.e., $v_L < v_R$, $C < 0$ and $f_3(z) = 0$ where

$$f_3(z) = u_L - u_R + \log\left(\frac{v_L}{z}\right) - \frac{z - v_R}{\sqrt{(v_R z)}},$$

for $v_L < z < v_R$. Observe from the above equation that

$$\frac{df_3}{dz} = -\frac{z + v_R}{2z\sqrt{(v_R z)}} - \frac{1}{z},$$

implying thereby that f_3 is decreasing. Since $C < 0$, the equation $f_3 = 0$ has a solution only when $f_3(v_L) > 0$ and $f_3(v_R) < 0$, i.e.,

$$f_3(v_L) > 0 \Rightarrow u_L - u_R - \frac{v_L - v_R}{\sqrt{(v_L v_R)}} > 0, \quad (2.5.15)$$

$$f_3(v_R) < 0 \Rightarrow u_L - u_R + \log\left(\frac{v_L}{v_R}\right) < 0, \quad (2.5.16)$$

If $u_L > u_R$ and $v_L < v_R$ then $A > 0$, $C < 0$ and the equation (2.5.15) is always true. Thus, the solution for $f_3(z) = 0$ over $[v_L, v_R]$ exists only when the equation (2.5.16) is true, i.e.,

$$u_L - u_R + \log\left(\frac{v_L}{v_R}\right) = A - C < 0. \quad (2.5.17)$$

Thus, $A > 0$, $C > 0$ and $A - C < 0$.

Further, if $v_L < v_R$ and $u_L < u_R$ then $C < 0$, $A > 0$ and the equation (2.5.16) is always true. Thus, the solution for $f_3(z) = 0$ over $[v_L, v_R]$ exists only when the equation (2.5.15) is true, i.e.,

$$u_L - u_R - \frac{v_L - v_R}{\sqrt{(v_L v_R)}} = A + 2 \sinh(C/2) > 0, \quad (2.5.18)$$

implying thereby that $A < 0$, $C > 0$ and $A + 2 \sinh(C/2) > 0$.

This situation is depicted in Figure 2.3. □

Lemma 2.5.4. *Let A and C be defined as in (2.5.10). If the solution to the Riemann problem for the system (2.3.1), with initial conditions (2.5.1), consists of 1-rarefaction wave and 2-rarefaction wave then $A > 0$ and $A - |C| > 0$.*

Proof. Let 1-wave and 2-wave be both rarefaction waves. In view of (2.5.6)-(2.5.9), the solutions given in the equations (2.5.6)-(2.5.7) and (2.5.8)-(2.5.9) should coincide in the region, $R_2(t) < x < R_3(t)$, i.e., $\tilde{v} = \hat{v} = z$ (say) and $f_4(z) = 0$ where

$$f_4(z) = u_L - u_R + \log\left(\frac{v_L}{z}\right) + \log\left(\frac{v_R}{z}\right),$$

with $\max\{v_L, v_R\} < z < \infty$. Observe from the above equation that

$$\frac{df_4}{dz} = -\frac{2}{z},$$

showing thereby f_4 is decreasing. Observe that $\lim_{z \rightarrow \infty} f_4(z) = -\infty$.

If $v_L < v_R$, $C > 0$, then $f_4 = 0$ has a solution provided $f_4(v_R) > 0$, i.e.,

$$u_L - u_R + \log\left(\frac{v_L}{v_R}\right) = A - C > 0. \quad (2.5.19)$$

Since $C > 0$ and $A - C > 0$ we have $A > 0$.

Similarly, if $v_R < v_L$ then $f_4 = 0$ has a solution if $f_4(v_L) > 0$, i.e.,

$$u_L - u_R + \log\left(\frac{v_R}{v_L}\right) = A + C > 0. \quad (2.5.20)$$

Since $C < 0$ and $A + C > 0$ we have $A > 0$.

Also observe that when $v_L > v_R$ (respectively, $v_R > v_L$) and $u_R > u_L$ then equation (2.5.19) (respectively, equation (2.5.20)) does not hold. This situation is depicted in Figure 2.4.

Further, when $v_R = v_L$, i.e., $C = 0$ from the equation $f_4(z) = 0$ yields $z = v_L e^{\frac{u_L - u_R}{2}}$ which on replacing $v_L = v_R = v_0$, $u_R = u_r$, $u_L = u_\ell$ and $z = V_m$ recovers the result in the equation (37) of Ref. [29]. $\square$

Lemma 2.5.5. *Let A and C be defined as in (2.5.10). If the solution to the Riemann problem for the system (2.3.1), with initial conditions (2.5.1), consists of only 1-rarefaction wave (respectively, 2-rarefaction wave) then $A > 0$, $C > 0$ and $A - C = 0$ (respectively, $A > 0$, $C < 0$ and $A + C = 0$).*

Proof. Let the solution be given through 1-wave as a rarefaction wave only, then in view of (2.5.6) – (2.5.7), we have $\tilde{v} > v_L$, $\tilde{v} = v_R$ and $\tilde{u} = u_R$, i.e., $u_L + \log\left(\frac{v_L}{v_R}\right) = u_R$, which implies that $A - C = 0$. Similarly, when solution is given through 2-rarefaction wave, it can be easily shown that $A + C = 0$. $\square$

Lemma 2.5.6. *Let A and C be defined as in (2.5.10). If the solution to the Riemann problem for the system (2.3.1), with initial conditions (2.5.1), consists of only 1-shock wave (respectively, 2-shock wave) then $A < 0$, $C < 0$ and $A - 2 \sinh(C/2) = 0$ (respectively, $A < 0$, $C > 0$ and $A + 2 \sinh(C/2) = 0$).*

Proof. Let solution be given through 1-wave as a shock wave only, then in view of (2.5.2) – (2.5.3), we have $\tilde{v} < v_L$, $\tilde{v} = v_R$ and $\tilde{u} = u_R$, i.e., $u_L - \frac{v_R - v_L}{\sqrt{v_L v_R}} = u_R$ which implies $A - 2 \sinh(C/2) = 0$. Similarly, when solution is given through 2-wave as a shock wave, it can be proved that $A + C = 0$. $\square$

We next give the following two theorems, which in fact, complete our discussion relating to the complete characterization of the solution of the Riemann problem under consideration.

Theorem 2.5.1. *Let A and C be defined as in (2.5.10). Consider the solution to the Riemann problem for the system (2.3.1), with initial conditions (2.5.1). Then 1-rarefaction wave (respectively, 1-shock wave) is a solution to the Riemann problem if and only if $A + \max(2 \sinh(C/2), C) > 0$ (respectively, $A + \max(2 \sinh(C/2), C) < 0$).*

Proof. Observe that if $C > 0$ (respectively; $C < 0$) then $\min(C, 2 \sinh(C/2)) = C$ (respectively; $\min(C, 2 \sinh(C/2)) = 2 \sinh(C/2)$).

Let the 1-wave be a rarefaction wave. Then, from Lemmas 2.5.3, 2.5.4 and 2.5.5, we have

1. $A < 0, C > 0, A + 2 \sinh(C/2) > 0 \Rightarrow A + \max(C, 2 \sinh(C/2)) > 0.$
2. $A > 0, C < 0, A + C > 0 \Rightarrow A + \max(C, 2 \sinh(C/2)) > 0.$
3. $A > 0, C > 0 \Rightarrow A + C > 0$ and $A + 2 \sinh(C/2) > 0.$

Thus, if the 1-wave is a rarefaction wave then $A + \max(C, 2 \sinh(C/2)) > 0.$

Let the 1-wave be a shock wave then from Lemmas 2.5.1, 2.5.2 and 2.5.6 we have

1. $A < 0, C > 0, A + 2 \sinh(C/2) < 0 \Rightarrow A + \max(C, 2 \sinh(C/2)) < 0.$
2. $A > 0, C < 0, A + C < 0 \Rightarrow A + \max(C, 2 \sinh(C/2)) < 0.$
3. $A < 0, C < 0 \Rightarrow A + C < 0$ and $A + \max(C, 2 \sinh(C/2)) < 0.$

Thus, if the 1-wave is a shock wave then $A + \max(C, 2 \sinh(C/2)) < 0.$

To prove the converse, let $A + \max(2 \sinh(C/2), C) > 0$, then we have one of the following possibilities

- $A > 0, C > 0.$
- $A > 0, C < 0, A + C > 0.$
- $A < 0, C > 0, A + 2 \sinh(C/2) > 0,$

which lead us to conclude that the 1-wave cannot be a shock wave as the above inequalities are contradicting the consequences of lemmas 2.5.1, 2.5.2 and 2.5.6. Hence, the 1-wave is a rarefaction wave.

Now, let $A + \max(2 \sinh(C/2), C) < 0$, then we have one of the following possibilities

- $A < 0, C < 0.$
- $A > 0, C < 0, A + C < 0.$
- $A < 0, C > 0, A + 2 \sinh(C/2) < 0,$

which imply that the 1-wave is not a rarefaction wave as the above inequalities are contradicting the consequences of lemmas 2.5.3, 2.5.4 and 2.5.5. Hence, the 1-wave is a shock wave. $\square$

Theorem 2.5.2. *Let A and C be defined as in (2.5.10). Consider the solution to the Riemann problem for the system (2.3.1), with initial conditions (2.5.1). Then, 2-rarefaction wave (respectively, 2-shock wave) is a solution to the Riemann problem if and only if $A - \min(2 \sinh(C/2), C) > 0$ (respectively, $A - \min(2 \sinh(C/2), C) < 0$).*

Proof. Observe that if $C > 0$ (respectively; $C < 0$) then $\min(C, 2 \sinh(C/2)) = C$ (respectively; $\min(C, 2 \sinh(C/2)) = 2 \sinh(C/2)$).

Let 2-wave be a rarefaction wave then from Lemmas 2.5.2, 2.5.4 and 2.5.5 we have

1. $A > 0, C < 0 \Rightarrow A - C > 0$ and $A - 2 \sinh(C/2) > 0$.
2. $A > 0, C > 0, A - C > 0 \Rightarrow A - \min(C, 2 \sinh(C/2)) > 0$.
3. $A < 0, C < 0, A - 2 \sinh(C/2) > 0 \Rightarrow A - \min(C, 2 \sinh(C/2)) > 0$.

Thus, if 2-wave is a rarefaction wave then $A - \min(C, 2 \sinh(C/2)) > 0$. However, if the 2-wave is a shock wave then from Lemmas 2.5.1, 2.5.3 and 2.5.6 we have

1. $A < 0, C > 0 \Rightarrow A - C < 0$ and $A - 2 \sinh(C/2) < 0$.
2. $A > 0, C > 0, A - C < 0 \Rightarrow A - \min(C, 2 \sinh(C/2)) < 0$.
3. $A < 0, C < 0, A - 2 \sinh(C/2) < 0 \Rightarrow A - \min(C, 2 \sinh(C/2)) < 0$.

Thus, if 2-wave is a shock wave then $A - \min(C, 2 \sinh(C/2)) < 0$.

To prove the converse, let $A - \min(2 \sinh(C/2), C) > 0$, then we have one of the following possibilities

- $A > 0, C < 0$.
- $A > 0, C > 0, A - C > 0$.
- $A < 0, C < 0, A - 2 \sinh(C/2) > 0$.

Assume that 2-wave is a shock wave, then the above inequalities are contradicting the consequences of lemmas 2.5.1, 2.5.3 and 2.5.6. Hence, the 2-wave is a rarefaction wave.

Now, let $A + \max(2 \sinh(C/2), C) < 0$; then we have one of the following possibilities

- $A < 0, C > 0$.

- $A > 0$, $C < 0$, $A - C < 0$.
- $A < 0$, $C < 0$, $A - 2 \sinh(C/2) < 0$.

Assume that 2-wave is a rarefaction wave, then the above possibilities are contradicting lemmas [2.5.2](#), [2.5.4](#) and [2.5.5](#). Hence, the 2-wave is a shock wave. $\square$

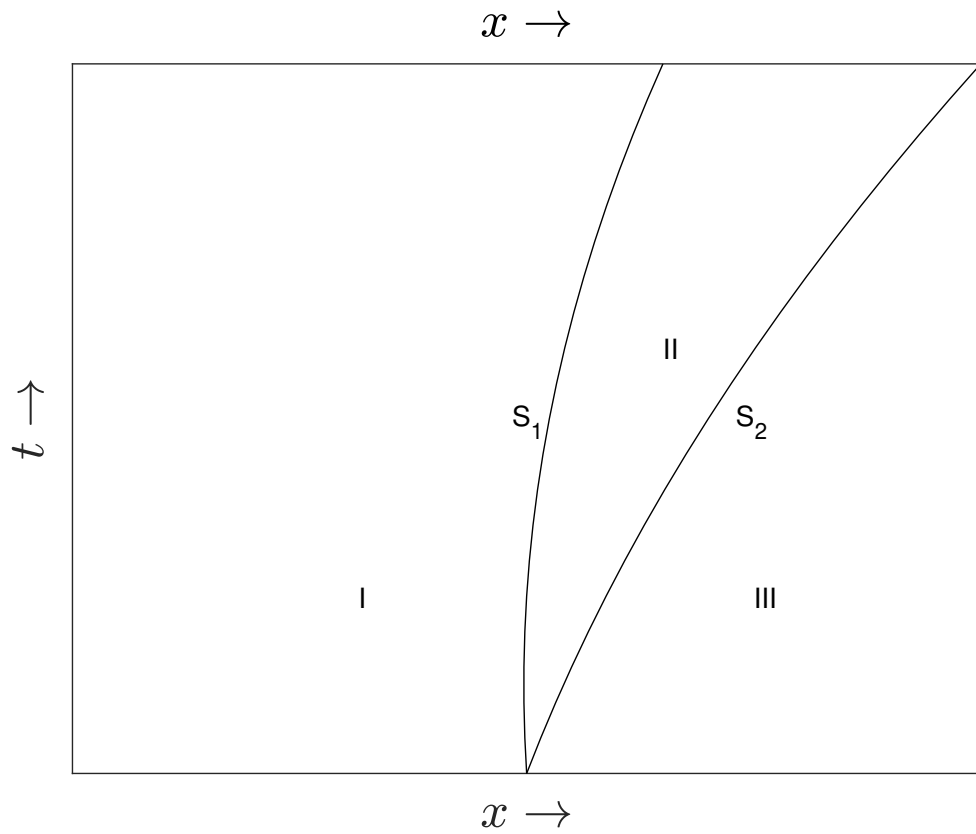


Figure 2.1: S_1 and S_2 are, respectively, the back-shock and the front shock; regions $x < S_1(t)$, $S_1(t) < x < S_2(t)$, and $x > S_2(t)$ are depicted as I, II and III respectively.

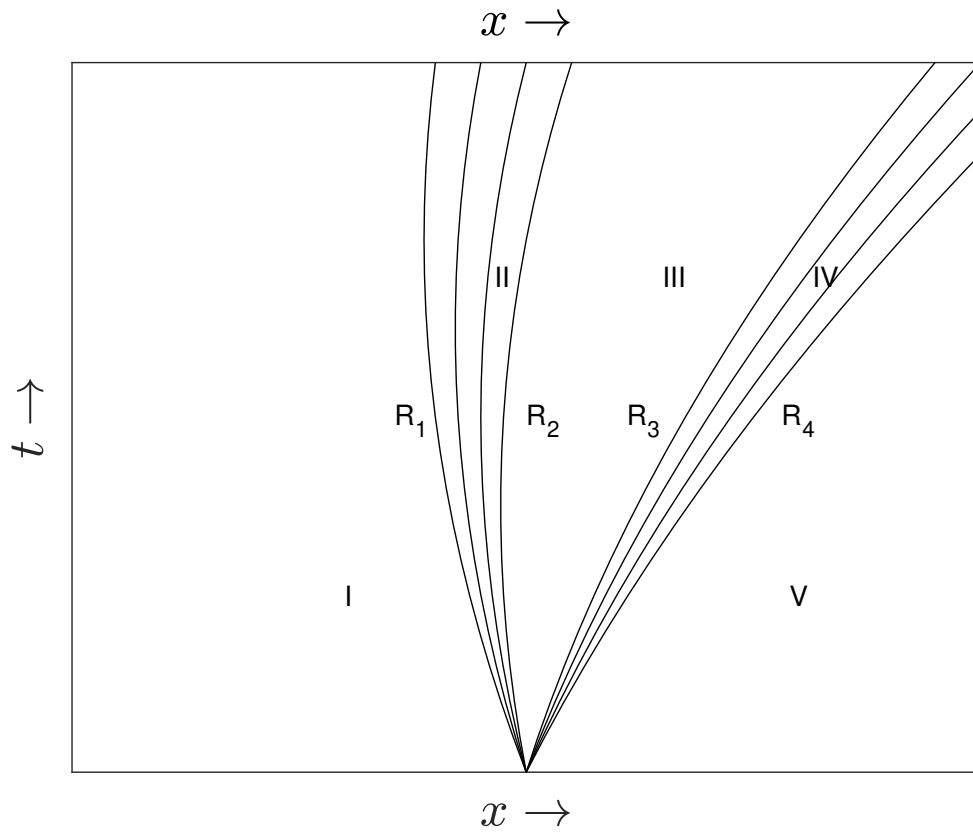


Figure 2.2: Region behind the back-shock S_1 is depicted as I; region $S_1(t) < x < R_3(t)$ between S_1 and the trail characteristic R_3 of the front rarefaction wave III is depicted as II; region $x > R_4(t)$ ahead of the front rarefaction is depicted as IV.

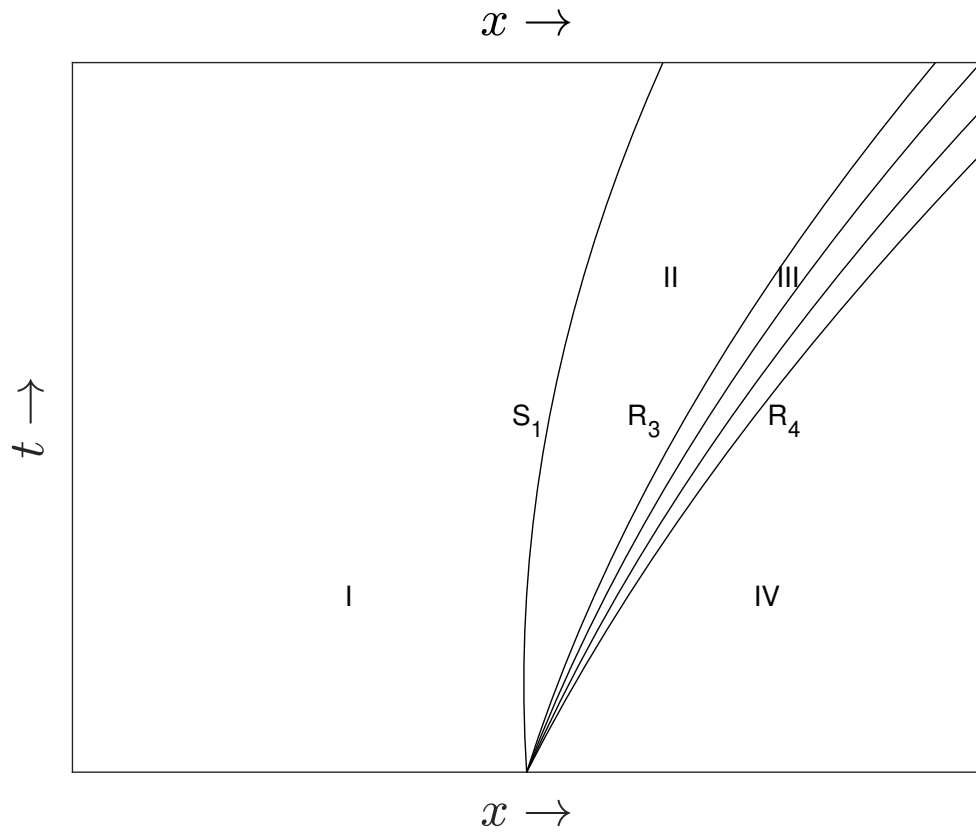


Figure 2.3: The region $x < R_1(t)$ is depicted as I; back rarefaction region $R_1(t) < x < R_2(t)$ is depicted as II; the region $R_2(t) \leq x \leq S_2(t)$ is depicted as III and the region $x > S_2(t)$ is depicted as IV.

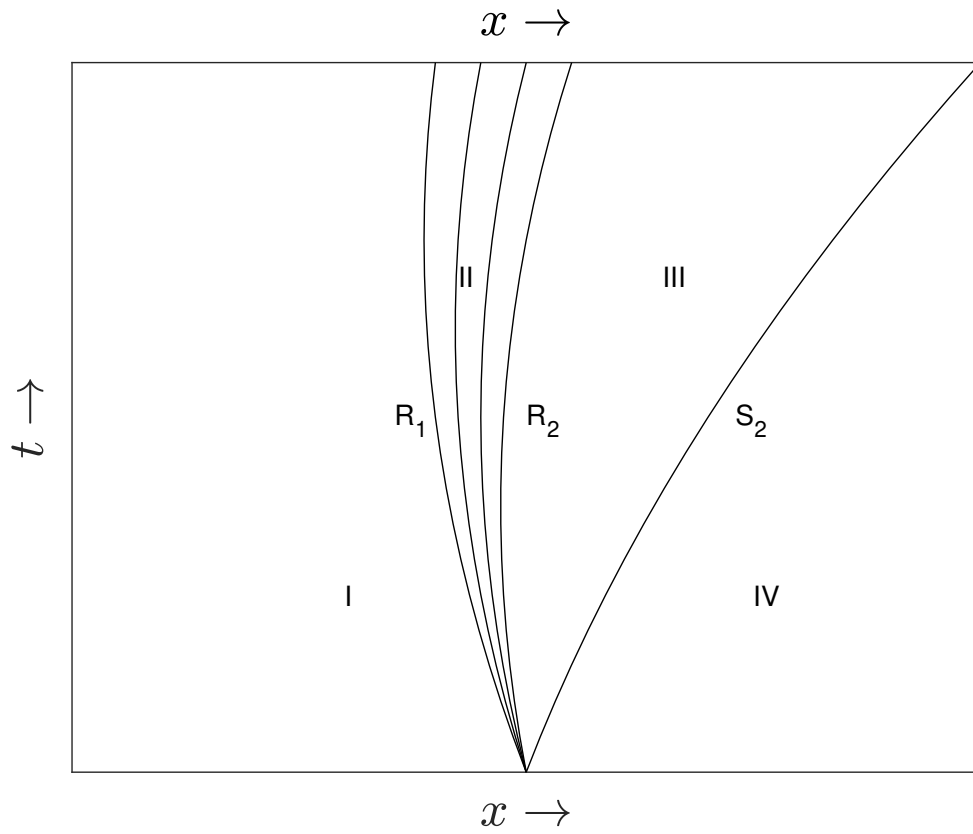


Figure 2.4: The region $x < R_1(t)$ is depicted as I; II is the back rarefaction wave region; region $R_2(t) \leq x \leq R_3(t)$ between front and back rarefaction is depicted as III; IV is the front rarefaction wave region and region $x > R_4(t)$ is depicted as V.

Chapter 3

Riemann problem for the Chaplygin gas equations for several classes of non-constant initial data

3.1 Introduction

In this chapter we considered a Chaplygin gas model

$$\rho_t + (\rho v)_x = 0, \tag{3.1.1}$$

$$(\rho v)_t + (\rho v^2 + p)_x = 0, \tag{3.1.2}$$

subject to the equation of state given by

$$p(\rho) = -\rho^{-1}, \tag{3.1.3}$$

where ρ and v are the density and velocity of the gas respectively. The model given by the equations (3.1.1), (3.1.2) describes the dark matter and dark energy in the unified form through an exotic background fluid proposed in [47–50] as a mathematical approximation to calculate the lifting force on a wing of an airplane in aerodynamics. Also it was a prototype of the unified model [51, 52] where dark energy and dark matter were depicted by a single fluid and Chaplygin cosmology provides an interesting possibility to account for current observations about the expansion of the universe. It was also predicted by them that the cosmological constant will increase (or that it was less in the past) and this could in principle be observed. In 1996, Hsiao and Serre [63], proved the global existence to the solutions for the system of compressible adiabatic

flow through porous media.

The differential constraint method is based upon appending a set of partial differential equations to a governing system of equations, first proposed by Janenko [30] to the gas dynamics model. The differential constraint equations play an important role to select classes of solutions of the system under interest. Within such a theoretical framework, recently, reduction procedures have been developed for studying soliton-like interaction for homogeneous and nonhomogeneous hyperbolic 2×2 systems [40, 41], for solving Riemann problems and generalized Riemann problems [39], for determining exact solution for the constant Astigmatism equation [55] and for a model of interest in chromatography [54].

3.2 Differential Constraint Method

In this section, to outline the method of differential constraints as demonstrated by Curró, Fusco and Manganaro [39], consider a system of non-homogeneous quasilinear partial differential equations as

$$\mathbf{U}_t + \mathbf{A}(x, t, \mathbf{U})\mathbf{U}_x = \mathbf{B}(x, t, \mathbf{U}), \quad (3.2.1)$$

where x and t are the space and the time coordinates, respectively; $\mathbf{U} \in \mathbb{R}^N$ denotes column vector of the dependent field variables, $\mathbf{A}(x, t, \mathbf{U})$ is $N \times N$ coefficient matrix and $\mathbf{B}(x, t, \mathbf{U})$ is the column vector related to the source terms. Hereafter, any subscript denotes the partial derivative with respect to the indicated variable. The system (3.2.1) is assumed to be strictly hyperbolic, i.e., the coefficient matrix $\mathbf{A}(x, t, \mathbf{U})$ has real distinct eigenvalues. Without loss of generality we assume that

$$\mathbf{l}^{(i)} \cdot \mathbf{d}^{(j)} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \quad (3.2.2)$$

where $\mathbf{l}^{(i)}$ and $\mathbf{d}^{(i)}$ represents the left and the right eigenvectors of the coefficient matrix $\mathbf{A}(x, t, \mathbf{U})$ corresponding to an eigenvalue $\lambda^{(i)}$, respectively. For strictly hyperbolic system (3.2.1), consider a set of first order differential constraints [8] as

$$\mathbf{l}^{(i)}(\mathbf{U}) \cdot \mathbf{U}_x = Q^{(i)}(x, t, \mathbf{U}), \quad i = 1, 2, \dots, N-1, \quad (3.2.3)$$

where $Q^{(i)}(x, t, \mathbf{U})$ are arbitrary functions to be determined subject to the compatibility conditions given in [39] and [30]. Owing to (3.2.3), we have

$$\mathbf{U}_x = \sum_{i=1} Q^{(i)} \mathbf{d}^{(i)} + \epsilon \mathbf{d}^{(N)}, \quad (3.2.4)$$

through which the equations (3.2.1) reduce to form

$$\mathbf{U}_t = \mathbf{B} - \sum_{i=1} Q^{(i)} \lambda^{(i)} \mathbf{d}^{(i)} - \epsilon \lambda^{(N)} \mathbf{d}^{(N)}. \quad (3.2.5)$$

For consistency of (3.2.4) and (3.2.5) to hold $\forall \epsilon$, we obtain

$$\begin{aligned} Q_t^{(i)} + \lambda^{(i)} Q_x^{(i)} + \nabla Q^{(i)} \left(\mathbf{B} - \sum_{j=1}^{N-1} Q^{(j)} (\lambda^{(j)} - \lambda^{(i)}) \mathbf{d}^{(j)} \right) \\ + \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} Q^{(j)} Q^{(k)} (\lambda^{(j)} - \lambda^{(k)}) \mathbf{l}^{(i)} \nabla \mathbf{d}^{(j)} \mathbf{d}^{(k)} \\ + \sum_{k=1}^{N-1} Q^{(k)} \left(\mathbf{l}^{(k)} \left(\nabla \mathbf{d}^{(k)} \mathbf{B} - \nabla \mathbf{B} \mathbf{d}^{(k)} \right) + Q^{(i)} \nabla \lambda^{(i)} \mathbf{d}^{(k)} \right) = 0, \end{aligned} \quad (3.2.6)$$

$$\begin{aligned} (\lambda^{(i)} - \lambda^{(N)}) \nabla Q^{(i)} \mathbf{d}^{(N)} + \sum_{k=1}^{N-1} Q^{(k)} (\lambda^{(k)} - \lambda^{(N)}) \mathbf{l}^{(i)} \left(\nabla \mathbf{d}^{(k)} \mathbf{d}^{(N)} - \nabla \mathbf{d}^{(N)} \mathbf{d}^{(k)} \right) \\ + \mathbf{l}^{(i)} \left(\nabla \mathbf{d}^{(N)} \mathbf{B} - \nabla \mathbf{B} \mathbf{d}^{(N)} \right) + Q^{(i)} \nabla \lambda^{(i)} \mathbf{d}^{(N)} = 0, \end{aligned} \quad (3.2.7)$$

where $i = 1, \dots, (N-1)$ and $\nabla = \partial/\partial \mathbf{U}$.

