Production and Characterisation of New Nonclassical States of Light

A Thesis
submitted for the Degree of
Doctor of Philosophy

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To my Parents

and

To Prasad

DECLARATION

I hereby declare that the matter embodied in this Thesis is the result of investigations carried out by me in the School of Physics, University of Hyderabad, Hyderabad, under the supervision of Prof. G.S. Agarwal.

Place: Hyderabad

Date: 30.8.1993

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CERTIFICATE

Certified that the work contained in this thesis entitled, **Production and Characterisation of New Nonclassical States of Light**, has been carried out by Ms. K. Tara, under my supervision and the same has not been submitted for the award of research degree of any University.

Place: Hyderabad

Date: 30.8.93

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CONTENTS

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ABSTRACT

Most part of this thesis is devoted to the study of nonclassical behavior of new states of radiation field and on how these states can be produced in practice. A summary of the important properties of the nonclassical states is presented in the introductory Chapter.

In Chapter 2, the following new class of states of the electromagnetic field are introduced:

i) $|\alpha, m> = \mathcal{N}\hat{a}^{\dagger m} | \alpha >$, where $|\alpha>$ is a coherent state and m is an integer and $\mathcal{N}$, the normalisation constant. This state can be called as photon added coherent state as the coherent state is being acted on by the photon creation operator $\hat{a}^{\dagger}$;

ii) the field in the photon added thermal state. The density operator for the field in such a state is given by $\hat{\rho} \propto a^{\dagger m} e^{-\lambda \hat{a}^{\dagger}\hat{a}} \hat{a}^m$ where λ is related to the average photon number and m is an integer;

iii) the state $|\psi, p>_+ = (\hat{a}^{\dagger}\hat{b}^{\dagger})^p \exp(\zeta^*\hat{a}\hat{b} - \zeta\hat{a}^{\dagger}\hat{b}^{\dagger}) | 0, 0 >$ which adds pairs of photons to the two mode squeezed vacuum state and the state $|\psi, p>_- = (\hat{a}\hat{b})^p \exp(\zeta^*\hat{a}\hat{b} - \zeta\hat{a}^{\dagger}\hat{b}^{\dagger}) | 0, 0 >$ which subtracts pairs of photons from the two mode squeezed vacuum state.

Various mathematical and physical properties of the field in the above mentioned states are studied in detail. Mandel's Q parameter, which is a measure of the sub-Poissonian statistics, the variance for one of field quadratures, the quadrature distribution, various quasiprobability distributions such as Glauber-Sudarshan P function, Wigner function and Q function, are studied.

Various schemes to generate these states are discussed in Chapter 3. These states can be produced in practice via state reduction methods. We consider the passage of the excited two level atoms through a micromaser cavity. The interaction of the atom with the cavity field is described by the Hamiltonian $\hbar g \mid e >< g \mid \hat{a} + h.c.$. Suppose after a short time we make the measurement and find the atom in the ground state, then the state of the field is reduced to photon added coherent state or to photon added thermal state depending on whether the field was initially in the coherent or in the thermal state. Alternatively these states can also be produced via parametric amplification in which the signal 'a' mode and the idler 'b' mode exhibit strong correlation. Assuming the signal field is initially in the coherent (thermal) state it is shown that the output signal field is reduced to photon added coherent (thermal) state when the idler mode is detected in the Fock state.

The concept of nonclassical properties of a radiation field can be best understood in terms of the P function, $P(\alpha)$, that gives a quasiprobability distribution in the phase space for a quantum state. The density operator of the radiation field can be expanded in terms of the diagonal coherent states as $\hat{\rho} = \int P(\alpha) \mid \alpha >< \alpha \mid d^2\alpha$. This quasiprobability distribution function $P(\alpha)$ can assume negative values, can become singular or may not exist. Therefore whenever this happens we say that the field is nonclassical. Unfortunately, we cannot measure the P functions directly. However the nonclassical nature of the P functions can manifest itself in many ways which can be detected in the laboratories. The two well known parameters which measures the nonclassical character of light are Mandel's Q parameter and squeezing parameter S. In Chapter 4, the question of finding new measures of nonclassical light when the usual methods based on Q and S parameter fail, are considered. A new criterion in terms of measurable quantities i.e. in terms of higher order moments of the field distribution $P(\alpha)$ is developed. The utility of this new criterion is demonstrated by applying it to the i) radiation field in photon added thermal state, ii) Schrödinger CAT state which is a coherent superposition of macroscopically distinguishable quantum mechanical states, iii) radiation field produced in a micromaser cavity.

It is well known that the statistical properties of a light field are modified by a linear optical amplifier. Thus when a signal passes through an amplifier, the amplitude of the signal gets amplified and also noise gets added to the signal. In particular, when the signal field possess nonclassical features such as sub-Poisson photon statistics or squeezing, these quantum properties can be lost after amplification when the gain is sufficiently high. The limits on the amplifier gain that are necessary and sufficient if sub-Poisson statistics or squeezing of the field are to be retained have been derived previously. Another limit in which all the nonclassical features are lost have also been derived. These limits never coincide. Thus there is a domain in which the quantum effects other than sub-Poissonian statistics or squeezing may persist. In Chapter 5, we show that the nonclassical behavior in this domain can be explained via the new criterion developed in the previous chapter.

The description of the quantum nature of optical phase has been a long standing problem. Classically, phase is a useful and easily understood concept. Quantum mechanically however one can find several descriptions of an operator corresponding to the phase of a quantum field. It is computationally advantageous to describe the phase properties of the field in terms of the phase distribution instead of phase operator. In Chapter 6, the state $\mid \theta >$ is defined in terms of number states as $\mid \theta >= \sum_{n=0}^{\infty} e^{\mathrm{i}n\theta} \mid n >$ and phase distribution $\mathcal{P}(\theta)$ is associated to a given density operator $\hat{\rho}$ as $\mathcal{P}(\theta) =< \theta \mid \hat{\rho} \mid \theta > /2\pi, 0 < \theta \leq 2\pi$. Using this definition the evolution of the phase distribution associated with the field as it propagates in a nonlinear medium, like an optical fiber, is examined. Classically, as the coherent field propagates through the fiber it is seen that the phase of the field shifts, the shift being proportional to the mean intensity of the field. Quantum mechanically, the numerical computations for the phase distribution shows that the phase not only shifts but also diffuses as it propagates through the fiber. The phase distribution of the field reflected from the phase conjugate mirror(PCM) is also investigated. Classically the PCM transforms the phase ϕ of the incident field to the phase $-\phi$ after reflection. Quantum mechanically it is found that the phase ϕ of the incident input field goes to

$-\phi$ after reflection and also the phase distribution becomes broader. This is attributed to the addition of the noise photon by PCM. The question of defining phase distribution via different quasiprobabilities associated with the field is also examined.

One of the outcomes of our study of the quantum phases is the discovery of the possibility of producing the discrete superposition of coherent states or so called Schrödinger kitten states by a coherent field propagating through a Kerr medium, like an optical fiber, provided that the propagation length L and the nonlinear susceptibility χ of the fiber are such that $\chi L/c = \pi/m$, where m is an integer. In Chapter 7, the evolution of the phase distribution associated with this state is examined. It is found that the distribution breaks up into multiple peaks. We investigate the squeezing properties and the phase space distributions like Q function and Wigner function for these states. We also discuss in this chapter the production of superpositions of squeezed coherent states.

The phase distribution of a classical oscillator is well known and is obtained by solving the diffusion equation. In Chapter 8, the question of what is the phase distribution for a quantum oscillator undergoing phase diffusion, is addressed. One can, for example consider a single mode laser well above threshold. Then the state of the system is given by coherent state $\mid \alpha >$ where the magnitude $\mid \alpha \mid$ is constant but phase diffuses i.e. $\alpha = \mid \alpha \mid e^{-i\phi(t)}$ where $\phi(t)$ is the solution of classical diffusion equation. That is, $\phi \equiv \omega(t)$ is treated as Gaussian random process which in addition is delta correlated. The quantum mechanical phase distribution is examined and conditions under which such a distribution goes over to classical distribution are obtained.

Finally an Appendix is devoted to the study of the correlations between the quadrature distributions of a field in a pair coherent state which exhibits strong nonclassical properties.

Chapter 1

INTRODUCTION

Quantum statistical properties of laser light have been extensively studied during the last 30 years. It is well known that an ideal laser pumped at well above the oscillation threshold generates a coherent state of light. The term coherent state was first coined by Glauber in his two seminal papers [1], where he defined them as an eigenstate of the annihilation operator of the electromagnetic field. Recently, a vast literature related to the coherent states and their applications has been exhaustively collected and catalogued by Klauder and Skagerstam [2]. The statistical properties of the electromagnetic field in a coherent state were found to be similar to those obtained from a fluctuating classical field. The photon number (counting) distribution of the field in a coherent state was shown to be Poissonian.

During the past two decades, with the development of more sophisticated experimental techniques, it has become possible to investigate in the laboratory a very fine scale phenomena in the nonlinear interaction of field with matter. It became possible to conceive new forms of radiation field that had never before been realised. The most prominent examples of such fields are squeezed and antibunched fields and fields with sub-Poisson statistics. The statistical properties of these fields cannot be described using a classical description but can be understood only by a quantum mechanical description. Thus the radiation field with certain characteristics that can be understood only by a quantum mechanical description are called nonclassical fields. Squeezing, antibunching and sub-Poisson statistics are the three manifestations of the nonclassical nature of the field. In any given radiation field these characteristics may, but need not accompany each other [3,4].

We now give a brief introduction to different types of nonclassical phenomena.

1.1. Squeezed States

A quantised single mode electromagnetic field of frequency ω can be expressed in terms of a photon creation operator $\hat{a}^\dagger$ and a photon annihilation operator $\hat{a}$, as

$$\hat{E}(t) = \epsilon_o(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}) \quad , \tag{1.1}$$

where ϵ_o contains the spatial factors. The operators $\hat{a}$ and $\hat{a}^\dagger$ obey the usual Boson commutation relationship.

$$[\hat{a}, \hat{a}^\dagger] = 1 \ . \tag{1.2}$$

Equation (1.1) can be rewritten in term of two quadrature operators $\hat{x}$ and $\hat{p}$ as

$$\hat{E}(t) = \epsilon_o\sqrt{2}\,[\hat{x}\cos(\omega t + \theta) + \hat{p}\sin(\omega t + \theta)] \tag{1.3}$$

where

$$\hat{x} = \frac{\hat{a}e^{i\theta} + \hat{a}^\dagger e^{-i\theta}}{\sqrt{2}} \ ; \ \hat{p} = \frac{\hat{a}e^{i\theta} - \hat{a}^\dagger e^{-i\theta}}{\sqrt{2}i} \quad . \tag{1.4}$$

It follows from the commutation relation (1.2) that

$$[\hat{x}, \hat{p}] = i \ . \tag{1.5}$$

In principle the quadrature operators $\hat{x}$ and $\hat{p}$ can be separately measured using phase sensitive detector techniques - such as homodyne detection. For a given state, the variance of the quadrature operators which corresponds to the fluctuations or quantum noise are defined as

$$(\Delta x)^2 \ = \ <\hat{x}^2> - <\hat{x}>^2 \ ; \ (\Delta p)^2 \ = \ <\hat{p}^2> - <\hat{p}>^2 \ , \tag{1.6}$$

where the brackets < . > denotes averaging with respect to a particular state of the field. From equation (1.5), the Heisenberg uncertainty relation for the quadratures is

$$(\Delta x)^2(\Delta p)^2 \geq \frac{1}{4} \quad , \tag{1.7}$$

and if the equality sign holds in the above equation i.e.,

$$(\Delta x)^2(\Delta p)^2 = \frac{1}{4} \quad , \tag{1.8}$$

then those states are called the minimum uncertainty states. For example, for the coherent state or for the vacuum state of the field, the variances are equal

$$(\Delta x)^2 = (\Delta p)^2 = \frac{1}{2} \quad . \tag{1.9}$$

These are the minimum uncertainty states. The squeezed states belongs to the broader class of minimum uncertainty states which have unequal variances in each quadrature, i.e.,

$$\text{either } (\Delta x)^2 \text{ or } (\Delta p)^2 < \frac{1}{2} \quad . \tag{1.10}$$

The term squeezed state was first used by Hollenhorst in 1979 [5]. Thus squeezed states are characterised by reduced quantum fluctuations in one quadrature component of the field at the expense of increased fluctuations in the other noncommuting component. This remarkable property of squeezed field has no classical interpretation and makes sense only in models where the nonlinear medium and the radiation fields are treated quantum mechanically. Theoretical predictions have shown that squeezing of quantum fluctuation can occur in a variety of nonlinear optical phenomena, in particular, four wave mixing [6,7], parametric amplification [8-10], harmonic generation [11-13], multiphoton absorption process [14,15], optical bistability [16] etc. Squeezed number states (introduced by Yuen [9]), squeezed coherent states and squeezed thermal states are some of the well known squeezed states. Over the past decade considerable effort has been put into experiments aiming at the generation and detection of squeezed states of the field. In 1985, the theoretically conceived squeezed light has been produced in the laboratories. Slusher et. al. in 1985 performed a four wave mixing experiment in sodium vapour to generate squeezing inside a resonant cavity [17]. Squeezed state generation has been reported by many other groups since then [18-25].

1.2. Photon Antibunching and sub-Poisson Statistics.

Fluctuations of the electromagnetic field at time t and $t+\tau$ are characterised by its first order (amplitude) and the second order (intensity) correlation functions. These correspond to the quantum mechanical expectation values

$$g^1(\tau) = \frac{Tr\{\hat{\rho}\hat{a}^\dagger(t)\hat{a}(t+\tau)\}}{[Tr\{\hat{\rho}\hat{a}^\dagger(t)\hat{a}(t)\}Tr\{\hat{\rho}\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\}]^{1/2}} , \quad (1.11)$$

$$g^2(\tau) = \frac{Tr\{\hat{\rho}\hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\hat{a}(t)\}}{Tr\{\hat{\rho}\hat{a}^\dagger(t)\hat{a}(t)\}Tr\{\hat{\rho}\hat{a}^\dagger(t+\tau)a(t+\tau)\}} , \quad (1.12)$$

respectively. Here $\hat{\rho}$ is the density operator of the field. Classically, the correlation functions satisfy the following identities.

$$\begin{aligned} 0 &\leq \left|g^1(\tau)\right| \leq 1 \\ g^2(0) &\geq 1 \\ g^2(\tau) &\leq \left[g^2(t)g^2(t+\tau)\right]^{1/2} . \end{aligned} \quad (1.13)$$

In general, radiation field cannot be unambiguously defined by a first order correlation function, and hence higher order correlation functions are often needed [26]. The first experiment to measure second order correlation function $g^2(\tau)$ was performed by Hanbury-Brown and Twiss in 1956 [27]. Generally photodetector is used to measure the intensity correlation function $g^2(\tau)$. Such a correlation represents the normalised coincidence rate of photon counts. More precisely, it gives the joint probability of detecting one photon at t and another photon at time $t + \tau$, normalised by the product of the marginal probabilities of occurrence of independent counts at t and $t + \tau$. In the limit when $\tau \rightarrow 0$, the normalised coincidence rate is measured by the function $g^2(0)$. The behavior of $g^2(0)$ enables us to determine whether the field exhibits photon bunching or photon antibunching. It is well known that the coincidence rate $g^2(0) = 1$ for a coherent field, signifying that the photon counts occur independently. Photon counts are then said to be unbunched. When $g^2(0) > 1$, occurrence of counts at the two points are positively correlated, i.e., when one occurs other is more likely to occur. Photon counts are then said to be bunched as photons have tendency to cluster together. For example, this effect occurs for light from a thermal source such as light bulb. Alternatively, when $g^2(0) < 1$, photon counts are anticorrelated, i.e. when one occurs, the other is less likely to occur. Photon counts are then said to be antibunched, i.e. they tend to be separated. The condition $g^2(0) < 1$ for photon antibunching, violates the inequality given by (1.13) for classical fields. Thus, photon antibunching is a phenomenon which can occur only for quantum fields.

It was first shown by Kimble and Mandel [28] and Carmichael and Walls [29] in 1976 that the light generated in resonance fluorescence from a single two level atom driven by a laser light would exhibit the features of photon antibunching and sub-Poisson photon statistics. It was observed experimentally by Kimble et. al [30] in 1977. Antibunching was the first characteristic of a nonclassical field to be observed in the laboratory. Since then these features have been observed in many systems [31-37].

Another measure of the quantum statistical properties of a radiation field is given by photon number distribution $P(N, \tau)$, the probability the detector registers N photons at time τ. This quantity is well characterised by a statistical parameter known as Mandel's

Q parameter [38]. It is used to compare the variance of the photon number with the mean value i.e., it provides a relative measure of the fluctuations in the photon count.

$$Q = \frac{<\hat{N}^2> - <\hat{N}>^2}{<\hat{N}>} - 1 \ , \tag{1.14}$$

where the number operator $\hat{N} = \hat{a}^\dagger \hat{a}$ and

$$<\hat{N}^k> = \sum_{N=0}^{\infty} N^k P(N,\tau) \ . \tag{1.15}$$

In terms of the creation and the annihilation operators, Q can be expressed as

$$Q = \frac{<\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}> - <\hat{a}^\dagger\hat{a}>^2}{<\hat{a}^\dagger\hat{a}>} = \frac{<:(\Delta N)^2:>}{<\hat{N}>} \ , \tag{1.16}$$

where :: denotes normal ordering of the operators, i.e, the annihilation operator is always to the right of the creation operator. In a classical field, we always have

$$(\Delta N)^2 \ = \ <\hat{N}^2> - <\hat{N}>^2 \ \geq <\hat{N}> \ . \tag{1.17}$$

Fields for which $(\Delta N)^2 \ > \ \langle\hat{N}\rangle$ are said to exhibit super-Poisson photon statistics. However, in a quantum field, it is possible to see the direction of the sign of the inequality (1.17) reversed. Thus fields for which

$$(\Delta N)^2 \ < \ \langle\hat{N}\rangle \ , \tag{1.18}$$

or equivalently $Q < 0$ is said to exhibit sub-Poisson photon statistics. It is well known that the photon number statistics of a coherent field produced by a highly stabilised laser has the Poisson distribution. It is characterised by the variance of the photon number equal to the mean value

$$(\Delta N)^2 \ = \ \langle\hat{N}\rangle \tag{1.19}$$

implying $Q = 0$ in (1.14). So a coherent state of the field stands on the borderline between classical and nonclassical states.

The reason that the squeezing in one of the quadrature phases of the field and sub-Poisson photon statistics are the manifestation of the nonclassical character of the electromagnetic field can be seen by considering the Glauber-Sudarshan P distribution function, $P(\alpha)$ [1,39]. This function gives the quasiprobability distribution in the phase space for a given quantum state. It is called quasiprobability distribution because it can take negative values which makes no sense if the distribution corresponds to a classical field. Therefore the condition for the field to be nonclassical is that its P distribution does not possess the properties of a classical probability distribution. It can be highly singular (more singular than a δ function) or even negative.

The density operator of the radiation field can be expanded in the diagonal coherent states as

$$\hat{\rho} = \int P(\alpha)|\alpha >< \alpha| d^2\alpha \quad , \tag{1.20}$$

where $d^2\alpha = d(Re\alpha)d(Im\alpha)$. The fluctuations in the quadrature operators $(\Delta x)^2$ and $(\Delta p)^2$ as given by Eqs. (1.6) and Mandel's Q parameter given by Eq.(1.14) can be written in terms of P distribution function

$$\begin{aligned} (\Delta x)^2 &= \frac{1}{2}\left\{1 + \int d^2\alpha P(\alpha)\left[(\alpha+\alpha^*) - (<\alpha> + <\alpha^*>)\right]^2\right\} \\ (\Delta p)^2 &= \frac{1}{2}\left\{1 + \int d^2\alpha P(\alpha)\left[\frac{(\alpha-\alpha^*)}{i} - \frac{(<\alpha> - <\alpha^*>)}{i}\right]^2\right\} \quad , \end{aligned} \tag{1.21}$$

and

$$Q = \frac{\int d^2\alpha P(\alpha)\left[|\alpha|^2 - <|\alpha|^2>\right]^2}{\int d^2\alpha P(\alpha)|\alpha|^2} \quad , \tag{1.22}$$

where $<\alpha> = \int d^2\alpha P(\alpha)\alpha$.

If $P(\alpha)$ had properties of a classical probability distribution then

$$(\Delta x)^2 > \frac{1}{2} \quad , \quad (\Delta p)^2 > \frac{1}{2} \quad , \quad Q > 0 \; .$$

The condition for squeezing (1.10) or sub-Poisson statistics (1.18) requires that the distribution function $P(\alpha)$ possess nonclassical properties.

An optical field behaves as a classical field when the probability distribution $P(\alpha)$ is always positive definite. For example if we consider a coherent state $|\beta>$ which corresponds as closely as possible to a classical state of definite complex amplitude we find that

$$P(\alpha) = \delta^2(\alpha - \beta) \tag{1.23}$$

where the two dimensional delta function is given by

$$\delta^2(\alpha - \beta) = \delta(Re(\alpha - \beta))\delta(Im(\alpha - \beta)) \; . \tag{1.24}$$

An example of a nonclassical state is a field in the Fock state $|n>$ whose P representation is given by

$$P(\alpha) = \frac{e^{|\alpha|^2}}{n!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\delta^2(\alpha) \tag{1.25}$$

which contains higher order derivatives of the delta function.

Recently the nonclassical fields have attracted increased attention, as such fields are connected with the prospects of solving a whole series of fundamental and applied problems of physics. Possible potential application for such fields are in optical communication [40, 41], interferometry [42-46] spectroscopy [47], noise free amplification [48]. Ref [49-56] gives a list of excellent review articles on the nonclassical states of the field.

The main aim of this thesis is to study the nonclassical features of various states of the electromagnetic field. We construct new types of states of the electromagnetic field and show that these states exhibit nonclassical behavior. We also discuss various schemes to generate these states. A new criterion, in terms of measurable quantities, to test the nonclassical nature of the field even if it does not exhibit squeezing and sub-Poisson statistics is developed. The utility of this criterion is demonstrated by successfully applying it to various states of the electromagnetic field. We study the evolution of the quantum phase distribution as it propagates through a nonlinear Kerr medium like an optical fiber. We also study the quantum phase distribution of the reflected field from a phase conjugate mirror. We show that under suitable conditions a field initially in the coherent state, propagating through an optical Kerr medium, can evolve into a discrete superposition of coherent states or playfully referred to as Schrödinger kitten state. We discuss the quantum mechanical distributions for the phase of an oscillator undergoing diffusion. Finally we study the correlations between the quadrature distribution of a field in a pair coherent state.

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Chapter 2

NONCLASSICAL PROPERTIES OF NEW TYPE OF STATES

In this chapter we introduce three new types of states of the electromagnetic field which can be produced by variety of nonlinear processes in the cavities. In Section 2.1 we construct a new class of states obtained by repeated application of the photon creation operator on the coherent state. The field in the photon added thermal state is considered in Section 2.2 and in Section 2.3 we define another type of state which can be obtained by addition or subtraction of pairs of photons from the two mode squeezed vacuum states. We study the mathematical and physical properties of these above mentioned states and show that they exhibit nonclassical properties such as squeezing and sub-Poisson statistics. We also calculate different quasiprobability distribution functions such as Wigner function, Q function and P function.

2.1. Photon Added Coherent State (PACS)

In this section we introduce the state obtained by the repeated application of the photon creation operator $\hat{a}^\dagger$ on the coherent state $|\alpha>$,

$$|\alpha, m> \;\equiv\; \sqrt{\mathcal{N}}\, \hat{a}^{\dagger m}|\alpha> \;, \quad m = \text{integer} \tag{2.1}$$

where $\mathcal{N}$ is the normalisation constant given by

$$\mathcal{N} \;=\; [<\alpha|\hat{a}^m \hat{a}^{\dagger m}|\alpha>]^{-1}. \tag{2.2}$$

This can be evaluated by using normal ordering of the operator $\hat{a}^m \hat{a}^{\dagger m}$.

$$\hat{a}^m \hat{a}^{\dagger m} \;=\; \sum_{p=o}^{m} \frac{(m!)^2}{(m-p)^2!p!} \hat{a}^{\dagger m-p} \hat{a}^{m-p} \quad . \tag{2.3}$$

Thus

$$< \alpha|\hat{a}^m \hat{a}^{\dagger m}|\alpha > \; = \; \sum_{p=o}^{m} \frac{m!^2}{(m-p)!^2 p!} |\alpha|^{2(m-p)}$$

$$= \; L_m(-|\alpha|^2) m! \; , \tag{2.4}$$

where $L_m(x)$ is the Laguerre polynomial of order m defined by [1]

$$L_m(x) = \sum_{n=0}^{m} \frac{(-1)^n x^n m!}{(n!)^2 (m-n)!} \; . \tag{2.5}$$

The state $|\alpha, m>$ can then be written as

$$|\alpha, m> \; \equiv \; \frac{\hat{a}^{\dagger m}|\alpha>}{[m! L_m(-|\alpha|^2)]^{1/2}} \; . \tag{2.6}$$

In the limit $\alpha \rightarrow 0$ this state reduces to the Fock state

$$|\alpha, m> = |0, m> \equiv |m> \quad ,$$

and in the limit $m \rightarrow 0$, it reduces to the coherent state

$$|\alpha, m> = |\alpha, 0> \; \equiv \; |\alpha> \quad .$$

Thus the state $|\alpha, m>$ is intermediate between the Fock state and the coherent state. This state is not to be confused with the eigenstate of the displaced harmonic oscillator [2-4] given by

$$D(\alpha)|m> \; \equiv \; \exp(\alpha\hat{a}^\dagger - \alpha^* \hat{a})|m> \quad . \tag{2.7}$$

Thus

$$\sqrt{\mathcal{N}} \hat{a}^{\dagger m} D(\alpha)|0> \; \neq \; \frac{D(\alpha)\hat{a}^{\dagger m}|0>}{\sqrt{m!}} \; ,$$

because the operators $D(\alpha)$ and $\hat{a}^{\dagger m}$ do not commute.

Using the unitarity property of $D(\alpha)$ the state $|\alpha, m>$ can be written as superposition of the eigenstates of the displaced harmonic oscillator as follows

$$\begin{aligned}
\hat{a}^{\dagger m}|\alpha> &\equiv \hat{a}^{\dagger m} D(\alpha)|0> \\
&= D(\alpha)D^{-1}(\alpha)\hat{a}^{\dagger m} D(\alpha)|0> \\
&= D(\alpha)(\hat{a}^{\dagger} + \alpha^{*})^{m}|0> \\
&= \sum_{p=o}^{m} \binom{m}{p} \alpha^{*m-p} D(\alpha)\hat{a}^{\dagger p}|0> \quad .
\end{aligned}$$

Hence

$$|\alpha, m> = \sqrt{\mathcal{N}} \sum_{p=o}^{m} \binom{m}{p} \alpha^{*m-p} \sqrt{p!} D(\alpha)|p> \quad , \tag{2.8}$$

where

$$\binom{m}{p} = \frac{m!}{(p-m)!p!} \ ,$$

and we have used the property $D^{-1}(\alpha)\hat{a}^{\dagger} D(\alpha) = \hat{a}^{\dagger} + \alpha^{*}$. In terms of Fock states, the state $|\alpha, m>$ looks like

$$\begin{aligned}
|\alpha, m> &= \sqrt{\mathcal{N}} \sum_{n=0}^{\infty} \frac{\hat{a}^{\dagger m} e^{-|\alpha|^2/2} \alpha^n |n>}{\sqrt{n!}} \\
&= \sqrt{\mathcal{N}} e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n \sqrt{(n+m)!}}{n!} |n+m> \quad .
\end{aligned} \tag{2.9}$$

A. Squeezing Properties

We next examine the fluctuations characteristics of this state. We consider the field quadrature operator $\hat{x}$ defined by

$$\hat{x} = \frac{\hat{a}e^{i\theta} + \hat{a}^{\dagger} e^{-i\theta}}{\sqrt{2}} \quad . \tag{2.10}$$

The mean value of $\hat{x}$ in the state $|\alpha, m>$ is

$$\begin{aligned}
<\hat{x}> &= \frac{\mathcal{N}}{\sqrt{2}} < \alpha, m|\hat{a}e^{i\theta} + \hat{a}^{\dagger} e^{-i\theta}|\alpha, m> \\
&= \frac{\mathcal{N}}{\sqrt{2}} \left[< \alpha|\hat{a}^{m+1}\hat{a}^{\dagger m}|\alpha > e^{i\theta} + < \alpha|\hat{a}^{m}\hat{a}^{\dagger m+1}|\alpha > e^{-i\theta} \right] \quad .
\end{aligned} \tag{2.11}$$

On using the normal ordering of $\hat{a}^m\hat{a}^{\dagger m}$ as given by Eq. (2.3) and the commutation relation,

$$[\hat{a}, \hat{a}^{\dagger n}] = n\hat{a}^{\dagger(n-1)} \ ; \ [\hat{a}^n, \hat{a}^\dagger] = n\hat{a}^{(n-1)} \ , \tag{2.12}$$

the Eq. (2.11) becomes

$$\begin{aligned} < \hat{x} > &= \frac{\mathcal{N}}{\sqrt{2}}(\alpha e^{i\theta} + \alpha^* e^{-i\theta}) \sum_{p=o}^{m} \frac{(m+1)!|\alpha|^{2(m-p)}}{(m-p)!p!(p+1)!} \\ &= \frac{\sqrt{2}|\alpha| \cos(\theta+\varphi) L_m^{(1)}(-|\alpha|^2)}{L_m(-|\alpha|^2)} \ \text{, where } \alpha = |\alpha|e^{i\varphi} \end{aligned} \tag{2.13}$$

and $L_m^{(k)}(x)$ is the associated Laguerre polynomials defined by [1]

$$L_m^{(k)}(x) = \sum_{n=0}^{m} \frac{(m+k)!}{(m-n)!n!(k+n)!}(-x)^n \ . \tag{2.14}$$

Similarly making use of the commutation relation $[\hat{a}^2, \hat{a}^{\dagger n}]$ and $[\hat{a}^n, \hat{a}^{\dagger 2}]$, the average of square of the field quadrature can be obtained as

$$\begin{aligned} < \hat{x}^2 > &= \langle \alpha, m | \hat{a}^2 e^{2i\theta} + \hat{a}^{\dagger 2} e^{-2i\theta} + 2\hat{a}\hat{a}^\dagger - 1 | \alpha, m \rangle \\ &= \frac{2L_m^{(2)}(-|\alpha|^2)|\alpha|^2 \cos 2(\theta+\phi) + 2(m+1)L_{m+1}(-|\alpha|^2) - L_m(-|\alpha|^2)}{2L_m(-|\alpha|^2)} \end{aligned} \tag{2.15}$$

Using Eqs.(2.13) and (2.15) we can calculate the variance of the quadrature operator in the state $|\alpha, m>$ as

$$\begin{aligned} (\Delta x)^2 &= < \hat{x}^2 > - < \hat{x}^2 > \\ &= (\{L_m^{(2)}(-|\alpha|^2)L_m(-|\alpha|^2) - [L_m^{(1)}(-|\alpha|^2)]^2\}2|\alpha|^2 \cos[2(\theta+\phi)] - 2[L_m^{(1)}(-|\alpha|^2)]^2 \\ &\quad \frac{|\alpha|^2 - [L_m(-|\alpha|^2)]^2 + 2(m+1)L_{m+1}(-|\alpha|^2)L_m(-|\alpha|^2))}{2[L_m(-|\alpha|^2)]^2} . \end{aligned} \tag{2.16}$$

For $(\theta + \varphi) = 0$ the fluctuation $(\Delta x)^2$ is found to be minimum. For $m \neq 0, \alpha = 0$ (Fock state) $(\Delta x)^2$ reduces to $m + 1/2$ as expected. For $m = 0$, i.e. when the state $|\alpha, m>$ reduces to the coherent state and also for $m = 0, \alpha = 0$ (vacuum state), the value of

$(\Delta x)^2$ in Eq.(2.16) becomes equal to $\frac{1}{2}$, as was discussed in Chapter 1. We define a quantity

$$S_x = 2(\Delta x)^2 \ . \tag{2.17}$$

The quadrature operator $\hat{x}$ is said to be squeezed for values of $(\Delta x)^2 < 1/2$ or equivalently for $S_x < 1$. In Fig.(2.1) we show how S_x varies as a function of the parameter $|\alpha|$ for different values of m. It is seen that S_x is almost equal to half for wide range of parameters implying that we get almost 50% squeezing.

B. Sub-Poissonian Character

We next calculate the number distribution of the field in the state $|\alpha, m>$. The probability of finding n photons in the field is given by

$$\begin{aligned} p(n) &= |<n|\alpha, m>|^2 \\ &= \frac{|<n-m|\alpha>|^2 n!}{L_m(-|\alpha|^2)m!(n-m)!} \ , \\ &= \frac{n!|\alpha|^{2(n-m)}e^{-|\alpha|^2}}{((n-m)!)^2 L_m(-|\alpha|^2)m!} \ , \end{aligned} \tag{2.18}$$

which is zero for $n < m$. This distribution is found to have variance which is less than that for Poisson distribution. To see this, we calculate Q parameter defined by

$$Q = \frac{(\Delta n^2)}{<\hat{n}>} - 1 \quad , \ \hat{n} = \hat{a}^\dagger \hat{a} \ , \tag{2.19}$$

introduced by Mandel [5] which is a good measure of the extent to which the photon statistics of a state is sub-Poissonian. When the value of Q is negative, the state is said to have sub-Poisson photon statistics. For Poissonian statistics Q is equal to zero.

The mean number of photons for the state $|\alpha, m>$ is given by

$$\begin{aligned}
\bar{n} &= <\hat{a}^\dagger\hat{a}> = <\hat{a}\hat{a}^\dagger> -1 \\
&= \frac{<\alpha|\hat{a}^{m+1}\hat{a}^{\dagger m+1}|\alpha>}{L_m(-|\alpha|^2)m!} - 1 \\
&= \frac{(m+1)L_{m+1}(-|\alpha|^2)}{L_m(-|\alpha|^2)} - 1 \ . \qquad (2.20)
\end{aligned}$$

The second moment $<(\hat{a}^\dagger\hat{a})^2>$ can be calculated by expressing $(\hat{a}^\dagger\hat{a})^2$ in terms of antinormal ordered form

$$\langle(\hat{a}^\dagger\hat{a})^2\rangle = \langle\hat{a}^2\hat{a}^{\dagger 2} - 3\hat{a}\hat{a}^\dagger + 1\rangle \ . \qquad (2.21)$$

The expectation values in Eq.(2.21) can be evaluated straight forward by using

$$\begin{aligned}
<\alpha, m|\hat{a}^n\hat{a}^{\dagger n}|\alpha, m> &= \frac{<\alpha|\hat{a}^{n+m}\hat{a}^{\dagger n+m}|\alpha>}{m!L_m(-|\alpha|^2)} \\
&= \frac{(n+m)!L_{n+m}(-|\alpha|^2)}{m!L_m(-|\alpha|^2)} \ . \qquad (2.22)
\end{aligned}$$

With the use of Eqs.(2.20)-(2.22) the Q parameter turns out to be

$$Q = \frac{\{[(m+2)L_{m+2}(-|\alpha|^2) - L_{m+1}(-|\alpha|^2)](m+1)L_m(-|\alpha|^2) - [(m+1)L_{m+1}(-|\alpha|^2)]^2\}}{L_m(-|\alpha|^2)[(m+1)L_{m+1}(-|\alpha|^2) - L_m(-|\alpha|^2)]} - 1 \qquad (2.23)$$

In Fig.(2.2) we show the mean number of photons (Eq.(2.20)) for different values of $|\alpha|^2$ and m. The Q parameter as a function of $|\alpha|$ for different values of m is displayed in Fig.(2.3). When the state reduces to coherent state, i.e., for $m = 0$, Q is equal to 0 and for Fock state ($\alpha = 0$), $Q = -1$ as expected. For $m \neq 0, \alpha \neq 0$ we see that the value of Q is negative implying that the state exhibits a significant amount of sub-Poissonian photon statistics.

