ON SOME ANALYTIC ASPECTS AND AVERAGE BEHAVIOR OF CERTAIN L-FUNCTIONS

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by

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CERTIFICATE

This is to certify that the thesis entitled "**On some analytic aspects and average behavior of certain L-functions**" by **Anubhav Sharma** bearing Reg. No. 19MMPP02 in partial fulfillment of the requirements for the award of Doctor of Philosophy in **Mathematics** is a bonafide work carried out by him under my supervision and guidance.

The thesis has not been submitted previously in part or in full to this or any other University or Institution for the award of any degree or diploma.

Parts of this thesis have been:

A. published in the following publications:

1. A. Sharma and A. Sankaranarayanan, Discrete mean square of the coefficients of symmetric square L-functions on certain sequence of positive numbers, *Res. Number Theory* **8(1)** (2022), 1-13.

2. A. Sharma and A. Sankaranarayanan, Higher moments of the Fourier coefficients of symmetric square L-functions on certain sequence, *Rend. Circ. Mat. Palermo (2)* (2022), 1-18.

3. A. Sharma and A. Sankaranarayanan, Average behavior of the Fourier coefficients of symmetric square L-function over some sequence of integers, *Integers* **22** (2022).

4. A. Sharma and A. Sankaranarayanan, On the average behavior of the Fourier coefficients of j^{th} symmetric power L-function over a certain sequences of positive integers, (accepted in *Czechoslovak Math. J.* on 02-01-2023).

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3. International Conference on Emerging Trends in Pure and Applied Mathematics, March 12-13, 2022, Tezpur University, Assam.

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2	MM803	Analysis	5	Pass
3	CS800	Research Methods in Computer Science	4	Pass
4	MM831	Research and Publications Ethics	2	Pass
5	MM899	Analytic Number Theory	5	Pass

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DECLARATION

I, **Anubhav Sharma**, hereby declare that this thesis entitled "**On some analytic aspects and average behavior of certain L-functions**" submitted by me under the guidance and supervision of **Prof. A. Sankaranarayanan** is a bonafide research work. I also declare that it has not been submitted previously in part or in full to this University or any other University or Institution for the award of any degree or diploma.

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ABSTRACT

This thesis consists of six chapters. For the convenience of the reader, we are now providing a quick summary of the work completed in each chapter.

- **Chapter 1:** In this chapter, we cover some elementary definitions and results of number theory and we give the necessary prerequisites which are important in order to understand the statement of the results and their proofs.

- **Chapter 2:** In this chapter, first we recall the recent work done by various authors which are used in our discussion and then we consider the behavior of discrete mean square of the n^{th} normalized Fourier coefficients of symmetric square L-function over certain sequence of positive integers.

- **Chapter 3:** In this chapter, we consider the higher discrete power moments of the n^{th} normalized Fourier coefficients of symmetric square L-functions on the same sequence of positive numbers.

- **Chapter 4:** In this chapter, we study the average behavior of n^{th} normalized Fourier coefficients of symmetric square L-function (i.e. $L(s, \text{sym}^2 f)$) on a higher dimension.

- **Chapter 5:** In this chapter, we improve as well as generalize the result of previous chapter. We investigate the average behavior of the n^{th} normalized Fourier coefficients of the j^{th} symmetric power L-function attached to a primitive holomorphic cusp form of weight k for the full modular group $SL(2, \mathbb{Z})$ over some sequence of positive integers and give a tightened error term.

- **Chapter 6:** In this chapter, we consider the integral power sums of coefficients of the Dedekind zeta-function of a non-normal cubic extension $\mathbb{K}_3$

of rational field $\mathbb{Q}$ and prove an asymptotic formula for the partial sum of coefficients of the k^{th} integral power of Dedekind zeta function for any integer $k \geq 1$.

SYNOPSIS

1 Introduction

The word "modular form" has a fairly broad definition. Although they play key roles in number theory's numerous branches and are most naturally found there, these objects also have a significant impact on other areas of mathematics. For instance, modular forms might evoke varied ideas like Fermat's Last Theorem, the Riemann Hypothesis, the Langlands programme, and applications of arithmetic or applications to string theory using L-functions and elliptic curves. Obviously, this is just a small selection, the roles that modular forms play; there are a plethora of other factors that keep research on modular forms quite current. The first half of the nineteenth century, during the time of Jacobi and Eisenstein, is when the definition of modular forms originally appeared. Since then, numerous generalisations have been identified and researched, and when combined with traditional modular forms, they help to explain many of the impressions that modular forms have left behind.

However, the history of modular forms starts with elliptic functions, which are doubly periodic meromorphic complex functions and thus a more distant relative of Jacobi's θ-functions. Elliptic functions, which date back to Gauss and were explored by Weierstrass, naturally led to the study of elliptic curves, which are closely related to modular forms. When it comes to modular forms for the full modular group $SL(2, \mathbb{Z})$, a cusp form can be identified by the disappearance of the constant coefficient c_0 in the Fourier series expansion.

The study of Dirichlet series with the form $\sum\limits_{n=1}^{\infty} \frac{f(n)}{n^s}$ has a long history and dates to the nineteenth century. This interest was primarily sparked by the prominent place that these series take in analytic number theory. Among others, Hadamard, Landau, Hardy, Riesz, and Bohr created the general theory of Dirichlet series.

A natural object to study in light of connections between the modular form and Dirichlet series formed with the same coefficients is the Fourier coefficients or Hecke eigenvalues.

Every primitive holomorphic cusp form $f(z)$ of weight k for the full modular group $SL(2,\mathbb{Z})$ has a Fourier expansion at the cusp ∞ of the type

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z},$$

where $\Im(z) > 0$. The coefficients $\lambda_f(n)$ satisfy the Deligne's bound, i.e.,

$$|\lambda_f(n)| \leq d(n) \ll n^{\varepsilon},$$

where $d(n)$ is the divisor function and ε denotes an arbitrarily small positive constant.

Understanding the behavior of Hecke eigenvalues $\lambda_f(n)$ is a key issue in the study of classical modular forms. Many authors have contributed in the divisor problems pertaining to these normalized n^{th} Fourier coefficients of Fourier expansion of $f(z)$ at the cusp ∞, for instance see [Fo 2, Zh]. Each cusp form f has a symmetric square L-function attached to it being defined as

$$\begin{aligned} L(s, \mathrm{sym}^2 f) &:= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^2 f}(n)}{n^s} \\ &= \prod_p \left(1 - \frac{\alpha^2(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta^2(p)}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-1}, \end{aligned}$$

where α and β are complex numbers, given by P. Deligne [De 1].

Our aim in this thesis is to investigate the average behavior of the Fourier coefficients associated to symmetric square L-functions, symmetric j^{th} power L-functions and Dedekind zeta-function under various interesting situations.

We will now explain a brief structure of our proposed thesis. There are six chapters in the thesis. In the first chapter, we will recall all the definitions and lemmas that will be of great essence to the soul of our thesis. Mainly, we will emphasize on arithmetic functions, characters of finite Abelian groups, modular and cusp forms, Perron's formula, Dirichlet L-functions, Dirichlet series and their convergence, representations of integers as sum of squares and properties of various L-functions such as symmetric j^{th} power L-functions, Dedekind zeta-function etc. We have also stated some elementary lemmas related to the bounds of Riemann zeta function and various L-functions that we are going to use frequently in the proof of our main theorems. Along with the analytic continuations of some L-functions, functional equations also have been discussed for the better understanding of the nature of these L-functions.

Second chapter deals with the discrete mean square of the n^{th} normalized Fourier coefficients of symmetric square L-function (i.e., $L(s, \text{sym}^2 f)$) over certain sequence of positive integers. To be more precise, we will see the average behavior of these Hecke eigenvalues over $r_4(n)$, where $r_4(n)$ is defined as the number of representations on n as sum of four squares. We will establish an asymptotic formula for the same. First, we prove the following Lemma, which is related to the decomposition of certain L-functions. From [Ha-Wr, pp. 415], we can write

$$\begin{aligned} r_4(n) &= 8 \sum_{d|n} \widetilde{\chi}_0(d) d \\ &=: 8r(n), \end{aligned}$$

where $\widetilde{\chi}_0$ is a character modulo 4, given by

$$\widetilde{\chi}_0(p^u) := \begin{cases} \chi_0(p^u) & \text{if } p > 2 \\ 3 & \text{if } p = 2 \end{cases},$$

and χ_0 is the principal character modulo 4.

Lemma 0.0.1 *Let f be a normalized primitive holomorphic cusp form of weight k for $SL(2, \mathbb{Z})$, and let $\lambda_{sym^2 f}(n)$ be the n^{th} normalized Fourier coefficient of the symmetric square L-function associated to f. If*

$$F_2(s) = \sum_{n=1}^{\infty} \frac{\lambda^2_{sym^2 f}(n) r(n)}{n^s},$$

for $\Re(s) > 2$, then

$$F_2(s) = G_2(s) H_2(s),$$

where

$$\begin{aligned} G_2(s) := &\zeta(s)L(s-1,\widetilde{\chi}_0)L(s,sym^2 f)L(s-1,sym^2 f\otimes\widetilde{\chi}_0) \\ &L(s,sym^4 f)L(s-1,sym^4 f\otimes\widetilde{\chi}_0), \end{aligned}$$

and $\widetilde{\chi}_0$ is a character modulo 4.
Here, $H_2(s)$ is a Dirichlet series which converges uniformly, and absolutely in the half plane $\Re(s) > \frac{3}{2}$, and $H_2(s) \neq 0$ on $\Re(s) = 2$.

With the help of above stated lemma, we will prove the key theorem of this chapter, i.e.,

Theorem 0.0.2 *For $x \geq x_o$ (sufficiently large), we have*

$$\sum_{\substack{a^2+b^2+c^2+d^2\leq x \\ (a,b,c,d)\in\mathbb{Z}^4}} \lambda^2_{sym^2 f}(a^2+b^2+c^2+d^2) = c_2 x^2 + O\left(x^{\frac{9}{5}+\varepsilon}\right),$$

where c_2 is an effective constant defined as

$$c_2 = (-2)\zeta(2)L(2,sym^2 f)L(1,sym^2 f\otimes\widetilde{\chi}_0)L(2,sym^4 f)L(1,sym^4 f\otimes\widetilde{\chi}_0)H_2(2),$$

and $H_2(2) \neq 0$, and $\widetilde{\chi}_0$ is a character modulo 4.

Remark 0.0.3 We observe from Lemma 0.0.1 that

$$F_2(s) = G_2(s)H_2(s),$$

where $G_2(s)$ is a product of certain L-functions and $H_2(s)$ has an Euler product, which is uniformly, and absolutely convergent in $\sigma \geq \frac{3}{2} + 2\varepsilon$ for any small positive constant ε. We also know good amount of analytic properties of $G_2(s)$, and each factor of $G_2(s)$ satisfies a functional equation of the Riemann zeta type. It will be clear from our demonstration of proof that we have only employed the previously mentioned known analytical features of $H_2(s)$. If one can find out more information of $H_2(s)$ in the region $\Re(s) \geq (1-10\varepsilon)$ then it may even lead to the following conjecture.

Conjecture 0.0.4 For sufficiently large x, we have

$$\sum_{\substack{n=a^2+b^2+c^2+d^2\leq x\\(a,b,c,d)\in\mathbb{Z}^4}} \lambda^2_{\mathrm{sym}^2 f}(a^2+b^2+c^2+d^2) = \tilde{c}_1x^2+\tilde{c}_2x+O(x^\theta),$$

where $\tilde{c}_1$, $\tilde{c}_2$ are effective constants, and θ is some positive constant satisfying $0<\theta<1$.

The higher discrete power moments is covered in the third chapter. More precisely, we study the behavior of the following sums:

$$\sum_{\substack{a^2+b^2+c^2+d^2\leq x\\(a,b,c,d)\in\mathbb{Z}^4}} \lambda^3_{\mathrm{sym}^2 f}(a^2+b^2+c^2+d^2)$$

and

$$\sum_{\substack{a^2+b^2+c^2+d^2\leq x\\(a,b,c,d)\in\mathbb{Z}^4}} \lambda^4_{\mathrm{sym}^2 f}(a^2+b^2+c^2+d^2).$$

and establish the Theorem 0.0.5 and Theorem 0.0.6.

Theorem 0.0.5 *For $x\geq x_0$ (sufficiently large), and $\varepsilon>0$ be any small constant, we have*

$$\sum_{\substack{a^2+b^2+c^2+d^2\leq x\\(a,b,c,d)\in\mathbb{Z}^4}} \lambda^3_{sym^2 f}(a^2+b^2+c^2+d^2) = c_3x^2+O\left(x^{\frac{27}{14}+\varepsilon}\right)$$

where c_3 is an effective constant defined as

$$\begin{aligned}c_3 &= (-2)\zeta(2)L^2(2,sym^2 f)L^2(1,sym^2 f\otimes\widetilde{\chi}_0)L(2,sym^4 f)L(1,sym^4 f\otimes\widetilde{\chi}_0)\\&\quad\times L(2,sym^2 f\otimes sym^4 f)L(1,sym^2 f\otimes sym^4 f\otimes\widetilde{\chi}_0)H_3(2),\end{aligned}$$

$H_3(s)$ is a Dirichlet series that converges uniformly, and absolutely in the half plane $\Re(s)>\frac{3}{2}$, and $H_3(s)\neq 0$ on $\Re(s)=2$, and $\widetilde{\chi}_0$ is a character modulo 4.

Theorem 0.0.6 *For $x\geq x_0$ (sufficiently large), and $\varepsilon>0$ be any small constant,* *we have*

$$\sum_{\substack{a^2+b^2+c^2+d^2\leq x\\(a,b,c,d)\in\mathbb{Z}^4}} \lambda_{sym^2 f}^4(a^2+b^2+c^2+d^2) = c_4 x^2\log x + O\left(x^{\frac{160}{81}+\varepsilon}\right),$$

where c_4 is an effective constant defined as

$$\begin{aligned}c_4 = {} & \zeta^2(2)L^3(2,sym^2 f)L^3(1,sym^2 f\otimes\widetilde{\chi}_0)L^3(2,sym^4 f)\\ & \times L^3(1,sym^4 f\otimes\widetilde{\chi}_0)L^2(2,sym^2 f\otimes sym^4 f)L^2(1,sym^2 f\otimes sym^4 f\otimes\widetilde{\chi}_0)\\ & \times L(2,sym^4 f\otimes sym^4 f)L(1,sym^4 f\otimes sym^4 f\otimes\widetilde{\chi}_0)H_4(2).\end{aligned}$$

$H_4(s)$ is a Dirichlet series that converges uniformly, and absolutely in the half plane $\Re(s)>\frac{3}{2}$, and $H_4(s)\neq 0$ on $\Re(s)=2$, and $\widetilde{\chi}_0$ is a character modulo 4.

In order to prove the above stated theorems, we will prove some lemmas related to the decomposition of corresponding L-functions but here it is much more complicated than the one we stated in chapter 2.

Lemma 0.0.7 *Let f be a normalized primitive holomorphic cusp form of weight k for $SL(2,\mathbb{Z})$ and let $\lambda_{sym^2 f}(n)$ be the n^{th} normalized Fourier coefficient of the symmetric square L-function associated to f. If*

$$F_3(s) = \sum_{n=1}^{\infty}\frac{\lambda_{sym^2 f}^3(n)r(n)}{n^s},$$

for $\Re(s)>2$, then

$$F_3(s) = G_3(s)H_3(s),$$

where

$$\begin{aligned}G_3(s) :={} & \zeta(s)L(s-1,\widetilde{\chi}_0)L^2(s,sym^2 f)L^2(s-1,sym^2 f\otimes\widetilde{\chi}_0)L(s,sym^4 f)\\ & L(s-1,sym^4 f\otimes\widetilde{\chi}_0)L(s,sym^2 f\otimes sym^4 f)L(s-1,sym^2 f\otimes sym^4 f\otimes\widetilde{\chi}_0),\end{aligned}$$

and $\widetilde{\chi}_0$ is a character modulo 4.
Here, $H_3(s)$ is a Dirichlet series which converges uniformly and absolutely in the half plane $\Re(s)>\frac{3}{2}$ and $H_3(s)\neq 0$ on $\Re(s)=2$.

Lemma 0.0.8 *Let f be a normalized primitive holomorphic cusp form of weight k for $SL(2,\mathbb{Z})$ and let $\lambda_{sym^2 f}(n)$ be the n^{th} normalized Fourier coefficient of the symmetric square L-function associated to f. If*

$$F_4(s) = \sum_{n=1}^{\infty} \frac{\lambda^4_{sym^2 f}(n) r(n)}{n^s}$$

for $\Re(s) > 2$, then

$$F_4(s) = G_4(s) H_4(s),$$

where

$$\begin{aligned} G_4(s) :=& \zeta^2(s) L^2(s-1, \widetilde{\chi}_0) L^3(s, sym^2 f) L^3(s-1, sym^2 f \otimes \widetilde{\chi}_0) L^3(s, sym^4 f) \\ & \times L^3(s-1, sym^4 f \otimes \widetilde{\chi}_0) L^2(s, sym^2 f \otimes sym^4 f) L^2(s-1, sym^2 f \otimes sym^4 f \otimes \widetilde{\chi}_0) \\ & \times L(s, sym^4 f \otimes sym^4 f) L(s-1, sym^4 f \otimes sym^4 f \otimes \widetilde{\chi}_0), \end{aligned}$$

and $\widetilde{\chi}_0$ is a character modulo 4.
Here, $H_4(s)$ is a Dirichlet series which converges uniformly and absolutely in the half plane $\Re(s) > \frac{3}{2}$ and $H_4(s) \neq 0$ on $\Re(s) = 2$.

From the proof of our theorems, we understand that there might be a possibility that the integration line could be shifted to the left of the line $\Re(s) = 1$. This prompts us to suggest:

Conjecture 0.0.9 For sufficiently large x and $\varepsilon > 0$ be any small constant, we have

$$\sum_{\substack{n=a^2+b^2+c^2+d^2 \leq x \\ (a,b,c,d) \in \mathbb{Z}^4}} \lambda^{\theta}_{sym^2 f}(a^2+b^2+c^2+d^2) = \begin{cases} \widetilde{c}_3(\theta) x^2 + \widetilde{c}_4(\theta) x + O(x^{\mu_1(\theta)+\varepsilon}) & \text{if } \theta = 3 \\ \hat{c}_3(\theta) x^2 \log x + \hat{c}_4(\theta) x \log x + O(x^{\mu_2(\theta)+\varepsilon}) & \text{if } \theta = 4 \end{cases}.$$

where $\widetilde{c}_3(\theta)$, $\widetilde{c}_4(\theta), \hat{c}_3(\theta), \hat{c}_4(\theta)$ are effective constants and $\mu_1(\theta), \mu_2(\theta)$ are some positive constants satisfying $0 < \mu_1(\theta), \mu_2(\theta) < 1$.

In the fourth chapter, we look into the possibility of enhancing the dimension from 4 to 6. We shall observe the average behavior of the symmetric square L-function. To be more specific, we investigate the nature of the following sum:

$$\sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\leq x\\(a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} \lambda^2_{\mathrm{sym}^2 f}(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2).$$

for sufficiently large x. More specifically, we demonstrate the following.

Theorem 0.0.10 *For sufficiently large x, and any $\varepsilon > 0$, we have*

$$\sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\leq x\\(a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} \lambda^2_{sym^2 f}(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2) = \mathfrak{c}_2 x^3 + O\left(x^{\frac{14}{5}+\varepsilon}\right).$$

Here, $\mathfrak{c}_2$ is an effective constant defined as

$$\mathfrak{c}_2 = \frac{16}{3}L(3,\chi)L(1,sym^2 f)L(3,sym^2 f\otimes\chi)L(1,sym^4 f)L(3,sym^4 f\otimes\chi)\mathfrak{H}_2(3),$$

and χ is the non-principal Dirichlet character modulo 4. Here, $\mathfrak{H}_2(s)$ is a Dirichlet series which converges uniformly and absolutely in the half plane $\Re(s) > \frac{5}{2}$, and $\mathfrak{H}_2(s) \neq 0$ on $\Re(s) = 3$.

The key idea is that the sum in question is being related to the sum involving $r_6(n)$. The main difference here from our earlier result 0.0.2 related to sum involving $r_4(n)$ is that $r_6(n)$ is not multiplicative. However, we will observe that $r_6(n)$ can be split into the sum of two multiplicative functions. Then the sum in question being split into two sums involving the corresponding multiplicative functions, and being dealt with independently and then we have to glue them to obtain our result.

The generalization and improvement of the result obtained in chapter four is discussed in chapter five. Very recently, Newton and Thorne proved the automorphy of the symmetric power lifting $\mathrm{sym}^n(f)$ for every $n \geq 1$, where f is a cuspidal Hecke eigenform of level 1 (for instance, see [Ne-Th 1, Ne-Th 2]). Due to these ground breaking results, we were enable to obtain the generalization. We have also incorporated better average or individual sub convexity bounds for the concerned L-functions. In this regard, we have proved the following theorem.

Theorem 0.0.11 *Let $j \geq 2$ be any fixed integer. For sufficiently large x, and*

$\varepsilon > 0$ any small constant, we have

$$\sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\leq x\\(a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} \lambda^2_{sym^j f}(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2) = c(j)x^3+O\left(x^{3-\frac{6}{3(j+1)^2+1}+\varepsilon}\right),$$

where $c(j)$ is an effective constant defined as

$$c(j) = \frac{16}{3}L(3,\chi)\prod_{n=1}^{j}L(1, sym^{2n}f)L(3, sym^{2n}f\otimes\chi)H_j(3),$$

and χ is the non-principal Dirichlet character modulo 4.

The key decomposition is given by the following lemmas. From [Ha-Wr, pp. 415], we can write

$$\begin{aligned} r_6(n) &= 16\sum_{d|n}\chi(d)\frac{n^2}{d^2} - 4\sum_{d|n}\chi(d)d^2 \\ &=: 16l(n) - 4v(n), \end{aligned}$$

where χ is the non-principal Dirichlet character modulo 4.

Lemma 0.0.12 *Let f be a normalized primitive holomorphic cusp form of weight k for $SL(2,\mathbb{Z})$, and let $\lambda_{sym^j f}(n)$ be the n^{th} normalized Fourier coefficient of the j^{th} symmetric power L-function associated to f. If*

$$F_j(s) = \sum_{n=1}^{\infty}\frac{\lambda^2_{sym^j f}(n)l(n)}{n^s},$$

for $\Re(s) > 3$, then

$$F_j(s) = G_j(s)H_j(s),$$

where

$$G_j(s) := \zeta(s-2)L(s,\chi)\prod_{n=1}^{j}L(s-2, sym^{2n}f)L(s, sym^{2n}f\otimes\chi),$$

and χ *is the non-principal character modulo* 4.
Here, $H_j(s)$ *is a Dirichlet series which converges uniformly, and absolutely in the half plane* $\Re(s) > \frac{5}{2}$*, and* $H_j(s) \neq 0$ *on* $\Re(s) = 3$.

Lemma 0.0.13 *Let* f *be a normalized primitive holomorphic cusp form of weight* k *for* $SL(2,\mathbb{Z})$*, and let* $\lambda_{sym^j f}(n)$ *be the* n^{th} *normalized Fourier coefficient of the* j^{th} *symmetric power* L*-function associated to* f*. If*

$$\widetilde{F}_j(s) = \sum_{n=1}^{\infty} \frac{\lambda^2_{sym^j f}(n) v(n)}{n^s},$$

for $\Re(s) > 3$*, then*

$$\widetilde{F}_j(s) = \widetilde{G}_j(s)\widetilde{H}_j(s),$$

where

$$\widetilde{G}_j(s) := \zeta(s)L(s-2,\chi)\prod_{n=1}^{j} L(s, sym^{2n} f)L(s-2, sym^{2n} f \otimes \chi),$$

and χ *is the non-principal character modulo* 4.
Here $\widetilde{H}_j(s)$ *is a Dirichlet series which converges uniformly, and absolutely in the half plane* $\Re(s) > \frac{5}{2}$*, and* $\widetilde{H}_j(s) \neq 0$ *on* $\Re(s) = 3$.

In our last chapter, we investigate a divisor problem related to a certain Dedekind zeta-function. It reveals the nature of integral power sums of coefficients of the Dedekind zeta-function of a non-normal cubic extension $\mathbb{K}_3$ of rational field $\mathbb{Q}$. The k^{th} integral power of Dedekind zeta function is defined as

$$(\zeta_{\mathbb{K}_3}(s))^k = \sum_{n=1}^{\infty} \frac{a_{k,\mathbb{K}_3}(n)}{n^s}$$

for $\Re(s) > 1$, where $a_{k,\mathbb{K}_3}(n) = \sum_{n=n_1 n_2 \ldots n_k} a_{\mathbb{K}_3}(n_1)a_{\mathbb{K}_3}(n_2)\ldots a_{\mathbb{K}_3}(n_k)$.

For any integer $k \geq 2$, writing,

$$\sum_{n \leq x} a_{k,\mathbb{K}_3}(n) = M_{k,\mathbb{K}_3}(x) + E_{k,\mathbb{K}_3}(x),$$

where $M_{k,\mathbb{K}_3}(x)$ is the main term which is of the form $xP_{k-1}(\log x)$, where $P_{k-1}(t)$ is a polynomial in t of degree $k-1$. We have proved the following theorem.

Theorem 0.0.14 *Let* $\varepsilon > 0$ *(be any small constant) and define* $\lambda_1 = 3\varepsilon$, $\lambda_2 = \min\left(2\mu, \frac{1}{4}\right)$, $\lambda_3 = \min\left(\mu + \frac{1}{2}, \frac{5}{8}\right)$, $\lambda_4 = \min\left(2\mu + \frac{3}{4}, 1\right)$, $\lambda_5 = \min\left(3\mu + 1, \frac{3}{2}\right)$ *and* $\lambda_k = \mu\,(k-6) + \frac{k}{3}$ *for* $k \geq 6$.
Then we have for any integer $k \geq 1$,

$$E_{k,\mathbb{K}_3}(x) \ll x^{1-\frac{1}{2(1+\lambda_k)}+3k\varepsilon}.$$

At the end of this chapter, we took the liberty to propose a conjecture that heavily depends on the famous Lindelöf hypothesis, which is yet to be settled.

Conjecture 0.0.15 For any integer $k \geq 2$ and any small positive constant ε, we have

$$E_{k,\mathbb{K}_3}(x) \ll_\varepsilon x^{\frac{1}{2}+c(k)\varepsilon},$$

where $c(k)$ is a positive constant depending only on k.

Publications concerning this thesis:

1. A. Sharma and A. Sankaranarayanan, Discrete mean square of the coefficients of symmetric square L-functions on certain sequence of positive numbers, *Res. Number Theory* **8(1)** (2022), 1-13.

2. A. Sharma and A. Sankaranarayanan, Higher moments of the Fourier coefficients of symmetric square L-functions on certain sequence, *Rend. Circ. Mat. Palermo (2)* (2022), 1-18.

3. A. Sharma and A. Sankaranarayanan, Average behavior of the Fourier coefficients of symmetric square L-function over some sequence of integers, *Integers* **22** (2022).

4. A. Sharma and A. Sankaranarayanan, On the average behavior of the Fourier coefficients of j^{th} symmetric power L-function over a certain sequences of positive integers, (accepted in *Czechoslovak Math. J.* on 02-01-2023).

5. A. Sharma and A. Sankaranarayanan, On a divisor problem related to a certain Dedekind zeta-function, (accepted in *J. Ramanujan Math. Soc.* on 27-09-2022).

In loving memory of my father & sister

Ravinder & Ankita

Contents

Remarks on Notation

- The letter p with or without indices will be reserved for prime numbers, unless otherwise explicitly stated.
- The letter $\mathbb{C}$ will denote the complex field, $\mathbb{R}$ the field of real numbers, $\mathbb{Q}$ the field of rationals, $\mathbb{Z}$ the ring of all rational integers and $\mathbb{N}$ the set of positive integers.
- The cardinality of a set A will be denoted by $\#A$.
- For a function f, we denote its image by $\mathrm{Im}(f)$.
- We will denote a complex variable by $s = \sigma + it$, $\sigma = \Re(s)$ and $t = \Im(s)$ being the real and complex part of s, respectively, where i is the fixed square root of -1.
- δ and ε always denote sufficiently small fixed positive constants.
- The parameters T and x are sufficiently large real numbers and $k \geq 1$ is an integer, except when explicitly stated.
- Let $k > 0$ be given. For each integer n, the congruence class or equivalence class of n modulo k is defined as

$$\hat{n} = \{x : x \equiv n \pmod{k}\}$$

  .
- We will denote the number of divisors of a positive integer n by $d(n)$, i.e.,

$$d(n) = \sum_{kl=n} 1.$$

- We will denote the Riemann-zeta function by $\zeta(s)$ (as usual), which is defined as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

for $\Re(s) > 1$. Also (for $\Re(s) > 1$), we can write $\zeta(s)$ as Euler product, namely,

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where the product runs over all primes p.

- **Big O.** Let a be any real number including the possibilities $\pm\infty$. Let $f(x)$ and $g(x)$ be two functions defined in some neighborhood of a and suppose that $g(x) > 0$. We say that f(x) is "big O of g(x)" and we write

$$f(x) = O(g(x)) \quad \text{or} \quad f(x) \ll g(x),$$

if there exists a constant $K > 0$ and a neighborhood $N(a)$ of a such that

$$|f(x)| \leq Kg(x)$$

for all x in $N(a)$. In particular, the notation

$$f(x) = O(1)$$

means that $f(x)$ is bounded in absolute value in a suitable neighborhood of a.

Some care must be exercised in its use and interpretation. For example, we frequently encounter a function $f(s)$ of the complex variable $s = \sigma + it$ and write

$$f(s) = O(g(t)) \qquad (t \to \infty).$$

The constant K whose existence is implied by the O is dependent upon σ, and the dependence may be such that $K = K(\sigma)$ is unbounded for σ in some neighborhood. Sometimes the dependence of K on the auxiliary variables or parameters is explicitly stated and sometimes it is implied by the context.

To allow for greater flexibility and to use the O symbol as effectively as possible, it is convenient to define $O(g(x))$ standing by itself. By $O(g(x))$, we shall mean the class of functions $C(g)$ such that $f \in C(g)$ if and only if

$$f(x) = O(g(x)).$$

- $f(x) \gg g(x)$ will mean $g(x) \ll f(x)$.
- **Little o.** Suppose that $f(x)$ and $g(x)$ are defined in a neighborhood of a, and suppose that $g(x) > 0$. Then we say that $f(x)$ is "little o of $g(x)$)" and we write

$$f(x) = o(g(x))$$

if

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0.$$

- **Asymptotic equality.** If f and g are two functions defined in a neighborhood of a, we say that f is asymptotic to g and write

$$f \sim g,$$

if

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 1.$$

The definition applies to both functions of real or complex variables. The relation is evidently symmetric and transitive.

- We will use the notation $f(s) \asymp g(x)$ to mean $g(x) \ll f(x) \ll g(x)$.
- (Unless stated otherwise) all our constants will be effective. In other words, they can be calculated explicitly.
- The constants γ_n are called the Stieltjes constants and can be defined by the limit

$$\gamma_n := \lim_{m \to \infty} \left[\left(\sum_{k=1}^{m} \frac{(\log k)^n}{k} \right) - \frac{(\log m)^{n+1}}{n+1} \right].$$

$\gamma_0 = \gamma$ is called Euler-Mascheroni constant or simply Euler's constant.

Introduction

Numerous researchers from various scientific fields have studied the L-functions extensively since it was first discovered. It will be impossible to recall everything that has been written in mathematics literature due to the vast implications of such discoveries. Although some of them will be covered in more detail later, we will only briefly touch on the main points and findings that are pertinent to our discussion in the latter chapters.

The study of L-functions is one of the central themes in number theory and the first and perhaps the most well known example is the Riemann zeta-function. These L-functions typically store fundamental arithmetic information, such as the distribution of prime numbers, that is not readily apparent from their descriptions and typically necessitates understanding of their poles, zeroes, functional equations, and other analytic features.

In this thesis, we study problems related to the average behavior of the Fourier coefficients of cusp forms in various situations and also a divisor problem related to a certain Dedekind zeta-function. There are six chapters in the thesis. The first chapter is devoted to a comprehensive study of the topics in number theory that are pertinent to our goal and, hopefully, reflect our efforts to make this thesis as self-sufficient as feasible.

In Chapter 1, some elementary definitions and concepts of number theory like arithmetic functions, characters of finite Abelian groups, modular and cusp forms, Perron's formula, Dirichlet L-functions, representations of integers as sum of Squares and properties of various L-functions etc. that are relevant for our purpose have been discussed. We have also stated some elementary lemmas related to the Riemann zeta-function and various L-functions that we are going to use frequently in the proof of our theorems.

In Chapter 2, we are concerned with the discrete mean square of the n^{th}

normalized Fourier coefficients of symmetric square L-function (i.e. $L(s, \mathrm{sym}^2 f)$) over certain sequence of positive integers and establish an asymptotic formula for the same.

In Chapter 3, we consider the higher discrete power moments of the Fourier coefficients of symmetric square L-functions on the same sequence of positive numbers.

We shall look into the prospect of the dimension being extended in chapter 4. We will see the average behavior of n^{th} normalized Fourier coefficients of symmetric square L-function (i.e. $L(s, \mathrm{sym}^2 f)$) over $r_6(n)$ (cf. section 1.4.2).

In Chapter 5, we improve as well as generalize the result of previous chapter, i.e., we investigate the average behavior of the n^{th} normalized Fourier coefficients of the j^{th} symmetric power L-function attached to a primitive holomorphic cusp form of weight k for the full modular group $SL(2, \mathbb{Z})$ over some sequence of positive integers.

In our last Chapter (cf. chapter 6), we discuss the concept of algebraic number fields and introduce the tools needed to define a special class of L-function, namely, the Dedekind zeta-function and then we consider the integral power sums of coefficients of the Dedekind zeta-function of a non-normal cubic extension $\mathbb{K}_3$ of rational field $\mathbb{Q}$ given by irreducible polynomial $f(x) = x^3 + ax^2 + bx + c$ and prove asymptotic formula for the sum $\sum\limits_{n \le x} a_{k,\mathbb{K}_3}(n)$ for any integer $k \ge 1$, where

$$a_{k,\mathbb{K}_3}(n) = \sum_{n = n_1 n_2 \ldots n_k} a_{\mathbb{K}_3}(n_1) a_{\mathbb{K}_3}(n_2) \ldots a_{\mathbb{K}_3}(n_k).$$

The synopsis provides a more thorough introduction to the thesis, along with explicit definitions, findings, and statements of theorems. (pp. **i**).

Chapter 1

A Flavor of Number Theory

We briefly examine a few number theory fundamentals in this chapter that are important to our goal. In order to make it as self-sufficient as feasible, we try to introduce nearly every concept that we will utilise in the next chapters.

We go into great detail regarding the arithmetic function, the relations between the arithmetic function and the Dirichlet series, and the connections between the Dirichlet series and cusp forms. In general instance, the Perron's formula and its variations have been explored. We also work with integer representations expressed as the sum of squares. We also discuss whether or not all positive integers may be written as sums of squares. In general, we provide an affirmative response to this query. We also spend some time discussing the functional equations related to various L-functions, including the symmetric j^{th} power L-functions. We will express a few key lemmas at the conclusion that we will refer frequently in the next chapters.

1.1 Elementary definitions

The concept of arithmetic functions is covered in this section along with a few examples and characteristics.

1.1.1 Number-theoretic functions

Definition 1.1.1 A complex or real valued function defined on the set of positive integers is known as *arithmetical function or number-theoretic function.*

Let $\mathfrak{A} = \{f : \mathbb{N} \to \mathbb{C}\}$ denote the set of number-theoretic function. For $f, g \in \mathfrak{A}$, we define the (multiplicative) Dirichlet convolution

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

$\mathfrak{A}$ forms a ring, which is commutative, together with the convolution and point-wise addition. The null function serves as the identity element with respect to addition, and the identity element with respect to the Dirichlet multiplication is

$$e(n) := \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Definition 1.1.2 We say that a number-theoretic function $f(\neq 0)$ is *multiplicative* if it satisfies

$$f(nm) = f(n)f(m),$$

for all co-prime natural numbers n and m. We say that f is *completely multiplicative* if the relation $f(nm) = f(n)f(m)$ holds for all $n, m \in \mathbb{N}$.