Using the differential constraint equations (3.2.3) and compatibility condition (3.2.6), (3.2.7), the equation (3.2.1) reduce to the form

$$\mathbf{U}_t + \lambda^{(N)} \mathbf{U}_x = \mathbf{B} + \sum_{i=1}^{N-1} Q^{(i)} (\lambda^{(N)} - \lambda^{(i)}) \mathbf{d}^{(i)}. \quad (3.2.8)$$

By determining $Q^{(i)}$ satisfying the equations (3.2.6), (3.2.7), the equations (3.2.8) can be integrated along the characteristics curves given by $\frac{dx}{dt} = \lambda^{(N)}$ with the given initial data.

3.3 Cauchy Problem

In this section, the system of conservation laws describing Chaplygin gas model is considered, given by the equations (3.1.1), (3.1.2), in the matrix form

$$\mathbf{U} = \begin{bmatrix} \rho \\ v \end{bmatrix}, \quad \mathbf{A}(x, t, \mathbf{U}) = \begin{bmatrix} v & \rho \\ \frac{1}{\rho^3} & v \end{bmatrix}, \quad \mathbf{B}(x, t, \mathbf{U}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The eigenvalues of the matrix $\mathbf{A}$ are given by

$$\lambda = v - \frac{1}{\rho} \quad \text{and} \quad \mu = v + \frac{1}{\rho}, \quad (3.3.1)$$

the corresponding left eigenvectors and right eigenvectors are as follows

$$\mathbf{l}^{(\lambda)} = \left[\frac{1}{2\rho}, \frac{-\rho}{2} \right], \quad \mathbf{l}^{(\mu)} = \left[\frac{1}{2\rho}, \frac{\rho}{2} \right], \quad (3.3.2)$$

$$\mathbf{d}^{(\lambda)} = \begin{bmatrix} \rho \\ -1 \\ \rho \end{bmatrix}, \quad \mathbf{d}^{(\mu)} = \begin{bmatrix} \rho \\ 1 \\ \rho \end{bmatrix}. \quad (3.3.3)$$

The authors have made an attempt to derive a nontrivial solution to the system of equations (3.1.1), (3.1.2) subject to the initial conditions

$$\rho(x, 0) = \rho_0(x), \quad v(x, 0) = v_0(x), \quad (3.3.4)$$

using the differential constraint

$$\rho v_x - \frac{\rho_x}{\rho} = q(x, t, \rho, v), \quad (3.3.5)$$

where q satisfies the following compatibility conditions derived (3.2.6) and (3.2.7)

$$q_t + \lambda q_x + q^2 \nabla \lambda \mathbf{d}^{(\lambda)} = 0, \quad (3.3.6)$$

$$(\lambda - \mu) \nabla q \mathbf{d}^{(\mu)} + q \nabla \lambda \mathbf{d}^{(\mu)} + q(\lambda - \mu) \mathbf{l}^{(\lambda)} \left(\nabla \mathbf{d}^{(\lambda)} \mathbf{d}^{(\mu)} - \nabla \mathbf{d}^{(\mu)} \mathbf{d}^{(\lambda)} \right) = 0. \quad (3.3.7)$$

Using the above analysis, Gupta et al. [46] derived the same equation as (3.3.6) which is numbered as equation (7) in their paper. However, instead of the equation (3.3.7), they had erroneously derived an equation numbered (8) in their paper which is the same as (3.3.7) but with the last term missing. Consequently, the equations (14) - (20) given in Gupta et al. [46] are in error, and hence, the solution to the Riemann problem

is not correct and does not satisfy the basic equations (3.1.1), (3.1.2).

In view of (3.3.1)-(3.3.3), the compatibility conditions (3.3.6), (3.3.7) reduce to

$$\begin{aligned} q_t + \lambda q_x &= 0, \\ \rho^2 q_\rho + q_v &= 2\rho q, \end{aligned}$$

which have a solution of the form

$$q = c_0 \rho^2, \quad (3.3.8)$$

where c_0 is an arbitrary constant. Thus, in view of the equations (3.3.5) and (3.3.8), the equations (3.1.1) and (3.1.2) can be written as

$$\begin{aligned} \rho_t + \mu \rho_x + q &= 0, \\ v_t + \mu v_x - q \rho^{-2} &= 0, \end{aligned}$$

which can be solved subject to the initial conditions (3.3.4), as

$$\rho(x, t) = \frac{\rho_0(\xi)}{1 + c_0 t \rho_0(\xi)}, \quad (3.3.9)$$

$$v(x, t) = v_0(\xi) - c_0 t, \quad (3.3.10)$$

$$x = \xi + \left(v_0(\xi) + \frac{1}{\rho_0(\xi)} \right) t - c_0 t^2. \quad (3.3.11)$$

Here, $\xi(x, t)$ denotes the point on x -axis that lies on the characteristic given by (3.3.11) passing through (x, t) with a speed of $\left(v + \frac{1}{\rho} \right)$, subject to the condition

$$v'_0(\xi) - \frac{1}{\rho_0^2(\xi)} \rho'_0(\xi) = c_0 \rho_0(\xi), \quad (3.3.12)$$

where ‘ $\prime$ ’ denotes derivative with respect to ξ .

The equations (3.3.9) and (3.3.10) admit unique values for ρ and v for a given x and t provided there exists a unique ξ satisfying (3.3.11); i.e., the existence of a unique solution is guaranteed for every x in $(-\infty, \infty)$ and for every $t > 0$ provided that

$$1 + \left(v'_0(\xi) - \frac{\rho'_0(\xi)}{\rho_0^2(\xi)} \right) t \neq 0. \quad (3.3.13)$$

Similarly, using the differential constraint

$$\frac{\rho_x}{2\rho} + \frac{\rho v_x}{2} = q(x, t, \rho, v),$$

and adopting the aforementioned procedure, we obtain a solution

$$\rho(x, t) = \frac{\rho_0(\zeta)}{1 + c_0 t \rho_0(\zeta)}, \quad (3.3.14)$$

$$v(x, t) = v_0(\zeta) + c_0 t, \quad (3.3.15)$$

$$x = \zeta + \left(v_0(\zeta) - \frac{1}{\rho_0(\zeta)} \right) t + c_0 t^2. \quad (3.3.16)$$

Here, $\zeta(x, t)$ denotes the point on x -axis that lies on the characteristic given by (3.3.16) passing through (x, t) with a speed of $\left(v - \frac{1}{\rho} \right)$, subject to the differential constraint

$$v'_0(\zeta) + \frac{1}{\rho_0^2(\zeta)} \rho'_0(\zeta) = c_0 \rho_0(\zeta), \quad (3.3.17)$$

where ‘ $\prime$ ’ denotes derivative with respect to ζ .

The equations (3.3.14) and (3.3.15) admit unique values for ρ and v for a given x and t provided there exists a unique ζ satisfying (3.3.16); i.e., the existence of a unique solution is guaranteed for every x in $(-\infty, \infty)$ and for every $t > 0$ provided that

$$1 + \left(v'_0(\zeta) + \frac{\rho'_0(\zeta)}{\rho_0^2(\zeta)} \right) t \neq 0. \quad (3.3.18)$$

Thus, the two solutions given in (3.3.9)-(3.3.13) and (3.3.14)-(3.3.18) of the equations (3.1.1), (3.1.2) subject to the initial conditions (3.3.4) are summarized as follows:

$$\begin{aligned}
\rho(x, t) &= \frac{\rho_0(\xi)}{1 + c_0 t \rho_0(\xi)}, \\
v(x, t) &= v_0(\xi) + \delta c_0 t, \\
x &= \xi + \left(v_0(\xi) + \frac{\delta}{\rho_0(\xi)} \right) t + \delta c_0 t^2, \\
v'_0(\xi) - \frac{\delta}{\rho_0^2(\xi)} \rho'_0(\xi) &= c_0 \rho_0(\xi), \\
1 + \left(v'_0(\xi) - \frac{\delta \rho'_0(\xi)}{\rho_0^2(\xi)} \right) t &\neq 0,
\end{aligned} \tag{3.3.19}$$

and $\delta = \pm 1$.

3.4 Characteristic Shocks

Let $x = X(t)$ be a curve, with speed $\frac{dX}{dt} = \sigma$, across which the flow variables are discontinuous then the R-H conditions [2] for the equations (3.1.1), (3.1.2)

$$\sigma (\rho^+(t) - \rho^-(t)) = (\rho^+(t)v^+(t) - \rho^-(t)v^-(t)), \tag{3.4.1}$$

$$\sigma (\rho^+(t)v^+(t) - \rho^-(t)v^-(t)) = \left(\rho^+(t) (v^+(t))^2 - \rho^-(t) (v^-(t))^2 - \frac{1}{\rho^+(t)} + \frac{1}{\rho^-(t)} \right), \tag{3.4.2}$$

where

$$\begin{aligned}
\rho^+(t) &= \lim_{x \rightarrow X(t)^+} \rho(x, t), \quad \rho^-(t) = \lim_{x \rightarrow X(t)^-} \rho(x, t), \\
v^+(t) &= \lim_{x \rightarrow X(t)^+} v(x, t), \quad v^-(t) = \lim_{x \rightarrow X(t)^-} v(x, t).
\end{aligned}$$

Solving the equations (3.4.1) and (3.4.2) we have $\sigma = v^+(t) + \frac{\delta}{\rho^+(t)} = v^-(t) + \frac{\delta}{\rho^-(t)}$, i.e., the speed of the discontinuity is equal to the speed of one of the characteristics given by $\frac{dx}{dt} = \lambda$ or $\frac{dx}{dt} = \mu$ which is a consequence of the result that the eigenvalues λ and μ are linearly-degenerate (because $\nabla \lambda \cdot \mathbf{d}^{(\lambda)} = 0$ and $\nabla \mu \cdot \mathbf{d}^{(\mu)} = 0$). Thus, the curve $x = X(t)$ with the speed $v^+(t) - \frac{1}{\rho^+(t)} \left(= v^-(t) - \frac{1}{\rho^-(t)} \right)$ is referred to as a 1- Characteristic shock across which the variables ρ and v are discontinuous and $v - \frac{1}{\rho}$ is continuous. Sim-

ilarly, the curve $x = X(t)$ with the speed $v^+(t) + \frac{1}{\rho^+(t)} \left(= v^-(t) + \frac{1}{\rho^-(t)} \right)$ is referred to as a 2- Characteristic shock across which the variables ρ and v are discontinuous and $v + \frac{1}{\rho}$ is continuous.

3.5 Riemann Problem for Non-Constant Initial Data

In this section, we consider the Riemann problem with non-constant initial discontinuous data:

$$(\rho(x, 0), v(x, 0)) = \begin{cases} (\rho_r(x), v_r(x)), & x \geq 0, \\ (\rho_l(x), v_l(x)), & x \leq 0, \end{cases} \quad (3.5.1)$$

where ρ_l, v_l, ρ_r and v_r are arbitrary functions of x such that

$$\begin{aligned} v_l'(x) - \frac{\alpha}{\rho_l^2(x)} \rho_l'(x) &= C_0 \rho_l(x), \\ v_r'(x) - \frac{\beta}{\rho_r^2(x)} \rho_r'(x) &= C_1 \rho_r(x). \end{aligned}$$

Here, $\alpha = \pm 1$, $\beta = \pm 1$ and C_0, C_1 are arbitrary constants. Let

$$\begin{aligned} \rho_L &= \lim_{x \rightarrow 0^-} \rho_l(x), & \rho_R &= \lim_{x \rightarrow 0^+} \rho_r(x), \\ v_L &= \lim_{x \rightarrow 0^-} v_l(x), & v_R &= \lim_{x \rightarrow 0^+} v_r(x), \end{aligned}$$

where $(\rho_L, v_L) \neq (\rho_R, v_R)$.

Theorem 3.5.1. *Let solution to the Riemann problem of the equations (3.1.1), (3.1.2) subject to (3.5.1) be connected through only 1-Characteristic shock then $v_L - \frac{1}{\rho_L} = v_R - \frac{1}{\rho_R}$ along with one of the following possibilities*

1. $\alpha = 1$, $\beta = 1$, and C_0, C_1 are arbitrary constants.
2. $\alpha = 1$, $\beta = -1$, $C_1 = 0$ and C_0 is an arbitrary constant.
3. $\alpha = -1$, $\beta = 1$, $C_0 = 0$ and C_1 is an arbitrary constant.
4. $\alpha = -1$, $\beta = -1$, $C_1 = C_0$ and C_0 is an arbitrary constant.

Proof. Given that the solution to the Riemann problem of the equations (3.1.1), (3.1.2) subject to (3.5.1) be connected through only 1- Characteristic shock we have the $x = X_1(t)$, such that $\frac{dX_1}{dt} = v - \frac{1}{\rho}$ across which the flow variables v and ρ are discontinuous and however, $\left(v - \frac{1}{\rho}\right)$ is continuous, i.e.,

$$\frac{dX_1}{dt} = \lim_{x \rightarrow X_1(t)^-} \left(v - \frac{1}{\rho}\right) = \lim_{x \rightarrow X_1(t)^+} \left(v - \frac{1}{\rho}\right). \quad (3.5.2)$$

In view of (3.3.19) when $\delta = \alpha$ and $c_0 = C_0$, the solution of the Riemann problem in the region $x < X_1(t)$ is given by

$$\rho(x, t) = \frac{\rho_l(\xi)}{1 + C_0 \rho_l(\xi)t}, \quad (3.5.3)$$

$$v(x, t) = v_l(\xi) + \alpha C_0 t, \quad (3.5.4)$$

$$x = \xi + \left(v_l(\xi) + \frac{\alpha}{\rho_l(\xi)}\right)t + \alpha C_0 t^2.$$

In the region $x \geq X_1(t)$, in view of (3.3.19) when $\delta = \beta$ and $c_0 = C_1$, the solution is given by

$$\rho(x, t) = \frac{\rho_r(\eta)}{1 + C_1 \rho_r(\eta)t}, \quad (3.5.5)$$

$$v(x, t) = v_r(\eta) + \beta C_1 t, \quad (3.5.6)$$

$$x = \eta + \left(v_r(\eta) + \frac{\beta}{\rho_r(\eta)}\right)t + \beta C_1 t^2,$$

Since, $\eta = \xi = 0$ as $x \rightarrow X_1(t)$, in view of (3.5.3)-(3.5.4) and (3.5.5)-(3.5.6), the equation (3.5.2) reduces to

$$\frac{dX_1}{dt} = \left(v_L - \frac{1}{\rho_L}\right) + (\alpha - 1)C_0 t = \left(v_R - \frac{1}{\rho_R}\right) + (\beta - 1)C_1 t,$$

which leads to upon integration

$$X_1(t) = \left(v_L - \frac{1}{\rho_L}\right)t + (\alpha - 1)C_0 \frac{t^2}{2} = \left(v_R - \frac{1}{\rho_R}\right)t + (\beta - 1)C_1 \frac{t^2}{2},$$

and holds for all values of t provided

$$\begin{aligned} v_L - \frac{1}{\rho_L} &= v_R - \frac{1}{\rho_R}, \\ (\alpha - 1)C_0 &= (\beta - 1)C_1. \end{aligned} \quad (3.5.7)$$

Thus, the equation (3.5.7) holds true for the following cases

1. $\alpha = 1$, $\beta = 1$, and C_0, C_1 are arbitrary constants.
2. $\alpha = 1$, $\beta = -1$, $C_1 = 0$ and C_0 is an arbitrary constant.
3. $\alpha = -1$, $\beta = 1$, $C_0 = 0$ and C_1 is an arbitrary constant.
4. $\alpha = -1$, $\beta = -1$, $C_1 = C_0$ and C_0 is an arbitrary constant.

Hence the proof. This is depicted in the Figure-3.1. □

Theorem 3.5.2. *Let solution of the Riemann problem of the equations (3.1.1), (3.1.2) subject to (3.5.1) be connected through only 2-Characteristic shock then $v_L + \frac{1}{\rho_L} = v_R + \frac{1}{\rho_R}$ along with one of the following cases*

1. $\alpha = 1$, $\beta = 1$, $C_1 = C_0$ and C_0 is an arbitrary constant.
2. $\alpha = 1$, $\beta = -1$, $C_0 = 0$ and C_1 is an arbitrary constant.
3. $\alpha = -1$, $\beta = 1$, $C_1 = 0$ and C_0 is an arbitrary constant.
4. $\alpha = -1$, $\beta = -1$, and C_0, C_1 are arbitrary constants.

Proof. Given that the solution to the Riemann problem of the equations (3.1.1), (3.1.2) subject to (3.5.1) be connected through only 2- Characteristic shock we have the the curve $x = X_2(t)$, such that $\frac{dX_2}{dt} = v + \frac{1}{\rho}$ across which the flow variables v and ρ are discontinuous and however, $\left(v + \frac{1}{\rho}\right)$ is continuous, i.e.,

$$\frac{dX_2}{dt} = \lim_{x \rightarrow X_2(t)^-} \left(v + \frac{1}{\rho}\right) = \lim_{x \rightarrow X_2(t)^+} \left(v + \frac{1}{\rho}\right). \quad (3.5.8)$$

In view of (3.3.19) when $\delta = \alpha$ and $c_0 = C_0$, the solution of the Riemann problem in the region $x < X_2(t)$ is

$$\rho(x, t) = \frac{\rho_l(\xi)}{1 + C_0 \rho_l(\xi) t}, \quad (3.5.9)$$

$$v(x, t) = v_l(\xi) + \alpha C_0 t, \quad (3.5.10)$$

$$x = \xi + \left(v_l(\xi) + \frac{\alpha}{\rho_l(\xi)} \right) t + \alpha C_0 t^2,$$

In the region $x \geq X_2(t)$, In view of (3.3.19) when $\delta = \beta$ and $c_0 = C_1$ the solution is given by

$$\rho(x, t) = \frac{\rho_r(\eta)}{1 + C_1 \rho_r(\eta) t}, \quad (3.5.11)$$

$$v(x, t) = v_r(\eta) + \beta C_1 t, \quad (3.5.12)$$

$$x = \eta + \left(v_r(\eta) + \frac{\beta}{\rho_r(\eta)} \right) t + \beta C_1 t^2,$$

Since, $\eta = \xi = 0$ as $x \rightarrow X_2(t)$, in view of (3.5.3)-(3.5.4) and (3.5.5)-(3.5.6), the equation (3.5.8) reduces to

$$\frac{dX_2}{dt} = \left(v_L + \frac{1}{\rho_L} \right) + (\alpha + 1)C_0 t = \left(v_R + \frac{1}{\rho_R} \right) + (\beta + 1)C_1 t,$$

which leads upon integration

$$X_2(t) = \left(v_L + \frac{1}{\rho_L} \right) t + (\alpha + 1)C_0 \frac{t^2}{2} = \left(v_R + \frac{1}{\rho_R} \right) t + (\beta + 1)C_1 \frac{t^2}{2},$$

and holds for all values of t provided

$$\begin{aligned} v_L + \frac{1}{\rho_L} &= v_R + \frac{1}{\rho_R}, \\ (\alpha + 1)C_0 &= (\beta + 1)C_1. \end{aligned} \quad (3.5.13)$$

Thus, the equation (3.5.13) holds true for the following cases

1. $\alpha = 1, \beta = 1, C_1 = C_0$ and C_0 is an arbitrary constant.
2. $\alpha = 1, \beta = -1, C_0 = 0$ and C_1 is an arbitrary constant.
3. $\alpha = -1, \beta = 1, C_1 = 0$ and C_0 is an arbitrary constant.

4. $\alpha = -1$, $\beta = -1$, and C_0, C_1 are arbitrary constants.

Hence the proof. This is depicted in the Figure-3.2.

□

Theorem 3.5.3. *Let solution of the Riemann problem of the equations (3.1.1), (3.1.2) subject to (3.5.1) be connected through both 1- and 2- Characteristic shocks then one of the following occurs*

1. $\alpha = 1$, $\beta = -1$ and C_0, C_1 are arbitrary constants.
2. $\alpha = 1$, $\beta = 1$, $C_1 = 0$ and C_0 is an arbitrary constant.
3. $\alpha = -1$, $\beta = -1$, $C_0 = 0$ and C_1 is an arbitrary constant.
4. $\alpha = -1$, $\beta = 1$ and $C_1 = C_0$ where C_0 is an arbitrary constant.

Proof. Given that the solution to the Riemann problem of the equations (3.1.1), (3.1.2) subject to (3.5.1) be connected through 1- and 2- Characteristic shocks, we have the the curve $x = X_1(t)$ (respectively; $x = X_2(t)$), such that $\frac{dX_1}{dt} = v - \frac{1}{\rho}$ (respectively; $\frac{dX_2}{dt} = v + \frac{1}{\rho}$) across which the flow variables v and ρ are discontinuous and however, $\left(v - \frac{1}{\rho}\right)$ (respectively; $\left(v + \frac{1}{\rho}\right)$) is continuous, i.e.,

$$\frac{dX_1}{dt} = \lim_{x \rightarrow X_1(t)^-} \left(v - \frac{1}{\rho}\right) = \lim_{x \rightarrow X_1(t)^+} \left(v - \frac{1}{\rho}\right), \quad (3.5.14)$$

$$\frac{dX_2}{dt} = \lim_{x \rightarrow X_2(t)^-} \left(v + \frac{1}{\rho}\right) = \lim_{x \rightarrow X_2(t)^+} \left(v + \frac{1}{\rho}\right). \quad (3.5.15)$$

In view of (3.3.19) when $\delta = \alpha$ and $c_0 = C_0$, the solution of the Riemann problem in the region $x < X_1(t)$ is given by

$$\rho(x, t) = \frac{\rho_l(\xi)}{1 + C_0 \rho_l(\xi) t}, \quad (3.5.16)$$

$$v(x, t) = v_l(\xi) + \alpha C_0 t, \quad (3.5.17)$$

$$x = \xi + \left(v_l(\xi) + \frac{\alpha}{\rho_l(\xi)}\right) t + \alpha C_0 t^2. \quad (3.5.18)$$

In the region $x \geq X_2(t)$, in view of (3.3.19) when $\delta = \beta$ and $c_0 = C_1$, the solution is given by

$$\rho(x, t) = \frac{\rho_r(\eta)}{1 + C_1 \rho_r(\eta)t}, \quad (3.5.19)$$

$$v(x, t) = v_r(\eta) + \beta C_1 t, \quad (3.5.20)$$

$$x = \eta + \left(v_r(\eta) + \frac{\beta}{\rho_r(\eta)} \right) t + \beta C_1 t^2. \quad (3.5.21)$$

Since, $\xi = 0$ as $x \rightarrow X_1(t)$ and $\eta = 0$ as $x \rightarrow X_2(t)$, in view of (3.5.16)-(3.5.17) and (3.5.19)-(3.5.20), the equations (3.5.14) and (3.5.15) reduces to

$$\begin{aligned} \frac{dX_1}{dt} &= \lim_{x \rightarrow X_1(t)^-} \left(v - \frac{1}{\rho} \right) = \left(v_L - \frac{1}{\rho_L} \right) + (\alpha - 1)C_0 t, \\ \frac{dX_2}{dt} &= \lim_{x \rightarrow X_2(t)^+} \left(v + \frac{1}{\rho} \right) = \left(v_R + \frac{1}{\rho_R} \right) + (\beta + 1)C_1 t, \end{aligned}$$

which leads to

$$X_1(t) = \left(v_L - \frac{1}{\rho_L} \right) t + (\alpha - 1)C_0 \frac{t^2}{2}, \quad (3.5.22)$$

$$X_2(t) = \left(v_R + \frac{1}{\rho_R} \right) t + (\beta + 1)C_1 \frac{t^2}{2}. \quad (3.5.23)$$

To solve the Riemann problem completely, let the solution be of the form in the region $X_1(t) < x < X_2(t)$,

$$\rho(x, t) = \frac{K_1}{K_2 t + K_3}, \quad (3.5.24)$$

$$v(x, t) = \frac{K_2 x + K_4}{K_2 t + K_3}, \quad (3.5.25)$$

which satisfy the equations (3.1.1) and (3.1.2) where K_1, K_2, K_3 and K_4 are arbitrary constants. In order to connect the solution to the Riemann problem, it is required that $\left(v - \frac{1}{\rho} \right)$ is continuous at $x = X_1(t)$ and $\left(v + \frac{1}{\rho} \right)$ is continuous at $x = X_2(t)$ for all time t .

$$\lim_{x \rightarrow X_1(t)^-} \left(v - \frac{1}{\rho} \right) = \lim_{x \rightarrow X_1(t)^+} \left(v - \frac{1}{\rho} \right), \quad (3.5.26)$$

$$\lim_{x \rightarrow X_2(t)^-} \left(v + \frac{1}{\rho} \right) = \lim_{x \rightarrow X_2(t)^+} \left(v + \frac{1}{\rho} \right). \quad (3.5.27)$$

Thus, in view of (3.5.16)-(3.5.18), (3.5.19)-(3.5.21) and (3.5.24)-(3.5.25), we have

$$\lim_{x \rightarrow X_1(t)^-} \left(v - \frac{1}{\rho} \right) = v_L + \alpha C_0 t - \frac{1 + C_0 \rho_L t}{\rho_L}, \quad (3.5.28)$$

$$\lim_{x \rightarrow X_1(t)^+} \left(v - \frac{1}{\rho} \right) = \frac{K_2 X_1(t) + K_4}{K_2 t + K_3} - \frac{K_2 t + K_3}{K_1}, \quad (3.5.29)$$

$$\lim_{x \rightarrow X_2(t)^-} \left(v + \frac{1}{\rho} \right) = v_R + \beta C_1 t + \frac{1 + C_1 \rho_R t}{\rho_R}, \quad (3.5.30)$$

$$\lim_{x \rightarrow X_2(t)^+} \left(v + \frac{1}{\rho} \right) = \frac{K_2 X_2(t) + K_4}{K_2 t + K_3} + \frac{K_2 t + K_3}{K_1}, \quad (3.5.31)$$

and hence, in view of the equations (3.5.28)-(3.5.31), the equations (3.5.26)-(3.5.27), reduce to

$$\begin{aligned} \left(v_L + \alpha C_0 t - \frac{1 + C_0 \rho_L t}{\rho_L} \right) (K_2 t + K_3) &= K_4 - \frac{(K_2 t + K_3)^2}{K_1} \\ &\quad + K_2 \left(\left(v_L - \frac{1}{\rho_L} \right) t + (\alpha - 1) C_0 \frac{t^2}{2} \right), \\ \left(v_R + \beta C_1 t + \frac{1 + C_1 \rho_R t}{\rho_R} \right) (K_2 t + K_3) &= K_4 + \frac{(K_2 t + K_3)^2}{K_1} \\ &\quad + K_2 \left(\left(v_R + \frac{1}{\rho_R} \right) t + (\beta + 1) C_1 \frac{t^2}{2} \right), \end{aligned}$$

which holds for all values of time when

$$\begin{aligned} \left(v_L - \frac{1}{\rho_L} \right) K_3 &= K_4 - \frac{(K_3)^2}{K_1}, \\ \left(v_L - \frac{1}{\rho_L} \right) K_2 + (\alpha C_0 - C_0) K_3 &= -\frac{2K_2 K_3}{K_1} + K_2 \left(v_L - \frac{1}{\rho_L} \right), \\ K_2 \left(\frac{K_2}{K_1} - \frac{(1 - \alpha) C_0}{2} \right) &= 0, \\ \left(v_R + \frac{1}{\rho_R} \right) K_3 &= K_4 + \frac{(K_3)^2}{K_1}, \\ \left(v_R + \frac{1}{\rho_R} \right) K_2 + (\beta C_1 + C_1) K_3 &= \frac{2K_2 K_3}{K_1} + K_2 \left(v_R + \frac{1}{\rho_R} \right), \\ K_2 \left(\frac{K_2}{K_1} - \frac{(\beta + 1) C_1}{2} \right) &= 0. \end{aligned} \quad (3.5.32)$$

Thus, after solving the equations (3.5.32), we have the following solutions:

1. $\alpha = 1, \beta = -1$ and C_0, C_1 are arbitrary constants then $\rho = \tilde{\rho}$ and $v = \tilde{v}$.
2. $\alpha = 1, \beta = 1, C_1 = 0$ and C_0 is an arbitrary constant then $\rho = \tilde{\rho}$ and $v = \tilde{v}$.
3. $\alpha = -1, \beta = -1, C_0 = 0$ and C_1 is an arbitrary constant then $\rho = \tilde{\rho}$ and $v = \tilde{v}$.
4. $\alpha = -1, \beta = 1$ and $C_1 = C_0$ where C_0 is an arbitrary constant then $\rho = \frac{\tilde{\rho}}{\tilde{\rho}C_0t + 1}$ and $v = \frac{\tilde{\rho}(C_0x + \tilde{v})}{\tilde{\rho}C_0t + 1}$, where

$$\begin{aligned}\frac{1}{\tilde{\rho}} &= \frac{1}{2} \left\{ v_R + \frac{1}{\rho_R} - v_L + \frac{1}{\rho_L} \right\}, \\ \tilde{v} &= \frac{1}{2} \left\{ v_R + \frac{1}{\rho_R} + v_L - \frac{1}{\rho_L} \right\}.\end{aligned}$$

Hence the proof. This is depicted in the Figure-3.3.