C. Quasi Probability Distribution Functions

In quantum optics quasiprobabiulity distribution functions such as the Glauber-Sudarshan P function, the Q function and the Wigner function play an important role. With the help of these functions expectation values of any products of creation operators $\hat{a}^\dagger$ and annihilation operators $\hat{a}$ can be calculated.

(i) P function

Glauber [6] and Sudarshan [7], independently, have introduced the P representation for the probability density. This function is defined by

$$\hat{\rho} = \int P(\alpha)|\alpha><\alpha|d^2\alpha \ , \tag{2.24}$$

where $|\alpha>$ is a coherent state. It can also be defined as the Fourier transform of the normally ordered characteristic function i.e.,

$$P(z) = \frac{1}{\pi^2}\int Tr\left[\hat{\rho}e^{\beta\hat{a}^\dagger}e^{-\beta^*\hat{a}}\right]\exp(z\beta^* - z^*\beta)d^2\beta \ . \tag{2.25}$$

The Glauber-Sudarshan P function associated with the state $|\alpha, m>$ can be calculated using the inversion [8] formula

$$\begin{aligned} P(z) &= \frac{\exp(|z|^2)}{\pi^2}\int d^2\beta <-\beta|\alpha,m><\alpha,m|\beta> \exp\left[|\beta|^2 - (\beta z^* - \beta^* z)\right] \\ &= \frac{\exp(|z|^2)}{\pi^2 L_m(-|\alpha|^2)m!}\int d^2\beta(-\beta\beta^*)^m \exp\left[-|\alpha|^2 + (z-\alpha)\beta^* - (z-\alpha)^*\beta\right] \\ &= \frac{\exp(|z|^2 - |\alpha|^2)}{m!L_m(-|\alpha|^2)}\ \frac{\partial^{2m}}{\partial z^{*m}\partial z^m}\delta^2(z-\alpha) \ . \end{aligned} \tag{2.26}$$

Thus the P function is highly singular which is quite typical of states exhibiting nonclassical nature.

(ii) Q function

The Q representation is defined as the Fourier transform of the antinormally ordered characteristic function

$$Q(z) = \frac{1}{\pi^2} \int Tr \left[\hat{\rho} e^{-\beta^* \hat{a}} e^{\beta \hat{a}^\dagger} \right] \exp(z\beta^* - z^*\beta) d^2\beta \ , \tag{2.27}$$

which can be shown to be the absolute magnitude squared of the projection of a state of the field onto a coherent state [9-11] i.e.

$$Q(z) = \frac{1}{\pi} < z|\hat{\rho}|z > \ . \tag{2.28}$$

This is particularly useful in calculating the antinormally ordered expectation values. For example

$$\langle \hat{a}\hat{a}^\dagger \rangle = \frac{1}{\pi} \int d^2\alpha |\alpha|^2 Q(\alpha). \tag{2.29}$$

Thus, the Q function for the field in the state $|\alpha, m>$ is

$$\begin{aligned} Q(z) &= \frac{1}{\pi} < z|\alpha, m >< \alpha, m|z > \\ &= \frac{|z|^{2m}}{m! L_m(-|\alpha|^2)} \exp(-|z-\alpha|^2) \ . \end{aligned} \tag{2.30}$$

which is no longer centered at $z = \alpha$.

(iii) Wigner function

Wigner was the first to propose the construction of a phase space distribution function from quantum mechanical wave function [12]. It is defined as the Fourier transform of the symmetrically ordered characteristic function.

$$\begin{aligned} W(z) &= \frac{1}{\pi^2} \int Tr[\hat{\rho} D(\beta)] \exp(z^*\beta - z\beta^*) d^2\beta \ , \\ &= \frac{1}{\pi^2} \int Tr \left[\hat{\rho} e^{(\beta\hat{a}^\dagger - \beta^*\hat{a})} \right] \exp(z^*\beta - z\beta^*) d^2\beta \ . \end{aligned} \tag{2.31}$$

It can also be evaluated in terms of coherent state matrix elements by using the formula [13].

$$W(z) = \frac{2}{\pi^2}\exp(2|z|^2)\int d^2\beta\langle-\beta|\alpha,m\rangle\langle\alpha,m|\beta\rangle\exp[2(\beta^* z - \beta z^*)] \quad , \tag{2.32}$$

which on simplification reduces to

$$\begin{aligned} W(z) &= \frac{2\exp(2|z|^2 - |\alpha|^2)}{\pi^2 m! L_m(-|\alpha|^2)} \\ &\times \int d^2\beta(-\beta^*\beta)^m \exp[-|\beta|^2 + \beta^*(2z-\alpha) - \beta(2z-\alpha)^*] \quad . \end{aligned} \tag{2.33}$$

and therefore the Wigner function for the state $|\alpha, m>$ is, with $\xi = 2z - \alpha$

$$\begin{aligned} W(z) &= \frac{2\exp(2|z|^2 - |\alpha|^2)}{\pi m! L_m(-|\alpha|^2)}\frac{\partial^{2m}}{\partial\xi^{*m}\partial\xi^m}\frac{1}{\pi}\int d^2\beta\exp(-|\beta|^2 + \beta^*\xi - \beta\xi^*) \\ &= \frac{2\exp(2|z|^2 - |\alpha|^2)}{\pi m! L_m(-|\alpha|^2)}\frac{\partial^{2m}}{\partial\xi^{*m}\partial\xi^m}\exp(-|\xi|^2) \\ &= \frac{2(-1)^m\exp(2|z|^2 - |\alpha|^2)}{\pi m! L_m(-|\alpha|^2)}\exp(-|\xi|^2)L_m(|\xi|^2)m! \, , \\ &= \frac{2(-1)^m \mathrm{L}_m(|2z-\alpha|^2)}{\pi L_m(-|\alpha|^2)}\exp(-2|z-\alpha|^2) \quad . \end{aligned} \tag{2.34}$$

It is obvious from Eq.(2.34) that the Wigner function can become negative. This crosses zero whenever $L_m(|2z - \alpha|^2) = 0$. For $m = 0(\alpha = 0)$, the expression in Eq.(2.34) reduces to that for a coherent state (number state). In Figs. 2.4a and 2.4b we show the Wigner function as function of $z = x + iy$ for $m = 1$ and $m = 5$ respectively. $W(z)$ shows minimum for some values of y in a fixed range of x. For example, $W(z)$ for $m = 1$ is minimum at $y = 0$ and for x in the range given by $0.1 < x < 1.87$ for $\alpha_1 = 2$ and $\alpha_2 = 0(\alpha = \alpha_1 + i\alpha_2)$. This is due to the Laguerre polynomial in the numerator in Eq.(2.33). Fig. 2.4b ($m = 5$) also shows regions where the Wigner function is negative.

D. The Distribution $p(x)$ of the Field Quadrature Operator $\hat{x}$

The probability distribution $p(x)$ associated with field quadrature operator $\hat{x}$ can be obtained from Eq.(2.34). The distribution $p(x)$ is defined by [14]

$$p(x) = \int_{o}^{\infty} W(x+iy)dy \quad . \tag{2.35}$$

On using Eqs.(2.34) and (2.35) and $\alpha = \alpha_1 + i\alpha_2$, we get

$$\begin{aligned} p(x) &= \frac{2\exp[-2(x-\alpha_1)^2](-1)^m}{\pi L_m(-|\alpha|^2)} \int_{-\infty}^{\infty} dy \exp[-2(y-\alpha_2)^2] \\ &\times \; L_m((2x-\alpha_1)^2 + (2y-\alpha_2)^2) \end{aligned} \tag{2.36}$$

The integral in Eq.(2.36) can be evaluated numerically. In Section A we saw that the field quadrature operator $\hat{x}$ of the field showed squeezing. This implies that the variance of the distribution $p(x)$ can be less than that for a coherent state. In Fig. (2.5) we show the distribution $p(x)$ for different values of m. We see that as m is increased, the width of the distribution becomes narrower and narrower compared to that for the coherent state. The distribution $p(x)$ corresponding to coherent state $m = 0$, is Gaussian with a width 1/2.

$$p(x) = \frac{2}{\sqrt{2\pi}} \exp[-2(x-\alpha_1)^2] \quad , \qquad m = 0 \quad . \tag{2.37}$$

For $\alpha_2 = 0$, the integral (2.36) can be written as

$$\begin{aligned} p(x) &= \frac{2\exp[-2(x-\alpha_1)^2](-1)^m}{\pi L_m(-|\alpha|^2)} \\ &\times \; \sum_{p=o}^{m}\sum_{q=o}^{p} \frac{(-4)^p m!(x-\alpha_1/2)^{2(p-q)}}{p!(m-p)!q!(p-q)!} \frac{\Gamma(q+1/2)}{2^{q+1/2}} \quad , \end{aligned} \tag{2.38}$$

where $\Gamma(q+\frac{1}{2})$ is the Gamma function given by

$$\Gamma(q+\frac{1}{2}) = \frac{2q!\sqrt{\pi}}{q!2^{2q}} \quad . \tag{2.39}$$

2.2. Photon Added Thermal States (PATS)

In this section we study the nonclassical properties of the state which is obtained by adding photons to the thermal state. The density operator for the field in the photon added thermal state is given by

$$\hat{\rho} \propto \hat{a}^{\dagger m} \hat{\rho}_{th} \hat{a}^{m} \quad , \tag{2.40}$$

where $\hat{\rho}_{th}$ is the thermal density operator of the field. Therefore (2.40) becomes

$$\hat{\rho} = \frac{1}{\mathcal{N}(1+\bar{n})} \hat{a}^{\dagger m} e^{-\lambda \hat{a}^{\dagger} \hat{a}} \hat{a}^{m} \quad , \tag{2.41}$$

where $\mathcal{N}$ is the normalisation constant, m is an integer and the average photon number of the thermal field $\bar{n} = (e^{\lambda} - 1)^{-1}$.

The normalisation constant is given by

$$\begin{aligned} \mathcal{N} = Tr(\hat{\rho}) &= \frac{1}{(1+\bar{n})\pi} \int < \alpha | \hat{a}^{\dagger m} e^{-\lambda \hat{a}^{\dagger} \hat{a}} \hat{a}^{m} | \alpha > d^2\alpha \\ &= \frac{1}{\pi(1+\bar{n})} \int |\alpha|^{2m} < \alpha | e^{-\lambda \hat{a}^{\dagger} \hat{a}} | \alpha > d^2\alpha \\ &= m!(1+\bar{n})^m \quad . \end{aligned} \tag{2.42}$$

A. Fluctuations in the Quadrature Operator $\hat{x}$

One can clearly see from Eq.(2.40) that the mean values of the quadrature $\hat{x}$ is equal to zero.

$$< \hat{x} > = 0 \quad , \tag{2.43}$$

and

$$\begin{aligned} < \hat{x}^2 > &= < \hat{a}^2 + \hat{a}^{\dagger 2} + 2\hat{a}\hat{a}^{\dagger} - 1 > \\ &= 2(m+1)(1+\bar{n}) - 1 \quad . \end{aligned} \tag{2.44}$$

From Eqs.(2.43) and (2.44) we find the variance of quadrature operator $\hat{x}$ as

$$\begin{aligned}(\Delta x)^2 &= <\hat{x}^2> - <\hat{x}>^2 \\ &= 2\bar{n}(1+m) + 2m + 1 \quad . \end{aligned} \tag{2.45}$$

Eq.(2.45) tells us that the state (2.40) does not exhibit any squeezing. For $m = 0$ (thermal state) this reduces to $2\bar{n}+1$ and $\bar{n} = 0$ (number state) this reduces to $2m+1$ as expected.

B. Sub-Poisson Character

The photon number distribution for the field in the photon added thermal state is given by

$$\begin{aligned} p(n) &= <n|\hat{\rho}|n> \\ &= \frac{<n|\hat{a}^{\dagger m} e^{-\lambda \hat{a}^{\dagger}\hat{a}} \hat{a}^m |n>}{m!(1+\bar{n})^{1+m}} \\ &= \binom{n}{m} \frac{(\bar{n})^{n-m}}{(1+\bar{n})^{1+m}} , \end{aligned} \tag{2.46}$$

which is zero for $n < m$. For $m = 0$, we get back the well known Bose-Einstein distribution for the thermal field given by

$$p(n) = \frac{\bar{n}^n}{(1+\bar{n})^{1+n}} \quad . \tag{2.47}$$

We now calculate Q parameter which measures the deviation from a Poisson statistics. The Q parameter, as given by Eq.(2.19) can be expressed in terms of antinormal ordered form $\hat{a}^{\dagger}\hat{a}$ as follows

$$Q = \frac{<\hat{a}^2\hat{a}^{\dagger 2}> - 2<\hat{a}\hat{a}^{\dagger}> - <\hat{a}\hat{a}^{\dagger}>^2 + 1}{<\hat{a}\hat{a}^{\dagger}> - 1} \quad . \tag{2.48}$$

In order to calculate this we need to know the expectation values of the antinormally ordered form which can be obtained as follows

$$\begin{aligned} < \hat{a}^p \hat{a}^{\dagger p} > &= Tr(\hat{\rho}\hat{a}^p\hat{a}^{\dagger p}) \\ &= Tr(\hat{a}^{\dagger p}\hat{\rho}\hat{a}^p) \\ &= Tr\left[\frac{1}{\pi}\int |\alpha>< \alpha| d^2\alpha \hat{a}^{\dagger p}\hat{\rho}\hat{a}^p\right] , \end{aligned}$$

where we have used the completeness relation $\frac{1}{\pi}\int |\alpha>< \alpha| d^2\alpha = 1$.

$$\begin{aligned} < \hat{a}^p \hat{a}^{\dagger p} > &= \frac{1}{m!(1+\bar{n})^{1+m}\pi}\int < \alpha|\hat{a}^{\dagger (m+p)} \epsilon^{-\lambda \hat{a}^\dagger \hat{a}} \hat{a}^{m+p}|\alpha > d^2\alpha \\ &= \frac{1}{m!(1+\bar{n})^{1+m}\pi}\int |\alpha|^{2(m+p)} \epsilon^{-|\alpha|^2(1-\epsilon^{-\lambda})} d^2\alpha \\ &= \frac{(m+p)!(1+\bar{n})^{m+p+1}}{m!(1+\bar{n})^{1+m}} \\ &= \frac{(m+p)!(1+\bar{n})^p}{m!} . \end{aligned} \tag{2.49}$$

Using (2.49) in (2.48) we obtain the Q parameter as

$$Q = \frac{\bar{n}^2(1+m) - m}{\bar{n}(1+m) + m} . \tag{2.50}$$

For $\bar{n} = 0$ (Fock state) the Q parameter becomes equal to -1 and for $m = 0$ (thermal state) Q is equal to $\bar{n}$ as expected. When $\bar{n} \neq 0$ and $m \neq 0$ the field is in the photon added thermal state. The Q parameter becomes negative when

$$m > \bar{n}^2(1+m)$$

$$\text{or} \qquad \bar{n} < \sqrt{\frac{m}{1+m}} . \tag{2.51}$$

Thus the field in PATS exhibits sub-Poisson stastistics as long as the condition as given by Eq.(2.51) is satisfied. When $\bar{n} > \sqrt{m/(1+m)}$ the field in PATS no longer exhibits sub-Poissonian statistics but still remains nonclassical. This is shown by considering a new criterion which is introduced in later chapter. We show the Q parameter as a function of $\bar{n}$ for different values of m in Fig.(2.6).

C. Quasiprobability Distributions

(i) P function

It is well known that the P function for the density matrix $\hat{\rho}_{th}$ for the thermal state is given by

$$P(\alpha) = \frac{1}{\pi\bar{n}} e^{-|\alpha|^2/\bar{n}} . \tag{2.52}$$

Hence $\hat{\rho}_{\rm th}$ can be written in the diagonal coherent state representation as

$$\hat{\rho}_{\rm th} = \frac{1}{\pi\bar{n}} \int e^{-|\alpha|^2/\bar{n}} |\alpha >< \alpha| d^2\alpha \quad . \tag{2.53}$$

Using the above equation in Eq.(2.40) we find that

$$\hat{\rho} = \frac{1}{\pi m!(1+\bar{n})^m \bar{n}} \int e^{-|\alpha|^2/\bar{n}} \hat{a}^{\dagger m} |\alpha >< \alpha| \hat{a}^m d^2\alpha \ . \tag{2.54}$$

This can be evaluated by noting that

$$\hat{a}|\alpha >= \alpha|\alpha > \ ; \ \hat{a}^\dagger|\alpha >= e^{-\frac{1}{2}|\alpha|^2} \frac{\partial e^{\frac{1}{2}|\alpha|^2}}{\partial\alpha} |\alpha > \ . \tag{2.55}$$

We can write Eq.(2.54), using Eq.(2.55) as

$$\hat{\rho} = \frac{1}{\pi m!\bar{n}(1+\bar{n})^m} \int e^{|\alpha|^2} \frac{\partial^{2m}}{\partial\alpha^{*m}\partial\alpha^m} e^{-|\alpha|^2(1+1/\bar{n})} |\alpha >< \alpha| d^2\alpha \ . \tag{2.56}$$

Therefore the P function for the field in photon added thermal state is

$$P(\alpha) = \frac{1}{\pi m!\bar{n}(1+\bar{n})^m} \int e^{|\alpha|^2} \frac{\partial^{2m}}{\partial\alpha^{*m}\partial\alpha^m} e^{-|\alpha|^2(1+1/\bar{n})} \ . \tag{2.57}$$

The derivatives in Eq.(2.57) can be expressed in the terms of Laguerre polynomials as [15]

$$\frac{\partial^{2m}}{\partial\alpha^{*m}\partial\alpha^{m}}e^{-\mu|\alpha|^2} = (-\mu)^m m! e^{-\mu|\alpha|^2} L_m(\mu|\alpha|^2) \ . \tag{2.58}$$

Hence the P function is

$$P(\alpha) = \frac{(-1)^m}{\pi(\bar{n})^{1+m}} e^{-|\alpha|^2/\bar{n}} L_m[(1+\frac{1}{\bar{n}})|\alpha|^2] \ . \tag{2.59}$$

Thus the quasiprobability distribution $P(\alpha)$ clearly becomes negative as Laguerre polynomials oscillate between positive and negative values. In Fig.(2.7) we show the nonclassical character of the field in PATS for a range of the values of $\bar{n}$ and for fixed m. It is seen that the P function takes negative values even when the condition given by Eq. (2.51) for the field to exhibit sub-Poisson statistics is violated. This indicates that there exists some other quantities which can be used to characterise the nonclassical nature of the field. This is discussed in detail in the next chapter.

(ii) Q function

The Q function for the state (2.40) can be obtained easily as

$$\begin{aligned} Q(\alpha) &= \frac{1}{\pi} < \alpha|\hat{\rho}|\alpha > \\ &= \frac{1}{\pi}\frac{< \alpha|\hat{a}^{\dagger m} e^{-\lambda \hat{a}^\dagger \hat{a}} \hat{a}^m|\alpha >}{m!(1+\bar{n})^{1+m}} \\ &= \frac{|\alpha|^{2m} e^{-|\alpha|^2/(1+\bar{n})}}{\pi m!(1+\bar{n})^{1+m}} \ . \end{aligned} \tag{2.60}$$

This distribution as a function of $|\alpha|^2$ is centered at $m(1+\bar{n})$ and is well behaved as shown in Fig.(2.8).

(iii) Wigner function

In order to obtain Wigner functior for the photon added thermal states we use the formula given by Eq. (2.32),

$$\begin{aligned} W(z) &= \frac{2}{\pi^2}\frac{\exp(2|z|^2)}{\mathcal{N}(1+\bar{n})}\int d^2\beta < -\beta|\hat{a}^{\dagger m}e^{-\lambda\hat{a}^\dagger\hat{a}}\hat{a}^m|\beta > \exp\left[2(\beta^* z - \beta z^*)\right] \\ &= \frac{2}{\pi^2}\frac{\exp(2|z|^2)}{m!(1+\bar{n})^{1+m}}\int d^2\beta(-1)^m|\beta|^{2m}\exp\left[-|\beta|^2(1+e^{-\lambda})\right]\exp\left[2(\beta^* z - \beta z^*)\right] \ , \end{aligned} \tag{2.61}$$

where we have used the relation

$$e^{-r\hat{a}^\dagger\hat{a}}|\alpha >= \exp\left[\frac{-|\alpha|^2}{2}(1-e^{-2r})\right]|\alpha e^{-r} > \ . \tag{2.62}$$

With $\xi = 2z$, the integral (2.61) can be rewritten as

$$\begin{aligned} W(z) &= \frac{2\exp(2|z|^2)}{\pi m!(1+\bar{n})^{1+m}}\frac{\partial^{2m}}{\partial\xi^{*m}\partial\xi^m}\frac{1}{\pi}\int d^2\beta\exp\left[-|\beta|^2(1+e^{-\lambda})+\beta^*\xi-\beta\xi^*\right] \\ &= \frac{2\exp(2|z|^2)}{\pi m!(1+\bar{n})^{1+m}(1+2\bar{n})}\ \frac{\partial^{2m}}{\partial\xi^{*m}\partial\xi^m}\exp\left[\frac{-|\xi|^2(1+\bar{n})}{(1+2\bar{n})}\right] \\ &= \frac{2(-1)^m}{\pi(1+2\bar{n})^{1+m}}\exp\left[\frac{-2|z|^2}{1+2\bar{n}}\right]L_m\left[\frac{4|z|^2(1+\bar{n})}{1+2\bar{n}}\right], \end{aligned} \tag{2.63}$$

where we have made use of the identity (2.58). For $m = 0(\bar{n} = 0)$, the above expression reduces to that for a thermal state (number state). The Wigner function for PATS is plotted in Fig. (2.9) for $m = 5$ and for average thermal photons $\bar{n} = 0.95$.

2.3. Addition or Subtraction of Pairs of Photons from a Two Mode Squeezed Vacuum State

In previous sections we considered single mode states of the quantized electromagnetic field. In this section we consider two mode state. We study the nonclassical properties of the states which are generated by addition or subtraction of pairs of photons from the two mode squeezed vacuum state. These states are defined as follows

$$|\psi, p>_+ \;=\; \sqrt{\mathcal{N}}(\hat{a}^\dagger \hat{b}^\dagger)^p S(r,\phi)|0,0> \,, \qquad (2.64a)$$

$$|\psi, p>_- \;=\; \sqrt{\mathcal{N}'}(\hat{a}\hat{b})^p S(r,\phi)|0,0> \,, \qquad (2.64b)$$

where $\mathcal{N}$ and $\mathcal{N}'$ are the normalisation constant, p is an integer, $\hat{a}, \hat{b}$ are the annihilation operators and $\hat{a}^\dagger, \hat{b}^\dagger$ are the creation operators of the two modes of the electromagnetic field. $S(r,\phi)$ is the two mode squeeze operator given by

$$S(r,\phi) = \exp\left[r(\hat{a}\hat{b}e^{-i\phi} - \hat{a}^\dagger \hat{b}^\dagger e^{i\phi})\right] \;, \qquad (2.65)$$

which is characterised by a real squeeze parameter r and an angle ϕ that determines the phase of squeezing.

We consider the state given by Eq. (2.64a). In order to obtain the normalisation constant we first factorise the squeeze operator $S(r,\phi)$ into a product of exponentials using disentangling theorem [16] as

$$S(r,\phi) = \frac{1}{\cosh r}\exp(-\hat{a}^\dagger \hat{b}^\dagger \tanh r e^{i\phi})\exp(-(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}) ln \cosh r)\exp(\hat{a}\hat{b}\tanh r e^{-i\phi})) \;. \qquad (2.66)$$

Substituting (2.66) in (2.64a), we get

$$\begin{aligned} |\psi, p>_+ &= \frac{\sqrt{\mathcal{N}}}{\cosh r}(\hat{a}^\dagger \hat{b}^\dagger)^p \exp(-\hat{a}^\dagger \hat{b}^\dagger \tanh r e^{i\phi})|0,0> \\ &= \frac{\sqrt{\mathcal{N}}}{\cosh r}\sum_{m=o}^{\infty}\frac{(-\tanh r e^{i\phi})^m}{m!}(\hat{a}^\dagger \hat{b}^\dagger)^p(\hat{a}^\dagger \hat{b}^\dagger)^m|0,0> \,, \end{aligned} \qquad (2.67)$$

and

$$_{+}< p,\psi|\psi,p>_{+} \;=\; \frac{\mathcal{N}}{\cosh^2 r}\sum_{m=o}^{\infty}\frac{(\tanh r)^{2m}}{n!^2}(n+p)!^2 \;. \tag{2.68}$$

Thus the normalisation constant $\mathcal{N}$ is

$$\mathcal{N} \;=\; \left[\frac{1}{\cosh^2 r}\sum_{n=o}^{\infty}\frac{(\tanh r)^{2n}}{n!^2}(n+p)!^2\right]^{-1} \;. \tag{2.69}$$

A. Squeezing Properties

A two mode quadrature operator is defined by [17]

$$\hat{x} = \frac{1}{2\sqrt{2}}[(\hat{a}+\hat{b})^{-i\theta/2} + (\hat{a}^\dagger + \hat{b}^\dagger)e^{i\theta/2}] \;. \tag{2.70}$$

From Eqs.(2.64) and (2.70) it is easy to show that the mean values of $\hat{x}$ is equal to zero and the fluctuations in $\hat{x}$

$$\begin{aligned}
\Delta\hat{x}^2 \;&\equiv\; <\hat{x}^2> - <\hat{x}>^2 \\
&=\; \frac{\mathcal{N}}{4\cosh^2 r}\sum_{n=o}^{\infty}\frac{(\tanh r)^{2n}}{n!^2}(n+p)!^2 \\
&\times\; \left[2(n+p) - \frac{2\tanh r(n+p+1)^2\cos(\phi-\theta)}{n+1} + 1\right] \;.
\end{aligned} \tag{2.71}$$

Squeezing occurs whenever $\Delta\hat{x}^2 < \frac{1}{4}$. For $p = 0$ that is when the state reduces to two photon squeezed vacuum state, the variance as given by Eq.(2.71) becomes equal to $\exp(-2r)/4$ as expected. The behaviour of $S_x = 4(\Delta\hat{x}^2)$ as function of r is shown in Fig.(2.10).

B. Sub-Poissonian Character

The joint probability to find n_a photons in mode $\hat{a}$ and n_b photons in mode $\hat{b}$ is given by

$$\begin{aligned}
p(n_a,n_b) \;&\equiv\; |\langle n_a,n_b|\psi,p\rangle_+|^2 \\
&=\; \frac{\mathcal{N}}{\cosh^2 r}\left|\sum_{m=o}^{\infty}\frac{(-1)^m(\tanh r)^m e^{i\phi m}(m+p)!\delta_{n_a,m+p}\delta_{n_b,m+p}}{m!}\right|^2 \;.
\end{aligned} \tag{2.72}$$

It is obvious from (2.72) that the joint distribution has only diagonal elements $n_a = n_b = n$. Thus

$$p(n,n) = \frac{\mathcal{N}}{\cosh^2 r}\frac{(\tanh r)^{2(n-p)} n!^2}{(n-p)!^2} \tag{2.73}$$

which is zero for $n < p$.

To test for sub-Poissonian character we calculate Mandel's parameter for modes $\hat{a}$ and $\hat{b}$, which are defined as

$$Q_a \equiv \frac{< (\hat{a}^\dagger \hat{a})^2 > - < \hat{a}^\dagger \hat{a} >^2}{< \hat{a}^\dagger \hat{a} >} - 1 \tag{2.74a}$$

$$Q_b \equiv \frac{< (\hat{b}^\dagger \hat{b})^2 > - < \hat{b}^\dagger \hat{b} >^2}{< \hat{b}^\dagger \hat{b} >} - 1 \tag{2.74b}$$

Calculations show that

$$Q_a = Q_b = \frac{\sum_{n=o}^{\infty} \frac{(\tanh r)^{2n}}{n!^2}(n+p)!^2(n+p)^2 - \frac{\mathcal{N}}{\cosh^2 r}\left[\sum_{n=o}^{\infty} \frac{(\tanh r)^{2n}(n+p)!^2(n+p)}{n!}\right]^2}{\sum_{n=o}^{\infty} \frac{(\tanh r)^{2n}(n+p)!^2(n+p)}{n!}} - 1 \tag{2.75}$$

We plot Q_a as a function of r in Fig.(2.11).

The correlation coefficient between $\hat{a}$ and $\hat{b}$ modes is defined by

$$\begin{aligned} C_{ab} &= < \hat{a}^\dagger \hat{a}\, \hat{b}^\dagger \hat{b} > - < \hat{a}^\dagger \hat{a} >< \hat{b}^\dagger \hat{b} > \\ &= < (\hat{a}^\dagger \hat{a})^2 > - < \hat{a}^\dagger \hat{a} >^2 \\ &= (\Delta n_a)^2 \text{ or } (\Delta n_b)^2 \ , \end{aligned} \tag{2.76}$$

since in this case

$$\langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle = \langle (\hat{a}^\dagger \hat{a})^2 \rangle \ .$$

We next consider the Cauchy Schwarz's inequality [18]. In the classical theory it is given by

$$|\langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle|^2 \le \langle (\hat{a}^\dagger \hat{a})^2 (\hat{b}^\dagger \hat{b})^2 \rangle \quad , \tag{2.77}$$

which states that the correlation of the photons of different field modes is less than the correlation of the photons of the same field mode. In order to measure the deviation from the classical inequality following Agarwal [19] we define a quantity

$$I_o = \frac{(\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle\langle \hat{b}^{\dagger 2}\hat{b}^2\rangle)^{1/2}}{|\langle \hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b}\rangle|} - 1 \ . \tag{2.78}$$

This becomes negative for states exhibiting strong quantum properties, thus violating the Cauchy Schwarz inequality (2.77). This is a nonclassical effect in such a manner that the P function for the field modes becomes non positive definite. In this case, I_o becomes

$$\begin{aligned} I_o &= \frac{\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle}{\langle \hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b}\rangle} - 1 \\ &= \frac{\langle(\hat{a}^\dagger\hat{a})^2\rangle - \langle\hat{a}^\dagger\hat{a}\rangle}{\langle(\hat{a}^\dagger\hat{a})^2\rangle} - 1 = \frac{-\langle\hat{a}^\dagger\hat{a}\rangle}{\langle(\hat{a}^\dagger\hat{a})^2\rangle} \quad , \end{aligned} \tag{2.79}$$

which is negative, thereby implying that the intermode correlation is larger than the correlation between the photons of the same mode.

C. Quasi Probability Distribution

The P function for the state does not exist.

The Q function for this state is

$$\begin{aligned} Q(\alpha_1,\alpha_2) &= \frac{1}{\pi^2}|\langle\alpha_1,\alpha_2|\psi,p\rangle_+|^2 \\ &= \frac{1}{\pi^2}\frac{\mathcal{N}}{\cosh^2 r}|\alpha_1\alpha_2|^{2p}\exp[-\tanh r(e^{i\phi}\alpha_1\alpha_2 + \alpha_1^*\alpha_2^*e^{-i\phi}) - |\alpha_1|^2 - |\alpha_2|^2] \ , \end{aligned} \tag{2.80}$$

which reduces to two mode squeezed vacuum state when $p = 0$.

$$Q(\alpha_1,\alpha_2) = \frac{1}{\pi\cosh^2 r}\exp\left[-\tanh r(e^{i\phi}\alpha_1\alpha_2 + \alpha_1^*\alpha_2^*e^{-i\phi}) - |\alpha_1|^2 - |\alpha_2|^2\right] \ . \tag{2.81}$$

In conclusion, we have introduced three new types of states of the electromagnetic field and studied their quantum properties. It is found that these states exhibit the nonclassical properties such as sub-Poissonian statistics, squeezing in one quadrature of the field, violation of the Cauchy-Schwarz inequality and the nonclassical behavior of the Glauber-Sudarshan P function. These states can be generated by nonlinear optical phenomena by using state reduction techniques which will be discussed in the following chapter.

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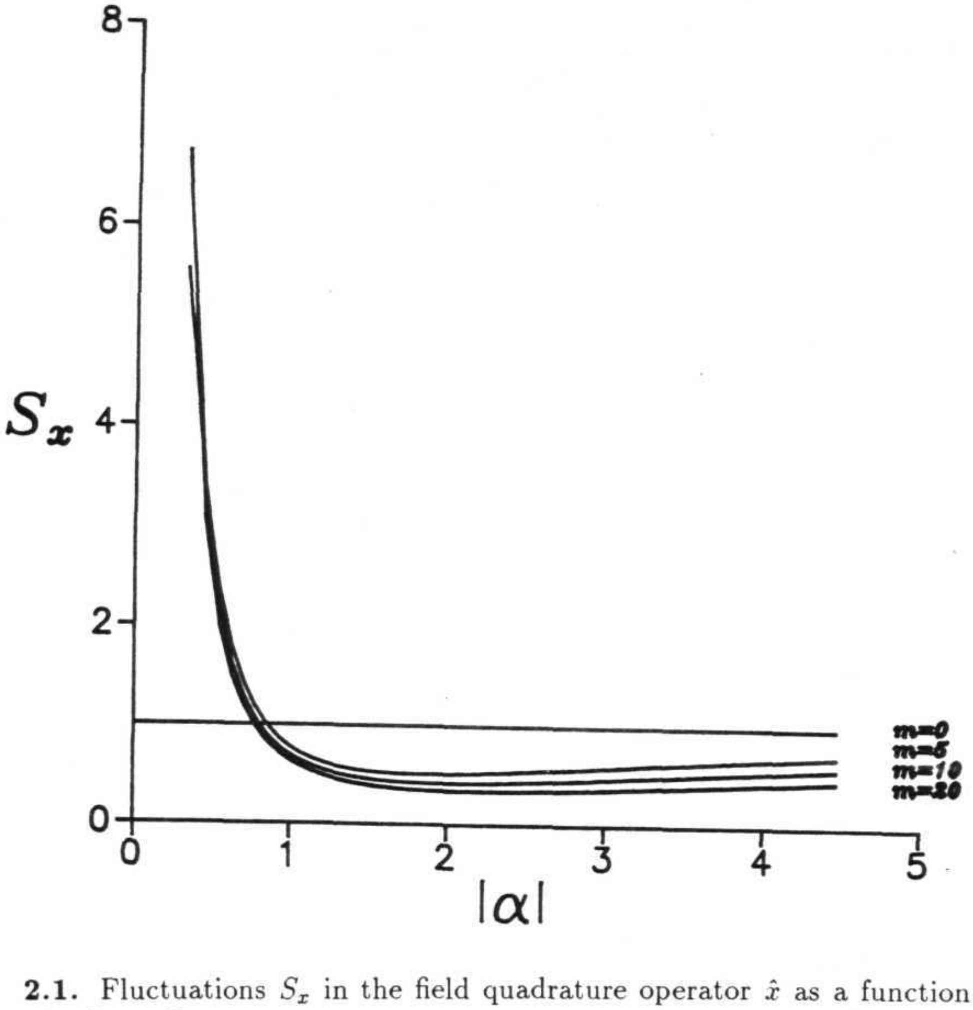


Fig. 2.1. Fluctuations S_x in the field quadrature operator $\hat{x}$ as a function of $|\alpha|$ for different values of m.