Evidently, the values of multiplicative functions on prime powers determine them entirely.

Definition 1.1.3 The *Euler totient function* $\varphi(n)$ is an arithmetic function given by

$$\varphi(n) := \sum_{\substack{1 \le k < n \\ (k,n)=1}} 1$$

for all $n \in \mathbb{N}$.

Definition 1.1.4 The *Möbius function* $\mu(n)$ is an arithmetic function given by

$$\mu(n) := \begin{cases} (-1)^k & \text{if } n = p_1 p_2 \dots p_k \text{ where } p_i\text{'s are distinct primes} \\ 1 & \text{if } n = 1 \\ 0 & \text{if } p^2 \mid n \text{ for some } p \text{ (prime)} \end{cases}.$$

Definition 1.1.5 The *von Mangoldt-function* $\Lambda(n)$ is an arithmetic function given by

$$\Lambda(n) := \begin{cases} \log p & \text{if } n = p^l \text{ for some prime } p \text{ and some } l \geq 1 \\ 0 & \text{otherwise} \end{cases}.$$

1.2 Characters of finite abelian groups

We now want to go into considerable detail about a few fundamental ideas in group theory. Knowing some arithmetical operations known as Dirichlet characters will be necessary for our following chapters. Although it is possible to learn Dirichlet characters without any prior knowledge of groups.

Definition 1.2.1 Let G be a group (arbitrary). A complex valued function f defined on G is known as a *character of* G if f has the multiplicative property

$$f(ba) = f(b)f(a)$$

for all $b, a \in G$, and $f(c) \neq 0$ for some c in G.

Remark 1.2.2 Every group G contains at least one character, namely, the function which is identically 1 on G. This is referred to as the *principal character*.

Definition 1.2.3 (Dirichlet characters) Let G be the group of reduced residue classes modulo k, then corresponding to each character f of G we define an arithmetical function $\chi = \chi_f$ as follows

$$\chi(n) := \begin{cases} f(\hat{n}) & \text{if } (n,k) = 1 \\ 0 & \text{otherwise} \end{cases},$$

where $\hat{n} = \{x : x \equiv n \pmod{k}\}$. The function χ is called a *Dirichlet character modulo* k. The *principal character* χ_1 is that which exhibits the property

$$\chi_1(n) := \begin{cases} 1 & \text{if } (n,k) = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Remark 1.2.4 There are $\varphi(k)$ distinct Dirichlet characters modulo k, each of which is completely multiplicative and periodic with period k, i.e.,

$$\chi(nm) = \chi(n)\chi(m) \quad \text{for all} \quad n, m$$

and

$$\chi(n+k) = \chi(n) \quad \text{for all} \quad n.$$

Remark 1.2.5 (a) A Dirichlet character modulo k is known as an even Dirichlet character if $\chi(-1) = 1$.

(b) A Dirichlet character modulo k is known as an odd Dirichlet character if $\chi(-1) = -1$.

(c) A Dirichlet character modulo k is known as a quadratic Dirichlet character if $\chi(n)$ takes only real values including -1.

Definition 1.2.6 (Induced modulus) Let χ be a Dirichlet character modulo k and d be any positive divisor of k. The number d is said to be an *induced modulus for* χ if we have

$$\chi(a) = 1 \quad \text{whenever} \quad (a, k) = 1$$

and

$$a \equiv 1 \pmod{d}.$$

To rephrase it, if the character χ modulo k acts like a character modulo d on the representatives of the residue class $\hat{1}$ modulo d which are relatively prime to k then d is an induced modulus . Note that, k itself is always an induced modulus for χ.

Definition 1.2.7 (Primitive character) A Dirichlet character χ modulo k is known as a *primitive modulo* k if it has no induced modulus $d < k$. To put it an another way, χ is primitive modulo k if and only if for every divisor d $(0 < d < k)$ of k, there exists an integer a such that

$$a \equiv 1 \pmod{d},$$

$$(a, k) = 1,$$

and

$$\chi(a) \neq 1.$$

Remark 1.2.8 The principal character, χ_1, is not primitive if $k > 1$ since it has 1 as an induced modulus.

1.3 Modular forms

In this section, we review the definitions and some fundamental properties of modular forms and Hecke operators. We next talk about how studying a modular form is not at that different from studying a Dirichlet series.

Since that modular forms are special holomorphic functions by definition, it is not surprising that they might be used in complex analysis. They also have connections to many other branches of mathematics, including algebraic and hyperbolic geometry, number theory, representation theory, mathematical physics, and combinatorics.

Let $\mathbb{H}$ be the upper half plane. There is a group action of $SL(2,\mathbb{R})$ on $\mathbb{H}$. If $\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (where $ad - bc = 1$), then τ acts by

$$\tau z = \frac{az+b}{cz+d}.$$

Clearly, $\tau z \in \mathbb{H}$, since

$$\begin{aligned}\Im\left(\frac{az+b}{cz+d}\right) &= \Im\left(\frac{(az+b)(c\bar{z}+d)}{|cz+d|^2}\right) \\ &= \frac{(ad-bc)\Im(z)}{|cz+d|^2}.\end{aligned}$$

These are the *linear fractional or Möbius transformations.* A Möbius transformations remains unaffected if we multiply all the coefficients a, b, c, d by some non zero constant. For each Möbius transformations with $ad - bc = 1$, we link a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

Then $\det A = ad - bc = 1$. If A and B are the matrices linked with the Möbius transformations f and g, respectively, then it is not difficult to see that the matrix product AB is linked with the composition $f \circ g$, where $(f \circ g)(z) = f(g(z))$.

We will focus our attention on a nice subgroup $SL(2,\mathbb{Z})$ of $SL(2,\mathbb{R})$. The set of Möbius transformations of the form

$$\tau^* = \frac{az+b}{cz+d},$$

where a, b, c, d are all integers with $ad - bc = 1$, is known as *Modular group* and is denoted by Γ. Understanding functions on $\mathbb{H}$ that transform well under $SL(2,\mathbb{Z})$ is our aim . For instance, we might need functions that have the property that $f(\gamma z) = f(z)\ \forall\ \gamma \in SL(2,\mathbb{Z})$. We might look for C^∞, meromorphic, or holomorphic functions that meet this requirement. We instead ask for $f(\gamma z) = j'(\gamma, z)f(z)$ for some multiplier systems j' because it turns out that there are no holomorphic functions for which this is true.

Note that we must need $j'(\gamma_1\gamma_2, z) = j'(\gamma_1, \gamma_2 z)j'(\gamma_2, z)$, in order to have $f(\gamma_1\gamma_2 z) = j'(\gamma_1\gamma_2, z)f(z) = j'(\gamma_1, \gamma_2 z)j'(\gamma_2, z)f(z)$. The first illustration that we will take into account is $j'(\gamma, z) = (cz+d)^k$ where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Clearly, it satisfies the above relation.

We will now look at holomorphic functions $f : \mathbb{H} \to \mathbb{C}$ satisfying

$$f(\gamma z) = (cz+d)^k f(z) \quad \forall \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}).$$

Observe that, I and $-I$ have the same action on $\mathbb{H}$, so therefore $f(-Iz) = f(z)$.

Definition 1.3.1 (Modular functions) A function f is said to be *modular function* if it satisfies the three conditions given below:

- f is meromorphic in the whole upper half plane $\mathbb{H}$.
- $f(A\tau) = f(\tau)$ for every A in the modular group Γ, i.e., f is invariant under all the transformation of Γ.
- The Fourier expansion of f has the form

$$f(\tau) = \sum_{n=-m}^{\infty} a(n)e^{2\pi i n\tau}.$$

The full modular group $\Gamma = SL(2, \mathbb{Z})$ has many subgroups of special interest in analytic number theory. We will talk about a special class of subgroups known as congruence subgroups.

Definition 1.3.2 (Congruence subgroups of the modular group) If $n \geq 1$ is an integer, there is a homomorphism

$$\pi_n : SL(2, \mathbb{Z}) \to SL(2, \mathbb{Z}/n\mathbb{Z}),$$

induced by the reduction modulo n morphism $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. The *principal congruence subgroup of level l* in Γ is the kernel of π_n, and it is usually denoted by $\Gamma(n)$. Explicitly, it is defined as:

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \colon a, d \equiv 1 \pmod{n} \;\; \& \;\; b, c \equiv 0 \pmod{n} \right\}.$$

If K is a subgroup contained in Γ then it is called a *congruence subgroup* if there exists a natural number $n \geq 1$ such that it contains the principal congruence subgroup $\Gamma(n)$. The level l of H is then the smallest such n.

From this definition it follows that:

- Congruence subgroups are of finite index in Γ.
- The congruence subgroups of level n are in one-to-one correspondence with the subgroups of $SL(2, \mathbb{Z}/l\mathbb{Z})$.

The subgroups $\Gamma_0(n)$ is known as the *Hecke congruence subgroup of level n* and defined as

$$\Gamma_0(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \colon c \equiv 0 \pmod{n} \right\}.$$

Clearly $\Gamma(n)$ is a normal subgroup of finite index in Γ.

Definition 1.3.3 (Modular form of weight k) A function f is said to be an *entire modular form of weight k* if it satisfies the following conditions:

- (Regularity) f is analytic in the whole upper half plane $\mathbb{H}$.
- (Modularity) $f\left(\dfrac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$ whenever $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

- (Growth condition) The Fourier expansion of f has the form

$$f(\tau) = \sum_{n=0}^{\infty} c(n) e^{2\pi i n \tau}.$$

Remark 1.3.4 Suppose $\tau \in \mathbb{H}$ and consider $q = e^{2\pi i \tau} \in D(0,1)$ (open disk with centre 0 and radius 1). Here, $i\infty$ corresponds to 0, so define "hole at ∞" to mean "$f(e^{2\pi i \tau})$ has a removable singularity at 0". Equivalently, the third condition means that f is bounded as $\Im(z) \to \infty$.

Remark 1.3.5 The cusps of Γ are the points of $\mathbb{Q} \cup \{\infty\}$. These cusps are finite in number. If $c' \in \mathbb{Q}$ then there is an element $\sigma_{c'} \in SL(2, \mathbb{Q})$ such that $\sigma_{c'} \cdot c' = \infty$. Therefore, locally all cusps seem to be the cusp at ∞.

The constant term $c(0)$ is called the value of f at $i\infty$, denoted by $f(i\infty)$. If we have $c(0) = 0$, then the function f is called a *cusp form*. The smallest r such that $c(r) \neq 0$ is called the order of the zero of f at $i\infty$.

In a more general sense a modular form is permitted to have poles at $i\infty$ or in $\mathbb{H}$. That's why the forms satisfying our conditions are called *entire* forms. Some trivial examples of entire modular forms are:

1. The zero function is a modular form of weight k for every integer k.

2. A non-zero constant function is a modular form of weight k only if $k = 0$.

We denote by M_k, the space of all entire modular forms of weight k and S_k the subspace of cusp forms.

Now, we observe a nice growth estimate for cusp forms. Let $f(z) \in S_k$ then we have the Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z},$$

from which we can obtain

$$|f(\sigma + it)| \ll e^{-2\pi t} \qquad \text{as} \quad t \to \infty,$$

uniformly in σ, with similar estimates at any given cusp. Therefore, cusp forms are rapidly decreasing at all cusps.

1.3.1 The Hecke operators T_n

Hecke obtained all of the entire modular forms with multiplicative coefficients by using a sequence of linear operators,

$$T_n : M_k \to M_k \qquad (n = 1, 2, 3, \dots).$$

These operators are known as *Hecke operators.*

Definition 1.3.6 For any fixed integer k and any $n = 1, 2, 3, \dots$, the operator T_n is defined on M_k by the equation

$$(T_n f)(\tau) := n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{n\tau + bd}{d^2}\right),$$

and when $n = p$,

$$(T_p f) = p^{k-1} f(p\tau) + \frac{1}{p} \sum_{b=0}^{p-1} f\left(\frac{\tau + b}{p}\right).$$

Clearly, T_n maps each f in M_k onto an another function in M_k.

Definition 1.3.7 A non-zero function f satisfying a relation of the type

$$T_n f = \lambda(n) f$$

for some scalar (complex) $\lambda(n)$ is called an *eigenfunction (eigenform) of the Hecke operator* T_n and the complex scalar $\lambda(n)$ is called an *eigenvalue of* T_n.

Remark 1.3.8 If f is an eigenform, then so is cf for all non-zero constant c.

Definition 1.3.9 If f is an eigenform for every Hecke operator T_n, where $n \geq 1$, then f is called a *simultaneous eigenform.*

Remark 1.3.10 An eigenform with the property $c(1) = 1$ is said to be a *normalized eigenform.*

Hecke found a remarkable relation between each modular form with Fourier series

$$f(\tau) = c(0) + \sum_{n=1}^{\infty} c(n) e^{2\pi i n \tau},$$

and the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s},$$

formed with the same coefficients. If $f \in M_{2k}$ then $c(n) = O(n^k)$ if f is a cusp form, and $c(n) = O(n^{2k-1})$ if f is not a cusp form. Thus, $F(s)$ converges absolutely for

$$\Re(s) > k + 1 \quad \text{,if } f \text{ is a cusp form,}$$

and for

$$\Re(s) > 2k \quad \text{,if } f \text{ is not a cusp form.}$$

Remark 1.3.11 For cusp forms, better bounds are available for $|c(n)|$ (by Kloosterman, Davenport, Rankin and Selberg (see [Se 2])). It has been shown that

$$c(n) = O\left(n^{k-\frac{1}{4}+\varepsilon}\right) \quad \text{if} \quad f \in M_{2k,0}$$

for every $\varepsilon > 0$.

Conjecture 1.3.12 If $f \in M_{2k,0}$ then

$$c(n) = O\left(n^{k-\frac{1}{2}+\varepsilon}\right)$$

for every $\varepsilon > 0$.

This conjecture is settled by Deligne [De 1] for f being a holomorphic cusp form. For maass forms, it is still open.

1.4 Representations of Integers as Sum of Squares

The representation of positive integers as sums of a fixed numbers of non-negative s^{th} powers has fascinated several generations of mathematicians, and its generalizations and analogues occupy a central place in number theory today. This problem is popularly known as Waring's problem. The following are the primary issues with representing an integer as a sum of squares:

Question 1.4.1 What positive integers can be represented as the sum of k squares, given a positive integer k?

Question 1.4.2 How many representations are there if an integer is so representable?

For any positive integer n, we define the function $r_k(n)$ as

$$r_k(n) := \#\{(n_1, n_2, \ldots, n_k) \in \mathbb{Z}^k : n_1^2 + n_2^2 + \cdots + n_k^2 = n\}$$

(allowing zeros, distinguishing signs, and order).

In this section, we only consider the situations when $k = 4$ and 6. For $k = 4$, the representational problem has been fully resolved using an exciting method which depends only on the Jacobi's-triple product identity by M. D. Hirschhorn [Hi].

1.4.1 Sum of Four Squares

Theorem 1.4.3 *[Hi, Jacobi's four-square theorem] The number of representations of a positive integer n as the sum of four squares, representations which differs only in sign or order being counted as distinctive, is eight times the sum of the divisors of n which are not multiples of 4, i.e.,*

$$r_4(n) = 8 \sum_{d|n, 4\nmid d} d.$$

Equivalently,

$$r_4(n) = \begin{cases} 8\sum_{m|n} m & \text{if } n \text{ is odd} \\ 24\sum_{m|n, m \text{ odd}} m & \text{if } n \text{ is even} \end{cases}.$$

Remark 1.4.4 We may also write this as

$$r_4(n) = 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right),$$

where $\sigma(n) = \sum_{d|n} d$. If n is not divisible by 4, then the second term is to be taken as 0. In particular, we have the explicit formula $r_4(p) = 8(p+1)$ for a prime number p.
Some values of $r_4(n)$ occur infinitely, since

$$r_4(n) = r_4(2^m n),$$

whenever n is even. The values of $r_4(n)$ can be arbitrarily large.

Lemma 1.4.5 *For any positive integer n, the function $r_4(n)$ is multiplicative.*

Proof. The principal character χ_0 modulo 4 is defined as

$$\chi_0(n) := \begin{cases} 1 & \text{if } (4,n) = 1 \\ 0 & \text{otherwise} \end{cases}.$$

We can rewrite $r_4(n) = 8r(n)$, where $r(n)$ is multiplicative and given by

$$r(p^u) := \begin{cases} \frac{1-p^{u+1}}{1-p} & \text{if } p > 2 \\ 3 & \text{if } p = 2 \end{cases}.$$

We write, $r_4(n) = 8\sum_{d|n} \widetilde{\chi}_0(d)d$, where $\widetilde{\chi}_0$ is a character modulo 4, given by

$$\widetilde{\chi}_0(p^u) := \begin{cases} \chi_0(p^u) & \text{if } p > 2 \\ 3 & \text{if } p = 2 \end{cases},$$

and χ_0 is the principal character modulo 4. We observe, if n has the prime power factorization, i.e., $n = 2^{a_2}3^{a_3}\cdots q^{a_q}$, then

$$\begin{aligned}\widetilde{\chi}_0(n) &= \widetilde{\chi}_0(2^{a_2})\widetilde{\chi}_0(3^{a_3})\cdots\widetilde{\chi}_0(q^{a_q})\\ &= 3\chi_0(3^{a_3}\cdots q^{a_q}).\end{aligned}$$

For any prime number p, we have

$$r(p) = \sum_{d|p}\widetilde{\chi}_0(d)d = 1 + p\widetilde{\chi}_0(p).$$

1.4.2 Sum of Six Squares

Now, we divert our attention to the representation of an integer as a sum of 6 squares. When $k = 6$, Jacobi's formula for $r_6(n)$ states that:

Theorem 1.4.6 *[Jacobi's six-square theorem] For any positive integer n, we have*

$$r_6(n) = 16\sum_{d|n}\chi(d')d^2 - 4\sum_{d|n}\chi(d)d^2, \tag{1.1}$$

where $dd' = n$, and χ is the non-principal Dirichlet character modulo 4, i.e.,

$$\chi(n) = \begin{cases}1 & \text{if } n \equiv 1 \pmod 4\\ -1 & \text{if } n \equiv -1 \pmod 4\\ 0 & \text{if } n \equiv 0 \pmod 2\end{cases}.$$

Although there have been numerous analytical demonstrations of Jacobi's formula, only one purely arithmetic proof of Jacobi's formula appears to be known in the literature, for instance, see [Car]. In his wonderful book on elementary methods in number theory [[Na], pp. 436-4391], Nathanson presents this argument.

Here, we furnish a reference [Mc-Wi], in which an entirely different elementary arithmetic proof of Jacobi's formula is achieved.

From [Ha-Wr, pp. 313], we observe that

$$\chi(n) = \begin{cases} 0 & \text{if } 2|n \\ (-1)^{\frac{n-1}{2}} & \text{if } 2 \nmid n \end{cases}.$$

It is clear that $\chi(n)$ is the non-principal character modulo 4 and can be defined as stated.

We can reframe the Equation (1.1) as

$$\begin{aligned} r_6(n) &= 16\sum_{d|n} \chi(d)\frac{n^2}{d^2} - 4\sum_{d|n} \chi(d)d^2 \\ &=: 16l(n) - 4v(n). \end{aligned}$$

We write $l_1(n) = 16l(n)$, and $v_1(n) = 4v(n)$.

The functions $\chi(d)$ and $\frac{n^2}{d^2}$ are completely multiplicative functions. This implies that $\chi(d)\frac{n^2}{d^2}$ is multiplicative. If $g(d)$ is any multiplicative function, then $\sum_{d|n} g(d)$ is also multiplicative. Therefore, $l(n)$ is a multiplicative function. Similarly, $v(n)$ is also multiplicative.

Note that, for a prime number p, we have

$$l(p) = p^2 + \chi(p),$$

$$l(p^2) = p^4 + p^2\chi(p) + \chi(p^2),$$

and

$$\begin{aligned} v(p) &= 1 + p^2\chi(p), \\ v(p^2) &= 1 + p^2\chi(p) + p^4\chi(p^2). \end{aligned}$$

One can easily see that $r_6(n)$ is not a multiplicative function. However, Theorem 1.4.6 demonstrates that $r_6(n)$ can be written as a sum of two multiplicative functions.

1.5 A Discussion on Perron's Formula

A crucial component in proving our theorems is the Perron's formula. We'll talk about its application in this chapter. In order to achieve this, we move away from simply elementary procedures and instead employ standard analytic techniques which are helpful across analytic number theory.

1.5.1 Derivation of Perron's formula

We have seen a lot of sums with a parameter n that goes up to x. Whether n lies below x or not can be expressed analytically using Perron's formula, and this statement opens the door for analysing such sums with the help of associated Dirichlet series analytic features.

Lemma 1.5.1 *Let x and a be two positive real numbers. Then*

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{x^s}{s}ds = \delta := \begin{cases} 1 & \textit{if } x > 1 \\ \frac{1}{2} & \textit{if } x = 1 \\ 0 & \textit{if } x < 1 \end{cases}. \tag{1.2}$$

where the conditionally convergent integral is to be understood as $\lim_{T\to\infty}\int_{a-iT}^{a+iT}$. *Quantitatively, for $x \neq 1$,*

$$\frac{1}{2\pi i}\int_{a-iT}^{a+iT}\frac{x^s}{s}ds = \delta + O\left(x^a \min\left(1, \frac{1}{T|\log x|}\right)\right) \tag{1.3}$$

When $x < 1$, moving the line of integration to the right; i.e., letting a tend to $+\infty$ and using Cauchy's theorem to justify that the integral does not change, formula (1.2) can be validated. By letting a tend to $-\infty$ and taking into account that we cross a pole at $s = 0$, which results in a residue of 1, we can shift the line of integration to the left when $x > 1$. Though the integral is not absolutely convergent, this argument can be made accurate with a little caution and it is standard.

Lemma 1.5.2 *Let x and a be two positive real numbers. Then*

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{x^s}{s^2}ds = \begin{cases}\log x & \textit{if } x \geq 1\\ 0 & \textit{if } x \leq 1\end{cases}. \tag{1.4}$$

Since the above integral is absolutely convergent, proof of (1.4) can be easily carried out.

Proof of Lemma 1.5.1. For $x \neq 1$, integration by parts gives

$$\int_{a-iT}^{a+iT}\frac{x^s}{s}ds = \int_{a-iT}^{a+iT}\frac{1}{s}d\left(\frac{x^s}{\log x}\right)$$

$$= \frac{1}{\log x}\left(\frac{x^{a+iT}}{a+iT} - \frac{x^{a-iT}}{a-iT}\right) + \frac{1}{\log x}\int_{a-iT}^{a+iT}\frac{x^s}{s^2}ds.$$

Since,

$$\int_{a-iT}^{a+iT}\frac{x^s}{s^2}ds = \int_{a-i\infty}^{a+i\infty}\frac{x^s}{s^2}ds + O\left(\frac{x^a}{T}\right),$$

using (1.4), we can conclude that for $x \neq 1$

$$\frac{1}{2\pi i}\int_{a-iT}^{a+iT}\frac{x^s}{s}ds = \delta + O\left(\frac{x^a}{T|\log x|}\right).$$

When $T|\log x| \geq 1$, this establishes (1.3). Now, we consider $T|\log x| \leq 1$. Here,

$$\frac{1}{2\pi i}\int_{a-iT}^{a+iT}\frac{x^s}{s}ds = \frac{1}{2\pi i}\int_{a-iT}^{a+iT}\frac{x^a}{s}(1 + O(|s|\log x))ds$$

$$= O(y^a),$$

and thus (1.3) holds again. Upon letting $T \to \infty$, the qualitative connection (1.3) (for $x \neq 1$) follows from the quantitative version (1.3). The case $x = 1$ can be easily checked.

1.5.2 Truncated Perron's formula

In this subsection, we state the well-known truncated Perron's formula (see [Ram-Sa]) that we will frequently use to prove our upcoming theorems.

Truncated Perron's formula: Suppose that the series $f(s)$ converges absolutely for $\sigma > 1$ and that $|a_n| \leq M(n)$, where $M(n) > 0$ is a monotonically increasing function and that

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} = O\left(\frac{1}{(\sigma-1)^\alpha}\right), \quad \alpha > 0$$

as $\sigma \to 1^+$. Also, suppose that for any $1 < c \leq c_0$, $T \geq 1$ and $x = N + \frac{1}{2}$, where $N \in \mathbb{N}$, then we have

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s) \frac{x^s}{s} ds + O\left(\frac{x^c}{T(c-1)^\alpha}\right) + O\left(\frac{xM(2x)\log x}{T}\right),$$

where the constants in the O symbol depends only on c_0.

1.5.3 Perron's formula

A well-known example is when $\Lambda(n) = f(n)$ is used to get

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{-\zeta'(s)}{\zeta(s)}\right) \frac{x^s}{s} ds.$$

The concept behind Riemann's method of proving the prime number theorem is to shift the contour to the left and use Cauchy's residue theorem to precisely determine the asymptotic nature of $\psi(x)$ in terms of the poles of $\left(\frac{-\zeta'(s)}{\zeta(s)}\right) \frac{x^s}{s}$ which contain the zeros of $\zeta(s)$. It is challenging to understand the zeros of $\zeta(s)$, and after 150 years, our comprehension is still rather rudimentary. In order to better grasp the integrand, we shall focus on this contour as well as the contours to the right of 1.

Because $x^s = x^c x^{it}$ has a mean value of 0 as t varies through any interval of size $\frac{2\pi}{\log x}$, we anticipate cancellation in the integral. $\frac{\mathbb{A}(s)}{s}$ must not vary significantly as t traverses this interval in order for us to acquire significant cancellation. We do succeed in obtaining the necessary cancellation if we integrate by parts, beginning with the x^s:

$$\int_{c-iT}^{c+iT} \mathbb{A}(s)\frac{x^s}{s}ds = \int_{c-iT}^{c+iT} \mathbb{A}(s)\frac{x^s}{s^2\log x}ds - \int_{c-iT}^{c+iT} \mathbb{A}'(s)\frac{x^s}{s\log x}ds + \left[\mathbb{A}(s)\frac{x^s}{s\log x}\right]_{c-iT}^{c+iT}.$$

The first and second terms are the results of applying Perron's formula to determine the values of $\frac{1}{\log x}\sum_{n\le x} f(n)\log\frac{x}{n}$ and $\frac{1}{\log x}\sum_{n\le x} f(n)\log n$, respectively and the third term is $O\left(\frac{x}{T}\right)$. Therefore, integration (by parts) here corresponds to the identity

$$\log x = \log\frac{x}{n} + \log n.$$

Observe that the first term is $O\left(\frac{x}{\log x}\right)$. Thus, we get

$$\int_{c-iT}^{c+iT} \mathbb{A}(s)\frac{x^s}{s}ds = -\frac{1}{\log x}\int_{c-iT}^{c+iT} \mathbb{A}'(s)\frac{x^s}{s}ds + O\left(\frac{x}{\log x}\right).$$

Therefore, we have

$$\sum_{n\le x} a_n = -\frac{1}{\log x}\int_{c-iT}^{c+iT} \frac{\mathbb{A}'(s)}{\mathbb{A}(s)}\mathbb{A}(s)\frac{x^s}{s}ds + O\left(\frac{x}{\log x}\right),$$

from which we can obtain

$$\sum_{n\le x} a_n \ll \frac{\max_{|t|\le T} \mathbb{A}\left(1+\frac{1}{\log x}+it\right)}{\log x} \times x\int_{-T}^{T} \left|\frac{\mathbb{A}'(c+iT)}{\mathbb{A}(c+iT)}\right| \left(\frac{1}{1+|t|}\right)dt + \frac{x}{\log x}.$$

Now that we have the desired $\frac{|\mathbb{A}|}{\log x}$, we must bound the integral. The integral over $\frac{\mathbb{A}'}{\mathbb{A}}$ is actually the main challenge, and we lack a method to approach it for general $\mathbb{A}$.

1.5.4 Estimation of an interesting asymptotic formula

In this subsection, we use the general method of Perron's formula to deal with k^{th}-divisor function $d_k(n)$ and find a good asymptotic formula.

Example 1.5.3 Let $d_k(n)$ be the k^{th} divisor function (where $k \in \mathbb{N}$), being defined as

$$\zeta^k(s) = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}.$$

Since $d_k(n) \ll n^{\varepsilon}$ for any $\varepsilon > 0$, the above series is absolutely convergent in $\Re(s) > 1$.

We can find $\sum_{n \leq x} d_k(n)$ by estimating

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta^k(s) \frac{x^s}{s} ds,$$

using Perron's formula. More precisely, we have

$$\sum_{n \leq x} d_k(n) = x P_k(\log x) + O\left(x^{1-\frac{1}{1+k}+\varepsilon}\right),$$

where the main term is obtained by using the Laurent expansion of $\zeta^k(s)\dfrac{x^s}{s}$ and the error term is obtained from combining the contribution of vertical and horizontal line integrals in absolute value.

Remark 1.5.4 For some better error term estimates, one may refer to [Ti-Ht, Chapter 12].

1.6 Properties of various L-functions

This chapter studies general properties of the Dirichlet series, Dirichlet L-functions and finally symmetric power L-functions.

The theory of Dirichlet series, when examined carefully for its own sake, involves many significant questions of convergence. One of the core areas of investigation in number theory is L-functions. These L-functions typically store fundamental arithmetic information, such as the distribution of prime numbers, that is not readily apparent from their descriptions and typically necessitates understanding of their poles, zeroes, functional equations, and other analytic properties.

1.6.1 Generation of arithmetical functions

Now, we provide a class of generating series for arithmetic functions.

Given an arithmetic function $f(n)$, the series

$$L(s,f)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s},$$

is known as the *Dirichlet series connected to f*. A Dirichlet series can be viewed as either a formal infinite series or as a function of the complex variable s, defined in the region where the series is convergent, where $\Re(s)=\sigma$ and $\Im(s)=t$.

Dirichlet series play a similar role to regular generating functions in combinatorics as a sort of generating function for arithmetic functions, tailored to the multiplicative structure of the integers. For instance, Dirichlet series can be used to find and show identities among number-theoretic functions, in the same way that combinatorial identities can be proved using regular generating functions.

On a more advanced level, it is possible to use the analytical properties of a Dirichlet series, which is thought of as a function of the complex variable s, to learn more about the behavior of the partial sum

$$\sum_{n\le x} f(n)$$

of arithmetic functions.

The Riemann zeta function $\zeta(s)$, often known as the Dirichlet series connected to the constant function 1, is the most well-known Dirichlet series, i.e.,

$$L(s,1)=\zeta(s) \qquad (\Re(s)>1).$$

Dirichlet series are the ideal technique to examine the behavior of arithmetic functions because the integers' multiplicative structure is preserved by Dirichlet series.

One of the most significant feature of the Dirichlet series of a multiplicative function is the representation as an infinite product over primes, known as Euler product.

Lemma 1.6.1 *Let f be a multiplicative function and assume that the series*

$L(s,f)$ converges absolutely for some $s \in \mathbb{C}$. Then, we've

$$L(s,f) = \prod_p \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}}. \tag{1.5}$$

Moreover, if f is completely multiplicative then we've

$$L(s,f) = \prod_p \frac{1}{1 - f(p)p^{-s}}. \tag{1.6}$$

Proof. Proof of (1.5) follows easily by using the fundamental theorem of arithmetic and (1.6) follows by the geometric series, noticing that $f(p^k) = (f(p))^k$.

1.7 Dirichlet L-functions

The concept of Dirichlet L-functions and symmetric power L-functions are reviewed in this section. It has also been discussed how their Dirichlet series and Euler product are related. While doing so, we discuss how certain series can be used to express the normalized Fourier coefficients.

Let χ be a Dirichlet character modulo k. The *Dirichlet L-function related with χ* is the function

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where $s = \sigma + it$.

Remark 1.7.1 If χ_0 is the principal character modulo k, then $L(s,\chi_0)$ is analytic in the half-plane $\sigma = \Re(s) > 1$, and if χ is a non-principal character modulo k, then $L(s,\chi)$ is analytic in the half-plane $\sigma = \Re(s) > 0$. Moreover, $L(1,\chi)$ is non-zero.

Theorem 1.7.2 *[Na, Theorem 10.3] Let χ be a Dirichlet character modulo k. Then, in the half-plane $\Re(s) = \sigma > 1$, the function $L(s,\chi)$ is analytic and has the Euler product*

$$L(s,\chi)=\prod_{p}\left(1-\frac{\chi(p)}{p^s}\right)^{-1}.$$

Moreover, $L(s,\chi)$ *is non-zero.*

Example 1.7.3 If χ_0 is the principal character modulo 3. Then, we have

$$L(s,\chi_0)=\prod_{p\geq 3}\left(1-\frac{1}{p^s}\right)^{-1}.$$

If χ is the non-principal character modulo 3. Then, we have

$$L(s,\chi)=\prod_{p\equiv 1 \pmod 3}\left(1-\frac{1}{p^s}\right)^{-1}\prod_{p\equiv 2 \pmod 3}\left(1+\frac{1}{p^s}\right)^{-1}.$$

For principle and non-principal characters, Dirichlet L-functions have different analytic properties. The Riemann zeta function is represented by the Dirichlet L-function $L(s,\chi_0)$ for $\sigma>1$ in the special case when χ_0 is the principal character modulo 1, i.e.,

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p}\left(1-\frac{1}{p^s}\right)^{-1}.$$

Let χ_0 be the principal character modulo m. For $\sigma>1$, we can write

$$\begin{aligned}
L(s,\chi_0)&=\prod_{p}\left(1-\frac{\chi_0(p)}{p^s}\right)^{-1}\\
&=\prod_{(p,m)=1}\left(1-\frac{1}{p^s}\right)^{-1}\\
&=\prod_{p}\left(1-\frac{1}{p^s}\right)^{-1}\prod_{p|m}\left(1-\frac{1}{p^s}\right)\\
&=\zeta(s)\prod_{p|m}\left(1-\frac{1}{p^s}\right)^{-1}.
\end{aligned}$$

1.7.1 Character modulo 4

There are two Dirichlet characters, principal and non-principal, when the modulus is 4. Let χ_0 be the principal character modulo 4, i.e.,

$$\chi_0(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.$$

We note that (for $\Re(s) > 1$),

$$\begin{aligned} L(s, \chi_0) &= \sum_{n=1}^{\infty} \frac{\chi_0(n)}{n^s} \\ &= \prod_{\substack{p \\ (p,4)=1}} \left(1 - \frac{1}{p^s}\right)^{-1} \\ &= \left(1 - \frac{1}{2^s}\right) \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1} \\ &= \left(1 - \frac{1}{2^s}\right) \zeta(s). \end{aligned}$$

But the series

$$L(1, \chi_0) = 1 + \frac{1}{3} + \frac{1}{5} + \cdots$$

diverges. Now, let χ be the principal character modulo 4, i.e.,

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 4 \\ -1 & \text{if } n \equiv -1 \pmod 4 \\ 0 & \text{if } n \equiv 0 \pmod 2 \end{cases}.$$

We note that for $\Re(s) > 0$,

$$\begin{aligned} L(s, \chi) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^s} \\ &= \prod_{p \equiv 1 \pmod 4} \left(1 - \frac{1}{p^s}\right)^{-1} \prod_{p \equiv 3 \pmod 4} \left(1 + \frac{1}{p^s}\right)^{1}. \end{aligned}$$

In fact,

$$\begin{aligned} L(1,\chi) &= \left(1-\frac{1}{3}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{9}-\frac{1}{11}\right)+\cdots \\ &> 0, \end{aligned}$$

and

$$\begin{aligned} L(1,\chi) &= 1-\left(\frac{1}{3}-\frac{1}{5}\right)-\left(\frac{1}{7}-\frac{1}{9}\right)-\cdots \\ &< 1. \end{aligned}$$

1.7.2 Symmetric power L-functions

Let χ be the Dirichlet character modulo N. If

$$f\left(\frac{az+b}{cz+d}\right)=\chi(d)(cz+d)^k f(z)$$

for all $z \in \mathbb{H}$ (upper half plane) and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, then f is known as a *modular form of weight k and level N with Nebentypus χ*. Here, $\Gamma_0(N)$ is the *congruence subgroup*, i.e.,

$$\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : c \equiv 0 \pmod{N}\right\}.$$

In 1974, P. Deligne [De 1] proved (as a consequence of the Riemann hypothesis for varities over finite field) that for any primitive holomorphic cusp form f and for any prime p, there exist complex numbers $\alpha(p)$ and $\beta(p)$ such that

$$\alpha(p)+\beta(p)=\lambda_f(p), \tag{1.7}$$

and

$$|\alpha(p)|=|\beta(p)|=1=\alpha(p)\beta(p). \tag{1.8}$$

Then $L(s,f)$ can be written as

$$L(s, f) = \prod_p \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1}.$$

Also, $|\lambda_f(n)| \leq d(n)$, where $d(n)$ is the divisor function.