Hence, the solution to the Riemann problem to the equations (3.1.1), (3.1.2) subject to (3.5.1) be connected through only 1- Characteristic shock (respectively; only 2- Characteristic shock) if and only if $v_L - \frac{1}{\rho_L} = v_R - \frac{1}{\rho_R}$ (respectively; $v_L + \frac{1}{\rho_L} = v_R + \frac{1}{\rho_R}$), otherwise, and be connected through 1- and 2- Characteristic shocks when $\alpha = \pm 1, \beta = \pm 1, C_0 = C_1 = 0$. $\square$

3.6 Conclusions

The solution to the generalized Riemann problem to the governing equations of the Chaplygin model is characterized by one 1- or/and 2- Characteristic shock(s) after determining the exact solution for a hyperbolic system of first-order PDEs describing a Chaplygin gas using the “Differential constraint method” for several class of initial conditions.

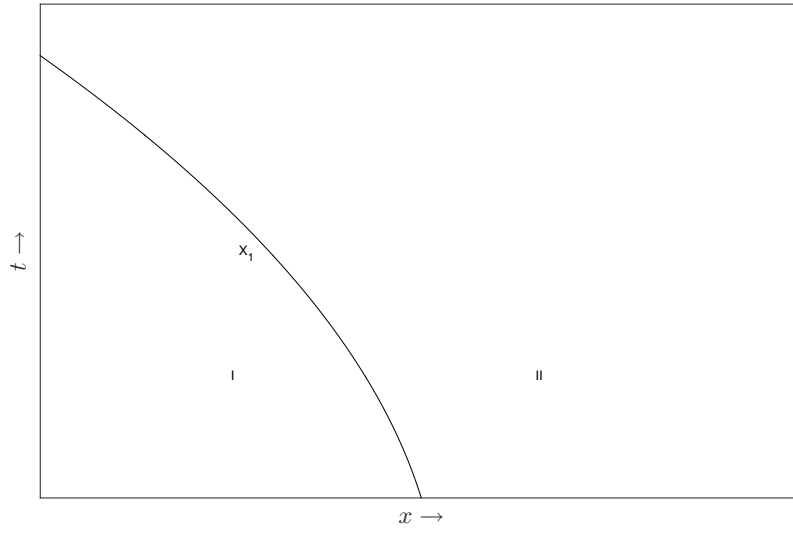


Figure 3.1: X_1 is the 1-Characteristic shock; regions $x < X_1(t)$ and $x > X_1(t)$ are depicted as I, II.

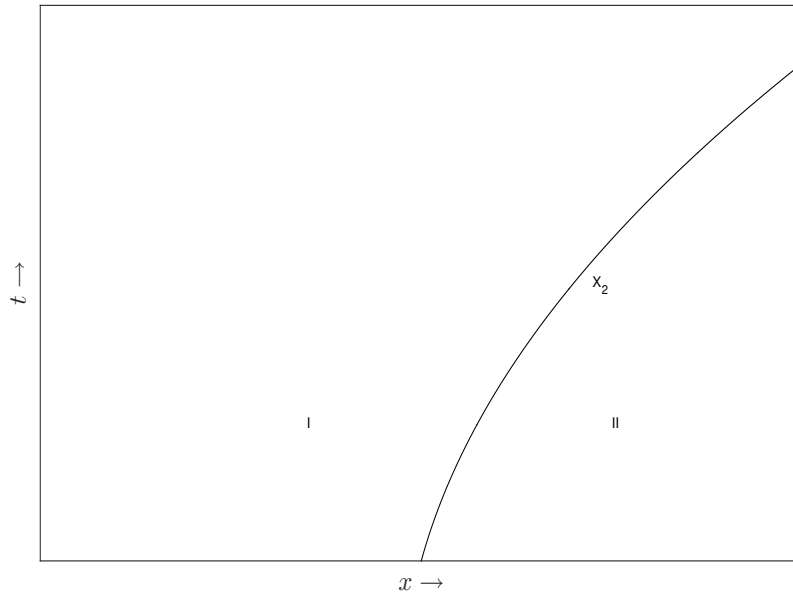


Figure 3.2: X_2 is the 2-Characteristic shock; regions $x < X_2(t)$ and $x > X_2(t)$ are depicted as I, II.

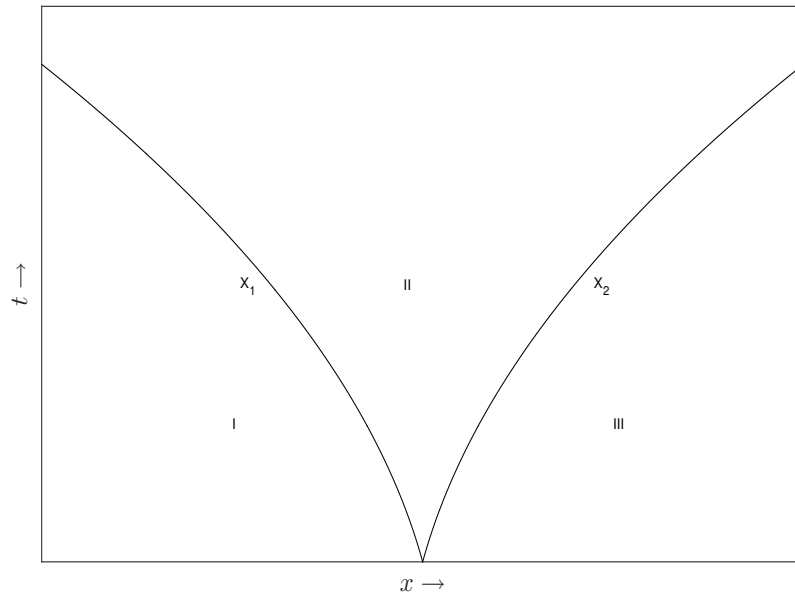


Figure 3.3: X_1 and X_2 are, respectively, the 1-Characteristic shock and the 2-Characteristic shock; regions $x < X_1(t)$, $X_1(t) < x < X_2(t)$, and $x > X_2(t)$ are depicted as I, II and III respectively.

Chapter 4

Riemann problems for Generalized gas dynamic equations

4.1 Introduction

It is well known that solving the Riemann problem plays a prominent role in quasilinear hyperbolic systems [2, 3, 6, 70–72] of balance laws. Moreover, finding an exact solution to a generalized Riemann problem is always more difficult. LeFloch and Raviart [56] obtained an approximate solution to the generalized Riemann problem in the form of an asymptotic expansion, which is explained further through an application to gas dynamics equations by Bourgeade et al. [57]. Although no general method is available for solving the Riemann problem with arbitrary initial data, the reduction method to determine the exact solutions of nonlinear partial differential equations [8] - [30] plays an important role by appending differential constraints to the governing systems of field equations under interest.

The differential constraints method, based upon appending a set of partial differential equations to a governing system of equations, was proposed by Janenko [30] and was applied to the gas dynamics model [33] - [32]. Within such a theoretical framework, a reduction procedure for the generalized Riemann problem was applied in different models, which can be seen in [39] - [59], and classes of exact solutions were parametrized in terms of arbitrary functions and were determined for ET6 model [60].

Among others, Radha et al., [61] used the differential constraints method to completely characterize the Riemann problem for non-constant initial data for rate-type materials. Furthermore, the method of differential constraints was also used to determine the exact solution of quasilinear systems depending on various applications

like p-systems with relaxation conditions (Curró, and Manganaro [62]), for Chaplygin gas model (Kumar and Radha [64]), for non-homogeneous shallow water equations (Sueet et al. [65]) and for the homogeneous p-system (Manganaro et al. [66]). Recently Meleskho et al. [67] discussed the generalized simple wave solution for a magnetic fluid using the differential constraint method and also applied it to systems of equations written in Riemann invariants [68]. Moreover, Chaiyasena et al. [69] discussed the generalized Riemann waves and their adjoinment through a shock wave.

The present paper deals with solving generalized Riemann problems for the system under investigation and completely characterizes the solution that connects the initial data to regions via shocks and/or rarefaction waves.

4.2 Generalized gas dynamic equations

The Euler equations modeling compressible gases in a conservative form are given as

$$\begin{aligned}\rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2 + p)_x &= \rho f(u, \rho, x, t),\end{aligned}\tag{4.2.1}$$

together with the equation of state given by p as a function of ρ alone such that $p'(\rho) > 0$, where ρ and u are, respectively, the density and velocity of the gas. The system (4.2.1) with the equation of state

$$p(\rho) = C\rho^\gamma, \quad C\gamma > 0,\tag{4.2.2}$$

represents the generalized Chaplygin gases when $-1 \leq \gamma < 0$ (see, [70], [71]), the isothermal gas when $\gamma = 1$ (see, [72]) and the isentropic gases when $\gamma > 1$.

The equations (4.2.1) can be re-written in a matrix form as

$$\mathbf{U}_t + \mathbf{A}(\mathbf{U})\mathbf{U}_x = \mathbf{B}(x, t, \mathbf{U}),\tag{4.2.3}$$

where

$$\mathbf{U} = \begin{bmatrix} \rho \\ u \end{bmatrix}, \quad \mathbf{A}(\mathbf{U}) = \begin{bmatrix} u & \rho \\ \rho^{-1}p'(\rho) & u \end{bmatrix}, \quad \mathbf{B}(x, t, \mathbf{U}) = \begin{bmatrix} 0 \\ f(u, \rho, x, t) \end{bmatrix}.$$

The eigenvalues of the coefficient matrix $\mathbf{A}(\mathbf{U})$ are,

$$\lambda^{(1)} = u - c(\rho), \quad \lambda^{(2)} = u + c(\rho),\tag{4.2.4}$$

where $c(\rho) = \sqrt{p'(\rho)}$ and the corresponding right eigenvectors are

$$\mathbf{R}^{(1)} = \begin{pmatrix} 1 \\ -\frac{c(\rho)}{\rho} \end{pmatrix}, \quad \mathbf{R}^{(2)} = \begin{pmatrix} 1 \\ \frac{c(\rho)}{\rho} \end{pmatrix}.$$

Thus, we have

$$\begin{aligned} \nabla \lambda^{(1)} \cdot \mathbf{R}^{(1)} &= -\frac{1}{2\rho c(\rho)} (\rho p''(\rho) + 2p'), \\ \nabla \lambda^{(2)} \cdot \mathbf{R}^{(2)} &= \frac{1}{2\rho c(\rho)} (\rho p''(\rho) + 2p'), \end{aligned} \tag{4.2.5}$$

where the symbol ∇ indicates the gradient operator with respect to (ρ, u) and thus, the system (4.2.1) is strictly hyperbolic and genuinely nonlinear in the sense of Lax [4] for all $p(\rho)$ except when $p = \frac{\hat{c}_1}{\rho} + \hat{c}_2$ where $\hat{c}_1$ and $\hat{c}_2$ are two arbitrary constants.

The Riemann invariant $W^{(1)}$ (respectively; $W^{(2)}$) along the characteristic field $\frac{dx}{dt} = \lambda^{(1)}$ (respectively; $\frac{dx}{dt} = \lambda^{(2)}$) determined from $\nabla W^{(1)} \cdot \mathbf{R}^{(1)} = 0$ (respectively; $\nabla W^{(2)} \cdot \mathbf{R}^{(2)} = 0$) as

$$W^{(1)} = u + \frac{2\sqrt{C}\gamma}{(\gamma-1)} \rho^{\frac{\gamma-1}{2}}, \quad W^{(2)} = u - \frac{2\sqrt{C}\gamma}{(\gamma-1)} \rho^{\frac{\gamma-1}{2}}. \tag{4.2.6}$$

4.3 Generalized Riemann invariants

In this section, generalized Riemann invariants compatibility conditions to the given system of partial differential equations are derived. For this, we consider a differential constraint in a more general form as

$$W_x^{(i)} = Q^{(i)}(x, t, W^{(1)}, W^{(2)}) \quad i = 1, 2, \tag{4.3.1}$$

where $W^{(1)}$ and $W^{(2)}$ denotes the Riemann invariants and $Q^{(i)}$ is to be determined.

Using this we can rewrite the hyperbolic system of partial differential differential equations in terms of generalized Riemann invariants as

$$\begin{aligned} W_t^{(1)} + \lambda(W^{(1)}, W^{(2)}) W_x^{(1)} &= f(W^{(1)}, W^{(2)}) \\ W_t^{(2)} + \mu(W^{(1)}, W^{(2)}) W_x^{(2)} &= g(W^{(1)}, W^{(2)}) \end{aligned} \tag{4.3.2}$$

To determine $Q^{(i)}$ we find the compatibility conditions for the system (4.3.2) and the differential constraints (4.3.1).

For this first we differentiating equation (4.3.1) with respect to x and t , we get

$$W_{xt}^{(1)} = Q_t^{(i)} + Q_{W^{(1)}}^{(i)} W_t^{(1)} + Q_{W^{(2)}}^{(i)} W_t^{(2)}, \quad (4.3.3)$$

$$W_{xx}^{(1)} = Q_x^{(i)} + Q_{W^{(1)}}^{(i)} W_x^{(1)} + Q_{W^{(2)}}^{(i)} W_x^{(2)}. \quad (4.3.4)$$

On differentiating equation (4.3.2) with respect to x and t , we get

$$W_{xt}^{(1)} + \left(\lambda_{W^{(1)}} W_x^{(1)} + \lambda_{W^{(2)} W_x^{(2)}} \right) W_x^{(1)} + \lambda W_{xx}^{(1)} = f_{W^{(1)}} W_x^{(1)} + f_{W^{(2)}} W_x^{(2)}, \quad (4.3.5)$$

$$W_{xt}^{(2)} + \left(\mu_{W^{(1)}} W_x^{(1)} + \mu_{W^{(2)} W_x^{(2)}} \right) W_x^{(1)} + \mu W_{xx}^{(1)} = g_{W^{(1)}} W_x^{(1)} + g_{W^{(2)}} W_x^{(2)}. \quad (4.3.6)$$

Compatibility conditions are

$$\begin{aligned} (\lambda - \mu) Q_{W^{(2)}}^{(i)} + \lambda_{W^{(2)}} Q^{(i)} + f_{W^{(2)}} &= 0, \\ Q_t^{(i)} + \lambda Q_x^{(i)} + \lambda_{W^{(1)}} (Q^{(i)})^2 + Q_{W^{(1)}}^{(i)} f + Q_{W^{(2)}}^{(i)} g - f_{W^{(1)}} Q^{(i)} &= 0. \end{aligned} \quad (4.3.7)$$

On solving (4.3.7), we determine Q^i , which enables us to solve the system under consideration along with a family of characteristic curves.

This methodology is used in the following section to a system of conservation laws describing generalized gas dynamics with source/sink.

4.4 Cauchy problem

Consider a Cauchy problem with the following initial data corresponding to the system (4.2.1)

$$\rho(x, 0) = \rho_0(x), \quad u(x, 0) = u_0(x). \quad (4.4.1)$$

To solve the above Cauchy problem, we use the differential constraints method as discussed in the previous section, for which we let

$$u_x = \frac{\delta \sqrt{p'(\rho)}}{\rho} \rho_x + Q, \quad \delta = \pm 1, \quad (4.4.2)$$

where the compatibility conditions for Q are obtained from (2.2.9) as

$$\begin{aligned}
2\delta\sqrt{p'(\rho)} \left\{ Q_\rho + \frac{\delta\sqrt{p'(\rho)}}{\rho} Q_u + \left(\frac{p''(\rho)}{4p'(\rho)} - \frac{1}{2\rho} \right) Q \right\} + f_\rho + \frac{\delta\sqrt{p'(\rho)}f_u}{\rho} &= 0, \\
Q_t + \left(u - \delta\sqrt{p'(\rho)} \right) Q_x + \left(f - \delta Q\sqrt{p'(\rho)} \right) Q_u - \rho Q Q_\rho - f_u Q + Q^2 - f_x &= 0.
\end{aligned} \tag{4.4.3}$$

Equations (4.4.3) has a solution of the form when $Q_t = 0$, $Q_x = 0$, $f_t = 0$ and $f_x = 0$,

$$Q = k_0 + c_0\rho, \tag{4.4.4}$$

$$f = k_0u - \delta(k_0 + c_0\rho)\sqrt{p'(\rho)} + c_1. \tag{4.4.5}$$

Here, k_0 , c_0 and c_1 are arbitrary constants. Using (4.4.2), (4.4.4) and (4.4.5), the system (4.2.1) reduces to

$$\begin{aligned}
\rho_t + \left(u + \delta\sqrt{p'(\rho)} \right) \rho_x + \rho(k_0 + c_0\rho) &= 0, \\
u_t + \left(u + \delta\sqrt{p'(\rho)} \right) u_x &= k_0u + c_1,
\end{aligned} \tag{4.4.6}$$

and can be integrated along the characteristic curves $\frac{dx}{dt} = u + \delta\sqrt{p'(\rho)}$ subject to the initial data (4.4.1).

Thus, we have

$$\begin{aligned}
\rho(x, t) &= F_1(\xi, t), \\
u(x, t) &= u_0(\xi)e^{k_0t} + \frac{c_1(e^{k_0t} - 1)}{k_0},
\end{aligned} \tag{4.4.7}$$

along the characteristics

$$x = \xi + \frac{u_0(\xi)}{k_0} (e^{k_0t} - 1) + \frac{c_1}{k_0} \left(\frac{e^{k_0t}}{k_0} - t \right) + \int_0^t \delta\sqrt{p'(F_1(\xi, s))} ds, \tag{4.4.8}$$

where F_1 is given as

$$F_1(\xi, t) = \frac{k_0 \rho_0(\xi)}{(k_0 + c_0 \rho_0(\xi)) e^{k_0 t} - c_0 \rho_0(\xi)}.$$

A few nontrivial solutions to the system (4.2.1) are determined in the closed form when $p(\rho) = C\rho^\gamma$, for $\gamma \in (-\infty, \infty)$ such that $C\gamma > 0$.

4.4.1 Case-1

Let $c_0 = 0$. Then, the solution to the system (4.2.1) with the initial data (4.4.1) is determined from (4.4.7), (4.4.8) as

$$\begin{aligned} \rho(x, t) &= \rho_0(\xi) e^{-k_0 t}, \\ u(x, t) &= u_0(\xi) e^{k_0 t} + \frac{c_1 (e^{k_0 t} - 1)}{k_0}, \end{aligned} \quad (4.4.9)$$

along the characteristics

$$x = \xi + \left(\frac{u_0(\xi)}{k_0} + \frac{c_1}{k_0^2} \right) (e^{k_0 t} - 1) - \frac{c_1 t}{k_0} - \frac{2\delta \sqrt{C\gamma} (\rho_0(\xi))^{\frac{\gamma-1}{2}}}{k_0(\gamma-1)} \left(e^{-\frac{k_0(\gamma-1)t}{2}} - 1 \right), \quad (4.4.10)$$

subject to the condition

$$\frac{du_0(x)}{dx} = \delta \sqrt{C\gamma} \rho_0(x)^{\frac{\gamma-3}{2}} \frac{d\rho_0(x)}{dx} + k_0. \quad (4.4.11)$$

4.4.2 Case-2

Let $k_0 = 0$. Then, the solution to the system (4.2.1) with the initial data (4.4.1) is determined from (4.4.7), (4.4.8) by taking the limit $k_0 \rightarrow 0$ as

$$\begin{aligned} \rho(x, t) &= \frac{\rho_0(\xi)}{1 + c_0 \rho_0(\xi) t}, \\ u(x, t) &= u_0(\xi) + c_1 t, \end{aligned} \quad (4.4.12)$$

along the characteristics

$$x = \xi + u_0(\xi)t + \frac{c_1 t^2}{2} + \frac{2\delta\sqrt{C\gamma\rho_0(\xi)^{\gamma-3}}}{c_0(3-\gamma)} \left\{ (1 + c_0\rho_0(\xi)t)^{(3-\gamma)/2} - 1 \right\}, \quad (4.4.13)$$

subject to the condition

$$\frac{du_0(x)}{dx} = \delta\sqrt{C\gamma\rho_0(x)^{\frac{\gamma-3}{2}}} \frac{d\rho_0(x)}{dx} + c_0\rho_0. \quad (4.4.14)$$

4.4.3 Case-3

Similarly, when $c_0 = k_0 = 0$, the solution to the system (4.2.1) with the initial data (4.4.1) is of the form

$$\begin{aligned} \rho(x, t) &= \rho_0(\xi), \\ u(x, t) &= u_0(\xi) + c_1 t, \end{aligned} \quad (4.4.15)$$

along the characteristics

$$x = \xi + \left(u_0(\xi) + \delta\sqrt{C\gamma\rho_0(\xi)^{\frac{\gamma-1}{2}}} \right) t + \frac{c_1 t^2}{t}, \quad (4.4.16)$$

subject to the condition

$$\frac{du_0(x)}{dx} = \delta\sqrt{C\gamma\rho_0(x)^{\frac{\gamma-3}{2}}} \frac{d\rho_0(x)}{dx}. \quad (4.4.17)$$

4.5 Shock waves

Let $x = X(t)$ be a curve, with the speed $\frac{dX}{dt} = \sigma$, across which ρ and u are discontinuous. Then using R-H conditions for the equations (4.2.1) we have

$$\begin{aligned} \sigma(\rho_r(t) - \rho_\ell(t)) &= (\rho_r(t)u_r(t) - \rho_\ell(t)u_\ell(t)), \\ \sigma(\rho_r(t)u_r(t) - \rho_\ell(t)u_\ell(t)) &= (\rho_r(t)(u_r(t))^2 - \rho_\ell(t)(u_\ell(t))^2) + (p(\rho_r(t)) - p(\rho_\ell(t))), \end{aligned} \quad (4.5.1)$$

where $\rho_r(t)$ and $u_r(t)$ (respectively; $\rho_\ell(t)$ and $u_\ell(t)$) are right hand side (respectively; left hand side) limits of $\rho(x, t)$ and $u(x, t)$ as $x \rightarrow X(t)$. The equations (4.5.1) can be solved for σ and $u_r(t)$ as

$$\begin{aligned}\sigma &= u_\ell(t) \pm \sqrt{\frac{(p(\rho_r(t)) - p(\rho_\ell(t))) \rho_r(t)}{(\rho_r(t) - \rho_\ell(t)) \rho_\ell(t)}}, \\ u_r(t) &= u_\ell(t) \pm (\rho_r(t) - \rho_\ell(t)) \sqrt{\frac{(p(\rho_r(t)) - p(\rho_\ell(t)))}{(\rho_r(t) - \rho_\ell(t)) \rho_r(t) \rho_\ell(t)}}.\end{aligned}\tag{4.5.2}$$

The curve $x = S_1(t)$ is said to be a 1-shock curve along which the flow variables evaluated in the left and right states must satisfy Lax conditions

$$u_r(t) - \sqrt{p'(\rho_r(t))} < \sigma < u_\ell(t) - \sqrt{p'(\rho_\ell(t))},\tag{4.5.3}$$

i.e., $\sigma - u_\ell(t) < -\sqrt{p'(\rho_\ell(t))} < 0$ due to which we have

$$\begin{aligned}\sigma &= u_\ell(t) - \sqrt{\frac{(p(\rho_r(t)) - p(\rho_\ell(t))) \rho_r(t)}{(\rho_r(t) - \rho_\ell(t)) \rho_\ell(t)}}, \\ u_r(t) &= u_\ell(t) - (\rho_r(t) - \rho_\ell(t)) \sqrt{\frac{(p(\rho_r(t)) - p(\rho_\ell(t)))}{(\rho_r(t) - \rho_\ell(t)) \rho_r(t) \rho_\ell(t)}}.\end{aligned}\tag{4.5.4}$$

Further, in view of the equations (4.5.4), the Lax conditions (4.5.3) imply that

$$\rho_\ell(t) \sqrt{p'(\rho_\ell(t))} < \rho_r(t) \sqrt{p'(\rho_r(t))},\tag{4.5.5}$$

However, when $p = C\rho^\gamma$ equation (4.5.5) reduces to $\rho_\ell^{\frac{\gamma+1}{2}} < \rho_r^{\frac{\gamma+1}{2}}$ which implies that $\rho_\ell < \rho_r$ when $\gamma > -1$ and $\rho_\ell > \rho_r$ when $\gamma < -1$. In view of (4.5.4), it implies that $u_r(t) < u_\ell(t)$ when $\gamma > -1$ and $u_r(t) > u_\ell(t)$ when $\gamma < -1$.