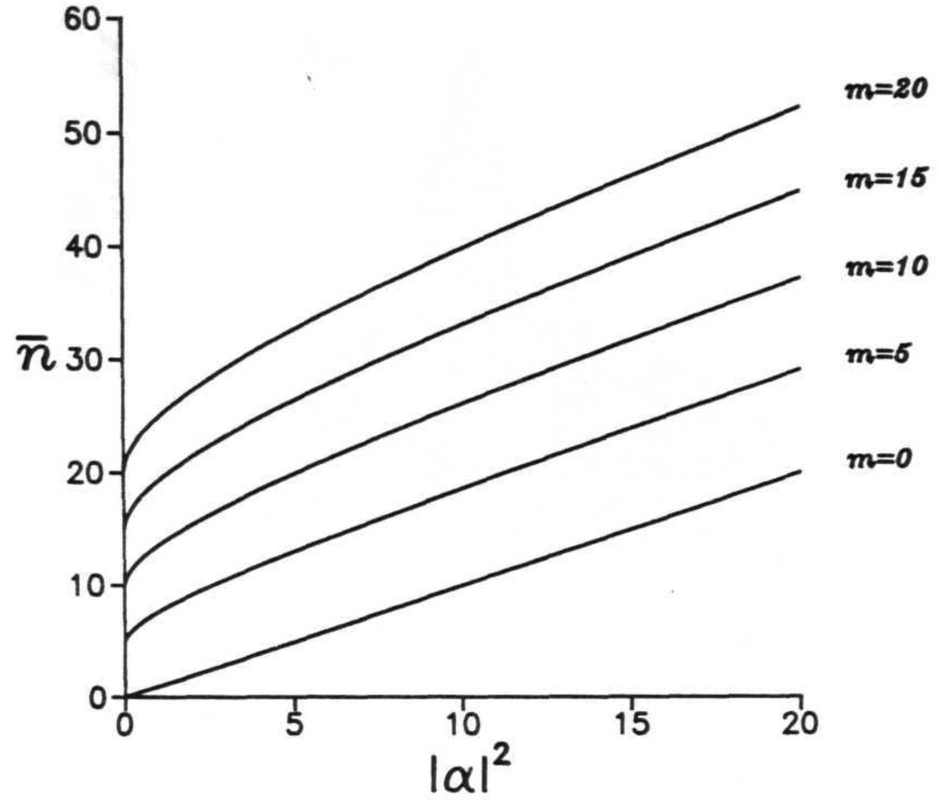


Fig. 2.2. Mean number of photons $\bar{n}$ as a function of $|\alpha|^2$ for different values of m.

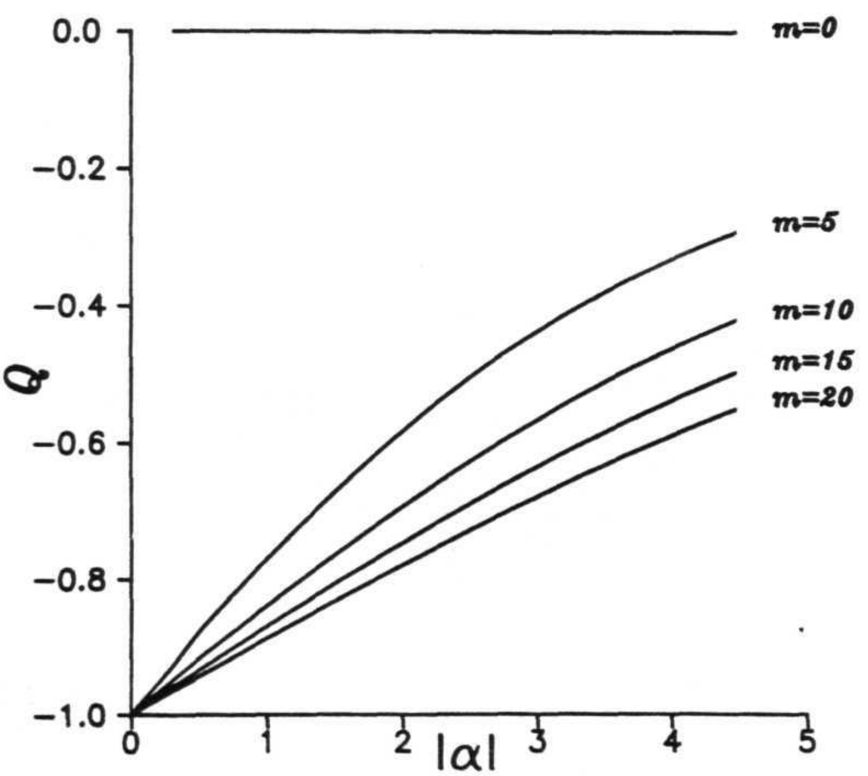
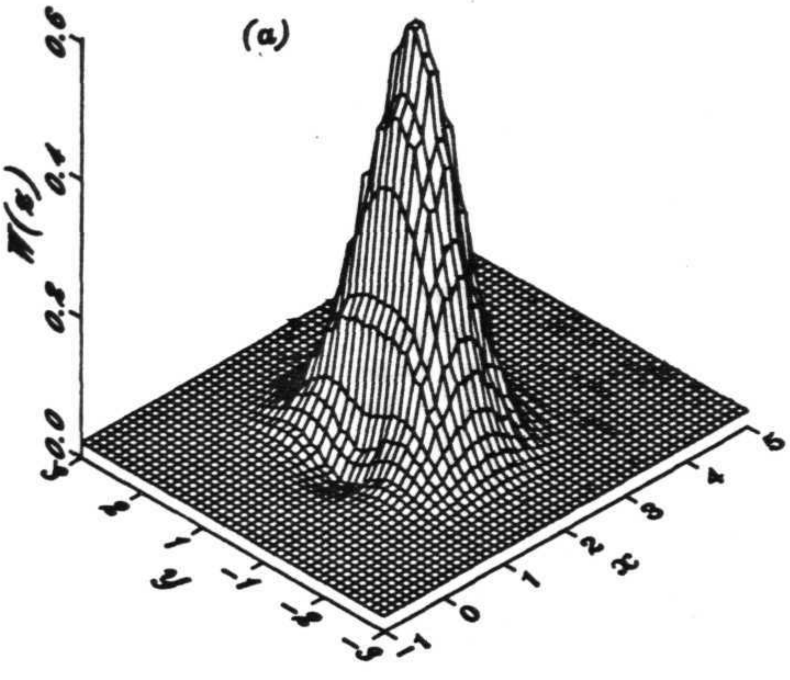


Fig. 2.3. Mandel's Q parameter as a function of $|\alpha|$ for different values of m.



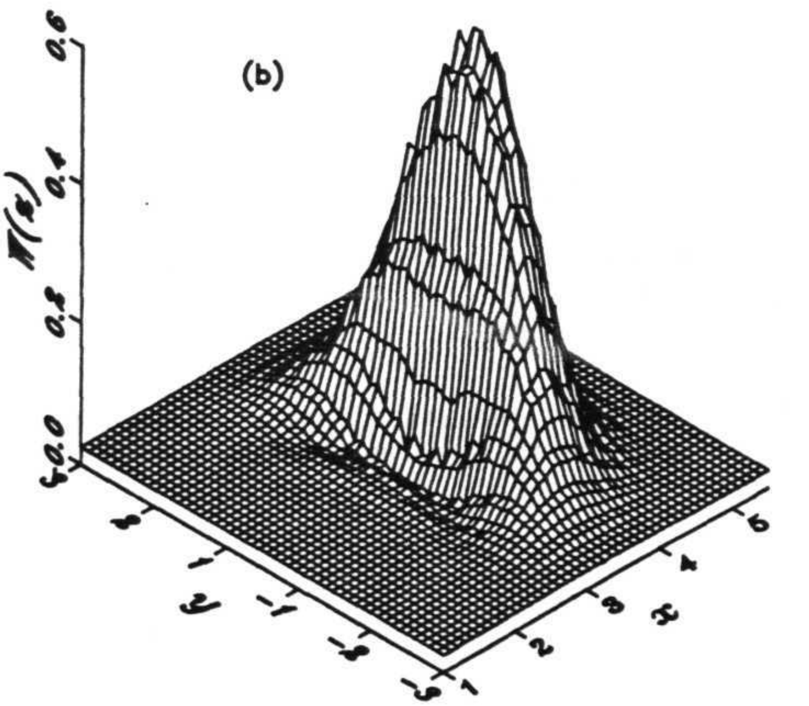


Fig. 2.4. Wigner function $W(z)$ as a given by Eq. (2.34) for $\alpha_1 = 2$, $\alpha_2 = 0$ and for (a) $m = 1$, (b) $m = 5$.

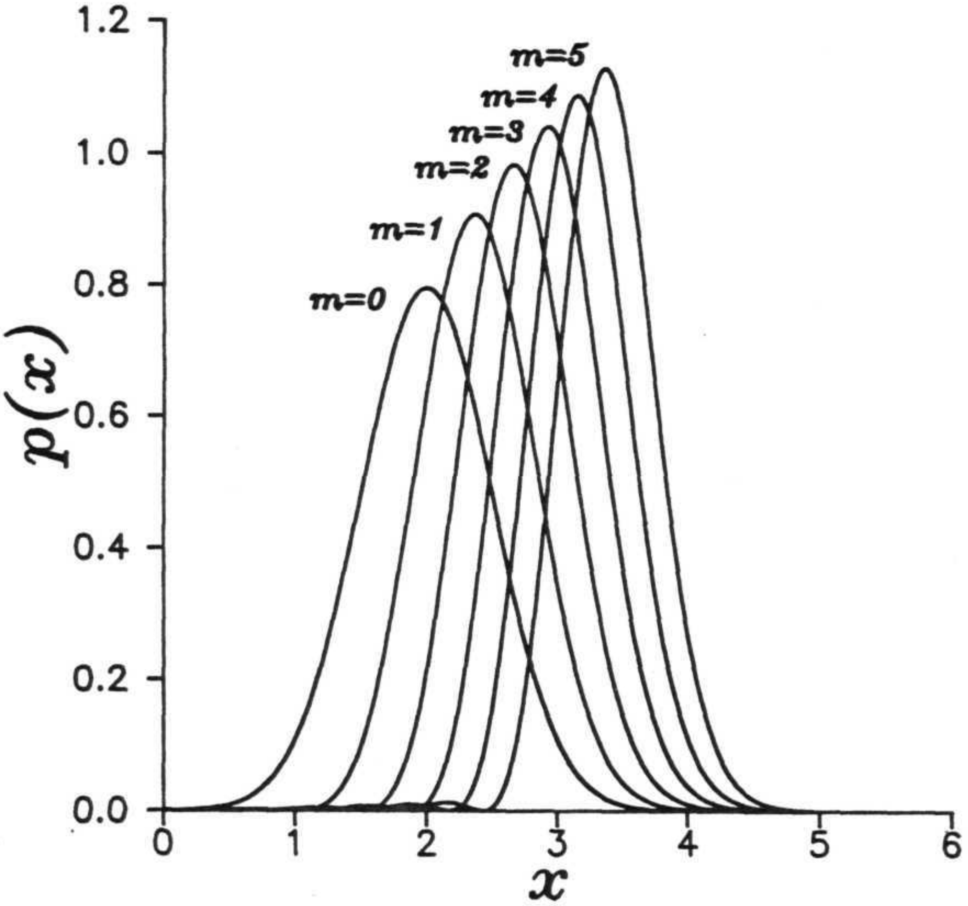


Fig. 2.5. Probability distribution $p(x)$ (Eq. (2.36)) for $\alpha_1 = 2$, $\alpha_2 = 0$ and for different values of m. Note $m = 0$ corresponds to a coherent state.

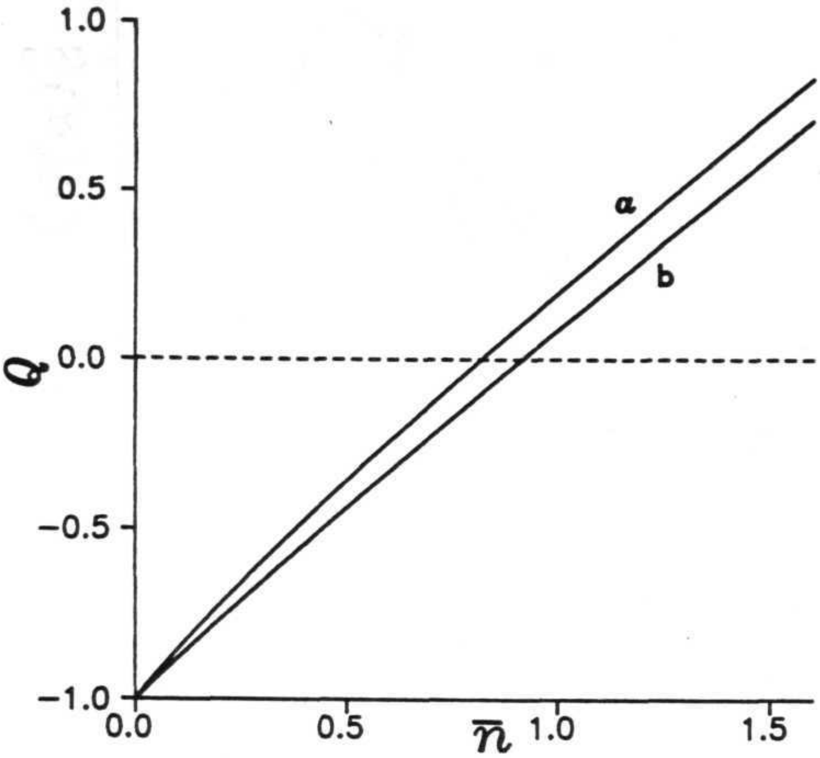


Fig. 2.6. Mandel's Q parameter as a function of average photon number $\bar{n}$ for (a) $m = 2$, (b) $m = 5$.

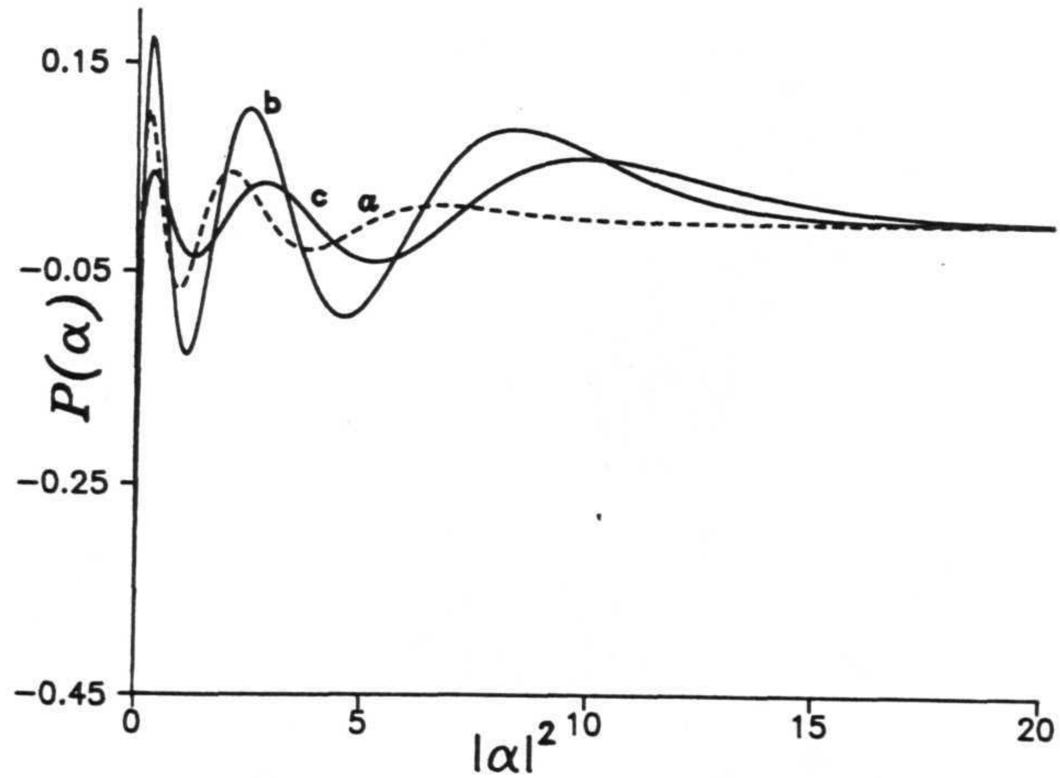


Fig. 2.7. P distribution function (Eq. (2.59)) with $m = 5$ and average photon number $\bar{n}$ (a) = 0.7, (b) = 0.95 and (c) = 1.2. For the case (a) the plotted values are 1/10-th of the actual values.

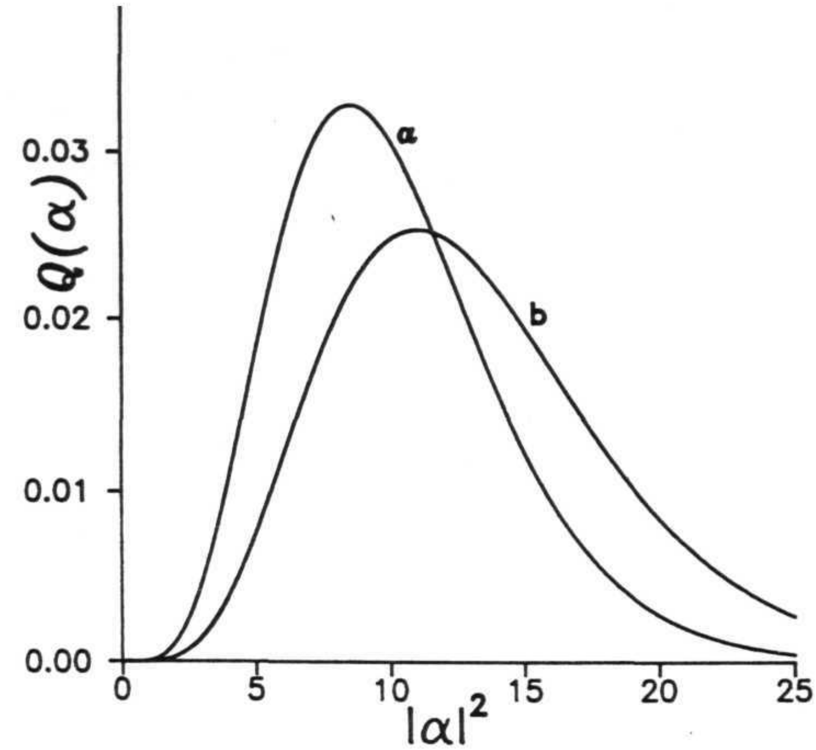


Fig. 2.8. Q distribution function as a function of $|\alpha|^2$ for $\bar{n}$ (a) =0.7 and (b)=1.2.

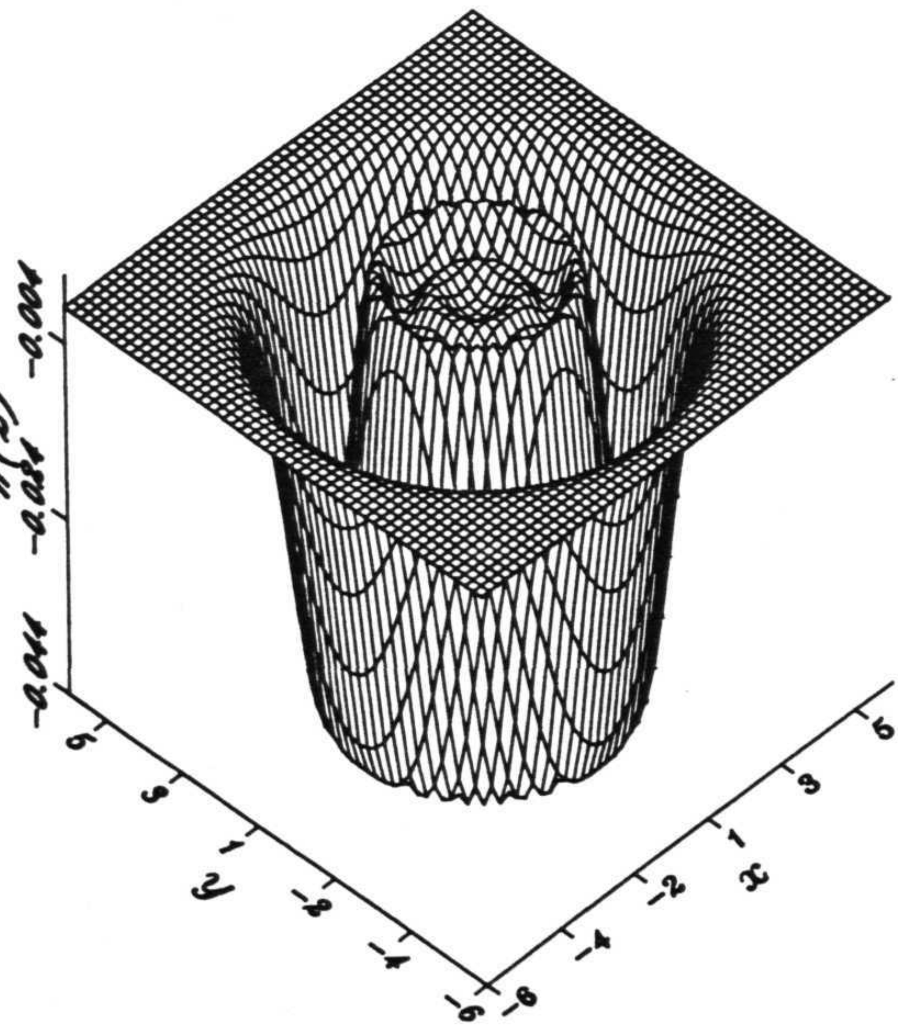


Fig. 2.9. Wigner distribution function as given by Eq. (2.63) for $m = 5$ and average photon number, $\bar{n}$=0.95.

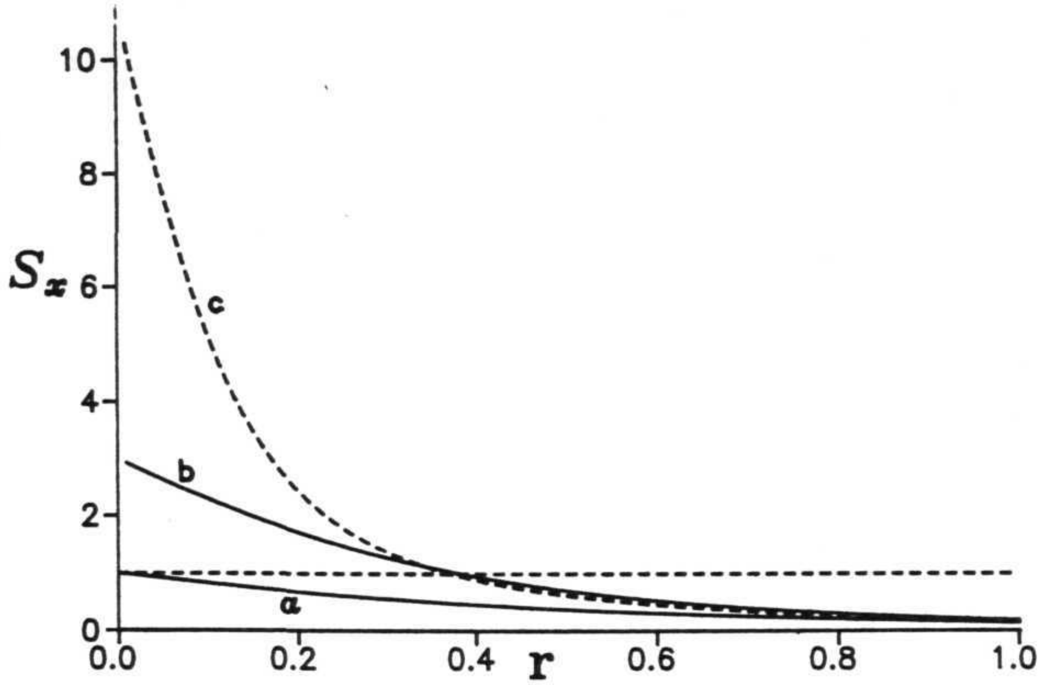


Fig. 2.10. The parameter S_x as a function of r for p (a)=0, (b)=1 and (c)=5.

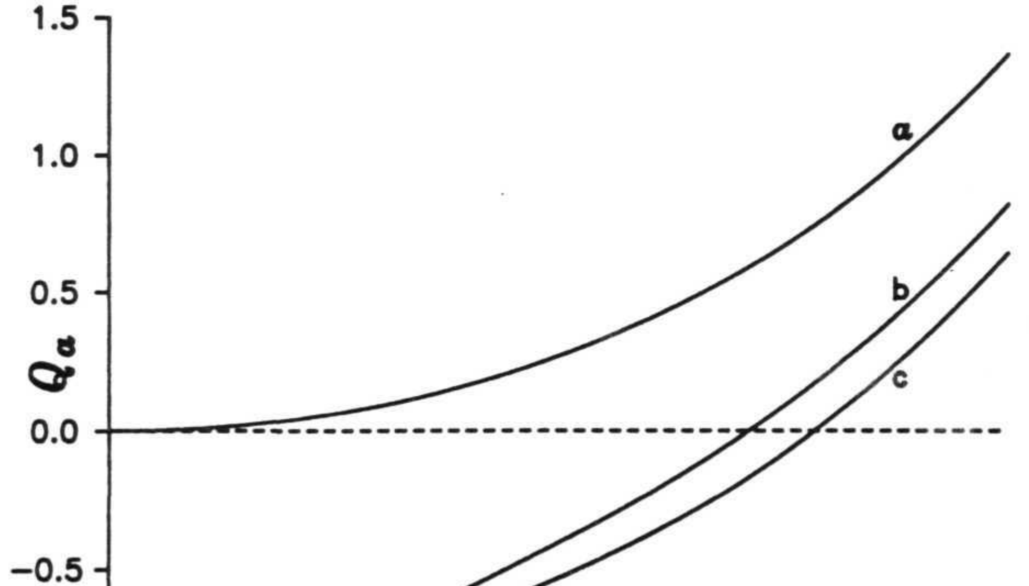


Fig. 2.11. The parameter Q_a given by Eq.(2.75) as a function of r for p (a)=0, (b)=1 and (c)=5.

Chapter 3

PRODUCTION OF PHOTON ADDED STATES

In the past few years there has been a large increase in the number of observations of the nonclassical behavior of light such as photon antibunching, sub-Poissonian statistics and squeezing of the vacuum fluctuations. We have already mentioned in the first chapter some of the successful experiments on resonance fluorescence light from single atoms, Franck Hertz light, second harmonic generation , four wave mixing, parametric processes and semiconductor lasers with negative feedback, where nonclassical effects of the field have been generated. There is at present considerable activity concerning theoretical [1-4] and experimental [5,6] aspects of micromaser operating on single atom injected into a very high Q-resonant cavity. Interesting quantum effects such as trapping states [1,7,8] and sub-Poissonian statistics of the field [1,3,5] have been investigated. It has been shown that the micromaser can be used to produce the photon number state using state reduction technique [2,9]. It has also been shown that the nonclassical photon states can be produced in a nonlinear parametric amplification of idler and signal waves which exhibit strong correlation. The state of the signal mode can be reduced to nonclassical state by appropriate measurements on the idler mode [10-13]. In this chapter we discuss the possible schemes for generating the new types of states introduced in Chapter 2.

We present in Section 3.1, how the photon added states can be produced in a micromaser cavity using state reduction technique. In Section 3.2, we discuss another method, using parametric amplifier, for producing these states. The signal and idler fields produced by a nondegenerate parametric amplifier are known to exhibit strong correlation. The state of one of the photons of the correlated pair can be reduced to photon added state by performing appropriate measurements on the other photon.

3.1 Production of Photon Added States in a Micromaser Cavity

The micromaser consists of a superconducting cavity with a quality factor as high as 3×10^{10}. We consider a beam of two level atoms injected into the cavity. All the atoms are initially prepared in their excited state $|e>$. The atomic flux is so low that only one atom is in the cavity at a time. We consider the field in the cavity to be in the coherent state $|\alpha>$. Then the atom field system will be initially in the state

$$|\psi(0)> \equiv |\alpha> |e> . \tag{3.1}$$

The interaction of the cavity field with an atom is described by the Hamiltonian

$$\hat{H}_{in} = \hbar(g\hat{S}^{+}\hat{a} + g^{*}\hat{S}^{-}\hat{a}^{\dagger}). \tag{3.2}$$

Here $\hat{a}$ and $\hat{a}^{\dagger}$ are the annihilation and the creation operators of the photons in the cavity mode, $\hat{S}^{\pm}$ are the standard spin $\frac{1}{2}$ operators, g is the coupling constant and $\hbar$ =Planck's constant$/2\pi$.

The time evolution operator $\hat{U}(t)$ of the atom field system in the interaction picture is given by

$$\hat{U}(t) = \exp[-it(g\hat{S}^{+}\hat{a} + g^{*}\hat{S}^{-}\hat{a}^{\dagger})] . \tag{3.3}$$

The state $|\psi(t)>$ evolves as

$$|\psi(t)> \equiv \hat{U}(t)|\psi(0)> \quad . \tag{3.4}$$

Now we consider the case of weak field interaction, that is the case when $gt \ll 1$. Then the state after time t can be approximated by

$$\begin{aligned} |\psi(t)> &\cong |\psi(0)> -it(g\hat{S}^{+}\hat{a} + \hat{S}^{-}\hat{a}^{\dagger})|\psi(0)> \\ &= |\alpha>|e> -ig^{*}t\hat{a}^{\dagger}|g> \ , \end{aligned} \tag{3.5}$$

where $|g>$ is the ground state of the atom.

After the atom leaves the cavity it is probed by a static electric field which ionises the atom if it is in the excited state. Thus if the atom is not ionised then it has emitted a photon in the cavity. Thus from Eq.(3.5) we observe that if the atom is detected to be in the ground state $|g>$ then the state of the field in the cavity is reduced to $\hat{a}^{\dagger}|\alpha>$ i.e., to $|\alpha, 1>$. Thus in principle we can produce the state $|\alpha, 1>$. Similarly we can produce the state $|\alpha, m>$ by passing atoms in their excited states successively though the cavity. After the atoms leave the cavity, if m atoms are detected to be in their ground state then the cavity field is in the state $\hat{a}^{\dagger m}|\alpha>$. Alternatively the states $\hat{a}^{\dagger m}|\alpha>$ can be produced in a multiphoton emission processes. For example, in a two-photon medium, Eq.(3.1) is replaced by a new Hamiltonian with $\hat{a} \to \hat{a}^2$ and g replaced by two photon matrix element. Thus, the above procedure for a two photon medium [14] will result in the state $|\alpha, 2>$. Similarly, the state $|\alpha, m>$ can be produced in the process of m-photon emission by an atom.

Now suppose the atoms are initially prepared in their ground state $|g>$ and we send these atoms through the cavity where the field is initially in the coherent state $|\alpha>$. Then the state of the combined atom-field system after time t for small gt can be approximated by

$$|\psi(t)> \cong |\alpha>|g> -igt\alpha|\alpha>|e> \ . \tag{3.6}$$

If we probe the atom after it has left the cavity and find that it is in the excited state but the cavity field is still in the coherent state $|\alpha>$ as can be seen from Eq.(3.6) indicating that the field has no nonclassical character. Thus we can generate nonclassical character in emission processes and not in absorption processes. Similarly photon added thermal states introduced in the previous chapter (Eq.(2.40)) can be produced in principle if the cavity field is initially in the thermal state instead of the coherent state. States obtained by addition of pairs of photon on two mode squeezed vacuum state can be produced in micromaser cavity by considering the two photon excitation of atom which is described by the interaction Hamiltonian

$$\hat{H}_{\text{int}} = \hbar \left(g^* \hat{S}^- \hat{a}^\dagger \hat{b}^\dagger + g \hat{S}^+ \hat{a} \hat{b} \right) \tag{3.7}$$

where $\hat{a}$, $\hat{b}\,(\hat{a}^\dagger, \hat{b}^\dagger)$ are the annihilation (creation) operators of the two modes.

3.2 Production of Photon Added States in Non-Degenerate Parametric Amplifier

In the nondegenerate optical parametric amplifier, one pump photon is destroyed and two photons, one the signal photon and other the idler photon are simultaneously created. The coupling between the two fields arises from a second order nonlinearity in the polarisability of certain crystals. The pairs of signal and idler photons are known to exhibit strong correlation which result in many nonclassical aspects of the radiation field.

The nondegenerate parametric amplifier consists of three interacting field modes, a pump, signal and idler mode within an optical cavity which is coherently driven at the pump frequency. Let $\hat{a}$, $(\hat{a}^\dagger)$, $\hat{b}$, $(\hat{b}^\dagger)$ denote the annihilation (creation) operators for the signal and the idler field respectively. These two cavity fields are coupled to a strong pump field which is treated classically with a complex amplitude $E(t)$. The Hamiltonian describing the interaction is

$$\hat{H}_{int} = \hbar[g\epsilon(t)\hat{a}^\dagger\hat{b}^\dagger + g^*\epsilon^*(t)\hat{a}\hat{b}] , \tag{3.8}$$

where $\epsilon(t) = E(t)e^{-i(\omega_a+\omega_b)t}$ with ω_a and ω_b being the frequencies of the signal and idler field, g is a coupling constant and $\hbar$ is the Planck's constant divided by 2π. We assume that the pump frequency is equal to $\omega_a + \omega_b$ The Eq.(3.8) can be rewritten as

$$\hat{H}_{int} = \hbar\kappa[e^{i\theta}\hat{a}^\dagger\hat{b}^\dagger + e^{-i\theta}\hat{a}\hat{b}] , \tag{3.9}$$

with $\kappa = |g\epsilon(t)|$.

The evolution operator $\hat{U}$ obeys the differential equation

$$i\hbar\frac{d\hat{U}}{dt} = \hat{H}_{int}\hat{U} . \tag{3.10}$$

The solution of this equation is given by

$$\hat{U} = \exp\left[-i\kappa t\left(e^{i\theta}\hat{a}^\dagger\hat{b}^\dagger + e^{-i\theta}\hat{a}\hat{b}\right)\right] . \tag{3.11}$$

On using disentangling theorem [15], it can be put in the normal ordered form as

$$\begin{aligned}\hat{U} &= \frac{1}{\cosh(\kappa t)}\exp[-ie^{i\theta}\tanh(\kappa t)\hat{a}^\dagger\hat{b}^\dagger]\exp[-\ln\cosh(\kappa t)\{\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}\}] \\ &\times \exp[-ie^{-i\theta}\tanh(\kappa t)\hat{a}\hat{b}] ,\end{aligned} \tag{3.12}$$

and the output density operator for signal and idler fields is given by

$$\hat{\rho}_{ab(out)} = \hat{U}\hat{\rho}_{ab(in)}\hat{U}^\dagger . \tag{3.13}$$

The output density operator with coherent state $|\alpha>$ as input signal state and input idler field in vacuum state $|0>$ is

$$\begin{aligned}\hat{\rho}_{ab} &= \frac{N^2e^{-|\alpha|^2}}{\cosh^2\kappa t}\sum_{p,q,r,s}^{\infty}\left\{\binom{p+r}{p}\binom{q+s}{q}\right\}^{1/2}(-ie^{i\theta}\tanh(\kappa t))^r(ie^{i\theta}\tanh\kappa t)^s \\ &\times \left(\frac{\alpha}{\cosh\kappa t}\right)^p\left(\frac{\alpha^*}{\cosh\kappa t}\right)^q\frac{1}{\sqrt{p!}}\frac{1}{\sqrt{q!}}|p+r>_a|r>_{b\,b}<s|_a<q+s| ,\end{aligned} \tag{3.14}$$

where

$$\binom{n}{m} = \frac{n!}{(n-m)!m!}$$

and N is the normalization constant.

The information regarding the signal state after parametric amplification, without performing any measurement, is given by the reduced density operator. The reduced density operator is obtained after taking trace over the idler variables in Eq.(3.14), i.e.,

$$\begin{aligned}\hat{\rho}_a^{\text{red}} &= Tr_b(\hat{\rho}_{ab}) = \frac{N^2 e^{-|\alpha|^2}}{\cosh^2 \kappa t} \sum_{p,q,r}^{\infty} \left\{ \binom{p+r}{p} \binom{q+r}{q} \right\}^{1/2} \tanh^{2r} \kappa t \\ &\times \left(\frac{\alpha}{\cosh \kappa t}\right)^p \left(\frac{\alpha^*}{\cosh \kappa t}\right)^q \frac{1}{\sqrt{p!}} \frac{1}{\sqrt{q!}} |p+r>_a {}_a<q+r| \quad . \end{aligned} \tag{3.15}$$

Watanabe and Yamamoto [13] have studied the effect of idler output measurement on signal output. After a measurement on idler output, if it is found to be in the photon number state then it can be shown that the state of the signal output is reduced to photon added coherent state or photon added thermal state depending on whether the input signal state was in the coherent state or in the thermal state.