The j^{th} *symmetric power* L*-function* is defined as

$$\begin{aligned} L(s, \mathrm{sym}^j f) &:= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s} \\ &= \prod_p \prod_{i=0}^{j} \left(1 - \frac{\alpha^{j-i}(p)\beta^i(p)}{p^s}\right)^{-1}, \end{aligned} \tag{1.9}$$

for $\Re(s) > 1$ and $j \geq 1$ (an integer), where $\lambda_{\mathrm{sym}^2 f}(n)$ is multiplicative. The Sato-Tate conjecture is connected to the analytical properties of j^{th} symmetric power L-functions. Additionally, it is established that the series $L(s, \mathrm{sym}^j f)$ can be analytically continued to the region $\Re(s) \geq 1$ for each integer $j \geq 1$ and it is non-vanishing in that region.

Now, we would like to mention some results related to the convergence of j^{th} symmetric power L-function, due to Kumar et al. [Ku-Me-Pu].

Theorem 1.7.4 *[Ku-Me-Pu, Theorem 1.2]*

Let $a(m)$ $(m \geq 1)$ be a sequence of complex numbers such that $a(m) \ll m^{\beta+\varepsilon}$ for any positive ε, and the infinite series $\sum_{m=1}^{\infty} \frac{|a(m)|^2}{m^s}$ has a singularity at $s = \alpha \geq 0$, where β, α are real numbers such that $2\beta + 1 \leq \alpha$. Then the series $\sum_{m=1}^{\infty} \frac{a(m)}{m^s}$ has abscissa of absolute convergence $\beta + 1$.

Proposition 1.7.5 *[Ku-Me-Pu, Proposition 3.1] The series*

$$\sum_{n=1}^{\infty} \frac{|\lambda_{sym^j f}(n)|^2}{n^s}$$

has a singularity at $s = 1$.

Using Theorem 1.7.4 and Proposition 1.7.5, the authors Kumar et al. have established the following theorem.

Theorem 1.7.6 *Let $f \in S_k$ and $L(s, sym^j f)$ be the j^{th} symmetric power L-function associated with f, then the series*

$$L(s, sym^j f) = \sum_{n=1}^{\infty} \frac{\lambda_{sym^j f}(n)}{n^s}$$

has abscissa of absolute convergence 1.

In particular, for $j = 2$, we have the *symmetric square L-function*, which is defined as

$$\begin{aligned} L(s, \mathrm{sym}^2 f) &:= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^2 f}(n)}{n^s} \\ &= \prod_p \left(1 - \frac{\alpha^2(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta^2(p)}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-1}, \end{aligned}$$

for $\Re(s) > 1$, where $\lambda_{\mathrm{sym}^2 f}(n)$ is multiplicative.

The *Rankin-Selberg convolution of L-function attached to $sym^i f$ and $sym^j f$* (for $j \geq 0$, and $\Re(s) > 1$) is defined as

$$\begin{aligned} L(s, \mathrm{sym}^i f \times \mathrm{sym}^j f) &:= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(n)}{n^s} \\ &= \prod_p \prod_{m=0}^{i} \prod_{u=0}^{j} \left(1 - \frac{\alpha^{i-m}(p)\beta^m(p)\alpha^{j-u}(p)\beta^u(p)}{p^s}\right)^{-1}. \end{aligned} \tag{1.10}$$

Observe that,

$$\lambda_{\mathrm{sym}^j f}(p) = \sum_{m=0}^{j} \alpha^{j-m}(p)\beta^m(p), \tag{1.11}$$

and

$$\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(p) = \sum_{m=0}^{i} \sum_{u=0}^{j} \alpha^{i-m}(p)\beta^m(p)\alpha^{j-u}(p)\beta^u(p) \tag{1.12}$$
$$= \left(\sum_{m=0}^{i} \alpha^{i-m}(p)\beta^m(p) \right) \left(\sum_{u=0}^{j} \alpha^{j-u}(p)\beta^u(p) \right)$$
$$= \lambda_{\mathrm{sym}^i f}(p)\lambda_{\mathrm{sym}^j f}(p).$$

Since $\lambda_{\mathrm{sym}^j f}(n)$ is a multiplicative function, and $|\lambda_{\mathrm{sym}^j f}(n)| \le d_{j+1}(n)$ (from (1.7), and (1.8)), where $d_{j+1}(n)$ is the number of ways of expressing n as a product of $j+1$ factors), we can write the Euler product of $L(s, \mathrm{sym}^j f)$ as

$$\prod_p \left(1 + \frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s} + \cdots + \frac{\lambda_{\mathrm{sym}^j f}(p^l)}{p^{ls}} + \cdots \right). \tag{1.13}$$

Comparing (1.9), and (1.13), we get (1.11).
Similarly, $\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(n)$ is a multiplicative function, and

$$|\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(n)| \le d_{(i+1)(j+1)}(n),$$

(from (1.7), and (1.8)), so we can write the Euler product of $L(s, \mathrm{sym}^i f \times \mathrm{sym}^j f)$ as

$$\prod_p \left(1 + \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(p)}{p^s} + \cdots + \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(p^l)}{p^{ls}} + \cdots \right). \tag{1.14}$$

Comparing (1.10) and (1.14), we get (1.12).

Theorem 1.7.7 *For any positive integers i and j, if*

$$L(s, sym^i f) = \sum_{n=1}^{\infty} \frac{\lambda_{sym^i f}(n)}{n^s}$$

and

$$L(s, sym^j f) = \sum_{n=1}^{\infty} \frac{\lambda_{sym^j f}(n)}{n^s},$$

then

$$L(s, sym^i f \times sym^j f) g(s) = \sum_{n=1}^{\infty} \frac{\lambda_{sym^i f \times sym^j f}(n)}{n^s},$$

where $g(s)$ is an absolutely convergent Dirichlet series in the half plane $\Re(s) > \frac{1}{2}$.

Proof of Theorem 1.7.7 follows in a similar fashion as of Theorem 6 of [Gu-Mu], since only Deligne's bound which is known to be true in the cases of symmetric power L-functions attached to holomorphic cusp forms is used in Theorem 6 of [Gu-Mu], not the automorphic properties of L-functions.

Lemma 1.7.8 *[Ku-Me-Pu, Lemma 3.3] The series*

$$\sum_{p(prime)} \frac{\lambda_{sym^j f \times sym^j f}(p)}{p^s},$$

is divergent.

1.8 Functional equations pertaining to some L-functions

There are numerous approaches to explain why analytic continuation should be studied. One can use them to derive functional equations. Let us illustrate this precisely.

1.8.1 Analytic continuation

Establishing the analytic continuation and a functional equation for various symmetric power L-functions has garnered a lot of attention recently. In this section, we will discuss some elementary functional equations associate to L-functions.

First, we provide an integral representation for the Riemann zeta function which will be valid in the half-plane $\Re(s) = \sigma > 0$ and gives an analytic continuation of $\zeta(s)$ to this half-plane.

Theorem 1.8.1 *The Riemann zeta function $\zeta(s)$ has an analytic continuation to a function which is defined on half-plane $\sigma > 0$ and is analytic in this half plane with the exception of a simple pole with residue 1 at $s = 1$, defined as*

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \{x\}x^{-s-1}dx \qquad (\sigma > 0). \tag{1.15}$$

We get an estimate for $\zeta(s)$ close to the point $s = 1$ as a direct consequence of the representation (1.15) for $\zeta(s)$.

Lemma 1.8.2 *We can write*

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|),$$

for $|s-1| \leq \frac{1}{2}$ and $s \neq 1$, where γ is the Euler's constant.

Proof.Observe that

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=0}^{\infty} c_n(s-1)^n \qquad (|s-1| < 1),$$

since the function $\zeta(s) - \dfrac{1}{s-1}$ is analytic in the disk $|s-1| < 1$. It implies that

$$\zeta(s) - \frac{1}{s-1} = c_0 + O(|s-1|)$$

in $|s-1| < \frac{1}{2}$. In the half plane $\sigma > 0$, using the integral representation of $\zeta(s)$ and letting $s \to 1$, we obtain

$$c_0 = \lim_{s\to 1}\left(\zeta(s) - \frac{1}{s-1}\right) = 1 - \int_1^\infty \{x\}x^{-2}dx.$$

From the harmonic sum estimate, we can conclude that

$$\gamma = 1 - \int_1^\infty \{x\}x^{-2}dx.$$

Thus, $c_0 = \gamma$.

1.8.2 Functional equations

The Riemann-zeta function $\zeta(s)$ satisfies the functional equation (see chapter 2 of [Ti-Ht])

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(\frac{1-s}{2})}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

so, we can write

$$\zeta(s) = \zeta(1-s)\chi(s),$$

where $\chi(s) = \pi^{s-\frac{1}{2}}\dfrac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$.

Using $\Gamma(s) = O\left(e^{-(\frac{\pi}{2})|t|}|t|^{\sigma-\frac{1}{2}}\right)$ (for proof, see pp. 37-38 of [Rad]), we will get

$$|\chi(s)| \asymp |t|^{\frac{1}{2}-\sigma},$$

as $|t| \to \infty$, and $a \le \sigma \le b$.

Let M_k be the normalized Hecke basis for the space of holomorphic cusp forms of weight k. For each $f \in M_k$, the associated L-function admits analytic continuation to the whole complex plane $\mathbb{C}$ as an entire function, and satisfies the functional equation

$$(2\pi)^{-s}\Gamma\left(\frac{k-1}{2}+s\right)L(s,f) = i^k(2\pi)^{1-s}\Gamma\left(\frac{k-1}{2}+1-s\right)L(1-s,f).$$

The L-function is also connected (analytically) to $f(z)$ by Mellin transform

$$\Gamma(s,f) = \left(\frac{1}{2\pi}\right)^s\Lambda(s,f)L(s,f) = \int_0^\infty f(iy)y^s dy,$$

providing an integral representation for the completed L-function, denoted by $\Lambda(s,f)$. Hecke was able to deduce the analytical properties of $\Lambda(s,f)$ from those of f using this integral representation.

For $s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we have

$$f(sz) = f\left(\frac{-1}{z}\right) = z^k f(z)$$

or

$$f\left(\frac{i}{y}\right) = i^k y^k f(iy).$$

Substituting this expression in the integral representation, we get

$$\begin{aligned}\Lambda(s,f) &= \int_0^1 f(iy)y^s dy + \int_1^\infty f(iy)y^s dy \\ &= \int_1^\infty f\left(\frac{i}{y}\right) y^{-s} dy + \int_1^\infty f(iy)y^s dy \\ &= i^k \int_1^\infty f(iy)y^{k-s} dy + \int_1^\infty f(iy)y^s dy \\ &= i^k \Lambda(k-s,f).\end{aligned}$$

The integral are all bounded in vertical strips and absolutely convergent since cusp forms are decreasing rapidly.

Theorem 1.8.3 *The completed L-function $\Lambda(s,f)$ is nice, that is, it converges absolutely in a half plane and*

(a) extends to an entire function of s,

(b) bounded in vertical strips,

(c) satisfies the functional equation $\Lambda(s,f) == i^k \Lambda(k-s,f)$.

Hecke was also successful in proving the converse of this theorem by inverting the integral representation using the Mellin inversion formula.

Theorem 1.8.4 *Suppose $A(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is absolutely convergent for $\Re(s) \gg 0$ and let*

$$\Lambda(s) = \left(\frac{1}{2\pi}\right)^s \Gamma(s)A(s),$$

where $\Lambda(s)$ is nice. Then

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

is a cusp form of weight k for the full modular group $SL(2,\mathbb{Z})$.

Similarly, the symmetric square L-function $L(s,\mathrm{sym}^2 f)$ also admits analytic continuation to the whole complex plane $\mathbb{C}$ as an entire function, and satisfies the functional equation (see Shimura[Su] and Gelbart-Jacquet[Ge-Ja])

$$\begin{aligned}\Lambda(s,\mathrm{sym}^2 f) =: (\pi)^{-\frac{3s}{2}}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)L(s,\mathrm{sym}^2 f)\\ = \Lambda(1-s,\mathrm{sym}^2 f).\end{aligned}$$

Let $H_k(\Gamma_0(N),\psi)$ be the space of cusp forms of weight k, with multiplier ψ, for the group $\Gamma_0(N)$, where ψ is a Dirichlet character modulo N.
For each $f \in H_k(\Gamma_0(N),\psi)$, associated normalized L-function has analytic continuation, and satisfies a functional equation.
For given $f \in H_k(\Gamma_0(N),\psi)$, and a primitive character χ modulo d with $(d,N) = 1$, the twisted cusp form is

$$f^{\chi}(z) = \sum_{n=1}^{\infty} a_n \chi(n) e^{2\pi i n z},$$

and the associated twisted L-function, i.e., $L(s, f\otimes\chi)$ is defined as

$$L(s, f\otimes\chi) = \sum_{n=1}^{\infty} \frac{a(n)\chi(n)}{n^s},$$

then Hecke proved that $f^{\chi}(z) \in H_k(\Gamma_0(Nd^2),\psi\chi^2)$, and satisfies the functional equation

$$\Lambda(s, f\otimes\chi) = \omega(\chi)\Lambda(1-s, f_{|\omega N}\otimes\overline{\chi}),$$

where $f_{|\omega N} = (\sqrt{N}z)^{-k} f\left(\frac{-1}{Nz}\right)$, and

$$\Lambda(s, f\otimes\chi) = \left(\frac{\sqrt{N}d}{2\pi}\right)\Gamma\left(s+\frac{k-1}{2}\right)L(s, f\otimes\chi),$$

and $\omega(\chi)$ is a complex number with modulus 1, depending upon k,ψ and χ.
Similar results are also available for the Rankin-Selberg L-functions attached

to $\mathrm{sym}^i f$ and $\mathrm{sym}^j f$ for $0 \le i, j \le 4$ (see [Ja-Sl 1],[Ja-Sl 2],[Ru-Sn],[Sh 1] and [Sh 2]).

1.9 Some prominent lemmas

In this section, we will introduce some elementary lemmas related to the Riemann zeta function and various L-functions that we are going to use frequently in the proof of our theorems.

Lemma 1.9.1 *[Ti-Ht, Theorem 7.2]*

For any positive number ε, we have

$$\int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt \ll T^{1+\varepsilon},$$

uniformly for $T \ge 1$.

Lemma 1.9.2 *[Iv 1, Theorem 5.1]*

For any positive number ε, we have

$$\int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt \sim \frac{T(\log T)^4}{2\pi},$$

uniformly for $T \ge 1$.

Lemma 1.9.3 *[Ht] For any positive number ε, we have*

$$\int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{12} dt \ll T^{2+\varepsilon},$$

uniformly for $T \ge 1$.

Lemma 1.9.4 *For any positive number ε, we have*

$$\zeta(\sigma+it) \ll_\varepsilon (|t|+1)^{\frac{1}{3}(1+\varepsilon-\sigma)+\varepsilon},$$

uniformly for $\frac{1}{2} \le \sigma \le 1+\varepsilon$, and $|t| \ge t_0$ (where t_0 is sufficiently large).

Proof. We get the result when we apply the maximum-modulus principle to $F(w) = \zeta(w)e^{(w-s)^2}x^{w-s}$ in a suitable rectangle and by using Hardy's estimate $\zeta\left(\frac{1}{2}+it\right) \ll (|t|+10)^{\frac{1}{6}}$. For instance, see [Sa].

Lemma 1.9.5 *[Bo] For any positive number ε, we have*

$$\zeta(\sigma+it) \ll_\varepsilon (|t|+1)^{\max\{\frac{13}{42}(1-\sigma),0\}+\varepsilon},$$

uniformly for $\frac{1}{2} \le \sigma \le 1+\varepsilon$, and $|t| \ge 1$.

Lemma 1.9.6 *[Gd] For any positive number ε, we have*

$$\int_1^T \left| L\left(\frac{1}{2}+it\right)\right|^2 dt \ll T\log T,$$

uniformly for $T \ge 1$.

Lemma 1.9.7 *[Ju] For any positive number ε and for any $T \ge 1$ uniformly, we have*

$$\int_1^T \left| L\left(\frac{1}{2}+it, f\right)\right|^6 dt \ll T^{2+\varepsilon}.$$

Lemma 1.9.8 *For any positive number ε, we have*

$$L(\sigma+it) \ll_\varepsilon (1+|t|)^{\frac{1}{3}(1+\varepsilon-\sigma)+\varepsilon},$$

uniformly for $\frac{1}{2} \le \sigma \le 1+\varepsilon$, and $|t| \ge t_0$ (where t_0 is sufficiently large).

Proof. We get the result by using the maximum-modulus principle in a suitable rectangle. For instance, see [Gd].

Lemma 1.9.9 *Let $L(s,f)$ be a Dirichlet series with Euler product of degree $m \ge 2$, i.e.,*

$$L(s,f) = \prod_{p<\infty}\prod_{i=0}^{m}\left(1-\frac{\alpha(p,i)}{p^s}\right)^{-1},$$

where $\alpha(p,i)$ are local parameters of $L(s,f)$ at prime p. If the Euler product converges absolutely for $\Re(s) > 1$, admits a meromorphic continuation to the whole complex plane $\mathbb{C}$, and satisfies a functional equation of Riemann-zeta type, then we have

$$\int_T^{2T} \left| L\left(\frac{1}{2}+\varepsilon+it, f\right)\right|^2 dt \ll T^{\frac{m}{2}+\varepsilon}, \tag{1.16}$$

for $T \geq 1$; and for $0 \leq \sigma \leq 1+\varepsilon$, we have

$$|L(\sigma+it,f)| \ll (|t|+1)^{\frac{m}{2}(1+\varepsilon-\sigma)+\varepsilon}. \tag{1.17}$$

Proof.The proof of Equation (1.16) is derived in a similar fashion as in [Lo-Sa, Theorem 4.1]. Equation (1.17) follows by the maximum-modulus principle.

Lemma 1.9.10 *[Ji-Lü] Let χ be a primitive character modulo q and $\mathfrak{L}^d_{m,n}(s,\chi)$ be a general L-function of degree $2A$. For any positive number ε, we have*

$$\int_T^{2T} \left|\mathfrak{L}^d_{m,n}(\sigma+it,\chi)\right|^2 dt \ll (qT)^{2A(1-\sigma)+\varepsilon},$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1+\varepsilon$, and $T \geq 1$. Also,

$$\mathfrak{L}^d_{m,n}(\sigma+it,\chi) \ll (q(|t|+1))^{\max\{A(1-\sigma),0\}+\varepsilon},$$

uniformly for $-\varepsilon \leq \sigma \leq 1+\varepsilon$.

Definition 1.9.11 The *residue of a complex-valued function* $f(z)$ at an isolated singularity c is the unique complex number s which makes $f(z) = \dfrac{s}{(z-c)}$ the derivative of a single valued analytic function in an annulus $0 < |z-c| < \delta$, for some $\delta > 0$. It is usually denoted by

$$s = \operatorname*{Res}_{z=c} f(z).$$

Alternatively, residues can be calculated from Laurent series expansions, and one can define the residue of a function $f(z)$ at an isolated singularity as the coefficient c_{-1} of the Laurent series.

Theorem 1.9.12 *[Ah, Theorem 17] Let $f(z)$ be an analytic function except for isolated singularities a_j in a region Ω. Then*

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_j n(\gamma,a_j) \operatorname*{Res}_{z=a_j} f(z),$$

where $n(\gamma,a_j)$ is the index of a_j with respect to γ for each j.

Definition 1.9.13 (Hölder's Inequalities) Let $\frac{1}{p}+\frac{1}{q}=1$ with $p,q>1$. Then *Hölder's inequality for integral* states that

$$\int_a^b |f(x)g(x)|dx \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}}.$$

Equality holds when

$$|g(x)| = c|f(x)|^{p-1}.$$

If $p=q=2$, this equality is known as *Cauchy-Schwarz's inequality.*

Similarly, *Hölder's inequality for sums* states that

$$\sum_{k=1}^{n} |a_k b_k| \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |b_k|^q\right)^{\frac{1}{q}},$$

with equality when

$$|b_k| = c|a_k|^{p-1}.$$

Lemma 1.9.14 *Let $\lambda_f(n)$ be the normalized n^{th} Fourier coefficient of the Fourier expansion of $f(z)$. Then, we have*

$$\lambda_{sym^j f}(p) = \lambda_f(p^j). \tag{1.18}$$

and

$$\lambda_f^2(p^j) = 1 + \sum_{l=1}^{j} \lambda_f(p^{2l}), \tag{1.19}$$

Proof. Equation (1.18) is the famous Hecke's identity. Now, we begin with the proof of Equation (1.19).

$$
\begin{aligned}
\lambda_f^2(p^j) &= \left(\sum_{m=0}^{j} \alpha^{j-m}(p)\beta^m(p)\right)^2 \\
&= \left(\sum_{m=0}^{j} \alpha^{j-m}(p)\beta^m(p)\right)\left(\sum_{m'=0}^{j} \alpha^{j-m'}(p)\beta^{m'}(p)\right) \\
&=\sum_{m=0}^{j}\sum_{m'=0}^{j}\left(\alpha^{2j-(m+m')}(p)\right)\left(\beta^{(m+m')}(p)\right)
\end{aligned}
$$

(We put $m+m'=t$. Observe that for every fixed integer t in the interval $[0,2l]$ and for every fixed integer m in the interval $[0,l]$, there is a unique integer m' in the interval $[0,l]$ satisfying $m+m'=t$ and thus,)

$$
\begin{aligned}
&=\sum_{l=0}^{j}\left(\sum_{t=0}^{2l} \alpha^{2j-t}(p)\beta^t(p)\right) \\
&=1+\sum_{l=1}^{j}\left(\sum_{t=0}^{2l} \alpha^{2j-t}(p)\beta^t(p)\right) \\
&=1+\sum_{l=1}^{j}\lambda_f(p^{2l}).
\end{aligned}
$$

Chapter 2

Discrete mean square of the coefficients of the symmetric square L-function on certain sequence of positive numbers

2.1 Introduction

The Fourier coefficients of modular forms (cf. Section 1.3) are important and interesting objects in number theory. In 1927, Hecke [Hec] proved that

$$\sum_{n\leq x}\lambda_f(n) \ll x^{\frac{1}{2}}.$$

Later, this upper bound was improved by a number of authors (see [Haf-Iv, Ja-Sl 2]). The best upper bound is due to Wu (see [Wu]), i.e.,

$$\sum_{n\leq x}\lambda_f(n) \ll x^{\frac{1}{3}}\log^{\rho} x \qquad \text{(where } \rho \approx -0.118).$$

Rankin [Ran] and Selberg [Se 1] considered the square moments of these Fourier coefficients and independently proved the following asymptotic formula

$$\sum_{n\leq x}\lambda_f^2(n)=cx+O\left(x^{\frac{3}{5}}\right), \tag{2.1}$$

where c is a positive constant. Recently, Huang [Hu] has improved the exponent in (2.1) to $\frac{3}{5}-\varepsilon$, where $\varepsilon\leq\frac{1}{560}$. This seems to be the best known result until now.

In 1999, Fomenko [Fo 1] established the following estimates for higher moments (motivated by the work of Moreno and Shahidi [Mo-Sh] concerning the symmetric power L-functions $L(s,\mathrm{sym}^j f)$ for $j=1,2,3,4$),

$$\sum_{n\leq x}\lambda_f^3(n)\ll x^{\frac{5}{6}+\varepsilon},$$

and

$$\sum_{n\leq x}\lambda_f^4(n)=\tilde{c}x\log x+\hat{c}x+O\left(x^{\frac{9}{10}+\varepsilon}\right),$$

where $\tilde{c}$ and $\hat{c}$ are some suitable constants. Later, Lü [Lü 1, Lü 2] improved and generalized the work of Fomenko by considering higher moments.

In 2011, Lü and Wu [Lü-Wu] proved that (for $3\leq j\leq 8$)

$$\sum_{n\leq x}\lambda_f^j(n)=xP_j(\log x)+O\left(x^{\theta_j+\varepsilon}\right),$$

where θ_j's and degree of the polynomials P_j's are given by the following table:

j	θ_j	degree of P_j
3	$\frac{7}{10}$	0
4	$\frac{151}{175}$	1
5	$\frac{40}{43}$	0
6	$\frac{175}{181}$	4
7	$\frac{176}{179}$	0
8	$\frac{2933}{2957}$	13

Under the assumption that $L(s, \mathrm{sym}^j f)$ is automorphic cuspidal for some positive integer l, Lau and Lü [La-Lü] established some general results for the sum

$$\sum_{n \leq x} \lambda_f^j(n),$$

for all $j \geq 2$. This is indeed the case, since very recently Newton and Thorne (see [Ne-Th 1, Ne-Th 2]) proved that $L(s, \mathrm{sym}^j f)$ is automorphic cuspidal for all $j \geq 1$.

In 2013, Zhai [Zh] proved that

$$\sum_{a^2+b^2 \leq x} \lambda_f^j(a^2+b^2) = xP_j'(\log x) + O\left(x^{\theta_j'+\varepsilon}\right),$$

where θ_j''s and degree of the polynomials P_j''s are given by the following table:

j	θ_j'	degree of P_j'
2	$\frac{8}{11}$	0
3	$\frac{17}{20}$	0
4	$\frac{43}{46}$	1
5	$\frac{83}{86}$	0
6	$\frac{184}{187}$	4
7	$\frac{355}{357}$	0
8	$\frac{752}{755}$	13

Using the recent breakthrough of Newton and Thorne [Ne-Th 1, Ne-Th 2] along with some nice analytic properties of the associated L-functions, Xu [Xu] refined and generalized the results of Zhai.

In 2006, Fomeko [Fo 2] was able to prove some results for symmetric square

L-functions. He showed that

$$\sum_{n \leq x} \lambda_{\mathrm{sym}^2 f}(n) \ll x^{\frac{1}{2}} \log^2 x,$$

and further he could establish that

$$\sum_{n \leq x} \lambda^2_{\mathrm{sym}^2 f}(n) = \mathfrak{c}x + O\left(x^{\theta}\right),$$

where $\theta < 1$. In 2019, Sankaranarayanan et al. [Sa-Si-Ss] proved some interesting results which are worth mentioning.

$$\sum_{n \leq x} \lambda^3_{\mathrm{sym}^2 f}(n) = c_1 x + O\left(x^{\frac{15}{17}+\varepsilon}\right)$$

and

$$\sum_{n \leq x} \lambda^4_{\mathrm{sym}^2 f}(n) = \hat{c_1} x + O\left(x^{\frac{12}{13}+\varepsilon}\right),$$

where c_1 and $\hat{c_1}$ are positive constants. Lately, Luo et al. [Lu-Lao-Zo] proved the following asymptotic formula.

$$\sum_{n \leq x} \lambda^2_{\mathrm{sym}^j f}(n) = \begin{cases} c_j x + O(x^{\alpha_j+\varepsilon}) & \text{if } 3 \leq j \leq 6 \\ c_j x + O(x^{\alpha_j}) & \text{if } 7 \leq j \leq 8 \end{cases},$$

where α_j's are given by the following table:

j	α_j
3	$\frac{551}{635}$
4	$\frac{929}{1013}$
5	$\frac{1391}{1475}$
6	$\frac{979}{1021}$
7	$\frac{63}{65}$
8	$\frac{40}{41}$

Our first result is motivated by the following question.

Question 2.1.1 Can we find an asymptotic formula for the average behavior of the Fourier coefficients of cusp forms in the 4-dimension in various situations with a good error term?

We recall (see subsection 1.7.2) the definition of n^{th} normalized coefficient of the Dirichlet expansion of the j^{th} symmetric power L-function $L(\text{sym}^j f, s)$ here for the convenience of the reader. With the help of these preliminaries we now give an affirmative answer to the question raised above. In this regard we have the following theorem.

Theorem 2.1.2 *For $x \geq x_o$ (sufficiently large), we have*

$$\sum_{\substack{a^2+b^2+c^2+d^2 \leq x \\ (a,b,c,d) \in \mathbb{Z}^4}} \lambda^2_{sym^2 f}(a^2+b^2+c^2+d^2) = c_2 x^2 + O\left(x^{\frac{9}{5}+\varepsilon}\right),$$

where c_2 is an effective constant defined as

$$c_2 = (-2)\zeta(2) L(2, sym^2 f) L(1, sym^2 f \otimes \widetilde{\chi}_0) L(2, sym^4 f) L(1, sym^4 f \otimes \widetilde{\chi}_0) H_2(2),$$

and $H_2(2) \neq 0$, and $\widetilde{\chi}_0$ is a character modulo 4.

Remark 2.1.3 For $\Re(s) > 1$, let

$$F_2(s) := \prod_p \left(1 + \frac{\lambda^2_{\text{sym}^2 f}(p) r(p)}{p^s} + \cdots + \frac{\lambda^2_{sym^2 f}(p^l) r(p^l)}{p^{ls}} + \cdots\right),$$

and for $\Re(s) > 2$,

$$G_2(s) := \zeta(s) L(s-1, \widetilde{\chi}_0) L(s, \text{sym}^2 f) L(s-1, \text{sym}^2 f \otimes \widetilde{\chi}_0) L(s, \text{sym}^4 f) L(s-1, \text{sym}^4 f \otimes \widetilde{\chi}_0).$$

We point out here that,

$$H_2(s) := \frac{F_2(s)}{G_2(s)},$$

which is a Dirichlet series that converges absolutely, and uniformly in the half plane $\Re(s) > \frac{3}{2}$, and $H_2(s) \neq 0$ on $\Re(s) = 2$.

Our main objective in this chapter is to establish Theorem 2.1.2. First, we develop the notions and tools needed to establish this chapter's main outcome. We prove Lemma 2.1.4, which is related to the decomposition of the related L-functions and will be crucial to the proof of our main theorem 2.1.2. The work presented in this chapter have already been published in [Sr-Sa 1]. From [Ha-Wr, pp. 415], we can write

$$\begin{aligned} r_4(n) &= 8\sum_{d|n} \widetilde{\chi}_0(d)d \\ &=: 8r(n), \end{aligned}$$

where $\widetilde{\chi}_0$ is a character modulo 4, given by

$$\widetilde{\chi}_0(p^u) := \begin{cases} \chi_0(p^u) & \text{if } p > 2 \\ 3 & \text{if } p = 2 \end{cases},$$

and χ_0 is the principal character modulo 4.

Lemma 2.1.4 *Let f be a normalized primitive holomorphic cusp form of weight k for $SL(2,\mathbb{Z})$, and let $\lambda_{sym^2 f}(n)$ be the n^{th} normalized Fourier coefficient of the symmetric square L-function associated to f. If*

$$F_2(s) = \sum_{n=1}^{\infty} \frac{\lambda^2_{sym^2 f}(n) r(n)}{n^s},$$

for $\Re(s) > 2$, then

$$F_2(s) = G_2(s)H_2(s),$$

where

$$\begin{aligned} G_2(s) :=& \zeta(s)L(s-1,\widetilde{\chi}_0)L(s, sym^2 f)L(s-1, sym^2 f \otimes \widetilde{\chi}_0) \\ & L(s, sym^4 f)L(s-1, sym^4 f \otimes \widetilde{\chi}_0), \end{aligned}$$

and $\widetilde{\chi}_0$ is a character modulo 4.
Here, $H_2(s)$ is a Dirichlet series which converges absolutely, and uniformly in the half plane $\Re(s) > \frac{3}{2}$, and $H_2(s) \neq 0$ on $\Re(s) = 2$.

2.2 Proof of Lemma 2.1.4

We observe that $\lambda^2_{\mathrm{sym}^2 f}(n)r(n)$ is multiplicative, and hence

$$F_2(s) = \prod_p \left(1 + \frac{\lambda^2_{\mathrm{sym}^2 f}(p)r(p)}{p^s} + \cdots + \frac{\lambda^2_{\mathrm{sym}^2 f}(p^l)r(p^l)}{p^{ls}} + \cdots\right). \tag{2.2}$$

Note that,

$$\begin{aligned}
\lambda^2_{\mathrm{sym}^2 f}(p)r(p) &= \left(\sum_{m=0}^{2} \alpha^{2-m}(p)\beta^m(p)\right)^2 (1 + \widetilde{\chi}_0(p)p) \\
&= \left(\alpha^4(p) + 2\alpha^2(p) + 3 + 2\beta^2(p) + \beta^4(p)\right)(1 + \widetilde{\chi}_0(p)p) \\
&= 1 + \widetilde{\chi}_0(p)p + (\alpha^2(p) + 1 + \beta^2(p))(1 + \widetilde{\chi}_0(p)p) \\
&\quad + \left(\alpha^4(p) + \alpha^2(p) + 1 + \beta^2(p) + \beta^4(p)\right)(1 + \widetilde{\chi}_0(p)p) \\
&= 1 + \widetilde{\chi}_0(p)p + \lambda_{\mathrm{sym}^2 f}(p)(1 + \widetilde{\chi}_0(p)p) + \lambda_{\mathrm{sym}^4 f}(p)(1 + \widetilde{\chi}_0(p)p) \\
&= 1 + \widetilde{\chi}_0(p)p + \lambda_{\mathrm{sym}^2 f}(p) + p\lambda_{\mathrm{sym}^2 f}(p)\widetilde{\chi}_0(p) \\
&\quad + \lambda_{\mathrm{sym}^4 f}(p) + p\lambda_{\mathrm{sym}^4 f}(p)\widetilde{\chi}_0(p) \\
&=: b_1(p).
\end{aligned}$$

From the structure of $b_1(p)$, we define the coefficients $b_1(n)$ as

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{b_1(n)}{n^s} &= \zeta(s)L(s-1, \widetilde{\chi}_0)L(s, \mathrm{sym}^2 f)L(s-1, \mathrm{sym}^2 f \otimes \widetilde{\chi}_0)L(s, \mathrm{sym}^4 f) \\
&\quad \times L(s-1, \mathrm{sym}^4 f \otimes \widetilde{\chi}_0),
\end{aligned}$$

which is absolutely convergent in $\Re(s) > 2$.

We also note that,

$$\begin{aligned}
&\prod_p \left(1+\frac{b_1(p)}{p^s}+\cdots+\frac{b_1(p^m)}{p^{ms}}+\cdots\right)\\
&\quad= \zeta(s)L(s-1,\widetilde{\chi}_0)L(s,\mathrm{sym}^2 f)L(s-1,\mathrm{sym}^2 f\otimes\widetilde{\chi}_0)\\
&\qquad\times L(s,\mathrm{sym}^4 f)L(s-1,\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)\\
&\quad=: G_2(s),
\end{aligned}$$

for $\Re(s) > 2$. Observe that $b_1(n) \ll_\varepsilon n^{1+\varepsilon}$ for any small positive constant ε.