Similarly, the curve $x = S_2(t)$ is said to be a 2-shock curve along which the flow variables evaluated in the left and right states must satisfy Lax conditions

$$u_r(t) + \sqrt{p'(\rho_r(t))} < \sigma < u_\ell(t) + \sqrt{p'(\rho_\ell(t))}.\tag{4.5.6}$$

Rewriting the equations (4.5.2) as

$$\begin{aligned}\sigma &= u_r(t) \pm \sqrt{\frac{(p(\rho_r(t)) - p(\rho_\ell(t))) \rho_\ell(t)}{(\rho_r(t) - \rho_\ell(t)) \rho_r(t)}}, \\ u_r(t) &= u_\ell(t) \mp \left(1 - \frac{\rho_r(t)}{\rho_\ell(t)}\right) \sqrt{\frac{(p(\rho_r(t)) - p(\rho_\ell(t))) \rho_\ell(t)}{(\rho_r(t) - \rho_\ell(t)) \rho_r(t)}}.\end{aligned}\tag{4.5.7}$$

From (4.5.6), $\sigma - u_r(t) > \sqrt{p'(\rho_r(t))} > 0$, we have

$$\begin{aligned}
\sigma &= u_r(t) + \sqrt{\frac{(p(\rho_r(t)) - p(\rho_\ell(t))) \rho_\ell(t)}{(\rho_r(t) - \rho_\ell(t)) \rho_r(t)}}, \\
u_r(t) &= u_\ell(t) - \left(1 - \frac{\rho_r(t)}{\rho_\ell(t)}\right) \sqrt{\frac{(p(\rho_r(t)) - p(\rho_\ell(t))) \rho_\ell(t)}{(\rho_r(t) - \rho_\ell(t)) \rho_r(t)}},
\end{aligned} \tag{4.5.8}$$

and

$$\rho_\ell(t) \sqrt{p'(\rho_\ell(t))} > \rho_r(t) \sqrt{p'(\rho_r(t))}. \tag{4.5.9}$$

However when $p = C\rho^\gamma$ equation (4.5.9) reduces to $\rho_\ell^{\frac{\gamma+1}{2}} > \rho_r^{\frac{\gamma+1}{2}}$ which implies that $\rho_\ell > \rho_r$ when $\gamma > -1$ and $\rho_\ell < \rho_r$ when $\gamma < -1$. As a consequence, we have from the equation (4.5.8)₂, that $u_r(t) < u_\ell(t)$ when $\gamma > -1$ and $u_r(t) > u_\ell(t)$ when $\gamma < -1$.

The summary of the above results is given in the following table.

1 – shock	$\gamma \in (-1, \infty)$	$u_r(t) < u_\ell(t)$	$\rho_\ell(t) < \rho_r(t)$
	$\gamma \in (-\infty, -1)$	$u_r(t) > u_\ell(t)$	$\rho_\ell(t) > \rho_r(t)$
2 – shock	$\gamma \in (-1, \infty)$	$u_r(t) < u_\ell(t)$	$\rho_\ell(t) > \rho_r(t)$
	$\gamma \in (-\infty, -1)$	$u_r(t) > u_\ell(t)$	$\rho_\ell(t) < \rho_r(t)$

Table 4.1: Shock waves

4.6 Rarefaction waves

In this section, when $p = C\rho^\gamma$ and $C\gamma > 0$, we derive continuous solutions in the rarefaction wave regions of the system (4.2.1) corresponding to each eigen mode $\lambda^{(1)}$ and $\lambda^{(2)}$. Let $x = R_1(t)$ and $x = R_2(t)$ be the curves that pass through (0,0) such that $R_1(t) \leq R_2(t)$ for all $t \geq 0$ with $R_1(t)$ and $R_2(t)$ satisfying

$$\begin{aligned}
\frac{dR_1}{dt} &= u(R_1(t), t) - \sqrt{C\gamma} (\rho(R_1(t), t))^{\frac{\gamma-1}{2}}, \\
\frac{dR_2}{dt} &= u(R_2(t), t) - \sqrt{C\gamma} (\rho(R_2(t), t))^{\frac{\gamma-1}{2}},
\end{aligned} \tag{4.6.1}$$

i.e., $\frac{dR_1}{dt} < \frac{dR_2}{dt}$ for $t \geq 0$. The region $R_1(t) \leq x \leq R_2(t)$ is referred to as 1- rarefaction region in which $\rho(x, t)$ and $u(x, t)$ are continuous and differentiable and however, the derivatives of $\rho(x, t)$ and $u(x, t)$ are discontinuous across the curves $x = R_1(t)$ and

$x = R_2(t)$.

Similarly, suppose there exist two curves $x = R_3(t)$ and $x = R_4(t)$ both passing through $(0, 0)$ such that $R_3(t) \leq R_4(t)$ for all $t \geq 0$ satisfying

$$\begin{aligned}\frac{dR_3}{dt} &= u(R_3(t), t) + \sqrt{C\gamma} (\rho(R_3(t), t))^{\frac{\gamma-1}{2}}, \\ \frac{dR_4}{dt} &= u(R_4(t), t) + \sqrt{C\gamma} (\rho(R_4(t), t))^{\frac{\gamma-1}{2}},\end{aligned}\tag{4.6.2}$$

i.e, $\frac{dR_3}{dt} < \frac{dR_4}{dt}$ for $t \geq 0$. Then the region $R_3(t) \leq x \leq R_4(t)$ is referred to as 2- rarefaction region in which $\rho(x, t)$ and $u(x, t)$ are continuous and differentiable. However, the derivatives of $\rho(x, t)$ and $u(x, t)$ are discontinuous across the curves $x = R_3(t)$ and $x = R_4(t)$.

In view of (4.2.6) and (4.4.2), the Riemann invariant $W^{(1)}$ (respectively; $W^{(2)}$) satisfies $\frac{\partial W^{(1)}}{\partial x} = Q$ (respectively; $\frac{\partial W^{(2)}}{\partial x} = Q$) in the 1- rarefaction region (respectively; 2-rarefaction region). Further, we have

$$u = \frac{(W^{(1)} + W^{(2)})}{2}, \quad \rho = \left(\frac{(\gamma - 1)(W^{(1)} - W^{(2)})}{4\sqrt{C\gamma}} \right)^{\frac{2}{\gamma-1}}.\tag{4.6.3}$$

Thus, in view of (4.4.4), (4.4.5), and (4.6.3), the system (4.2.1) can be written in terms of Riemann invariants $W^{(1)}$ and $W^{(2)}$ as

$$\begin{aligned}\frac{dW^{(1)}}{dt} &= \frac{k_0(3-\gamma)}{4}W^{(1)} + \frac{k_0(\gamma+1)}{4}W^{(2)} - c_0\sqrt{C\gamma} \left(\frac{(\gamma-1)(W^{(1)}-W^{(2)})}{4\sqrt{C\gamma}} \right)^{\frac{\gamma+1}{\gamma-1}} + c_1, \\ \frac{dW^{(2)}}{dt} &= \frac{k_0(\gamma+1)}{4}W^{(1)} + \frac{k_0(3-\gamma)}{4}W^{(2)} + c_0\sqrt{C\gamma} \left(\frac{(\gamma-1)(W^{(1)}-W^{(2)})}{4\sqrt{C\gamma}} \right)^{\frac{\gamma+1}{\gamma-1}} + c_1,\end{aligned}\tag{4.6.4}$$

along the characteristics $\frac{dx}{dt} = \lambda^{(1)}$ and $\frac{dx}{dt} = \lambda^{(2)}$.

The solution in the 1-rarefaction region (respectively; 2-rarefaction region) can be obtained by solving the equations (4.6.4) for $W^{(1)}$ and $W^{(2)}$ along the characteristic curves $\frac{dx}{dt} = \lambda^{(1)}$ (respectively; $\frac{dx}{dt} = \lambda^{(2)}$) subject to the conditions $W^{(1)} = \alpha_1$ and $W^{(2)} = \alpha_2$ at $t = 0$ where α_1 and α_2 are arbitrary constants.

4.6.1 Case-1

Let $c_0 = 0$. Then, the differential constraint equation (4.4.2) for the initial data (4.4.1) reduces to (4.4.11) such that $(\rho_L, u_L) \neq (\rho_R, u_R)$ where $(\rho_L, u_L) = \lim_{x \rightarrow 0^-} (\rho_0(x), u_0(x))$ and $(\rho_R, u_R) = \lim_{x \rightarrow 0^+} (\rho_0(x), u_0(x))$. The 1-rarefaction region (respectively; 2-rarefaction region) is bounded by $R_1(t) \leq x \leq R_2(t)$ (respectively; $R_3(t) \leq x \leq R_4(t)$) where $x = R_1(t)$ and $x = R_2(t)$ (respectively; $x = R_3(t)$ and $x = R_4(t)$) are solutions of the equations (4.6.1) (respectively; (4.6.2)) subject to $(\rho, u) = (\rho_L, u_L)$ and $(\rho, u) = (\rho_R, u_R)$ at $t = 0$ respectively. Here,

$$\begin{aligned} R_1(t) &= \left(\frac{u_L}{k_0} + \frac{c_1}{k_0^2} \right) (e^{k_0 t} - 1) - \frac{c_1 t}{k_0} + \frac{2\sqrt{C}\gamma}{k_0(\gamma-1)} \sqrt{(\rho_L)^{(\gamma-1)}} \left(e^{-\frac{k_0(\gamma-1)t}{2}} - 1 \right), \\ R_2(t) &= \left(\frac{u_R}{k_0} + \frac{c_1}{k_0^2} \right) (e^{k_0 t} - 1) - \frac{c_1 t}{k_0} + \frac{2\sqrt{C}\gamma}{k_0(\gamma-1)} \sqrt{(\rho_R)^{(\gamma-1)}} \left(e^{-\frac{k_0(\gamma-1)t}{2}} - 1 \right), \\ R_3(t) &= \left(\frac{u_L}{k_0} + \frac{c_1}{k_0^2} \right) (e^{k_0 t} - 1) - \frac{c_1 t}{k_0} - \frac{2}{k_0(\gamma-1)} \sqrt{C\gamma(\rho_L)^{(\gamma-1)}} \left(e^{-\frac{k_0(\gamma-1)t}{2}} - 1 \right), \\ R_4(t) &= \left(\frac{u_R}{k_0} + \frac{c_1}{k_0^2} \right) (e^{k_0 t} - 1) - \frac{c_1 t}{k_0} - \frac{2}{k_0(\gamma-1)} \sqrt{C\gamma(\rho_R)^{(\gamma-1)}} \left(e^{-\frac{k_0(\gamma-1)t}{2}} - 1 \right). \end{aligned} \quad (4.6.5)$$

In the 1- rarefaction region (respectively; 2-rarefaction region), let $(\rho, u) = (R, \eta)$ at $t = 0$ along the characteristic curves $\frac{dx}{dt} = \lambda^{(1)}$ (respectively; $\frac{dx}{dt} = \lambda^{(2)}$), and thus we have $\alpha_1 = \eta + \frac{2\sqrt{C}\gamma}{\gamma-1} R^{\frac{\gamma-1}{2}}$ and $\alpha_2 = \eta - \frac{2\sqrt{C}\gamma}{\gamma-1} R^{\frac{\gamma-1}{2}}$.

Thus, when $c_0 = 0$, the equations (4.6.4) are integrated to get

$$\begin{aligned} W^{(1)}(x, t) &= \eta e^{k_0 t} + \frac{2\sqrt{C}\gamma}{\gamma-1} R^{\frac{\gamma-1}{2}} e^{\frac{(1-\gamma)k_0 t}{2}} + \frac{c_1}{k_0} (e^{k_0 t} - 1), \\ W^{(2)}(x, t) &= \eta e^{k_0 t} - \frac{2\sqrt{C}\gamma}{\gamma-1} R^{\frac{\gamma-1}{2}} e^{\frac{(1-\gamma)k_0 t}{2}} + \frac{c_1}{k_0} (e^{k_0 t} - 1). \end{aligned} \quad (4.6.6)$$

In view of equations (4.6.3) and (4.6.6), the characteristics satisfying $\frac{dx}{dt} = \lambda^{(1)}$ in 1-rarefaction region are given by

$$x = \left(\frac{\eta}{k_0} + \frac{c_1}{k_0^2} \right) (e^{k_0 t} - 1) - \frac{c_1 t}{k_0} + \frac{2\sqrt{C}\gamma R^{(\gamma-1)}}{k_0(\gamma-1)} \left(e^{-\frac{k_0(\gamma-1)t}{2}} - 1 \right). \quad (4.6.7)$$

Since $\frac{\partial W^{(1)}}{\partial x} = Q$ holds in the 1-rarefaction region, in view of equations (4.6.3) and

(4.6.6), at $t = 0$, we have $u = \eta$, $\rho = R$ and $W^{(1)} = \eta + \frac{2\sqrt{C\gamma}}{\gamma-1}R^{\frac{\gamma-1}{2}}$ along the characteristic curve $\frac{dx}{dt} = \lambda^{(1)}$. In particular, along the left most (respectively; right most) characteristic curve of the 1-rarefaction region, $\eta = u_L$ and $R = \rho_L$ (respectively; $\eta = u_R$ and $R = \rho_R$) and hence we have

$$\eta + \frac{2\sqrt{C\gamma}}{\gamma-1}R^{\frac{\gamma-1}{2}} = u_L + \frac{2\sqrt{C\gamma}}{\gamma-1}(\rho_L)^{\frac{\gamma-1}{2}} = u_R + \frac{2\sqrt{C\gamma}}{\gamma-1}(\rho_R)^{\frac{\gamma-1}{2}}. \quad (4.6.8)$$

Similarly, the characteristics satisfying $\frac{dx}{dt} = \lambda^{(2)}$ in 2-rarefaction region are given by

$$x = \left(\frac{\eta}{k_0} + \frac{c_1}{k_0^2} \right) (e^{k_0 t} - 1) - \frac{c_1 t}{k_0} - \frac{2\sqrt{C\gamma R^{(\gamma-1)}}}{k_0(\gamma-1)} \left(e^{-\frac{k_0(\gamma-1)t}{2}} - 1 \right), \quad (4.6.9)$$

and since $\frac{\partial W^{(2)}}{\partial x} = Q$ holds in the 2-rarefaction region we have,

$$\eta - \frac{2\sqrt{C\gamma}}{\gamma-1}R^{\frac{\gamma-1}{2}} = u_L - \frac{2\sqrt{C\gamma}}{\gamma-1}(\rho_L)^{\frac{\gamma-1}{2}} = u_R - \frac{2\sqrt{C\gamma}}{\gamma-1}(\rho_R)^{\frac{\gamma-1}{2}}. \quad (4.6.10)$$

Thus, the solution with a 1-rarefaction wave can be written as

$$\rho(x, t) = \begin{cases} \rho_0(\xi)e^{-k_0 t}, & x < R_1(t) \text{ and } x > R_2(t), \\ R e^{-k_0 t}, & R_1(t) \leq x \leq R_2(t), \end{cases} \quad (4.6.11)$$

$$u(x, t) = \begin{cases} u_0(\xi)e^{k_0 t} + \frac{c_1(e^{k_0 t} - 1)}{k_0}, & x < R_1(t) \text{ and } x > R_2(t), \\ \eta e^{k_0 t} + \frac{c_1(e^{k_0 t} - 1)}{k_0}, & R_1(t) \leq x \leq R_2(t), \end{cases} \quad (4.6.12)$$

where, for a given x and t , ξ is to be determined from equation (4.4.10), when $\delta = -1$, and η is to be determined from (4.6.7) and (4.6.8). Since, $\frac{dR_1}{dt} < \frac{dR_2}{dt}$ for $t \geq 0$, in particular at $t = 0$ we have

$$u_L - \sqrt{C\gamma\rho_L^{\gamma-1}} < u_R - \sqrt{C\gamma\rho_R^{\gamma-1}},$$

which in turn, in view of (4.6.8), can be written as

$$\frac{\sqrt{C\gamma}(\gamma+1)}{(\gamma-1)} \left(\sqrt{\rho_L^{\gamma-1}} - \sqrt{\rho_R^{\gamma-1}} \right) > 0. \quad (4.6.13)$$

Hence, in view of (4.6.8) and (4.6.13), for a 1- rarefaction wave, we have $\rho_R < \rho_L$, $u_R > u_L$, when $\gamma > -1$, and $\rho_R > \rho_L$, $u_R < u_L$ when $\gamma < -1$.

Similarly, the solution with a 2-rarefaction wave can be written as

$$\rho(x, t) = \begin{cases} \rho_0(\xi)e^{-k_0 t}, & x < R_3(t) \text{ and } x > R_4(t), \\ R e^{-k_0 t}, & R_3(t) \leq x \leq R_4(t), \end{cases} \quad (4.6.14)$$

$$u(x, t) = \begin{cases} u_0(\xi)e^{k_0 t} + \frac{c_1(e^{k_0 t}-1)}{k_0}, & x < R_3(t) \text{ and } x > R_4(t), \\ \eta e^{k_0 t} + \frac{c_1(e^{k_0 t}-1)}{k_0}, & R_3(t) \leq x \leq R_4(t), \end{cases} \quad (4.6.15)$$

where for a given x and t , ξ and η is to be determined from the equations (4.4.10) when $\delta = 1$, and (4.6.9) subject to the relation between R and η given by the equation (4.6.10). Since, $\frac{dR_3}{dt} < \frac{dR_4}{dt}$ for $t \geq 0$. In particular when $t = 0$ we have

$$u_L + \sqrt{C\gamma\rho_L^{\gamma-1}} < u_R + \sqrt{C\gamma\rho_R^{\gamma-1}},$$

which in turn, in view of (4.6.10), can be written as

$$\frac{\sqrt{C\gamma}(\gamma+1)}{(\gamma-1)} \left(\sqrt{\rho_L^{\gamma-1}} - \sqrt{\rho_R^{\gamma-1}} \right) < 0. \quad (4.6.16)$$

Hence, in view of (4.6.10) and (4.6.16), for a 2- rarefaction wave, we have $\rho_R > \rho_L$, $u_R < u_L$ when $\gamma > -1$ and $\rho_R < \rho_L$, $u_R > u_L$ when $\gamma < -1$.

4.6.2 Case-2

Let $k_0 = 0$. Then, the differential constraint equation (4.4.2) for the initial data (4.4.1) reduces to (4.4.14) such that $(\rho_L, u_L) \neq (\rho_R, u_R)$ where $(\rho_L, u_L) = \lim_{x \rightarrow 0^-} (\rho_0(x), u_0(x))$ and $(\rho_R, u_R) = \lim_{x \rightarrow 0^+} (\rho_0(x), u_0(x))$.

Following a similar procedure used in Case-1 of the current section, the solution with a 1-rarefaction wave obtained from (4.4.12) for the governing system (4.2.1) along with (4.4.1), given by

$$\begin{aligned}\rho(x, t) &= \begin{cases} \frac{\rho_0(\xi)}{1+c_0\rho_0(\xi)t}, & x < R_1(t) \text{ and } x > R_2(t), \\ \frac{R}{1+c_0Rt}, & R_1(t) \leq x \leq R_2(t), \end{cases} \\ u(x, t) &= \begin{cases} u_0(\xi) + c_1t, & x < R_1(t) \text{ and } x > R_2(t), \\ \eta + c_1t, & R_1(t) \leq x \leq R_2(t), \end{cases}\end{aligned}\quad (4.6.17)$$

where

$$R_1(t) = u_L t + c_1 t^2 / 2 - \frac{2\sqrt{C\gamma\rho_L^{\gamma-3}}}{c_0(3-\gamma)} \left\{ (1 + c_0\rho_L t)^{(3-\gamma)/2} - 1 \right\}, \quad (4.6.18)$$

$$R_2(t) = u_R t + c_1 t^2 / 2 - \frac{2\sqrt{C\gamma\rho_R^{\gamma-3}}}{c_0(3-\gamma)} \left\{ (1 + c_0\rho_R t)^{(3-\gamma)/2} - 1 \right\}, \quad (4.6.19)$$

$$R = \left\{ \frac{-(\gamma-1)}{2\sqrt{C\gamma}} (\eta - u_L) + \rho_L^{\frac{(\gamma-1)}{2}} \right\}^{\frac{2}{(\gamma-1)}}, \quad (4.6.20)$$

$$u_R = u_L - \frac{2\sqrt{C\gamma}}{(\gamma-1)} \left(\rho_R^{\frac{(\gamma-1)}{2}} - \rho_L^{\frac{(\gamma-1)}{2}} \right), \quad (4.6.21)$$

such that $\rho_R < \rho_L$, $u_R > u_L$ when $\gamma > -1$ and $\rho_R > \rho_L$, $u_R < u_L$ when $\gamma < -1$. Further, for a given x and t , the values of ξ and η are to be determined from the implicit equations

$$x = u_0(\xi)t + c_1 t^2 / 2 - \frac{2\sqrt{C\gamma\rho_0(\xi)^{\gamma-3}}}{c_0(3-\gamma)} \left\{ (1 + c_0\rho_0(\xi)t)^{(3-\gamma)/2} - 1 \right\}, \quad (4.6.22)$$

$$x = \eta t + c_1 t^2 / 2 - \frac{2\sqrt{C\gamma(R)^{\gamma-3}}}{c_0(3-\gamma)} \left\{ (1 + c_0Rt)^{(3-\gamma)/2} - 1 \right\}.$$

Similarly, the solution with a 2-rarefaction wave is written as

$$\begin{aligned}\rho(x, t) &= \begin{cases} \frac{\rho_0(\xi)}{1+c_0\rho_0(\xi)t}, & x < R_3(t) \text{ and } x > R_4(t), \\ \frac{R}{1+c_0Rt}, & R_3(t) \leq x \leq R_4(t) \end{cases} \\ u(x, t) &= \begin{cases} u_0(\xi) + c_1t, & x < R_3(t) \text{ and } x > R_4(t), \\ \eta + c_1t, & R_3(t) \leq x \leq R_4(t), \end{cases}\end{aligned}\quad (4.6.23)$$

where

$$\begin{aligned}
R &= \left\{ \frac{(\gamma-1)}{2\sqrt{C\gamma}} (\eta - u_R) + \rho_R^{\frac{(\gamma-1)}{2}} \right\}^{\frac{2}{(\gamma-1)}}, \\
u_L &= u_R + \frac{2\sqrt{C\gamma}}{(\gamma-1)} \left(\rho_L^{\frac{(\gamma-1)}{2}} - \rho_R^{\frac{(\gamma-1)}{2}} \right), \\
R_3(t) &= u_L t + c_1 t^2/2 + \frac{2\sqrt{C\gamma\rho_L^{\gamma-3}}}{c_0(3-\gamma)} \{ (1 + c_0 \rho_L t)^{(3-\gamma)/2} - 1 \}, \\
R_4(t) &= u_R t + c_1 t^2/2 + \frac{2\sqrt{C\gamma\rho_R^{\gamma-3}}}{c_0(3-\gamma)} \{ (1 + c_0 \rho_R t)^{(3-\gamma)/2} - 1 \},
\end{aligned} \tag{4.6.24}$$

such that $\rho_R > \rho_L$, $u_R > u_L$ when $\gamma > -1$ and $\rho_R < \rho_L$, $u_R < u_L$ when $\gamma < -1$. Further, for a given x and t , the values of ξ and η are to be determined from the implicit equations

$$x = u_0(\xi)t + c_1 t^2/2 + \frac{2\sqrt{C\gamma\rho_0(\xi)^{\gamma-3}}}{c_0(3-\gamma)} \{ (1 + c_0 \rho_0(\xi)t)^{(3-\gamma)/2} - 1 \}, \tag{4.6.25}$$

$$x = \eta t + c_1 t^2/2 + \frac{2\sqrt{C\gamma(R)^{\gamma-3}}}{c_0(3-\gamma)} \{ (1 + c_0 R t)^{(3-\gamma)/2} - 1 \}. \tag{4.6.26}$$

4.6.3 Case-3

Let $p(\rho) = C\rho^\gamma$, for $\gamma \in (-\infty, \infty)$ and $C\gamma > 0$. Also, let $c_0 = k_0 = 0$. Then, the differential constraint equation (4.4.2) for the initial data (4.4.1) reduces to (4.4.17) such that $(\rho_L, u_L) \neq (\rho_R, u_R)$ where $(\rho_L, u_L) = \lim_{x \rightarrow 0^-} (\rho_0(x), u_0(x))$ and $(\rho_R, u_R) = \lim_{x \rightarrow 0^+} (\rho_0(x), u_0(x))$.

Following a similar procedure used in Case-1 of the current section, the solution with a 1-rarefaction wave is obtained, from the solution (4.4.15) for the governing system (4.2.1) along with (4.4.1), given by

$$\rho(x, t) = \begin{cases} \rho_0(\xi), & x < R_1(t) \text{ and } x > R_2(t), \\ R, & R_1(t) \leq x \leq R_2(t), \end{cases} \tag{4.6.27}$$

$$u(x, t) = \begin{cases} u_0(\xi) + c_1 t, & x < R_1(t) \text{ and } x > R_2(t), \\ \eta + c_1 t & R_1(t) \leq x \leq R_2(t), \end{cases} \tag{4.6.28}$$

where

$$\begin{aligned}
R_1(t) &= \left(u_L - \sqrt{C\gamma} \rho_L^{\frac{(\gamma-1)}{2}} \right) t + \frac{c_1 t^2}{2}, \\
R_2(t) &= \left(u_R - \sqrt{C\gamma} \rho_R^{\frac{(\gamma-1)}{2}} \right) t + \frac{c_1 t^2}{2}, \\
R &= \left\{ -\frac{(\gamma-1)}{2\sqrt{C\gamma}} (\eta - u_L) + \rho_L^{\frac{(\gamma-1)}{2}} \right\}^{\frac{\gamma-1}{2}}, \\
u_R &= u_L - \frac{2\sqrt{C\gamma}}{(\gamma-1)} \left(\rho_R^{\frac{(\gamma-1)}{2}} - \rho_L^{\frac{(\gamma-1)}{2}} \right).
\end{aligned} \tag{4.6.29}$$

such that $\rho_R < \rho_L$, $u_R > u_L$ when $\gamma > -1$ and $\rho_R > \rho_L$, $u_R < u_L$ when $\gamma < -1$. Further, for a given x and t , the values of ξ and η are to be determined from the implicit equations

$$x = \xi + \left(u_0(\xi) - \sqrt{C\gamma} \rho_0(\xi)^{\frac{(\gamma-1)}{2}} \right) t + \frac{c_1 t^2}{2}, \tag{4.6.30}$$

$$x = \left(\eta - \sqrt{C\gamma} \rho^{\frac{(\gamma-1)}{2}} \right) t + \frac{c_1 t^2}{2}. \tag{4.6.31}$$

Similarly, the solution with a 2-rarefaction wave is obtained as

$$\rho(x, t) = \begin{cases} \rho_0(\xi), & x < R_3(t) \text{ and } x > R_4(t), \\ R, & R_3(t) \leq x \leq R_4(t), \end{cases} \tag{4.6.32}$$

$$u(x, t) = \begin{cases} u_0(\xi) + c_1 t, & x < R_2(t) \text{ and } x > R_4(t), \\ \eta + c_1 t, & R_3(t) \leq x \leq R_4(t), \end{cases} \tag{4.6.33}$$

where

$$\begin{aligned}
R &= \left\{ \frac{(\gamma-1)}{2\sqrt{C\gamma}} (\eta - u_R) + \rho_R^{\frac{(\gamma-1)}{2}} \right\}^{\frac{\gamma-1}{2}}, \\
u_L &= u_R + \frac{2\sqrt{C\gamma}}{(\gamma-1)} \left(\rho_L^{\frac{(\gamma-1)}{2}} - \rho_R^{\frac{(\gamma-1)}{2}} \right), \\
R_3(t) &= \left(u_L + \sqrt{C\gamma} \rho_L^{\frac{(\gamma-1)}{2}} \right) t + \frac{c_1 t^2}{2}, \\
R_4(t) &= \left(u_R + \sqrt{C\gamma} \rho_R^{\frac{(\gamma-1)}{2}} \right) t + \frac{c_1 t^2}{2},
\end{aligned} \tag{4.6.34}$$

such that $\rho_R > \rho_L$, $u_R < u_L$ when $\gamma > -1$ and $\rho_R < \rho_L$, $u_R > u_L$ when $\gamma < -1$.

Further, for a given x and t , the values of ξ and η are to be determined from the implicit equations

$$x = \xi + \left(u_0(\xi) + \sqrt{C\gamma\rho_0(\xi)}^{\frac{(\gamma-1)}{2}} \right) t + \frac{c_1 t^2}{2}, \quad (4.6.35)$$

$$x = \left(\eta + \sqrt{C\gamma R}^{\frac{(\gamma-1)}{2}} \right) t + \frac{c_1 t^2}{2}. \quad (4.6.36)$$

The inequalities obtained for rarefaction wave regions are given in the following table.