If m photons are recorded in a measurement on idler field then this can described by the projection operator

$$|m>_b {}_b<m| \quad . \tag{3.16}$$

The density operator of the signal field conditioned on this measurement is

$$\hat{\rho}_a^{meas,m} = Tr_b(\hat{\rho}_b \hat{\rho}_{ab}) \quad , \tag{3.17}$$

where $\hat{\rho}_b \equiv I_s \otimes |m>_b {}_b<m|$ and I_s is an identity operator for a signal field. With coherent state $|\alpha>$ as the input signal state, the output conditional density operator is found to be from Eqs.(3.14) and (3.16).

$$\begin{aligned}\hat{\rho}_a^{\text{meas,m}} &= \frac{N^2}{\cosh^2(\kappa t)}\frac{\tanh^{2m}(\kappa t)}{m!} \\ &\times \ \hat{a}^{\dagger m}\exp\left[-\ln\cosh(\kappa t)\hat{a}^\dagger\hat{a}\right]|\alpha><\alpha|\exp\left[-\ln\cosh(\kappa t)\hat{a}^\dagger\hat{a}\right]\hat{a}^m . \quad (3.18)\end{aligned}$$

Using

$$\exp\left(-\lambda\hat{a}^\dagger\hat{a}\right)|\alpha>=\exp\left[\frac{-|\alpha|^2}{2}(1-e^{-2\lambda})\right]|\alpha e^{-\lambda}> , \quad (3.19)$$

Eq. (3.18) can be rewritten as

$$\begin{aligned}\hat{\rho}_a^{\text{meas,m}} &= \frac{N^2}{\cosh^2(\kappa t)}\frac{\tanh^{2m}(\kappa t)}{m!}\exp\left[-|\alpha|^2(1-e^{-2\ln\cosh\kappa t})\right] \\ &\times \ \hat{a}^{\dagger m}|\alpha e^{-\ln\cosh\kappa t}><\alpha e^{-\ln\cosh\kappa t}|\hat{a}^m \\ &= \frac{N^2}{\cosh^2(\kappa t)}\frac{\tanh^{2m}(\kappa t)}{m!}\exp\left[-|\alpha|^2\tanh^2(\kappa t)\right]\hat{a}^{\dagger m}|\alpha\sec h(\kappa t)><\alpha\sec h\kappa t|\hat{a}^m . \quad (3.20)\end{aligned}$$

Thus the output signal field is found to be in the photon added coherent state. Similarly if the input signal is in the thermal state then one can show that the output signal field will be in the the photon added thermal state.

Note that the density operator $\hat{\rho}_a^{\text{meas,m}}$ obtained after the state reduction is different from the reduced density operator $\hat{\rho}_a^{\text{red}}$ (Eq.(3.15)). $\hat{\rho}_a^{\text{meas,m}}$ corresponds to the density operator of the output signal field when m photons are recorded in a measurement on the idler field. Suppose in another measurement n photons are recorded then the conditional density operator of the output signal field is given by $\hat{\rho}_a^{\text{meas,n}}$. Thus, if we want to know the quantum statistical properties of all samples over all possible counts it is nothing but the reduced density operator obtained in Eq.(3.15). Feedforward technique is used to overcome this difficulty and produce a photon added state for specific m continuously. A feedforward process operates on the output signal such that all the conditional density operators are translated back to the same mean value $< \hat{a}^{\dagger}\hat{a} >_m$. Thus the output signal field is always reduced to the same photon added state irrespective of number of photons recorded during the measurement on the idler field.

In conclusion, we have discussed in detail how the photon added states introduced in previous chapter can be produced in principle via state reduction technique. We have shown that these states can be produced in a micromaser cavity by passing single atoms successively through the cavity. These states can also be produced in a parametric amplifier which generates pairs of highly correlated signal and idler photons.

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Chapter 4

NEW CRITERION TO TEST NONCLASSICAL NATURE OF LIGHT

As was discussed in the first two chapters, the concept of nonclassical properties of a radiation field can be best understood in terms of the P representation introduced in 1963 by Glauber [1] and by Sudarshan [2] independently. This P representation gives a quasiprobability distribution function in the phase space for a quantum state. It is called a quasiprobability distribution function as the P function can be highly singular or even take negative values which does not make any sense in the classical world. The classical states of the electromagnetic field are the states whose P representation is positive definite. Such states can be described in terms of classical stochastic fields. On the other hand, nonclassical states have P representation that is non positive definite. Unfortunately, we cannot measure the P functions directly. However, the nonclassical nature of the P functions can manifest itself in many ways that can be detected in the laboratories. Nonclassical field has been characterised in a quantitative way by examining the degree of squeezing and sub-Poissonian photon statistics. The squeezing parameter S and the Mandel's Q parameter [3] which measures deviation from Poissonian statistics are defined as

$$S_{x_1} = 2\langle(:\Delta x_1:)^2\rangle = \langle:(\hat{a}e^{i\theta} + \hat{a}^\dagger e^{-i\theta})^2:\rangle - \langle:(\hat{a}e^{i\theta} + \hat{a}^\dagger e^{-i\theta}):\rangle^2 \ , \tag{4.1a}$$

$$S_{x_2} = 2\langle(:\Delta x_2:)^2\rangle = \langle:(\frac{\hat{a}e^{i\theta} - \hat{a}^\dagger e^{-i\theta}}{i})^2:\rangle - \langle:(\frac{\hat{a}e^{i\theta} - \hat{a}^\dagger e^{-i\theta}}{i}):\rangle^2 \ , \tag{4.1b}$$

$$Q = \frac{\langle:(\Delta n)^2:\rangle}{\langle\hat{n}\rangle} = \frac{\langle\hat{a}^{\dagger 2}\hat{a}^2\rangle - \langle\hat{a}^\dagger\hat{a}\rangle^2}{\langle\hat{a}^\dagger\hat{a}\rangle} \ . \tag{4.2}$$

These can be expressed in terms of P function as

$$S_{x_1} = 2\langle :(\Delta x_1)^2 :\rangle = \int d^2\alpha P(\alpha)\left[(\alpha e^{i\theta} + \alpha^* e^{-i\theta}) - (\langle \alpha e^{i\theta}\rangle + \langle \alpha^* e^{-i\theta}\rangle)\right]^2 , \quad (4.3a)$$

$$S_{x_2} = 2\langle :(\Delta x_2)^2 :\rangle = \int d^2\alpha P(\alpha)\left[\frac{(\alpha^{i\theta} - \alpha^* e^{-i\theta})}{i} - \frac{(\langle \alpha e^{i\theta}\rangle - \langle \alpha^* e^{-i\theta}\rangle)}{i}\right]^2 , \quad (4.3b)$$

$$Q = \frac{\langle :(\Delta n)^2 :\rangle}{\langle \hat{n}\rangle} = \frac{\int d^2\alpha P(\alpha)[\,|\alpha|^2 - \langle |\alpha|^2\rangle]^2}{\int d^2\alpha |\alpha|^2 P(\alpha)} . \quad (4.4)$$

where $\langle \alpha \rangle = \int P(\alpha)\alpha d^2\alpha$. These corresponds to the lower order moments of the P function.

From the above equations it is obvious that the condition for squeezing or sub-Poisson statistics i.e., S or Q being negative requires that $P(\alpha)$ be singular or nonpositive definite.

Now a question that arises is: Can the light be still nonclassical even if both S and Q are positive? If yes then, what is the counterpart of Q or S which can quantitatively characterise the nonclassical light? Hong and Mandel [4] have introduced the concept of higher order squeezing, the existence of which implies that the P function is nonpositive definite. A related concept of amplitude squared squeezing was introduced by Hillery [5].

In Section 4.1 we present the definitions of higher order squeezing and amplitude squared squeezing. A criterion for testing the nonclassical character of the field even if it does not exhibit squeezing or sub-Poisson statistics, in terms of higher order moments of P function, is developed in Section 4.2. In Section 4.3 we show the utility of this criterion by applying it to the (i) radiation field in photon added thermal state (ii) Schrödinger cat state which is a coherent superposition of distinguishable quantum mechanical states, (iii) radiation field in a micromaser cavity.

4.1. Higher Order Squeezing and Amplitude Squared Squeezing

The squeezing conditions similar to Eqs.(4.3a) and (4.3b) can be obtained for higher order variances of the field quadratures. Second order squeezing was generalized to N-th order squeezing by Hong and Mandel. A state is said to be squeezed in N-th order if $\langle : (\Delta x_i)^N : \rangle$ is lower than its corresponding coherent state value. The condition for squeezing can be obtained by expanding both sides of Campbell-Baker-Hausdroff identity

$$\langle \exp\{\lambda \Delta x_i\} \rangle = \langle : \exp\{\lambda \Delta x_i\} : \rangle \exp\{\frac{\lambda^2}{2}\} \tag{4.6}$$

and comparing the coefficients of $\lambda^N / N!$.

$$\begin{aligned} \langle (\Delta x_i)^N \rangle &= \langle : (\Delta x_i)^N : \rangle + N(N-1)\frac{1}{2}\langle : (\Delta x_i)^{N-2} : \rangle \\ &+ \frac{N(N-1)(N-2)(N-3)}{2!}(\frac{1}{2})^2 \langle : (\Delta x_i)^{N-4} : \rangle + \cdots \\ &\begin{cases} + \ (N-1)!! \ ; & \text{for even } N \\ + \ \frac{N!}{3! 2^{(N-3)/2}} \langle : (\Delta x)^3 : \rangle \ ; & \text{for odd } N, \end{cases} \end{aligned} \tag{4.7}$$

where $[\hat{a}, \hat{a}^\dagger] = 1$. For a coherent state all the normally ordered variances $\langle : (\Delta x_i)^N : \rangle$ become zero and the condition for squeezing for any even order N is

$$\langle (\Delta x_i)^N \rangle - (N-1)!! < 0 \ . \tag{4.8}$$

A convenient parameter for measuring the degree of N-th order squeezing is

$$q_N = \frac{\langle (\Delta x_i)^N \rangle - (N-1)!!}{(N-1)!!} \ . \tag{4.9}$$

q_N is negative whenever there is N-th order squeezing and the maximum negative value it can take is minus one. Note that the definition is not very meaningful for odd powers of N because $\langle (\Delta x_i)^{2N+1} \rangle$ vanishes for a coherent state.

Another type of higher order squeezing called amplitude squared squeezing was defined by Hillery [5]. The two variances in this case correspond to the real and imaginary parts of the square of the complex amplitude of the fields defined by

$$\hat{Y}_1 = \frac{1}{2}(\hat{a}^2 + \hat{a}^{\dagger 2}) \; ; \; \hat{Y}_2 = \frac{1}{2i}(\hat{a}^2 - \hat{a}^{\dagger 2}) \; . \tag{4.10}$$

These operators obey the commutation relation

$$[\hat{Y}_1, \hat{Y}_2] = i(2\hat{n} + 1) \; . \tag{4.11}$$

where the number operator $\hat{n} = \hat{a}^\dagger \hat{a}$. This leads to the uncertainty relation

$$\Delta Y_1 \Delta Y_2 \geq \langle \hat{n} + \frac{1}{2} \rangle \; , \tag{4.12}$$

and the squeezing condition is

$$\langle (\Delta Y_i)^2 \rangle \; < \langle \hat{n} + \frac{1}{2} \rangle \; , \; i = 1 \text{ or } 2 \; . \tag{4.13}$$

4.2. A New Criterion to Test Nonclassical Character of the Field

In order to obtain other quantum measures to characterise the nonclassical states of light when both S and Q are positive, we have examined the inequalities in terms of the higher order moments of P distribution. We first consider phase insensitive nonclassical characteristics. We derive the inequalities which the higher order moments of a classical distribution $P_{cl}(\alpha)$ obey. Let m_n be the n-th order moment of the intensity distribution

$$m_n = \langle \alpha^{*n} \alpha^n \rangle \equiv \int_{-\infty}^{\infty} P(\alpha) \alpha^{*n} \alpha^n \; . \tag{4.14}$$

Let $f(\alpha)$ be a function

$$f(\alpha) = c_o + c_1 \alpha^* \alpha + c_2 \alpha^{*2} \alpha^2 \; , \tag{4.15}$$

where c's are arbitrary coefficients. We construct the quadratic form

$$F(\{c\}) \equiv \langle |f(\alpha)|^2 \rangle = \sum_{i,j=0}^{2} m_{i+j} c_i^* c_j \; . \tag{4.16}$$

The quadratic form(4.16) is always found to be positive which follows from the fact that $P_{cl}(\alpha)$ is a classical probability distribution. Eq.(4.15) can be written in a matrix form as

$$m^{(3)} = \begin{bmatrix} 1 & m_1 & m_2 \\ m_1 & m_2 & m_3 \\ m_2 & m_3 & m_4 \end{bmatrix} \tag{4.17}$$

The positiveness of the quadratic form implies that the above matrix should be positive definite. Now the matrix $m^{(3)}$ is positive definite if

$$\det \begin{bmatrix} 1 & m_1 \\ m_1 & m_2 \end{bmatrix} \geq 0 \text{ ,i.e. } Q \geq 0 \ , \tag{4.18a}$$

$$\det m^{(3)} \geq 0 \ . \tag{4.18b}$$

Thus the positivity of $P_{cl}(\alpha)$ is a new condition that $\det m^{(3)} \geq 0$, where the higher order moments of $P_{cl}(\alpha)$ are involved. A generalization of the above arguments shows that for a classical distribution all the matrices $m^{(n)}$ defined by

$$m^{(n)} = \begin{pmatrix} 1 & m_1 & m_2 & \ldots & m_{n-1} \\ m_1 & & & & \\ m_2 & & & & \\ \vdots & & & & \\ m_{n-1} & & & & m_{2n-2} \end{pmatrix} \tag{4.19}$$

should be *positive definite*

Now we apply the above results to the quantized electromagnetic fields. Here the moments m_n will be expressed in terms of expectation values of normally ordered field annihilation $\hat{a}$ and creation $\hat{a}^\dagger$ operators.

$$m_n = \langle \hat{a}^{\dagger n} \hat{a}^n \rangle = \int_{-\infty}^{\infty} P(\alpha) |\alpha|^{2n} d^2\alpha \ . \tag{4.20}$$

Applicability of the result (4.18) implies that the underlying distribution $P(\alpha)$ will exhibit nonclassical properties

$$\text{if} \quad Q \geq 0 \quad but \quad \det m^{(3)} < 0 \; . \tag{4.21}$$

Supposing one has a situation such that

$$Q \geq 0 \;\; , \; \det m^{(3)} \geq 0 \; . \tag{4.22}$$

Then one can examine inequalities in terms of moments which are still higher than those given in (4.21),i.e.,

$$\det m^{(4)} < 0 \; , \tag{4.23}$$

to say that the field has nonclassical properties. When even this does not work then one has to examine the negativity of still higher order matrices $m^{(5)}, m^{(6)}$ etc.

We consider the case when $n = 3$.

For a coherent state, the $\det m^{(3)}$ turns out to be

$$\det m^{(3)} = \begin{vmatrix} 1 & |\alpha|^2 & |\alpha|^4 \\ |\alpha|^2 & |\alpha|^4 & |\alpha|^6 \\ |\alpha|^4 & |\alpha|^6 & |\alpha|^8 \end{vmatrix} \; , \tag{4.24}$$

which is equal to zero, whereas for the most nonclassical state namely the Fock state $|N\rangle$

$$\det m^{(3)} = \begin{vmatrix} 1 & N & N(N-1) \\ N & N(N-1) & N(N-1)(N-2) \\ N(N-1) & N(N-1)(N-2) & N(N-1)(N-2)(N-3) \end{vmatrix} = -2N^2(N-1) \, , \tag{4.25}$$

which does not have a lower bound. Thus it would be useful to introduce a normalized quantity so that it is bounded by -1. For this purpose we introduce μ_n, the analog of m_n, formed from the moments of the photon number distribution $p(m)$, which gives the probability of finding m photons in the field, rather than the moments of the $P(\alpha)$ distribution.

$$\mu_{(n)} = \langle(\hat{a}^{\dagger}\hat{a})^{n}\rangle = \sum_{n=o}^{\infty} p(m)m^{n} \quad , \tag{4.26}$$

and we construct the matrix $\mu^{(n)}$ similar to matrix $m^{(n)}$ given by (4.19) by the replacement $m_n \rightarrow \mu_n$. Clearly $\mu^{(n)}$ are positive definite matrices. As the photon number distribution $p(m)$ for any state $|\psi\rangle$ is given by

$$p(m) = |\langle m|\psi\rangle|^{2} \quad , \tag{4.27}$$

which is always positive.

For a Fock state $|N\rangle, \det\mu^{(n)} = 0$ if $N \geq 1$. As a measure of nonclassical property we introduce the quantity

$$\mathcal{A}_n = \frac{\det m^{(n)}}{\det\mu^{(n)} - \det m^{(n)}} \quad . \tag{4.28}$$

Clearly it can be seen that for a coherent state $|\alpha\rangle$, $\mathcal{A}_n$ is equal to zero and for a Fock state $|N\rangle$, $\mathcal{A}_n = -1$. It follows from this that the nonclassical region corresponds to

$$0 \leq \mathcal{A}_n \leq -1 \ . \tag{4.29}$$

Therefore we could use the parameter $\mathcal{A}_n$ to test the nonclassical character whenever the parameter Q fails to signify nonclassical character.

In an analogous way we can obtain quantitative measures for studying the nonclassical properties which are phase sensitive. We introduce a quantity

$$q_n = \langle: (\hat{a}e^{i\theta} + \hat{a}^{\dagger}e^{-i\theta})^{n} :\rangle \quad , \tag{4.30}$$

and define the matrices $q^{(n)}$ which can be obtained from the matrices $m^{(n)}$ in Eq.(4.19) by replacing m_n by q_n which is defined in the above equation. For a classical distribution, the matrices $q^{(n)}$ should be positive definite. For $n = 2$

$$q^{(2)} = \begin{pmatrix} 1 & \langle : (\hat{a}e^{i\theta} + \hat{a}^{\dagger}e^{-i\theta}) : \rangle \\ \langle : (\hat{a}e^{i\theta} + \hat{a}^{\dagger}e^{-i\theta}) : \rangle & \langle : (\hat{a}e^{i\theta} + \hat{a}^{\dagger}e^{-i\theta})^2 : \rangle \end{pmatrix} . \quad (4.31)$$

The above equation is non positive definite when

$$\det q^{(2)} < 0 \quad ,$$

i.e.,

$$\langle : (\hat{a}e^{i\theta} + \hat{a}^{\dagger}e^{-i\theta})^2 : \rangle - \langle : (\hat{a}e^{i\theta} + \hat{a}^{\dagger}e^{-i\theta}) : \rangle^2 < 0 \ , \quad (4.32)$$

which is equivalent to the requirement that the parameter S is negative. Thus the nonpositive definiteness of the $q^{(n)}$ matrices implies that the field possess phase sensitive nonclassical properties.

4.3. Examples

In this section we present few examples of states of the radiation field for which the new criterion developed in the previous section can be successfully applied for characterising the nonclassical properties.

(i) Radiation field in the photon added thermal states

The density matrix for this state is given by (see Chapter 2, Eq.2.40)

$$\hat{\rho} = \frac{1}{m!(1+\bar{n})^{1+m}} \hat{a}^{\dagger m} e^{-\lambda \hat{a}^{\dagger}\hat{a}} \hat{a}^{m} \quad , \quad (4.33)$$

where m is an integer and $\bar{n}$, the average photon number the thermal field, $\bar{n} = (e^{\lambda} - 1)^{-1}$.

In Chapter 2 we have calculated the Q parameter and the P function for the field in the above state. It is found to be (see Chapter 2, Eqs. (2.50) and (2.59) respectively).

$$Q = \frac{\bar{n}^2(1+m) - m}{\bar{n}(1+m) + m} \quad , \quad (4.34)$$

$$P(\alpha) = \frac{(-1)^m e^{-|\alpha|^2/\bar{n}}}{\pi(\bar{n})^{1+m}} L_m[(1+\frac{1}{\bar{n}})|\alpha|^2] \quad . \tag{4.35}$$

The condition for the existence of sub-Poissonian statistics can be deduced straight forwardly from Eq.(4.34). It is found to be

$$\bar{n} < \sqrt{\frac{m}{1+m}} \quad . \tag{4.36}$$

The behavior of Q parameter as a function of $\bar{n}$ for different values of m and P function for range of values of $\bar{n}$ is given in Figs.(2.6) and (2.7) respectively of Chapter 2. We can see from the Fig. (2.7) that $P(\alpha)$ still takes negative values even if the above condition is violated. Also note that the state (4.33) does not show any squeezing (see Chapter 2 Eq.(2.45)). Thus for $\bar{n} > \sqrt{\frac{m}{1+m}}$; $Q > 0$ and $S > 0$ implying that there should be some other quantity which tells that the field is nonclassical. We calculate the parameter $\mathcal{A}_3$ defined in previous section, for the state (4.33). In Fig.(4.1) we show the behavior of both Q and $\mathcal{A}_3$. One can see from the figure that eventhough Q becomes positive, $\mathcal{A}_3$ is still negative indicating that the field still possesses nonclassical characteristics.

(ii) Schrödinger Cat State

Here we consider a state

$$|\psi\rangle = \mathcal{N}(|\alpha e^{i\theta/2}\rangle + |\alpha e^{-i\theta/2}\rangle) \ , \tag{4.37}$$

formed by the superposition of two coherent states which has a phase difference of θ between them as shown in the Fig. (4.2). The coherent states $|\alpha e^{\pm i\theta/2}\rangle$ are given by

$$|\alpha e^{\pm i\theta/2}\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{m=o}^{\infty} \frac{\alpha^m}{\sqrt{m!}} e^{\pm im\theta/2} |m\rangle \quad , \tag{4.38}$$

where $|m\rangle$ is a Fock state. The normalisation constant $\mathcal{N}$ has the form

$$\mathcal{N}^2 = [2 + e^{-|\alpha|^2}(\exp(|\alpha|^2 e^{i\theta}) + c.c.)]^{-1} \quad . \tag{4.39}$$

The above state $|\psi\rangle$ can be called as Schrödinger cat state which is a quantum superposition of macroscopically distinguishable states. Due to the quantum interference the properties of such superposition of coherent states are very different from the properties of the individual coherent states. For instance, the Schrödinger cat state exhibits squeezing [6,7], higher order squeezing [8], sub-Poissonian photon statistics [7] and oscillations in the photon number distributions [7]. There have already been proposed several schemes to produce the state. The nonlinear interaction of the field in a coherent state with a Kerr-like medium can produce Schrödinger cat state [9, 10]. Another possible way would be through the unitary evolution of the field initially in a coherent state interacting with the two level atoms [11-13].

The normally ordered moments $\langle \hat{a}^{\dagger n}\hat{a}^n\rangle$ for the field in the state (4.37) are calculated.

$$\begin{aligned}
\langle \hat{a}^{\dagger}\hat{a}\rangle &= \langle\psi|\hat{a}^{\dagger}\hat{a}|\psi\rangle \\
&= \mathcal{N}^2|\alpha|^2[2+e^{-|\alpha|^2}\{exp(i\theta+|\alpha|^2e^{i\theta})+c.c\}] \ , \qquad (4.40)\\
\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle &= \mathcal{N}^2|\alpha|^4[2+e^{-|\alpha|^2}\{exp(2i\theta+|\alpha|^2e^{i\theta})+c.c.\}] \ , \qquad (4.41)\\
\langle \hat{a}^{\dagger 3}\hat{a}^3\rangle &= \mathcal{N}^2|\alpha|^6[2+e^{-|\alpha|^2}\{exp(3i\theta+|\alpha|^2e^{i\theta})+c.c.\}] \ , \qquad (4.42)\\
\langle \hat{a}^{\dagger 4}\hat{a}^4\rangle &= \mathcal{N}^2|\alpha|^8[2+e^{-|\alpha|^2}\{exp(4i\theta+|\alpha|^2e^{i\theta})+c.c.\}] \ . \qquad (4.43)
\end{aligned}$$

The relation between the normally ordered moments $\langle \hat{a}^{\dagger n}\hat{a}^n\rangle$ and the moments of the number distribution $\langle(\hat{a}^{\dagger}\hat{a})^n\rangle$ can be expressed by means of their respective generating functions which are related by [14]

$$\langle\exp(x\hat{a}^{\dagger}\hat{a})\rangle = \langle:\exp(\hat{a}^{\dagger}\hat{a}(e^x-1)):\rangle \ . \qquad (4.44)$$

On comparing the coefficients of x in the power series expansion of the above equation we get the relation between $\langle(\hat{a}^{\dagger}\hat{a})^k\rangle$ and $\langle:(\hat{a}^{\dagger}\hat{a})^k:\rangle$. Thus

$$\mu_1 = m_1 \ , \qquad (4.45)$$

$$\mu_2 = m_1 + m_2 \quad , \tag{4.46}$$

$$\mu_3 = m_1 + 3m_2 + m_3 \quad , \tag{4.47}$$

$$\mu_4 = m_1 + 7m_2 + 6m_3 + m_4 \quad . \tag{4.48}$$

We calculate both the Q and the $\mathcal{A}_3$ parameters for the state (4.37) using equations (4.40)-(4.48). We show in Fig.(4.3) how these parameters vary with change in θ for $\alpha = 8$. We see that the quantity $\mathcal{A}_3$ becomes negative for the range of θ values where Q is positive thus indicating the nonclassical character of the state (4.37).

(iii) Radiation Field in the Micromaser Cavity

The micromaser [15-18] consists of a superconducting microwave cavity (cavity made from pure niobium) with a high quality factor, $Q(3 \times 10^{10})$. A low velocity beam of Rydberg atoms (atoms with a valence electron in a highly excited orbit having extremely large dipole moments for transitions to neighbouring levels) are injected inside the cavity at such a low rate that at most one atom at a time is present inside the cavity. While an atom is inside the cavity the coupled atoms field system is described by Jaynes Cummings Hamiltonian and for the intervals between the successive atoms the evolution of the field is governed by the master equation of a harmonic oscillator interacting with a thermal bath. This is under the assumption that the transit time is much smaller than the relaxation time in the cavity.

At time t_i, an atom with density matrix $\hat{\rho}_a$ enters the cavity containing the field in the state $\hat{\rho}_f(t_i)$ so that the density operator of the combined system is simply the tensor product of $\hat{\rho}_f(t_i)$ and $\hat{\rho}_a$ i.e., $\hat{\rho}(t_i) = \hat{\rho}_a \otimes \hat{\rho}_f(t_i)$. After the interaction, the state of the system is

$$\hat{\rho}(t_i + \tau) = \hat{U}(\tau)\hat{\rho}(t_i)\hat{U}^{\dagger}(\tau) \quad . \tag{4.49}$$

The time evolution operator $\hat{U}(t) = \exp(-i\hat{H}t/\hbar)$ is easily found from the Jaynes Cummings Hamiltonian for the interaction of an atom with a single mode field

$$\hat{H} = \frac{1}{2}\hbar\omega_o\hat{S}_z + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar g(\hat{S}^+\hat{a} + \hat{S}^-\hat{a}^\dagger) \ , \tag{4.50}$$

where $\hbar\omega_o$ is the energy difference between the two atomic levels, g is the atom-cavity coupling, $\hat{a}$ and $\hat{a}^\dagger$ the field annihilation and creation operators, ω the frequency of the cavity field mode and $\hat{S}_z, \hat{S}^\pm$ are the Pauli spin operators. Here we assume that cavity mode frequency coincides with atomic transition frequency, i.e., $\omega = \omega_o$.

After the interaction time τ, when the atom leaves the cavity the field is left in the state described by the reduced density operator.

$$\begin{aligned} \hat{\rho}_f(t_i + \tau) &= Tr_a[\hat{U}(\tau)\hat{\rho}(t_i)U^\dagger(\tau)] \\ &= T(\tau)\hat{\rho}_f(t_i) \ , \end{aligned} \tag{4.51}$$

where Tr_a denotes trace over the atomic states and $T(\tau)$ is the time evolution operator of the field over the time τ the atom stays in the cavity. In the interval $t_i + \tau$ at which the next atom is injected, the field evolves at the rate $\gamma = \omega/Q$ (photon damping rate) towards a thermal steady state which can be described by the standard master equation [14]

$$\begin{aligned} \dot{\hat{\rho}}_f \equiv L\hat{\rho}_f &= (\frac{\gamma}{2})(1 + n_b)(2\hat{a}\hat{\rho}_f\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}_f - \hat{\rho}_f\hat{a}\hat{a}^\dagger) \\ &+ (\frac{\gamma}{2})n_b(2\hat{a}^\dagger\hat{\rho}_f\hat{a} - \hat{a}\hat{a}^\dagger\hat{a}\hat{\rho}_f - \hat{\rho}_f\hat{a}\hat{a}^\dagger) \ . \end{aligned} \tag{4.52}$$

where L, is the Liouville operator, has a simple structure $L = -i[\hat{H}, \ ]$ and n_b is the average number of thermal photons in the cavity.

At time t_{i+1} the field is therefore described by density matrix

$$\hat{\rho}_f(t_{i+1}) = \exp(Lt_p)T(\tau)\hat{\rho}_f(t_i) \ , \tag{4.53}$$

where $t_p = t_{i+1} - t_i - \tau$ is the time interval between atom i leaving the cavity and atom $i+1$ entering it. Suppose the field density matrix is initially diagonal in the Fock state and the atoms without initial coherence are injected inside the cavity, then the reduced density operator of the field remains diagonal during the interaction. The probability distribution $\langle n|\hat{\rho}_f|n'\rangle = p_n\delta_{n,n'}$ can be obtained from Eq.(4.51) as

$$p_n(t_i+\tau) = (1-\lambda_{n+1})p_n(t_i) + \lambda_n p_{n-1}(t_i) \ , \tag{4.54}$$

where $\lambda_n = \sin^2(g\sqrt{n}\tau)$. The field remains diagonal during its decay so that the master equation (4.52) can be restricted to its diagonal elements.

$$\dot{p}_n = \gamma(n_b+1)[(n+1)p_{n+1} - np_n] + \gamma n_b[np_{n-1} - (n+1)p_n] \ . \tag{4.55}$$

The equation (4.53) can be further simplified by assuming that the time intervals of the injected atoms obey Poissonian statistics with mean spacing $1/r$ between events, where r is the atomic flux. The distribution is given by

$$P(t_p) = r\exp(-rt_p) \ . \tag{4.56}$$

The field density matrix as given by Eq.(4.53) is then averaged over the above distribution, so that it can now be written as

$$\bar{\hat{\rho}}_f(t_{i+1}) = (1-\frac{L}{r})^{-1}T(\tau)\bar{\hat{\rho}}_f(t_i) \ . \tag{4.57}$$

At steady state $\bar{\hat{\rho}}_f(t_{i+1}) = \bar{\hat{\rho}}_f(t_i) \equiv \hat{\rho}_{f,st}$. Then the above equation becomes

$$(1-\frac{L}{r})\hat{\rho}_{f,st} = T(\tau)\hat{\rho}_{f,st} \ . \tag{4.58}$$

Using (4.54),(4.55) and (4.58) we get a three term recursion relation for the probability distribution $\bar{p}_n = \langle n|\hat{\rho}_{f,st}|n\rangle$, as follows

$$S_n = S_{n+1} \ , \tag{4.59}$$

with

$$S_n \equiv \frac{r}{\gamma}\lambda_n \bar{p}_{n-1} + n_b \bar{p}_{n-1} - (n_b + 1) n \bar{p}_n \quad . \tag{4.60}$$

It is well known that the physical condition for the density matrix of the field be normalisable implies that the probability distribution $\bar{p}_n \to 0$ faster than $1/n$ (divergent series) for $n \to \infty$. Hence, in the above equation $S_n \to 0$ for $n \to \infty$ and from Eq.(4.54) it is obvious that $S_n = 0$ for all n. The ratio of the successive probability number distribution $\bar{p}_n$ can then be obtained from (4.60), i.e.,

$$\frac{r}{\gamma}\lambda_n \bar{p}_{n-1} + n_b n \bar{p}_{n-1} - (n_b + 1) n \bar{p}_n = 0 \quad , \tag{4.61}$$

or

$$\bar{p}_n = \frac{n_b}{1+n_b}\left[1 + \frac{r\lambda_n}{\gamma n n_b}\right]\bar{p}_{n-1} \quad , n = 1, 2, \cdots \quad . \tag{4.62}$$

Eq.(4.62) can also be written as

$$\bar{p}_n = \left(\frac{n_b}{1+n_b}\right)^n \prod_{m=1}^{n}\left[1 + \frac{r\lambda_m}{\gamma n_b m}\right]\bar{p}_o \quad , \tag{4.63}$$

for $n \geq 1$, where the probability $\bar{p}_o$ is determined by the normalization condition

$$\sum_{n=0}^{\infty} \bar{p}_n = 1 \quad .$$

The Q parameter has been calculated for this field [16]. It is found that for certain range of parameter values the Q parameter is positive. We calculate the parameter $\mathcal{A}_3$ as defined in Section 4.2 (Eq.(4.28)), using (4.63). The normally ordered moments $\langle \hat{a}^{\dagger k}\hat{a}^k \rangle$ can be expressed in terms of moments of the number distributions using Eqs.(4.45)-(4.48). Fig.(4.4) shows the Q parameter and the parameter $\mathcal{A}_3$ as a function of $g\tau\sqrt{r/\gamma}$. We find that the parameter $\mathcal{A}_3$ is negative for $g\tau\sqrt{r/\gamma}$ in the range π to 5π eventhough the Q parameter becomes positive. Thus the field remains nonclassical for $g\tau\sqrt{r/\gamma}$ in the range π to 5π eventhough there is no signature of the sub-Poissonian nature of the field.

The intrinsic quantum nature of a radiation field manifests itself in the violation of some classical inequalities. This violation is possible only if we allow the P distribution function to assume negative values, which is only possible in the quantum world. Squeezing and sub-Poisson photon statistics are the two well known features which characterises the nonclassical field. These are expressed in terms of lower order moments of P function. We have developed a new criterion involving higher order moments of P function to characterise the nonclassical properties of the field which is especially useful when the field does not exhibit squeezing and sub-Poisson statistics. We have demonstrated the utility of this criterion by applying it successfully to study the nonclassical properties of the field in the photon added thermal state, in Schrödinger cat state and the field produced in the micromaser cavity. Other applications of this criterion will be considered in Chapter 5.

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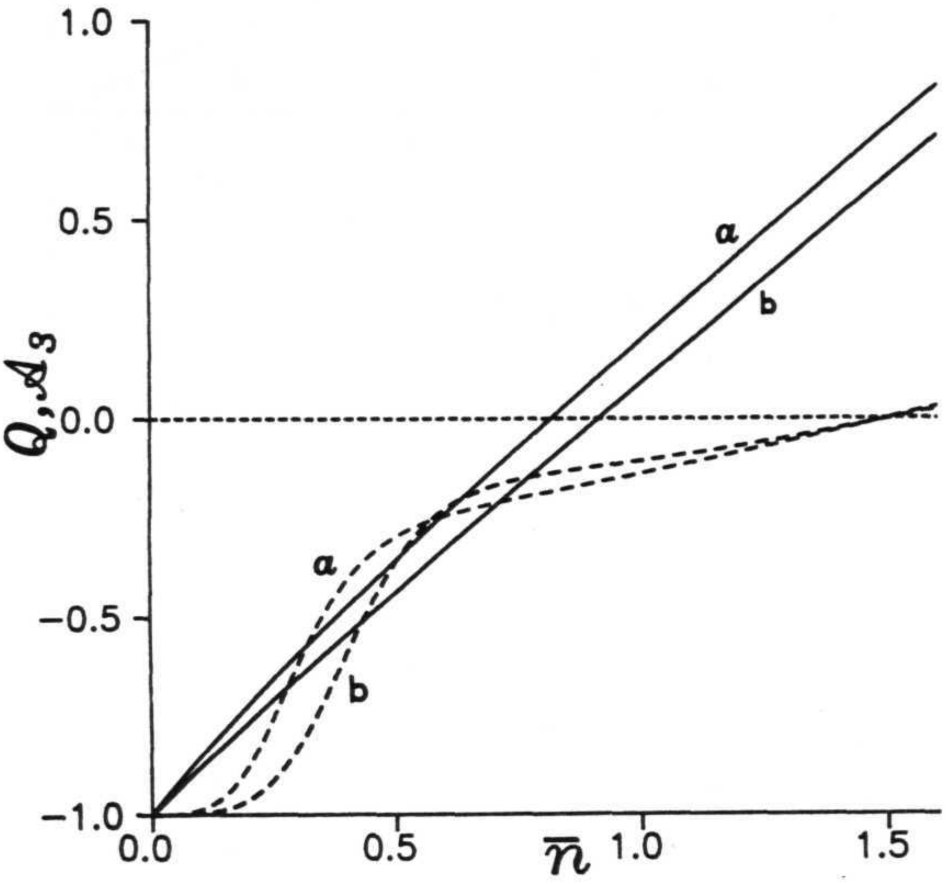

Fig. 4.1. The parameter Q (solid curves) and $\mathcal{A}_3$ (dashed curves) as functions of average photon number $\bar{n}$ and for m (a)=2, (b)=5. For $\bar{n} > \sqrt{m/(1+m)}$, the parameter Q becomes positive but the parameter $\mathcal{A}_3$ still continues to be negative implying that the field is still nonclassical.