Now, we note that in the half plane $\Re(s) \geq 2+2\varepsilon$, we have

$$\begin{aligned}
\left|\frac{b_1(p)}{p^s}+\frac{b_1(p^2)}{p^{2s}}+\cdots+\frac{b_1(p^m)}{p^{ms}}+\cdots\right| &\ll \sum_{m=1}^{\infty}\frac{p^{(1+\varepsilon)m}}{p^{m\sigma}}\\
&\leq \sum_{m=1}^{\infty}\frac{p^{(1+\varepsilon)m}}{p^{(2+2\varepsilon)m}}\\
&= \sum_{m=1}^{\infty}\frac{1}{p^{(1+\varepsilon)m}}\\
&= \frac{\frac{1}{p^{1+\varepsilon}}}{1-\frac{1}{p^{1+\varepsilon}}}\\
&= \frac{1}{p^{1+\varepsilon}-1}\\
&< 1.
\end{aligned}$$

Let us write

$$A_1 = \frac{\lambda^2_{\mathrm{sym}^2 f}(p)r(p)}{p^s}+\cdots+\frac{\lambda^2_{\mathrm{sym}^2 f}(p^m)r(p^m)}{p^{ms}}+\cdots,$$

and

$$B_1 = \frac{b_1(p)}{p^s}+\cdots+\frac{b_1(p^m)}{p^{ms}}+\cdots.$$

From the above calculations, we observe that $|B_1| < 1$ in $\Re(s) \geq 2+2\varepsilon$.

We note that in the half plane $\Re(s) \geq 2 + 2\varepsilon$, we have

$$\begin{aligned} \frac{1+A_1}{1+B_1} &= (1+A_1)(1 - B_1 + B_1^2 - B_1^3 + \cdots) \\ &= 1 + A_1 - B_1 - A_1 B_1 + \text{higher terms} \\ &= 1 + \frac{\lambda^2_{\mathrm{sym}^2 f}(p^2) r(p^2) - b_1(p^2)}{p^{2s}} + \cdots + \frac{c_m(p^m)}{p^{ms}} + \cdots , \end{aligned}$$

with $c_m(n) \ll_\varepsilon n^{1+\varepsilon}$. So, we have (in the half plane $\Re(s) > \frac{3}{2}$)

$$\begin{aligned} \prod_p \left(\frac{1+A_1}{1+B_1} \right) &= \prod_p \left(1 + \frac{\lambda^2_{\mathrm{sym}^2 f}(p^2) r(p^2) - b_1(p^2)}{p^{2s}} + \cdots + \frac{c_m(p^m)}{p^{ms}} + \cdots \right) \\ &\ll_\varepsilon 1. \end{aligned}$$

Thus, we have (in the half plane $\Re(s) > \frac{3}{2}$)

$$\begin{aligned} H_2(s) &:= \frac{F_2(s)}{G_2(s)} \\ &= \prod_p \left(\frac{1+A_1}{1+B_1} \right) \\ &\ll_\varepsilon 1, \end{aligned}$$

and also $H_2(s) \neq 0$ on $\Re(s) = 2$.

Remark 2.2.1 We observe that the L-functions appearing in $G_2(s)$ satisfy a functional equation of the Riemann-zeta type and its corresponding conversion factor behaves like $\asymp (|t| + 10)^{m\left(\frac{1}{2} - \sigma\right)}$ (where m is the degree of $G_2(s)$) in any fixed vertical strip $a \leq \sigma \leq b$ and $|t| \geq t_0$.

Now we are at a stage where we can prove our main Theorem 2.1.2.

2.3 Proof of Theorem 2.1.2

We note that

$$\begin{aligned}\sum_{\substack{a^2+b^2+c^2+d^2\leq x\\(a,b,c,d)\in\mathbb{Z}^4}} \lambda^2_{sym^2 f}(a^2+b^2+c^2+d^2) &= \sum_{n\leq x}\lambda^2_{sym^2 f}(n)\sum_{\substack{n=a^2+b^2+c^2+d^2\\(a,b,c,d)\in\mathbb{Z}^4}} 1\\ &= \sum_{n\leq x}\lambda^2_{sym^2 f}(n)r_4(n)\\ &= 8\sum_{n\leq x}\lambda^2_{sym^2 f}(n)r(n),\end{aligned}$$

where $r(n) = \sum_{d|n}\widetilde{\chi}_0(d)d.$

Also,

$$r(p) = \sum_{d|p}\widetilde{\chi}_0(d)d = 1 + p\widetilde{\chi}_0(p),$$

where $\widetilde{\chi}_0$ is a character modulo 4.

Now, we begin by applying the Perron's formula (see section 1.5) to $F_2(s)$ with $\eta = 2+\varepsilon$ and $10 \leq T \leq x$. Thus, we have

$$\begin{aligned}\sum_{n\leq x}\lambda^2_{\text{sym}^2 f}(n)r_4(n) &= 8\sum_{n\leq x}\lambda^2_{\text{sym}^2 f}(n)r(n)\\ &= \frac{8}{2\pi i}\int_{\eta-iT}^{\eta+iT} F_2(s)\frac{x^s}{s}ds + O\left(\frac{x^{2+2\varepsilon}}{T}\right).\end{aligned}$$

We note that for $\Re(s) > 2$,

$$\begin{aligned}
L(s-1, \widetilde{\chi}_0) &= \sum_{n=1}^{\infty} \frac{\widetilde{\chi}_0(n)}{n^{s-1}} \\
&= \prod_p \left(1 - \frac{\widetilde{\chi}_0(p)}{p^{s-1}}\right)^{-1} \\
&= \left(1 - \frac{\widetilde{\chi}_0(2)}{2^{s-1}}\right)^{-1} \prod_{p>2} \left(1 - \frac{\chi_0(p)}{p^{s-1}}\right)^{-1} \\
&= \left(1 - \frac{3}{2^{s-1}}\right)^{-1} \left(1 - \frac{1}{2^{s-1}}\right) \prod_p \left(1 - \frac{\chi_0(p)}{p^{s-1}}\right)^{-1} \\
&= \left(1 - \frac{3}{2^{s-1}}\right)^{-1} \left(1 - \frac{1}{2^{s-1}}\right) L(s-1, \chi_0) \\
&= \left(1 - \frac{3}{2^{s-1}}\right)^{-1} \left(1 - \frac{1}{2^{s-1}}\right)^2 \zeta(s-1),
\end{aligned}$$

since

$$\begin{aligned}
L(s-1, \chi_0) &= \sum_{n=1}^{\infty} \frac{\chi_0(n)}{n^{s-1}} \\
&= \prod_{\substack{p \\ (p,4)=1}} \left(1 - \frac{1}{p^{s-1}}\right)^{-1} \\
&= \left(1 - \frac{1}{2^{s-1}}\right) \prod_p \left(1 - \frac{1}{p^{s-1}}\right)^{-1} \\
&= \left(1 - \frac{1}{2^{s-1}}\right) \zeta(s-1).
\end{aligned}$$

Thus,

$$\begin{aligned}
F_2(s) := \zeta(s) L(s-1, \widetilde{\chi}_0) L(s, \mathrm{sym}^2 f) L(s-1, \mathrm{sym}^2 f \otimes \widetilde{\chi}_0) L(s, \mathrm{sym}^4 f) \\
\times L(s-1, \mathrm{sym}^4 f \otimes \widetilde{\chi}_0) H_2(s) \\
= \zeta(s) \left(1 - \frac{3}{2^{s-1}}\right)^{-1} \left(1 - \frac{1}{2^{s-1}}\right)^2 \zeta(s-1) L(s, \mathrm{sym}^2 f) L(s-1, \mathrm{sym}^2 f \otimes \widetilde{\chi}_0) \\
\times L(s, \mathrm{sym}^4 f) L(s-1, \mathrm{sym}^4 f \otimes \widetilde{\chi}_0) H_2(s).
\end{aligned}$$

We move the line of integration to $\Re(s) = \frac{3}{2} + \varepsilon$. By Cauchy's residue theorem there is only one simple pole at $s = 2$, coming from the factor $\zeta(s-1)$. This contributes a residue, which is $c_2 x^2$, where c_2 is an effective constant depending on the values of various L-functions appearing in $G_2(s)$ at $s = 2$.

More precisely,

$$\begin{aligned}
c_2 &= 8 \lim_{s\to 2}(s-2)\frac{F_2(s)}{s} \\
&= 8\zeta(2)\frac{(-1)}{2}L(2, \mathrm{sym}^2 f)L(1, \mathrm{sym}^2 f \otimes \widetilde{\chi}_0) \\
&\quad \times L(2, \mathrm{sym}^4 f)L(1, \mathrm{sym}^4 f \otimes \widetilde{\chi}_0)\frac{1}{2}H_2(2) \\
&= (-2)\zeta(2)L(2, \mathrm{sym}^2 f)L(1, \mathrm{sym}^2 f \otimes \widetilde{\chi}_0)L(2, \mathrm{sym}^4 f)L(1, \mathrm{sym}^4 f \otimes \widetilde{\chi}_0)H_2(2).
\end{aligned}$$

So, we obtain

$$\begin{aligned}
\sum_{n\le x}\lambda^2_{\mathrm{sym}^2 f}(n)r_4(n) &= c_2 x^2 + \frac{8}{2\pi i}\left\{\int_{\frac{3}{2}+\varepsilon-iT}^{\frac{3}{2}+\varepsilon+iT} + \int_{2+\varepsilon-iT}^{\frac{3}{2}+\varepsilon-iT} + \int_{\frac{3}{2}+\varepsilon+iT}^{2+\varepsilon+iT}\right\} F_2(s)\frac{x^s}{s}ds \\
&\quad + O\left(\frac{x^{2+2\varepsilon}}{T}\right) \\
&=: c_2 x^2 + \frac{8}{2\pi i}(J_1^{(1)} + J_2^{(1)} + J_3^{(1)}) + O\left(\frac{x^{2+2\varepsilon}}{T}\right).
\end{aligned}$$

The contribution of horizontal line integrals ($J_2^{(1)}$ and $J_3^{(1)}$), in absolute value (using Lemmas 2.1.4, 1.9.4 and 1.9.9) is

$$\ll \int_{\frac{3}{2}+\varepsilon}^{2+\varepsilon} \frac{|\zeta(\sigma-1+iT)L(\sigma-1+iT, \mathrm{sym}^2 f\otimes\widetilde{\chi}_0)L(\sigma-1+iT, \mathrm{sym}^4 f\otimes\widetilde{\chi}_0)|}{T}x^\sigma d\sigma$$

$$\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} \frac{|\zeta(\sigma+iT)L(\sigma+iT, \mathrm{sym}^2 f\otimes\widetilde{\chi}_0)L(\sigma+iT, \mathrm{sym}^4 f\otimes\widetilde{\chi}_0)|}{T}x^{\sigma+1} d\sigma$$

$$\begin{aligned}
&\ll \left(\frac{x}{T}\right)\max_{\frac{1}{2}+\varepsilon\le\sigma\le 1+\varepsilon} x^\sigma T^{\frac{1}{3}(1+\varepsilon-\sigma)}T^{\frac{8}{2}(1+\varepsilon-\sigma)} \\
&\ll \left(\frac{x^{1+\varepsilon}}{T}\right)\max_{\frac{1}{2}+\varepsilon\le\sigma\le 1+\varepsilon}\left(\frac{x}{T^{\frac{13}{3}}}\right)^\sigma T^{\frac{13}{3}}.
\end{aligned}$$

Clearly $\left(\frac{x}{T^{\frac{13}{3}}}\right)^{\sigma}$ is monotonic as a function of σ for $\frac{1}{2}+\varepsilon \leq \sigma \leq 1+\varepsilon$ and hence the maximum is attained at the extremities of the interval $\left[\frac{1}{2}+\varepsilon, 1+\varepsilon\right]$. Thus,

$$\begin{aligned} J_2^{(1)} + J_3^{(1)} &\ll x^{1+\varepsilon}\left(x^{\frac{1}{2}+\varepsilon}T^{\frac{13}{6}-1+\varepsilon} + \frac{x^{1+\varepsilon}}{T}\right) \\ &\ll x^{\frac{3}{2}+2\varepsilon}T^{\frac{7}{6}+\varepsilon} + \frac{x^{2+2\varepsilon}}{T}. \end{aligned}$$

The contribution of left vertical line integral ($J_1^{(1)}$), in absolute value (using Cauchy-Schwarz inequality, Lemma 2.1.4, 1.9.1 and 1.9.9) is

$$\ll \int_{\frac{3}{2}+\varepsilon-iT}^{\frac{3}{2}+\varepsilon+iT} \frac{|\zeta(\frac{1}{2}+\varepsilon+it)L(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^2 f\otimes\widetilde{\chi}_0)L(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^4 f\otimes\widetilde{\chi}_0)|}{\left|\frac{3}{2}+\varepsilon+it\right|} x^{\frac{3}{2}+\varepsilon}dt$$

$$\begin{aligned} \ll x^{\frac{3}{2}+\varepsilon} + x^{\frac{3}{2}+\varepsilon}&\left(\int_{10\leq|t|\leq T} \frac{|\zeta(\frac{1}{2}+\varepsilon+it)|^2}{t}dt\right)^{\frac{1}{2}} \\ &\times\left(\int_{10\leq|t|\leq T} \frac{|L(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^2 f\otimes\widetilde{\chi}_0)L(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^4 f\otimes\widetilde{\chi}_0)|^2}{t}dt\right)^{\frac{1}{2}} \end{aligned}$$

$$\ll x^{\frac{3}{2}+\varepsilon} + x^{\frac{3}{2}+\varepsilon}\left(T^{\frac{\varepsilon}{2}}T^{\frac{3}{2}+\frac{\varepsilon}{2}}\right)$$

$$\ll x^{\frac{3}{2}+2\varepsilon}T^{\frac{3}{2}+2\varepsilon}.$$

Note that, $10 \leq T \leq x$. Thus, we obtain

$$\sum_{n\leq x} \lambda^2_{\mathrm{sym}^2 f}(n) r_4(n) = c_2 x^2 + O\left(x^{\frac{3}{2}+2\varepsilon}T^{\frac{3}{2}+2\varepsilon}\right) + O\left(\frac{x^{2+2\varepsilon}}{T}\right).$$

We choose T such that $x^{\frac{3}{2}}T^{\frac{3}{2}} \asymp \frac{x^2}{T}$ i.e., $T^{\frac{5}{2}} \asymp x^{\frac{1}{2}}$. Therefore, $T \asymp x^{\frac{1}{5}}$.

Thus, we get

$$\sum_{n\leq x}\lambda^2_{\mathrm{sym}^2 f}(n)r_4(n) = c_2x^2 + O\left(x^{\frac{9}{5}+\varepsilon}\right).$$

This proves the theorem.

2.3.1 Concluding Remarks

We observe from Lemma 2.1.4 that

$$F_2(s) = G_2(s)H_2(s),$$

where

$$\begin{aligned} G_2(s) := {} & \zeta(s)L(s-1,\widetilde{\chi}_0)L(s,\mathrm{sym}^2 f)L(s-1,\mathrm{sym}^2 f\otimes\widetilde{\chi}_0) \\ & \times L(s,\mathrm{sym}^4 f)L(s-1,\mathrm{sym}^4 f\otimes\widetilde{\chi}_0), \end{aligned}$$

and $H_2(s)$ has an Euler product, which is uniformly, and absolutely convergent in $\sigma \geq \frac{3}{2} + 2\varepsilon$ for any small positive constant ε. We also know good amount of analytic properties of $G_2(s)$, and each factor of $G_2(s)$ satisfies a functional equation of the Riemann zeta type. From our proof, it is evident that we have used only the known analytic properties of $H_2(s)$ said above. More information of $H_2(s)$ in the region $\Re(s) \geq (1-10\varepsilon)$ may even lead to the following conjecture.

Conjecture 2.3.1 For sufficiently large x, we have

$$\sum_{\substack{n=a^2+b^2+c^2+d^2\leq x\\ (a,b,c,d)\in\mathbb{Z}^4}} \lambda^2_{\mathrm{sym}^2 f}(a^2+b^2+c^2+d^2) = \tilde{c}_1x^2 + \tilde{c}_2x + O(x^\theta),$$

where $\tilde{c}_1$, $\tilde{c}_2$ are effective constants, and θ is some positive constant satisfying $0<\theta<1$.

However, currently we do not have any idea how to proceed towards the above proposed conjecture.

Chapter 3

Higher moments of the Fourier coefficients of symmetric square L-functions on certain sequence

3.1 Introduction

In this chapter, we consider the discrete higher power moments of the Fourier coefficients of symmetric square L-functions on the same sequence of positive numbers.

More precisely, we study the behavior of the following sums:

$$\sum_{\substack{a^2+b^2+c^2+d^2\leq x \\ (a,b,c,d)\in\mathbb{Z}^4}} \lambda_{\mathrm{sym}^2 f}^3(a^2+b^2+c^2+d^2)$$

and

$$\sum_{\substack{a^2+b^2+c^2+d^2\leq x \\ (a,b,c,d)\in\mathbb{Z}^4}} \lambda_{\mathrm{sym}^2 f}^4(a^2+b^2+c^2+d^2).$$

Remark 3.1.1 Very recently, Newton and Thorne proved the automorphy of the symmetric power lifting $\mathrm{sym}^n(f)$ for every $n \geq 1$, where f is a cuspidal Hecke eigenform of level 1. They also establish the same result for a more general class of cuspidal Hecke eigenforms, including all those associated to semistable elliptic curves over $\mathbb{Q}$, see [Ne-Th 1, Ne-Th 2]. This enables us to study the higher moments, namely, for an integer $l \geq 3$,

$$\sum_{\substack{a^2+b^2+c^2+d^2 \leq x \\ (a,b,c,d)\in\mathbb{Z}^4}} \lambda^l_{\mathrm{sym}^2 f}(a^2+b^2+c^2+d^2),$$

where $x \geq x_o$ (sufficiently large). However, in this chapter, we concentrate on this sum with symmetric power L-functions for $l = 3, 4$.

More precisely, we prove the following theorems:

Theorem 3.1.2 *For $x \geq x_0$ (sufficiently large), and $\varepsilon > 0$ be any small constant, we have*

$$\sum_{\substack{a^2+b^2+c^2+d^2 \leq x \\ (a,b,c,d)\in\mathbb{Z}^4}} \lambda^3_{sym^2 f}(a^2+b^2+c^2+d^2) = c_3 x^2 + O\left(x^{\frac{27}{14}+\varepsilon}\right)$$

where c_3 is an effective constant defined as

$$\begin{aligned} c_3 = & (-2)\zeta(2)L^2(2, sym^2 f)L^2(1, sym^2 f \otimes \widetilde{\chi}_0)L(2, sym^4 f)L(1, sym^4 f \otimes \widetilde{\chi}_0) \\ & \times L(2, sym^2 f \otimes sym^4 f)L(1, sym^2 f \otimes sym^4 f \otimes \widetilde{\chi}_0)H_3(2), \end{aligned}$$

$H_3(s)$ is a Dirichlet series that converges uniformly, and absolutely in the half plane $\Re(s) > \frac{3}{2}$, and $H_3(s) \neq 0$ on $\Re(s) = 2$, and $\widetilde{\chi}_0$ is a character modulo 4.

Theorem 3.1.3 *For $x \geq x_0$ (sufficiently large), and $\varepsilon > 0$ be any small constant, we have*

$$\sum_{\substack{a^2+b^2+c^2+d^2 \leq x \\ (a,b,c,d)\in\mathbb{Z}^4}} \lambda^4_{sym^2 f}(a^2+b^2+c^2+d^2) = c_4 x^2 \log x + O\left(x^{\frac{160}{81}+\varepsilon}\right),$$

where c_4 *is an effective constant defined as*

$$\begin{aligned} c_4 = &\zeta^2(2)L^3(2, sym^2 f)L^3(1, sym^2 f \otimes \widetilde{\chi}_0)L^3(2, sym^4 f) \\ &\times L^3(1, sym^4 f \otimes \widetilde{\chi}_0)L^2(2, sym^2 f \otimes sym^4 f)L^2(1, sym^2 f \otimes sym^4 f \otimes \widetilde{\chi}_0) \\ &\times L(2, sym^4 f \otimes sym^4 f)L(1, sym^4 f \otimes sym^4 f \otimes \widetilde{\chi}_0)H_4(2). \end{aligned}$$

$H_4(s)$ *is a Dirichlet series that converges uniformly, and absolutely in the half plane* $\Re(s) > \frac{3}{2}$, *and* $H_4(s) \neq 0$ *on* $\Re(s) = 2$, *and* $\widetilde{\chi}_0$ *is a character modulo 4.*

Our main objective in this chapter is to establish Theorem 3.1.2 and Theorem 3.1.3. The published version of the work discussed in this chapter can be seen in [Sr-Sa 3].

First we will prove Lemma 3.1.4 which is related to the decomposition of the relative L-functions but here it is much more complicated than the one we stated in chapter 2. This lemma is essential in order to prove Theorem 3.1.2.

Lemma 3.1.4 *Let* f *be a normalized primitive holomorphic cusp form of weight* k *for* $SL(2, \mathbb{Z})$ *and let* $\lambda_{sym^2 f}(n)$ *be the* n^{th} *normalized Fourier coefficient of the symmetric square* L*-function associated to* f*. If*

$$F_3(s) = \sum_{n=1}^{\infty} \frac{\lambda^3_{sym^2 f}(n) r(n)}{n^s},$$

for $\Re(s) > 2$, *then*

$$F_3(s) = G_3(s)H_3(s),$$

where

$$\begin{aligned} G_3(s) :=&\zeta(s)L(s-1, \widetilde{\chi}_0)L^2(s, sym^2 f)L^2(s-1, sym^2 f \otimes \widetilde{\chi}_0)L(s, sym^4 f) \\ &L(s-1, sym^4 f \otimes \widetilde{\chi}_0)L(s, sym^2 f \otimes sym^4 f)L(s-1, sym^2 f \otimes sym^4 f \otimes \widetilde{\chi}_0), \end{aligned}$$

and $\widetilde{\chi}_0$ *is a character modulo* 4.
Here, $H_3(s)$ *is a Dirichlet series which converges uniformly and absolutely in the half plane* $\Re(s) > \frac{3}{2}$ *and* $H_3(s) \neq 0$ *on* $\Re(s) = 2$.

3.2 Proof of Lemma 3.1.4

We observe that $\lambda^3_{\mathrm{sym}^2 f}(n)r(n)$ is multiplicative, and hence

$$F_3(s) = \prod_p \left(1 + \frac{\lambda^3_{\mathrm{sym}^2 f}(p)r(p)}{p^s} + \cdots + \frac{\lambda^3_{\mathrm{sym}^2 f}(p^l)r(p^l)}{p^{ls}} + \dots\right). \tag{3.1}$$

Note that,

$$\begin{aligned}
\lambda^3_{\mathrm{sym}^2 f}(p)r(p) &= \left(\sum_{m=0}^{2} \alpha^{2-m}(p)\beta^m(p)\right)^3 (1 + \widetilde{\chi}_0(p)p) \\
&= \left(\alpha^6(p) + 2\alpha^4(p) + 6\alpha^2(p) + 7 + 6\beta^2(p) + 2\beta^4(p) + \beta^6(p)\right)(1 + \widetilde{\chi}_0(p)p) \\
&= (1 + 2\alpha^2(p) + 2 + 2\beta^2(p) + \alpha^4(p) + \alpha^2(p) + 1 + \beta^2(p) + \beta^4(p) \\
&\quad + \alpha^6(p) + \alpha^4(p) + 3\alpha^2(p) + 3 + 3\beta^2(p) + \beta^4(p) + \beta^6(p))\,(1 + \widetilde{\chi}_0(p)p) \\
&= (1 + 2(\alpha^2(p) + 1 + \beta^2(p)) + (\alpha^4(p) + \alpha^2(p) + 1 + \beta^2(p) + \beta^4(p)) \\
&\quad + (\alpha^6(p) + \alpha^4(p) + 3\alpha^2(p) + 3 + 3\beta^2(p) + \beta^4(p) + \beta^6(p)))\,(1 + \widetilde{\chi}_0(p)p) \\
&= 1 + \widetilde{\chi}_0(p)p + 2\lambda_{2f}(p)(1 + \widetilde{\chi}_0(p)p) + \lambda_{\mathrm{sym}^4 f}(p)(1 + \widetilde{\chi}_0(p)p) \\
&\quad + \lambda_{\mathrm{sym}^2 f \otimes \mathrm{sym}^4 f}(p)(1 + \widetilde{\chi}_0(p)p) \\
&= 1 + \widetilde{\chi}_0(p)p + 2\lambda_{\mathrm{sym}^2 f}(p) + 2p\lambda_{\mathrm{sym}^2 f}(p)\widetilde{\chi}_0(p) + \lambda_{\mathrm{sym}^4 f}(p) \\
&\quad + p\lambda_{\mathrm{sym}^4 f}(p)\widetilde{\chi}_0(p) + \lambda_{\mathrm{sym}^2 f \otimes \mathrm{sym}^4 f}(p) + p\lambda_{\mathrm{sym}^2 f \otimes \mathrm{sym}^4 f}(p)\widetilde{\chi}_0(p) \\
&=: b_2(p).
\end{aligned}$$

From the structure of $b_2(p)$, we define the coefficients $b_2(n)$ as

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{b_2(n)}{n^s} &= \zeta(s)L(s-1, \widetilde{\chi}_0)L^2(s, \mathrm{sym}^2 f)L^2(s-1, \mathrm{sym}^2 f \otimes \widetilde{\chi}_0)L(s, \mathrm{sym}^4 f) \\
&\quad \times L(s-1, \mathrm{sym}^4 f \otimes \widetilde{\chi}_0)L(s, \mathrm{sym}^2 f \otimes \mathrm{sym}^4 f)L(s-1, \mathrm{sym}^2 f \otimes \mathrm{sym}^4 f \otimes \widetilde{\chi}_0),
\end{aligned}$$

which is absolutely convergent in $\Re(s) > 2$. We also note that

$$\begin{aligned}
&\prod_p \left(1 + \frac{b_2(p)}{p^s} + \dots + \frac{b_2(p^l)}{p^{ls}} + \dots\right) \\
&\quad = \zeta(s)L(s-1,\widetilde{\chi}_0)L^2(s,\mathrm{sym}^2 f)L^2(s-1,\mathrm{sym}^2 f\otimes\widetilde{\chi}_0)L(s,\mathrm{sym}^4 f) \\
&\qquad \times L(s-1,\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)L(s,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f)L(s-1,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0) \\
&\quad =: G_3(s),
\end{aligned}$$

for $\Re(s) > 2$. Observe that $b_2(n) \ll_\varepsilon n^{1+\varepsilon}$ for any small positive constant ε.

Now, we note that in the half plane $\Re(s) \geq 2 + 2\varepsilon$, we have

$$\begin{aligned}
\left|\frac{b_2(p)}{p^s} + \frac{b_2(p^2)}{p^{2s}} + \dots + \frac{b_2(p^l)}{p^{ls}} + \dots\right| &\ll \sum_{l=1}^{\infty} \frac{p^{(1+\varepsilon)l}}{p^{l\sigma}} \\
&\leq \sum_{l=1}^{\infty} \frac{p^{(1+\varepsilon)l}}{p^{(2+2\varepsilon)l}} \\
&= \sum_{l=1}^{\infty} \frac{1}{p^{(1+\varepsilon)l}} \\
&= \frac{\frac{1}{p^{1+\varepsilon}}}{1 - \frac{1}{p^{1+\varepsilon}}} \\
&= \frac{1}{p^{1+\varepsilon} - 1} \\
&< 1.
\end{aligned}$$

Let us write

$$A_2 = \frac{\lambda^3_{\mathrm{sym}^2 f}(p)r(p)}{p^s} + \dots + \frac{\lambda^3_{\mathrm{sym}^2 f}(p^l)r(p^l)}{p^{ls}} + \dots,$$

and

$$B_2 = \frac{b_2(p)}{p^s} + \dots + \frac{b_2(p^l)}{p^{ls}} + \dots.$$

From the above calculation, we observe that $|B_2| < 1$ in $\Re(s) \geq 2 + 2\varepsilon$.

We note that in the half plane $\Re(s) \geq 2 + 2\varepsilon$, we have

$$\begin{aligned}\frac{1+A_2}{1+B_2} &= (1+A_2)(1 - B_2 + B_2^2 - B_2^3 + \dots) \\ &= 1 + A_2 - B_2 - A_2B_2 + \text{higher terms} \\ &= 1 + \frac{\lambda^3_{\mathrm{sym}^2 f}(p^2)r(p^2) - b_2(p^2)}{p^{2s}} + \dots + \frac{c_l(p^l)}{p^{ls}} + \dots,\end{aligned}$$

with $c_l(n) \ll_\varepsilon n^{1+\varepsilon}$. So, we have (in the half plane $\Re(s) > \frac{3}{2}$)

$$\begin{aligned}\prod_p \left(\frac{1+A_2}{1+B_2}\right) &= \prod_p \left(1 + \frac{\lambda^3_{\mathrm{sym}^2 f}(p^2)r(p^2) - b_2(p^2)}{p^{2s}} + \dots + \frac{c_l(p^l)}{p^{ls}} + \dots\right) \\ &\ll_\varepsilon 1.\end{aligned}$$

Thus, we have (in the half plane $\Re(s) > \frac{3}{2}$)

$$\begin{aligned}H_3(s) := \frac{F_3(s)}{G_3(s)} &= \prod_p \left(\frac{1+A_2}{1+B_2}\right) \\ &\ll_\varepsilon 1,\end{aligned}$$

and also $H_3(s) \neq 0$ on $\Re(s) = 2$.

We can now proceed with proving our main theorem 3.1.2.

First, we note that,

$$\begin{aligned}\sum_{\substack{a^2+b^2+c^2+d^2 \leq x \\ (a,b,c,d)\in\mathbb{Z}^4}} \lambda^\theta_{\mathrm{sym}^2 f}(a^2+b^2+c^2+d^2) &= \sum_{n\leq x} \lambda^\theta_{\mathrm{sym}^2 f}(n) \sum_{\substack{n=a^2+b^2+c^2+d^2 \\ (a,b,c,d)\in\mathbb{Z}^4}} 1 \\ &= \sum_{n\leq x} \lambda^\theta_{\mathrm{sym}^2 f}(n) r_4(n) \\ &= 8\sum_{n\leq x} \lambda^\theta_{\mathrm{sym}^2 f}(n) r(n)\end{aligned}$$

where $r(n) = \sum_{d|n, 4\nmid d} d$, with $\theta = 3$ or 4.

For the reader's convenience, we recall some definitions (cf. section 1.4) here. We note that $r(n)$ is multiplicative and given by

$$r(p^u) := \begin{cases} \dfrac{1-p^{u+1}}{1-p} & \text{if } p > 2 \\ 3 & \text{if } p = 2 \end{cases}.$$

We write $r_4(n) = 8\sum_{d|n} \widetilde{\chi}_0(d)d$, where $\widetilde{\chi}_0$ is a character modulo 4, given by

$$\widetilde{\chi}_0(p^u) := \begin{cases} \chi_0(p^u) & \text{if } p > 2 \\ 3 & \text{if } p = 2 \end{cases},$$

and χ_0 is the principal character modulo 4.
Note that,

$$\begin{aligned} r(p) &= \sum_{d|p} \widetilde{\chi}_0(d)d \\ &= 1 + p\widetilde{\chi}_0(p). \end{aligned}$$

3.3 Proof of Theorem 3.1.2

We begin with the Perron's formula (see section 1.5), applying to $F_3(s)$ with $\eta = 2 + \varepsilon$ and $10 \le T \le x$. Thus, we have

$$\begin{aligned} \sum_{n \le x} \lambda^3_{\mathrm{sym}^2 f}(n) r_4(n) &= 8 \sum_{n \le x} \lambda^3_{\mathrm{sym}^2 f}(n) r(n) \\ &= \frac{8}{2\pi i} \int_{\eta - iT}^{\eta + iT} F_3(s) \frac{x^s}{s} ds + O\left(\frac{x^{2+2\varepsilon}}{T}\right). \end{aligned}$$

We note that for $\Re(s) > 2$ (cf. section ref),

$$L(s-1, \widetilde{\chi}_0) = \left(1 - \frac{3}{2^{s-1}}\right)^{-1} \left(1 - \frac{1}{2^{s-1}}\right)^2 \zeta(s-1),$$

Thus,

$$\begin{aligned}F_3(s) &= \zeta(s)L(s-1,\widetilde{\chi}_0)L^2(s,\mathrm{sym}^2 f)L^2(s-1,\mathrm{sym}^2 f\otimes\widetilde{\chi}_0)L(s,\mathrm{sym}^4 f)\\ &\quad\times L(s-1,\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)L(s,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f)L(s-1,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)H_3(s)\\ &= \zeta(s)\left(1-\frac{3}{2^{s-1}}\right)^{-1}\left(1-\frac{1}{2^{s-1}}\right)^{2}\zeta(s-1)L^2(s,\mathrm{sym}^2 f)L^2(s-1,\mathrm{sym}^2 f\otimes\widetilde{\chi}_0)\\ &\quad\times L(s,\mathrm{sym}^4 f)L(s-1,\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)L(s,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f)\\ &\quad\times L(s-1,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)H_3(s).\end{aligned}$$

We move the line of integration to $\Re(s)=\frac{3}{2}+\varepsilon$ and by Cauchy's residue theorem there is only one simple pole at $s=2$ coming from the factor $\zeta(s-1)$. This contributes a residue, which is c_3x^2, where c_3 is an effective constant depending on the values of various L-functions appearing in $G_3(s)$ at $s=2$.
More precisely,

$$\begin{aligned}c_3 &= 8\lim_{s\to 2}(s-2)\frac{F_3(s)}{s}\\ &= 8\zeta(2)\frac{(-1)}{2}\frac{1}{2}L^2(2,\mathrm{sym}^2 f)L^2(1,\mathrm{sym}^2 f\otimes\widetilde{\chi}_0)L(2,\mathrm{sym}^4 f)L(1,\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)\\ &\quad\times L(2,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f)L(1,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)H_3(2).\end{aligned}$$

So, we obtain

$$\begin{aligned}\sum_{n\le x}\lambda^3_{\mathrm{sym}^2 f}(n)r_4(n) &= c_3x^2+\frac{8}{2\pi i}\left\{\int_{\frac{3}{2}+\varepsilon-iT}^{\frac{3}{2}+\varepsilon+iT}+\int_{2+\varepsilon-iT}^{\frac{3}{2}+\varepsilon-iT}+\int_{\frac{3}{2}+\varepsilon+iT}^{2+\varepsilon+iT}\right\}F_3(s)\frac{x^s}{s}ds\\ &\quad+O\left(\frac{x^{2+2\varepsilon}}{T}\right)\\ &=: c_3x^2+\frac{8}{2\pi i}(J_1^{(2)}+J_2^{(2)}+J_3^{(2)})+O\left(\frac{x^{2+2\varepsilon}}{T}\right).\end{aligned}$$

The contribution of horizontal line integrals ($J_2^{(2)}$ and $J_3^{(2)}$) in absolute value (using Lemmas 1.9.4, 3.1.4 and 1.9.9) is

$$\ll \int_{\frac{3}{2}+\varepsilon}^{2+\varepsilon} \left[\frac{|\zeta(\sigma-1+iT)L^2(\sigma-1+iT,\mathrm{sym}^2 f\otimes\widetilde{\chi}_0)L(\sigma-1+iT,\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)|}{T} \right.$$
$$\left. \times |L(\sigma-1+iT,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)|x^{\sigma}\right] d\sigma$$

$$\ll \int_{\frac{3}{2}+\varepsilon}^{2+\varepsilon} \left[\frac{|\zeta(\sigma-1+iT)L^2(\sigma-1+iT,\mathrm{sym}^2 f\otimes\widetilde{\chi}_0)L(\sigma-1+iT,\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)|}{T} \right.$$
$$\left. \times\ |L(\sigma-1+iT,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)|x^{\sigma}\right] d\sigma$$

$$\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} \left[\frac{|\zeta(\sigma+iT)L^2(\sigma+iT,\mathrm{sym}^2 f\otimes\widetilde{\chi}_0)L(\sigma+iT,\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)|}{T} \right.$$
$$\left. \times\ |L(\sigma-1+iT,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)|x^{\sigma+1}\right] d\sigma$$

$$\ll \left(\frac{x}{T}\right) \max_{\frac{1}{2}+\varepsilon\le\sigma\le1+\varepsilon} x^{\sigma} T^{\frac{1}{3}(1+\varepsilon-\sigma)} T^{\frac{26}{2}(1+\varepsilon-\sigma)}$$

$$\ll \left(\frac{x^{1+\varepsilon}}{T}\right) \max_{\frac{1}{2}+\varepsilon\le\sigma\le1+\varepsilon} \left(\frac{x}{T^{\frac{40}{3}}}\right)^{\sigma} T^{\frac{40}{3}}.$$

Clearly $\left(\frac{x}{T^{\frac{40}{3}}}\right)^{\sigma}$ is monotonic as a function of σ for $\frac{1}{2}+\varepsilon\le\sigma\le 1+\varepsilon$ and hence the maximum is attained at the extremities of the interval $[\frac{1}{2}+\varepsilon, 1+\varepsilon]$. Thus,

$$\begin{aligned} J_2^{(2)} + J_3^{(2)} &\ll x^{1+\varepsilon}\left(x^{\frac{1}{2}+\varepsilon}T^{\frac{20}{3}-1+\varepsilon} + \frac{x^{1+\varepsilon}}{T}\right) \\ &\ll x^{\frac{3}{2}+2\varepsilon}T^{\frac{17}{3}+\varepsilon} + \frac{x^{2+2\varepsilon}}{T}. \end{aligned}$$

The contribution of left vertical line integral ($J_1^{(2)}$) in absolute value (using Cauchy-Schwarz inequality, Lemmas 1.9.1, 1.9.2, 3.1.4 and 1.9.9) is

$$\begin{aligned}
&\ll \int_{\frac{3}{2}+\varepsilon-iT}^{\frac{3}{2}+\varepsilon+iT} \Bigg[\frac{\left|\zeta\left(\frac{1}{2}+\varepsilon+it\right) L^2(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^2 f\otimes\widetilde{\chi}_0)L(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)\right|}{|\frac{3}{2}+\varepsilon+it|} \\
&\qquad \times \left| L\left(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0\right)\right| x^{\frac{3}{2}+\varepsilon}\Bigg] dt \\
&\ll x^{\frac{3}{2}+\varepsilon} + x^{\frac{3}{2}+\varepsilon}\left(\int_{10\le|t|\le T}\frac{|\zeta(\frac{1}{2}+\varepsilon+it)|^2}{t}dt\right)^{\frac{1}{2}} \\
&\qquad \times \Bigg(\int_{10\le|t|\le T}\frac{|L^2(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^2 f\otimes\widetilde{\chi}_0)L(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)|^2}{t} \\
&\qquad \times \left| L\left(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0\right)\right| dt\Bigg)^{\frac{1}{2}} \\
&\ll x^{\frac{3}{2}+\varepsilon} + x^{\frac{3}{2}+\varepsilon}\left(T^{\frac{\varepsilon}{2}}T^{\frac{1}{2}(\frac{26}{2}-1+\varepsilon)}\right) \\
&\ll x^{\frac{3}{2}+2\varepsilon}T^{6+2\varepsilon}.
\end{aligned}$$

Note that $10 \le T \le x$. Thus, we obtain

$$\sum_{n\le x}\lambda^3_{\mathrm{sym}^2 f}(n)r_4(n) = c_3x^2 + O(x^{\frac{3}{2}+2\varepsilon}T^{6+2\varepsilon}) + O\left(x^{\frac{3}{2}+2\varepsilon}T^{\frac{17}{3}+2\varepsilon}+\frac{x^{2+2\varepsilon}}{T}\right).$$

We choose T such that $x^{\frac{3}{2}}T^6 \asymp \frac{x^2}{T}$, i.e., $T^7 \asymp x^{\frac{1}{2}}$.