1 – rarefaction waves	$\gamma \in (-1, \infty)$	$u_R > u_L$	$\rho_R < \rho_L$
	$\gamma \in (-\infty, -1)$	$u_R < u_L$	$\rho_R > \rho_L$
2 – rarefaction waves	$\gamma \in (-1, \infty)$	$u_R > u_L$	$\rho_R > \rho_L$
	$\gamma \in (-\infty, -1)$	$u_R < u_L$	$\rho_R < \rho_L$

Table 4.2: Rarefaction waves

4.7 Riemann problem

Consider the initial profile

$$(\rho(x, 0), u(x, 0)) = (\rho_0(x), u_0(x)) = \begin{cases} (\rho_L, u_L), & \text{if } x < 0, \\ (\rho_R, u_R), & \text{if } x \geq 0, \end{cases} \quad (4.7.1)$$

where ρ_L, ρ_R, u_L and u_R are constants.

In this section, we construct the solution for the Riemann problem to the system of equations (4.2.1) when $f \equiv c_1$ subject to the initial data (4.7.1) consisting of shock and/or rarefaction waves.

For this, if 1-wave is a shock wave, in view of the equations (4.4.15) we have

$$\rho(x, t) = \begin{cases} \rho_L, & \text{if } x \leq S_1(t), \\ \bar{\rho}, & \text{if } x > S_1(t), \end{cases} \quad (4.7.2)$$

$$u(x, t) = \begin{cases} u_L + c_1 t, & \text{if } x \leq S_1(t), \\ \bar{u} + c_1 t, & \text{if } x > S_1(t), \end{cases} \quad (4.7.3)$$

where $x = S_1(t)$ is the 1-shock curve determined from the equation (4.5.4)₁, given as

$$S_1(t) = \left(u_L - \sqrt{\frac{C(\bar{\rho}^\gamma - \rho_L^\gamma)\bar{\rho}}{(\bar{\rho} - \rho_L)\rho_L}} \right) t + \frac{c_1 t^2}{2}.$$

Since, $u_\ell(t) = u_L + c_1 t$, $\rho_\ell(t) = \rho_L$, $u_r(t) = \bar{u} + c_1 t$ and $\rho_r(t) = \bar{\rho}$. In view of (4.5.4)₂ and Table-1, we have $\bar{u} = u_L - (\bar{\rho} - \rho_L) \sqrt{\frac{C(\bar{\rho}^\gamma - \rho_L^\gamma)}{(\bar{\rho} - \rho_L)\bar{\rho}\rho_L}}$, $\rho_L < \bar{\rho}$ and $u_L > \bar{u}$.

Similarly, if 2-wave is a shock wave, in view of the equations (4.4.15), we have

$$\rho(x, t) = \begin{cases} \rho_R, & \text{if } x \geq S_2(t), \\ \check{\rho}, & \text{if } x < S_2(t), \end{cases} \quad (4.7.4)$$

$$u(x, t) = \begin{cases} u_R + c_1 t, & \text{if } x \geq S_2(t), \\ \check{u} + c_1 t, & \text{if } x < S_2(t), \end{cases} \quad (4.7.5)$$

where $x = S_2(t)$ is the 2-shock curve determined from the equation (4.5.8)₁, given as

$$S_2(t) = \left(u_R + \sqrt{\frac{C(\rho_R^\gamma - \check{\rho}^\gamma)\check{\rho}}{(\rho_R - \check{\rho})\rho_R}} \right) t + \frac{c_1 t^2}{2}.$$

In view of (4.5.8)₂ and Table-1, we have $\check{u} = u_R - (\rho_R - \check{\rho}) \sqrt{\frac{C(\rho_R^\gamma - \check{\rho}^\gamma)}{(\rho_R - \check{\rho})\rho_R\check{\rho}}}$ with, $\check{\rho} > \rho_R$ and $\check{u} > u_R$.

If 1-wave is a rarefaction wave, in view of (4.6.27), (4.6.28) and Table-2, we have

$$\rho(x, t) = \begin{cases} \rho_L, & \text{if } x < R_1(t), \\ R, & \text{if } R_1(t) \leq x \leq R_2(t), \\ \bar{\rho}, & \text{if } x > R_2(t), \end{cases} \quad (4.7.6)$$

$$u(x, t) = \begin{cases} u_L + c_1 t, & \text{if } x < R_1(t), \\ \eta + c_1 t, & \text{if } R_1(t) \leq x \leq R_2(t), \\ \bar{u} + c_1 t, & \text{if } x > R_2(t), \end{cases} \quad (4.7.7)$$

where $u_L < \eta < \bar{u}$ and

$$\begin{aligned}
x &= \left(\eta - \sqrt{C\gamma} R^{\frac{(\gamma-1)}{2}} \right) t + \frac{c_1 t^2}{2}, \\
R &= \left\{ -\frac{(\gamma-1)}{2\sqrt{C\gamma}} (\eta - u_L) + \rho_L^{\frac{(\gamma-1)}{2}} \right\}^{\frac{2}{\gamma-1}}, \\
R_1(t) &= \left(u_L - \sqrt{C\gamma} \rho_L^{\frac{(\gamma-1)}{2}} \right) t + \frac{c_1 t^2}{2}, \\
R_2(t) &= \left(u_R - \sqrt{C\gamma} \rho_R^{\frac{(\gamma-1)}{2}} \right) t + \frac{c_1 t^2}{2}.
\end{aligned}$$

Similarly, if 2-wave is a rarefaction wave, in view of (4.6.32), (4.6.33) and Table-2, we have

$$\rho(x, t) = \begin{cases} \check{\rho}, & \text{if } x < R_3(t), \\ R, & \text{if } R_3(t) \leq x \leq R_4(t), \\ \rho_R, & \text{if } x > R_4(t), \end{cases} \quad (4.7.8)$$

$$u(x, t) = \begin{cases} \check{u} + c_1 t, & \text{if } x < R_3(t), \\ \eta + c_1 t, & \text{if } R_3(t) \leq x \leq R_4(t), \\ u_R + c_1 t, & \text{if } x > R_4(t), \end{cases} \quad (4.7.9)$$

where $\check{u} < \eta < u_R$ and

$$\begin{aligned}
x &= \left(\eta + \sqrt{C\gamma} R^{\frac{(\gamma-1)}{2}} \right) t + \frac{c_1 t^2}{2}, \\
R &= \left\{ \frac{(\gamma-1)}{2\sqrt{C\gamma}} (\eta - u_L) + \rho_L^{\frac{(\gamma-1)}{2}} \right\}^{\frac{2}{\gamma-1}}, \\
R_3(t) &= \left(u_L + \sqrt{C\gamma} \rho_L^{\frac{(\gamma-1)}{2}} \right) t + \frac{c_1 t^2}{2}, \\
R_4(t) &= \left(u_R + \sqrt{C\gamma} \rho_R^{\frac{(\gamma-1)}{2}} \right) t + \frac{c_1 t^2}{2}.
\end{aligned}$$

We let γ belong to $(-1, \infty)$ such that $C\gamma > 0$ for the following Lemmas 1 to 6 and Theorems 1 to 4.

Lemma 4.7.1. *Let the solution to the Riemann problem for the system (4.2.1) subject to the initial conditions 4.7.1 be given as a 1-shock wave and a 2-shock wave. Then the constants ρ_L , ρ_R , u_L and u_R satisfy the following inequalities*

$$u_L - u_R > |\rho_L - \rho_R| \sqrt{\frac{C(\rho_L^\gamma - \rho_R^\gamma)}{(\rho_L - \rho_R)\rho_L\rho_R}}. \quad (4.7.10)$$

Proof. If 1-wave is a shock wave, $x = S_1(t)$, implies that $\rho_L < \bar{\rho}$ and $\bar{u} < u_L$. Similarly, if 2-wave is a shock wave, $x = S_2(t)$, then $\check{\rho} > \rho_R$ and $\check{u} > u_R$. In the region, $S_1(t) < x < S_2(t)$ the solution given in the equations (4.7.2) and (4.7.3), through 1-shock, and the solution (4.7.4) and (4.7.5), through 2-shock, should coincide, that is, $\bar{\rho} = \check{\rho} = z$ (say), $\bar{u} = \check{u}$, i.e., $u_L > u_R$ and $f_1(z) = 0$ where

$$f_1(z) = u_L - (z - \rho_L) \sqrt{\frac{C(z^\gamma - \rho_L^\gamma)}{(z - \rho_L)z\rho_L}} - u_R - (z - \rho_R) \sqrt{\frac{C(\rho_R^\gamma - z^\gamma)}{(\rho_R - z)\rho_R z}},$$

with $\max\{\rho_L, \rho_R\} < z < \infty$. Observe that $\lim_{z \rightarrow \infty} f_1(z) = -\infty$ and

$$\frac{df_1}{dz} = -\frac{C}{2z^{\frac{3}{2}}} \left(\frac{\gamma z^\gamma(z - \rho_L) + \rho_L(z^\gamma - \rho_L^\gamma)}{\sqrt{C\rho_L(z^\gamma - \rho_L^\gamma)(z - \rho_L)}} + \frac{\gamma z^\gamma(z - \rho_R) + \rho_R(z^\gamma - \rho_R^\gamma)}{\sqrt{C\rho_R(z^\gamma - \rho_R^\gamma)(z - \rho_R)}} \right) < 0,$$

which implies that f_1 is a decreasing function in z .

In order to have a unique solution for $f_1(z) = 0$ in $[\rho_L, \infty)$ (respectively; in $[\rho_R, \infty)$) when $\rho_L > \rho_R$ (respectively; when $\rho_R > \rho_L$), it is required that $f_1(\rho_L) > 0$ (respectively; $f_1(\rho_R) > 0$), i.e.,

$$u_L - u_R - |\rho_L - \rho_R| \sqrt{\frac{C(\rho_L^\gamma - \rho_R^\gamma)}{(\rho_L - \rho_R)\rho_L\rho_R}} > 0. \quad (4.7.11)$$

Hence the proof. This is depicted in Figure-4.1. □

Lemma 4.7.2. *Let the solution to the Riemann problem consist of a 1-shock wave and a 2-rarefaction wave for the system (4.2.1) subject to the initial conditions (4.7.1). Then ρ_L, ρ_R, u_L and u_R satisfy the inequalities $\rho_L < \rho_R$ and*

$$u_L - u_R < \sqrt{\frac{C(\rho_R^\gamma - \rho_L^\gamma)(\rho_R - \rho_L)}{\rho_L \rho_R}}, \quad \text{when } u_L > u_R, \quad (4.7.12)$$

$$u_L - u_R > -\frac{2\sqrt{C}\gamma}{\gamma - 1} \left(\rho_R^{\frac{\gamma-1}{2}} - \rho_L^{\frac{\gamma-1}{2}} \right), \quad \text{when } u_L < u_R. \quad (4.7.13)$$

Proof. If the 1-wave is a shock wave and the 2-wave is a rarefaction wave, it implies that $\rho_L < \bar{\rho}$, $\bar{u} < u_L$ and $\check{\rho} < \rho_R$, $\check{u} < u_R$. In view of (4.7.2), (4.7.3), (4.7.8) and (4.7.9), it follows that the solutions given by (4.7.2)-(4.7.3) and (4.7.8)-(4.7.9) should coincide in the region $S_1(t) < x < R_3(t)$. This means that $\bar{\rho} = \check{\rho} = z$ (say), i.e., $\rho_L < z < \rho_R$ and $f_2(z) = 0$ where

$$f_2(z) = u_L - (z - \rho_L) \sqrt{\frac{C(z^\gamma - \rho_L^\gamma)}{(z - \rho_L)z\rho_L}} - u_R + \frac{2\sqrt{C}\gamma}{(\gamma - 1)} \left(\rho_R^{\frac{\gamma-1}{2}} - z^{\frac{\gamma-1}{2}} \right),$$

for $\rho_L < z < \rho_R$. Observe from the above equation that

$$\frac{df_2}{dz} = -\frac{C}{2z^{\frac{3}{2}}} \left(\frac{\gamma z^\gamma (z - \rho_L) + \rho_L (z^\gamma - \rho_L^\gamma)}{\sqrt{C\rho_L(z^\gamma - \rho_L^\gamma)(z - \rho_L)}} \right) - \sqrt{C}\gamma z^{\frac{\gamma-3}{2}} < 0,$$

implying that f_2 is decreasing. Since, $\rho_L < \rho_R$, the equation $f_2(z) = 0$ has a solution only when $f_2(\rho_L) > 0$ and $f_2(\rho_R) < 0$, i.e.,

$$f_2(\rho_L) > 0 \Rightarrow u_L - u_R + \frac{2\sqrt{C}\gamma}{(\gamma - 1)} \left(\rho_R^{\frac{\gamma-1}{2}} - \rho_L^{\frac{\gamma-1}{2}} \right) > 0, \quad (4.7.14)$$

$$f_2(\rho_R) < 0 \Rightarrow u_L - u_R - \sqrt{\frac{C(\rho_R^\gamma - \rho_L^\gamma)(\rho_R - \rho_L)}{\rho_L \rho_R}} < 0. \quad (4.7.15)$$

Since $\rho_L < \rho_R$, and if $u_L > u_R$, the condition (4.7.14) always holds. Thus the solution for $f_2(z) = 0$ over $[\rho_L, \rho_R]$ is possible if (4.7.15) holds, i.e.,

$$u_L - u_R < \sqrt{\frac{C(\rho_R^\gamma - \rho_L^\gamma)(\rho_R - \rho_L)}{\rho_L \rho_R}}.$$

Similarly, if $u_L < u_R$ then the condition (4.7.15) always holds. Thus the solution for $f_2(z) = 0$ over $[\rho_L, \rho_R]$ is possible only if (4.7.14) holds, i.e.,

$$u_L - u_R > -\frac{2\sqrt{C\gamma}}{(\gamma-1)} \left(\rho_R^{\frac{\gamma-1}{2}} - \rho_L^{\frac{\gamma-1}{2}} \right).$$

Hence the proof. This information is depicted in Figure-4.2. $\square$

Lemma 4.7.3. *Let the solution to the Riemann problem consist of a 1-rarefaction wave and a 2-shock wave for the system (4.2.1) subject to the initial conditions (4.7.1). Then ρ_L , ρ_R , u_L and u_R satisfy the inequalities $\rho_L > \rho_R$ and*

$$u_L - u_R > \frac{2\sqrt{C\gamma}}{(\gamma-1)} \left(\rho_R^{\frac{\gamma-1}{2}} - \rho_L^{\frac{\gamma-1}{2}} \right), \quad \text{when } u_L < u_R, \quad (4.7.16)$$

$$u_L - u_R < \sqrt{\frac{C(\rho_R^\gamma - \rho_L^\gamma)(\rho_R - \rho_L)}{\rho_R \rho_L}}, \quad \text{when } u_L > u_R. \quad (4.7.17)$$

Proof. Suppose that the 1-wave is a rarefaction wave. It implies that $\bar{\rho} < \rho_L$, $\bar{u} > u_L$. Similarly, let the 2-wave be a shock wave. Then $\check{\rho} > \rho_R$ and $\check{u} > u_R$. In view of the equations (4.7.4) - (4.7.7), the solutions given by the equations (4.7.4)-(4.7.5) and (4.7.6)-(4.7.7) should coincide in the region $R_2(t) < x < S_2(t)$, i.e., $\bar{\rho} = \check{\rho} = z$ (say), $\bar{u} = \check{u}$, i.e., $\rho_R < z < \rho_L$, and $f_3(z) = 0$ where

$$f_3(z) = u_L - u_R - \frac{2\sqrt{C\gamma}}{(\gamma-1)} \left(z^{\frac{\gamma-1}{2}} - \rho_L^{\frac{\gamma-1}{2}} \right) - (z - \rho_R) \sqrt{\frac{C(\rho_R^\gamma - z^\gamma)}{(\rho_R - z)\rho_R z}},$$

for $\rho_R < z < \rho_L$. Observe from the above equation that

$$\frac{df_3}{dz} = -\sqrt{C\gamma} z^{\frac{\gamma-3}{2}} - \frac{C}{2z^{\frac{3}{2}}} \left(\frac{\gamma z^\gamma (z - \rho_R) + \rho_R (z^\gamma - \rho_R^\gamma)}{\sqrt{C\rho_R(z^\gamma - \rho_R^\gamma)}(z - \rho_R)} \right) < 0,$$

implying thereby that f_3 is decreasing. The equation $f_3(z) = 0$ has a solution only when $f_3(\rho_R) > 0$ and $f_3(\rho_L) < 0$, i.e.,

$$f_3(\rho_L) < 0 \Rightarrow u_L - u_R - \sqrt{\frac{C(\rho_R^\gamma - \rho_L^\gamma)(\rho_R - \rho_L)}{\rho_R \rho_L}} < 0, \quad (4.7.18)$$

$$f_3(\rho_R) > 0 \Rightarrow u_L - u_R - \frac{2\sqrt{C\gamma}}{(\gamma-1)} \left(\rho_R^{\frac{\gamma-1}{2}} - \rho_L^{\frac{\gamma-1}{2}} \right) > 0, \quad (4.7.19)$$

If $u_L < u_R$ then the condition (4.7.18) is always true. Thus, the solution for $f_3(z) = 0$ over $[\rho_R, \rho_L]$ exists only when the condition (4.7.19) is true, i.e.,

$$u_L - u_R - \frac{2\sqrt{C}\gamma}{(\gamma - 1)} \left(\rho_R^{\frac{\gamma-1}{2}} - \rho_L^{\frac{\gamma-1}{2}} \right) > 0. \quad (4.7.20)$$

Further, when $u_L > u_R$, the condition (4.7.19) is always true. Thus, the solution for $f_3(z) = 0$ over $[\rho_R, \rho_L]$ exists only when the condition (4.7.18) is true, i.e.,

$$u_L - u_R - \sqrt{\frac{C(\rho_R^\gamma - \rho_L^\gamma)(\rho_R - \rho_L)}{\rho_R \rho_L}} < 0. \quad (4.7.21)$$

Hence the proof. This is depicted in Figure-4.3. □

Lemma 4.7.4. *Let the solution to the Riemann problem consist of a 1-rarefaction wave and a 2-rarefaction wave for the system (4.2.1) subject to the initial conditions (4.7.1). Then ρ_L , ρ_R , u_L and u_R satisfy the inequalities $u_L < u_R$ and*

$$-\frac{2\sqrt{C}\gamma}{(\gamma - 1)} \left(\rho_R^{\frac{\gamma-1}{2}} + \rho_L^{\frac{\gamma-1}{2}} \right) < u_L - u_R < -\left| \frac{2\sqrt{C}\gamma}{(\gamma - 1)} \left(\rho_R^{\frac{\gamma-1}{2}} - \rho_L^{\frac{\gamma-1}{2}} \right) \right|. \quad (4.7.22)$$

Proof. Suppose that the 1-wave is a rarefaction wave. It implies that $\bar{\rho} < \rho_L$, $\bar{u} > u_L$. Similarly, let the 2-wave be a rarefaction wave. Then $\check{\rho} < \rho_R$ and $\check{u} < u_R$. In view of the equations (4.7.4)-(4.7.7), the solutions given by the equations (4.7.4)-(4.7.5) and (4.7.6)-(4.7.7) should coincide in the region $R_2(t) < x < R_3(t)$, i.e., $\bar{\rho} = \check{\rho} = z$ (say), $u_L < \bar{u} = \check{u} < u_R$ and $f_4(z) = 0$ where

$$f_4(z) = u_L - u_R - \frac{2\sqrt{C}\gamma}{(\gamma - 1)} \left(2z^{\frac{\gamma-1}{2}} - \rho_L^{\frac{\gamma-1}{2}} - \rho_R^{\frac{\gamma-1}{2}} \right),$$

for $0 < z < \min\{\rho_R, \rho_L\}$. Observe from the above equation that

$$\frac{df_4}{dz} = -2\sqrt{C}\gamma z^{\frac{\gamma-3}{2}} < 0,$$

implying thereby that f_4 is decreasing. The equation $f_4(z) = 0$ has a solution only when $f_4(0) > 0$ and $f_4(\rho_L) < 0$ (respectively; $f_4(\rho_R) < 0$) when $\rho_L < \rho_R$ (respectively; $\rho_R < \rho_L$), i.e., ρ_L , ρ_R , u_L and u_R should satisfy

$$u_L - u_R + \frac{2\sqrt{C\gamma}}{(\gamma-1)} \left(\rho_R^{\frac{\gamma-1}{2}} + \rho_L^{\frac{\gamma-1}{2}} \right) > 0, \quad (4.7.23)$$

and

$$u_L - u_R + \frac{2\sqrt{C\gamma}}{(\gamma-1)} \left(\rho_R^{\frac{\gamma-1}{2}} - \rho_L^{\frac{\gamma-1}{2}} \right) < 0, \quad \text{when } \rho_L < \rho_R, \quad (4.7.24)$$

$$u_L - u_R + \frac{2\sqrt{C\gamma}}{(\gamma-1)} \left(\rho_L^{\frac{\gamma-1}{2}} - \rho_R^{\frac{\gamma-1}{2}} \right) < 0, \quad \text{when } \rho_R < \rho_L. \quad (4.7.25)$$

Hence the proof. This is depicted in Figure-4.4. $\square$

Lemma 4.7.5. *Let the solution to the Riemann problem consist of only a 1-rarefaction wave (respectively; a 2-rarefaction wave) for the system (4.2.1) subject to the initial conditions (4.7.1) then $\rho_L > \rho_R$ (respectively; $\rho_L < \rho_R$) and $u_L - u_R = -\frac{\sqrt{C\gamma}}{(\gamma-1)} \left| \rho_L^{\frac{\gamma-1}{2}} - \rho_R^{\frac{\gamma-1}{2}} \right|$.*

Proof. Suppose that the solution is given through a 1-rarefaction wave only. Then in view of (4.7.6)-(4.7.7), we have $\bar{\rho} < \rho_L$, $\check{\rho} = \rho_R$ and $\check{u} = u_R$, i.e., $u_L - \frac{2\sqrt{C\gamma}}{(\gamma-1)} \left(\rho_R^{\frac{\gamma-1}{2}} - \rho_L^{\frac{\gamma-1}{2}} \right) = u_R$.

Similarly, when the solution is given only through a 2-rarefaction wave, in view of (4.7.8) and (4.7.9), it can be easily shown that $\rho_L < \rho_R$ and $u_L + \frac{2\sqrt{C\gamma}}{(\gamma-1)} \left(\rho_R^{\frac{\gamma-1}{2}} - \rho_L^{\frac{\gamma-1}{2}} \right) = u_R$. Hence the proof. $\square$

Lemma 4.7.6. *Let the solution to the Riemann problem consist of only a 1-shock wave (respectively, a 2-shock wave) for the system (4.2.1) subject to the initial conditions (4.7.1) then $\rho_L < \rho_R$ (respectively; $\rho_L > \rho_R$) and $u_L - u_R = |\rho_R - \rho_L| \sqrt{\frac{C(\rho_R^\gamma - \rho_L^\gamma)}{(\rho_R - \rho_L)\rho_R\rho_L}}$.*

Proof. Suppose that the solution is given only through a 1-shock wave. Then in view of (4.7.2) and (4.7.3), we have $\rho_L < \bar{\rho}$, $\check{\rho} = \rho_R$ and $\bar{u} = u_R$, i.e., $u_L - \sqrt{\frac{(\rho_R^\gamma - \rho_L^\gamma)(\rho_R - \rho_L)}{\rho_L\rho_R}} = u_R$.

Similarly, when the solution is given through a 2-wave as a shock wave, in view of (4.7.4) and (4.7.5), it can be proved that $u_L + \sqrt{\frac{(\rho_R^\gamma - \rho_L^\gamma)(\rho_R - \rho_L)}{\rho_L\rho_R}} = u_R$. Hence the proof. $\square$

The following four theorems give the complete characterization of the solution of the Riemann problem under consideration.

Theorem 4.7.1. *Let $u_L > u_R$. If $\rho_L > \rho_R$ (respectively; $\rho_L < \rho_R$), then a 1-shock wave (respectively, a 2-shock wave) is a solution to the Riemann problem for*

the system (4.2.1) subject to the initial conditions (4.7.1) if and only if $(u_L - u_R) \geq |\rho_L - \rho_R| \sqrt{\frac{C(\rho_L^\gamma - \rho_R^\gamma)}{(\rho_L - \rho_R)\rho_L\rho_R}}$.

Theorem 4.7.2. *Let $u_L < u_R$. If $\rho_L > \rho_R$ (respectively; $\rho_L < \rho_R$) then a 1-rarefaction wave (respectively, a 2-rarefaction wave) is a solution to the Riemann problem for the system (4.2.1) subject to the initial conditions (4.7.1) if and only if $(u_L - u_R) \leq |\rho_L - \rho_R| \sqrt{\frac{C(\rho_L^\gamma - \rho_R^\gamma)}{(\rho_L - \rho_R)\rho_L\rho_R}}$.*

Theorem 4.7.3. *Let $u_L > u_R$. If $\rho_L > \rho_R$ (respectively; $\rho_L < \rho_R$), then a 1-shock wave (respectively, a 2-shock wave) is a solution to the Riemann problem for the system (4.2.1) subject to the initial conditions (4.7.1) if and only if $(u_L - u_R) \geq -\frac{2\sqrt{C\gamma}}{\gamma-1} \left| \rho_L^{\frac{\gamma-1}{2}} - \rho_R^{\frac{\gamma-1}{2}} \right|$.*

Theorem 4.7.4. *Let $-\frac{2\sqrt{C\gamma}}{\gamma-1} \left(\rho_L^{\frac{\gamma-1}{2}} + \rho_R^{\frac{\gamma-1}{2}} \right) < (u_L - u_R) < 0$. If $\rho_L > \rho_R$ (respectively; $\rho_L < \rho_R$), then a 1-rarefaction wave (respectively, a 2-rarefaction wave) is a solution to the Riemann problem for the system (4.2.1) subject to the initial conditions (4.7.1) if and only if $(u_L - u_R) \leq -\frac{2\sqrt{C\gamma}}{\gamma-1} \left| \rho_L^{\frac{\gamma-1}{2}} - \rho_R^{\frac{\gamma-1}{2}} \right|$.*

4.8 Riemann problem with non-constant initial data

In this section, we construct the solution for the Riemann problem to the system of equations (4.2.1) when $f = k_0 u$ and $p(\rho) = C\rho$ subject to the initial profile

$$(\rho(x, 0), u(x, 0)) = (\rho_0(x), u_0(x)) = \begin{cases} (\rho_L, k_0 x + u_L), & \text{if } x < 0, \\ (\rho_R, k_0 x + u_R), & \text{if } x \geq 0, \end{cases} \quad (4.8.1)$$

where ρ_L, ρ_R, u_L and u_R are constants, consisting of shock and/or rarefaction waves. Then, the solution is given in the following form.

Let the 1-wave be a shock wave. Then

$$\rho(x, t) = \begin{cases} \rho_L e^{-k_0 t}, & \text{if } x \leq S_1(t), \\ \tilde{\rho} e^{-k_0 t}, & \text{if } x > S_1(t), \end{cases} \quad (4.8.2)$$

$$u(x, t) = \begin{cases} u_L + k_0 x, & \text{if } x \leq S_1(t), \\ \tilde{u} + k_0 x, & \text{if } x > S_1(t), \end{cases} \quad (4.8.3)$$

where $S_1(t) = \left(u_L - \sqrt{\frac{C\tilde{\rho}}{\rho_L}}\right) \left(\frac{e^{k_0 t} - 1}{k_0}\right)$.

In view of (4.5.4)₂ we have $\tilde{u} = u_L - \frac{(\tilde{\rho} - \rho_L)\sqrt{C}}{\sqrt{(\rho_L\tilde{\rho})}}$, $\rho_L < \tilde{\rho}$ and $u_L > \tilde{u}$.