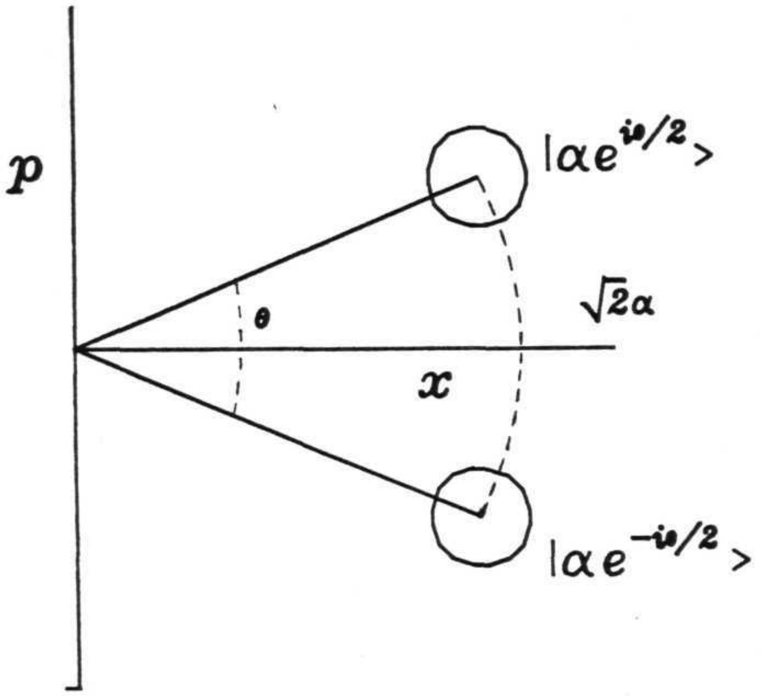

Fig. 4.2. The Schrodinger cat state $|\psi>$ (Eq.(4.37)) can be visualised by two circles of radius unity displaced by an amount $\sqrt{2}\alpha$ from the origin and having an angle θ between them.

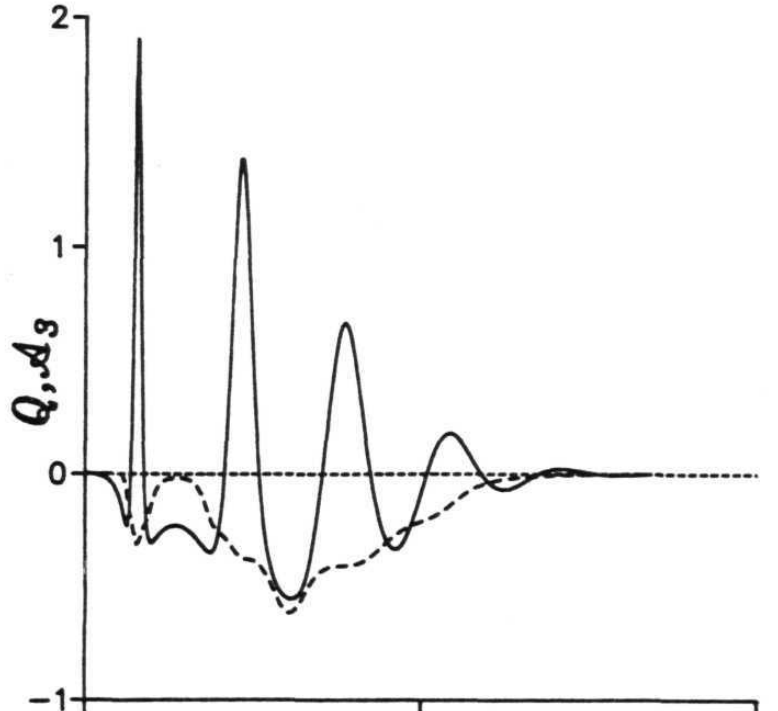

Fig. 4.3. The parameter Q (solid line) and $\mathcal{A}_3$ (dashed line) as a function of θ. Schrödinger cat state for α=8. It is interesting to note that $\mathcal{A}_3$ is negative in the range of θ values where Q is positive.

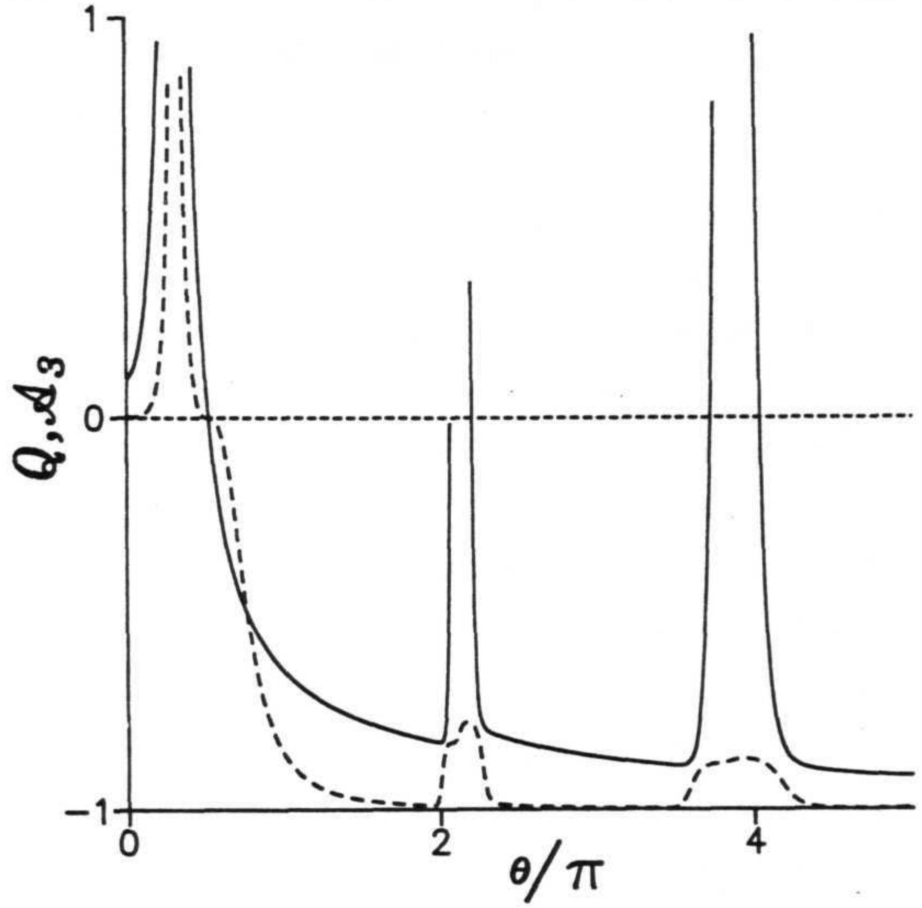

Fig. 4.4. The parameter Q (solid line) and $\mathcal{A}_3$ (dashed line) as a function of the pump parameter $\theta = g\tau\sqrt{r/\gamma}$ and for r/γ=200 and black body photon number $n_b = 0.1$.

Chapter 5

TRANSFORMATIONS OF THE NONCLASSICAL STATES BY AN OPTICAL AMPLIFIER

It is well known that the statistical properties of a light field are modified by a linear amplifier [1]. When a signal passes through an amplifier it is modified in two ways: Firstly the amplitude of the signal gets amplified, Secondly, noise is added to the signal which can be either phase sensitive or phase insensitive. Here we shall consider only phase insensitive optical amplifier. The ultimate lower limit on the amount of noise which is added is dictated by the quantum mechanics. This has led to considerable amount of work on the quantum properties of optical amplifiers [2-6]. The added noise is of particular concern when one wants to amplify a nonclassical state of the electromagnetic field. If too much phase insensitive noise is added to such a state it loses its nonclassical character. One finds that for sufficiently high gain all nonclassical input states will become classical at the output, because the amount of noise added increases with the amplifier gain. In particular, when the incoming field exhibits nonclassical properties such as sub-Poissonian statistics or squeezing, then these features can be lost after amplification when the gain is too high.

The question of transformations of the nonclassical states of the electromagnetic field by a phase insensitive optical amplifier has been examined in detail by Hong et al. [2] They have derived limits on the amplifier gain that are necessary and sufficient for the output of the amplifier such that

(a) it continues to exhibit sub-Poissonian statistics if the input field is sub-Poissonian.

(b) it continues to exhibit squeezing if the input field is squeezed.

They have also obtained another limit in which all the nonclassical features of the output state vanish i.e., the P function of the output has the properties of a classical probability distribution. These two limits never coincide. Thus there is a domain of the amplifier characteristics for which the output is neither squeezed nor exhibits sub-Poissonian statistics but has a P function which is still nonclassical implying the existence of the quantum effects other than sub-Poissonian statistics and squeezing. In such a situation one has to search for physical quantities that have nonclassical characteristics. This is what is done in this chapter.

In Section 5.1, we summarise the known characteristics of the optical amplifiers. In Section 5.2 we consider an input field to be in nonclassical state like Fock state and examine the limits on the amplifier gain for which the output remains nonclassical, by using the new criterion which was developed in the previous chapter. We present physical quantities which remain nonclassical in the region in which sub-Poissonian statistics and squeezing have already disappeared, amplifier gain remaining below the threshold value for all nonclassical features of the output field of the amplifier to be lost.

5.1 Linear Optical Amplifier

We consider a simple model of an optical amplifier. It can be modelled by a collection of identical two level atoms with level spacing $\hbar\omega$ of which N_1 atoms are in the excited state and N_2 atoms are in the ground state with $N_2 \leq N_1$. The atomic transition has a width which is large. The atoms interact through their electric dipoles with a single mode electromagnetic field. The propagation of the field through the amplifier can be described by the density matrix equation [7-9].

$$\begin{aligned}\frac{\partial\hat{\rho}}{\partial t} &= -\kappa N_1\left(\hat{a}\hat{a}^\dagger\hat{\rho} - 2\hat{a}^\dagger\hat{\rho}\hat{a} + \hat{\rho}\hat{a}\hat{a}^\dagger\right)\\ &- \kappa N_2\left(\hat{a}^\dagger\hat{a}\hat{\rho} - 2\hat{a}\hat{\rho}\hat{a}^\dagger + \hat{\rho}\hat{a}^\dagger\hat{a}\right) \ ,\end{aligned} \tag{5.1}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are photon annihilation and creation operators and κ is a coupling constant. The field mode has been taken on resonance with the atomic transition. The evolution of the field amplitude and number of photons can be obtained directly from the equation (5.1) as

$$< \hat{a}(t) > \ = \ \exp\{(N_1 - N_2)\kappa t\} < \hat{a}(0) > \ \equiv \ G < \hat{a}(0) > \ , \tag{5.2}$$

$$\begin{aligned}< \hat{a}^\dagger(t)\hat{a}(t) > &= \exp\{2(N_1 - N_2)\kappa t\} < \hat{a}^\dagger(0)\hat{a}(0) >\\ &+ \frac{N_1(\exp\{2(N_1 - N_2)\kappa t\} - 1)}{(N_1 - N_2)} \ .\end{aligned} \tag{5.3}$$

The quantity G gives the gain of the amplifier

$$G(t) = \exp\{(N_1 - N_2)\kappa t\} \ , \tag{5.4}$$

and the last term in (5.3) represents the noise photon added by the amplifier. Note that G depends on time t or the propagation length.

The time evolution of the density matrix can be obtained by solving Eq.(5.1). It is however more convenient to work with quasiprobabilities like the P function and the Q function. The behavior of the P function tells us when the state is classical or nonclassical as discussed in the first chapter. We write the solution of (5.1) as the input-output relation

$$\begin{aligned}\hat{\rho} &= \int P(\alpha)|\alpha >< \alpha| d^2\alpha \ ,\\ P_{out}(\alpha) &= \int d^2\alpha_o P_{in}(\alpha_o)\frac{1}{\pi\bar{n}}\exp\left\{-\frac{|\alpha - \alpha_o G|^2}{\bar{n}}\right\} \ ,\end{aligned} \tag{5.5}$$

$$
\begin{aligned}
Q &= \frac{1}{\pi} < \alpha|\hat{\rho}|\alpha > , \\
Q_{out}(\alpha) &= \int d^2\alpha_o Q_{in}(\alpha_o) \frac{1}{\pi\bar{m}} \left\{ -\frac{|\alpha - \alpha_o G|^2}{\bar{m}} \right\} ,
\end{aligned} \tag{5.6}
$$

where

$$
\bar{n} = \left(\frac{N_1}{N_1 - N_2} \right) (G^2 - 1) \ , \bar{m} = \frac{N_2}{(N_1 - N_2)} (G^2 - 1) \ . \tag{5.7}
$$

Note that if all the atoms are in the excited state, then $\bar{m} \rightarrow 0$. In such a case (5.6) reduces to

$$
Q_{out}(\alpha) = \frac{1}{G^2} Q_{in}(\alpha/G) = \frac{1}{\pi G^2} \langle \frac{\alpha}{G} | \hat{\rho}_{in} | \frac{\alpha}{G} \rangle \ , \tag{5.8}
$$

and hence a simple scaling of the input Q function gives the Q function for the output.

Hong et al [4] have shown that the output of the amplifier exhibits sub-Poissonian (squeezing) character if the input has the corresponding character provided the gain of the amplifier satisfies

$$
G^2 < \frac{2N_1}{(N_1 + N_2)} \leq 2 \ . \tag{5.9}
$$

They have further shown that all nonclassical features of the optical field at the output are lost if

$$
\bar{n}/G^2 \geq 1 \ , \ G^2 \geq \frac{N_1}{N_2} \ . \tag{5.10}
$$

The condition (5.10) is a sufficient condition. In the special case when $N_2 = 0$, then Eq.(5.10) leads to $G \rightarrow \infty$ whereas (5.9) becomes $G^2 < 2$. Therefore from Eqs. (5.9) and (5.10) one can see that there is a range of the amplifier gain below which the well known nonclassical features such as sub-Poissonian statistics and squeezing disappear but which itself lies below the gain at which the amplifier output must be nonclassical. A question then arises as to whether there are any nonclassical properties other than sub-Poissonian statistics or squeezing which survive in this range. We explore this point further in the following section.

5.2. Nonclassical Properties at the Output when Q Parameter is Positive

We answer the question raised in previous section by considering the input field in a Fock state $|n>$. Note that this state is in a sense maximally nonclassical as far as the photon number fluctuations are concerned. The Q parameter for the field in a Fock state takes the minimum possible value equal to minus one. No functional form exists for a P function for a Fock state, however it can be written in terms of derivatives of delta function [10].

$$P(\alpha) = \frac{1}{n!} e^{|\alpha|^2} \left(\frac{\partial^2}{\partial\alpha\partial\alpha^*} \right)^n \delta^2(\alpha) \ . \tag{5.11}$$

The Q function however is well behaved and is given by

$$Q_{in}(\alpha) = \frac{1}{\pi} \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!} \ . \tag{5.12}$$

The output Q function is obtained from (5.6) as

$$Q_{out}(\alpha) = \frac{1}{\pi^2 \bar{m} n!} \int d^2\alpha_o |\alpha_o|^{2n} \exp\left\{ -|\alpha_o|^2 - \frac{|\alpha - \alpha_o G|^2}{\bar{m}} \right\} \ . \tag{5.13}$$

The antinormally ordered moments of the output distribution can be easily obtained using (5.13) as

$$\begin{aligned} < \hat{a}^p \hat{a}^{\dagger p} > &= \int d^2\alpha Q_{out}(\alpha) |\alpha|^{2p} \\ &= \frac{1}{\pi^2 \bar{m}} \int d^2\alpha_o \int d^2\alpha \frac{|\alpha_o|^{2n} |\alpha|^{2p}}{n!} \exp\left\{ -|\alpha_o|^2 - \frac{|\alpha - \alpha_o G|^2}{\bar{m}} \right\} \ . \end{aligned} \tag{5.14}$$

The integration over a plane can be done by making a transformation as $\beta = \alpha - \alpha_o G$ and using

$$\int \beta^n \beta^{*m} \exp(-|\beta|^2/s) d^2\beta = \frac{\pi n!}{s^{n+1}} \delta_{nm} \ . \tag{5.15}$$

Then (5.14) reduces to

$$\begin{aligned} < \hat{a}^p \hat{a}^{\dagger p} > &= \frac{1}{\pi} \sum_{q=o}^{p} \binom{p}{q}^2 G^{2(p-q)} \frac{\bar{m}^q q!}{n!} \int |\alpha_o|^{2(p-q+n)} e^{-|\alpha_o|^2} d^2\alpha_o \\ &= \sum_{q=o}^{p} \frac{(p!)^2}{q!((p-q)!)^2} G^{2(p-q)} \bar{m}^q \frac{(n+p-q)!}{n!} \ . \end{aligned} \tag{5.16}$$

The normally ordered moments can be obtained by using the relation between normally and antinormally ordered moments [7]

$$\hat{a}^{\dagger l}\hat{a}^{l} = \sum_{r=o}^{l} \frac{(-1)^{l+r}}{r!}\binom{l}{r}\hat{a}^{r}\hat{a}^{\dagger r} l! \ . \tag{5.17}$$

We next evaluate the P function of the output. The P function for the input is

$$P_{in}(\alpha_o) = \frac{e^{|\alpha_o|^2}}{n!}\frac{\partial^{2n}}{\partial\alpha_o^n\partial\alpha_o^{*n}}\delta^{(2)}(\alpha_o) \ . \tag{5.18}$$

On substituting (5.18) in (5.5) we obtain

$$P_{out}(\alpha) = \frac{1}{\pi\bar{n}n!}\frac{\partial^{2n}}{\partial\xi^n\partial\xi^{*n}}\exp\left\{|\xi|^2 - |\frac{\alpha-\xi G|^2}{\bar{n}}\right\}\Bigg|_{\xi=o} , \tag{5.19}$$

which can be written as

$$P_{out}(\alpha) = \frac{e^{-|\alpha|^2/\bar{n}}}{\pi\bar{n}n!}\frac{\partial^{2n}}{\partial\xi^n\partial\xi^{*n}}\exp\left\{\left(\frac{G}{\bar{n}}\alpha\right)\xi^* + \left(\frac{G}{\bar{n}}\alpha^*\right)\xi + \left(1-\frac{G^2}{\bar{n}}\right)\xi\xi^*\right\}\Bigg|_{\xi=o} . \tag{5.20}$$

On using the identity [11]

$$\exp(\lambda W + \mu z + \nu W z) = \sum_{i,j=o}^{\infty}\frac{\nu^j\mu^{i-j}}{i!}L_j^{i-j}\left(-\frac{\mu\lambda}{\nu}\right)W^j z^i , \tag{5.21}$$

where L_j^{i-j} is the associated Laguerre polynomial [12]

$$L_j^{i-j}(x) = \sum_{k=o}^{j}\frac{i!(-x)^k}{(j-k)!k!(i-j+k)!} \ . \tag{5.22}$$

Eq.(5.20) is then reduced to

$$\begin{aligned}
P_{out}(\alpha) &= \frac{e^{-|\alpha|^2/\bar{n}}}{\pi\bar{n}n!}\frac{\partial^{2n}}{\partial\xi^n\partial\xi^{*n}}\left[\sum_{i,j=o}^{\infty}\left(1-\frac{G^2}{\bar{n}}\right)^j\left(\frac{G}{\bar{n}}\alpha^*\right)^{i-j}L_j^{i-j}\left[\frac{-|\alpha|^2}{\bar{n}(\bar{n}/G^2-1)}\right]\frac{\xi^j\xi^{*i}}{i!}\right]\Bigg|_{\xi=o} \\
&= \frac{e^{-|\alpha|^2/\bar{n}}}{\pi\bar{n}n!}\sum_{i,j=o}^{\infty}\frac{1}{i!}\left(1-\frac{G^2}{\bar{n}}\right)^j\left(\frac{G}{\bar{n}}\alpha^*\right)^{i-j}L_j^{i-j}\left[\frac{-|\alpha|^2}{\bar{n}(\bar{n}/G^2-1)}\right]n!n!\delta_{jn}\delta_{in} \\
&= \frac{e^{-|\alpha|^2/\bar{n}}}{\pi\bar{n}}L_n\left[\frac{-|\alpha|^2}{\bar{n}(\bar{n}/G^2-1)}\right]\left(1-\frac{G^2}{\bar{n}}\right)^n \ .
\end{aligned} \tag{5.23}$$

Note that if $\bar{n}/G^2 < 1$ then $P_{out}(\alpha)$ is an oscillatory function of $|\alpha|^2$ and thus is negative in certain regions. For $\bar{n}/G^2 > 1$, the function (5.23) has all the properties of a classical probability distribution and is thus well behaved. This is in conformity with the result of Hong et al. It is interesting to examine how the negative regions in (5.23) shrink as $\bar{n}/G^2$ approaches unity. For this purpose we examine the zeroes of the Laguerre polynomials. The zeroes of $L_n(x)$ are given by [13].

$$x_m^{(n)} \cong \frac{j_{om}^2}{2(2n+1)}\left[1 + \frac{j_{om}^2 - 2}{12(2n+1)^2}\right] \quad , \tag{5.24}$$

where j_{om} is the m-th positive zero of the Bessel function $J_o(x)$ [13]. The first few zeros of $J_o(x)$ are given by 2.40482, 5.52007, 8.65372 et. Clearly the zeros of the P function get more and more separated as $\bar{n}/G^2$ approaches unity. In the Fig. (5.1) we present the behavior of the P function of the output for several values of the parameters $N_1/(N_1 - N_2)$ and the amplifier gain so that output distribution is both nonclassical and classical.

We next answer the question - what is nonclassical in between the domain defined by the inequalities (5.9) and (5.10). We make use of the new criterion developed in the previous chapter. For this purpose we need to study the higher order moments of the output distribution which is related to antinormally ordered moments as given by Eq.(5.17). We calculate the parameter $\mathcal{A}_3$ defined by

$$\mathcal{A}_3 = \frac{\det m^{(3)}}{\det m^{(3)} - \det \mu^{(3)}} \quad , \tag{5.25}$$

where $m^{(3)}$ is the matrix

$$m^{(3)} = \begin{bmatrix} 1 & m_1 & m_2 \\ m_1 & m_2 & m_3 \\ m_2 & m_3 & m_4 \end{bmatrix} \quad , m_p = < \hat{a}^{\dagger p}\hat{a}^p > \tag{5.26}$$

and $\mu^{(3)}$ is defined by (5.26) with m_p replaced by $\langle(\hat{a}^\dagger\hat{a})^p\rangle$ of the number distribution. In previous chapter it was shown that the nonclassical region corresponds to $0 \geq \mathcal{A}_3 \geq -1$.

We also calculate Mandel's Q parameter

$$Q = \frac{< \hat{a}^{\dagger 2}\hat{a}^2 > - < \hat{a}^\dagger \hat{a} >^2}{< \hat{a}^\dagger \hat{a} >} . \tag{5.27}$$

In Figs. (5.2) and (5.3) we show the behavior of the parameter Q and $\mathcal{A}_3$ for a range of the amplifier characteristics. These figures show the utility of the parameter $\mathcal{A}_3$ when Q is positive but the underlying P function is still nonclassical.

In conclusion we have studied the transformations of the nonclassical states by a phase insensitive optical amplifier. We have derived the physical characteristics which are nonclassical in the region where the amplifier output does not exhibit either squeezing or sub-Poissonian statistics. This characteristics is obtained by using the new criterion developed in the previous chapter. We considered the transformation properties of the Fock state which is maximally nonclassical. The other nonclassical states of the field can be treated similarly because it is well known that for an input field in a state such that its Q function is Gaussian then the output of the amplifier is also characterised by a Gaussian Q function.

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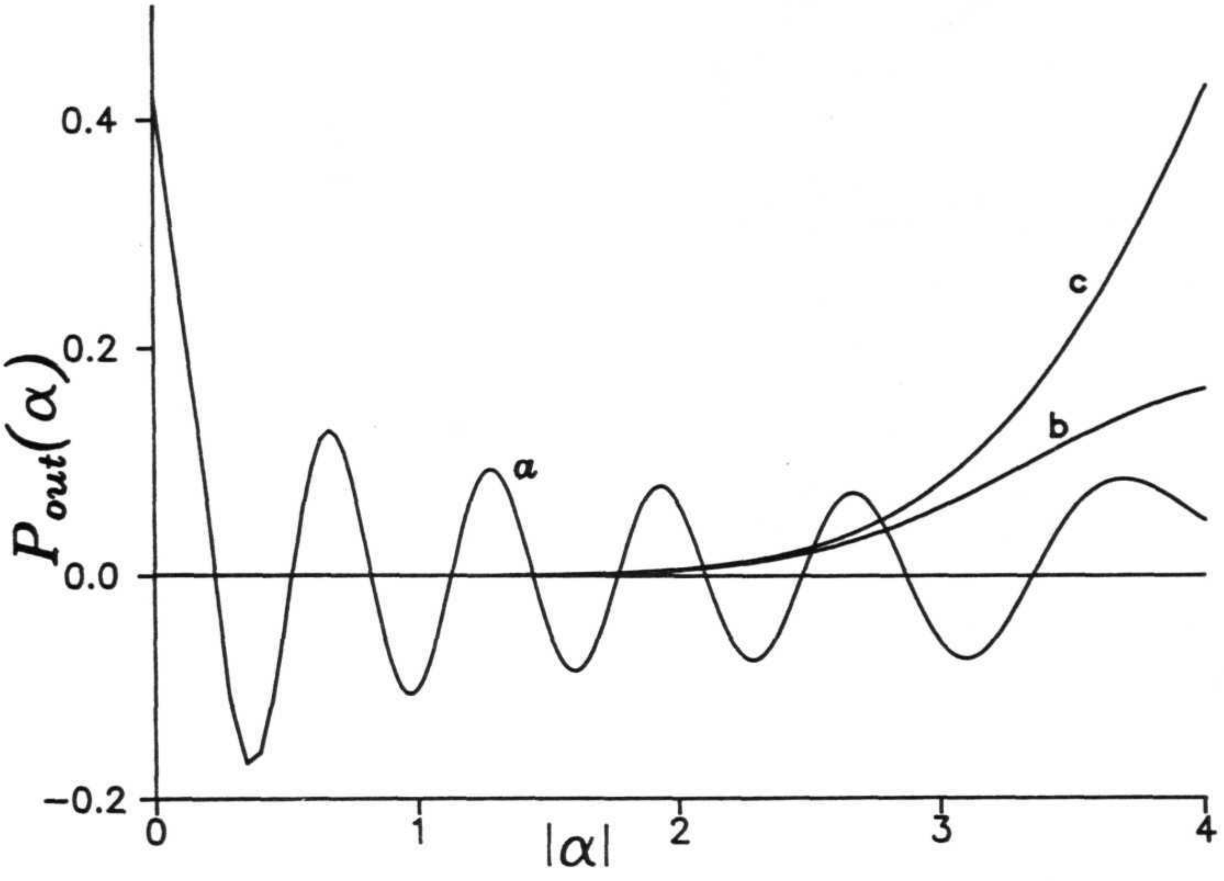

Fig. 5.1. The distribution $P_{out}(x)$ at the input of the amplifier (Eq.(5.23)) as a function of $|\alpha|$ with (a) $\frac{N_1}{N_1 - N_2}$=1.5, G^2=1.5, (b) $\frac{N_1}{N_1 - N_2}$=2, G^2=3. Note that the actual values in the curves (b) and (c) are 1/10-th and 1/100-th times respectively of those shown. The input field is considered to be in a Fock state with 10 photons.

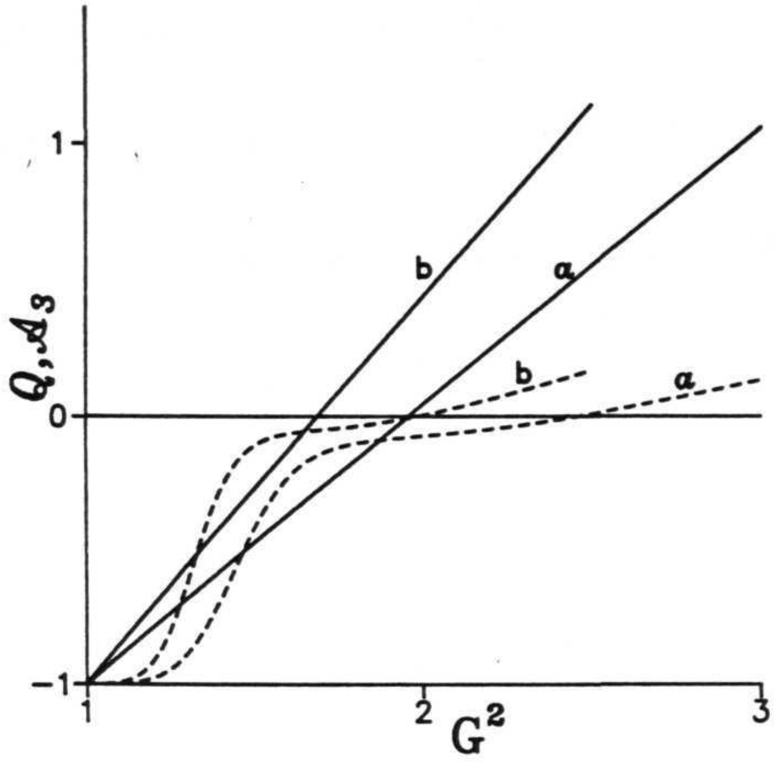

Fig. 5.2. The parameter Q (solid curves) and $\mathcal{A}_3$ (dashed curves) as a function of the amplifier gain G^2 for $\frac{N_1}{N_1 - N_2}$ (a)=1, (b)=1.2. The number of photons in the input field is taken to be 10.

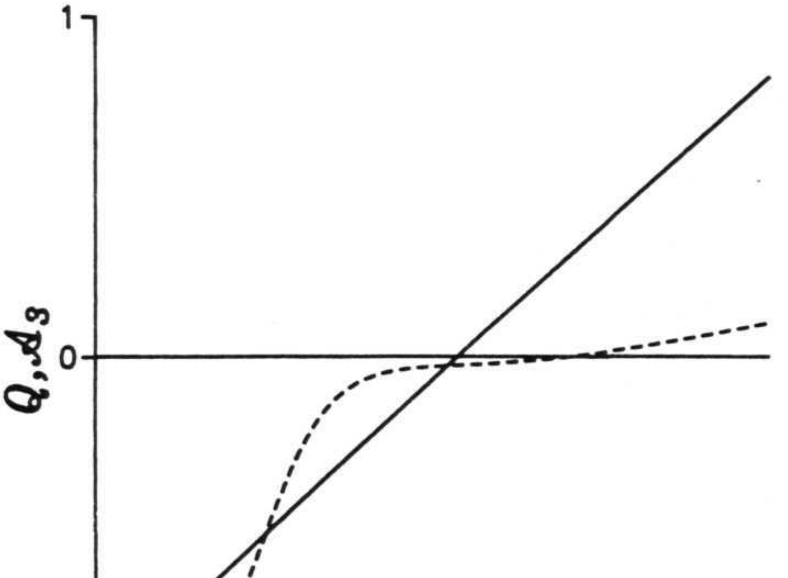

Fig. 5.3. Same as in Fig. 5.2 except for $\frac{N_1}{N_1 - N_2}=2$.

Chapter 6

QUANTUM PHASE DISTRIBUTION IN NONLINEAR OPTICAL PHENOMENA

In quantum theory the problem of correct definition of phase operator of a single mode electromagnetic field has a long and contentious history. It has provoked many discussions and controversies. In classical coherence theory, to deal with fluctuating fields, one introduces a distribution $P_{cl}(|\alpha|e^{i\varphi})$ for the complex field amplitude a.

$$a = \sqrt{I}e^{i\varphi} \ , \tag{6.1}$$

where $I(= |\alpha|^2)$ is the intensity of the field and φ its phase. The phase distribution is then naturally obtained from $P_{cl}(|\alpha|e^{i\varphi})$ by integrating over the amplitude $|\alpha|$ of the field.

When the single mode classical field is quantised, a becomes an operator. Thus the analogue of (6.1) in quantum mechanics is, the separation of the operator $\hat{a}$ into product of amplitude and phase operators. The amplitude of the field is proportional to the square root of the photon number operator. But the question is how to define the phase operator? In the published literature one can find several descriptions of the phase operator [1-5]. The main considerations are that the quantum mechanical phase should have the same significance as the classical phase in the appropriate limit and that the phase should be associated with a Hermetian operator so that in principle it is an observable quantity.

Dirac's [1] attempt to construct a Hermetian operator φ in the analogy with classical phase through a polar decomposition of the annihilation operator as $\hat{a} = e^{i\varphi}\sqrt{\hat{N}}$ did not succeed because of the one sided boundedness of the eigenvalue spectrum of the photon number operator $\hat{N}$. Since then Susskind and Glogower [2], Carruthers and Neito [3], Pegg and Barnett [4] and Shapiro and coworkers [5] have made significant contributions to this topic. Susskind and Glogower introduced a one sided unitary decomposition of the phase operator which has been extensively used in variety of problems in quantum optics. Shapiro et al introduced phase measurement statistics through the quantum estimation theory. Pegg and Barnett carried out a polar decomposition of the annihilation operator a la Dirac in a truncated Hilbert space of dimension $s+1$ and defined a Hermetian phase operator $\hat{\theta}_s$ in this finite dimensional space.

$$\hat{\theta}_s = \sum_{k=o}^{s} \theta_k^s |\theta_k^s >< \theta_k^s| \, , \tag{6.2}$$

where the state $|\theta_k^s >$ is given by

$$|\theta_k^s > = \frac{1}{\sqrt{s+1}} \sum_{n=o}^{s} \exp(in\theta_k^s|n > \ . \tag{6.3}$$

In this procedure, given a state $|\psi >$ in the infinite dimensional Hilbert space, one first computes, the expectation value $< \psi, s|f(\hat{\theta})|\psi, s >$, where the state $|\psi, s >$ denotes the restriction of the state $|\psi >$ to the $s+1$ dimensional space. The limit $s \to \infty$ is taken at the end. However, it does not turn out to be possible to interpret the resulting expression as the expectation value of a function f of a Hermetian operator $\hat{\theta}$ in the state $|\psi >$ in the infinite dimensional Hilbert space [6] i.e.,

$$\langle f(\hat{\theta}_\infty)\rangle_\psi \neq \lim_{s\to\infty} \langle f(\hat{\theta}_s)\rangle_\psi \ . \tag{6.4}$$

What does seem to be possible is to associate a phase distribution $\mathcal{P}(\theta)$ to a given state $|\psi >$ such that the average of $f(\hat{\theta})$ with respect to $\mathcal{P}(\theta)$ reproduces the results computed via the Pegg-Barnett scheme.

$$\lim_{s\to\infty}\langle f(\hat{\theta}_s)\rangle_\psi = \frac{1}{2\pi}\int_o^{2\pi} d\theta f(\theta) P_\psi(\theta) \ . \qquad (6.5)$$

Thus, it appears to be more sensible and computationally advantageous to describe the quantum mechanical phase via a phase distribution rather than through a phase operator, a view which has been strengthened by the work of Shapiro et al [5,7] where it has been shown that phase measurements can be properly described via the positive operator valued measures investigated by Helstrom [8]. It is this point of view which we adopt in this chapter in investigating the quantum mechanical phase properties of the nonlinear phenomena like phase conjugation and propagation of field through an optical fiber.