Therefore, $T \asymp x^{\frac{1}{14}}$. Thus, we get

$$\sum_{n\le x}\lambda^3_{\mathrm{sym}^2 f}(n)r_4(n) = c_3x^2 + O\left(x^{\frac{27}{14}+2\varepsilon}\right),$$

where

$$\begin{aligned}
c_3 &= (-2)\zeta(2)L^2(2,\mathrm{sym}^2 f)L^2(1,\mathrm{sym}^2 f\otimes\widetilde{\chi}_0)L(2,\mathrm{sym}^4 f)L(1,\mathrm{sym}^4 f\otimes\widetilde{\chi}_0) \\
&\quad\times L(2,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f)L(1,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)H_3(2).
\end{aligned}$$

This proves the theorem 3.1.2.

To establish Theorem 3.1.3, we shall first prove Lemma 3.3.1.

Lemma 3.3.1 *Let f be a normalized primitive holomorphic cusp form of weight k for $SL(2,\mathbb{Z})$ and let $\lambda_{sym^2 f}(n)$ be the n^{th} normalized Fourier coefficient of the symmetric square L-function associated to f. If*

$$F_4(s) = \sum_{n=1}^{\infty} \frac{\lambda^4_{sym^2 f}(n) r(n)}{n^s}$$

for $\Re(s) > 2$, then

$$F_4(s) = G_4(s) H_4(s),$$

where

$$\begin{aligned} G_4(s) :=& \zeta^2(s) L^2(s-1, \widetilde{\chi}_0) L^3(s, sym^2 f) L^3(s-1, sym^2 f \otimes \widetilde{\chi}_0) L^3(s, sym^4 f) \\ &\times L^3(s-1, sym^4 f \otimes \widetilde{\chi}_0) L^2(s, sym^2 f \otimes sym^4 f) L^2(s-1, sym^2 f \otimes sym^4 f \otimes \widetilde{\chi}_0) \\ &\times L(s, sym^4 f \otimes sym^4 f) L(s-1, sym^4 f \otimes sym^4 f \otimes \widetilde{\chi}_0), \end{aligned}$$

and $\widetilde{\chi}_0$ is a character modulo 4.

Here, $H_4(s)$ is a Dirichlet series which converges uniformly and absolutely in the half plane $\Re(s) > \frac{3}{2}$ and $H_4(s) \neq 0$ on $\Re(s) = 2$.

3.4 Proof of Lemma 3.3.1

We observe that $\lambda^4_{\mathrm{sym}^2 f}(n) r(n)$ is multiplicative, and hence

$$F_4(s) = \prod_p \left(1 + \frac{\lambda^4_{\mathrm{sym}^2 f}(p) r(p)}{p^s} + \cdots + \frac{\lambda^4_{\mathrm{sym}^2 f}(p^l) r(p^l)}{p^{ls}} + \dots \right). \tag{3.2}$$

Note that,

$$\begin{aligned}
\lambda^4_{\mathrm{sym}^2 f}(p) r(p) &= \left(\sum_{m=0}^{2} \alpha^{2-m}(p)\beta^m(p)\right)^4 (1+\widetilde{\chi}_0(p)p)\\
&= \left(\alpha^4(p) + 2\alpha^2(p) + 3 + 2\beta^2(p) + \beta^4(p)\right)^2 (1+\widetilde{\chi}_0(p)p)\\
&= (\alpha^8(p) + 4\alpha^6(p) + 10\alpha^4(p) + 4\alpha^2(p) + 4\beta^2(p) + 12\alpha^2(p) + 19 + \beta^8(p)\\
&\quad + 4\beta^6(p) + 10\beta^4(p) + 12\beta^2(p))\,(1+\widetilde{\chi}_0(p)p)\\
&= 2 + 2\widetilde{\chi}_0(p)p + 3(\alpha^2(p) + 1 + \beta^2(p))(1+\widetilde{\chi}_0(p)p)\\
&\quad + 3(\alpha^4(p) + \alpha^2(p) + 1 + \beta^2(p) + \beta^4(p))\,(1+\widetilde{\chi}_0(p)p)\\
&\quad + 2(\alpha^6(p) + \alpha^4(p) + 3\alpha^2(p) + 3 + 3\beta^2(p) + \beta^4(p) + \beta^6(p))\,(1+\widetilde{\chi}_0(p)p)\\
&\quad + (\alpha^8(p) + 2\alpha^6(p) + 3\alpha^4(p) + 4\alpha^2(p) + 5 + 4\beta^2(p)\\
&\quad + 3\beta^4(p) + 2\beta^6(p) + \beta^8(p))\,(1+\widetilde{\chi}_0(p)p)\\
&= 2 + 2\widetilde{\chi}_0(p)p + 3\lambda_{\mathrm{sym}^2 f}(p)(1+\widetilde{\chi}_0(p)p) + 3\lambda_{\mathrm{sym}^4 f}(p)(1+\widetilde{\chi}_0(p)p)\\
&\quad + 2\lambda_{\mathrm{sym}^2 f\otimes \mathrm{sym}^4 f}(p)(1+\widetilde{\chi}_0(p)p) + \lambda_{\mathrm{sym}^4 f\otimes \mathrm{sym}^4 f}(p)(1+\widetilde{\chi}_0(p)p)\\
&=: b_3(p).
\end{aligned}$$

From the structure of $b_3(p)$, we define the coefficients $b_3(n)$ as

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{b_3(n)}{n^s} &= \zeta^2(s)L^2(s-1,\widetilde{\chi}_0)L^3(s,\mathrm{sym}^2 f)L^3(s-1,\mathrm{sym}^2 f\otimes\widetilde{\chi}_0)L^3(s,\mathrm{sym}^4 f)\\
&\quad\times L^3(s-1,\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)L^2(s,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f)L^2(s-1,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)\\
&\quad\times L(s,\mathrm{sym}^4 f\otimes\mathrm{sym}^4 f)L(s-1,\mathrm{sym}^4 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0),
\end{aligned}$$

which is absolutely convergent in $\Re(s) > 2$. We also note that

$$
\begin{aligned}
&\prod_p \left(1 + \frac{b_3(p)}{p^s} + \dots + \frac{b_3(p^l)}{p^{ls}} + \dots\right) \\
&\quad = \zeta^2(s) L^2(s-1, \widetilde{\chi}_0) L^3(s, \mathrm{sym}^2 f) L^3(s-1, \mathrm{sym}^2 f \otimes \widetilde{\chi}_0) L^3(s, \mathrm{sym}^4 f) \\
&\qquad \times L^3(s-1, \mathrm{sym}^4 f \otimes \widetilde{\chi}_0) L^2(s, \mathrm{sym}^2 f \otimes \mathrm{sym}^4 f) L^2(s-1, \mathrm{sym}^2 f \otimes \mathrm{sym}^4 f \otimes \widetilde{\chi}_0) \\
&\qquad \times L(s, \mathrm{sym}^4 f \otimes \mathrm{sym}^4 f) L(s-1, \mathrm{sym}^4 f \otimes \mathrm{sym}^4 f \otimes \widetilde{\chi}_0) \\
&\quad =: G_4(s),
\end{aligned}
$$

for $\Re(s) > 2$. Observe that $b_3(n) \ll_\varepsilon n^{1+\varepsilon}$ for any small positive constant ε. Now, we note that in the half plane $\Re(s) \geq 2 + 2\varepsilon$, we have

$$
\begin{aligned}
\left| \frac{b_3(p)}{p^s} + \frac{b_3(p^2)}{p^{2s}} + \dots + \frac{b_3(p^l)}{p^{ls}} + \dots \right| &\ll \sum_{l=1}^{\infty} \frac{p^{(1+\varepsilon)l}}{p^{l\sigma}} \\
&\leq \sum_{l=1}^{\infty} \frac{p^{(1+\varepsilon)l}}{p^{(2+2\varepsilon)l}} \\
&= \sum_{l=1}^{\infty} \frac{1}{p^{(1+\varepsilon)l}} \\
&= \frac{\frac{1}{p^{1+\varepsilon}}}{1 - \frac{1}{p^{1+\varepsilon}}} \\
&= \frac{1}{p^{1+\varepsilon} - 1} \\
&< 1.
\end{aligned}
$$

Let us write

$$
A_3 = \frac{\lambda^4_{\mathrm{sym}^2 f}(p) r(p)}{p^s} + \dots + \frac{\lambda^4_{\mathrm{sym}^2 f}(p^l) r(p^l)}{p^{ls}} + \dots,
$$

and

$$B_3 = \frac{b_3(p)}{p^s} + \cdots + \frac{b_3(p^l)}{p^{ls}} + \cdots.$$

From the above calculation, we observe that $|B_3| < 1$ in $\Re(s) \geq 2 + 2\varepsilon$.

We note that in the half plane $\Re(s) \geq 2 + 2\varepsilon$, we have

$$\begin{aligned}\frac{1 + A_3}{1 + B_3} &= (1 + A_3)(1 - B_3 + B_3^2 - B_3^3 + \dots) \\ &= 1 + A_3 - B_3 - A_3 B_3 + \text{higher terms} \\ &= 1 + \frac{\lambda^4_{\mathrm{sym}^2 f}(p^2) r(p^2) - b_3(p^2)}{p^{2s}} + \cdots + \frac{\widetilde{c}_l(p^l)}{p^{ls}} + \dots,\end{aligned}$$

with $\widetilde{c}_l(n) \ll_\varepsilon n^{1+\varepsilon}$. So, we have (in the half plane $\Re(s) > \frac{3}{2}$)

$$\begin{aligned}\prod_p \left(\frac{1 + A_3}{1 + B_3}\right) &= \prod_p \left(1 + \frac{\lambda^4_{\mathrm{sym}^2 f}(p^2) r(p^2) - b_3(p^2)}{p^{2s}} + \cdots + \frac{\widetilde{c}_l(p^l)}{p^{ls}} + \dots\right) \\ &\ll_\varepsilon 1.\end{aligned}$$

Thus, we have (in the half plane $\Re(s) > \frac{3}{2}$)

$$\begin{aligned}H_4(s) := \frac{F_4(s)}{G_4(s)} &= \prod_p \left(\frac{1 + A_3}{1 + B_3}\right) \\ &\ll_\varepsilon 1,\end{aligned}$$

and also $H_4(s) \neq 0$ on $\Re(s) = 2$.

Now, we proceed to establish our key theorem 3.1.3.

3.5 Proof of Theorem 3.1.3

We begin with the Perron's formula (see section 1.5), applying to $F_4(s)$ with $\eta = 2+\varepsilon$ and $10 \le T \le x$. Thus, we have

$$\begin{aligned}\sum_{n\le x}\lambda^4_{\mathrm{sym}^2 f}(n)r_4(n) &= 8\sum_{n\le x}\lambda^4_{\mathrm{sym}^2 f}(n)r(n)\\ &= \frac{8}{2\pi i}\int_{\eta-iT}^{\eta+iT} F_4(s)\frac{x^s}{s}ds + O\left(\frac{x^{2+2\varepsilon}}{T}\right).\end{aligned}$$

We note that, for $\Re(s) > 2$,

$$L^2(s-1,\widetilde{\chi}_0) = \left(1-\frac{3}{2^{s-1}}\right)^{-2}\left(1-\frac{1}{2^{s-1}}\right)^4\zeta^2(s-1),$$

since

$$L^2(s-1,\chi_0) = \left(1-\frac{1}{2^{s-1}}\right)^2\zeta^2(s-1).$$

Thus,

$$\begin{aligned}F_4(s) &:= \zeta^2(s)L^2(s-1,\widetilde{\chi}_0)L^3(s,\mathrm{sym}^2 f)L^3(s-1,\mathrm{sym}^2 f\otimes\widetilde{\chi}_0)L^3(s,\mathrm{sym}^4 f)\\ &\quad\times L^3(s-1,\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)L^2(s,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f)L^2(s-1,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)\\ &\quad\times L(s,\mathrm{sym}^4 f\otimes\mathrm{sym}^4 f)L(s-1,\mathrm{sym}^4 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)H_4(s)\\ &= \zeta^2(s)\left(1-\frac{3}{2^{s-1}}\right)^{-2}\left(1-\frac{1}{2^{s-1}}\right)^4\zeta^2(s-1)L^3(s,\mathrm{sym}^2 f)L^3(s-1,\mathrm{sym}^2 f\otimes\widetilde{\chi}_0)\\ &\quad\times L^3(s-1,\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)L^2(s,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f)L^2(s-1,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)\\ &\quad\times L(s,\mathrm{sym}^4 f\otimes\mathrm{sym}^4 f)L(s-1,\mathrm{sym}^4 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)L^3(s,\mathrm{sym}^4 f)H_4(s).\end{aligned}$$

We move the line of integration to $\Re(s) = \dfrac{3}{2}+\varepsilon$. There is only one second order pole at $s=2$ coming from the factor $\zeta^2(s-1)$, so by Cauchy's residue theorem, we obtain

$$\begin{aligned}\sum_{n\le x}\lambda^4_{\mathrm{sym}^2 f}(n)r_4(n) &= 8\operatorname*{Res}_{s=2}\left\{F_4(s)\frac{x^s}{s}\right\}+\frac{8}{2\pi i}\left\{\int_{\frac32+\varepsilon-iT}^{\frac32+\varepsilon+iT}+\int_{2+\varepsilon-iT}^{\frac32+\varepsilon-iT}+\int_{\frac32+\varepsilon+iT}^{2+\varepsilon+iT}\right\}F_4(s)\frac{x^s}{s}ds\\ &\quad+O\left(\frac{x^{2+2\varepsilon}}{T}\right)\\ &= 8\operatorname*{Res}_{s=2}\left\{F_4(s)\frac{x^s}{s}\right\}+\frac{8}{2\pi i}(J_1^{(3)}+J_2^{(3)}+J_3^{(3)})+O\left(\frac{x^{2+2\varepsilon}}{T}\right).\end{aligned}$$

More precisely,

$$\begin{aligned}\operatorname*{Res}_{s=2}\left\{F_4(s)\frac{x^s}{s}\right\} &= \operatorname*{Res}_{s=2}\{\zeta^2(s)L^2(s-1,\widetilde{\chi}_0)L^3(s,\mathrm{sym}^2 f)L^3(s-1,\mathrm{sym}^2 f\otimes\widetilde{\chi}_0)\\ &\quad\times L^3(s,\mathrm{sym}^4 f)L^3(s-1,\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)L^2(s,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f)\\ &\quad\times L^2(s-1,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)L(s,\mathrm{sym}^4 f\otimes\mathrm{sym}^4 f)\\ &\quad\times L(s-1,\mathrm{sym}^4 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)H_4(s)\frac{x^s}{s}\Big\}\\ &= \zeta^2(2)L^3(2,\mathrm{sym}^2 f)L^3(1,\mathrm{sym}^2 f\otimes\widetilde{\chi}_0)L^3(2,\mathrm{sym}^4 f)\\ &\quad\times L^3(1,\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)L^2(2,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f)L^2(1,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)\\ &\quad\times L(2,\mathrm{sym}^4 f\otimes\mathrm{sym}^4 f)L(1,\mathrm{sym}^4 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)H_4(2)\\ &\quad\times\operatorname*{Res}_{s=2}\left\{L^2(s-1,\widetilde{\chi}_0)\frac{x^s}{s}\right\},\end{aligned}$$

where

$$\begin{aligned}\operatorname*{Res}_{s=2}\left\{L^2(s-1,\widetilde{\chi}_0)\frac{x^s}{s}\right\} &= \frac{x^2}{2}\left(1-\frac32\right)^{-2}\left(1-\frac12\right)^4\operatorname*{Res}_{s=2}\left\{\left(\frac{1}{s-2}+\gamma+O(|s-2|)\right)^2x^{s-2}\right\}\\ &= \frac{x^2}{2}\times\frac14\times\operatorname*{Res}_{s=2}\left\{\left(\frac{1}{s-2}+\gamma+O(|s-2|)\right)^2e^{(s-2)\log x}\right\}\\ &= \frac{x^2\log x}{8},\end{aligned}$$

and γ is Euler's constant. Hence,

$$\begin{aligned}\operatorname*{Res}_{s=2}\left\{F_4(s)\frac{x^s}{s}\right\} =&\zeta^2(2)L^3(2,\mathrm{sym}^2 f)L^3(1,\mathrm{sym}^2 f\otimes\widetilde{\chi}_0)L^3(2,\mathrm{sym}^4 f)\\ &\times L^3(1,\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)L^2(2,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f)L^2(1,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)\\ &\times L(2,\mathrm{sym}^4 f\otimes\mathrm{sym}^4 f)L(1,\mathrm{sym}^4 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)H_4(2)\frac{x^2\log x}{8}.\end{aligned}$$

The contribution of horizontal line integrals ($J_2^{(3)}$ and $J_3^{(3)}$) in absolute value (using Lemmas 1.9.4, 3.3.1 and 1.9.9) is

$$\begin{aligned}\ll \int_{\frac{3}{2}+\varepsilon}^{2+\varepsilon}\Bigg[&\frac{|\zeta^2(\sigma-1+iT)L^3(\sigma-1+iT,\mathrm{sym}^2 f\otimes\widetilde{\chi}_0)L^3(\sigma-1+iT,\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)|}{T}\\ &\times|L^2(\sigma-1+iT,\mathrm{sym}^2 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)L(\sigma-1+iT,\mathrm{sym}^4 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)|x^\sigma\Bigg]\,d\sigma\end{aligned}$$

$$\begin{aligned}\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon}\Bigg[&\frac{|\zeta^2(\sigma+iT)L^3(\sigma+iT,\mathrm{sym}^2 f\otimes\widetilde{\chi}_0)L^3(\sigma+iT,\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)|}{T}\\ &\times|L^2(\sigma+iT,\mathrm{sym}^2\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)L(\sigma+iT,\mathrm{sym}^4 f\otimes\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)|x^{\sigma+1}\Bigg]\,d\sigma\end{aligned}$$

$$\ll \left(\frac{x}{T}\right)\max_{\frac{1}{2}+\varepsilon\le\sigma\le1+\varepsilon} x^\sigma T^{\frac{2}{3}(1+\varepsilon-\sigma)}T^{\frac{79}{2}(1+\varepsilon-\sigma)}$$

$$\ll \left(\frac{x^{1+\varepsilon}}{T}\right)\max_{\frac{1}{2}+\varepsilon\le\sigma\le1+\varepsilon}\left(\frac{x}{T^{\frac{241}{6}}}\right)^\sigma T^{\frac{241}{6}}.$$

Clearly $\left(\frac{x}{T^{\frac{241}{6}}}\right)^\sigma$ is monotonic as a function of σ for $\frac{1}{2}+\varepsilon\le\sigma\le1+\varepsilon$ and hence the maximum is attained at the extremities of the interval $[\frac{1}{2}+\varepsilon,1+\varepsilon]$. Thus,

$$\begin{aligned}J_2^{(3)}+J_3^{(3)} &\ll x^{1+\varepsilon}\left(x^{\frac{1}{2}+\varepsilon}T^{\frac{241}{12}-1+\varepsilon}+\frac{x^{1+\varepsilon}}{T}\right)\\ &\ll x^{\frac{3}{2}+2\varepsilon}T^{\frac{229}{12}+\varepsilon}+\frac{x^{2+2\varepsilon}}{T}.\end{aligned}$$

The contribution of left vertical line integral ($J_1^{(3)}$) in absolute value (using Cauchy-Schwarz inequality, Lemmas 1.9.1, 1.9.2, 3.3.1 and 1.9.9) is

$$\ll \int_{\frac{3}{2}+\varepsilon-iT}^{\frac{3}{2}+\varepsilon+iT} \Bigg[\frac{|\zeta^2(\frac{1}{2}+\varepsilon+it)L^3(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^2 f \otimes \widetilde{\chi}_0)L^3(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^4 f \otimes \widetilde{\chi}_0)|}{|\frac{3}{2}+\varepsilon+it|}$$

$$\times \left| L^2\left(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^2 f \otimes \mathrm{sym}^4 f \otimes \widetilde{\chi}_0\right) L\left(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^4 f \otimes \mathrm{sym}^4 f \otimes \widetilde{\chi}_0\right)\right| x^{\frac{3}{2}+\varepsilon} \Bigg] d\sigma$$

$$\ll x^{\frac{3}{2}+\varepsilon} + x^{\frac{3}{2}+\varepsilon} \left\{ \int_{10\le|t|\le T} \frac{|\zeta(\frac{1}{2}+\varepsilon+it)|^2}{t} dt \right\}^{\frac{1}{2}} \Bigg\{ \int_{10\le|t|\le T} \frac{|L^2(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^2 f \otimes \widetilde{\chi}_0)|^2}{t}$$

$$\times \left| L\left(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^4 f \otimes \widetilde{\chi}_0\right) L\left(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^2 f \otimes \mathrm{sym}^4 f \otimes \widetilde{\chi}_0\right)\right|^2$$

$$\times \left| L\left(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^4 f \otimes \mathrm{sym}^4 f \otimes \widetilde{\chi}_0\right)\right|^2 dt \Bigg\}^{\frac{1}{2}}$$

$$\ll x^{\frac{3}{2}+\varepsilon} + x^{\frac{3}{2}+\varepsilon}\left(T^{\frac{\varepsilon}{2}} T^{\frac{1}{2}(\frac{79}{2}-1+\varepsilon)}\right)$$

$$\ll x^{\frac{3}{2}+2\varepsilon} T^{\frac{77}{4}+2\varepsilon}.$$

Note that $10 \le T \le x$. Thus, we obtain

$$\sum_{n\le x} \lambda^4_{\mathrm{sym}^2 f}(n) r_4(n) = c_4 x^2 \log x + O\left(x^{\frac{3}{2}+2\varepsilon} T^{\frac{77}{4}+2\varepsilon}\right) + O\left(x^{\frac{3}{2}+2\varepsilon} T^{\frac{229}{12}+2\varepsilon} + \frac{x^{2+2\varepsilon}}{T}\right).$$

We choose T such that $x^{\frac{3}{2}} T^{\frac{77}{4}} \asymp \frac{x^2}{T}$ i.e., $T^{\frac{81}{4}} \asymp x^{\frac{1}{2}}$.

$\therefore T \asymp x^{\frac{2}{81}}$.

Thus, we get

$$\sum_{n\le x} \lambda^4_{\mathrm{sym}^2 f}(n) r_4(n) = c_4 x^2 \log x + O(x^{\frac{160}{81}+2\varepsilon}),$$

where

$$\begin{aligned}c_4 = \zeta^2(2)&L^3(2, \mathrm{sym}^2 f)L^3(1, \mathrm{sym}^2 f \otimes \widetilde{\chi}_0)L^3(2, \mathrm{sym}^4 f)\\ &\times L^3(1, \mathrm{sym}^4 f \otimes \widetilde{\chi}_0)L^2(2, \mathrm{sym}^2 f \otimes \mathrm{sym}^4 f)L^2(1, \mathrm{sym}^2 f \otimes \mathrm{sym}^4 f \otimes \widetilde{\chi}_0)\\ &\times L(2, \mathrm{sym}^4 f \otimes \mathrm{sym}^4 f)L(1, \mathrm{sym}^4 f \otimes \mathrm{sym}^4 f \otimes \widetilde{\chi}_0)H_4(2).\end{aligned}$$

This proves the theorem.

3.6 Concluding Remarks

From the proof of the theorems above, we observe that if we know some more analytic properties of $H_3(s)$ and $H_4(s)$ in the region $\Re(s) \geq (1-\varepsilon)$ then possibly we may move the line of integration to the left of line $\Re(s) = 1$. This leads us to propose:

Conjecture 3.6.1 For sufficiently large x and $\varepsilon > 0$ be any small constant, we have

$$\sum_{\substack{n=a^2+b^2+c^2+d^2\leq x\\(a,b,c,d)\in\mathbb{Z}^4}} \lambda^{\theta}_{sym^2 f}(a^2+b^2+c^2+d^2) = \begin{cases}\widetilde{c}_3(\theta)x^2 + \widetilde{c}_4(\theta)x + O(x^{\mu_1(\theta)+\varepsilon}) & \text{if } \theta = 3\\ \hat{c}_3(\theta)x^2\log x + \hat{c}_4(\theta)x\log x + O(x^{\mu_2(\theta)+\varepsilon}) & \text{if } \theta = 4\end{cases}.$$

where $\widetilde{c}_3(\theta)$, $\widetilde{c}_4(\theta), \hat{c}_3(\theta), \hat{c}_4(\theta)$ are effective constants and $\mu_1(\theta), \mu_2(\theta)$ are some positive constants satisfying $0 < \mu_1(\theta), \mu_2(\theta) < 1$.

Remark 3.6.2 Though the above conjecture seems to be reasonable, we do not have any idea currently how to tackle and conclude our conjectural estimates.

Chapter 4

Average behavior of the Fourier coefficients of the symmetric square L-function over some sequence of integers

4.1 Introduction

In this chapter, we will explore the possibility of an extension of the dimension. It seems natural to ask for similar results related to higher dimensions? We investigate the following question in this chapter:

Question 4.1.1 Can we come up with an asymptotic formula that adequately reflects the average behaviour of the Fourier coefficients of cusp form in the 6-dimension under diverse conditions?

Our main objective in this chapter is to establish Theorem 4.1.2. We first notice the behavior of these Fourier coefficients of cusp form on $r_6(n)$. To be more specific, we investigate the nature of the following sum:

$$\sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2 \leq x \\ (a_1,a_2,a_3,a_4,a_5,a_6) \in \mathbb{Z}^6}} \lambda^2_{\mathrm{sym}^2 f}(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2).$$

for sufficiently large x. More precisely, we prove the following.

Theorem 4.1.2 *For sufficiently large x, and any $\varepsilon > 0$, we have*

$$\sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\leq x\\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} \lambda^2_{sym^2 f}(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2) = \mathfrak{c}_2 x^3 + O\left(x^{\frac{14}{5}+\varepsilon}\right).$$

Here, $\mathfrak{c}_2$ is an effective constant defined as

$$\mathfrak{c}_2 = \frac{16}{3} L(3,\chi)L(1, sym^2 f)L(3, sym^2 f\otimes\chi)L(1, sym^4 f)L(3, sym^4 f\otimes\chi)\mathfrak{H}_2(3),$$

and χ is the non-principal Dirichlet character modulo 4. Here, $\mathfrak{H}_2(s)$ is a Dirichlet series which converges uniformly and absolutely in the half plane $\Re(s) > \frac{5}{2}$, and $\mathfrak{H}_2(s) \neq 0$ on $\Re(s) = 3$.

Remark 4.1.3 The main idea of the proof here is that the sum in Theorem 4.1.2 is being related to the sum involving $r_6(n)$. The main difference from our earlier result (see 2.1.2) related to a sum involving $r_4(n)$ is that $r_6(n)$ is not multiplicative (cf. section 1.4.2). However, Theorem 1.4.6 demonstrates that $r_6(n)$ can be written as a sum of two multiplicative functions. The sum in Theorem 4.1.2 is split into two sums involving the corresponding multiplicative functions, which are dealt with independently. The two sums are then combined suitably to obtain the result.

In order to prove Theorem 4.1.2, we need to first prove Lemma 4.1.4 and Lemma 4.1.5, which will play a pivotal role in the proof. This chapter's findings have been published and can be accessed at [Sr-Sa 2]. From [Ha-Wr, pp. 415], we can write

$$\begin{aligned} r_6(n) &= 16\sum_{d|n}\chi(d)\frac{n^2}{d^2} - 4\sum_{d|n}\chi(d)d^2 \\ &=: 16l(n) - 4v(n), \end{aligned}$$

where χ is the non-principal Dirichlet character modulo 4.

Lemma 4.1.4 *Let f be a normalized primitive holomorphic cusp form of weight k for $SL(2,\mathbb{Z})$. Let $\lambda_{sym^2 f}(n)$ be the n^{th} normalized Fourier coefficient of the* *symmetric square L-function associated with f. If*

$$\mathfrak{F}_2(s) = \sum_{n=1}^{\infty} \frac{\lambda^2_{sym^2 f}(n) l(n)}{n^s},$$

for $\Re(s) > 3$, then

$$\mathfrak{F}_2(s)(s) = \mathfrak{G}_2(s)\mathfrak{H}_2(s),$$

where

$$\begin{aligned}\mathfrak{G}_2(s) :=& \zeta(s-2)L(s,\chi)L(s-2, sym^2 f)L(s, sym^2 f \otimes \chi)\\ &\times L(s-2, sym^4 f)L(s, sym^4 f \otimes \chi),\end{aligned}$$

and χ is the non-principal character modulo 4. Here, $\mathfrak{H}_2(s)$ is a Dirichlet series which converges uniformly and absolutely in the half plane $\Re(s) > \frac{5}{2}$, and $\mathfrak{H}_2(s) \neq 0$ on $\Re(s) = 3$.

Lemma 4.1.5 *Let f be a normalized primitive holomorphic cusp form of weight k for $SL(2, \mathbb{Z})$. Let $\lambda_{sym^2 f}(n)$ be the n^{th} normalized Fourier coefficient of the symmetric square L-function associated with f. If*

$$\widetilde{\mathfrak{F}}_2(s) = \sum_{n=1}^{\infty} \frac{\lambda^2_{sym^2 f}(n) v(n)}{n^s},$$

for $\Re(s) > 3$, then

$$\widetilde{\mathfrak{F}}_2(s) = \widetilde{\mathfrak{G}}_2(s)\widetilde{\mathfrak{H}}_2(s),$$

where

$$\begin{aligned}\widetilde{\mathfrak{G}}_2(s) :=& \zeta(s)L(s-2,\chi)L(s, sym^2 f)L(s-2, sym^2 f \otimes \chi)\\ &\times L(s, sym^4 f)L(s-2, sym^4 f \otimes \chi),\end{aligned}$$

and χ is the non-principal character modulo 4. Here, $\widetilde{\mathfrak{H}}_2(s)$ is a Dirichlet series which converges uniformly and absolutely in the half plane $\Re(s) > \frac{5}{2}$, and $\widetilde{\mathfrak{H}}_2(s) \neq 0$ on $\Re(s) = 3$.

4.2 Proof of Lemma 4.1.4

We observe that $\lambda^2_{\mathrm{sym}^2 f}(n)l(n)$ is multiplicative, and hence

$$\mathfrak{F}_2(s) = \prod_p \left(1 + \frac{\lambda^2_{\mathrm{sym}^2 f}(p)l(p)}{p^s} + \cdots + \frac{\lambda^2_{\mathrm{sym}^2 f}(p^m)l(p^m)}{p^{ms}} + \cdots\right).$$

Note that

$$\begin{aligned}
\lambda^2_{\mathrm{sym}^2 f}(p)l(p) &= \left(\sum_{m=0}^{2} \alpha^{2-m}(p)\beta^m(p)\right)^2 \left(p^2 + \chi(p)\right) \\
&= (\alpha^4(p) + 2\alpha^2(p) + 3 + 2\beta^2(p) + \beta^4(p)) \left(p^2 + \chi(p)\right) \\
&= p^2 + \chi(p) + (\alpha^2(p) + 1 + \beta^2(p))(p^2 + \chi(p)) \\
&\quad + (\alpha^4(p) + \alpha^2(p) + 1 + \beta^2(p) + \beta^4(p)) \left(p^2 + \chi(p)\right) \\
&= p^2 + \chi(p) + \lambda_{\mathrm{sym}^2 f}(p)(p^2 + \chi(p)) + \lambda_{\mathrm{sym}^4 f}(p)(p^2 + \chi(p)) \\
&= p^2 + \chi(p) + p^2\lambda_{\mathrm{sym}^2 f}(p) + \chi(p)\lambda_{\mathrm{sym}^2 f}(p) \\
&\quad + p^2\lambda_{\mathrm{sym}^4 f}(p) + \chi(p)\lambda_{\mathrm{sym}^4 f}(p) \\
&=: b_4(p).
\end{aligned}$$

From the structure of $b_4(p)$, we define the coefficients $b_4(n)$ as

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{b_4(n)}{n^s} =& \zeta(s-2)L(s,\chi)L(s-2,\mathrm{sym}^2 f)L(s,\mathrm{sym}^2 f \otimes \chi)L(s-2,\mathrm{sym}^4 f) \\
&\times L(s,\mathrm{sym}^4 f \otimes \chi),
\end{aligned}$$

which is absolutely convergent in $\Re(s) > 3$. We also note that

$$\begin{aligned}
&\prod_p \left(1 + \frac{b_4(p)}{p^s} + \cdots + \frac{b_4(p^m)}{p^{ms}} + \cdots\right) \\
&= \zeta(s-2)L(s,\chi)L(s-2,\mathrm{sym}^2 f)L(s,\mathrm{sym}^2 f \otimes \chi) \\
&\quad \times L(s-2,\mathrm{sym}^4 f)L(s,\mathrm{sym}^4 f \otimes \chi) \\
&=: \mathfrak{G}_2(s),
\end{aligned}$$

for $\Re(s) > 3$. Observe that $b_4(n) \ll_\varepsilon n^{2+\varepsilon}$ for any small positive constant ε.