Similarly, if the 2-wave is a shock wave, then

$$\rho(x, t) = \begin{cases} \rho_R e^{-k_0 t}, & \text{if } x \geq S_2(t), \\ \hat{\rho} e^{-k_0 t}, & \text{if } x < S_2(t), \end{cases} \quad (4.8.4)$$

$$u(x, t) = \begin{cases} u_R + k_0 x, & \text{if } x \geq S_2(t), \\ \hat{u} + k_0 x, & \text{if } x < S_2(t), \end{cases} \quad (4.8.5)$$

where $S_2(t) = \left(\hat{u} + \sqrt{\frac{C\hat{\rho}}{\rho_R}}\right) \left(\frac{e^{k_0 t} - 1}{k_0}\right)$.

In view of (4.5.8)₂ we have $\hat{u} = u_R + \frac{(\hat{\rho} - \rho_R)\sqrt{C}}{\sqrt{\rho_R\hat{\rho}}}$ with, $\hat{\rho} > \rho_R$ and $\hat{u} > u_R$.

If the 1-wave is a rarefaction wave, then

$$\rho(x, t) = \begin{cases} \rho_L e^{-k_0 t}, & \text{if } x < R_1(t), \\ \rho_L e^{\left(-k_0 t - \frac{1}{\sqrt{C}}\left(\frac{x k_0}{e^{k_0 t} - 1} + \sqrt{C} - u_L\right)\right)}, & \text{if } R_1(t) \leq x \leq R_2(t), \\ \tilde{\rho} e^{-k_0 t}, & \text{if } x > R_2(t), \end{cases} \quad (4.8.6)$$

$$u(x, t) = \begin{cases} u_L + k_0 x, & \text{if } x < R_1(t), \\ \left(\frac{x k_0 e^{k_0 t}}{e^{k_0 t} - 1} + \sqrt{C}\right), & \text{if } R_1(t) \leq x \leq R_2(t), \\ \tilde{u} + k_0 x, & \text{if } x > R_2(t), \end{cases} \quad (4.8.7)$$

where $\tilde{u} = u_L - \sqrt{C} \log\left(\frac{\tilde{\rho}}{\rho_L}\right)$, $R_1(t) = \left(u_L - \sqrt{C}\right) \left(\frac{e^{k_0 t} - 1}{k_0}\right)$, $R_2(t) = \left(\tilde{u} - \sqrt{C}\right) \left(\frac{e^{k_0 t} - 1}{k_0}\right)$ and $\rho_L > \tilde{\rho}$.

Similarly, if the 2-wave is a rarefaction wave, then

$$\rho(x, t) = \begin{cases} \hat{\rho}e^{-k_0 t}, & \text{if } x < R_3(t), \\ \rho_R e^{\left(-k_0 t + \frac{1}{\sqrt{C}} \left(\frac{x k_0}{e^{k_0 t} - 1} - \sqrt{C} - u_R \right)\right)}, & \text{if } R_3(t) \leq x \leq R_4(t), \\ \rho_R e^{-k_0 t}, & \text{if } x > R_4(t), \end{cases} \quad (4.8.8)$$

$$u(x, t) = \begin{cases} \hat{u} + k_0 x, & \text{if } x < R_3(t), \\ \left(\frac{x k_0 e^{k_0 t}}{e^{k_0 t} - 1} - \sqrt{C} \right), & \text{if } R_3(t) \leq x \leq R_4(t), \\ u_R + k_0 x, & \text{if } x > R_4(t), \end{cases} \quad (4.8.9)$$

where $\hat{u} = u_R - \sqrt{C} \log \left(\frac{\rho_R}{\hat{\rho}} \right)$, $R_3(t) = \left(\hat{u} + \sqrt{C} \right) \left(\frac{e^{k_0 t} - 1}{k_0} \right)$, $R_4(t) = \left(u_R + \sqrt{C} \right) \left(\frac{e^{k_0 t} - 1}{k_0} \right)$ and $\hat{\rho} < \rho_R$. Here, $\tilde{u}$ and $\hat{u}$ are arbitrary constants.

Using a similar analysis as done in Section 4.7, we state the following theorems.

Theorem 4.8.1. *Let $u_L > u_R$. If $\rho_L > \rho_R$ (respectively; $\rho_L < \rho_R$) then a 1-shock wave (respectively, a 2-shock wave) is a solution to the Riemann problem for the system (4.2.1) with $f = k_0 u$ and $p(\rho) = C\rho$ subject to the initial conditions (4.8.1) if and only if $(u_L - u_R) \geq |\rho_L - \rho_R| \sqrt{\frac{C}{\rho_L \rho_R}}$.*

Theorem 4.8.2. *Let $u_L > u_R$. If $\rho_L > \rho_R$ (respectively; $\rho_L < \rho_R$) then a 1-rarefaction wave (respectively, a 2-rarefaction wave) is a solution to the Riemann problem for the system (4.2.1) with $f = k_0 u$ and $p(\rho) = C\rho$ subject to the initial conditions (4.8.1) if and only if $(u_L - u_R) \leq |\rho_L - \rho_R| \sqrt{\frac{C}{\rho_L \rho_R}}$.*

Theorem 4.8.3. *Let $u_L < u_R$. If $\rho_L > \rho_R$ (respectively; $\rho_L < \rho_R$) then a 1-shock wave (respectively, a 2-shock wave) is a solution to the Riemann problem for the system (4.2.1) with $f = k_0 u$ and $p(\rho) = C\rho$ subject to the initial conditions (4.8.1) if and only if $(u_L - u_R) \geq -\sqrt{C} \left| \log \left(\frac{\rho_R}{\rho_L} \right) \right|$.*

Theorem 4.8.4. *If $\rho_L > \rho_R$ (respectively; $\rho_L < \rho_R$) then a 1-rarefaction wave (respectively, a 2-rarefaction wave) is a solution to the Riemann problem for the system (4.2.1) with $f = k_0 u$ and $p(\rho) = C\rho$ subject to the initial conditions (4.8.1) if and only if $(u_L - u_R) \leq -\sqrt{C} \left| \log \left(\frac{\rho_R}{\rho_L} \right) \right|$.*

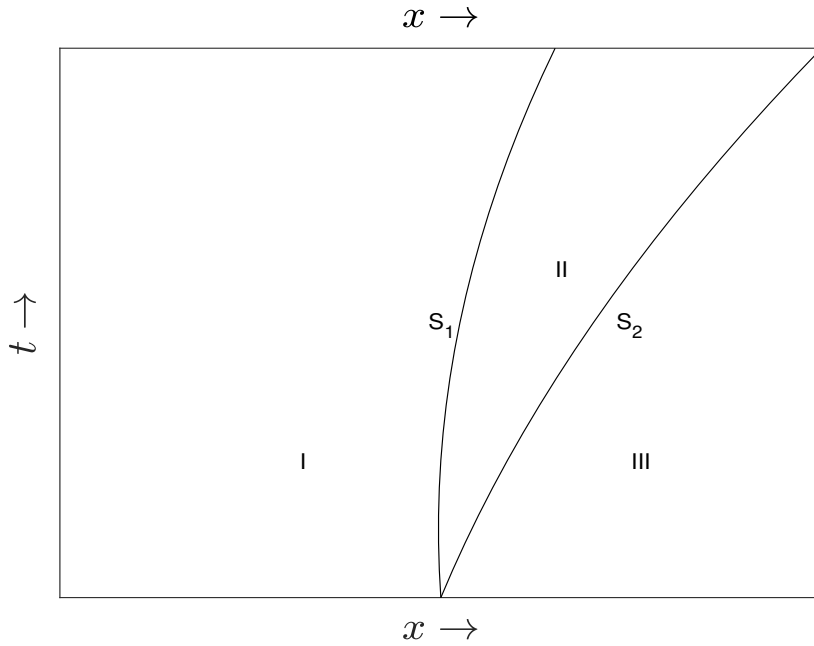


Figure 4.1: S_1 and S_2 are, respectively, the back-shock and the front shock; regions $x \leq S_1(t)$, $S_1(t) < x < S_2(t)$, and $x \geq S_2(t)$ are depicted as I, II and III respectively.

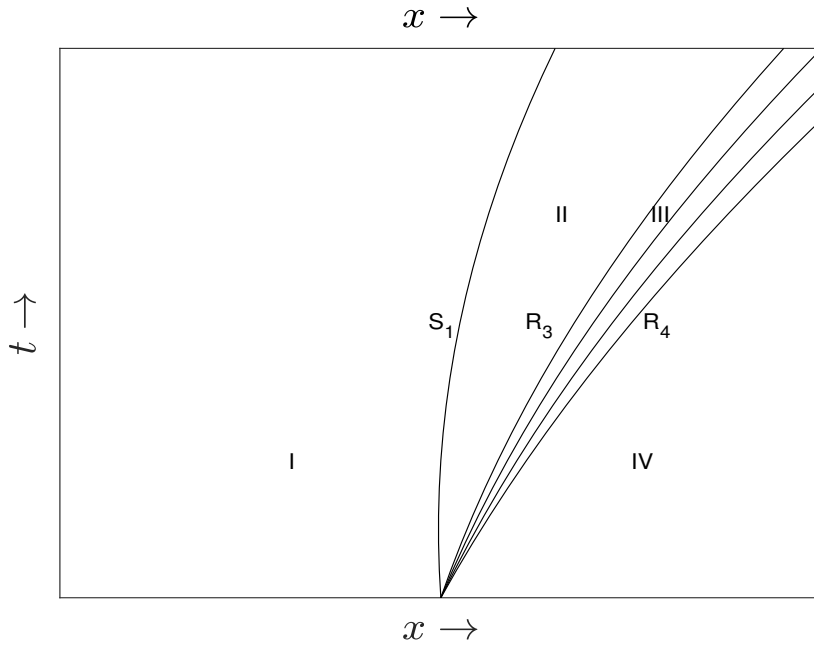


Figure 4.2: Region behind the back-shock S_1 is depicted as I; region $S_1(t) < x < R_3(t)$ between S_1 and the trail characteristic R_3 of the front rarefaction wave III is depicted as II; region $x > R_4(t)$ ahead of the front rarefaction is depicted as IV.

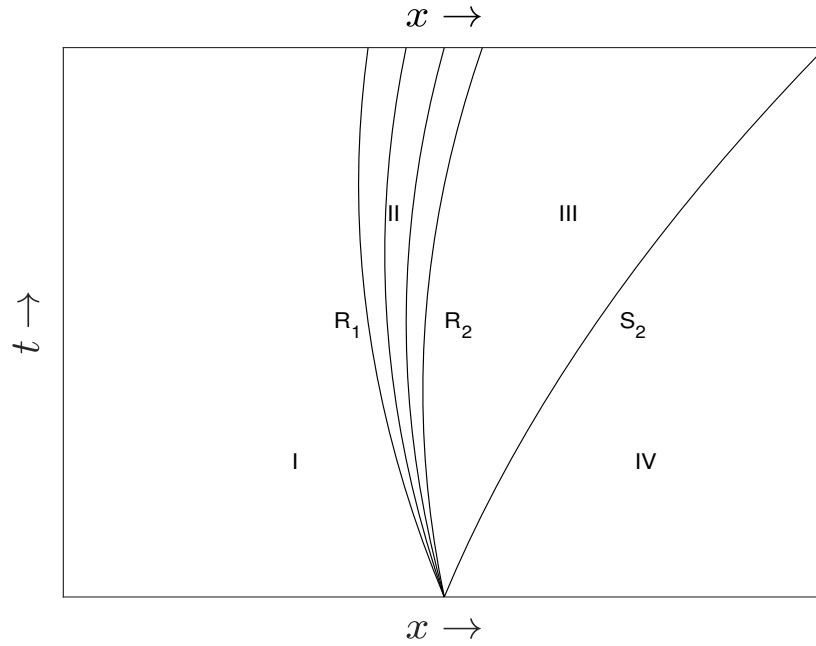


Figure 4.3: The region $x < R_1(t)$ is depicted as I; back rarefaction region $R_1(t) \leq x \leq R_2(t)$ is depicted as II; the region $R_2(t) < x < S_2(t)$ is depicted as III and the region $x \geq S_2(t)$ is depicted as IV.

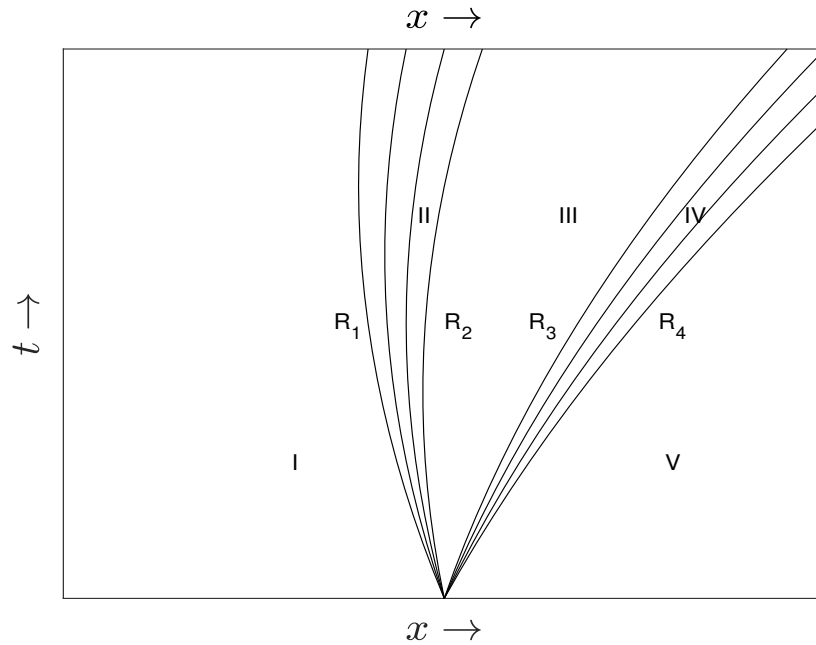


Figure 4.4: The region $x < R_1(t)$ is depicted as I; II is the back rarefaction wave region; region $R_2(t) \leq x \leq R_3(t)$ between front and back rarefaction is depicted as III; IV is the front rarefaction wave region and region $x > R_4(t)$ is depicted as V.

Chapter 5

Wave interactions of a double Riemann problem for rate-type material with non-constant initial data

5.1 Introduction

In the recent decade, researchers have shown a growing interest in exploring the interactions of elementary waves in hyperbolic systems of partial differential equations [2, 5, 6, 8, 9, 15, 91–94], owing to their extensive practical applications such as stability of Riemann solution with respect to small perturbations in the initial data.

However, to discuss interactions with the perturbed Riemann problem for non-constant data, first it is required to understand completely the Riemann problem for the system under consideration. Over the years, numerous mathematical techniques have been proposed to obtain the exact solutions to these systems, as documented in various references from [40, 61, 85]. In recent years, the approach based on differential constraints, as introduced by Janenko in [30], [31] and [32]), has garnered significant interest. Further, the contributions towards the development of this approach may be found in [40, 41, 43].

Finding out an exact solution to a generalized Riemann problem is a difficult task. However, LeFloch and Raviart [56] obtained an approximate solution to the generalized Riemann problem in the form of an asymptotic expansion, which was employed for gas dynamics equations given in [57].

Interactions of elementary waves for perturbed Riemann problem was discussed by many researchers. In particular, the interaction problem for p-system which represents isentropic gas dynamic in Lagrangian co-ordinate, was discussed by Smoller [2]. Luo and Yang [89] discussed the interactions of elementary waves for compressible Euler equations with frictional damping. Raja Sekhar and Sharma [90] delved into the existence of a vacuum state and briefly discussed wave interactions within the realm of isentropic magnetogasdynamics and the wave interactions were explained by them in shallow water waves in [76]. Liu and Sun [86] carried out an in-depth analysis of elementary wave interactions for the Aw-Rascle model with generalized Chaplygin gas. Ji and Shen [88] constructed the global solution to the perturbed Riemann problem and discussed the interactions of elementary waves, and gave the phenomena of coalesce for shock waves of the same family. Moreover, Wei and Sun [81] completely discussed about the Riemann problem and interaction for a class of strictly hyperbolic systems of conservation laws.

The aforementioned works motivated us to investigate the conditions under which the Riemann solution given by shock waves interacts, and also to discuss the structure of the solution, which is ultimately characterized through elementary waves for the system under consideration.

In this chapter, the shock wave interactions of the double Riemann problem are considered. The results derived in Chapter 2 to solve the Riemann problem for non-constant data for rate-type material enable us to discuss the wave interactions arising due to two discontinuities in the initial data.

5.2 Solutions to the Cauchy problem

We consider the balance laws describing rate-type materials ([30] - [32]) as follows

$$\begin{aligned}\frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left(\frac{1}{v} \right) &= 0, \\ \frac{\partial v}{\partial t} + \frac{\partial u}{\partial x} &= 1 - v,\end{aligned}\tag{5.2.1}$$

where u is the Lagrangian velocity and $1/v$ with $v \neq 0$ denotes the stress in the material that is undergoing loading/unloading processes.

The eigenvalues $\lambda^{(i)}$, $i = 1, 2$ representing the characteristic speeds of the system (5.2.1) are given by

$$\lambda^{(1)} = -\frac{1}{v}, \quad \lambda^{(2)} = \frac{1}{v},$$

and the corresponding right eigenvectors $\mathbf{R}^{(i)}$

$$\mathbf{R}^{(1)} = \begin{bmatrix} -v^{-1} \\ 1 \end{bmatrix}, \quad \mathbf{R}^{(2)} = \begin{bmatrix} v^{-1} \\ 1 \end{bmatrix}.$$

Further, both the eigenvalues are real and distinct, the system (5.2.1) is strictly hyperbolic and genuinely nonlinear (since $\mathbf{R}^{(i)} \cdot \nabla \lambda^{(i)} \neq 0$ for $i = 1, 2$).

In Chapter-2, [61], the system was considered with smooth initial data

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad (5.2.2)$$

and the solution of the Cauchy problem (5.2.1), (5.2.2) was obtained using the differential constraints method as follows

$$\begin{aligned} v(x, t) &= v_0(\xi) e^{-t}, \\ u(x, t) &= u_0(\xi) + \delta \left(\frac{e^t - 1}{v_0(\xi)} \right), \\ x &= \xi + \delta \left(\frac{e^t - 1}{v_0(\xi)} \right), \quad \delta = \pm 1. \end{aligned}$$

Here, $\xi(x, t)$ is determined for a given x and t from the equation (2.3.9)₃, and

$$\frac{du_0(x)}{dx} = \frac{\delta}{v_0(x)} \frac{dv_0}{dx} + 1. \quad (5.2.3)$$

Observe that for a given x and t , the equations (2.3.9)₁ and (2.3.9)₂ admit unique values for v and u provided there exists a unique ξ satisfying (2.3.9)₃; in other words, the existence of a unique solution is guaranteed for every x in $(-\infty, \infty)$ and for every $t > 0$ provided that

$$\delta \left(\frac{e^t - 1}{(v_0(\xi))^2} \right) \frac{dv_0}{d\xi} + 1 \neq 0. \quad (5.2.4)$$

Further, in Chapter-2, [61], the solution of the generalized Riemann problem of the

system (5.2.1) subject to the initial conditions

$$(u(x, 0), v(x, 0)) = (u_0(x), v_0(x)) = \begin{cases} (x + u_L, v_L), & \text{if } x < 0, \\ (x + u_R, v_R), & \text{if } x \geq 0, \end{cases} \quad (5.2.5)$$

is provided with shock and/or rarefaction waves, the results of which can be summarized as follows.

Theorem 5.2.1. *The solution to the generalized Riemann problem for the system (5.2.1) subject to the initial data (5.2.5) consists of*

- a 1-shock wave if and only if one of the following conditions hold:

$$\begin{aligned} (a.) \quad & u_L - u_R + \frac{v_R - v_L}{\sqrt{v_L v_R}} < 0, \text{ when } v_R > v_L \text{ and } u_L < u_R. \\ (b.) \quad & u_L - u_R + \log\left(\frac{v_R}{v_L}\right) < 0, \text{ when } v_R < v_L \text{ and } u_L > u_R. \\ (c.) \quad & v_R < v_L \text{ and } u_L < u_R. \end{aligned}$$

- a 2-shock wave if and only if one of the following conditions hold:

$$\begin{aligned} (a.) \quad & u_L - u_R - \log\left(\frac{v_R}{v_L}\right) < 0, \text{ when } v_R > v_L \text{ and } u_L > u_R. \\ (b.) \quad & u_L - u_R - \frac{v_R - v_L}{\sqrt{v_L v_R}} < 0, \text{ when } v_R < v_L \text{ and } u_L < u_R. \\ (c.) \quad & v_R > v_L \text{ and } u_L < u_R. \end{aligned}$$

- a 1-rarefaction wave if and only if one of the following conditions hold

$$\begin{aligned} (a.) \quad & u_L - u_R + \frac{v_R - v_L}{\sqrt{v_L v_R}} > 0, \text{ when } v_R > v_L \text{ and } u_L < u_R. \\ (b.) \quad & u_L - u_R + \log\left(\frac{v_R}{v_L}\right) > 0, \text{ when } v_R < v_L \text{ and } u_L > u_R. \\ (c.) \quad & v_R > v_L \text{ and } u_L > u_R. \end{aligned}$$

- a 2-rarefaction wave if and only if one of the following conditions hold:

$$\begin{aligned} (a.) \quad & u_L - u_R - \log\left(\frac{v_R}{v_L}\right) > 0, \text{ when } v_R > v_L \text{ and } u_L > u_R. \\ (b.) \quad & u_L - u_R - \frac{v_R - v_L}{\sqrt{v_L v_R}} > 0, \text{ when } v_R < v_L \text{ and } u_L < u_R. \\ (c.) \quad & v_R < v_L \text{ and } u_L > u_R. \end{aligned}$$

5.3 Elementary waves interactions

In this section, an attempt is made to discuss and study the interaction of shock waves for the system (5.2.1), which may arise due to two jump discontinuities in the initial data, one at $x = 0$ and the second one at $x = x_0 > 0$, named as a double Riemann problem and is given as follows

$$(u(x, 0), v(x, 0)) = (u_0(x), v_0(x)) = \begin{cases} (x + u_L, v_L), & \text{if } x \leq 0, \\ (x + u_M, v_M), & \text{if } 0 < x \leq x_0, \\ (x + u_R, v_R), & \text{if } x_0 < x, \end{cases} \quad (5.3.1)$$

where u_L, u_R, u_M, v_L and v_R are given constants but arbitrary. As discussed in Chapter 1, the solution of the system due to the discontinuity in the initial data at $x = 0$ is given through either a 1-shock curve, $x = \tilde{S}_1(t)$, or a 1-rarefaction region, $\tilde{R}_1(t) < x < \tilde{R}_2(t)$, or a 2-shock curve, $x = \tilde{S}_2(t)$ or a 2-rarefaction region, $\tilde{R}_3(t) < x < \tilde{R}_4(t)$. Similarly, due to the discontinuity in the initial data at $x = x_0$ the solution is given through either a 1-shock curve, $x = \bar{S}_1(t)$, or a 1-rarefaction region, $\bar{R}_1(t) < x < \bar{R}_2(t)$, or a 2-shock curve, $x = \bar{S}_2(t)$, or a 2-rarefaction region, $\bar{R}_3(t) < x < \bar{R}_4(t)$. Due to the presence of two discontinuities in the initial data, the shock/rarefaction curves of different families of characteristics emanate from $x = 0$ and $x = x_0$, and hence there exists a $T > 0$ such that interaction of shock curves and/or rarefaction region may take place for $t \geq T$, which is discussed in the following subsections.

5.3.1 Interaction of a 1-shock wave ($x = \tilde{S}_1(t)$) with a 1-shock wave ($x = \bar{S}_1(t)$)

Due to the discontinuity at $x = 0$ in the initial data, when

$$u_L = u_M + \frac{v_M - v_L}{\sqrt{(v_L v_M)}}, \quad (5.3.2)$$

with $v_M < v_L$ and $u_M > u_L$, the solution is given through only a 1-shock wave as

$$v(x, t) = \begin{cases} v_L e^{-t}, & \text{if } x \leq \tilde{S}_1(t), \\ v_M e^{-t}, & \text{if } x > \tilde{S}_1(t), \end{cases} \quad (5.3.3)$$

$$u(x, t) = \begin{cases} u_L + x, & \text{if } x \leq \tilde{S}_1(t), \\ u_M + x, & \text{if } x > \tilde{S}_1(t), \end{cases} \quad (5.3.4)$$

where $x = \tilde{S}_1(t)$ is the shock location with $\tilde{S}_1(t) = \frac{(1 - e^t)}{\sqrt{(v_L v_M)}}$.

Similarly, due to the discontinuity in the initial data at $x = x_0$, when

$$u_R = u_M - \frac{v_R - v_M}{\sqrt{(v_M v_R)}}, \quad (5.3.5)$$

with $v_R < v_M$ and $u_R > u_M$, the solution is given through only a 1- shock wave as

$$v(x, t) = \begin{cases} v_M e^{-t}, & \text{if } x \leq \bar{S}_1(t), \\ v_R e^{-t}, & \text{if } x > \bar{S}_1(t), \end{cases} \quad (5.3.6)$$

$$u(x, t) = \begin{cases} u_M + x, & \text{if } x \leq \bar{S}_1(t), \\ u_R + x, & \text{if } x > \bar{S}_1(t), \end{cases} \quad (5.3.7)$$

where $x = \bar{S}_1(t)$ is the shock location with $\bar{S}_1(t) = x_0 + \frac{(1 - e^t)}{\sqrt{(v_R v_M)}}$.

Thus, in view of the equations (5.3.2) and (5.3.5), the solution of the double Riemann problem is obtained through only 1-shock waves at $x = 0$ and at $x = x_0$ when

$$v_R < v_M < v_L, \quad u_L < u_M < u_R, \quad u_L - u_R - \frac{v_M - v_L}{\sqrt{(v_L v_M)}} - \frac{v_R - v_M}{\sqrt{(v_M v_R)}} = 0. \quad (5.3.8)$$

Since $v_R < v_L$ the shock curve, $x = \tilde{S}_1(t)$, overtakes the shock curve, $x = \bar{S}_1(t)$, in a finite time, $t = t_1 > 0$, where t_1 is given as

$$t_1 = \log \left(1 + \frac{x_0 \sqrt{v_R v_L v_M}}{\sqrt{v_L} - \sqrt{v_R}} \right), \quad (5.3.9)$$

which implies that the solution of the double Riemann problem given in equations (5.3.3) - (5.3.7) is valid for $t \leq t_1$. The solution, due to the interaction of a 1-shock wave emanated at $x = 0$ with a 1-shock wave emanated at $x = x_0$, for $t > t_1$, can be obtained by solving the Riemann problem with the following initial data

$$(u(x, t_1), v(x, t_1)) = \begin{cases} (x + u_L, v_L e^{-t_1}), & \text{if } x < x_1, \\ (x + u_R, v_R e^{-t_1}), & \text{if } x \geq x_1, \end{cases} \quad (5.3.10)$$

where $x_1 = \frac{(1 - e^{t_1})}{\sqrt{(v_L v_M)}} = x_0 + \frac{(1 - e^{t_1})}{\sqrt{(v_R v_M)}}$.

Since, $v_R < v_L$ and $u_L < u_R$, using Theorem 5.2.1, the possible forms for the solution to the Riemann problem subject to the initial data (5.3.10) are

- I. given through a 1-shock wave, $x = S_1(t)$, and a 2-shock wave, $x = S_2(t)$, emanating at the point $(x, t) = (x_1, t_1)$.
- II. given through a 1-shock wave, $x = S_1(t)$, and a 2-rarefaction wave in the region, $R_3(t) \leq x \leq R_4(t)$, emanating at the point $(x, t) = (x_1, t_1)$.
- III. given through only a 1-shock wave, $x = S_1(t)$, emanating at the point $(x, t) = (x_1, t_1)$.