This chapter is organised as follows. The definition of the phase distribution $\mathcal{P}(\theta)$ which one may associate to a given density operator is given in Section 6.1. We briefly discuss its relation to the work of Susskind and Glogower and also consider some illustrative example in this section. There are numerous nonlinear optical phenomena [9] which involve changes only in the phase of the electric field, and practically no changes in the amplitude. Two well known phenomena which fall in this category are phase conjugation and propagation in an optical fiber. While the phenomena of phase conjugation involves reversal of the phase of the input field, an optical fiber produces a continuous change in the phase of the signal as it propagates along the fiber. The evolution of the phase distribution associated with the field as it propagates through an optical fiber is considered in Section 6.2. Section 6.3 contains an analysis of the phase conjugation phenomena with $\mathcal{P}(\theta)$ taken as the basis for a quantum mechanical description of phase. In Section 6.4, we briefly discuss another probability distribution for phase, $\mathcal{P}^W(\theta)$, based on the Wigner function which can be used for discussing the quantum mechanical phase and compare it with $\mathcal{P}(\theta)$ defined in Section 6.1.

6.1. Quantum Mechanical Phase Distributions

We consider a single mode electromagnetic field. Let $\hat{a}$ and $\hat{a}^\dagger$ be the annihilation and creation operators satisfying the usual bosonic commutation relations $[\hat{a}, \hat{a}^\dagger] = 1$. We introduce the state $|\theta >$ defined, in terms of the number states $|n >$, as

$$|\theta >= \sum_{n=o}^{\infty} e^{in\theta}|n> \quad . \tag{6.6}$$

These states play an important role in what follows. They are related to the Susskind-Glogower phase operator $\hat{b} = e^{i\widehat{\theta}}$ as follows

$$e^{i\widehat{\theta}} = \hat{b} = (1 + \hat{a}^\dagger\hat{a})^{-1/2}\hat{a} \ . \tag{6.7}$$

Using number ket expansions of $\hat{a}^\dagger\hat{a}$ and $\hat{a}$ we have

$$\hat{b} = \sum_{n=o}^{\infty} |n><n+1| \ . \tag{6.8}$$

The eigenstates of $\hat{b}$

$$\hat{b}|\beta >= \beta|\beta > \ , \tag{6.9}$$

is easily obtained by substituting (6.8) in (6.9). So that

$$|\beta >= (1 - |\beta|^2)^{1/2} \sum_{n=o}^{\infty} \beta^n|n> \quad . \tag{6.10}$$

These states are normalised, nonorthogonal and with finite average photon number,

$$\begin{aligned} <\hat{n}> &= \langle\beta|\hat{a}^\dagger\hat{a}|\beta\rangle = (1 - |\beta|^2)\sum_{n=o}^{\infty} n|\beta|^{2n} \\ &= \frac{|\beta|^2}{1 - |\beta|^2} \ . \end{aligned} \tag{6.11}$$

Eq.(6.11) indicates that there will be singularity in the limit $|\beta| \to 1$. But if we suppress the normalisation factor $(1 - |\beta|^2)^{1/2}$ from Eq.(6.10), then Eq.(6.10) goes over to Eq.(6.6) in the limit $|\beta| \to 1$,

$$|\beta >= \sum_{n=o}^{\infty} e^{in\theta}|n>= |\theta> \quad , \text{with}\, \beta = |\beta|e^{i\theta} \ . \tag{6.12}$$

The states $|\theta >$ are non normalisable and nonorthogonal.

$$\langle\theta'|\theta\rangle = \sum_{n=o}^{\infty} e^{-in(\theta'-\theta)} = \frac{1}{2}\left[\sum_{n=-\infty}^{\infty} e^{-in(\theta'-\theta)} + \sum_{n=-\infty}^{\infty} sgn(n)e^{-in(\theta'-\theta)} + 1\right]$$
$$= \pi\delta(\theta'-\theta) - \frac{i}{2}\cot(\theta'-\theta) + \frac{1}{2} , \tag{6.13}$$

where

$$sgn(n) \equiv \begin{cases} -1, & n < 0 \\ 0, & n = 0 \\ 1, & n > 0 \end{cases}$$

is the signum function.

The states $|\theta >$ form a overcomplete set and yield the following resolution of the identity

$$\frac{1}{2\pi}\int_o^{2\pi} d\theta|\theta >< \theta| = \frac{1}{2\pi}\sum_{n=o}^{\infty}\sum_{m=o}^{\infty}\int_o^{2\pi} d\theta e^{i(n-m)\theta}|n >< m|$$
$$= \sum_{m=o}^{\infty} |n >< n| = I . \tag{6.14}$$

The time evolution of the states $|\theta >$ under the free evolution of the radiation mode is rather simple

$$\exp\left\{-i\omega t\hat{a}^\dagger\hat{a}\right\}|\theta > = |\theta - \omega t > . \tag{6.15}$$

Further, the expectation value of the electric field

$$\hat{E} = \frac{1}{2}\left(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right) , \tag{6.16}$$

in the state $|\theta >$ is found to diverge.

$$< \theta|\hat{E}|\theta > = \frac{1}{2}\sum_{m=o}^{\infty}\sum_{n=o}^{\infty} < n|\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}|m > e^{i(m-n)\theta}$$
$$= \sum_{n=o}^{\infty}\sqrt{n}\cos(\theta - \omega t) . \tag{6.17}$$

Thus the wave is a superposition of an infinite number of waves of different amplitudes corresponding to all the different values of n. However each supplementary wave has the same frequency ω and phase θ so that the nodes of the composite wave occur at well defined position (nodes occur whenever $\cos(\theta - \omega t) = 0$ in 6.17).

We use the states $|\theta>$, to associate, to a given density operator $\hat{\rho}$, a phase distribution $\mathcal{P}(\theta)$ as follows

$$\mathcal{P}(\theta) = \frac{1}{2\pi}\langle\theta|\hat{\rho}|\theta\rangle \ , \ 0 \leq \theta \leq 2\pi \ , \tag{6.18a}$$

$$= \frac{1}{2\pi}\sum_{m,n=o}^{\infty} \rho_{mn} e^{i(m-n)\theta} \ , \tag{6.18b}$$

where we assume that the sum in (6.18b) converges. The $\mathcal{P}(\theta)$ as defined in (6.18) is manifestly *positive*, owing to the positivity of $\hat{\rho}$, and is normalized

$$\int_o^{2\pi} d\theta \mathcal{P}(\theta) = 1 \ , \mathcal{P}(\theta) \geq 0 \ . \tag{6.19}$$

Examples

We now consider several states of the field and compute the corresponding phase distributions.

(a) Incoherent States

For the density operator

$$\hat{\rho} = \sum_{n=o}^{\infty} p_n |n><n| \ , \tag{6.20}$$

which includes Fock states as special cases, $\mathcal{P}(\theta)$ is found to be

$$\mathcal{P}(\theta) = \frac{1}{2\pi} \ . \tag{6.21}$$

Thus the phase distribution is, as expected, uniform.

(b) Coherent States

For the coherent states $|\alpha>, \alpha = |\alpha|e^{i\theta_o}$, the phase distribution $\mathcal{P}(\theta)$ is found to be

$$\mathcal{P}(\theta) = \frac{1}{2\pi}\left|\sum_{n=o}^{\infty} e^{in(\theta-\theta_o)}\frac{|\alpha|^n}{\sqrt{n!}}e^{-|\alpha|^2/2}\right|^2 \ , \tag{6.22}$$

which may alternatively be written as a Fourier series

$$\mathcal{P}(\theta) = \frac{1}{2\pi}\left(1 + 2\sum_{p=1}^{\infty} c_p \cos p(\theta - \theta_o)\right) \quad , \tag{6.23}$$

where the coefficients c_p are given by

$$c_p = \sum_{k=p}^{\infty} \frac{|\alpha|^{2k-p}}{\sqrt{k!(k-p)!}} e^{-|\alpha|^2} \quad . \tag{6.24}$$

The summation in (6.24) involves the square root of a Poisson distribution. For large $|\alpha|^2$, we may use the Gaussian approximation for a Poisson distribution [10]

$$\frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2} \approx (2\pi|\alpha|^2)^{1/2} \exp\left(\frac{-(n - |\alpha|^2)}{2|\alpha|^2}\right) \quad , \tag{6.25}$$

and evaluate the sum in (6.22) to obtain the following approximate Gaussian form for $\mathcal{P}(\theta)$

$$\mathcal{P}(\theta) \approx (2|\alpha|^2\pi)^{-1/2} \exp(-2|\alpha|^2(\theta - \theta_o)^2) \quad . \tag{6.26}$$

In Fig.(6.1), we have plotted $\mathcal{P}(\theta)$ given by (6.22) for various values of $|\alpha|^2$. It has a peak at $\theta = \theta_o$ and, as expected, becomes progressively narrower as $|\alpha|^2$ increases. The phase distribution for $\mathcal{P}(\theta)$ for the two photon coherent state has been calculated and studied by Schleich et al [11]. It is found to be

$$\mathcal{P}(\theta) = \frac{\sqrt{s}}{\pi(s+1)} \exp\left[\frac{-2sa^2}{s+1}\right] \left|\sum_{m=o}^{\infty} \left(\frac{s-1}{s+1}\right)^m H_m\left(\frac{s}{(s^2-1)^{1/2}}\sqrt{2}a\right) \frac{e^{-im\theta}}{2^m m!}\right|^2 \quad . \tag{6.27}$$

Here $s(>0)$ and a denotes the squeeze and displacement parameter, respectively. Finally note that Shapiro et al [7] have constructed special state of the field given by

$$|\psi> = A \sum_{n=o}^{M} \frac{1}{n+1} |n> \quad , \tag{6.28}$$

which gives minimal errors in phase measurements.

(c) Pair Coherent States

Next we consider the pair coherent states [12-14] which are produced in a nonlinear medium under conditions involving competition between several nonlinear processes like four-wave mixing, two photon absorption and amplified spontaneous emission [14].

The pair coherent state $|\zeta, 0>$ with $\zeta = |\zeta|e^{i\theta_o}$, are defined by

$$|\zeta, 0> = \mathcal{N} \sum_{n=o}^{\infty} \frac{\zeta^n}{n!} |n, n> , \mathcal{N} = [I_o(2|\zeta|)]^{-1/2} \quad , \tag{6.29}$$

where I_o is the modified Bessel function [15] of order zero and $|n, m>$ are the Fock states for the two modes. The phase distribution $\mathcal{P}(\theta_1, \theta_2)$ defined by the two mode generalisation of (6.18)

$$\mathcal{P}(\theta_1, \theta_2) = \frac{1}{(2\pi)^2} \langle \theta_1, \theta_2 | \hat{\rho} | \theta_1, \theta_2 \rangle \quad , \tag{6.30}$$

is found to be

$$\begin{aligned} \mathcal{P}(\theta_1, \theta_2) &= \frac{\mathcal{N}^2}{(2\pi)^2} \sum_{n,m} \frac{|\zeta|^n |\zeta|^m}{n! m!} e^{i(n-m)(\theta_o - \theta_1 - \theta_2)} \\ &= \frac{\mathcal{N}^2}{(2\pi)^2} \exp\{2|\zeta| \cos(\theta_o - \theta_1 - \theta_2)\} \quad . \end{aligned} \tag{6.31}$$

Considered as a function of $\theta_1 + \theta_2$, this distribution is known in statistical literature as the von Mises distribution [16]. Its range extends from 0 to 4π. A noteworthy feature of $\mathcal{P}(\theta_1, \theta_2)$ given by (6.31) is that the sum of the phases $\theta_1 + \theta_2$ is locked to the phase of ζ. Further, each mode by itself is uniformly distributed.

$$\mathcal{P}(\theta_1) = \int_o^{2\pi} d\theta_2 \mathcal{P}(\theta_1, \theta_2) = \frac{1}{2\pi} \quad . \tag{6.32}$$

The phase distribution for the pair coherent states $|\zeta, q>$ has also been recently discussed by Ganstog and Tanas [17] using the Pegg-Barnett scheme.

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(d) Entangled States

We next consider the entangled state produced by a down converter [18]

$$|\psi> = \alpha|0,0> + \beta|1,1>, \ |\alpha|^2 + |\beta|^2 = 1 \ , \tag{6.33}$$

which has been extensively studied in the context of two photon interference [19]. The phase distribution $\mathcal{P}(\theta_1, \theta_2)$ corresponding to this turns out to be

$$\mathcal{P}(\theta_1, \theta_2) = \frac{1}{(2\pi)^2}\left[1 + 2|\alpha\beta|\cos(\varphi - \theta_1 - \theta_2)\right] \ , \frac{\alpha^*\beta}{|\alpha\beta|} = e^{-i\varphi} \ . \tag{6.34}$$

For the entangled state

$$|\psi> = \frac{1}{\sqrt{2}}(|1,0> + e^{i\varphi}|0,1>) \ , \tag{6.35}$$

encountered in the context of quantum beats, the phase distribution $\mathcal{P}(\theta_1, \theta_2)$ is found to be

$$\mathcal{P}(\theta_1, \theta_2) = \frac{2}{(2\pi)^2}\cos^2\left[(\varphi + \theta_1 - \theta_2)/2\right] \ . \tag{6.36}$$

It should be borne in mind that the entangled states and the pair coherent states have no classical analogues. Yet the phase distribution $\mathcal{P}(\theta_1, \theta_2)$ in both the cases is well defined. However, a distribution like (6.34) as a function of $\theta_1 + \theta_2$ is double peaked with the peaks located at $\theta_1 + \theta_2 = \varphi$ and $\varphi + 2\pi$. Furthermore, there are strong correlations between the two phase angles θ_1 and θ_2.

6.2. Field Propagation Through an Optical Fiber

In this section we consider the dynamic evolution of the phase distribution associated with the field as it propagates through a nonlinear dispersive element like an optical fiber.

Let $\hat{a}$ and $\hat{a}^\dagger$ denote the annihilation and the creation operators for the field. The interaction Hamiltonian is given by

$$\hat{H}_{in} = \hbar\chi\hat{a}^{\dagger 2}\hat{a}^2 \ , \tag{6.37}$$

where the nonlinear coupling constant χ is related to the third order nonlinearity of the medium. It is apparent from Eq.(6.37) that the photon number $\hat{n} = \hat{a}^\dagger\hat{a}$ is constant of motion. The Heisenberg equation for $\hat{a}$ is obtained as

$$\dot{\hat{a}}(t) = -2i\chi\hat{a}^\dagger\hat{a}\,\hat{a} \quad . \tag{6.38}$$

Since $\hat{n}$ is a constant of motion, the solution to the above equation is

$$\hat{a}(t) = \exp(-2i\chi t\hat{n})\hat{a}(0) \quad . \tag{6.39}$$

The classical equation corresponding to (6.39) is

$$\alpha(t) = \exp(-2i\chi t|\alpha|^2)\alpha_o \quad . \tag{6.40}$$

Thus classically the phase shift $\varphi_c = 2\chi t|\alpha|^2$ is proportional to the intensity of the electromagnetic field. In order to compute the corresponding quantum result, we assume that the initial state of the field is in the coherent state $|\alpha>$. Then the mean amplitude is

$$\begin{aligned} <\alpha|\hat{a}(t)|\alpha> &= \alpha <\alpha|\exp(-2i\chi t\hat{n})|\alpha> \\ &= \alpha\langle\alpha|\alpha e^{-2i\chi t}\rangle \\ &= \alpha\exp[-|\alpha|^2(1-e^{-2i\chi t)}] \\ &= \alpha\exp[-|\alpha|^2(2\sin^2\chi t + i\sin 2\chi t)] \quad , \end{aligned} \tag{6.41}$$

where we have used the relation

$$e^{i\theta\hat{n}}|\alpha> = |\alpha e^{i\theta}> \quad . \tag{6.42}$$

There are two interesting features in this result. First, there is a decay $(= |\alpha|^2\sin^2\chi t)$ of the mean amplitude. Second, there is a phase shift $\phi = |\alpha|^2\sin 2\chi t$. For small χ this reduces to the classical result $\varphi_c = 2\chi t|\alpha|^2$ as given in (6.40). *Thus quantum mechanically, one finds that not only does the peak of the associated phase distribution shift, as expected classically, but its width also progressively increases.* This can be seen from the following considerations.

The dynamical evolution of the density operator for the situation under consideration is given by [20-23]

$$\hat{\rho} = \exp\left\{-i\chi(\hat{a}^{\dagger 2}\hat{a}^2)t\right\}\hat{\rho}(0)\exp\left\{i\chi(\hat{a}^{\dagger 2}\hat{a}^2)t\right\} \quad , \tag{6.43}$$

where χ is related to the Kerr constant of the medium. Kitagawa and Yamamoto [20] have studied, at length, the phase space evolution of the Q function of the field as it propagates through the fiber. The time evolution of the corresponding phase distribution is given by

$$\mathcal{P}(\theta,t) = \frac{1}{2\pi}Tr\left[\hat{\rho}(0)\exp\left\{i\chi\left(\hat{a}^{\dagger 2}\hat{a}^2\right)t\right\}|\theta><\theta|\exp\left\{-i\chi\left(\hat{a}^{\dagger 2}\hat{a}^2\right)t\right\}\right] \quad . \tag{6.44}$$

For an initial coherent state

$$\hat{\rho}(0) = |\alpha><\alpha|,\ \alpha = |\alpha|e^{i\theta_o} \quad , \tag{6.45}$$

$\mathcal{P}(\theta,t)$ is given by

$$\begin{aligned}\mathcal{P}(\theta,\tau) &= \frac{1}{2\pi}\left|<\alpha|\exp\left\{i(\hat{a}^{\dagger 2}\hat{a}^2)\tau\right\}|\theta>\right| \\ &= \frac{1}{2\pi}\left|\sum_{n=o}^{\infty}\frac{|\alpha|^2}{\sqrt{n!}}\exp\left[\frac{-|\alpha|^2}{2}+in(\theta-\theta_o-\tau)+i\tau n^2\right]\right|^2 \quad ,\end{aligned} \tag{6.46}$$

where $\tau = \chi t$. We have numerically calculated $\mathcal{P}(\theta,\tau)$ in (6.46) for small values of τ. The numerical results are shown in Fig. (6.2).

We observe that the phase distribution is single valued and also that the distribution shifts and broadens as the field propagates through the fiber. Thus, quantum mechanically the phase not only shifts but also diffuses. Agarwal [23] has shown how the phase diffusion can be measured by a Mach-Zehnder interferometer.

The behavior of the phase distribution in the long time limit is discussed in the next chapter. It is found that the phase distribution splits into separate peaks indicating that the field is in a discrete superposition of coherent states.

6.3. Optical Phase Conjugation

We next consider the phenomena of reflection from a phase conjugating mirror (PCM). Classically, the electromagnetic field of phase φ and frequency ω incident on the PCM is represented as

$$E_i(r,t) = \mathcal{E}(r,t)e^{i\varphi-i\omega t} + c.c. \, . \tag{6.47}$$

Then the field after reflection from the PCM is transformed to

$$E_r(r,t) = \lambda\mathcal{E}^*(r,t)e^{-i\varphi-i\omega t} + c.c. \, , \tag{6.48}$$

where λ denotes the amplitude reflectivity of the PCM. Thus in classical treatment the amplitude $\mathcal{E}$ becomes $\mathcal{E}^*$ after reflection from the PCM and phase φ goes to $-\varphi$.

In quantum theory, a single mode field of frequency ω incident on PCM is given by

$$\hat{E}_i(r,t) = c\hat{a}e^{i\varphi-i\omega t} + h.a. \, , \tag{6.49}$$

where $\hat{a}$ denotes the photon annihilation operator of the incident field mode and $c = -i(\hbar\omega/2\epsilon_o V)^{1/2}$ where ϵ_o is the permittivity of free space and V is the quantisation volume. The field after reflection from the PCM is represented as

$$\hat{E}_r(r,t) = c^*\hat{b}e^{-i\varphi-i\omega t} + h.a. \, , \tag{6.50}$$

where $\hat{b}$ denotes the annihilation operator for the reflected mode. The operators $\hat{a}$ and $\hat{b}$ and their Hermetian adjoint $\hat{a}^\dagger$ and $\hat{b}^\dagger$ obey the Bosonic commutation relations

$$[\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1 \ . \tag{6.51}$$

Thus, quantum mechanically the phase conjugation process cannot be described via the relation $\hat{b} = \lambda \hat{a}^\dagger$, since this relationship is incompatible with the commutation relation. Gaeta and Boyd [24] postulated that the field operators $\hat{a}$ and $\hat{b}$ are related through

$$\hat{b} = \lambda \hat{a}^\dagger + \hat{F} \ , \tag{6.52}$$

where $\hat{F}$ represents a noise operator that obeys the commutation relation

$$\left[\hat{F}, \hat{F}^\dagger\right] = |\lambda|^2 + 1 \ , \tag{6.53}$$

and satisfies the condition

$$\langle \hat{F} \rangle = \langle \hat{F}^\dagger \rangle = 0 \ . \tag{6.54}$$

Thus an input field in a coherent state $|\alpha_o>$ is transformed by the PCM into a field with a coherent amplitude α_o^* and an additional noise photon. This noise photon added by the PCM also affects the performance of an interferometer in which one of the mirrors is replaced by a PCM[25]. Gaeta and Boyd have calculated the statistical properties of the field generated by the PCM. They have shown that the P function for the light reflected from a PCM is the same as the Q function for the input light i.e.,

$$P_{out}(\alpha) = Q_{in}(\alpha^*) \tag{6.55}$$

$$\hat{\rho}_{out} = \int P_{out}(\alpha) |\alpha><\alpha| d^2\alpha \ , \tag{6.56}$$

$$Q_{in}(\alpha^*) = \langle \alpha^* | \hat{\rho}_{in} | \alpha^* \rangle \ . \tag{6.57}$$

Thus if the input field is in the coherent state $|\alpha_o>$, then

$$Q_{in}(\alpha) = \frac{1}{\pi} \exp\{-|\alpha - \alpha_o|^2\}; \ P_{in}(\alpha) = \delta^2(\alpha - \alpha_o) \ , \tag{6.58}$$

and hence

$$P_{out}(\alpha) = \frac{1}{\pi}\exp\{-|\alpha-\alpha_o^*|^2\} \ . \tag{6.59}$$

Recall from the Chapter 2 that the P function associated with a mixture of a thermal field with a mean number of photons $\bar{n}$ and a coherent field $|\alpha_o>$ is given by

$$P(\alpha) = \frac{1}{\pi\bar{n}}\exp\left\{\frac{-|\alpha-\alpha_o|^2}{\bar{n}}\right\} \ . \tag{6.60}$$

Thus the field reflected from a PCM can be viewed as a mixture of a thermal field with $\bar{n}=1$ and a coherent field with amplitude α_o. The density matrix associated with (6.60) can be written as

$$\hat{\rho} = \sum_{n=o}^{\infty}\frac{(\bar{n})^n}{(1+\bar{n})^{1+n}}\hat{D}(\alpha_o)|n><n|\hat{D}^\dagger(\alpha_o) \ , \tag{6.61}$$

where $\hat{D}(\alpha_o)$ is the displacement operator given by

$$\hat{D}(\alpha_o) = \exp(\alpha_o\hat{a}^\dagger - \alpha_o^*\hat{a}) \ . \tag{6.62}$$

Thus the density matrices for the input (6.58) and the reflected fields (6.59) are

$$\hat{\rho}_{in} = |\alpha_o><\alpha_o| \ ; \ \ \hat{\rho}_{out} = \sum_{n=o}^{\infty}\frac{1}{2^{n+1}}\hat{D}(\alpha_o^*)|n><n|\hat{D}^\dagger(\alpha_o^*) \ . \tag{6.63}$$

The Wigner function for the state (6.61) is also given by (6.60) with $\bar{n}$ replaced by $\bar{n}+\frac{1}{2}$. Thus the Wigner functions, denoted by ϕ, for the input field ($\bar{n}=0$) and the output field ($\bar{n}=1$) are

$$\phi_{in}(\alpha) = \frac{2}{\pi}\exp(-2|\alpha-\alpha_o|^2) \ , \tag{6.64a}$$

$$\phi_{out}(\alpha) = \frac{2}{3\pi}\exp(-\frac{2}{3}|\alpha-\alpha_o^*|^2) \ . \tag{6.64b}$$

Having obtained the density matrix for the output field, we can now calculate the corresponding $\mathcal{P}(\theta)$. The phase distribution $\mathcal{P}_{in}(\theta)$ corresponding to $\hat{\rho}_{in}$ has already been calculated in Section 6.1. The phase distribution $\mathcal{P}_{out}(\theta)$ for the output field is given by

$$\mathcal{P}_{out}(\theta) = \frac{1}{2\pi}\langle\theta|\hat{\rho}_{out}|\theta\rangle \tag{6.65a}$$

$$= \frac{1}{2\pi}\sum_{n=o}^{\infty}\frac{1}{2^{n+1}}\left|\sum_{m=o}^{\infty} e^{-im\theta}\langle m|\hat{D}(\alpha_o^*)|n\rangle\right|^2 . \tag{6.65b}$$

Writing $\alpha_o^* = |\alpha_o|e^{-i\theta_o}$ and using the relation

$$e^{i\beta\hat{a}^\dagger\hat{a}}\hat{D}(\alpha_o^*)e^{-i\beta\hat{a}^\dagger\hat{a}} = \hat{D}(\alpha_o^* e^{i\beta}) , \tag{6.66}$$

with $\beta = \theta_o$, we may reexpress (6.65b) as

$$\mathcal{P}_{out}(\theta) = \frac{1}{2\pi}\sum_{n=o}^{\infty}\frac{1}{2^{n+1}}\left|\sum_{m=o}^{\infty} e^{-im(\theta+\theta_o)}\langle m|\hat{D}(|\alpha_o|)|n\rangle\right|^2 , \tag{6.67}$$

where the matrix elements of $\hat{D}(|\alpha|)$ are given by

$$\langle m|(\hat{D}|\alpha|)|n\rangle = e^{-|\alpha|^2/2}\sum_{p=o}^{\infty}\sum_{q=o}^{\infty}\sqrt{\frac{m!n!}{(n-q)!(m-p)!}}\;\frac{(-1)^q|\alpha|^{p+q}}{p!q!}\;\delta_{m-p,n-q} . \tag{6.68}$$

The relation (6.68) is easily proved by expressing $\hat{D}(|\alpha|)$ in the normally ordered form and then expanding the exponential in a power series.

Using (6.67) we have numerically computed $\mathcal{P}_{out}(\theta)$ for various values of $|\alpha|$. The results are shown in Fig. (6.3) where we have also displayed the phase distribution of the input field for comparison. One finds that, after reflection from a PCM, the peak at $\theta = \theta_o$ goes to $\theta = -\theta_o$ and at the same time the *output phase distribution also becomes broader. The broadening is due to the photon added by the phase conjugate device.*

For moderately large values $|\alpha_o|$, numerical computations reveal the following approximate relation between the output phase distribution and the phase distribution $\mathcal{P}_{coh}(\theta,\alpha)$ for the coherent state characterized by α

$$\mathcal{P}_{out}(\theta,\theta_o,|\alpha_o|) \approx \mathcal{P}_{coh}(\theta,-\theta_o,|\alpha_o|/\sqrt{3}) . \tag{6.69}$$

Thus, relative to the input phase distribution, the peak height of the output phase distribution is suppressed by a factor of $\sqrt{3}$ and its width correspondingly becomes larger by the same factor.

Using the Glauber-Sudarshan P representation (6.59) for $\hat{\rho}_{out}$ given by (6.56) and (6.65a), one obtains, for a coherent input field, the following alternative expression for $\mathcal{P}_{out}(\theta)$.

$$\mathcal{P}_{out}(\theta) = \frac{1}{2\pi^2} \int d^2\alpha \exp\left\{-|\alpha - \alpha_o^*|^2\right\} |\langle\theta|\alpha\rangle|^2 . \tag{6.70}$$

Noting that $|\langle\theta|\alpha\rangle|^2$ is the phase distribution associated with the coherent state $|\alpha>$, we may rewrite (6.70) as

$$\mathcal{P}_{out}(\theta) = \frac{1}{\pi} \int d^2\alpha \exp\left\{-|\alpha - \alpha_o^*|^2\right\} \mathcal{P}_{coh}(\theta, \alpha) . \tag{6.71}$$

This relation would be used in the next section to establish (6.69).

6.4. Comparison with a Phase Distribution defined via Wigner function

As noted in the introduction, the phase distribution for a classical field can be obtained from the probability distribution $P_{cl}(\alpha)$ for the complex field amplitude by integrating over the strength of the field i.e., by

$$\mathcal{P}_{cl}(\theta) = \int_o^\infty |\alpha| d|\alpha| P_{cl}(|\alpha| e^{i\theta}) . \tag{6.72}$$

The relation (6.72) can not be directly taken over into quantum theory owing to the non-commutativity of $\hat{a}$ and $\hat{a}^\dagger$. This has led to various choices of quasiprobability distributions that can be used to characterise a quantum field. It might be tempting to define a phase distribution in the quantum theory by using (6.72) with $P_{cl}(\theta)$ replaced by one of the quasiprobability distributions. Then the question that naturally arises is which quasiprobability distributions is the most appropriate one to be used in (6.72). In order to examine this we go back to the Susskind-Glogower phase operator and consider a situation where the field consists of a larger number of photons. In such a case, an expansion of $\hat{a}$ and $\hat{a}^\dagger$ around $(\bar{n})^{1/2}$ yields the following

$$\hat{\phi} = (\hat{a} - \hat{a}^\dagger)/2i(\bar{n})^{1/2} . \tag{6.73}$$

To this approximation, the phase operator is essentially the "momentum operator". This suggests that a phase distribution can be defined in terms of the Wigner function since it is natural to use the latter whenever the momentum distribution is to be studied. We, therefore, examine the consequences of defining a phase distribution $\mathcal{P}^W(\theta)$ via the Wigner function [26] $\Phi(\alpha)$ as follows

$$\mathcal{P}^W(\theta) = \int_o^\infty |\alpha| d|\alpha| \Phi(|\alpha| e^{i\theta}) . \tag{6.74}$$

Since the Wigner function is normalized over the entire complex plane

$$\int d^2\alpha \Phi(\alpha) = \int_o^\infty \int_o^{2\pi} |\alpha| d|\alpha| d\theta \Phi(|\alpha| e^{i\theta}) = 1 , \tag{6.75}$$

one has

$$\int_o^{2\pi} d\theta \mathcal{P}^W(\theta) = 1 . \tag{6.76}$$

In fact, the phase distribution defined as above has often been used in quantum optics in discussions of phase properties. It should, however, be borne in mind that like the Wigner function $\mathcal{P}^W(\theta)$ is not necessarily positive. Thus, for instance, for the entangled states (6.33), our calculations lead to

$$\mathcal{P}^W(\theta_1, \theta_2) = \frac{1}{2\pi} [1 + \pi |\alpha\beta| \cos(\varphi - \theta_1 - \theta_2)] , \tag{6.77}$$

which is evidently not always positive. Inspite of this, we expect $\mathcal{P}^W(\theta)$ to provide a description fairly close to that given by $\mathcal{P}(\theta)$ in situations where a large number of photons are involved. For a coherent state $|\alpha_o>$, (6.74) and (6.64a) yield

$$\mathcal{P}^W(\theta) = \frac{2}{\pi} \int |\alpha| d|\alpha| \exp\{-2|\alpha - \alpha_o|^2\} \tag{6.78a}$$

$$= \frac{1}{2\pi} \exp\left\{-2|\alpha_o|^2 \left(1 - \frac{1}{4}\cos^2(\theta - \theta_o)\right)\right\} D_{-2}\left(|\alpha_p| \cos(\theta - \theta_o)\right) \ , \tag{6.78b}$$

where $D_{-p}(z)$'s denote parabolic cylinder functions [27].

Using (6.78a) we have numerically calculated the $\mathcal{P}^W(\theta)$ for a coherent state. The results for moderately large values $|\alpha_o|$ are shown in Fig.(6.4). In Fig. (6.5), we compare $\mathcal{P}(\theta)$ and $\mathcal{P}^W(\theta)$ for a coherent state. We find that, while for large values of $|\alpha_o|$, the two distributions are practically identical, differences arise when $|\alpha_o|$ becomes small. We would also like to note here that if instead of the Wigner function, we choose the Q function $Q(\alpha) =< \alpha|\hat{\rho}|\alpha >$, then we obtain a phase distribution $\mathcal{P}^Q(\theta)$ which, though guaranteed to be positive, is found, on numerical evaluation, to differ considerably from $\mathcal{P}(\theta)$ both for large as well as for small photon numbers.

Turning our attention to reflection of a coherent input field from PCM, using (6.64) and (6.74), we can establish the following exact relation between $\mathcal{P}^W_{in}(\theta)$ and $\mathcal{P}^W_{out}(\theta)$

$$\mathcal{P}^W_{out}(\theta, \theta_o, |\alpha_o|) = \mathcal{P}^W_{in}(\theta, -\theta_o, |\alpha_o|/\sqrt{3}) \ . \tag{6.79}$$

The numerical results of Fig.(6.5) are of course consistent with the above relation.

We next give an alternate proof of (6.79) which also enables us to see the approximate validity of (6.69). We note that the density matrix of the output can be written as

$$\hat{\rho}_{out} = \frac{1}{\pi} \int d^2\beta \exp\{-|\beta - \alpha_o^*|^2\} |\beta >< \beta| \ , \tag{6.80}$$

and thus the Wigner function of the output field is related to the Wigner function $\Phi(\alpha, \beta)$ for a coherent state $|\beta >$ via

$$\Phi_{out}(\alpha) = \frac{1}{\pi} \int d^2\beta \exp\{-|\beta - \alpha_o^*|^2\} \Phi(\alpha, \beta) \ . \tag{6.81}$$

The phase distribution $\mathcal{P}^W_{out}(\theta)$ can be obtained by substituting (6.81) in (6.74) which then leads to

$$\mathcal{P}^W_{out}(\theta) = \frac{1}{\pi}\int d^2\alpha \exp\{-|\alpha-\alpha_o^*|^2\}\mathcal{P}^W_{coh}(\theta,\alpha) \ , \tag{6.82}$$

where $\mathcal{P}^W_{coh}(\theta,\alpha)$ is the phase distribution associated with the coherent state $|\alpha>$. Then the relation (6.79) may be easily obtained from (6.82). That $\mathcal{P}_{coh}(\theta)$ and $\mathcal{P}_{out}(\theta)$ should be related as in (6.69) can be understood from (6.82) as follows. Comparing (6.71) and (6.82) we find that $\mathcal{P}^W_{out}(\theta)$ is related to $\mathcal{P}^W_{coh}(\theta)$ in exactly the same way as $\mathcal{P}_{out}(\theta)$ is related to $\mathcal{P}_{coh}(\theta)$. Since the exponential in the integrand in (6.69) is peaked at $\alpha=\alpha_o^*$, the dominant contribution to the integral on the R.H.S. of (6.69) comes from $\mathcal{P}_{coh}(\theta)$ with α close to α_o^*. Now, for reasonably large values of $|\alpha_o|$, we know that $\mathcal{P}_{coh}(\theta)$ and $\mathcal{P}^W_{coh}(\theta)$ are practically identical. This then implies that, for large values of $|\alpha_o|$, owing to the relation (6.79), $\mathcal{P}_{out}(\theta)$ must be related to $\mathcal{P}_{in}(\theta)$ as in (6.69).

In conclusion, we have defined a phase distribution $\mathcal{P}(\theta)$ which one may associate to a given density operator. We have looked at the phase distribution associated with various states of the field which have no classical analogues. We have investigated the quantum mechanical phase distribution for the nonlinear optical phenomena like reflection from a phase conjugate mirror and propagation of a coherent field through an optical fiber where the classical theory leads to changes in the phase of the field. It is found that while the peaks of the respective phase distributions exhibit the expected classical behavior, the widths of the phase distributions undergo substantial changes due to quantum nature of the field. We have also examined the question of defining phase distribution via the quasidistribution like Wigner distribution. It is found that, for large photon numbers, phase distribution defined via the Wigner function comes rather close to those defined via the Susskind-Glogower decomposition. Experimental observation of phase distribution would be quite interesting. Shapiro and coworkers [5,7] have suggested methods which allows one to measure $\mathcal{P}(\theta)$ directly by relating it to a Hermetian observable in an extended space consisting of the field and the detector system.