Now, we note that in the half plane $\Re(s) \geq 3 + 2\varepsilon$, we have

$$\begin{aligned}
\left|\frac{b_4(p)}{p^s} + \frac{b_4(p^2)}{p^{2s}} + \cdots + \frac{b_4(p^m)}{p^{ms}} + \cdots\right| &\ll \sum_{m=1}^{\infty} \frac{p^{(2+\varepsilon)m}}{p^{m\sigma}} \\
&\leq \sum_{m=1}^{\infty} \frac{p^{(2+\varepsilon)m}}{p^{(3+2\varepsilon)m}} \\
&= \sum_{m=1}^{\infty} \frac{1}{p^{(1+\varepsilon)m}} \\
&= \frac{\frac{1}{p^{1+\varepsilon}}}{1 - \frac{1}{p^{1+\varepsilon}}} \\
&= \frac{1}{p^{1+\varepsilon} - 1} \\
&< 1.
\end{aligned}$$

Let us write

$$A_4 = \frac{\lambda^2_{\mathrm{sym}^2 f}(p)l(p)}{p^s} + \cdots + \frac{\lambda^2_{\mathrm{sym}^2 f}(p^m)l(p^m)}{p^{ms}} + \cdots,$$

and

$$B_4 = \frac{b_4(p)}{p^s} + \cdots + \frac{b_4(p^m)}{p^{ms}} + \cdots.$$

From the above calculations, we observe that $|B_4| < 1$ in $\Re(s) \geq 3 + 2\varepsilon$.

We note that in the half plane $\Re(s) \geq 3 + 2\varepsilon$, we have

$$\begin{aligned}\frac{1+A_4}{1+B_4} &= (1+A_4)(1 - B_4 + B_4^2 - B_4^3 + \cdots) \\ &= 1 + A_4 - B_4 - A_4B_4 + \text{higher terms} \\ &= 1 + \frac{\lambda^2_{\mathrm{sym}^2 f}(p^2) l(p^2) - b_4(p^2)}{p^{2s}} + \cdots + \frac{c_m(p^m)}{p^{ms}} + \cdots,\end{aligned}$$

with $c_m(n) \ll_\varepsilon n^{2+\varepsilon}$. So, we have (in the half plane $\Re(s) > \frac{5}{2}$)

$$\begin{aligned}\prod_p \left(\frac{1+A_4}{1+B_4}\right) &= \prod_p \left(1 + \frac{\lambda^2_{\mathrm{sym}^2 f}(p^2) l(p^2) - b_4(p^2)}{p^{2s}} + \cdots + \frac{c_m(p^m)}{p^{ms}} + \cdots\right) \\ &\ll_\varepsilon 1.\end{aligned}$$

Thus, we have (in the half plane $\Re(s) > \frac{5}{2}$)

$$\begin{aligned}\mathfrak{H}_2(s) &:= \frac{\mathfrak{F}_2(s)}{\mathfrak{G}_2(s)} \\ &= \prod_p \left(\frac{1+A_4}{1+B_4}\right) \\ &\ll_\varepsilon 1,\end{aligned}$$

and also $\mathfrak{H}_2(s) \neq 0$ on $\Re(s) = 3$.

4.3 Proof of Lemma 4.1.5

We observe that $\lambda^2_{\mathrm{sym}^2 f}(n) v(n)$ is multiplicative, and hence

$$\widetilde{\mathfrak{F}}_2(s) = \prod_p \left(1 + \frac{\lambda^2_{\mathrm{sym}^2 f}(p) v(p)}{p^s} + \cdots + \frac{\lambda^2_{\mathrm{sym}^2 f}(p^m) v(p^m)}{p^{ms}} + \cdots\right).$$

Note that

$$\begin{aligned}
\lambda_{\mathrm{sym}^2 f}^2(p)v(p) &= \left(\sum_{m=0}^{2} \alpha^{2-m}(p)\beta^m(p)\right)^2 \big(1+p^2\chi(p)\big)\\
&= (\alpha^4(p)+2\alpha^2(p)+3+2\beta^2(p)+\beta^4(p))\big(1+p^2\chi(p)\big)\\
&= 1+p^2\chi(p)+(\alpha^2(p)+1+\beta^2(p))(1+p^2\chi(p))\\
&\quad +(\alpha^4(p)+\alpha^2(p)+1+\beta^2(p)+\beta^4(p))\big(1+p^2\chi(p)\big)\\
&= 1+p^2\chi(p)+\lambda_{\mathrm{sym}^2 f}(p)(1+p^2\chi(p))+\lambda_{\mathrm{sym}^4 f}(p)(1+p^2\chi(p))\\
&= 1+p^2\chi(p)+\lambda_{\mathrm{sym}^2 f}(p)+p^2\chi(p)\lambda_{\mathrm{sym}^2 f}(p)\\
&\quad +\lambda_{\mathrm{sym}^4 f}(p)+p^2\chi(p)\lambda_{\mathrm{sym}^4 f}(p)\\
&=: b_5(p).
\end{aligned}$$

From the structure of $b_5(p)$, we define the coefficients $b_5(n)$ as

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{b_5(n)}{n^s} =&\zeta(s)L(s-2,\chi)L(s,\mathrm{sym}^2 f)L(s-2,\mathrm{sym}^2 f\otimes\chi)L(s,\mathrm{sym}^4 f)\\
&\times L(s-2,\mathrm{sym}^4 f\otimes\chi),
\end{aligned}$$

which is absolutely convergent in $\Re(s)>3$. We also note that

$$\begin{aligned}
&\prod_p\left(1+\frac{b_5(p)}{p^s}+\cdots+\frac{b_5(p^m)}{p^{ms}}+\cdots\right)\\
&=\zeta(s)L(s-2,\chi)L(s,\mathrm{sym}^2 f)L(s-2,\mathrm{sym}^2 f\otimes\chi)L(s,\mathrm{sym}^4 f)L(s-2,\mathrm{sym}^4 f\otimes\chi)\\
&=: \widetilde{\mathfrak{G}}_2(s),
\end{aligned}$$

for $\Re(s) > 3$. Observe that $b_5(n) \ll_\varepsilon n^{2+\varepsilon}$ for any small positive constant ε. Now, we note that in the half plane $\Re(s) \geq 3 + 2\varepsilon$, we have

$$\begin{aligned}\left|\frac{b_5(p)}{p^s} + \frac{b_5(p^2)}{p^{2s}} + \cdots + \frac{b_5(p^m)}{p^{ms}} + \cdots\right| &\ll \sum_{m=1}^{\infty} \frac{p^{(2+\varepsilon)m}}{p^{m\sigma}} \\ &< 1.\end{aligned}$$

Let us write

$$A_5 = \frac{\lambda^2_{\mathrm{sym}^2 f}(p)v(p)}{p^s} + \cdots + \frac{\lambda^2_{\mathrm{sym}^2 f}(p^m)v(p^m)}{p^{ms}} + \cdots,$$

and

$$B_5 = \frac{b_5(p)}{p^s} + \cdots + \frac{b_5(p^m)}{p^{ms}} + \cdots.$$

From the above calculations, we observe that $|B_5| < 1$ in $\Re(s) \geq 3 + 2\varepsilon$.

We note that in the half plane $\Re(s) \geq 3 + 2\varepsilon$, we have

$$\begin{aligned}\frac{1 + A_5}{1 + B_5} &= (1 + A_5)(1 - B_5 + B_5^2 - B_5^3 + \cdots) \\ &= 1 + A_5 - B_5 - A_5 B_5 + \text{higher terms} \\ &= 1 + \frac{\lambda^2_{\mathrm{sym}^2 f}(p^2)v(p^2) - b_5(p^2)}{p^{2s}} + \cdots + \frac{\widetilde{c}_m(p^m)}{p^{ms}} + \cdots,\end{aligned}$$

with $\widetilde{c}_m(n) \ll_\varepsilon n^{2+\varepsilon}$. So, we have (in the half plane $\Re(s) > \frac{5}{2}$)

$$\begin{aligned}\prod_p \left(\frac{1 + A_5}{1 + B_5}\right) &= \prod_p \left(1 + \frac{\lambda^2_{\mathrm{sym}^2 f}(p^2)v(p^2) - b_5(p^2)}{p^{2s}} + \cdots + \frac{\widetilde{c}_m(p^m)}{p^{ms}} + \cdots\right) \\ &\ll_\varepsilon 1.\end{aligned}$$

Thus, we have (in the half plane $\Re(s) > \frac{5}{2}$)

$$\begin{aligned}\widetilde{\mathfrak{H}}_2(s) &:= \frac{\widetilde{\mathfrak{F}}_2(s)}{\widetilde{\mathfrak{G}}_2(s)} \\ &= \prod_p \left(\frac{1+A_5}{1+B_5}\right) \\ &\ll_\varepsilon 1,\end{aligned}$$

and also $\widetilde{\mathfrak{H}}_2(s) \neq 0$ on $\Re(s) = 3$.

We are now in a position to prove our core theorem of this chapter.

4.4 Proof of Theorem 4.1.2

We can write

$$\begin{aligned}&\sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2 \le x \\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} \lambda_{\mathrm{sym}^2 f}^2(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2) \\ &= \sum_{n\le x} \lambda_{\mathrm{sym}^2 f}^2(n) \sum_{\substack{n=a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2 \\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} 1 \\ &= \sum_{n\le x} \lambda_{\mathrm{sym}^2 f}^2(n) r_6(n) \\ &= \sum_{n\le x} \lambda_{\mathrm{sym}^2 f}^2(n) \left(l_1(n) - v_1(n)\right) \qquad (4.1) \\ &= 16\sum_{n\le x} \lambda_{\mathrm{sym}^2 f}^2(n) l(n) - 4\sum_{n\le x} \lambda_{\mathrm{sym}^2 f}^2(n) v(n),\end{aligned}$$

where $l(n) = \sum_{d|n} \chi(d) \frac{n^2}{d^2}$, and $v(n) = \sum_{d|n} \chi(d) d^2$.

From Equation (4.1), we can write

$$\sum_{n\leq x}\lambda^2_{\mathrm{sym}^2 f}(n)r_6(n)=\sum_{n\leq x}\lambda^2_{\mathrm{sym}^2 f}(n)l_1(n)-\sum_{n\leq x}\lambda^2_{\mathrm{sym}^2 f}(n)v_1(n).$$

Firstly, we consider the sum $\sum_{n\leq x}\lambda^2_{\mathrm{sym}^2 f}(n)l_1(n)$. We begin by applying Perron's formula (see section 1.5) to $\mathfrak{F}_2(s)$ with $\eta=3+\varepsilon$ and $10\leq T\leq x$. Thus, we have

$$\begin{aligned}\sum_{n\leq x}\lambda^2_{\mathrm{sym}^2 f}(n)l_1(n)&=16\sum_{n\leq x}\lambda^2_{\mathrm{sym}^2 f}(n)l(n)\\&=\frac{16}{2\pi i}\int_{\eta-iT}^{\eta+iT}\mathfrak{F}_2(s)\frac{x^s}{s}ds+O\left(\frac{x^{3+3\varepsilon}}{T}\right).\end{aligned}$$

We move the line of integration to $\Re(s)=\frac{5}{2}+\varepsilon$. By Cauchy's residue theorem there is only one simple pole at $s=3$, coming from the factor $\zeta(s-2)$. This contributes a residue, which is $\mathfrak{c}_2x^3$, where $\mathfrak{c}_2$ is an effective constant depending on the values of various L-functions appearing in $\mathfrak{G}_2(s)$ at $s=3$.

More precisely,

$$\begin{aligned}\mathfrak{c}_2&=16\lim_{s\to 3}(s-3)\frac{\mathfrak{F}_2(s)}{s}\\&=\frac{16}{3}L(3,\chi)L(1,\mathrm{sym}^2 f)L(3,\mathrm{sym}^2 f\otimes\chi)\\&\quad\times L(1,\mathrm{sym}^4 f)L(3,\mathrm{sym}^4 f\otimes\chi)\mathfrak{H}_2(3).\end{aligned}$$

So, we obtain

$$\begin{aligned}\sum_{n\leq x}\lambda^2_{\mathrm{sym}^2 f}(n)l_1(n)&=\mathfrak{c}_2x^3+\frac{16}{2\pi i}\left\{\int_{\frac{5}{2}+\varepsilon-iT}^{\frac{5}{2}+\varepsilon+iT}+\int_{3+\varepsilon-iT}^{\frac{5}{2}+\varepsilon-iT}+\int_{\frac{5}{2}+\varepsilon+iT}^{3+\varepsilon+iT}\right\}\mathfrak{F}_2(s)\frac{x^s}{s}ds\\&\quad+O\left(\frac{x^{3+3\varepsilon}}{T}\right)\\&=:\mathfrak{c}_2x^3+\frac{16}{2\pi i}(J_1^{(4)}+J_2^{(4)}+J_3^{(4)})+O\left(\frac{x^{3+3\varepsilon}}{T}\right).\end{aligned}$$

The contribution of the horizontal line integrals (using Lemma 1.9.4 and Lemma 1.9.9) is

$$J_2^{(4)} + J_3^{(4)} \ll \int_{\frac{5}{2}+\varepsilon}^{3+\varepsilon} \frac{|\zeta(\sigma-2+iT)L(\sigma-2+iT,\mathrm{sym}^2 f)L(\sigma-2+iT,\mathrm{sym}^4 f)|}{T} x^\sigma d\sigma$$

$$\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} \frac{|\zeta(\sigma+iT)L(\sigma+iT,\mathrm{sym}^2 f)L(\sigma+iT,\mathrm{sym}^4 f)|}{T} x^{\sigma+2} d\sigma$$

$$\ll \left(\frac{x^2}{T}\right) \max_{\frac{1}{2}+\varepsilon\le\sigma\le 1+\varepsilon} x^\sigma T^{\frac{1}{3}(1+\varepsilon-\sigma)} T^{\frac{8}{2}(1+\varepsilon-\sigma)}$$

$$\ll \left(\frac{x^{2+2\varepsilon}}{T}\right) \max_{\frac{1}{2}+\varepsilon\le\sigma\le 1+\varepsilon} \left(\frac{x}{T^{\frac{13}{3}}}\right)^\sigma T^{\frac{13}{3}}.$$

Clearly, $\left(\frac{x}{T^{\frac{13}{3}}}\right)^\sigma$ is monotonic as a function of σ for $\frac{1}{2}+\varepsilon \le \sigma \le 1+\varepsilon$, and hence the maximum is attained at the extremities of the interval $[\frac{1}{2}+\varepsilon, 1+\varepsilon]$. Thus,

$$\begin{aligned} J_2^{(4)} + J_3^{(4)} &\ll x^{2+2\varepsilon}\left(x^{\frac{1}{2}+\varepsilon} T^{\frac{13}{6}-1+\varepsilon} + \frac{x^{1+\varepsilon}}{T}\right) \\ &\ll x^{\frac{5}{2}+3\varepsilon} T^{\frac{7}{6}+\varepsilon} + \frac{x^{3+3\varepsilon}}{T}. \end{aligned}$$

The contribution of the left vertical line integral (using the Cauchy-Schwarz inequality, Lemma 1.9.1, and Lemma 1.9.9) is

$$J_1^{(4)} \ll \int_{\frac{5}{2}+\varepsilon-iT}^{\frac{5}{2}+\varepsilon+iT} \frac{|\zeta(\frac{1}{2}+\varepsilon+it)L(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^2 f)L(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^4 f)|}{|\frac{5}{2}+\varepsilon+it|} x^{\frac{5}{2}+\varepsilon} dt$$

$$\ll x^{\frac{5}{2}+\varepsilon} + x^{\frac{5}{2}+\varepsilon} \left(\int_{10\le|t|\le T} \frac{|\zeta(\frac{1}{2}+\varepsilon+it)|^2}{t} dt \right)^{\frac{1}{2}}$$
$$\times \left(\int_{10\le|t|\le T} \frac{|L(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^2 f)L(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^4 f)|^2}{t} dt \right)^{\frac{1}{2}}$$

$$\ll x^{\frac{5}{2}+\varepsilon} + x^{\frac{5}{2}+\varepsilon} \left(T^{\frac{\varepsilon}{2}} T^{\frac{3}{2}+\frac{\varepsilon}{2}} \right)$$

$$\ll x^{\frac{5}{2}+2\varepsilon} T^{\frac{3}{2}+2\varepsilon}.$$

Note that $10 \le T \le x$. Thus, we obtain

$$\sum_{n\le x} \lambda^2_{\mathrm{sym}^2 f}(n) l_1(n) = \mathfrak{c}_2 x^3 + O\left(x^{\frac{5}{2}+2\varepsilon} T^{\frac{3}{2}+2\varepsilon}\right) + O\left(\frac{x^{3+3\varepsilon}}{T}\right).$$

We choose T such that $x^{\frac{5}{2}} T^{\frac{3}{2}} \asymp \frac{x^3}{T}$ i.e., $T^{\frac{5}{2}} \asymp x^{\frac{1}{2}}$. Therefore, $T \asymp x^{\frac{1}{5}}$. Thus, we get

$$\sum_{n\le x} \lambda^2_{\mathrm{sym}^2 f}(n) l_1(n) = \mathfrak{c}_2 x^3 + O(x^{\frac{14}{5}+\varepsilon}). \tag{4.2}$$

Similarly, we apply Perron's formula (see section 1.5) to $\widetilde{\mathfrak{F}}_2(s)$ with $\eta = 3+\varepsilon$ and $10 \le T \le x$. Thus, we have

$$\begin{aligned}\sum_{n\le x} \lambda^2_{\mathrm{sym}^2 f}(n) v_1(n) &= 4\sum_{n\le x} \lambda^2_{\mathrm{sym}^2 f}(n) v(n)\\ &= \frac{4}{2\pi i}\int_{\eta-iT}^{\eta+iT} \widetilde{\mathfrak{F}}_2(s) \frac{x^s}{s} ds + O\left(\frac{x^{3+3\varepsilon}}{T}\right).\end{aligned}$$

We move the line of integration to $\Re(s) = \frac{5}{2}+\varepsilon$. There is no singularity in the rectangle obtained and the function $\widetilde{\mathfrak{F}}_2(s)\dfrac{x^s}{s}$ is analytic in this region. Thus, using Cauchy's theorem for rectangles pertaining to analytic functions, we get

$$\begin{aligned}\sum_{n\le x}\lambda^2_{\mathrm{sym}^2 f}(n)v_1(n) &= \frac{4}{2\pi i}\left\{\int_{\frac{5}{2}+\varepsilon-iT}^{\frac{5}{2}+\varepsilon+iT}+\int_{3+\varepsilon-iT}^{\frac{5}{2}+\varepsilon-iT}+\int_{\frac{5}{2}+\varepsilon+iT}^{3+\varepsilon+iT}\right\}\widetilde{\mathfrak{F}}_2(s)\frac{x^s}{s}ds\\ &\quad + O\left(\frac{x^{3+3\varepsilon}}{T}\right)\\ &=: \frac{4}{2\pi i}(J_1^{(5)}+J_2^{(5)}+J_3^{(5)})+O\left(\frac{x^{3+3\varepsilon}}{T}\right).\end{aligned}$$

The contribution of the horizontal line integrals (using Lemma 1.9.8 and Lemma 1.9.9) is

$$\begin{aligned}J_2^{(5)}+J_3^{(5)} &\ll \int_{\frac{5}{2}+\varepsilon}^{3+\varepsilon}\frac{|L(\sigma-2+iT,\chi)L(\sigma-2+iT,\mathrm{sym}^2 f\otimes\chi)L(\sigma-2+iT,\mathrm{sym}^4 f)\otimes\chi|}{T}x^\sigma d\sigma\\ &\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon}\frac{|L(\sigma+iT,\chi)L(\sigma+iT,\mathrm{sym}^2 f\otimes\chi)L(\sigma+iT,\mathrm{sym}^4 f\otimes\chi)|}{T}x^{\sigma+2}d\sigma.\end{aligned}$$

Thus,

$$\begin{aligned}J_2^{(5)}+J_3^{(5)} &\ll \left(\frac{x^2}{T}\right)\max_{\frac{1}{2}+\varepsilon\le\sigma\le 1+\varepsilon} x^\sigma T^{\frac{1}{3}(1+\varepsilon-\sigma)}T^{\frac{8}{2}(1+\varepsilon-\sigma)}\\ &\ll \left(\frac{x^{2+2\varepsilon}}{T}\right)\max_{\frac{1}{2}+\varepsilon\le\sigma\le 1+\varepsilon}\left(\frac{x}{T^{\frac{13}{3}}}\right)^\sigma T^{\frac{13}{3}}.\end{aligned}$$

Clearly, $\left(\frac{x}{T^{\frac{13}{3}}}\right)^\sigma$ is monotonic as a function of σ for $\frac{1}{2}+\varepsilon\le\sigma\le 1+\varepsilon$, and hence the maximum is attained at the extremities of the interval $\left[\frac{1}{2}+\varepsilon, 1+\varepsilon\right]$. Thus,

$$\begin{aligned}J_2^{(5)}+J_3^{(5)} &\ll x^{2+2\varepsilon}\left(x^{\frac{1}{2}+\varepsilon}T^{\frac{13}{6}-1+\varepsilon}+\frac{x^{1+\varepsilon}}{T}\right)\\ &\ll x^{\frac{5}{2}+3\varepsilon}T^{\frac{7}{6}+\varepsilon}+\frac{x^{3+3\varepsilon}}{T}.\end{aligned}$$

The contribution of the left vertical line integral (using the Cauchy-Schwarz inequality, Lemma 1.9.6, and Lemma 1.9.9) is

$$J_1^{(5)} \ll \int_{\frac{5}{2}+\varepsilon-iT}^{\frac{5}{2}+\varepsilon+iT} \frac{|L(\frac{1}{2}+\varepsilon+it,\chi)L(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^2 f\otimes\chi)L(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^4 f\otimes\chi)|}{|\frac{5}{2}+\varepsilon+it|} x^{\frac{5}{2}+\varepsilon} dt$$

$$\ll x^{\frac{5}{2}+\varepsilon} + x^{\frac{5}{2}+\varepsilon}\left(\int_{10\le|t|\le T} \frac{|L(\frac{1}{2}+\varepsilon+it,\chi)|^2}{t} dt\right)^{\frac{1}{2}}$$
$$\times \left(\int_{10\le|t|\le T} \frac{|L(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^2 f\otimes\chi)L(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^4 f\otimes\chi)|^2}{t} dt\right)^{\frac{1}{2}}$$

$$\ll x^{\frac{5}{2}+\varepsilon} + x^{\frac{5}{2}+\varepsilon}\left(T^{\frac{\varepsilon}{2}} T^{\frac{3}{2}+\frac{\varepsilon}{2}}\right)$$

$$\ll x^{\frac{5}{2}+2\varepsilon} T^{\frac{3}{2}+2\varepsilon}.$$

Note that $10 \le T \le x$. Thus, we obtain

$$\sum_{n\le x} \lambda^2_{\mathrm{sym}^2 f}(n) v_1(n) = O\left(x^{\frac{5}{2}+2\varepsilon} T^{\frac{3}{2}+2\varepsilon}\right) + O\left(\frac{x^{3+3\varepsilon}}{T}\right).$$

We choose T such that $x^{\frac{5}{2}} T^{\frac{3}{2}} \asymp \frac{x^3}{T}$ i.e., $T^{\frac{5}{2}} \asymp x^{\frac{1}{2}}$. Therefore, $T \asymp x^{\frac{1}{5}}$.

Thus, we get

$$\sum_{n\le x} \lambda^2_{\mathrm{sym}^2 f}(n) v_1(n) = O\left(x^{\frac{14}{5}+\varepsilon}\right). \tag{4.3}$$

Combining Equations (4.2) and (4.3), we get

$$\sum_{n\le x} \lambda^2_{\mathrm{sym}^2 f}(n) r_6(n) = \mathfrak{c}_2 x^3 + O\left(x^{\frac{14}{5}+\varepsilon}\right),$$

where $\mathfrak{c}_2$ is an effective constant given by

$$\mathfrak{c}_2 = \frac{16}{3} L(3,\chi)L(1,\mathrm{sym}^2 f)L(3,\mathrm{sym}^2 f\otimes\chi)L(1,\mathrm{sym}^4 f)L(3,\mathrm{sym}^4 f\otimes\chi)\mathfrak{H}_2(3),$$

and χ is the non-principal Dirichlet character modulo 4.
This proves the theorem.

Chapter 5

On the average behavior of the Fourier coefficients of j^{th} symmetric power L-function over a certain sequences of positive integers

5.1 Introduction

In the previous chapter, we investigated the average behavior of the n^{th} normalized Fourier coefficients of the symmetric square L-function attached to a primitive holomorphic cusp form of weight k for the full modular group $SL(2,\mathbb{Z})$ over some sequence of integers. To be more precise, we proved the following asymptotic formula:

For sufficiently large x, and any $\varepsilon > 0$, we have

$$\sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\leq x\\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} \lambda_{\mathrm{sym}^2 f}^2(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2) = \mathfrak{c}_2 x^3 + O\left(x^{\frac{14}{5}+\varepsilon}\right).$$

Here, $\mathfrak{c}_2$ is an effective constant defined as

$$\mathfrak{c}_2 = \frac{16}{3}L(3,\chi)L(1,\mathrm{sym}^2 f)L(3,\mathrm{sym}^2 f\otimes\chi)L(1,\mathrm{sym}^4 f)L(3,\mathrm{sym}^4 f\otimes\chi)H_2(3),$$

and χ is the non-principal Dirichlet character modulo 4.

Now, our goal is to improve as well as generalize the above result. Generalization was possible due to recent celebrated work of Newton and Thorne (see 3.1.1) and for improvement, we use better average or individual subconvexity bounds for the related L-functions. The findings presented in this chapter will shortly be published in [Sr-Sa 5].

Our main aim in this chapter is to establish an asymptotic formula for the following sum:

$$\sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\leq x \\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} \lambda^2_{\mathrm{sym}^j f}(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2),$$

where $j\geq 2$ be any fixed integer, x is sufficiently large, and $\varepsilon>0$ any small constant. We have the following theorem in this regard.

Theorem 5.1.1 *Let $j\geq 2$ be any fixed integer. For sufficiently large x, and $\varepsilon>0$ any small constant, we have*

$$\sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\leq x \\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} \lambda^2_{sym^j f}(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2) = c(j)x^3+O\left(x^{3-\frac{6}{3(j+1)^2+1}+\varepsilon}\right),$$

where $c(j)$ is an effective constant defined as

$$c(j)=\frac{16}{3}L(3,\chi)\prod_{n=1}^{j}L(1,sym^{2n}f)L(3,sym^{2n}f\otimes\chi)H_j(3),$$

and χ is the non-principal Dirichlet character modulo 4.

From [Ha-Wr, pp. 415], we can write

$$\begin{aligned} r_6(n) &= 16\sum_{d|n}\chi(d)\frac{n^2}{d^2}-4\sum_{d|n}\chi(d)d^2 \\ &=: 16l(n)-4v(n), \end{aligned}$$

where χ is the non-principal Dirichlet character modulo 4.

Lemma 5.1.2 *Let f be a normalized primitive holomorphic cusp form of weight k for $SL(2,\mathbb{Z})$, and let $\lambda_{sym^j f}(n)$ be the n^{th} normalized Fourier coefficient of the j^{th} symmetric power L-function associated to f. If*

$$F_j(s) = \sum_{n=1}^{\infty} \frac{\lambda^2_{sym^j f}(n) l(n)}{n^s},$$

for $\Re(s) > 3$, then

$$F_j(s) = G_j(s) H_j(s),$$

where

$$G_j(s) := \zeta(s-2) L(s,\chi) \prod_{n=1}^{j} L(s-2, sym^{2n} f) L(s, sym^{2n} f \otimes \chi),$$

and χ is the non-principal character modulo 4.
Here, $H_j(s)$ is a Dirichlet series which converges uniformly, and absolutely in the half plane $\Re(s) > \frac{5}{2}$, and $H_j(s) \neq 0$ on $\Re(s) = 3$.

Lemma 5.1.3 *Let f be a normalized primitive holomorphic cusp form of weight k for $SL(2,\mathbb{Z})$, and let $\lambda_{sym^j f}(n)$ be the n^{th} normalized Fourier coefficient of the j^{th} symmetric power L-function associated to f. If*

$$\widetilde{F}_j(s) = \sum_{n=1}^{\infty} \frac{\lambda^2_{sym^j f}(n) v(n)}{n^s},$$

for $\Re(s) > 3$, then

$$\widetilde{F}_j(s) = \widetilde{G}_j(s) \widetilde{H}_j(s),$$

where

$$\widetilde{G}_j(s) := \zeta(s) L(s-2,\chi) \prod_{n=1}^{j} L(s, sym^{2n} f) L(s-2, sym^{2n} f \otimes \chi),$$

and χ is the non-principal character modulo 4.
Here $\widetilde{H}_j(s)$ is a Dirichlet series which converges uniformly, and absolutely in the half plane $\Re(s) > \frac{5}{2}$, and $\widetilde{H}_j(s) \neq 0$ on $\Re(s) = 3$.

Lemma 5.1.2 and Lemma 5.1.3 are crucial in the proof of Theorem 5.1.1 and are proved before we proceed to the theorem.

5.2 Proof of Lemma 5.1.2

We observe that $\lambda^2_{\mathrm{sym}^j f}(n)l(n)$ is multiplicative, and hence

$$F_j(s) = \prod_p \left(1 + \frac{\lambda^2_{\mathrm{sym}^j f}(p)l(p)}{p^s} + \cdots + \frac{\lambda^2_{\mathrm{sym}^j f}(p^m)l(p^m)}{p^{ms}} + \cdots\right).$$

Using Lemma 1.9.14, we note that,

$$\begin{aligned}
\lambda^2_{\mathrm{sym}^j f}(p)l(p) &= \lambda_f^2(p^j)\left(p^2 + \chi(p)\right) \\
&= \left(1 + \sum_{l=1}^{j} \lambda_f(p^{2l})\right)\left(p^2 + \chi(p)\right) \\
&= \left(1 + \sum_{l=1}^{j} \lambda_{\mathrm{sym}^{2l} f}(p)\right)\left(p^2 + \chi(p)\right) \\
&= p^2 + \chi(p) + \sum_{l=1}^{j} \lambda_{\mathrm{sym}^{2l} f}(p)p^2 + \sum_{l=1}^{j} \lambda_{\mathrm{sym}^{2l} f}(p)\chi(p) \\
&=: b_6(p).
\end{aligned}$$

From the structure of $b_6(p)$, we define the coefficients $b_6(n)$ as

$$\sum_{n=1}^{\infty} \frac{b_6(n)}{n^s} = \zeta(s-2)L(s,\chi)\prod_{n=1}^{j} L(s-2, \mathrm{sym}^{2n} f)L(s, \mathrm{sym}^{2n} f \otimes \chi),$$

which is absolutely convergent in $\Re(s) > 3$. We also note that,

$$\begin{aligned}
&\prod_p \left(1 + \frac{b_6(p)}{p^s} + \cdots + \frac{b_6(p^m)}{p^{ms}} + \cdots\right) \\
&\quad = \zeta(s-2)L(s,\chi)\prod_{n=1}^{j} L(s-2, \mathrm{sym}^{2n} f)L(s, \mathrm{sym}^{2n} f \otimes \chi) \\
&\quad =: G_j(s),
\end{aligned}$$

for $\Re(s) > 3$. Observe that $b_6(n) \ll_\varepsilon n^{2+\varepsilon}$ for any small positive constant ε. Now, we note that in the half plane $\Re(s) \geq 3 + 2\varepsilon$, we have

$$\begin{aligned}
\left| \frac{b_6(p)}{p^s} + \frac{b_6(p^2)}{p^{2s}} + \cdots + \frac{b_6(p^m)}{p^{ms}} + \cdots \right| &\ll \sum_{m=1}^{\infty} \frac{p^{(2+\varepsilon)m}}{p^{m\sigma}} \\
&\leq \sum_{m=1}^{\infty} \frac{p^{(2+\varepsilon)m}}{p^{(3+2\varepsilon)m}} \\
&= \sum_{m=1}^{\infty} \frac{1}{p^{(1+\varepsilon)m}} \\
&= \frac{\frac{1}{p^{1+\varepsilon}}}{1 - \frac{1}{p^{1+\varepsilon}}} \\
&= \frac{1}{p^{1+\varepsilon} - 1} \\
&< 1.
\end{aligned}$$

Let us write

$$A_6 = \frac{\lambda^2_{\mathrm{sym}^j f}(p) l(p)}{p^s} + \cdots + \frac{\lambda^2_{\mathrm{sym}^j f}(p^m) l(p^m)}{p^{ms}} + \cdots,$$

and

$$B_6 = \frac{b_6(p)}{p^s} + \cdots + \frac{b_6(p^m)}{p^{ms}} + \cdots.$$

From the above calculations, we observe that $|B_6| < 1$ in $\Re(s) \geq 3 + 2\varepsilon$.

We note that in the half plane $\Re(s) \geq 3 + 2\varepsilon$, we have

$$\begin{aligned}\frac{1+A_6}{1+B_6} &= (1+A_6)(1 - B_6 + B_6^2 - B_6^3 + \cdots) \\ &= 1 + A_6 - B_6 - A_6 B_6 + \text{higher terms} \\ &= 1 + \frac{\lambda^2_{\mathrm{sym}^j f}(p^2) l(p^2) - b_6(p^2)}{p^{2s}} + \cdots + \frac{c_m(p^m)}{p^{ms}} + \cdots ,\end{aligned}$$

with $c_m(n) \ll_\varepsilon n^{2+\varepsilon}$. So, we have (in the half plane $\Re(s) > \frac{5}{2}$)

$$\begin{aligned}\prod_p \left(\frac{1+A_6}{1+B_6}\right) &= \prod_p \left(1 + \frac{\lambda^2_{\mathrm{sym}^j f}(p^2) l(p^2) - b_6(p^2)}{p^{2s}} + \cdots + \frac{c_m(p^m)}{p^{ms}} + \cdots\right) \\ &\ll_\varepsilon 1.\end{aligned}$$

Thus, we have (in the half plane $\Re(s) > \frac{5}{2}$)

$$\begin{aligned}H_j(s) &:= \frac{F_j(s)}{G_j(s)} \\ &= \prod_p \left(\frac{1+A_6}{1+B_6}\right) \\ &\ll_\varepsilon 1,\end{aligned}$$

and also $H_j(s) \neq 0$ on $\Re(s) = 3$.

Now, we will see the proof of Lemma 5.1.3.

5.3 Proof of Lemma 5.1.3

We observe that $\lambda^2_{\mathrm{sym}^j f}(n) v(n)$ is multiplicative, and hence

$$\widetilde{F}_j(s) = \prod_p \left(1 + \frac{\lambda^2_{\mathrm{sym}^j f}(p) v(p)}{p^s} + \cdots + \frac{\lambda^2_{\mathrm{sym}^j f}(p^m) v(p^m)}{p^{ms}} + \cdots\right). \tag{5.1}$$

Using Lemma 1.9.14, we note that,

$$\begin{aligned}
\lambda^2_{\mathrm{sym}^j f}(p)v(p) &= \lambda_f^2(p^j)\left(1+p^2\chi(p)\right) \\
&= \left(1+\sum_{l=1}^{j}\lambda_f(p^{2l})\right)\left(1+p^2\chi(p)\right) \\
&= \left(1+\sum_{l=1}^{j}\lambda_{\mathrm{sym}^{2l} f}(p)\right)\left(1+p^2\chi(p)\right) \\
&= 1+p^2\chi(p)+\sum_{l=1}^{j}\lambda_{\mathrm{sym}^{2l} f}(p)+\sum_{l=1}^{j}\lambda_{\mathrm{sym}^{2l} f}(p)p^2\chi(p) \\
&=: b_7(p).
\end{aligned}$$

From the structure of $b_7(p)$, we define the coefficients $b_7(n)$ as

$$\sum_{n=1}^{\infty}\frac{b_7(n)}{n^s} = \zeta(s)L(s-2,\chi)\prod_{n=1}^{j}L(s,\mathrm{sym}^{2n}f)L(s-2,\mathrm{sym}^{2n}f\otimes\chi),$$

which is absolutely convergent in $\Re(s)>3$. We also note that,

$$\begin{aligned}
&\prod_p\left(1+\frac{b_7(p)}{p^s}+\cdots+\frac{b_7(p^m)}{p^{ms}}+\cdots\right) \\
&\quad= \zeta(s)L(s-2,\chi)\prod_{n=1}^{j}L(s,\mathrm{sym}^{2n}f)L(s-2,\mathrm{sym}^{2n}f\otimes\chi) \\
&\quad=: \widetilde{G}_j(s),
\end{aligned}$$

for $\Re(s)>3$. Observe that $b_7(n)\ll_\varepsilon n^{2+\varepsilon}$ for any small positive constant ε.