5.3.1.1 Case-I

Since, $v_R < v_L$ and $u_L < u_R$, using Theorem 5.2.1, if the solution to the Riemann problem subject to the initial data (5.3.10) is given through a 1-shock wave, $x = S_1(t)$ and a 2-shock wave, $x = S_2(t)$ only emanating at the point $(x, t) = (x_1, t_1)$, then

$$u_L - u_R + \frac{v_L - v_R}{\sqrt{v_L v_R}} < 0, \quad (5.3.11)$$

and the solution for $t > t_1$ is given as follows

$$v(x, t) = \begin{cases} v_L e^{-t}, & \text{if } x \leq S_1(t), \\ z_1 e^{-t}, & \text{if } S_1(t) < x < S_2(t), \\ v_R e^{-t}, & \text{if } x \geq S_2(t), \end{cases} \quad (5.3.12)$$

$$u(x, t) = \begin{cases} u_L + x, & \text{if } x \leq S_1(t), \\ u_L - \frac{z_1 - v_L e^{-t_1}}{\sqrt{(v_L e^{-t_1} z_1)}} + x, & \text{if } S_1(t) < x < S_2(t), \\ u_R + x, & \text{if } x > S_2(t), \end{cases} \quad (5.3.13)$$

where the 1-shock curve $x = S_1(t)$ is given by $S_1(t) = \frac{(e^{t_1} - e^t)}{\sqrt{(v_L z_1)}} + x_1$ and the 2-shock curve $x = S_2(t)$ is given by $S_2(t) = \frac{(e^t - e^{t_1})}{\sqrt{(v_R z_1)}} + x_1$. Here, $z = z_1$ is the solution to the equation

$$u_L - u_R - \frac{z - v_L e^{-t_1}}{\sqrt{(v_L e^{-t_1} z)}} - \frac{v_R e^{-t_1} - z}{\sqrt{(z v_R e^{-t_1})}} = 0, \quad z \in (v_R, v_L). \quad (5.3.14)$$

Thus, the necessary and sufficient condition for the solution to the double Riemann problem is ultimately given through a 1-shock wave and only a 2-shock wave is obtained, in view of (5.3.8) and (5.3.11), in the form of an inequality satisfied by the constants v_L, v_M and v_R as

$$f_1(v_L, v_M, v_R) < 0,$$

when $v_R < v_M < v_L$, $u_L < u_M < u_R$, and

$$f_1(v_L, v_M, v_R) = \frac{v_L - v_R}{\sqrt{v_L v_R}} - \frac{v_M - v_R}{\sqrt{v_M v_R}} - \frac{v_L - v_M}{\sqrt{v_L v_M}}. \quad (5.3.15)$$

.

5.3.1.2 Case-II

Similarly, since $v_R < v_L$ and $u_L < u_R$, using Theorem 5.2.1, it can be concluded that the solution to the Riemann problem subject to the initial data (5.3.10) is given through a 1-shock wave, $x = S_1(t)$, and a 2-rarefaction wave in the region, $R_3(t) < x < R_4(t)$, emanating from the point $(x, t) = (x_1, t_1)$ only if

$$u_L - u_R + \frac{v_L - v_R}{\sqrt{v_L v_R}} > 0, \quad (5.3.16)$$

and the solution for $t > t_1$ is given as follows

$$v(x, t) = \begin{cases} v_L e^{-t}, & \text{if } x \leq S_1(t), \\ z_2 e^{-t}, & \text{if } S_1(t) < x \leq R_3(t), \\ \left(\frac{1 - e^{(t_1 - t)}}{x - x_1} \right), & \text{if } R_3(t) < x < R_4(t), \\ v_R e^{-t}, & \text{if } x \geq R_4(t), \end{cases} \quad (5.3.17)$$

$$u(x, t) = \begin{cases} u_L + x, & \text{if } x \leq S_1(t), \\ u_L - \frac{z_2 - v_L e^{-t_1}}{\sqrt{(v_L e^{-t_1} z_2)}} + x, & \text{if } S_1(t) < x \leq R_3(t), \\ u_R - \log \left(\frac{(x - x_1) v_R}{(e^t - e^{t_1})} \right) + x, & \text{if } R_3(t) < x < R_4(t), \\ u_R + x, & \text{if } x \geq R_4(t), \end{cases} \quad (5.3.18)$$

where the 1-shock curve $x = S_1(t)$ is given by $S_1(t) = \frac{(e^{t_1} - e^t)}{\sqrt{(v_L z_2)}} + x_1$ and the 2-rarefaction region, $R_3(t) < x < R_4(t)$, where $R_3(t) = \frac{(e^t - e^{t_1})}{z_2} + x_1$ and $R_4(t) = \frac{(e^t - e^{t_1})}{v_R} + x_1$ which is shown in Figure 5.1. Here $z = z_2$ is the solution to the equation

$$u_L - u_R - \frac{z - v_L e^{-t_1}}{\sqrt{(v_L e^{-t_1} z)}} + \log \left(\frac{v_R e^{-t_1}}{z} \right) = 0, \quad z \in (v_R, v_L). \quad (5.3.19)$$

Further, the necessary and sufficient condition for the solution to the double Riemann problem ultimately given through a 1-shock wave and a 2-rarefaction wave, in view of (5.3.8) and (5.3.16), is in the form of an inequality satisfied by the constants v_L, v_M and v_R as

$$f_1(v_L, v_M, v_R) > 0$$

when $v_R < v_M < v_L$, $u_L < u_M < u_R$ and $f_1(v_L, v_M, v_R)$ is defined as in (5.3.15).

5.3.1.3 Case-III

Similarly, since $v_R < v_L$ and $u_L < u_R$, using Theorem 5.2.1, it can be concluded that the solution to the Riemann problem subject to the initial data (5.3.10) is given through only a 1-shock wave, $x = S_1(t)$, emanating from the point $(x, t) = (x_1, t_1)$ only if

$$u_L - u_R + \frac{v_L - v_R}{\sqrt{v_L v_R}} = 0, \quad (5.3.20)$$

and the solution for $t > t_1$ is given as follows

$$v(x, t) = \begin{cases} v_L e^{-t}, & \text{if } x \leq S_1(t), \\ v_R e^{-t}, & \text{if } x > S_1(t), \end{cases} \quad (5.3.21)$$

$$u(x, t) = \begin{cases} u_L + x, & \text{if } x \leq S_1(t), \\ u_R + x, & \text{if } x > S_1(t), \end{cases} \quad (5.3.22)$$

where the 1-shock curve $x = S_1(t)$ is given by $S_1(t) = \frac{(e^{t_1} - e^t)}{\sqrt{(v_L v_R)}} + x_1$.

Further, the necessary and sufficient condition for the solution to the double Riemann problem ultimately given through only a 1-shock wave is obtained, in view of (5.3.8)

and (5.3.20), is in the form of an equality satisfied by the constants v_L, v_M and v_R as

$$f_1(v_L, v_M, v_R) = 0,$$

when $v_R < v_M < v_L$, $u_L < u_M < u_R$, and $f_1(v_L, v_M, v_R)$ is defined as in (5.3.15)

Lemma 5.3.1. *Consider the function $f_1(v_L, v_M, v_R)$ defined as in (5.3.15). Then f_1 is a positive function when $v_R < v_M < v_L$.*

Proof. Observe that $\lim_{v_R \rightarrow v_M} f_1 = 0$ and

$$\frac{\partial f_1}{\partial v_R} = \frac{(v_R - \sqrt{v_L v_M})(\sqrt{v_L} - \sqrt{v_M})}{2v_R \sqrt{v_R v_L v_M}} < 0,$$

since $v_R < v_M < v_L$. This inequality implies that f_1 is monotonically decreasing as v_R increases to v_M , and since the function $f_1(v_L, v_M, v_M) = 0$, it follows that f_1 is a positive function when $v_R < v_M < v_L$. $\square$

Thus, in view of Lemma 5.3.1, the solution due to the interaction of a 1-shock wave, $x = \tilde{S}_1(t)$, originated from $(x, t) = (0, 0)$ and a 1-shock wave, $x = \bar{S}_1(t)$, originated from $(x, t) = (x_0, 0)$, is given through only 1-shock and 2-rarefaction waves as given in Case-II of Section 5.3.1, for which the solution is given by equations (5.3.17) and (5.3.18) and is depicted in Figure 5.1.

5.3.2 Interaction of a 2-shock wave ($x = \tilde{S}_2(t)$) with a 1-shock wave ($x = \bar{S}_1(t)$)

Due to the discontinuity at $x = 0$ in the initial data, when

$$u_L = u_M + \frac{v_L - v_M}{\sqrt{(v_L v_M)}}, \quad (5.3.23)$$

with $v_L < v_M$ and $u_L < u_M$, the solution is given through a 2-shock wave only as

$$v(x, t) = \begin{cases} v_M e^{-t}, & \text{if } x \geq \tilde{S}_2(t), \\ v_L e^{-t}, & \text{if } x < \tilde{S}_2(t), \end{cases} \quad (5.3.24)$$

$$u(x, t) = \begin{cases} u_M + x, & \text{if } x \geq \tilde{S}_2(t), \\ u_L + x, & \text{if } x < \tilde{S}_2(t), \end{cases} \quad (5.3.25)$$

where $x = \tilde{S}_2(t)$ is the location of the 2-shock wave with $\tilde{S}_2(t) = \frac{(e^t - 1)}{\sqrt{(v_L v_M)}}$.

Similarly, due to the discontinuity in the initial data at $x = x_0$, when

$$u_R = u_M + \frac{v_M - v_R}{\sqrt{(v_R v_M)}}, \quad (5.3.26)$$

with $v_M > v_R$ and $u_M < u_R$, the solution is given through only a 1-shock wave as

$$v(x, t) = \begin{cases} v_M e^{-t}, & \text{if } x \leq \bar{S}_1(t), \\ v_R e^{-t}, & \text{if } x > \bar{S}_1(t), \end{cases} \quad (5.3.27)$$

$$u(x, t) = \begin{cases} u_M + x, & \text{if } x \leq \bar{S}_1(t), \\ u_R + x, & \text{if } x > \bar{S}_1(t), \end{cases} \quad (5.3.28)$$

where $x = \bar{S}_1(t)$ is the location of the 1-shock with $\bar{S}_1(t) = x_0 + \frac{(1 - e^t)}{\sqrt{(v_R v_M)}}$.

Thus, in view of the equations (5.3.23) and (5.3.26), the solution of the double Riemann problem is obtained through only a 2-shock wave originating at $x = 0$ and through only a 1-shock wave originated at $x = x_0$ when $v_R < v_M$, $v_L < v_M$, $u_L < u_M < u_R$ and

$$u_L - u_R - \frac{v_L - v_M}{\sqrt{(v_L v_M)}} - \frac{v_R - v_M}{\sqrt{(v_R v_M)}} = 0. \quad (5.3.29)$$

It is easy to observe that the shock speed $\frac{d\tilde{S}_2}{dt} > \frac{d\bar{S}_1}{dt}$. Hence, the 2-shock wave interacts with the 1-shock wave in a finite time, $t = t_2 > 0$, where t_2 is given as

$$t_2 = \log \left(1 + \frac{x_0 \sqrt{v_R v_L v_M}}{\sqrt{v_R} + \sqrt{v_L}} \right), \quad (5.3.30)$$

which implies that the solution of the double Riemann problem given in equations (5.3.24)- (5.3.28) is valid for $t \leq t_2$. The solution after the interaction, i.e. $t > t_2$, can be obtained by solving the Riemann problem with the following initial data

$$(u(x, t_2), v(x, t_2)) = \begin{cases} (x + u_L, v_L e^{-t_2}), & \text{if } x < x_2, \\ (x + u_R, v_R e^{-t_2}), & \text{if } x \geq x_2, \end{cases} \quad (5.3.31)$$

where $x_2 = \frac{(e^{t_2} - 1)}{\sqrt{(v_L v_M)}} = x_0 + \frac{(1 - e^{t_2})}{\sqrt{(v_R v_M)}}$.

Since, $u_L < u_R$ using Theorem 5.2.1, the possible forms for the solution to the Riemann problem subject to the initial data (5.3.10) are

- I. given through a 1-shock wave, $x = S_1(t)$, and a 2-shock wave, $x = S_2(t)$, emanating at the point $(x, t) = (x_2, t_2)$.
- II. given through a 1-shock wave, $x = S_1(t)$, and a 2-rarefaction wave in the region, $R_3(t) \leq x \leq R_4(t)$, emanating at the point $(x, t) = (x_2, t_2)$ for $v_L > v_R$.
- III. given through a 1-rarefaction wave in the region, $R_1(t) \leq x \leq R_2(t)$, and a 2-shock wave, $x = S_2(t)$, emanating at the point $(x, t) = (x_2, t_2)$ for $v_L < v_R$.
- IV. given through only a 1-shock wave, $x = S_1(t)$, emanating at the point $(x, t) = (x_2, t_2)$.
- V. given through only a 2-shock wave, $x = S_2(t)$, emanating at the point $(x, t) = (x_2, t_2)$.

5.3.2.1 Case-I

After the interaction of the waves $x = \tilde{S}_2(t)$ and $x = \bar{S}_1(t)$, since $u_L < u_R$ hold, using Theorem-5.2.1, it can be concluded that the solution to the Riemann problem subject to the initial data (5.3.31) is given through a 1-shock wave, $x = S_1(t)$, and a 2-shock wave, $x = S_2(t)$, emanating from the point $(x, t) = (x_2, t_2)$, if

$$u_L - u_R + \frac{|v_L - v_R|}{\sqrt{v_L v_R}} < 0, \quad u_L < u_R, \quad (5.3.32)$$

and the solution after the interaction, i.e., $t > t_2$, is given as

$$v(x, t) = \begin{cases} v_L e^{-t}, & \text{if } x \leq S_1(t), \\ z_3 e^{-t}, & \text{if } S_1 < x < S_2(t), \\ v_R e^{-t}, & \text{if } x \geq S_2(t), \end{cases} \quad (5.3.33)$$

$$u(x, t) = \begin{cases} u_L + x, & \text{if } x \leq S_1(t), \\ u_L - \frac{z_3 - v_L e^{-t_2}}{\sqrt{(v_L e^{-t_2} z_3)}} + x, & \text{if } S_1(t) < x < S_2(t), \\ u_R + x, & \text{if } x \geq S_2(t), \end{cases} \quad (5.3.34)$$

where $x = S_1(t)$ is the location of the 1-shock wave with $S_1(t) = \frac{(e^{t_2} - e^t)}{\sqrt{(v_L z_3)}} + x_2$ and

$x = S_2(t)$ is the location the 2-shock wave with $S_2(t) = \frac{(e^t - e^{t_2})}{\sqrt{v_R z_3}} + x_2$, both emanating from (x_2, t_2) as shown in Figure 5.2. Here $z = z_3$ is the solution to the equation

$$u_L - u_R - \frac{z - v_L e^{-t_2}}{\sqrt{(v_L e^{-t_2} z)}} - \frac{v_R e^{-t_2} - z}{\sqrt{(v_R e^{-t_2} z)}} = 0, \quad z \in (\min\{v_L, v_R\}, \max\{v_L, v_R\}). \quad (5.3.35)$$

Further, the necessary and sufficient condition for the double Riemann problem is ultimately given through only a 1-shock and a 2-shock, in view of (5.3.29) and (5.3.26), is in the form of an inequality satisfied by the constants v_L, v_M and v_R as

$$\begin{aligned} f_2(v_L, v_M, v_R) &< 0, \quad \text{when } v_L > v_R, \\ f_3(v_L, v_M, v_R) &< 0, \quad \text{when } v_L < v_R, \end{aligned}$$

when $v_R < v_M, v_L < v_M, u_L < u_M < u_R$, and

$$f_2(v_L, v_M, v_R) = \frac{v_L - v_R}{\sqrt{v_L v_R}} - \frac{v_M - v_L}{\sqrt{v_M v_L}} - \frac{v_M - v_R}{\sqrt{v_R v_M}}, \quad (5.3.36)$$

$$f_3(v_L, v_M, v_R) = \frac{v_R - v_L}{\sqrt{v_L v_R}} - \frac{v_M - v_L}{\sqrt{v_M v_L}} - \frac{v_M - v_R}{\sqrt{v_R v_M}}. \quad (5.3.37)$$

Lemma 5.3.2. *If $f_2(v_L, v_M, v_R)$ and $f_3(v_L, v_M, v_R)$ are defined as in equation (5.3.36) and in the equation (5.3.37) then f_2 is a negative function when $v_R < v_L < v_M$, and f_3 is a negative function when $v_L < v_R < v_M$.*

Proof. Observe that

$$\frac{\partial f_2}{\partial v_R} = \frac{(\sqrt{v_M} - \sqrt{v_L})(\sqrt{v_L v_M} - v_R)}{2v_R \sqrt{v_R v_L v_M}} > 0$$

due to the condition $v_R < v_L < v_M$. This implies that f_2 is monotonically increasing as v_R increases to v_L and since the function $f_2(v_L, v_M, v_L) = 0$, we have f_2 is a negative function when $v_R < v_L < v_M$.

Similarly, when $v_L < v_R < v_M$, we have

$$\frac{\partial f_3}{\partial v_R} = \frac{v_R + v_L}{2v_R \sqrt{v_R v_L}} + \frac{v_R + v_M}{2v_R \sqrt{v_R v_M}} > 0.$$

This implies that f_3 is monotonically increasing as v_R increases to v_M and since the

function $f_3(v_L, v_M, v_M) = 0$, we have f_3 to be a negative function when $v_L < v_R < v_M$. $\square$

Thus, in view of Lemma 5.3.2, the solution due to the interaction of a 2-shock wave, $x = \tilde{S}_2(t)$ originated from $(x, t) = (0, 0)$ and a 1-shock wave, $x = \bar{S}_1(t)$ originated from $(x, t) = (x_0, 0)$, is given through only 1-shock wave and a 2-shock wave as given in the equations (5.3.33) and (5.3.34) which is depicted in the Figure 5.2.

5.3.2.2 Case-II

Similarly, since $u_L < u_R$, using Theorem 5.2.1, it can be concluded that the solution to the Riemann problem subject to the initial data (5.3.31) is given through a 1-shock wave, $x = S_1(t)$, and a 2-rarefaction wave in the region, $R_3(t) < x < R_4(t)$, emanating from the point $(x, t) = (x_2, t_2)$ only if

$$u_L - u_R + \frac{v_L - v_R}{\sqrt{v_L v_R}} > 0, \quad \text{and} \quad v_L > v_R, \quad u_L < u_R, \quad (5.3.38)$$

and the solution for $t > t_2$ is given by

$$v(x, t) = \begin{cases} v_L e^{-t}, & \text{if } x \leq S_1(t), \\ z_4 e^{-t}, & \text{if } S_1(t) < x \leq R_3(t), \\ \left(\frac{1 - e^{(t_2 - t)}}{x - x_2} \right), & \text{if } R_3(t) < x < R_4(t), \\ v_R e^{-t}, & \text{if } x \geq R_4(t), \end{cases} \quad (5.3.39)$$

$$u(x, t) = \begin{cases} u_L + x, & \text{if } x \leq S_1(t), \\ u_L - \frac{z_4 - v_L e^{-t_2}}{\sqrt{(v_L e^{-t_2} z_4)}} + x, & \text{if } S_1(t) < x \leq R_3(t), \\ u_R - \log \left(\frac{(x - x_2) v_R}{(e^t - e^{t_2})} \right) + x, & \text{if } R_3(t) < x < R_4(t), \\ u_R + x, & \text{if } x \geq R_4(t), \end{cases} \quad (5.3.40)$$

where the 1-shock curve $x = S_1(t)$ is given by $S_1(t) = \frac{(e^{t_2} - e^t)}{\sqrt{(v_L z_4)}} + x_2$ and the 2-rarefaction region is given by $R_3(t) < x < R_4(t)$ where $R_3(t) = \frac{(e^t - e^{t_2})}{z_4} + x_2$ and $R_4(t) = \frac{(e^t - e^{t_2})}{v_R} + x_2$. Here $z = z_4$ is the solution to the equation

$$u_L - u_R - \frac{z - v_L e^{-t_2}}{\sqrt{(v_L e^{-t_2} z)}} + \log \left(\frac{v_R e^{-t_2}}{z} \right) = 0, \quad z \in (v_R, v_L). \quad (5.3.41)$$

Further, the necessary and sufficient condition for the solution to the double Riemann problem, ultimately given through only 1-shock and 2-rarefaction waves, is obtained, in view of (5.3.29) and (5.3.38), is in the form of inequalities satisfied by the constants v_L, v_M and v_R as

$$f_2(v_L, v_M, v_R) > 0, \text{ and } v_L > v_R$$

when $v_R < v_M$, $v_L < v_M$, $u_L < u_M < u_R$, and $f_2(v_L, v_M, v_R)$ is exactly the same as defined in the equation (5.3.36).

Thus, in view of Lemma 5.3.2, the solution due to interaction of a 2-shock wave, $x = \tilde{S}_2(t)$ originated from $(x, t) = (0, 0)$ and a 1-shock wave, $x = \bar{S}_1(t)$ originated from $(x, t) = (x_0, 0)$, cannot be given through a 1-shock wave and a 2-rarefaction wave.

5.3.2.3 Case- III

Similarly, since $u_L < u_R$, using Theorem 5.2.1, it can be concluded that the solution to the Riemann problem subject to the initial data (5.3.31) is given through a 1-rarefaction wave in the region, $R_1(t) < x < R_2(t)$, and a 2-shock wave, $x = S_2(t)$, both emanating from the point $(x, t) = (x_2, t_2)$ only if

$$u_L - u_R + \frac{v_R - v_L}{\sqrt{v_L v_R}} > 0 \quad \text{and} \quad v_L < v_R, \quad (5.3.42)$$

and the solution for $t > t_2$ is given by

$$v(x, t) = \begin{cases} v_L e^{-t}, & \text{if } x \leq R_1(t), \\ \left(\frac{e^{(t_2-t)} - 1}{x - x_2} \right), & \text{if } R_1(t) < x < R_2(t), \\ z_5 e^{-t}, & \text{if } R_2(t) \leq x < S_2(t), \\ v_R e^{-t}, & \text{if } x \geq S_2(t), \end{cases} \quad (5.3.43)$$

$$u(x, t) = \begin{cases} u_L + x, & \text{if } x \leq R_1(t), \\ u_L - \frac{z_5 - v_L e^{-t_2}}{\sqrt{(v_L e^{-t_2} z_5)}} + x, & \text{if } R_1(t) < x < R_2(t), \\ u_R - \log \left(\frac{(x - x_2) v_R}{(e^t - e^{t_2})} \right) + x, & \text{if } R_2(t) \leq x < S_2(t), \\ u_R + x, & \text{if } x \geq S_2(t), \end{cases} \quad (5.3.44)$$

where the 1-rarefaction region is given by, $R_1(t) < x < R_2(t)$, with $R_1(t) = \frac{(e^t - e^{t_2})}{z_5} + x_2$ and $R_2(t) = \frac{(e^t - e^{t_2})}{v_R} + x_2$, and the 2-shock curve $x = S_2(t)$ is given by $S_2(t) = x_2 + \frac{e^t - e^{t_2}}{\sqrt{v_R z_5}}$. Here $z = z_5$ is the solution to the equation

$$u_L - u_R + \log \left(\frac{v_L e^{-t_2}}{z} \right) - \frac{z - v_R e^{-t_2}}{\sqrt{(v_R e^{-t_2} z)}} = 0, \quad z \in (v_L, v_R). \quad (5.3.45)$$

Further, the necessary and sufficient condition for the solution to the double Riemann problem is ultimately given through only a 1-rarefaction wave and a 2-shock wave, in view of (5.3.29) and (5.3.42), is in the form of an inequality satisfied by the constants v_L, v_M and v_R as follows

$$f_3(v_L, v_M, v_R) > 0 \quad \text{and} \quad v_L < v_R,$$

when $v_R < v_M$, $v_L < v_M$ and $u_L < u_M < u_R$, and $f_3(v_L, v_M, v_R)$ is exactly the same as in the equation (5.3.37).

Thus, in view of Lemma 5.3.2, the solution due to the interaction of a 2-shock wave, $x = \tilde{S}_2(t)$ originated from $(x, t) = (0, 0)$ and a 1-shock wave, $x = \bar{S}_1(t)$ originated from $(x, t) = (x_0, 0)$, cannot be given through a 1-rarefaction wave and a 2-shock wave.

5.3.2.4 Case-IV

Similarly, since $v_R < v_L$ and $u_L < u_R$, using Theorem 5.2.1, it can be concluded that the solution to the Riemann problem subject to the initial data (5.3.10) is given through only a 1-shock wave, $x = S_1(t)$, emanating from the point $(x, t) = (x_2, t_2)$ only if

$$u_L - u_R + \frac{v_L - v_R}{\sqrt{v_L v_R}} = 0, \quad (5.3.46)$$

and the solution for $t > t_2$ is given as

$$v(x, t) = \begin{cases} v_L e^{-t}, & \text{if } x \leq S_1(t), \\ v_R e^{-t}, & \text{if } x > S_1(t), \end{cases} \quad (5.3.47)$$

$$u(x, t) = \begin{cases} u_L + x, & \text{if } x \leq S_1(t), \\ u_R + x, & \text{if } x > S_1(t), \end{cases} \quad (5.3.48)$$

where the 1-shock curve $x = S_1(t)$ is given by $S_1(t) = \frac{(e^{t_1} - e^t)}{\sqrt{(v_L v_R)}} + x_1$.

Further, the necessary and sufficient condition for the solution to the double Riemann problem is ultimately given through only a 1-shock wave, in view of (5.3.29) and (5.3.46), in the form of equality satisfied by the constants v_L, v_M and v_R as

$$f_2(v_L, v_M, v_R) = 0,$$

when $v_R < v_M$, $v_L < v_M$, $u_L < u_M < u_R$, and $f_2(v_L, v_M, v_R)$ is defined as in (5.3.36).

Thus, in view of Lemma 5.3.2, the solution due to interaction of a 2-shock wave, $x = \tilde{S}_2(t)$ originated from $(x, t) = (0, 0)$ and a 1-shock wave, $x = \bar{S}_1(t)$ originated from $(x, t) = (x_0, 0)$, cannot be given through only a 1-shock wave.

5.3.2.5 Case-V

Similarly, since $v_L < v_R$ and $u_L < u_R$, using Theorem 5.2.1 it can be concluded that the solution to the Riemann problem subject to the initial data (5.3.60) is given through only a 2-shock curve, $x = S_2(t)$, emanating from the point $(x, t) = (x_2, t_2)$ only if

$$u_L - u_R + \frac{v_R - v_L}{\sqrt{v_L v_R}} = 0, \quad (5.3.49)$$

and the solution for $t > t_2$ is given as

$$v(x, t) = \begin{cases} v_L e^{-t}, & \text{if } x \leq S_2(t), \\ v_R e^{-t}, & \text{if } x > S_2(t), \end{cases} \quad (5.3.50)$$

$$u(x, t) = \begin{cases} u_L + x, & \text{if } x \leq S_2(t), \\ u_R + x, & \text{if } x > S_2(t), \end{cases} \quad (5.3.51)$$

where the 2-shock curve $x = S_2(t)$ is given by $S_2(t) = \frac{(e^t - e^{t_3})}{\sqrt{(v_L v_R)}} + x_2$.

Thus, the necessary and sufficient condition for the solution to the double Riemann problem is ultimately given through only a 2-shock wave, in view of (5.3.29) and (5.3.49), is in the form of equality satisfied by the constants v_L, v_M and v_R as

$$f_3(v_L, v_M, v_R) = 0,$$

when $v_R < v_M$, $v_L < v_M$, $u_L < u_M < u_R$, and $f_3(v_L, v_M, v_R)$ is defined as in (5.3.37).