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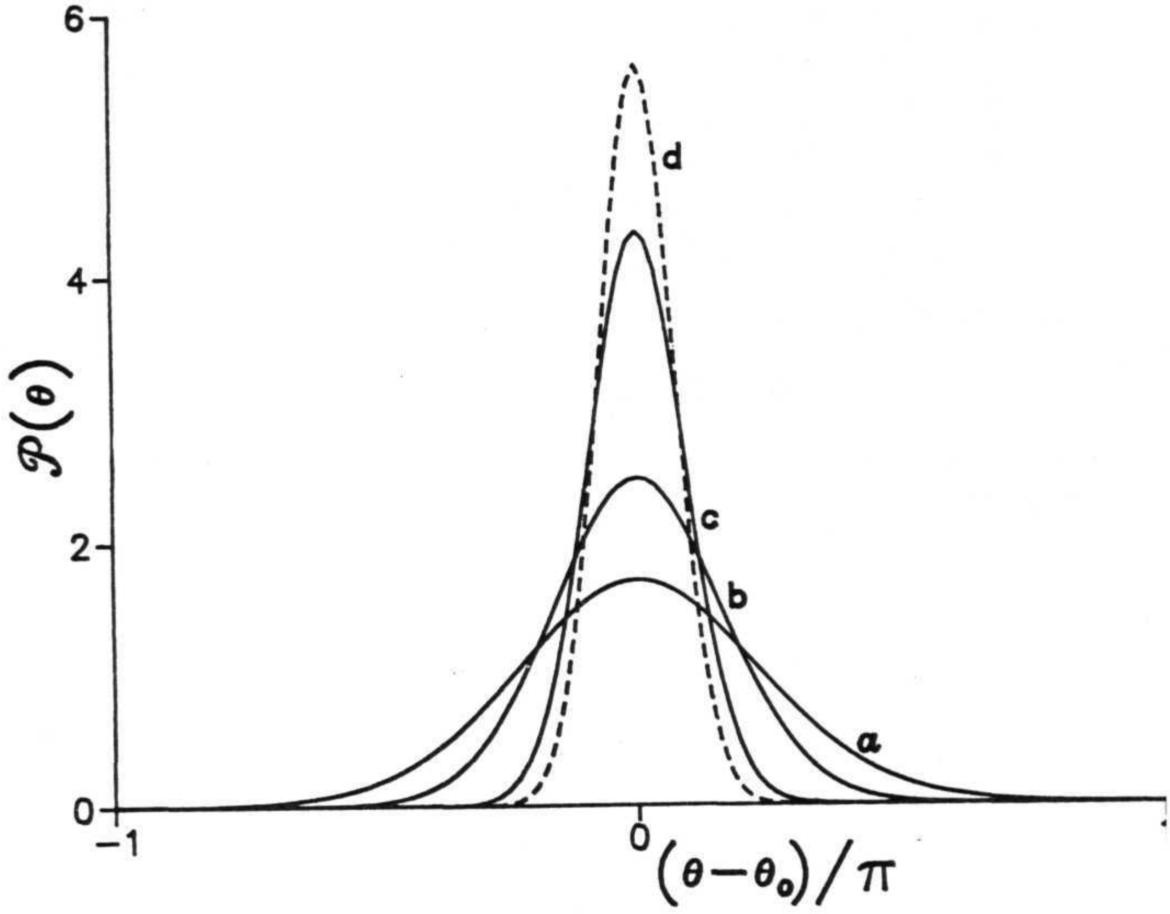

Fig. 6.1. Plot of $\mathcal{P}(\theta)$ as a function of $\theta - \theta_o$ for a coherent state (Eq.(6.22) with the mean number of photon $|\alpha|^2$ (a) = 5, (b)=10 (c)=30 and (d)=50.

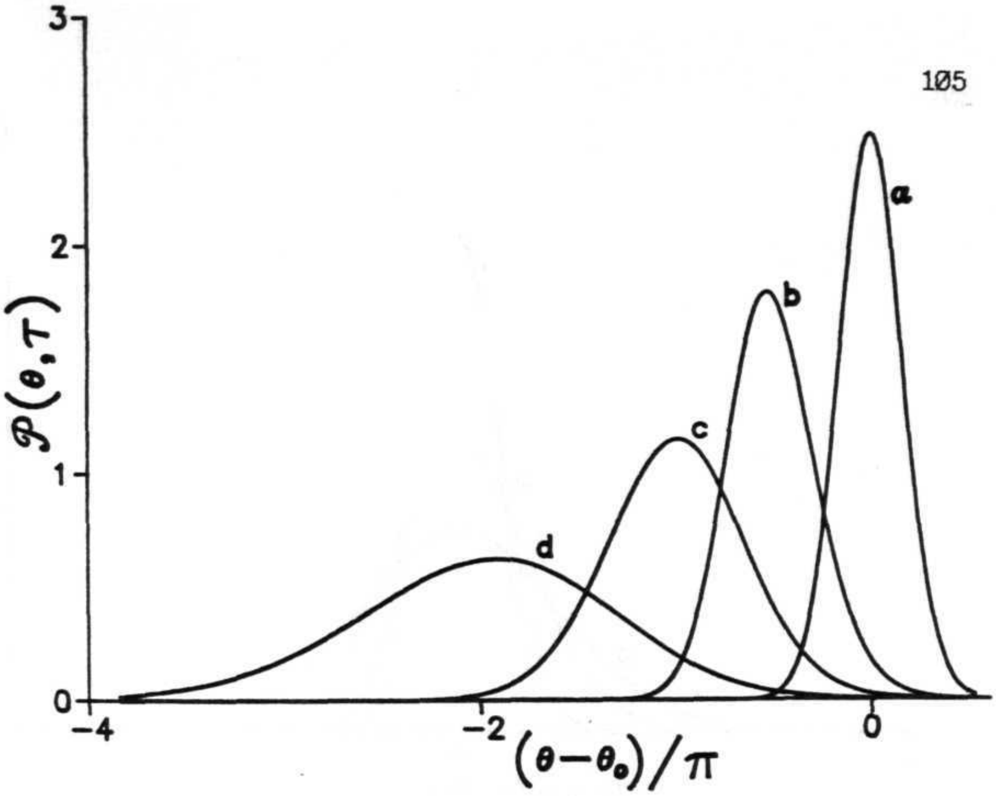

Fig. 6.2. The phase distribution $\mathcal{P}(\theta,\tau)$ (Eq.(6.46)) for a coherent signal with the mean number of photons $|\alpha|^2$=10, propagating through an optical fiber for the propagation distances corresponding to τ (a)=0, (b)=0.025, (c)=0.05 and (d)=0.1.

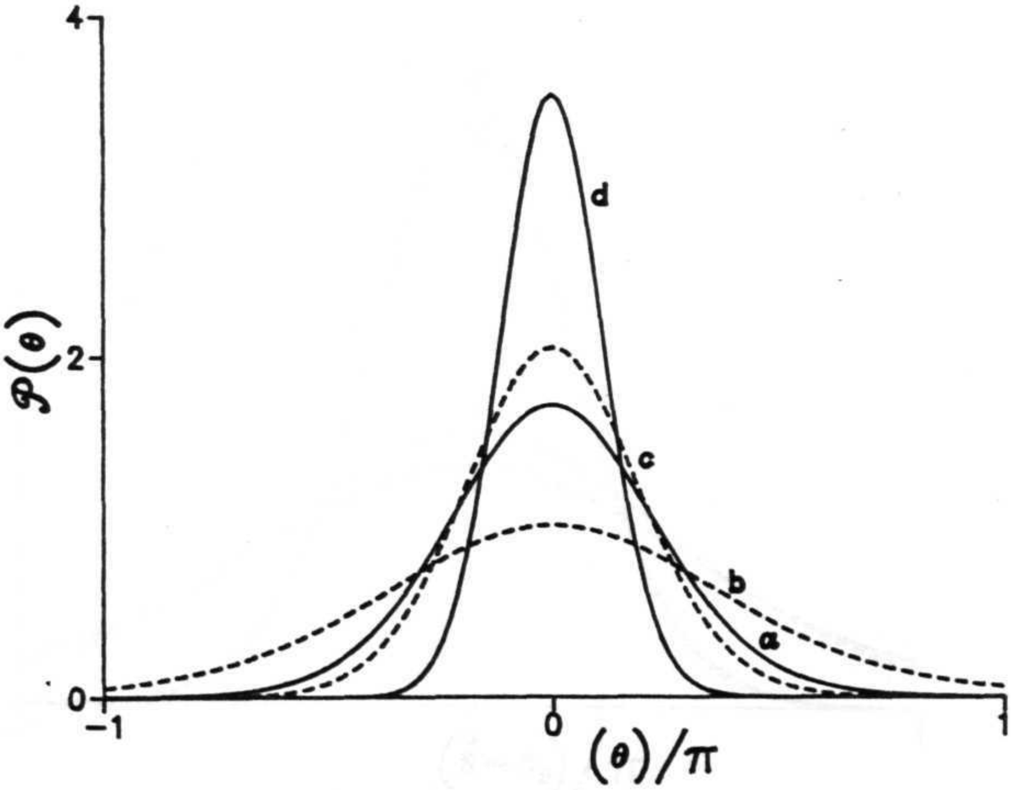

Fig. 6.3. The input $\mathcal{P}_{in}(\theta)$ (Eq.(6.22)) (solid curves) and the output $P_{out}(\theta)$ (Eq.(6.67)) (dashed curves) phase distributions for the reflection of a coherent input for a PCM. The mean number of input photons has been chosen as 5 (curves (a) and (b)) and 20 (curves (c) and (d)). The orgin of the θ axis is taken to be at θ_o for $\mathcal{P}_{in}(\theta)$ and at $-\theta_o$ for $\mathcal{P}_{out}(\theta)$. Here $\theta_o = 0$.

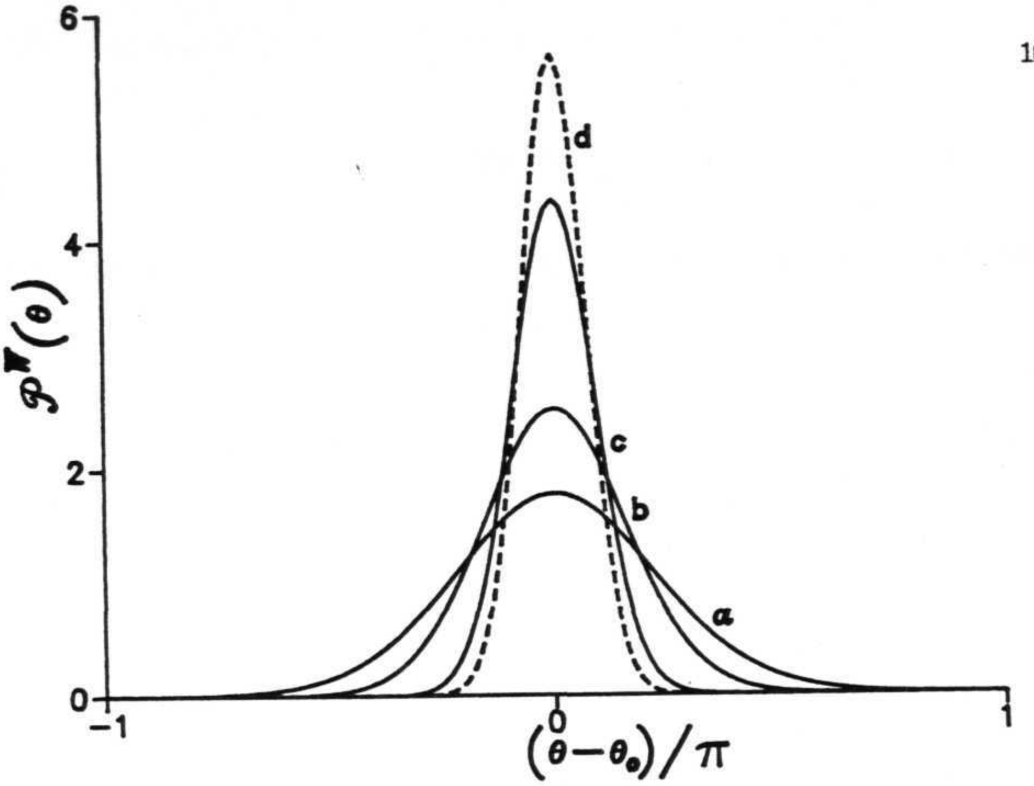

Fig. 6.4. Plot of phase distribution $\mathcal{P}^W(\theta)$ defined via Wigner function (Eq.(6.78)) for a coherent state with mean number of photons $|\alpha|^2$ (a)=5, (b)=10, (c)=30 and (d)=50.

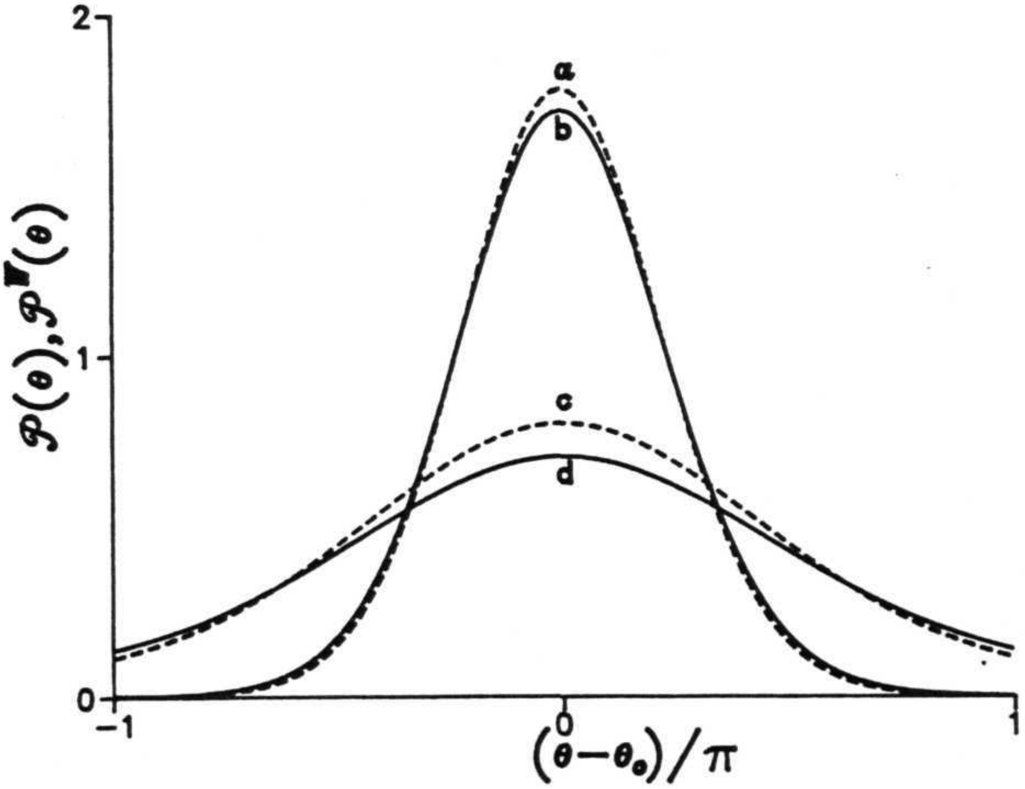

Fig. 6.5. A comparison of $\mathcal{P}(\theta)$ (solid curves) and $\mathcal{P}^W(\theta)$ (dashed curves) for a coherent state with mean number of photons $|\alpha|^2=5$ (curves (a) and (b)), and $|\alpha|^2=1$ (curves (c) and (d)).

Chapter 7

PRODUCTION OF SCHRÖDINGER KITTEN STATES IN A KERR MEDIUM

Recently there has been a lot of interest in the properties of quantum states of the electromagnetic field consisting of superposition of macroscopically distinguishable states which are well known as Schrödinger cat states [1-5]. For example a state

$$|\psi> = C_1|\alpha e^{i\theta}> + C_2|\alpha e^{-i\theta}> \tag{7.1}$$

has properties which are very different from the properties of the individual coherent states. The state (7.1) is known to exhibit squeezing [5], sub-Poissonian statistics [5] and other nonclassical properties of the field which were discussed in Chapter 4.. Several schemes were proposed recently for the generation of the superposition states in various nonlinear processes. It has now been realised that a state like (7.1) can be produced by quantum non-demolition measurement [6], in two photon resonant Jaynes-Cummings model (JCM) [7]. Brune et al [8] have discussed the possibilities of producing such states with Rydberg atoms in microwave cavities. It is also known that the state (7.1) can be produced by a single mode radiation field initially in a coherent state propagating through an optical fiber[1,2,9,10].

In this chapter we consider a state which is a superposition of several coherent states i.e., a Schrödinger 'kitten' state and discuss how this state can be produced in nonlinear optical processes. In particular we consider a special class of states by superposing equally spaced states on a ring.

$$|\psi> = \sum_{m=o}^{N-1} C_m |\alpha e^{2\pi i m/N}> \quad . \tag{7.2}$$

Here N is an integer greater than one. Note that coherent states in (7.2) differ in their amplitudes by phase factors given by the N-th roots of unity. One can easily verify that the state $|\psi>$ in Eq.(7.2) is an eigenstate of $\hat{a}^N$ for all values of C_m, i.e.,

$$\begin{aligned}\hat{a}^N|\psi> &= \sum_{m=o}^{N-1} C_m \alpha^N e^{2\pi i m} |\alpha e^{2\pi i m/N}> \\ &= \alpha^N |\psi> \text{ , as } e^{2\pi i m} = 1 \; , \end{aligned} \tag{7.3}$$

where m is an integer. Thus the Schrödinger kitten state (7.2) is an eigenstate of $\hat{a}^N$. These eigenstates are highly degenerate since C_m's are arbitrary [11]. The superposition of states in (7.2) involves states which are nonorthogonal to each other which gives rise to interference terms. Thus this nonorthogonality is responsible for many of the quantum characteristics of the state (7.2) which in turn depends on the expansion coefficients C_m.

In Section 7.1, we show that the state (7.2) with specific values of C_m's can be produced by a single mode coherent field propagating in a Kerr medium. The properties such as phase distribution, quasiprobability distribution like Wigner function and Q function, squeezing, quadrature distribution are discussed in Section. 7.2. In Section 7.3, we show that discrete superposition of squeezed coherent states can be produced if the input state is in a squeezed coherent state.

7.1. Superposition of Coherent States

The effective Hamiltonian [9] for the propagation of the single mode electric field in a Kerr medium can be written as

$$\hat{H} = \hbar\chi \hat{a}^{\dagger 2}\hat{a}^2 \quad , \tag{7.4}$$

where $\hbar$ Planck's constant divided by 2π, χ is the third order nonlinear susceptibility of the medium. Let the field initially be in a coherent state $|\alpha>$. Then at time t the field will be in a state.

$$\begin{aligned} |\psi(t)> &= \exp\left(\frac{-i\hat{H}t}{\hbar}\right)|\alpha> \\ &= \exp\left\{-i\chi t\hat{N}(\hat{N}-1)\right\}|\alpha> , \end{aligned} \tag{7.5}$$

where $\hat{N}$ is the number operator $\hat{a}^{\dagger}\hat{a}$. We evaluate the state of the system at time $t = \pi/m\chi$

$$|\psi(\pi/m\chi)>\equiv|\psi_m>=\exp\left\{\frac{-i\pi}{m}\hat{N}(\hat{N}-1)\right\}|\alpha> . \tag{7.6}$$

where m is an integer.

The evolution operator

$$\hat{U}^{(m)}(t=\pi/m)\equiv\exp\left\{\frac{-i\pi}{m}\hat{N}(\hat{N}-1)\right\} , \tag{7.7}$$

has a very interesting periodic property which arises from the fact that the eigenvalues of the number operator $\hat{N}$ are integers. Thus the following results can be proved straightforwardly using Eq.(7.7).

$$\exp\left\{\frac{-i\pi}{m}(\hat{N}+m)(\hat{N}+m+1)\right\} = (-1)^{m-1}\exp\left\{\frac{-i\pi}{m}\hat{N}(\hat{N}-1)\right\} , \tag{7.8}$$

$$\exp\left\{\frac{-i\pi}{m}(\hat{N}+m)^2\right\} = (-1)^m\exp\left\{\frac{-i\pi}{m}\hat{N}^2\right\} . \tag{7.9}$$

For odd (even) m we will make use of Eq. (7.8)((7.9)). The periodicity property can be used to expand $\hat{U}^m(t)$ as a Fourier series with $\exp((-2\pi iq)/m)$ as base functions i.e.,

$$\exp\left\{\frac{-i\pi}{m}\hat{N}(\hat{N}-1)\right\} = \sum_{q=o}^{m-1} f_q^{(o)} \exp\left[\frac{-2\pi iq\hat{N}}{m}\right] \ , \qquad (7.10)$$

for odd m and

$$\exp\left\{\frac{-i\pi\hat{N}^2}{m}\right\} = \sum_{q=o}^{m-1} f_q^{(e)} \exp\left[\frac{-2\pi iq\hat{N}}{m}\right] \ , \qquad (7.11)$$

for even m.

The coefficients in Eq.(7.10) and Eq.(7.11) are obtained by inverting these relations by noting that

$$\sum_{q=o}^{m-1} \exp\left[\frac{2\pi iqN}{m}\right] = m\delta_{N0} \ , \qquad (7.12)$$

and hence

$$f_q^{(o)} = \frac{1}{m}\sum_{N=0}^{m-1} \exp\left[\frac{2\pi iq}{m}N\right] \exp\left\{\frac{-i\pi}{m}N(N-1)\right\} \ , \qquad (7.13)$$

$$f_q^{(e)} = \frac{1}{m}\sum_{N=0}^{m-1} \exp\left[\frac{2\pi iq}{m}N\right] \exp\left\{\frac{-i\pi}{m}N^2\right\} \ . \qquad (7.14)$$

Using Eqs. (7.6), (7.10) and (7.11) and the property

$$\exp(i\chi\hat{a}^\dagger\hat{a})|\alpha> = |\alpha e^{i\chi}> \ , \qquad (7.15)$$

we can obtain the dynamical evolution of the coherent state.

$$|\psi_m> = \sum_{q=o}^{m-1} f_q^{(o)}|\alpha \exp\{-2\pi iq/m\}> \quad m = \text{odd} \ , \qquad (7.16)$$

$$= \sum_{q=o}^{m-1} f_q^{(e)}|\alpha \exp\{(-2i\pi q + i\pi)/m\}> \quad m = \text{even} \ . \qquad (7.17)$$

These are the special cases of (7.2) with expansion coefficients given by (7.13) and (7.14). We have thus shown that superposition of coherent states or in other words Schrödinger kitten state can be produced by the propagation of a coherent field through a Kerr medium provided that the propagation length and the nonlinear susceptibility of the Kerr medium are such that

$$\frac{\chi L}{c} = \frac{\pi}{m} . \tag{7.18}$$

The coefficients $f_q^{(o)}$ and $f_q^{(e)}$ can be evaluated from (7.13) and (7.14) and these, in the special cases, are given by

(a) m=3

$$f_o^{(o)} = \left(\frac{3-i\sqrt{3}}{6}\right) \; ; f_1^{(o)} = \frac{i\sqrt{3}}{3} \; ; f_2^{(o)} = \left(\frac{3-i\sqrt{3}}{6}\right) . \tag{7.19}$$

(b) m=4

$$f_o^{(e)} = \frac{1-i}{2\sqrt{2}} \; ; f_1^{(e)} = \frac{1}{2} \; ; f_2^{(e)} = \frac{-(1-i)}{2\sqrt{2}} \; ; f_3^{(e)} = \frac{1}{2} . \tag{7.20}$$

7.2. Nonclassical Properties of the Schrödinger Kitten States

The photon number distribution does not change i.e., it still remains Poissonian at the exit of the Kerr medium. This follows from (7.5)

$$p(n) = |\langle n|\psi(t)\rangle|^2 = |e^{-i\chi t n(n-1)}\langle n|\alpha\rangle|^2 = |\langle n|\alpha\rangle|^2 , \tag{7.21}$$

thus the state (7.5) will exhibit Poissonian statistics and its Mandel's Q parameter will be zero. Hence we examine the phase distribution $\mathcal{P}(\theta)$ associated with (7.5). The phase distribution was discussed in detail in the previous chapter. This distribution is defined by [12]

$$\mathcal{P}(\theta) = \frac{1}{2\pi}|\langle\theta|\psi(t)\rangle|^2 , \; |\theta> = \sum_{n=0}^{\infty} e^{in\theta}|n> , \tag{7.22}$$

where $|\psi(t)>$ is given by equation (7.16) and (7.17). The phase distribution of the output field will have multiple peaks. For example, using (7.16) we have

$$\mathcal{P}(\theta) = \frac{1}{2\pi}\left|\sum_{q=o}^{m-1} f_q^{(o)}\langle\theta|\alpha\exp\{-2\pi iq/m\}\rangle\right|^2 , \tag{7.23}$$

where

$$< \theta|\alpha >= \sum_{n=o}^{\infty} e^{-in\theta} e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \quad .$$

In the Fig. (7.1) we show the behavior of the phase distribution for $m = 2, 3, 10$. The initial phase distribution breaks up into multiple peaks with peak positions located at

$$\theta = \frac{-2\pi q}{m} + \arg \alpha \ , \ \text{for } m = \text{odd} \ , \qquad (7.24a)$$

$$\theta = \frac{-2\pi q}{m} + \arg \alpha + \frac{\pi}{m} \ , \ \text{for } m = \text{even} \ . \qquad (7.24b)$$

The width of each of the peaks is determined by $|\alpha|^2$. In Fig. 7.2, we investigate how sensitive the production of such a state is, to the precise value of the parameter χt. We find (Fig. 7.2a) that the phase distribution in the immediate vicinity of say $\chi t = \pi/3$ remains multipeaked. Further deviation from the value $\chi t = \pi/3$ makes the phase distribution oscillate rapidly around $1/2\pi$ (Fig. 7.2b). In Fig. (7.3) we show the behavior of the quadrature distribution $P(x)$ for $m = 3$. Let $|x >$ be the eigenstate of the quadrature $(\hat{a} + \hat{a}^\dagger)/\sqrt{2}$. Then the quadrature distribution

$$P(x) = |\langle x|\psi(t)\rangle|^2 \ , \qquad (7.25a)$$

with

$$\begin{aligned} < x|\psi(t) >= \ & f_o \exp\left[-(x - \sqrt{(2)}\alpha)^2/2\right] \\ + & f_1 \exp\left[-(x + \alpha/\sqrt{2})^2/2 - \sqrt{(3/2)} i\alpha x - (\sqrt{3}i\alpha)^2/4\right] \\ + & f_2 \exp\left[-(x + \alpha/\sqrt{2})^2/2 - \sqrt{(3/2)} i\alpha x + (\sqrt{3}i\alpha)^2/4\right] \ , \end{aligned} \qquad (7.25b)$$

where we have used (7.16) and the coordinate representation of the coherent state

$$< x|\alpha e^{i\theta} >= \frac{1}{(\pi)^{1/4}} \exp\left[-\frac{x^2}{2} + \sqrt{2}\alpha e^{i\theta} x - \frac{1}{2}|\alpha|^2 - \frac{\alpha^2}{2} e^{2i\theta}\right] \quad . \qquad (7.26)$$

The coefficients f_o, f_1, f_2 are given by (7.19). The main peaks in Fig.(7.3) correspond to $\sqrt{2}\alpha$ and $-\alpha/\sqrt{2}$. The modulation around $-\alpha/\sqrt{2}$ arises from the complex exponential.

We now turn to examine of the squeezing properties of the Schrödinger kitten state. The field quadratures of the single mode field are given by

$$\hat{x} = \frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}} \ , \ \hat{p} = \frac{\hat{a}-\hat{a}^\dagger}{\sqrt{2}i} \ . \tag{7.27}$$

Considering only the $\hat{x}$ quadrature, we calculate its fluctuations. When the field is in the Schrödinger kitten state, i.e., we calculate the quantity

$$S_x \equiv 2(\Delta x)^2 = <\hat{x}^2> - <\hat{x}>^2 \ , \tag{7.28}$$

using (7.27), (7.16) and (7.17). For values of $S_x < 1$, the state (7.5) exhibits squeezing. In Fig. (7.4), we plot S_x as a function of $|\alpha|^2$ for various values of m. It is found that the state for $m = 2$ exhibits maximum amount of squeezing when compared to the states obtained when $m > 2$.

Next we examine the phase space characteristics of the state (7.16) and (7.17). We calculate the Q function defined as

$$Q(\beta,\beta^*) = |\langle\beta|\psi_m\rangle|^2 \ . \tag{7.29}$$

For the state (7.16), we find

$$\begin{aligned} Q(\beta,\beta^*) &= \left|\sum_{q=o}^{m-1} f_q^{(o)}\langle\beta|\alpha\exp\{-2\pi iq/m\}\rangle\right|^2 \\ &= \left|\sum_{q=o}^{m-1} f_q^{(o)}\exp\left[-\frac{|\beta|^2}{2}-\frac{|\alpha|^2}{2}+\beta^*\alpha e^{-(2\pi iq)/m}\right]\right|^2 \ , \end{aligned} \tag{7.30a}$$

and for the state (7.17) we find

$$\begin{aligned} Q(\beta,\beta^*) &= \left|\sum_{q=o}^{m-1} f_q^{(e)}\langle\beta|\alpha\exp\{(-2\pi iq+i\pi)/m\}\rangle\right|^2 \\ &= \left|\sum_{q=o}^{m-1} f_q^{(e)}\exp\left[-\frac{|\beta|^2}{2}-\frac{|\alpha|^2}{2}+\beta^*\alpha\exp\{(-2\pi iq+i\pi)/m\}\right]\right|^2 \ . \end{aligned} \tag{7.30b}$$

The result for $m = 6$ and $|\alpha|^2 = 9$ is displayed in the Fig.(7.5). We show both the Q function as well as its contour plots. Similar contour plots have been obtained by Tanás et al[10]. The contour plot shows the production of Schrödinger kitten states.

Next we calculate the Wigner function for the state given by Eqs.(7.16) and (7.17). The Wigner function $W(\beta, \beta^*)$ is defined as the Fourier transform of the characteristic function, $C(\xi)$ [13],

$$C(\xi) = Tr\left[\hat{\rho}\exp(\xi\hat{a}^\dagger - \xi^*\hat{a})\right] \ , \tag{7.31}$$

$$W(\beta, \beta^*) = \frac{1}{\pi^2}\int C(\xi)\exp(\beta\xi^* - \beta^*\xi)d^2\xi \ . \tag{7.32}$$

Substituting (7.31) in (7.32) and performing the integral using Eqs.(7.16) and (7.17) we obtain

$$W(\beta, \beta^*) = \frac{2}{\pi}\sum_{q=o}^{m-1} f_p^{(o)*} f_q^{(o)} \exp\Big\{-|\alpha|^2\Big(e^{(2\pi i(p-q))/m} + 1\Big)$$

$$+2\Big(\alpha\beta^* e^{(-2\pi i q)/m} + \alpha^*\beta e^{(2\pi i p)/m} - |\beta|^2\Big)\Big\} \quad m = \text{odd} \ , \tag{7.33a}$$

$$= \frac{2}{\pi}\sum_{q=o}^{m-1} f_p^{(e)*} f_q^{(e)} \exp\Big\{-|\alpha|^2\Big(e^{(2\pi i(p-q))/m} + 1\Big)$$

$$+2\Big(\alpha\beta^* e^{(-2\pi i q + i\pi)/m} + \alpha^*\beta e^{(2\pi i p - i\pi)/m} - |\beta|^2\Big)\Big\} \quad m = \text{even} \ . \tag{7.33b}$$

In Fig. (7.6) we plot the Wigner function for two values of m with $|\alpha|^2 = 4$. This figure shows the regions where the Wigner function is negative which arises because of the quantum interference effects between the superposition of coherent states. The Wigner function taking the negative values signifies the nonclassical behavior of the Schrödinger kitten state.

7.3 Superposition of Squeezed Coherent States

We have so far considered the propagation of a coherent field through a Kerr medium. Other states of the field can be similarly studied using the decomposition (7.10) and

(7.11). Using our Eqs.(7.5-7.15) and considering squeezed coherent state as the input we show below that the superposition of squeezed coherent states can also be produced in a Kerr medium [14].

For example for odd m, applying the operator (7.10) on the squeezed coherent state

$$|\alpha, r> \equiv S(r)|\alpha> \quad , \tag{7.34}$$

where

$$S(r) = \exp\left(\frac{1}{2}r\hat{a}^{\dagger 2} - \frac{1}{2}r\hat{a}^2\right) \quad . \tag{7.35}$$

We obtain

$$|\psi_m>_s = \sum_{q=o}^{m-1} f_q^{(o)} \exp\left\{\frac{-2\pi iq}{m}\hat{a}^\dagger\hat{a}\right\} \exp\left\{\frac{1}{2}r\hat{a}^{\dagger 2} - \frac{1}{2}r\hat{a}^2\right\}|\alpha> \quad . \tag{7.36}$$

Making use of (7.5) and the transformation

$$\exp(-i\theta\hat{a}^\dagger\hat{a})S(r)\exp(i\theta\hat{a}^\dagger\hat{a}) = S(re^{-2i\theta}) \quad , \tag{7.37}$$

Eq.(7.36) can be rewritten in an elegant form as

$$|\psi_m>_s = \sum_{q=o}^{m-1} f_q^{(o)}|\alpha e^{(-2\pi iq)/m} \quad , re^{(-4\pi iq)/m}> \quad . \tag{7.38}$$

Thus in the superposition of squeezed coherent state (7.38), not only the phase of the coherent component change but the direction of squeezing also changes. In a similar fashion one can write for $|\psi_m>_s$ for even m.

$$|\psi_m>_s = \sum_{q=o}^{m-1} f_q^{(e)}|\alpha e^{(-2\pi iq+i\pi)/m}, re^{(-4\pi iq+2\pi i)/m}> \quad . \tag{7.39}$$

We calculate the Q function for the state (7.39).

$$\begin{aligned} Q(\beta,\beta^*) &= |<\beta|\psi_m>_s|^2 \\ &= \left|\sum_{q=o}^{m-1} f_q^{(e)} <\beta|\alpha e^{(-2\pi iq+i\pi)/m}, re^{(-4\pi iq+2\pi i)/m}>\right|^2 \\ &= \left|\sum_{q=o}^{m-1} f_q^{(e)} <\beta|S(re^{-2i\mu})|\alpha e^{-i\mu}>\right|^2 \quad , \end{aligned} \tag{7.40}$$

where $\mu = \pi(2q-1)/m$.

Using disentangling theorem [15], $S(re^{-2i\mu})$ can be written in the normal order form as

$$\exp\left[\frac{r}{2}(e^{-2i\mu}\hat{a}^{\dagger 2} - e^{2i\mu}\hat{a}^2)\right] \equiv \frac{1}{\sqrt{\cosh r}}\exp\left[\frac{\tanh re^{-2i\mu}\hat{a}^{\dagger 2}}{2}\right] \times \sum_{n=o}^{\infty}\frac{(\sec hr-1)^n}{n!}\hat{a}^{\dagger n}\hat{a}^n \exp\left[\frac{-\tanh re^{2i\mu}\hat{a}^2}{2}\right] .(7.41)$$

Substituting (7.41) in (7.40) and simplifying we find

$$\begin{aligned} Q(\beta,\beta^*) &= \frac{1}{\cosh r}\left|\sum_{q=o}^{m-1} f_q^{(e)}\exp\left[\frac{\tanh r}{2}(e^{-2i\mu}\beta^{*2}-\alpha^2)\right]\sum_{n=o}^{\infty}\frac{(\sec hr-1)^n}{n!}\right. \\ &\times \left.(\beta^*\alpha e^{-i\mu})^n\exp\left[\frac{-|\beta|^2}{2}-\frac{|\alpha|^2}{2}+\beta^*\alpha e^{-i\mu}\right]\right|^2 \\ &= \frac{1}{\cosh r}\exp\left[-|\alpha|^2-|\beta|^2-\tanh r\alpha^2\right] \\ &\times \left|\sum_{q=o}^{m-1} f_q^{(e)}\exp\{\frac{1}{2}\tanh re^{-2i\mu}\beta^{*2}+\beta^*\alpha e^{-i\mu}\sec hr\}\right|^2 . \qquad (7.42) \end{aligned}$$

In Fig. (7.7) we display the behavior of the Q function for the superposition of squeezed coherent states for $m = 6$ and for various values of squeezing parameter r. The squeezing characteristics of the individual states in the superposition are apparent from the contour plots.