Now, we note that in the half plane $\Re(s) \geq 3 + 2\varepsilon$, we have

$$\begin{aligned}\left| \frac{b_7(p)}{p^s} + \frac{b_7(p^2)}{p^{2s}} + \cdots + \frac{b_7(p^m)}{p^{ms}} + \cdots \right| &\ll \sum_{m=1}^{\infty} \frac{p^{(2+\varepsilon)m}}{p^{m\sigma}} \\ &< 1.\end{aligned}$$

Let us write,

$$A_7 = \frac{\lambda^2_{\mathrm{sym}^j f}(p)v(p)}{p^s} + \cdots + \frac{\lambda^2_{\mathrm{sym}^j f}(p^m)v(p^m)}{p^{ms}} + \cdots,$$

and

$$B_7 = \frac{b_7(p)}{p^s} + \cdots + \frac{b_7(p^m)}{p^{ms}} + \cdots.$$

From the above calculations, we observe that $|B_7| < 1$ in $\Re(s) \geq 3 + 2\varepsilon$.

We note that in the half plane $\Re(s) \geq 3 + 2\varepsilon$, we have

$$\begin{aligned}\frac{1 + A_7}{1 + B_7} &= (1 + A_7)(1 - B_7 + B_7^2 - B_7^3 + \cdots) \\ &= 1 + A_7 - B_7 - A_7B_7 + \text{higher terms} \\ &= 1 + \frac{\lambda^2_{\mathrm{sym}^j f}(p^2)v(p^2) - b_7(p^2)}{p^{2s}} + \cdots + \frac{\widetilde{c}_m(p^m)}{p^{ms}} + \cdots,\end{aligned}$$

with $\widetilde{c}_m(n) \ll_\varepsilon n^{2+\varepsilon}$. So, we have (in the half plane $\Re(s) > \frac{5}{2}$)

$$\begin{aligned}\prod_p \left(\frac{1 + A_7}{1 + B_7}\right) &= \prod_p \left(1 + \frac{\lambda^2_{\mathrm{sym}^j f}(p^2)v(p^2) - b_7(p^2)}{p^{2s}} + \cdots + \frac{\widetilde{c}_m(p^m)}{p^{ms}} + \cdots\right) \\ &\ll_\varepsilon 1.\end{aligned}$$

Thus, we have (in the half plane $\Re(s) > \frac{5}{2}$)

$$\begin{aligned}
\widetilde{H}_j(s) &:= \frac{\widetilde{F}_j(s)}{\widetilde{G}_j(s)} \\
&= \prod_p \left(\frac{1+A_7}{1+B_7}\right) \\
&\ll_\varepsilon 1,
\end{aligned}$$

and also $\widetilde{H}_j(s) \neq 0$ on $\Re(s) = 3$.

Now that we have reached this point, we may demonstrate the chapter's main theorem.

5.4 Proof of Theorem 5.1.1

Observe that,

$$\begin{aligned}
&\sum_{\substack{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2 \le x \\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} \lambda^2_{\mathrm{sym}^j f}(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2) \\
&= \sum_{n\le x} \lambda^2_{\mathrm{sym}^j f}(n) \sum_{\substack{n=a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2 \\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} 1 \\
&= \sum_{n\le x} \lambda^2_{\mathrm{sym}^j f}(n) r_6(n) \\
&= \sum_{n\le x} \lambda^2_{\mathrm{sym}^j f}(n) \left(l_1(n) - v_1(n)\right) \qquad (5.2) \\
&= 16\sum_{n\le x} \lambda^2_{\mathrm{sym}^j f}(n) l(n) - 4\sum_{n\le x} \lambda^2_{\mathrm{sym}^j f}(n) v(n),
\end{aligned}$$

where $l(n) = \sum_{d|n} \chi(d)\frac{n^2}{d^2}$, and $v(n) = \sum_{d|n}\chi(d)d^2$.

Now, from (5.2), we can write

$$\sum_{n\le x}\lambda^2_{\mathrm{sym}^j f}(n)r_6(n)=\sum_{n\le x}\lambda^2_{\mathrm{sym}^j f}(n)l_1(n)-\sum_{n\le x}\lambda^2_{\mathrm{sym}^j f}(n)v_1(n).$$

Firstly, we consider the sum $\sum_{n\le x}\lambda^2_{\mathrm{sym}^j f}(n)l_1(n)$. We begin by applying the Perron's formula (see section 1.5) to $F_j(s)$ with $\eta=3+\varepsilon$ and $10\le T\le x$. Thus, we have

$$\begin{aligned}\sum_{n\le x}\lambda^2_{\mathrm{sym}^j f}(n)l_1(n)&=16\sum_{n\le x}\lambda^2_{\mathrm{sym}^j f}(n)l(n)\\&=\frac{16}{2\pi i}\int_{\eta-iT}^{\eta+iT}F_j(s)\frac{x^s}{s}ds+O\left(\frac{x^{3+3\varepsilon}}{T}\right).\end{aligned}$$

We move the line of integration to $\Re(s)=\frac{5}{2}+\varepsilon$, and by Cauchy's residue theorem there is only one simple pole at $s=3$, coming from the factor $\zeta(s-2)$. This contributes a residue, which is $c(j)x^3$, where $c(j)$ is an effective constant depending on the values of various L-functions appearing in $G_j(s)$ at $s=3$.

More precisely,

$$\begin{aligned}c(j)&=16\lim_{s\to3}(s-3)\frac{F_j(s)}{s}\\&=\frac{16}{3}L(3,\chi)\prod_{n=1}^{j}L(1,\mathrm{sym}^{2n}f)L(3,\mathrm{sym}^{2n}f\otimes\chi)H_j(3).\end{aligned}$$

So, we obtain

$$\begin{aligned}\sum_{n\le x}\lambda^2_{\mathrm{sym}^j f}(n)l_1(n)=c(j)x^3&+\frac{16}{2\pi i}\left\{\int_{\frac{5}{2}+\varepsilon-iT}^{\frac{5}{2}+\varepsilon+iT}+\int_{3+\varepsilon-iT}^{\frac{5}{2}+\varepsilon-iT}+\int_{\frac{5}{2}+\varepsilon+iT}^{3+\varepsilon+iT}\right\}F_j(s)\frac{x^s}{s}ds\\&+O\left(\frac{x^{3+3\varepsilon}}{T}\right)\\&=:c(j)x^3+\frac{16}{2\pi i}(J_1^{(6)}+J_2^{(6)}+J_3^{(6)})+O\left(\frac{x^{3+3\varepsilon}}{T}\right).\end{aligned}$$

The contribution of horizontal line integrals ($J_2^{(6)}$ and $J_3^{(6)}$) in absolute value (using Lemmas 5.1.2, 1.9.3 and 1.9.10) is

$$\begin{aligned}
&\ll \int_{\frac{5}{2}+\varepsilon}^{3+\varepsilon} \frac{|\zeta(\sigma-2+iT)\prod_{n=1}^{j} L(\sigma-2+iT,\mathrm{sym}^{2n}f)|}{T} x^{\sigma}\,d\sigma \\
&\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} \frac{|\zeta(\sigma+iT)\prod_{n=1}^{j} L(\sigma+iT,\mathrm{sym}^{2n}f)|}{T} x^{\sigma+2}\,d\sigma \\
&\ll \left(\frac{x^2}{T}\right) \max_{\frac{1}{2}+\varepsilon\le\sigma\le 1+\varepsilon} x^{\sigma} T^{\left(\frac{(j+1)^2}{2}-\frac{4}{21}\right)(1-\sigma)+\varepsilon} \\
&\ll \left(\frac{x^{2+2\varepsilon}}{T}\right) \max_{\frac{1}{2}+\varepsilon\le\sigma\le 1+\varepsilon} \left(\frac{x}{T^{\left(\frac{(j+1)^2}{2}-\frac{4}{21}\right)}}\right)^{\sigma} T^{\left(\frac{(j+1)^2}{2}-\frac{4}{21}\right)}.
\end{aligned}$$

Clearly, $\left(\frac{x}{T^{\left(\frac{(j+1)^2}{2}-\frac{4}{21}\right)}}\right)^{\sigma}$ is monotonic as a function of σ for $\frac{1}{2}+\varepsilon \le \sigma \le 1+\varepsilon$, and hence the maximum is attained at the extremities of the interval $\left[\frac{1}{2}+\varepsilon, 1+\varepsilon\right]$. Thus,

$$\begin{aligned}
J_2^{(6)} + J_3^{(6)} &\ll x^{2+2\varepsilon}\left(x^{\frac{1}{2}+\varepsilon} T^{\left(\frac{(j+1)^2}{4}-\frac{2}{21}-1\right)} + \frac{x^{1+\varepsilon}}{T}\right) \\
&\ll x^{\frac{5}{2}+3\varepsilon} T^{\left(\frac{(j+1)^2}{4}-\frac{23}{21}+\varepsilon\right)} + \frac{x^{3+3\varepsilon}}{T}.
\end{aligned}$$

The contribution of the left vertical line integral ($J_1^{(6)}$) in absolute value (using Lemmas 5.1.2, 1.9.3, 1.9.10 and Hölder's inequality) is

$$\ll \int_{\frac{5}{2}+\varepsilon-iT}^{\frac{5}{2}+\varepsilon+iT} \frac{\left|\zeta(\frac{1}{2}+\varepsilon+it)\prod_{n=1}^{j} L(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^{2n} f)\right|}{\left|\frac{5}{2}+\varepsilon+it\right|} x^{\frac{5}{2}+\varepsilon} dt$$

$$\ll x^{\frac{5}{2}+\varepsilon} + x^{\frac{5}{2}+\varepsilon}\frac{1}{T}\left(\int_{10\leq|t|\leq T}\left|\zeta(\frac{1}{2}+\varepsilon+it)\right|^{12} dt\right)^{\frac{1}{12}}$$
$$\times\left(\int_{10\leq|t|\leq T}\left|L(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^2 f)\right|^{2} dt\right)^{\frac{1}{2}}$$

$$\times\left\{\max_{10\leq|t|\leq T}\left|\prod_{n=2}^{j} L(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^{2n} f)\right|^{\frac{2}{5}}\left(\int_{10\leq|t|\leq T}\left|\prod_{n=2}^{j} L(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^{2n} f)\right|^{2} dt\right)\right\}^{\frac{5}{12}}$$

$$\ll x^{\frac{5}{2}+\varepsilon} + x^{\frac{5}{2}+\varepsilon}\left(T^{-1+2.\frac{1}{12}+3.\frac{1}{2}.\frac{1}{2}+\left((j+1)^2-4\right)\left(\frac{1}{2}.\frac{1}{2}.\frac{2}{5}.\frac{5}{12}\right)+\left((j+1)^2-4\right)\left(\frac{1}{2}.\frac{5}{12}\right)}\right)$$

$$\ll x^{\frac{5}{2}+\varepsilon} T^{\left(\frac{(j+1)^2}{4}-\frac{13}{12}+\varepsilon\right)}.$$

Note that $10 \leq T \leq x$. Thus, we obtain

$$\sum_{n\leq x} \lambda^2_{\mathrm{sym}^j f}(n) l_1(n) = c(j)x^3 + O(x^{\frac{5}{2}+\varepsilon} T^{\left(\frac{(j+1)^2}{4}-\frac{13}{12}+\varepsilon\right)}) + O\left(\frac{x^{3+3\varepsilon}}{T}\right).$$

We choose T such that $x^{\frac{5}{2}} T^{\left(\frac{(j+1)^2}{4}-\frac{13}{12}\right)} \asymp \frac{x^3}{T}$ i.e., $T^{\left(\frac{3(j+1)^2-1}{12}\right)} \asymp x^{\frac{1}{2}}$.

Therefore, $T \asymp x^{\frac{6}{3(j+1)^2-1}}$.

Thus, we get

$$\sum_{n\le x}\lambda^2_{\mathrm{sym}^j f}(n)l_1(n) = c(j)x^3 + O\left(x^{3-\frac{6}{3(j+1)^2-1}+3\varepsilon}\right). \tag{5.3}$$

Similarly, we apply the Perron's formula (see section 1.5) to $\widetilde{F}_j(s)$ with $\eta = 3+\varepsilon$ and $10 \le T \le x$. Thus, we have

$$\begin{aligned}\sum_{n\le x}\lambda^2_{\mathrm{sym}^j f}(n)v_1(n) &= 4\sum_{n\le x}\lambda^2_{\mathrm{sym}^j f}(n)v(n)\\ &= \frac{4}{2\pi i}\int_{\eta-iT}^{\eta+iT}\widetilde{F}_j(s)\frac{x^s}{s}ds + O\left(\frac{x^{3+3\varepsilon}}{T}\right).\end{aligned}$$

We move the line of integration to $\Re(s) = \frac{5}{2}+\varepsilon$. Note that, there is no singularity in the rectangle obtained, and the function $\widetilde{F}_j(s)\dfrac{x^s}{s}$ is analytic in this region. Thus, using Cauchy's theorem for rectangle pertaining to analytic functions, we get

$$\begin{aligned}\sum_{n\le x}\lambda^2_{\mathrm{sym}^j f}(n)v_1(n) &= \frac{4}{2\pi i}\left\{\int_{\frac{5}{2}+\varepsilon-iT}^{\frac{5}{2}+\varepsilon+iT} + \int_{3+\varepsilon-iT}^{\frac{5}{2}+\varepsilon-iT} + \int_{\frac{5}{2}+\varepsilon+iT}^{3+\varepsilon+iT}\right\}\widetilde{F}_j(s)\frac{x^s}{s}ds\\ &\quad + O\left(\frac{x^{3+3\varepsilon}}{T}\right)\\ &=: \frac{4}{2\pi i}(J_1^{(7)} + J_2^{(7)} + J_3^{(7)}) + O\left(\frac{x^{3+3\varepsilon}}{T}\right).\end{aligned}$$

The contribution of horizontal line integrals ($J_2^{(7)}$ and $J_3^{(7)}$) in absolute value (using Lemmas 5.1.3, 1.9.8 and 1.9.10) is

$$\ll \int_{\frac{5}{2}+\varepsilon}^{3+\varepsilon} \frac{|L(\sigma-2+iT)\prod_{n=1}^{j} L(\sigma-2+iT, \mathrm{sym}^{2n} f \otimes \chi)|}{T} x^{\sigma} d\sigma$$

$$\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} \frac{|L(\sigma+iT)\prod_{n=1}^{j} L(\sigma+iT, \mathrm{sym}^{2n} f \otimes \chi)|}{T} x^{\sigma+2} d\sigma$$

$$\ll \left(\frac{x^2}{T}\right) \max_{\frac{1}{2}+\varepsilon\leq\sigma\leq 1+\varepsilon} x^{\sigma} T^{\left(\frac{(j+1)^2}{2}-\frac{1}{6}\right)(1-\sigma)+\varepsilon}$$
$$\ll \left(\frac{x^{2+2\varepsilon}}{T}\right) \max_{\frac{1}{2}+\varepsilon\leq\sigma\leq 1+\varepsilon} \left(\frac{x}{T^{\left(\frac{(j+1)^2}{2}-\frac{1}{6}\right)}}\right)^{\sigma} T^{\left(\frac{(j+1)^2}{2}-\frac{1}{6}\right)}.$$

Clearly, $\left(\frac{x}{T^{\left(\frac{(j+1)^2}{2}-\frac{1}{6}\right)}}\right)^{\sigma}$ is monotonic as a function of σ for $\frac{1}{2}+\varepsilon \leq \sigma \leq 1+\varepsilon$, and hence the maximum is attained at the extremities of the interval $[\frac{1}{2}+\varepsilon, 1+\varepsilon]$. Thus,

$$\begin{aligned} J_2^{(7)} + J_3^{(7)} &\ll x^{2+2\varepsilon}\left(x^{\frac{1}{2}+\varepsilon} T^{\left(\frac{(j+1)^2}{4}-\frac{1}{12}-1+\varepsilon\right)} + \frac{x^{1+\varepsilon}}{T}\right) \\ &\ll x^{\frac{5}{2}+3\varepsilon} T^{\left(\frac{(j+1)^2}{4}-\frac{13}{12}+\varepsilon\right)} + \frac{x^{3+3\varepsilon}}{T}. \end{aligned}$$

The contribution of the left vertical line integral ($J_1^{(7)}$) in absolute value (using Lemmas 5.1.3, 1.9.7, 1.9.10 and Hölder's inequality) is

$$\ll \int_{\frac{5}{2}+\varepsilon-iT}^{\frac{5}{2}+\varepsilon+iT} \frac{\left| L(\frac{1}{2}+\varepsilon+it)\prod_{n=1}^{j} L(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^{2n} f \otimes \chi)\right|}{\left|\frac{5}{2}+\varepsilon+it\right|} x^{\frac{5}{2}+\varepsilon} dt$$

$$\ll x^{\frac{5}{2}+\varepsilon} + x^{\frac{5}{2}+\varepsilon} \frac{1}{T} \left\{ \int_{10\le|t|\le T} \left| L(\frac{1}{2}+\varepsilon+it)\right|^6 dt \right\}^{\frac{1}{6}}$$
$$\times \left\{ \int_{10\le|t|\le T} \left| L(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^2 f\otimes\chi)\right|^2 dt \right\}^{\frac{1}{2}}$$

$$\times \left\{ \max_{10\le|t|\le T} \left| \prod_{n=2}^{j} L(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^{2n} f\otimes \chi)\right| \right.$$
$$\times \left. \left(\int_{10\le|t|\le T} \left| \prod_{n=2}^{j} L(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^{2n} f\otimes \chi)\right|^2 dt \right) \right\}^{\frac{1}{3}}$$

$$\ll x^{\frac{5}{2}+\varepsilon} + x^{\frac{5}{2}+\varepsilon} \left(T^{-1+2.\frac{1}{6}+3.\frac{1}{2}\cdot\frac{1}{2}+\left((j+1)^2-4\right)\left(\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{3}\right)+\left((j+1)^2-4\right)\left(\frac{1}{2}\cdot\frac{1}{3}\right)} \right)$$

$$\ll x^{\frac{5}{2}+\varepsilon} T^{\left(\frac{(j+1)^2}{4}-\frac{11}{12}\right)}.$$

Note that $10 \le T \le x$. Thus, we obtain

$$\sum_{n\le x} \lambda^2_{\mathrm{sym}^j f}(n) v_1(n) = O\left(x^{\frac{5}{2}+\varepsilon} T^{\left(\frac{(j+1)^2}{4}-\frac{11}{12}\right)}\right) + O\left(\frac{x^{3+3\varepsilon}}{T}\right).$$

We choose T such that $x^{\frac{5}{2}} T^{\left(\frac{(j+1)^2}{4}-\frac{11}{12}\right)} \asymp \frac{x^3}{T}$ i.e., $T^{\left(\frac{(j+1)^2}{4}+\frac{1}{12}\right)} \asymp x^{\frac{1}{2}}$.

Therefore, $T \asymp x^{\frac{6}{3(j+1)^2+1}}$.

Thus, we get

$$\sum_{n\leq x}\lambda^2_{\mathrm{sym}^j f}(n)v_1(n) = O\left(x^{3-\frac{6}{3(j+1)^2+1}+\varepsilon}\right). \tag{5.4}$$

Combining (5.3) and (5.4), we get

$$\sum_{n\leq x}\lambda^2_{\mathrm{sym}^j f}(n)r_6(n) = c(j)x^3 + O\left(x^{3-\frac{6}{3(j+1)^2+1}+\varepsilon}\right),$$

where $c(j)$ is an effective constant given by

$$c(j) = \frac{16}{3}L(3,\chi)\prod_{n=1}^{j}L(1,\mathrm{sym}^{2n}f)L(3,\mathrm{sym}^{2n}f\otimes\chi)H_j(3),$$

and χ is the non-principal Dirichlet character modulo 4.
This proves the theorem.

Remark 5.4.1 When $j = 2$, Theorem 5.1.1 gives the error term $O\left(x^{\frac{39}{14}+\varepsilon}\right)$, which improves the error term in 4.1.2.

5.4.1 Concluding Remarks

Note that we have the expected upper bounds, namely,

$$\int_T^{2T}\left|\zeta\left(\frac{5}{7}+it\right)\right|^{12}dt \ll T^{1+\varepsilon}$$

and

$$\int_T^{2T}\left|L\left(\frac{5}{8}+it,f\right)\right|^{4}dt \ll T^{1+\varepsilon},$$

uniformly for $T \geq 1$ (see [Iv 2, Gd]). Even if we move the line of integration to $\Re(s) = \frac{5}{7}$ and $\Re(s) = \frac{5}{8}$ pertaining to $l_1(n)$ and $v_1(n)$ respectively, and using the arguments of this paper, we end up with the same error term as stated in the Theorem 5.1.1.

Chapter 6

On a divisor problem related to a certain Dedekind zeta-function

6.1 Introduction

We begin this chapter by discussing algebraic number fields and introduce the tools needed to define a special class of L-function, namely, the Dedekind zeta-function.

Definition 6.1.1 An extension field $\mathbb{K}$ of a field $\mathbb{F}$ is an *algebraic extension of* $\mathbb{F}$ if every element in $\mathbb{K}$ is algebraic over $\mathbb{F}$.

Definition 6.1.2 An extension field $\mathbb{K}$ of a field $\mathbb{F}$ is said to be a *Galois extension field* if every irreducible polynomial over $\mathbb{F}$ which has a root in $\mathbb{K}$ factors into linear factors in $\mathbb{K}$. Also, $\mathbb{K}$ must be a separable extension.

Definition 6.1.3 A finite field extension of $\mathbb{Q}$ is known as *algebraic number field.*

In other words, a field that includes $\mathbb{Q}$ as a sub-field. The fact that an algebraic number field is a finite-dimensional vector space over the field $\mathbb{Q}$, however, provides a more thorough explanation. The dimension of this vector space refers to its degree.

Definition 6.1.4 An *algebraic integer* is a number that is the root of some monic polynomial with integer coefficients.

The algebraic integers of a given algebraic number field constitute a ring within the field, just as the set of rational integers $\mathbb{Z}$ forms a ring in $\mathbb{Q}$.

Definition 6.1.5 The collection of algebraic integers of an algebraic number field $\mathbb{K}$ is known as the *ring of integers*, and denoted by $\mathcal{O}_{\mathbb{K}}$.

Definition 6.1.6 An *ideal* α is a subset of $\mathcal{O}_{\mathbb{K}}$, which satisfies the following properties:

1. if $a, b \in \alpha$ then so will $a + b$ and $a.b$.

2. for every $c \in \mathcal{O}_{\mathbb{K}}$ and $a \in \alpha$, $ca \in \alpha$.

Now that we have the necessary resources, we can introduce a new L-function. Let $\mathbb{K}$ be an algebraic extension of degree m of rational field $\mathbb{Q}$. Define (for $\Re(s) > 1$)

$$\zeta_{\mathbb{K}}(s) := \sum_{\alpha} \frac{1}{N(\alpha)^s},$$

where the summation is running over all the integral ideals α of $\mathbb{K}$, and norm of integral ideal α is denoted by $N(\alpha)$.

We demonstrate that Dedekind zeta-function can also be represented as an infinite product of the norms of prime ideals $\mathfrak{p}$, much like the Riemann zeta-function, by using the fundamental theorem of ideal theory. For $\Re(s) > 1$, we have

$$\zeta_{\mathbb{K}}(s) = \prod_{\mathfrak{p} \subseteq \mathcal{O}_{\mathbb{K}}} \frac{1}{1 - \left(\frac{1}{N(\mathfrak{p})}\right)^s}$$

Remark 6.1.7 When $\mathbb{K} = \mathbb{Q}$, the Dedekind zeta-function becomes the Riemann zeta-function.

One can easily observe that the function $\zeta_{\mathbb{K}}(s)$ can also be written as:

$$\zeta_{\mathbb{K}}(s) = \sum_{n=1}^{\infty} \frac{a_{\mathbb{K}}(n)}{n^s},$$

where $a_{\mathbb{K}}(n)$ denotes the number of integral ideals of $\mathbb{K}$ with norm m.

Lemma 6.1.8 *$\zeta_{\mathbb{K}}(s)$ has an analytic continuation to $\mathbb{C} \backslash \{1\}$ with a simple pole at $s = 1$.*

The coefficients $a_{\mathbb{K}}$ plays a very vital role in number theory. It is shown (by Chandrasekharan and Good [Ch-Gd]) that these coefficients are multiplicative and satisfies the upper bound

$$a_{\mathbb{K}}(n) \leq d(n)^m,$$

where m is the degree of extension, i.e., $m = [\mathbb{K} : \mathbb{Q}]$ and $d(n)$ is the number of divisors of n.

In 1949, Landau [Lan] showed that

$$\sum_{n \leq x} a_{\mathbb{K}}(n) = c'x + O\left(x^{1-\frac{2}{m+1}+\varepsilon}\right),$$

where c' is the residue of $\zeta_{\mathbb{K}}(s)$ at its simple pole at $s = 1$, which is further improved to

$$\sum_{n \leq x} a_{\mathbb{K}}(n) = c'x + O\left(x^{\frac{23}{73}} \log^{\frac{315}{146}} x\right),$$

for quadratic field by Huxley and Watt [Hx-Wa]. Some further improvement is also available for cubic fields by Müller [Mü]. In 1993, W.G. Nowak [No] established that

$$\sum_{n \leq x} a_{\mathbb{K}}(n) = c''x + \begin{cases} O\left(x^{1-\frac{2}{m}+\frac{8}{m(5m+2)}} \log^{\frac{10}{5m+2}} x\right) & \text{if } 3 \leq m \leq 6 \\ \\ O\left(x^{1-\frac{2}{m}+\frac{3}{2m^2}} \log^{\frac{2}{m}} x\right) & \text{if } m \geq 7 \end{cases}.$$

We also have some significant results (by Chandrasekharan and Narasimhan [Ch-Nr] and by Chandrasekharan and Good [Ch-Gd]) of $\displaystyle\sum_{n \leq x} a_{\mathbb{K}}(n)^k$ for some higher powers k, if $\mathbb{K}$ is the Galois extension of $\mathbb{Q}$.

If h is the class number of $\mathbb{K}$ and $[\mathbb{K} : \mathbb{Q}] = r_1 + 2r_2$, where r_1 is the number of real conjugate fields and $2r_2$ is the number of complex conjugate fields, then we can write

$$\sum_{n \le x} a_{\mathbb{K}}(n) = h\lambda x + E(x),$$

where

$$\lambda := \frac{2^{r_1+r_2}\pi^{r_2}R}{w|\Delta|^{\frac{1}{2}}}.$$

Here, w is the number of roots of unity in $\mathbb{K}$; R is the regulator of $\mathbb{K}$ and Δ is the discriminant of $\mathbb{K}$.

When $[\mathbb{K}:\mathbb{Q}] = m \ge 10$, B. Paul and A. Sankaranarayanan proved that

$$E(x) \ll x^{1-\frac{3}{m+6}+\varepsilon},$$

where implied constants depend only on $\mathbb{K}$ and ε (see [Pa-Sa]).
Also, if $\mathbb{K} = \mathbb{Q}(\zeta_l)$, where l is some positive integer and $[\mathbb{K}:\mathbb{Q}] = m \ge 8$, then,

$$E(x) \ll x^{1-\frac{3}{m+5}+\varepsilon},$$

where the implied constants depend only on $\mathbb{K}$ and ε (see [Pa-Sa]).

It is of great interest to study the L-functions related to primitive holomorphic cusp forms. For many years, it has been a profound area in which many authors have contributed.

Let $L(s,f)$ be the L-function connected with the primitive holomorphic cusp form f of weight w for the full modular group $SL(2,\mathbb{Z})$ and $\lambda_f(n)$ are the normalized n^{th} Fourier coefficients of Fourier expansion of $f(z)$ at the cusp ∞. Then, we can write

$$L^k(s,f) = \sum_{n=1}^{\infty} \frac{\lambda_{k,f}(n)}{n^s},$$

where

$$\lambda_{k,f}(n) = \sum_{n=n_1n_2\ldots n_k} \lambda_f(n_1)\lambda_f(n_2)\ldots\lambda_f(n_k).$$

In 2012, Kanemitsu, Sankaranarayanan and Tanigawa [Ka-Sa-Tn] proved that for $k \geq 2$,

$$\sum_{n \leq x} \lambda_{k,f}(n) \ll x^{1-\frac{3}{2k+2}+\varepsilon},$$

where implied constant depends only on f and ε, which is further improved by Lü in [Lü 3].

For such divisor problems connected to holomorphic cusp forms, see the work of H.F. Liu [Li], [Li-Za] and Lü [Lü 3].

Let $\mathbb{K}_3$ be a non-normal cubic extension of a rational field $\mathbb{Q}$. It is natural to study the k^{th} integral power of Dedekind zeta function, i.e.,

$$(\zeta_{\mathbb{K}_3}(s))^k = \sum_{n=1}^{\infty} \frac{a_{k,\mathbb{K}_3}(n)}{n^s}$$

for $\Re(s) > 1$, where $a_{k,\mathbb{K}_3}(n) = \sum_{n=n_1 n_2 \ldots n_k} a_{\mathbb{K}_3}(n_1) a_{\mathbb{K}_3}(n_2) \ldots a_{\mathbb{K}_3}(n_k)$.

In 2012, Lü [Lü 4] was able to refine the previously known results (by Fomenko [Fo 3]) of mean square and third power of $a_{\mathbb{K}_3}(n)$ to

$$\sum_{n \leq x} a_{\mathbb{K}_3}(n)^2 = a_1 \log x + a_2 + O\left(x^{\frac{23}{31}+\varepsilon}\right)$$

where a_1 and a_2 are constants and

$$\sum_{n \leq x} a_{\mathbb{K}_3}(n)^3 = x P_3(\log x) + O\left(x^{\frac{235}{259}+\varepsilon}\right)$$

where $P_3(t)$ is a suitable polynomial in t of degree 4.

Question 6.1.9 Do we have such an asymptotic formula for higher mean values of $a_{\mathbb{K}}(n)$ for any non-normal extension $\mathbb{K}$ of $\mathbb{Q}$ and for the sum related to the coefficients of the k-fold generating series of $\sum_{n=1}^{\infty} \frac{a_{\mathbb{K}}(n)}{n^s}$?

Throughout this chapter, we restrict our attention to non-normal cubic extension $\mathbb{K}_3$ of rational field $\mathbb{Q}$. Let

$$\zeta_{\mathbb{K}_3}(s) = \sum_{n=1}^{\infty} \frac{a_{\mathbb{K}_3}(n)}{n^s}$$

is the Dedekind zeta-function which is absolutely convergent in $\Re(s) > 1$, continuable as a meromorphic function to the whole complex plane $\mathbb{C}$ with a pole at $s = 1$.
We are interested in the asymptotic formula for the sum

$$\sum_{n\leq x} a_{k,\mathbb{K}_3}(n),$$

for any integer $k \geq 1$, where

$$a_{k,\mathbb{K}_3}(n) = \sum_{n=n_1n_2\ldots n_k} a_{\mathbb{K}_3}(n_1)a_{\mathbb{K}_3}(n_2)\ldots a_{\mathbb{K}_3}(n_k).$$

Note that, $a_{1,\mathbb{K}_3}(n) = a_{\mathbb{K}_3}(n)$.

Our primary purpose in this chapter is to investigate the above partial sum and establish an asymptotic formula for the same, with a tightened error term. In order to make our discussion somewhat self-sufficient, we will briefly review some of the essential definitions and associated results that will be referenced frequently to support our finding.

Definition 6.1.10 For each σ such that $0 \leq \sigma \leq 1$, define

$$\mu(\sigma) := \inf\{\xi : \zeta(\sigma + it) \ll |t|^{\xi}\}.$$

As a function of σ, $\mu(\sigma)$ is continuous, non-increasing and convex downwards in the sense that no arc of the curve $y = \mu(\sigma)$ has any point above its chord. Also, $\mu(\sigma)$ is never negative. Therefore,

$$\mu(\sigma) = \begin{cases} 0 & \text{if } \sigma > 1 \\ \frac{1}{2} - \sigma & \text{if } \sigma < 0 \end{cases}.$$

By continuity of the function $\mu(\sigma)$, we obtain

$$\mu(1) = 0 \quad \text{and} \quad \mu(0) = \frac{1}{2}.$$

Hypothesis 6.1.11 (Lindelöf hypothesis) For any $\varepsilon > 0$, we have

$$\zeta\left(\frac{1}{2} + it\right) \ll t^{\varepsilon}.$$

This is equivalent to saying that

$$\zeta\left(\sigma + it\right) \ll t^{\varepsilon} \quad \text{for all} \quad \varepsilon > 0 \quad \text{and} \quad \sigma \geq \frac{1}{2},$$

i.e.,

$$\mu(\sigma) = 0 \quad \text{for} \quad \sigma \geq \frac{1}{2}.$$

First, we make the following hypothesis:

Hypothesis 6.1.12 Let $|t| \geq 1$ and $\varepsilon > 0$ be any small constant. Then, we have

$$\zeta\left(\frac{1}{2} + it\right) \ll (|t| + 1)^{\mu+\varepsilon},$$

where $\mu = \mu\left(\frac{1}{2}\right)$.

Remark 6.1.13 Phragmén Lindelöf principle leads to

$$\zeta(\sigma + it) \ll (|t| + 1)^{2\mu(1-\sigma)+\varepsilon},$$

uniformly for $\frac{1}{2} \leq \sigma \leq 2$, $|t| \geq 1$, under the assumption of our hypothesis 6.1.12. Unconditionally, the hypothesis 6.1.12 is true with $\mu = \frac{13}{84}$, see Bourgain [Bo].

For any integer $k \geq 2$, writing,

$$\sum_{n \leq x} a_{k,\mathbb{K}_3}(n) = M_{k,\mathbb{K}_3}(x) + E_{k,\mathbb{K}_3}(x),$$

where $M_{k,\mathbb{K}_3}(x)$ is the main term which is of the form $xP_{k-1}(\log x)$, where $P_{k-1}(t)$ is a polynomial in t of degree $k-1$. We prove the following theorem.

Theorem 6.1.14 *Let $\varepsilon > 0$ (be any small constant) and define $\lambda_1 = 3\varepsilon$, $\lambda_2 = \min\left(2\mu, \frac{1}{4}\right)$, $\lambda_3 = \min\left(\mu + \frac{1}{2}, \frac{5}{8}\right)$, $\lambda_4 = \min\left(2\mu + \frac{3}{4}, 1\right)$, $\lambda_5 = \min\left(3\mu + 1, \frac{3}{2}\right)$ and $\lambda_k = \mu\,(k-6) + \frac{k}{3}$ for $k \geq 6$.*
Then we have for any integer $k \geq 1$,

$$E_{k,\mathbb{K}_3}(x) \ll x^{1-\frac{1}{2(1+\lambda_k)}+3k\varepsilon}.$$

Conjecture 6.1.15 (Strong Artin Conjecture) Let G be the Galois group of an extension $\mathbb{K}/\mathbb{F}$ of number fields. Let ρ be an n-dimensional complex representation of G. There exists an automorphic representation π of $GL(n, \mathbb{A}_{\mathbb{F}})$, such that the L-functions agree almost everywhere, i.e., except at a finite number of places ν, $L(s, \rho_\nu) = L(s, \pi_\nu)$. Moreover, if ρ is irreducible, then π is cuspidal.