Thus, in view of Lemma 5.3.2, the solution due to the interaction of a 2-shock wave, $x = \tilde{S}_2(t)$ originated from $(x, t) = (0, 0)$ and a 1-shock wave, $x = \bar{S}_1(t)$ originated from $(x, t) = (x_0, 0)$, cannot be given through only a 2-shock wave.

5.3.3 Interaction of a 2-shock wave ($x = \tilde{S}_2(t)$) with a 2-shock wave ($x = \bar{S}_2(t)$)

Due to the discontinuity at $x = 0$ for $t = 0$ in the initial data, when

$$u_L = u_M + \frac{v_L - v_M}{\sqrt{(v_L v_M)}}, \quad (5.3.52)$$

with $v_L < v_M$ and $u_L < u_M$, the solution is given through only a 2-shock wave as follows

$$v(x, t) = \begin{cases} v_L e^{-t}, & \text{if } x \leq \tilde{S}_2(t), \\ v_M e^{-t}, & \text{if } x > \tilde{S}_2(t), \end{cases} \quad (5.3.53)$$

$$u(x, t) = \begin{cases} u_L + x, & \text{if } x \leq \tilde{S}_2(t), \\ u_M + x, & \text{if } x > \tilde{S}_2(t), \end{cases} \quad (5.3.54)$$

where $x = \tilde{S}_2(t)$ is the location of the 2-shock wave emanated from $(x, t) = (0, 0)$, with $\tilde{S}_2(t) = \frac{(e^t - 1)}{\sqrt{(v_L v_M)}}$.

Similarly, due to the discontinuity at $x = x_0$ for $t = 0$ in the initial data, when

$$u_R = u_M + \frac{v_R - v_M}{\sqrt{(v_M v_R)}}, \quad (5.3.55)$$

with $v_M < v_R$ and $u_R > u_M$, the solution is given through only a 2-shock wave as follows

$$v(x, t) = \begin{cases} v_M e^{-t}, & \text{if } x \leq \bar{S}_2(t), \\ v_R e^{-t}, & \text{if } x > \bar{S}_2(t), \end{cases} \quad (5.3.56)$$

$$u(x, t) = \begin{cases} u_M + x, & \text{if } x \leq \bar{S}_2(t), \\ u_R + x, & \text{if } x > \bar{S}_2(t), \end{cases} \quad (5.3.57)$$

where $x = \bar{S}_2(t)$ is the location of the 2-shock wave originated at $(x, t) = (x_0, 0)$ with $\bar{S}_2(t) = x_0 + \frac{(e^t - 1)}{\sqrt{(v_R v_M)}}$.

Thus, in view of the equations (5.3.52) and (5.3.55), the solution of the double Riemann problem is obtained through only a 2-shock wave originating at $(x, t) = (0, 0)$ and at $(x, t) = (x_0, 0)$ when

$$v_L < v_M < v_R, \quad u_L < u_M < u_R, \quad u_L - u_R + \frac{v_M - v_L}{\sqrt{(v_L v_M)}} + \frac{v_R - v_M}{\sqrt{(v_M v_R)}} = 0. \quad (5.3.58)$$

As $v_L < v_R$ the shock curve $x = \tilde{S}_2(t)$ overtakes the shock curve $x = \bar{S}_2(t)$ in a finite time $t = t_3 > 0$, where t_3 is given as

$$t_3 = \log \left(1 + \frac{x_0 \sqrt{v_R v_L v_M}}{\sqrt{v_R} - \sqrt{v_L}} \right). \quad (5.3.59)$$

which implies that the solution of the double Riemann problem given in equations (5.3.53) - (5.3.56) is valid for $t < t_3$ till the interaction takes place. The solution after the interaction, i.e. $t > t_3$, can be obtained by solving the Riemann problem with the following initial data

$$(u(x, t_3), v(x, t_3)) = \begin{cases} (x + u_L, v_L e^{-t_3}), & \text{if } x < x_3, \\ (x + u_R, v_R e^{-t_3}), & \text{if } x \geq x_3, \end{cases} \quad (5.3.60)$$

where $x_3 = \frac{(e^{t_3} - 1)}{\sqrt{(v_L v_M)}}$.

Since, $v_L < v_R$ and $u_L < u_R$ using the Theorem 5.2.1, the possible forms for the solution to the Riemann problem subject to the initial data (5.3.10) are

- I. given through a 1-shock wave, $x = S_1(t)$, and a 2-shock wave, $x = S_2(t)$, emanating at the point $(x, t) = (x_3, t_3)$.

- II. given through a 1-rarefaction wave in the region, $R_1(t) \leq x \leq R_2(t)$, and a 2-shock wave, $x = S_2(t)$, emanating at the point $(x, t) = (x_3, t_3)$.
- III. given through only a 2-shock wave, $x = S_2(t)$, emanating at the point $(x, t) = (x_3, t_3)$.

5.3.3.1 Case-I

Since, $v_L < v_R$ and $u_L < u_R$, using Theorem 5.2.1, it can be concluded that the solution to the Riemann problem subject to the initial data (5.3.60) is given through a 1-shock curve, $x = S_1(t)$ and a 2-shock curve, $x = S_2(t)$ emanating from the point $(x, t) = (x_3, t_3)$ only if

$$u_L - u_R + \frac{v_R - v_L}{\sqrt{v_L v_R}} < 0, \quad (5.3.61)$$

and the solution for $t > t_3$ is given as follows

$$v(x, t) = \begin{cases} v_L e^{-t}, & \text{if } x \leq S_1(t), \\ z_6 e^{-t}, & \text{if } S_1(t) < x < S_2(t), \\ v_R e^{-t}, & \text{if } x \geq S_2(t), \end{cases} \quad (5.3.62)$$

$$u(x, t) = \begin{cases} u_L + x, & \text{if } x \leq S_1(t), \\ u_L - \frac{z_6 - v_L e^{-t_3}}{\sqrt{(v_L e^{-t_3} z_6)}} + x, & \text{if } S_1(t) < x < S_2(t), \\ u_R + x, & \text{if } x \geq S_2(t), \end{cases} \quad (5.3.63)$$

where the 1-shock curve, $x = S_1(t)$, is given by $S_1(t) = \frac{(e^{t_3} - e^t)}{\sqrt{(v_L z_6)}} + x_3$, and the 2-shock curve, $x = S_2(t)$, is given by $S_2(t) = \frac{(e^t - e^{t_3})}{\sqrt{(v_R z_6)}} + x_3$. Here $z = z_6$ is the solution to the equation

$$u_L - u_R - \frac{z - v_L e^{-t_3}}{\sqrt{(v_L e^{-t_3} z)}} - \frac{v_R e^{-t_3} - z}{\sqrt{(v_R e^{-t_3} z)}} = 0, \quad z \in (v_L, v_R). \quad (5.3.64)$$

Thus, the necessary and sufficient condition for the solution to the double Riemann problem is ultimately given through only a 1-shock wave and a 2-shock wave, in view of (5.3.58) and (5.3.61), is in the form of an inequality satisfied by the constants v_L, v_M and v_R as follows

$$f_4(v_L, v_M, v_R) < 0,$$

when $v_R < v_M < v_L$, $u_L < u_M < u_R$, and

$$f_4(v_L, v_M, v_R) = \frac{v_L - v_M}{\sqrt{(v_L v_M)}} + \frac{v_M - v_R}{\sqrt{(v_M v_R)}} + \frac{v_R - v_L}{\sqrt{(v_R v_L)}}. \quad (5.3.65)$$

5.3.3.2 Case-II

Similarly, since $v_L < v_R$ and $u_L < u_R$, using Theorem 5.2.1, it can be concluded that the solution to the Riemann problem subject to the initial data (5.3.60) is given through a 1-rarefaction wave in the region $R_1(t) < x < R_2(t)$ for $t > t_3$ and a 2-shock wave, $x = S_2(t)$ emanating from the point $(x, t) = (x_3, t_3)$ only if

$$u_L - u_R + \frac{v_R - v_L}{\sqrt{v_L v_R}} > 0, \quad (5.3.66)$$

and the solution for $t > t_3$ is given as follows

$$v(x, t) = \begin{cases} v_L e^{-t}, & \text{if } x \leq R_1(t), \\ z_7 e^{-t}, & \text{if } R_1(t) < x < R_2(t), \\ \left(\frac{e^{(t_3-t)} - 1}{x - x_3} \right), & \text{if } R_2(t) \leq x < S_2(t), \\ v_R e^{-t}, & \text{if } x \geq S_2(t), \end{cases} \quad (5.3.67)$$

$$u(x, t) = \begin{cases} u_L + x, & \text{if } x \leq R_1(t), \\ u_L - \frac{z_7 - v_L e^{-t_3}}{\sqrt{(v_L e^{-t_3} z_7)}} + x, & \text{if } R_1(t) < x < R_2(t), \\ u_R - \log \left(\frac{(x - x_3) v_R}{(e^t - e^{t_3})} \right) + x, & \text{if } R_2(t) \leq x < S_2(t), \\ u_R + x, & \text{if } x \geq S_2(t), \end{cases} \quad (5.3.68)$$

where the 1-rarefaction region, $R_1(t) < x < R_2(t)$, with $R_1(t) = \frac{(e^{t_3} - e^t)}{v_L} + x_3$ and $R_2(t) = \frac{(e^{t_3} - e^t)}{z_7} + x_3$, and the 2-shock curve, $x = S_2(t)$, is given by $S_2(t) = \frac{(e^t - e^{t_3})}{\sqrt{(v_L z_7)}} + x_3$ which is shown in Figure 5.3. Here $z = z_7$ is the solution to the equation

$$u_L - u_R + \log \left(\frac{v_L e^{-t_3}}{z} \right) - \frac{z - v_R e^{-t_3}}{\sqrt{(v_R e^{-t_3} z)}} = 0, \quad z \in (v_L, v_R). \quad (5.3.69)$$

Thus, the necessary and sufficient condition for the solution to the double Riemann problem is ultimately given through only 1-rarefaction wave and 2-shock wave, in view of (5.3.58) and (5.3.66), is in the form of an inequality satisfied by the constants v_L, v_M and v_R as follows

$$f_4(v_L, v_M, v_R) > 0,$$

when $v_L < v_M < v_R$, $u_L < u_M < u_R$, and $f_4(v_L, v_M, v_R)$ is defined as in (5.3.65).

5.3.3.3 Case-III

Similarly, since $v_L < v_R$ and $u_L < u_R$, using Theorem 5.2.1 it can be concluded that the solution to the Riemann problem subject to the initial data (5.3.60) is given through only a 2-shock curve, $x = S_2(t)$ emanating from the point $(x, t) = (x_3, t_3)$, only if

$$u_L - u_R + \frac{v_R - v_L}{\sqrt{v_L v_R}} = 0, \quad (5.3.70)$$

and the solution for $t > t_3$ is given as follows

$$v(x, t) = \begin{cases} v_L e^{-t}, & \text{if } x \leq S_2(t), \\ v_R e^{-t}, & \text{if } x > S_2(t), \end{cases} \quad (5.3.71)$$

$$u(x, t) = \begin{cases} u_L + x, & \text{if } x \leq S_2(t), \\ u_R + x, & \text{if } x > S_2(t), \end{cases} \quad (5.3.72)$$

where the 2-shock curve, $x = S_2(t)$, is given by $S_2(t) = \frac{(e^t - e^{t_3})}{\sqrt{(v_L v_R)}} + x_3$.

Thus, the necessary and sufficient condition for the solution to the double Riemann problem is ultimately given through only a 2-shock wave, in view of (5.3.58) and (5.3.70), is in the form an equality satisfied by the constants v_L, v_M and v_R as follows

$$f_4(v_L, v_M, v_R) = 0,$$

when $v_L < v_M < v_R$, $u_L < u_M < u_R$, and $f_4(v_L, v_M, v_R)$ is defined as in (5.3.65).

Lemma 5.3.3. *Consider the function $f_4(v_L, v_M, v_R)$ defined as in (5.3.15). Then f_4 is a positive function when $v_L < v_M < v_R$.*

Proof. Observe that $\lim_{v_R \rightarrow v_M} f_4 = 0$ and

$$\frac{\partial f_4}{\partial v_R} = \frac{(v_R - \sqrt{v_L v_M})(\sqrt{v_M} - \sqrt{v_L})}{2v_R \sqrt{v_R v_L v_M}} > 0,$$

since $v_R < v_M < v_L$. This implies that f_4 is monotonically increasing as v_R increases to v_M and since the function $f_4(v_L, v_M, v_M) = 0$, it follows that f_4 is a positive function when $v_L < v_M < v_R$. $\square$

Thus, in view of Lemma 5.3.3, the solution due to the interaction of a 2-shock wave, $x = \tilde{S}_2(t)$ originated from $(x, t) = (0, 0)$ and a 2-shock wave, $x = \bar{S}_2(t)$, originated from $(x, t) = (x_0, 0)$, is given through a 1-rarefaction wave and a 2-shock wave as given in Case-II of Section 5.3.3, for which the solution is given in the equations (5.3.67) and (5.3.68) and is depicted in Figure 5.3.

5.4 Conclusions

A double generalized Riemann problem is considered with two discontinuities in the initial data at $x = 0$ and $x = x_0$, for which the solution is given through two shocks either in 1-shock waves or 2-shock waves from the points $(x, t) = (0, 0)$ and $(x, t) = (x_0, 0)$. As the two shocks propagate into the medium at different speeds, the possibilities of overtaking these shocks are analysed. Further, owing to the interaction of shock waves, whenever possible, it is discussed about the behavior of the solution after the interaction of the two shocks, whether 1-shock/1-rarefaction wave or 2-shock/2-rarefaction waves are present in the solution.

The solution to the double Riemann problem is ultimately given through only a 1-shock wave and only a 2-rarefaction wave, when $u_L < u_M < u_R$ and the constants v_L, v_M and v_R satisfy the following conditions:

$$\bullet \frac{v_M - v_L}{\sqrt{(v_L v_M)}} + \frac{v_R - v_M}{\sqrt{(v_R v_M)}} + \frac{v_L - v_R}{\sqrt{(v_R v_L)}} > 0, \text{ when } v_R < v_M < v_L.$$

Similarly, the solution to the double Riemann problem is ultimately given through a 1-shock wave and a 2-shock wave when $u_L < u_M < u_R$ and when the constants v_L, v_M and v_R satisfy the following conditions:

$$\bullet \frac{v_L - v_M}{\sqrt{(v_L v_M)}} + \frac{v_R - v_M}{\sqrt{(v_R v_M)}} < \frac{v_L - v_R}{\sqrt{(v_R v_L)}} < \frac{v_M - v_L}{\sqrt{(v_L v_M)}} + \frac{v_M - v_R}{\sqrt{(v_R v_M)}},$$

when $v_R < v_M$, $v_L < v_M$.

Similarly, the solution to the double Riemann problem is ultimately given through a 1-rarefaction wave and a 2-shock wave when $u_L < u_M < u_R$, and the constants v_L, v_M and v_R satisfy the following conditions:

$$\bullet \frac{v_L - v_M}{\sqrt{(v_L v_M)}} + \frac{v_M - v_R}{\sqrt{(v_M v_R)}} + \frac{v_R - v_L}{\sqrt{(v_R v_L)}} > 0, \text{ when } v_L < v_M < v_R.$$

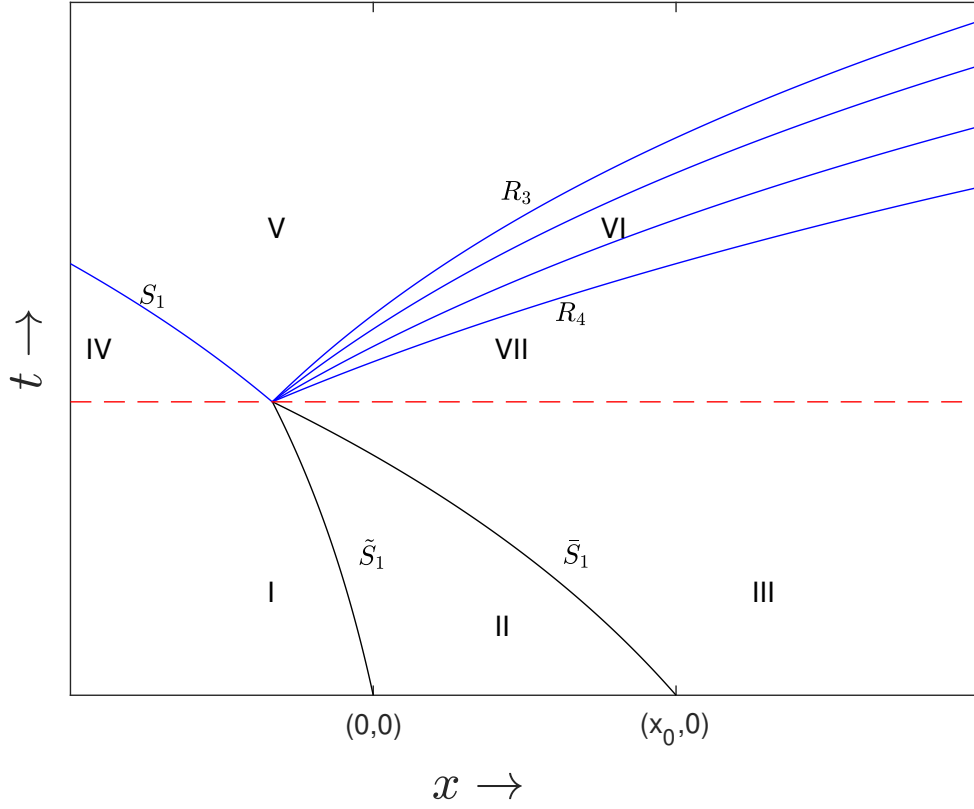


Figure 5.1: Before interaction, i.e., for $0 < t < t_1$, subject to the initial condition (5.3.1), the solutions in the region I, $x < \tilde{S}_1(t)$, in the region II, $\tilde{S}_1(t) \leq x \leq \bar{S}_1(t)$ and in the region III, $x > \bar{S}_1(t)$, are given in the equations (5.3.3) - (5.3.7). After the interaction, $t > t_1$, subject to the initial condition (5.3.10), the solution in the region IV, $x < S_1(t)$, in the region V, $S_1(t) \leq x \leq R_3(t)$, in the region VI, $R_3(t) < x < R_4(t)$ and in the region VII, $x \geq R_4(t)$, are given in equations (5.3.17) and (5.3.18).

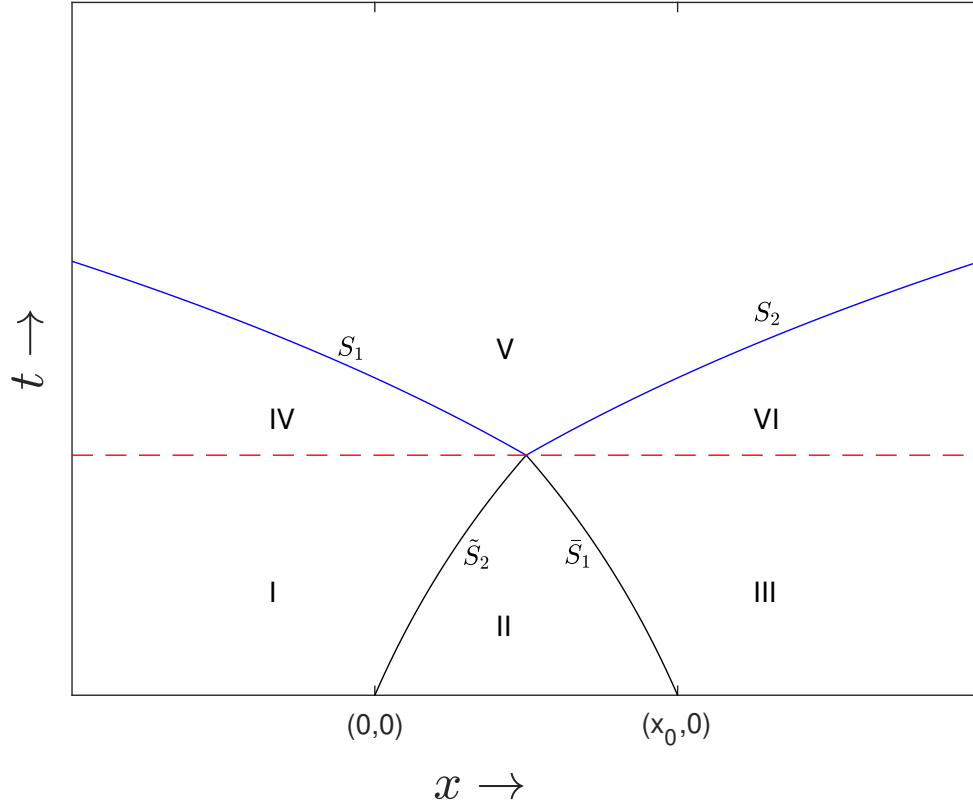


Figure 5.2: Before interaction, i.e., for $0 < t < t_2$, subject to the initial condition (5.3.1), the solutions in the region I, $x < \tilde{S}_2(t)$, in the region II, $\tilde{S}_2(t) \leq x \leq \bar{S}_1(t)$ and in the region III, $x > \bar{S}_1(t)$, are given by the equations (5.3.24)- (5.3.28). After the interaction, $t > t_2$, subject to the initial condition (5.3.31), the solution in the region IV, $x < S_1(t)$, in the region V, $S_1(t) \leq x \leq S_2(t)$ and in the region VI, $S_2(t) < x$ are given in equations (5.3.33) and (5.3.34).

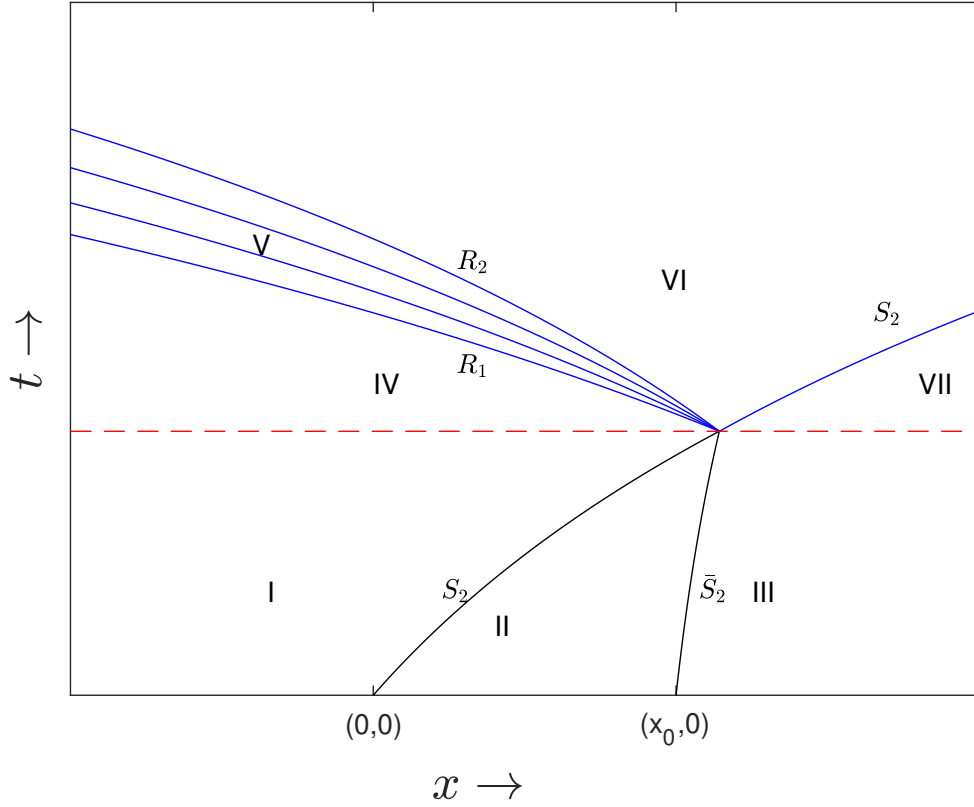


Figure 5.3: Before interaction, i.e., for $0 < t < t_3$, subject to initial condition (5.3.1), the solutions in the region I, $x < \tilde{S}_2(t)$, in the region II, $\tilde{S}_2(t) \leq x \leq \bar{S}_2(t)$ and in the region III, $x > \bar{S}_1(t)$, are given by the equations (5.3.53)- (5.3.57). After the interaction, $t > t_3$, subject to the initial condition (5.3.60), the solution in the region IV, $x < R_1(t)$, in the region V, $R_1(t) \leq x \leq R_2(t)$, in the region VI, $R_2(t) < x < S_2(t)$ and in the region VII, $x \geq S_2(t)$, are given in equations (5.3.67) and (5.3.68).

Chapter 6

Summary

The Riemann problem with non-constant initial data and the interaction of shock waves to several strictly hyperbolic systems is discussed using the differential constraints method, describing several physical phenomena in the thesis.

In Chapter 2, solutions to the Riemann problem with non-constant initial data for the rate-type material were determined by introducing differential constraints. Also, the elementary waves associated with the genuinely nonlinear characteristic fields were presented in explicit forms. The complete characterization of Riemann solutions of the rate-type model was presented in the form of Lemmas.

In Chapter 3, a first-order hyperbolic system of partial differential equations that represents the Chaplygin gas model was investigated. The method of differential constraints was employed to provide a characterization of the solutions for this particular model. Within the framework of this approach, the compatibility conditions between the differential constraints and the governing model were derived. Further, the solution to the Riemann problem for the governing model was obtained, which is characterized by characteristic shocks with initial data satisfying the differential constraints an enable to determine the exact solution for the generalized Riemann problem.

In Chapter 4, the solution structure of the generalized Riemann problem for generalized gas dynamic equations was focused, particularly when it involves non-constant initial data. It was observed that the solutions to the generalized Riemann problem can be effectively determined by the introduction of differential constraints. Additionally, explicit representations of the elementary waves associated with genuinely nonlinear characteristic fields were provided. The rarefaction solution was determined with the help of generalized Riemann invariants. The comprehensive description of Riemann so-

lutions for the generalized gas dynamic equations was presented in the form of Lemmas.

In Chapter 5, the interaction of the double generalized Riemann problem for rate-type material was examined. Essentially, explicit solutions were provided before and after the interaction of shock waves in different cases. Moreover, the conditions were derived that guide in determining the structure of the new Riemann solution after the interaction.

Overall, the generalized Riemann problem extends the principles of the classic Riemann problem to handle more complex and realistic scenarios, making it a crucial tool in numerical simulations and modeling of hyperbolic systems in various scientific domains.

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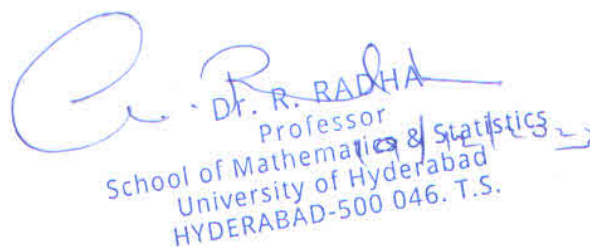
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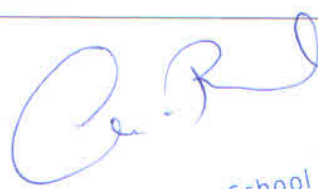
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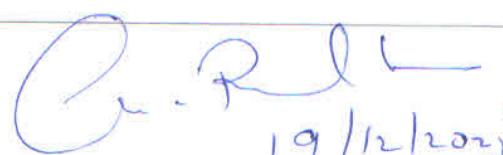
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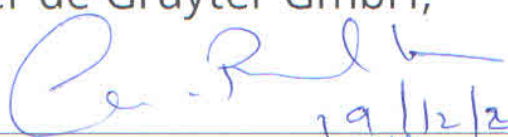
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