Lemma 6.1.16 *For $\Re(s) > 1$, we have*

$$\zeta_{\mathbb{K}_3}(s) = \zeta(s)L(s, f).$$

Proof. Proof of this lemma is strongly motivated by the fact that the strong Artin conjecture is true for the group S_3 and S_3 is the Galois group of normal closure (i.e. $\mathbb{K}_6$) of $\mathbb{K}_3$ over $\mathbb{Q}$. Note that, $\mathbb{K}_6$ is a non abelian extension of degree 6. For the sake of completeness we give the details (see [Lü 4]). There are three conjugacy classes of S_3, namely,

$$\rho_1 : (1),$$

$$\rho_2 : (123)(132) \quad \text{and}$$

$$\rho_3 : (12)(13)(23).$$

Hence, there are three simple characters: the principal character φ_1; character determined by the subgroup $\rho_2 \cup \rho_3$, say φ_2; and the 2-dimensional character φ_3. Let D be the discriminant of $f(x) = x^3 + ax^2 + bx + c$ (i.e., $D = a^2b^2 + 18abc - 4b^3 - 4a^3c - 27c^2$) and $\mathbb{K}_2 = \mathbb{Q}(\sqrt{D})$. The extensions $\mathbb{K}_2/\mathbb{Q}$, $\mathbb{K}_6/\mathbb{K}_2$ and $\mathbb{K}_6/\mathbb{K}_3$ are abelian. The Dedekind zeta function satisfies the following relations:

$$\zeta_{\mathbb{K}_6}(s) = L_{\varphi_1}L_{\varphi_2}L_{\varphi_3},$$

$$\zeta_{\mathbb{K}_2}(s) = L_{\varphi_1}L_{\varphi_2},$$

$$\zeta_{\mathbb{K}_3}(s) = L_{\varphi_1} L_{\varphi_3},$$

where

$$L_{\varphi_1} = \zeta(s),$$

$$L_{\varphi_2} = L(s, \varphi_2, \mathbb{K}_6/\mathbb{Q}),$$

$$L_{\varphi_3} = L(s, \varphi_3, \mathbb{K}_6/\mathbb{Q}).$$

Here $L(s, \varphi_2, \mathbb{K}_6/\mathbb{Q})$ and $L(s, \varphi_3, \mathbb{K}_6/\mathbb{Q})$ are Artin L-functions, (see pp. 226-227 of [Cas-Fr]).

Since, the strong Artin conjecture holds true in this situation, the function $L(s, \varphi_3, \mathbb{K}_6/\mathbb{Q})$ also can be interpreted in another way (see [De 2]). Let

$$\Phi : S_3 \longrightarrow GL(2, \mathbb{C})$$

be the irreducible 2-dimensional representation. Then Φ gives a cuspidal representation π of $GL(2, \mathbb{A}_{\mathbb{Q}})$. Let

$$L(s, \pi) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}.$$

In particular, if Φ is odd, i.e., $D < 0$ then

$$L(s, \pi) = L(s, f),$$

where f is a holomorphic cusp form of weight 1 for the congruence group $\Gamma_0(|D|)$, i.e.,

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}.$$

Thus, we have $L_{\varphi_3} = L(s, f)$ and hence

$$\zeta_{\mathbb{K}_3}(s) = \zeta(s) L(s, f).$$

This proves the lemma.

Here after we consider only those $\mathbb{K}_3$ for which $\zeta_{\mathbb{K}_3}(s)$ has such a lift stated in Lemma 6.1.16.

Now, we have all the necessary tools required to prove the key theorem 6.1.14 of this chapter. The work presented in this chapter will soon appear in [Sr-Sa 4].

6.2 Proof of Theorem 6.1.14

Let $k \geq 1$ be an integer. We start with the Perron's formula (see section 1.5), applying to $(\zeta_{\mathbb{K}_3}(s))^k$ with $\eta = 1 + \varepsilon$ and $1 \leq T \leq x$. Thus, we obtain

$$\sum_{n \leq x} a_{k,\mathbb{K}_3}(n) = \frac{1}{2\pi i} \int_{\eta - iT}^{\eta + iT} (\zeta_{\mathbb{K}_3}(s))^k \frac{x^s}{s} ds + O\left(\frac{x^{1+\varepsilon}}{T}\right).$$

Note that $(\zeta_{\mathbb{K}_3}(s))^k$ has a pole at $s = 1$ of order k so that by moving line of integration to $\Re(s) = \frac{1}{2}$, we obtain

$$\begin{aligned}
\sum_{n \leq x} a_{k,\mathbb{K}_3}(n) &= \operatorname*{Res}_{s=1}\left\{(\zeta_{\mathbb{K}_3}(s))^k \frac{x^s}{s}\right\} + \frac{1}{2\pi i}\left\{\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}-iT}^{\eta+iT} + \int_{\eta-iT}^{\frac{1}{2}-iT}\right\}(\zeta_{\mathbb{K}_3}(s))^k \frac{x^s}{s} ds \\
&\quad + O\left(\frac{x^{1+\varepsilon}}{T}\right) \\
&=: xP_{k-1}(\log x) + J_1(k) + J_2(k) + J_3(k) + O\left(\frac{x^{1+\varepsilon}}{T}\right),
\end{aligned}$$

where $P_{k-1}(t)$ is a polynomial in t of degree $k-1$.
Note that, the horizontal lines ($J_2(k)$ and $J_3(k)$) contribute (for any fixed integer $k \geq 1$), using Lemma 6.1.16

$$\begin{aligned}
J_2(k) + J_3(k) &\ll \max_{\frac{1}{2} \leq \sigma \leq \eta} x^{\sigma} T^{(2k\mu + \frac{2k}{3})(1-\sigma)+\varepsilon} T^{-1} \\
&\ll \max_{\frac{1}{2} \leq \sigma \leq \eta} \left(\frac{x}{T^{2k\mu + \frac{2k}{3}}}\right)^{\sigma} T^{2k\mu + \frac{2k}{3} - 1 + \varepsilon}.
\end{aligned}$$

For any fixed k and $\mu(> 0)$, $\left(\frac{x}{T^{2k\mu + \frac{2k}{3}}}\right)^{\sigma}$ is monotonic as a function of σ for $\frac{1}{2} \leq \sigma \leq \eta$ and hence the maximum is attained at the extremities of the interval $\left[\frac{1}{2}, \eta\right]$.
Thus,

$$J_2(k) + J_3(k) \ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}+\varepsilon} T^{\frac{1}{2}(2k\mu + \frac{2k}{3})-1}.$$

Vertical line contributions:

0. **For $\boldsymbol{k}$=1**,

$$J_1(1) := \frac{1}{2\pi i} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \zeta_{\mathbb{K}_3}(s) \frac{x^s}{s} ds.$$

Using Lemma 6.1.16, Lemma 1.9.6, Lemma 1.9.7 and Cauchy-Schwarz inequality,

$$\begin{aligned}
J_1(1) &\ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \log T \left\{ \max_{1 \le U \le T} \ \frac{1}{U} \int_{\frac{U}{2}}^{U} \left| \zeta\left(\frac{1}{2}+it\right) \right| \left| L\left(\frac{1}{2}+it, f\right) \right| dt \right\} \\
&\ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \log T \left\{ \max_{1 \le U \le T} \ \frac{1}{U} \left(\int_{\frac{U}{2}}^{U} \left| \zeta\left(\frac{1}{2}+it\right) \right|^2 dt \right)^{\frac{1}{2}} \left(\int_{\frac{U}{2}}^{U} \left| L\left(\frac{1}{2}+it, f\right) \right|^2 dt \right)^{\frac{1}{2}} \right\} \\
&\ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \log T \max_{1 \le U \le T} \left\{ \frac{1}{U} U^{\frac{1}{2}+\varepsilon} U^{\frac{1}{2}+\varepsilon} \right\} \\
&\ll x^{\frac{1}{2}} + x^{\frac{1}{2}} T^{3\varepsilon}.
\end{aligned}$$

Note that, with $k=1$,

$$J_2(1)+J_3(1) \ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}+\varepsilon}T^{\frac{1}{2}\left(2\mu+\frac{2}{3}\right)-1}$$

$$\ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}+\varepsilon}T^{\mu-\frac{2}{3}}$$

$$\ll \frac{x^{1+\varepsilon}}{T} + \frac{x^{\frac{1}{2}+\varepsilon}}{T^{\frac{1}{2}}} \qquad \left(\text{since } \mu < \tfrac{1}{6}\right)$$

$$\ll \frac{x^{1+\varepsilon}}{T} \qquad (\text{as long as } 10 \le T \le x).$$

Thus, $J_1(1)$ dominates over $J_2(1)+J_3(1)$.

Now,

$$\sum_{n\le x} a_{1,\mathbb{K}_3}(n) = xP_0(\log x) + E_{1,\mathbb{K}_3}(x),$$

where $E_{1,\mathbb{K}_3}(x) \ll \dfrac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}} + x^{\frac{1}{2}}T^{3\varepsilon}$, i.e.,

$$E_{1,\mathbb{K}_3}(x) \ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}} + x^{\frac{1}{2}}T^{\lambda_1}.$$

We choose T such that $\dfrac{x}{T} \sim x^{\frac{1}{2}}$, i.e., $T \sim x^{\frac{1}{2}}$.

So finally, we have

$$E_{1,\mathbb{K}_3}(x) \ll x^{1-\frac{1}{2(1+\lambda_1)}+3\varepsilon}.$$

1. **For k=2,**

$$J_1(2) := \frac{1}{2\pi i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} (\zeta_{\mathbb{K}_3}(s))^2 \frac{x^s}{s}\,ds.$$

Using Lemma 6.1.16, hypothesis 6.1.12 and Lemma 1.9.6 and Lemma 1.9.7

$$J_1(2) \ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \log T \left\{ \max_{1 \le U \le T} \frac{1}{U} \int_{\frac{U}{2}}^{U} \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \left| L\left(\frac{1}{2} + it, f\right) \right|^2 dt \right\}$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \log T \max_{1 \le U \le T} \left\{ \frac{1}{U} U^{2\mu + 2\varepsilon} U \log U \right\}$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}} T^{2\mu + 4\varepsilon}.$$

Note that, by Lemma 1.9.6 and Lemma 1.9.7,

$$\int_{\frac{U}{2}}^{U} \left| L\left(\frac{1}{2} + it, f\right) \right|^4 dt \ll \left(\int_{\frac{U}{2}}^{U} \left| L\left(\frac{1}{2} + it, f\right) \right|^2 dt \right)^{\frac{1}{2}} \left(\int_{\frac{U}{2}}^{U} \left| L\left(\frac{1}{2} + it, f\right) \right|^6 dt \right)^{\frac{1}{2}}$$

$$\ll U^{\frac{3}{2} + \varepsilon}.$$

Also, we have (using Lemmas 1.9.2, 1.9.3 and above observation)

$$J_1(2) \ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \log T \left\{ \max_{1 \le U \le T} \frac{1}{U} \int_{\frac{U}{2}}^{U} \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \left| L\left(\frac{1}{2} + it, f\right) \right|^2 dt \right\}$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \log T \left\{ \max_{1 \le U \le T} \frac{1}{U} \left(\int_{\frac{U}{2}}^{U} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt \right)^{\frac{1}{2}} \left(\int_{\frac{U}{2}}^{U} \left| L\left(\frac{1}{2} + it, f\right) \right|^4 dt \right)^{\frac{1}{2}} \right\}$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \log T \max_{1 \le U \le T} \left\{ \frac{1}{U} U^{\frac{1}{2} + \varepsilon} U^{\frac{1}{2}\left(\frac{3}{2} + \varepsilon\right)} \right\}$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}} T^{\frac{1}{4} + 2\varepsilon}.$$

Thus, we have

$$J_1(2) \ll x^{\frac{1}{2}} + x^{\frac{1}{2} + 4\varepsilon} T^{\min\left(2\mu, \frac{1}{4}\right)}.$$

Note that, with $k = 2$,

$$\frac{1}{2}\left(2k\mu + \frac{2k}{3}\right) - 1 = \frac{1}{2}\left(4\mu + \frac{4}{3}\right) - 1$$
$$= 2\mu - \frac{1}{3}.$$

Case (i) If $0 \leq \mu < \frac{1}{8}$, then $\min\left(2\mu, \frac{1}{4}\right) = 2\mu$, so that $2\mu - \frac{1}{3} \leq 2\mu$ is true.

Case (ii) If $\mu \geq \frac{1}{8}$, then $\min\left(2\mu, \frac{1}{4}\right) = \frac{1}{4}$, then $2\mu - \frac{1}{3} \leq \frac{1}{4}$ happens when

$$\mu \leq \frac{1}{8} + \frac{1}{6} = \frac{7}{24},$$

which is anyway true, since we know $\mu \leq \frac{13}{84} (\leq \frac{7}{24})$ (by Bourgain [Bo]). Thus,

$$J_1(2) \ll x^{\frac{1}{2}} + x^{\frac{1}{2}+4\varepsilon} T^{\min\left(2\mu, \frac{1}{4}\right)}$$

holds good, which dominates over $J_2(2) + J_3(2)$.

Now,

$$\sum_{n \leq x} a_{2,\mathbb{K}_3}(n) = xP_1(\log x) + E_{2,\mathbb{K}_3}(x),$$

where $E_{2,\mathbb{K}_3}(x) \ll \dfrac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}} + x^{\frac{1}{2}+4\varepsilon} T^{\min\left(2\mu, \frac{1}{4}\right)}$, i.e.,

$$E_{2,\mathbb{K}_3}(x) \ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}} + x^{\frac{1}{2}+4\varepsilon} T^{\lambda_2}.$$

We choose T such that $\dfrac{x}{T} \sim x^{\frac{1}{2}} T^{\lambda_2}$, i.e., $T^{1+\lambda_2} \sim x^{\frac{1}{2}}$, i.e.,

$$T \sim x^{\frac{1}{2(1+\lambda_2)}}.$$

So finally, we have

$$E_{2,\mathbb{K}_3}(x) \ll x^{1-\frac{1}{2(1+\lambda_2)}+6\varepsilon}.$$

2. **For k=3,**

$$J_1(3) := \frac{1}{2\pi i} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} (\zeta_{\mathbb{K}_3}(s))^3 \frac{x^s}{s} ds.$$

Using Lemma 6.1.16, Cauchy-Schwarz Inequality and on the assumption of our hypothesis 6.1.12,

$$J_1(3) \ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \log T \left\{ \max_{1\le U\le T} \frac{1}{U} \int_{\frac{U}{2}}^{U} \left|\zeta\left(\frac{1}{2}+it\right)\right|^3 \left|L\left(\frac{1}{2}+it, f\right)\right|^3 dt \right\}$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \log T \left\{ \max_{1\le U\le T} \frac{1}{U} U^{\mu+\varepsilon} \left(\int_{\frac{U}{2}}^{U} \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt \right)^{\frac{1}{2}} \times \left(\int_{\frac{U}{2}}^{U} \left|L\left(\frac{1}{2}+it, f\right)\right|^6 dt \right)^{\frac{1}{2}} \right\}$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \log T \max_{1\le U\le T} \left\{ \frac{1}{U} U^{\mu+\varepsilon} U^{\frac{1}{2}(1+\varepsilon)} U^{\frac{1}{2}(2+\varepsilon)} \right\}$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}+4\varepsilon} T^{\mu+\frac{1}{2}}.$$

Also, we have (using Lemmas 1.9.2, 1.9.3 and above observation)

$$J_1(3) \ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \log T \left\{ \max_{1\le U\le T} \frac{1}{U} \left(\int_{\frac{U}{2}}^{U} \left|\zeta\left(\frac{1}{2}+it\right)\right|^{12} dt \right)^{\frac{1}{4}} \left(\int_{\frac{U}{2}}^{U} \left|L\left(\frac{1}{2}+it, f\right)\right|^4 dt \right)^{\frac{3}{4}} \right\}$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \log T \max_{1\le U\le T} \left\{ \frac{1}{U} U^{\frac{1}{2}+\varepsilon} U^{\frac{3}{4}\left(\frac{3}{2}+\varepsilon\right)} \right\}$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}+5\varepsilon} T^{\frac{5}{8}}.$$

Thus, we have

$$J_1(3) \ll x^{\frac{1}{2}} + x^{\frac{1}{2}+5\varepsilon} T^{\min(\mu+\frac{1}{2},\frac{5}{8})}.$$

Note that with $k = 3$,

$$\begin{aligned}\frac{1}{2}\left(2k\mu+\frac{2k}{3}\right)-1&=\frac{1}{2}\left(6\mu+2\right)-1\\&=3\mu.\end{aligned}$$

Case (i) If $0 \leq \mu < \frac{1}{8}$, then $\min\left(\mu+\frac{1}{2},\frac{5}{8}\right)=\mu+\frac{1}{2}$.

We observe that $3\mu \leq \mu+\frac{1}{2}$ provided $2\mu \leq \frac{1}{2}$, which is anyway true (by Bourgain [Bo]).

Case (ii) If $\mu \geq \frac{1}{8}$, then $\min\left(\mu+\frac{1}{2},\frac{5}{8}\right)=\frac{5}{8}$, then $3\mu \leq \frac{5}{8}$ holds only when $\mu \leq \frac{5}{24}$,
which is anyway true, since we know $\mu \leq \frac{13}{84}\left(\leq \frac{5}{24}\right)$ (by Bourgain [Bo]).
Thus,

$$J_1(3) \ll x^{\frac{1}{2}}+x^{\frac{1}{2}+5\varepsilon}T^{\min\left(\mu+\frac{1}{2},\frac{5}{8}\right)}$$

holds good, which dominates over $J_2(3)+J_3(3)$.

Now,

$$\sum_{n\leq x} a_{3,\mathbb{K}_3}(n)=xP_2(\log x)+E_{3,\mathbb{K}_3}(x),$$

where $E_{3,\mathbb{K}_3}(x) \ll \dfrac{x^{1+\varepsilon}}{T}+x^{\frac{1}{2}}+x^{\frac{1}{2}+5\varepsilon}T^{\min\left(\mu+\frac{1}{2},\frac{5}{8}\right)}$, i.e.,

$$E_{3,\mathbb{K}_3}(x) \ll \frac{x^{1+\varepsilon}}{T}+x^{\frac{1}{2}}+x^{\frac{1}{2}+5\varepsilon}T^{\lambda_3}.$$

We choose T such that $\dfrac{x}{T}\sim x^{\frac{1}{2}}T^{\lambda_3}$, i.e., $T^{1+\lambda_3}\sim x^{\frac{1}{2}}$, i.e., $T\sim x^{\frac{1}{2(1+\lambda_3)}}$.

So finally, we have

$$E_{3,\mathbb{K}_3}(x) \ll x^{1-\frac{1}{2(1+\lambda_3)}+9\varepsilon}.$$

3. **For k=4,**

First we observe, (using Lemmas 1.9.2, 1.9.3 and Cauchy-Schwarz inequality)

$$\int_{\frac{U}{2}}^{U}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{8}dt \ll \left(\int_{\frac{U}{2}}^{U}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{4}dt\right)^{\frac{1}{2}}\left(\int_{\frac{U}{2}}^{U}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{12}dt\right)^{\frac{1}{2}}$$

$$\ll U^{\frac{1}{2}(1+\varepsilon)}U^{\frac{1}{2}(2+\varepsilon)}$$

$$\ll U^{\frac{3}{2}+\varepsilon}.$$

Now,

$$\int_{\frac{U}{2}}^{U}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{6}dt \ll \left(\int_{\frac{U}{2}}^{U}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{4}dt\right)^{\frac{1}{2}}\left(\int_{\frac{U}{2}}^{U}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{8}dt\right)^{\frac{1}{2}}$$

$$\ll U^{\frac{1}{2}(1+\varepsilon)}T^{\frac{1}{2}(\frac{3}{2}+\varepsilon)}$$

$$\ll U^{\frac{5}{4}+\varepsilon}.$$

Now,

$$J_1(4) := \frac{1}{2\pi i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}(\zeta_{\mathbb{K}_3}(s))^4\frac{x^s}{s}ds.$$

Using Lemma 6.1.16, Lemma 1.9.6, Lemma 1.9.7, Hölder's inequality and on the assumption of our hypothesis 6.1.12,

$$J_1(4) \ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \log T \left\{ \max_{1 \le U \le T} \frac{1}{U} \int_{\frac{U}{2}}^{U} \left| \zeta\left(\frac{1}{2}+it\right)\right|^4 \left| L\left(\frac{1}{2}+it, f\right)\right|^4 dt \right\}$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \log T \left\{ \max_{1 \le U \le T} \frac{1}{U} U^{2\mu+2\varepsilon} \left(\int_{\frac{U}{2}}^{U} \left| \zeta\left(\frac{1}{2}+it\right)\right|^6 dt \right)^{\frac{1}{3}} \right.$$
$$\left. \times \left(\int_{\frac{U}{2}}^{U} \left| L\left(\frac{1}{2}+it, f\right)\right|^6 dt \right)^{\frac{2}{3}} \right\}$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \log T \max_{1 \le U \le T} \left\{ \frac{1}{U} U^{2\mu+2\varepsilon} U^{\frac{1}{3}(\frac{5}{4}+\varepsilon)} U^{\frac{2}{3}(2+\varepsilon)} \right\} \text{(using above observation)}$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}+5\varepsilon} T^{2\mu+\frac{3}{4}}.$$

Also, we have (using Lemma 1.9.2, Lemma 1.9.3, Lemma 1.9.6, Lemma 1.9.7 and Hölder's inequality)

$$J_1(4) \ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \log T \left\{ \max_{1 \le U \le T} \frac{1}{U} \left(\int_{\frac{U}{2}}^{U} \left| \zeta\left(\frac{1}{2}+it\right)\right|^{12} dt \right)^{\frac{1}{3}} \left(\int_{\frac{U}{2}}^{U} \left| L\left(\frac{1}{2}+it, f\right)\right|^6 dt \right)^{\frac{2}{3}} \right\}$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \log T \max_{1 \le U \le T} \left\{ \frac{1}{U} U^{\frac{1}{3}(2+\varepsilon)} U^{\frac{2}{3}(2+\varepsilon)} \right\}$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}+5\varepsilon} T.$$

Thus, we have

$$J_1(4) \ll x^{\frac{1}{2}} + x^{\frac{1}{2}+5\varepsilon} T^{\min\left(2\mu+\frac{3}{4}, 1\right)}.$$

Note that, with $k = 4$,

$$\frac{1}{2}\left(2k\mu+\frac{2k}{3}\right)-1=\frac{1}{2}\left(8\mu+\frac{8}{3}\right)-1$$
$$=4\mu+\frac{1}{3}.$$

Case (i) If $0\le\mu<\frac{1}{8}$, then $\min\left(2\mu+\frac{3}{4},1\right)=2\mu+\frac{3}{4}$.

We observe that $4\mu+\frac{1}{3}\le 2\mu+\frac{3}{4}$ provided $2\mu\le\frac{3}{4}-\frac{1}{3}=\frac{5}{12}$, i.e., $\mu\le\frac{5}{24}$, which is anyway true (see Bourgain [Bo]).

Case (ii) If $\mu\ge\frac{1}{8}$, then $\min(2\mu+\frac{3}{4},1)=1$, then $4\mu+\frac{1}{3}\le 1$ holds only when $\mu\le\frac{1}{6}$,
which is anyway true, since we know $\mu\le\frac{13}{84}$ (by Bourgain [Bo]).
Thus,

$$J_1(4)\ll x^{\frac{1}{2}}+x^{\frac{1}{2}+5\varepsilon}T^{\min\left(2\mu+\frac{3}{4},1\right)}$$

holds good, which dominates over $J_2(4)+J_3(4)$.

Now,

$$\sum_{n\le x}a_{4,\mathbb{K}_3}(n)=xP_3(\log x)+E_{4,\mathbb{K}_3}(x),$$

where $E_{4,\mathbb{K}_3}(x)\ll\dfrac{x^{1+\varepsilon}}{T}+x^{\frac{1}{2}}+x^{\frac{1}{2}+5\varepsilon}T^{\min\left(2\mu+\frac{3}{4},1\right)}$, i.e.,

$$E_{4,\mathbb{K}_3}(x)\ll\frac{x^{1+\varepsilon}}{T}+x^{\frac{1}{2}}+x^{\frac{1}{2}+5\varepsilon}T^{\lambda_4}.$$

We choose T such that $\dfrac{x}{T}\sim x^{\frac{1}{2}}T^{\lambda_4}$, i.e., $T^{1+\lambda_4}\sim x^{\frac{1}{2}}$, i.e., $T\sim x^{\frac{1}{2(1+\lambda_4)}}$.

So finally, we have

$$E_{4,\mathbb{K}_3}(x)\ll x^{1-\frac{1}{2(1+\lambda_4)}+12\varepsilon}.$$

4. **For k=5,**

$$J_1(5):=\frac{1}{2\pi i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}(\zeta_{\mathbb{K}_3}(s))^5\frac{x^s}{s}ds.$$

Using Lemma 6.1.16, Hölder's inequality and on the assumption of our hypothesis

6.1.12,

$$J_1(5) \ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \log T \left\{ \max_{1\le U\le T} \frac{1}{U} \int_{\frac{U}{2}}^{U} \left|\zeta\left(\frac{1}{2}+it\right)\right|^5 \left|L\left(\frac{1}{2}+it,f\right)\right|^5 dt \right\}$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \log T \left\{ \max_{1\le U\le T} \frac{1}{U} U^{3\mu+3\varepsilon} \left(\int_{\frac{U}{2}}^{U} \left|\zeta\left(\frac{1}{2}+it\right)\right|^{12} dt \right)^{\frac{1}{6}} \right.$$

$$\left. \times \left(\int_{\frac{U}{2}}^{U} \left|L\left(\frac{1}{2}+it,f\right)\right|^{6} dt \right)^{\frac{5}{6}} \right\}$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \log T \max_{1\le U\le T} \left\{ \frac{1}{U} U^{3\mu+3\varepsilon} U^{\frac{1}{6}(2+\varepsilon)} U^{\frac{5}{6}(2+\varepsilon)} \right\}$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}+6\varepsilon} T^{3\mu+1}.$$

Also, we have

$$J_1(5) \ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \log T \left\{ \max_{1\le U\le T} \frac{1}{U} \left(\int_{\frac{U}{2}}^{U} \left|\zeta\left(\frac{1}{2}+it\right)\right|^{12} dt \right)^{\frac{5}{12}} \left(\int_{\frac{U}{2}}^{U} \left|L\left(\frac{1}{2}+it,f\right)\right|^{5.\frac{12}{7}} dt \right)^{\frac{7}{12}} \right\}$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \log T \left\{ \max_{1\le U\le T} \frac{1}{U} U^{\frac{5}{12}(2+\varepsilon)} U^{\frac{7}{12}(2+\varepsilon)} U^{\frac{18}{7}\cdot\frac{1}{3}\cdot\frac{7}{12}+2\varepsilon} \right\}$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}+5\varepsilon} T^{\frac{3}{2}}.$$

Thus, we have

$$J_1(5) \ll x^{\frac{1}{2}} + x^{\frac{1}{2}+5\varepsilon} T^{\min\left(3\mu+1,\frac{3}{2}\right)}.$$

Observe that, with $k = 5$,

$$\frac{1}{2}\left(2k\mu + \frac{2k}{3}\right) - 1 = \frac{1}{2}\left(10\mu + \frac{10}{3}\right) - 1$$
$$= 5\mu + \frac{2}{3}.$$

Case (i) If $0 \leq \mu < \frac{1}{6}$, then $\min(3\mu + 1, \frac{3}{2}) = 3\mu + 1$.

We observe that $5\mu + \frac{2}{3} \leq 3\mu + 1$ provided $2\mu \leq \frac{1}{3}$, i.e., $\mu \leq \frac{1}{6}$ which is anyway true (see Bourgain [Bo]).

Case (ii) If $\mu \geq \frac{1}{6}$, then $\min\left(3\mu + 1, \frac{3}{2}\right) = \frac{3}{2}$, then $5\mu + \frac{2}{3} \leq \frac{3}{2}$ holds

only when $5\mu \leq \frac{5}{6}$, i.e., $\mu \leq \frac{1}{6}$, which is anyway true (Bourgain [Bo]).

Therefore,

$$J_1(5) \ll x^{\frac{1}{2}} + x^{\frac{1}{2}+5\varepsilon} T^{\min\left(3\mu+1, \frac{3}{2}\right)}$$

holds good, which dominates over $J_2(5) + J_3(5)$.

Now,

$$\sum_{n \leq x} a_{5,\mathbb{K}_3}(n) = xP_4(\log x) + E_{5,\mathbb{K}_3}(x),$$

where $E_{5,\mathbb{K}_3}(x) \ll \dfrac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}} + x^{\frac{1}{2}+5\varepsilon} T^{\min\left(3\mu+1, \frac{3}{2}\right)}$, i.e.,

$$E_{5,\mathbb{K}_3}(x) \ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}} + x^{\frac{1}{2}+5\varepsilon} T^{\lambda_5}.$$

We choose T such that $\dfrac{x}{T} \sim x^{\frac{1}{2}} T^{\lambda_5}$, i.e., $T^{1+\lambda_5} \sim x^{\frac{1}{2}}$, i.e., $T \sim x^{\frac{1}{2(1+\lambda_5)}}$.

So finally, we have

$$E_{5,\mathbb{K}_3}(x) \ll x^{1-\frac{1}{2(1+\lambda_5)}+15\varepsilon}.$$

5. **For $k \geq 6$,**

$$J_1(k) := \frac{1}{2\pi i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} (\zeta_{\mathbb{K}_3}(s))^k \frac{x^s}{s} ds.$$

Using Lemma 6.1.16, Cauchy-Schwarz Inequality, Lemmas 1.9.2, 1.9.3, 1.9.6 and 1.9.7 we get

$$J_1(k) \ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \log T \left\{ \max_{1 \leq U \leq T} \frac{1}{U} \int_{\frac{U}{2}}^{U} \left|\zeta\left(\frac{1}{2}+it\right)\right|^k \left|L\left(\frac{1}{2}+it, f\right)\right|^k dt \right\}$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}+\varepsilon} \left\{ \max_{1 \leq U \leq T} \frac{1}{U} U^{(k-6)(\mu+\varepsilon)} U^{(k-3)(\frac{1}{3}+\varepsilon)} \int_{\frac{U}{2}}^{U} \left|\zeta\left(\frac{1}{2}+it\right)\right|^6 \left|L\left(\frac{1}{2}+it, f\right)\right|^3 dt \right\}$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}+2k\varepsilon} \left\{ \max_{1 \leq U \leq T} U^{\mu(k-6)+\frac{1}{3}(k-3)-1} \left(\int_{\frac{U}{2}}^{U} \left|\zeta\left(\frac{1}{2}+it\right)\right|^{12} dt \right)^{\frac{1}{2}} \right.$$

$$\left. \times \left(\int_{\frac{U}{2}}^{U} \left|L\left(\frac{1}{2}+it, f\right)\right|^6 \right)^{\frac{1}{2}} dt \right\}$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}+2k\varepsilon} \left\{ \max_{1 \leq U \leq T} U^{\mu(k-6)+\frac{1}{3}(k-3)-1} U^{\frac{1}{2}(2+\varepsilon)} U^{\frac{1}{2}(2+\varepsilon)} \right\}$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}+3k\varepsilon} T^{\mu(k-6)+\frac{k}{3}}.$$

Define

$$\lambda_k := \mu(k-6) + \frac{k}{3},$$

for $k \geq 6$, then

$$J_1(k) \ll x^{\frac{1}{2}} + x^{\frac{1}{2}+3k\varepsilon} T^{\lambda_k}.$$

Here, we observe that for $k \geq 6$,

$$E_{k,\mathbb{K}_3}(x) \ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}} + x^{\frac{1}{2}+3k\varepsilon} T^{\lambda_k}.$$

We choose T such that $\frac{x}{T} \sim x^{\frac{1}{2}} T^{\lambda_k}$, i.e., $T^{1+\lambda_k} \sim x^{\frac{1}{2}}$, i.e,. $T \sim x^{\frac{1}{2(1+\lambda_k)}}$.

So finally, we have

$$E_{k,\mathbb{K}_3}(x) \ll x^{1-\frac{1}{2(1+\lambda_k)}+3k\varepsilon}.$$

This proves the theorem.

6.2.1 Remarks and Conjecture

Remark 6.2.1 From [Bo] of bourgain, we can very well take $\mu = \frac{13}{84}$. We notice that $\frac{1}{7} < \frac{13}{84} < \frac{1}{6}$. Thus, the theorem is unconditional with $\mu = \frac{13}{84}$. However if we assume the Lindelöf hypothesis for the Riemann zeta function $\zeta(s)$, namely $\mu = 0$, then from this theorem, we obtain the conditional estimates for the error term $E_{k,\mathbb{K}_3}(x)$ for each $k = 2, 3, 4, \ldots$.
It should be pointed out that for $k = 1$, the estimate in Theorem 6.1.14 is weaker than the result of Nowak [No] with $m = 3$.

Conjecture 6.2.2 For any integer $k \geq 2$ and any small positive constant ε, we have

$$E_{k,\mathbb{K}_3}(x) \ll_\varepsilon x^{\frac{1}{2}+c(k)\varepsilon},$$

where $c(k)$ is a positive constant depending only on k.

Remark 6.2.3 If we assume the Riemann hypothesis for both the L-functions $\zeta(s)$ and $L(s,f)$ (in turn the growth estimates $\zeta(\frac{1}{2}+it) \ll (|t|+10)^\varepsilon$, $L(\frac{1}{2}+it, f) \ll_f (|t|+10)^\varepsilon$), then the proof of our theorem suggests that we can even get

$$E_{k,\mathbb{K}_3}(x) \ll_\varepsilon x^{\frac{1}{2}+c(k)\varepsilon},$$

for any fixed integer $k \geq 2$ and for any small positive constant ε, where the constant $c(k)$ depending only on k. However, this seems to be far out of reach with the current knowledge of the L-function theory and hence we proposed the above conjecture.

Conclusion

In this chapter, we will wrap up by briefly addressing the topics that can be researched in the same spirit as what we tried to study in our thesis.

We have investigated the discrete mean square of the n^{th} normalized Fourier coefficients of symmetric square L-function over certain sequence of positive integers and establish an asymptotic formula for the same, in our initial work (cf. Chapter 2). It is apparent from our argument of the proof that we have only exploited the previously mentioned known analytical properties of concerned functions. With some additional information, one can even extend the result in a relatively preponderant region. In this regard we have proposed a conjecture 2.3.1.

In our subsequent work (cf. Chapter 3), we have established an asymptotic formula for the third and fourth power moments of the n^{th} normalized Fourier coefficients of symmetric square L-functions over $r_4(n)$. At the end of this chapter an analogous Conjecture 3.6.1 has been proposed, with the same idea in mind.

We have also investigated the possibility of the dimension increasing in Chapter 4 (cf. Chapter 4). Earlier, we were only concerned with $r_4(n)$. With some detailed in-depth analysis, we have developed an asymptotic formula that accurately captures the average behavior of the Fourier coefficients of the cusp form in the six-dimensional space under various circumstances.

In the very next chapter (cf. Chapter 5), we have generalized our previous work that we have done in chapter 4. The recent, well-known work of Newton and Thorne [Ne-Th 1, Ne-Th 2] made it possible. We have also employed better average or individual sub convexity bounds for the associated L-functions which lead us to the improvement of our work 4.1.2.

In our last chapter (cf. Chapter 6), we have moved our attention to a special class of L-function, popularly known as Dedekind zeta-function and worked on a divisor problem pertaining to Dedekind zeta-function. We have demonstrated an asymptotic formula for the same, with a tightened error term. Eventually, we have demonstrated that if we assume the Riemann hypothesis for both the L-functions $\zeta(s)$ and $L(s, f)$, then we can even settle the problem in question with much better error terms. With the current knowledge of the L-function theory, this appears to be very hard, which is why we put forth the conjecture 6.2.2.

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