In conclusion, we have shown how a special class of Schrödinger kitten states can be generated by a coherent field propagating through a Kerr medium. We have also shown how superposition of squeezed coherent states can be produced if the field is initially in a squeezed coherent state. As the kitten states are eigenstates of the m-th power of the photon annihilation operator ($\hat{a}^m$), hence by analogy to Ref. 7 where they have analysed a scheme for production of Schrödinger cat state based on two photon Jaynes Cummings model, the kitten states can be produced by considering multiphoton Jaynes-Cummings model. It is also possible to consider higher order interactions like $\hbar\chi(\hat{a}^\dagger\hat{a})^3$ [16], or even two mode interactions [17]. These would also produce Schrödinger kitten states.

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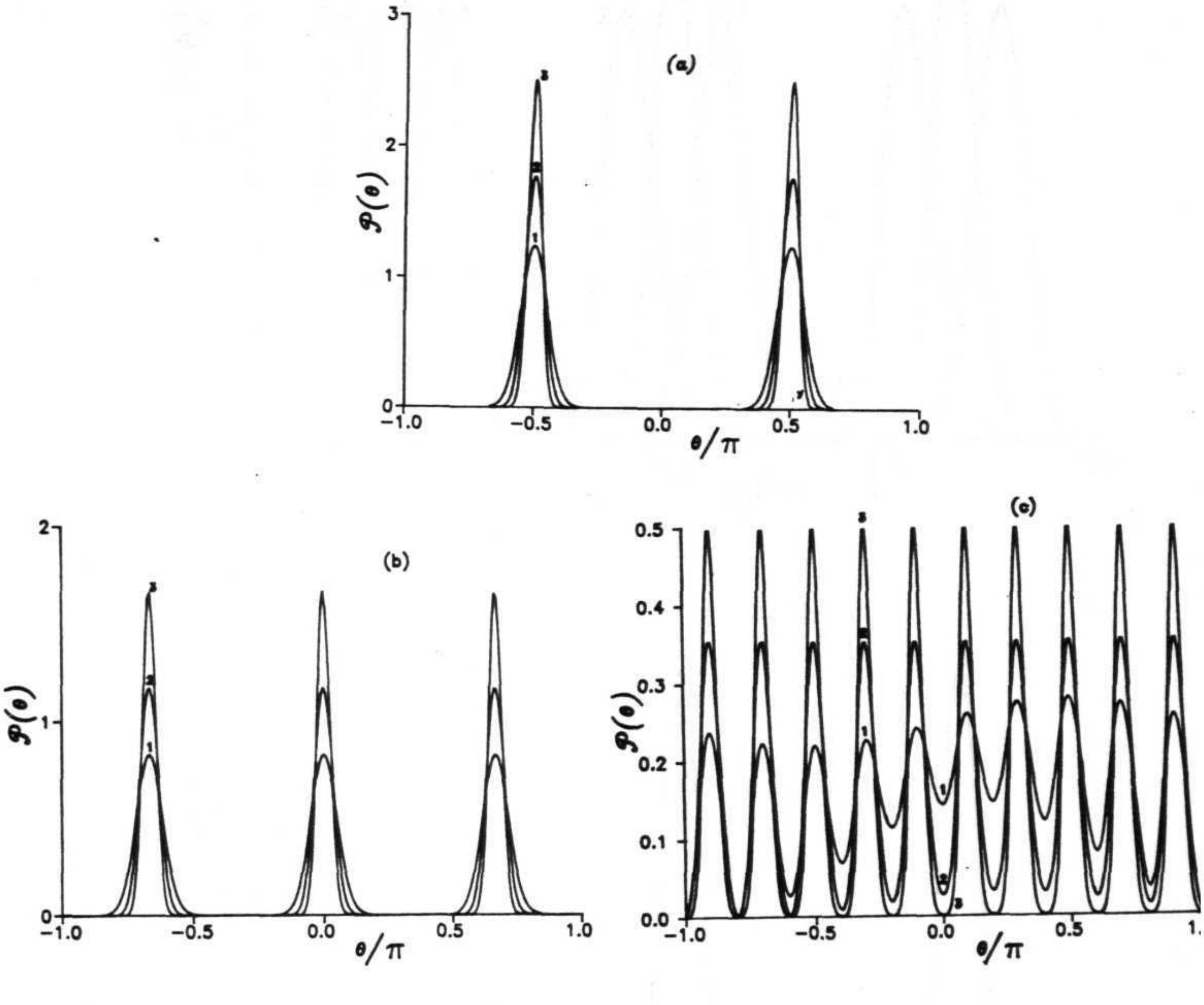

Fig. 7.1. The behavior of the phase distribution, $\mathcal{P}(\theta)$ for
(a) m=2, $|\alpha|^2$=(1) 10, (2) 20, (3) 40,
(b) m=3 and for the same values $|\alpha|^2$ as in (a),
(a) m=10 and for the same values of $|\alpha|^2$ as in (a).

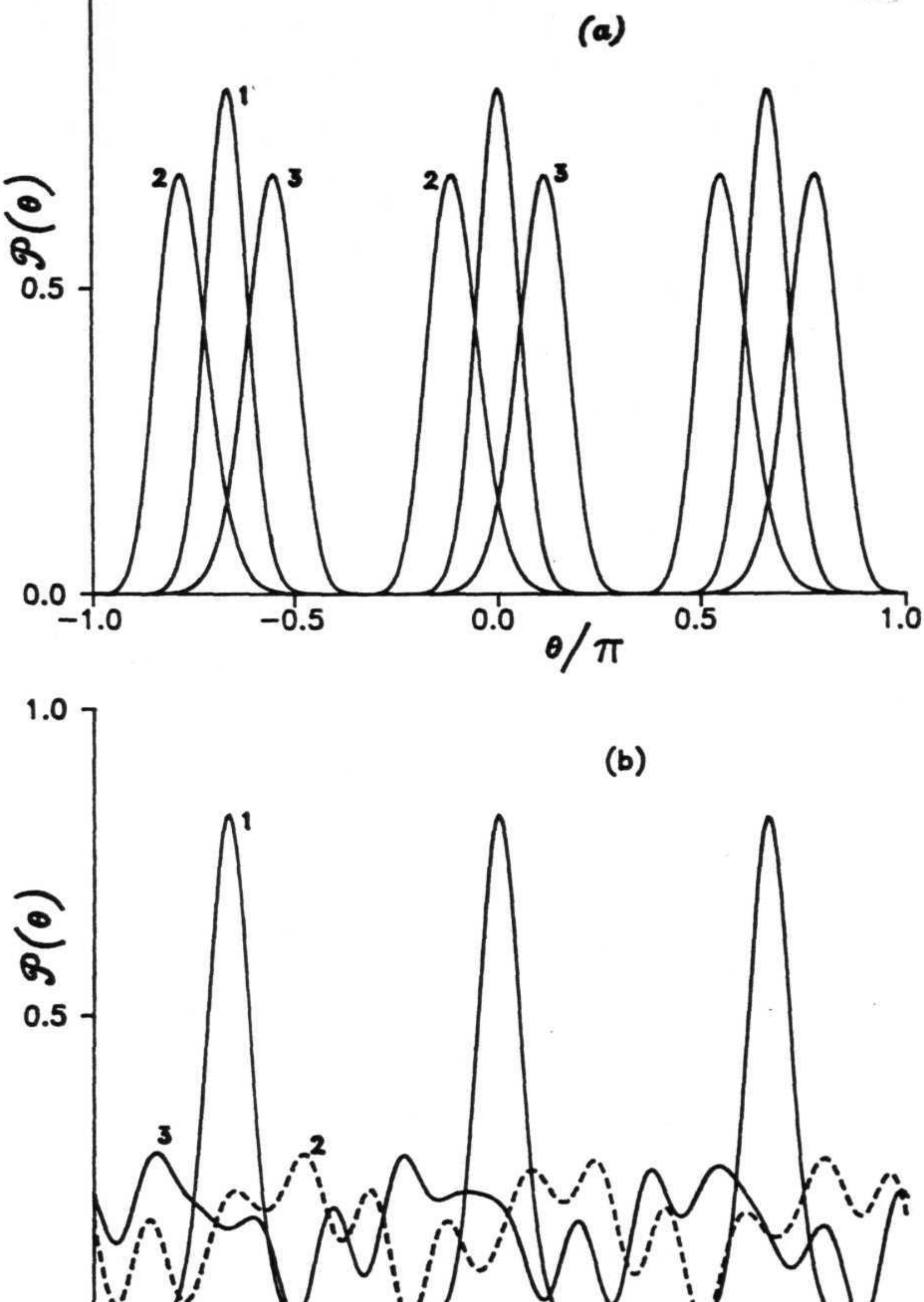

Fig. 7.2. The dependence of the phase distribution, $\mathcal{P}(\theta)$ on the parameter χt for m=3 and $|\alpha|^2=10$.
(a) For small deviation from $\pi/3$, χt=(1) $\pi/3$, (2) $\pi/3+0.017$ and (3) $\pi/3$-0.017. Note that $\mathcal{P}(\theta)$ still retains it shape.
(b) For large deviation from $\pi/3$, χt=(1)$\pi/3$, (2)$\pi/3+0.085$ and (3) $\pi/3$-0.085: $\mathcal{P}(\theta)$ oscillates rapidly around $1/2\pi$.

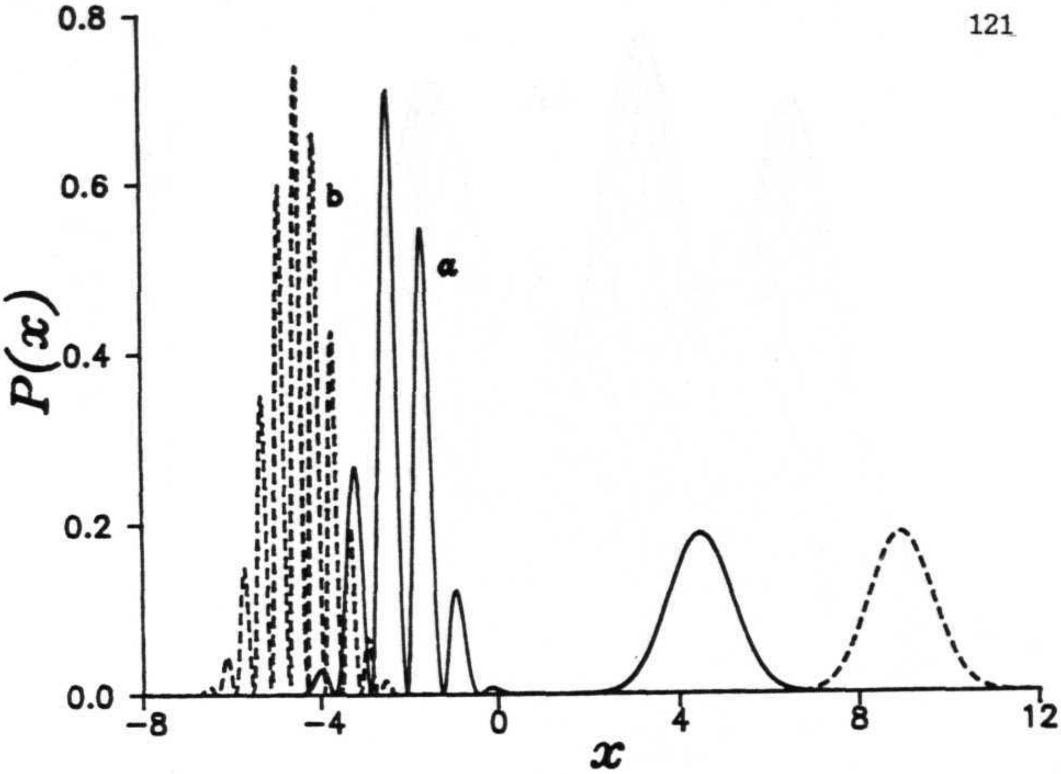

Fig. 7.3. The quadrature distributions $P(x)$ (Eq.(7.25)) for m=3 and $|\alpha|^2$=(a) 10 and (b) 40.

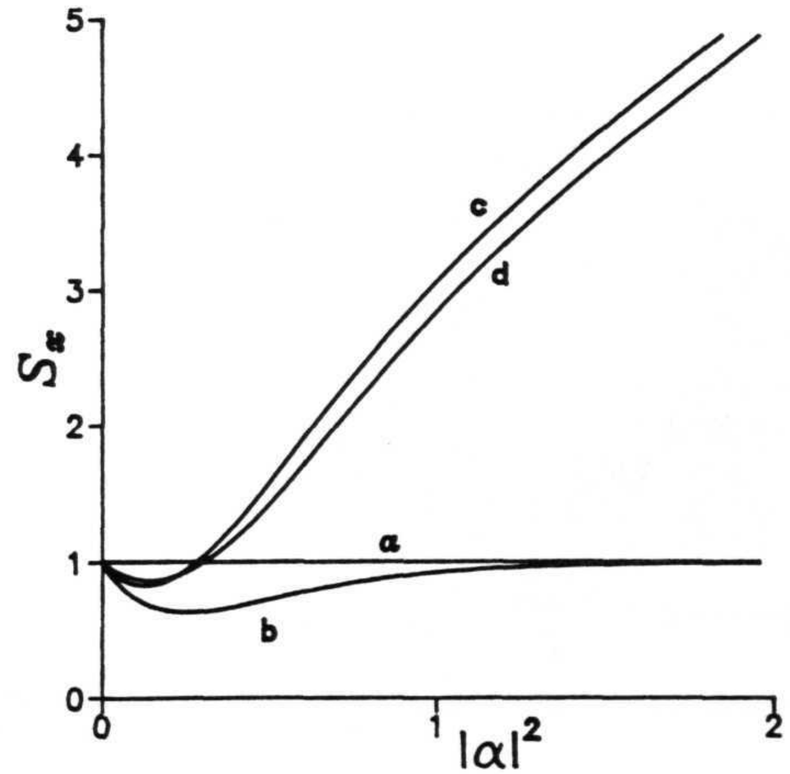

Fig. 7.4. The fluctuations S_x in the x quadrature as a function of $|\alpha|^2$ for m=(a)1, (b)2, (c)3 and (d)4. The phase $\theta - \varphi$ is taken to be 0. φ corresponds to the phase of α.

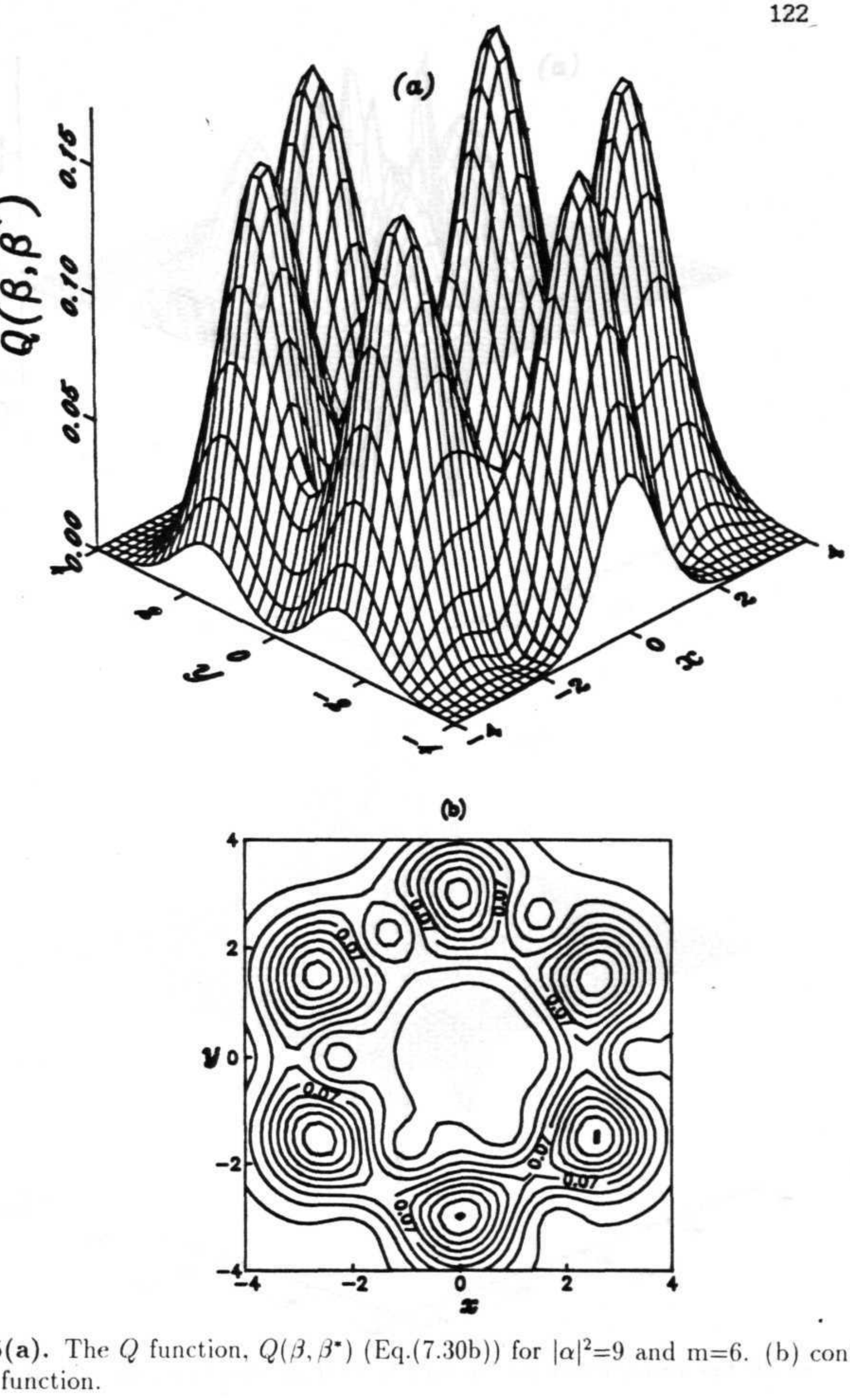

Fig. 7.5(a). The Q function, $Q(\beta,\beta^*)$ (Eq.(7.30b)) for $|\alpha|^2$=9 and m=6. (b) contour of the Q function.

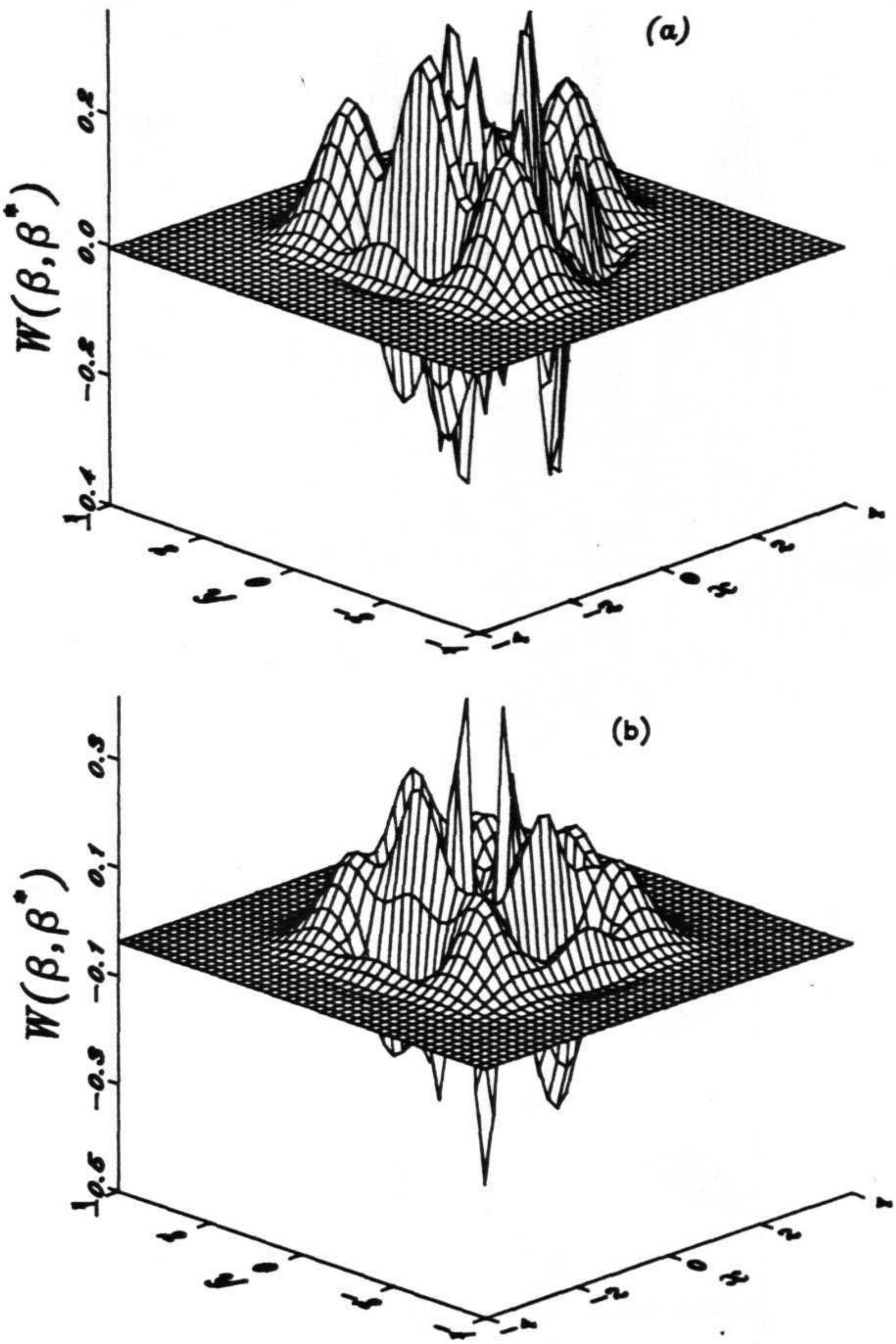

Fig. 7.6. The Wigner function, $W(\beta, \beta^*)$ for $|\alpha|^2=4$ and m(a)=3 and (b)=4.

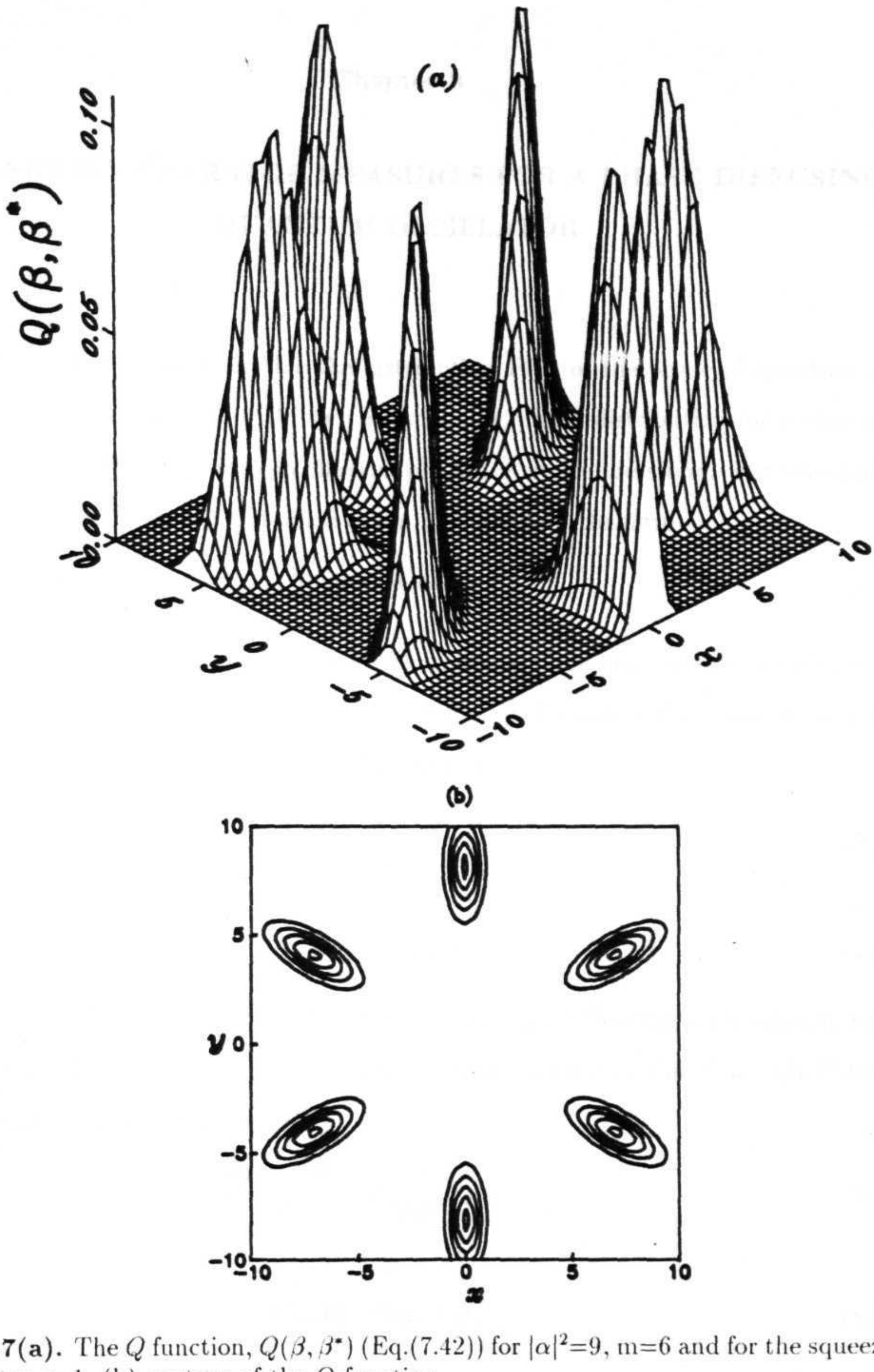

Fig. 7.7(a). The Q function, $Q(\beta, \beta^*)$ (Eq.(7.42)) for $|\alpha|^2=9$, m=6 and for the squeezing parameter $r=1$, (b) contour of the Q function.

Chapter 8

PROBABILITY OPERATOR MEASURES FOR A PHASE DIFFUSING QUANTUM OSCILLATOR

In this short chapter, we determine the distribution $P(\theta)$ of the phase of quantum oscillator and then compare it with the corresponding phase distribution for a classical oscillator. The diffusion process in classical physics is well understood. For example a classical field of frequency ω and phase φ can be written in the form

$$\vec{E} = \vec{E}_o e^{i\vec{k}.\vec{r} - i\omega t - i\varphi(t)} + c.c. \quad . \tag{8.1}$$

Here the phase φ is a fluctuating quantity. For a single mode laser operating well above threshold we assume that the amplitude is stabilised and phase φ fluctuates with time, φ can be taken to be Gaussian random processes with

$$\langle \dot{\varphi}(t) \rangle = 0 \quad , \tag{8.2}$$

and

$$\langle \dot{\varphi}(t) \dot{\varphi}(t') \rangle = 2\gamma \delta(t - t') \quad . \tag{8.3}$$

where γ is the diffusion constant. The spectral line shape of the field is Lorentzian with a half width γ [1]. Let $P_{cl}(\varphi, t)$ be the classical phase distribution function. The Fokker Planck equation is then given by [1]

$$\frac{\partial P_{cl}(\varphi, t)}{\partial t} = 2\gamma \frac{\partial^2}{\partial \varphi^2} P_{cl}(\varphi, t) \quad , \tag{8.4}$$

At $t = 0$,

$$P(\varphi, 0) = \delta(\varphi - \varphi_o) \quad . \tag{8.5}$$

Then using (8.5) in (8.4) we obtain the solution $P_{cl}(\varphi,t)$

$$P_{cl}(\varphi,t) = \exp\left(2\gamma t\frac{\partial^2}{\partial\varphi^2}\right)\delta(\varphi-\varphi_o)$$

$$= \frac{1}{2\pi}\exp\left(2\gamma t\frac{\partial^2}{\partial\varphi^2}\right)\sum_{n=-\infty}^{\infty} e^{in(\varphi-\varphi_o)} \ ,0\leq\varphi\leq 2\pi \tag{8.6}$$

where we have used

$$\delta(\varphi-\varphi_o) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in(\varphi-\varphi_o)} \ . \tag{8.7}$$

Eq.(8.6) can be rewritten in a more neater form as

$$P_{cl}(\varphi,t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\exp(in(\varphi-\varphi_o)-2\gamma tn^2) \ . \tag{8.8}$$

We next consider a quantized single mode field with diffusing phase and steady amplitude. We thus consider a quantum oscillator with Hamiltonian

$$\hat{H} = \omega(t)\hat{a}^\dagger\hat{a} \quad , \tag{8.9}$$

with $\omega(t) = \dot{\varphi}(t)$ and in units of $\hbar = 1$. Here φ has the same properties as in Eq.(8.2) and (8.3). That is, it is treated as a random quantity and it is delta correlated. If the field at some initial time is in a coherent state $|\alpha>$, then the coherent state evolves with respect to time as

$$|\alpha> \longrightarrow |\alpha\exp -i\int_o^t\omega(t)dt> \quad . \tag{8.10}$$

The density operator of the field is then given by

$$\hat{\rho}(t) = |\alpha e^{-i\varphi(t)}><\alpha e^{-i\varphi(t)}| \ . \tag{8.11}$$

Since $\varphi(t)$ is fluctuating, it is meaningful to work with the ensemble average of $\hat{\rho}(t)$ over these fluctuations of the phase φ. Expressing $|\alpha e^{-i\varphi(t)}>$ in terms of Fock states $|n>$, Eq.(8.11) can be written as

$$\hat{\rho}(t) = \sum_{n,m=o}^{\infty} e^{-|\alpha|^2}\frac{\alpha^{*m}\alpha^n}{\sqrt{n!m!}}\exp[-i\varphi(t)(n-m)]|n><m| \ . \tag{8.12}$$

On using the Gaussian property of φ and Eqs.(8.2) and (8.3), the ensemble average of $\hat{\rho}(t)$ becomes

$$< \hat{\rho}(t) > \ = \ \bar{\hat{\rho}}(t) = \sum_{n,m=o}^{\infty} e^{-|\alpha|^2} \frac{\alpha^{*m}\alpha^n}{\sqrt{n!m!}} \ \exp\{-(n-m)^2\gamma t\} \ |m><n| \ , \tag{8.13}$$

where we have used

$$\begin{aligned} \langle \exp\left[-i(n-m)\int_o^t dt'\omega(t')\right]\rangle &= \exp\left[\frac{(n-m)^2}{2}\int_o^t\int_o^{t'} dtdt' < \omega(t)\omega(t') >\right] \\ &= \exp[-(n-m)^2\gamma t] \ . \end{aligned} \tag{8.14}$$

Having obtained the density operator of the quantum oscillator which is undergoing phase diffusion, we now proceed to determine the distribution $P(\theta,\tau)$, with $\tau = \gamma t$, of the phase of the quantum oscillator. The phase distribution as defined in Chapter 6 is given by

$$P(\theta,\tau) = \frac{1}{2\pi}\langle\theta|\bar{\hat{\rho}}|\theta\rangle \ , \tag{8.15}$$

with

$$|\theta> = \sum_{n=0}^{\infty} e^{in\theta}|n> \ . \tag{8.16}$$

And hence

$$P(\theta,\tau) = \frac{1}{2\pi}\sum_{n,m=0}^{\infty} e^{im\theta-in\theta}\bar{\rho}_{nm} \ , \tag{8.17}$$

which on using (8.13) reduces to

$$P(\theta,\tau) = \frac{e^{-|\alpha|^2}}{2\pi}\sum_{n,m=o}^{\infty}\frac{|\alpha|^{n+m}}{\sqrt{n!m!}}\exp\{-i(n-m)(\theta-\theta_o)-(n-m)^2\tau\} \ , \tag{8.18}$$

where $\alpha = |\alpha|e^{i\theta_o}$. This is the result for the phase distribution for a quantum oscillator. The question arises as to how the phase distribution (8.18) compares with the distribution (8.8) for a classical oscillator. Note that the classical phase distribution (8.8) is independent of the magnitude of the classical field whereas the quantum phase distribution depends on the average excitation in the field. For large values of $|\alpha|$, using Gaussian approximation for a Poisson distribution [2]

$$\frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2} \approx (2\pi|\alpha|^2)^{-1/2}\exp\left[\frac{-(n-|\alpha|^2)^2}{2|\alpha|^2}\right] \quad , \tag{8.19}$$

The expression (8.18) can be approximated as

$$P_G(\theta,t) \approx \frac{1}{[2\pi\sigma^2(t)]^{1/2}}\exp\left\{\frac{-(\theta-\theta_\circ)^2}{2\sigma^2(t)}\right\} \quad , \tag{8.20}$$

with $\sigma^2(t) = 1/4|\alpha|^2 + 2\gamma t$.

In Fig.(8.1) we plot quantum mechanical phase distribution given by Eq. (8.17). In Figs.(8.2) and (8.3) we plot the classical phase distribution (8.8) and the quantum mechanical phase distribution(8.18) for a fixed $|\alpha|^2$ and for different values of $\tau(=\gamma t)$ and for specific τ and different values of $|\alpha|^2$ respectively. In Fig.(8.4) we show the quantum phase distribution and compare it with the Gaussian approximation given by Eq.(8.19) and the distribution (8.8) for the classical oscillator phase.

In conclusion, we have made a comparison of the distribution of the classical and quantum oscillator phase. It should be borne in mind that the diffusion process itself is classical in its origin. One may also consider what happens if the diffusion process itself is of quantum origin. Such a situation indeed arises in the propagation of a single mode electromagnetic field through a Kerr medium. This has been discussed at length in the previous chapters (Chapter 6 and 7).

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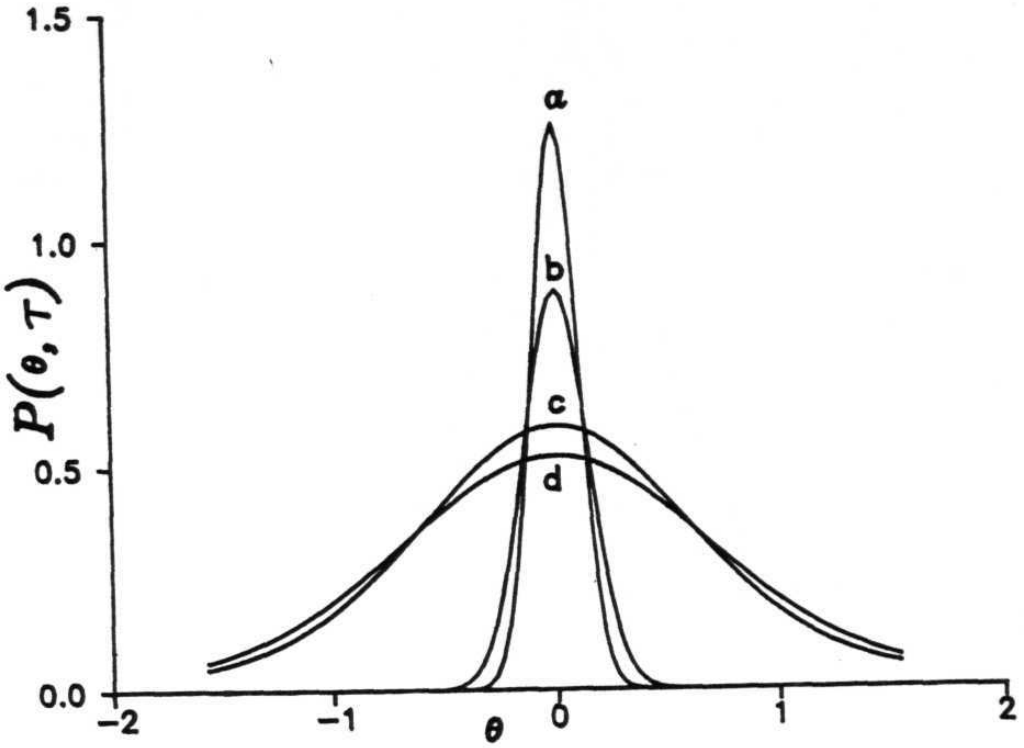

Fig. 8.1. Quantum mechanical phase distribution (Eq.(8.18)) for $|\alpha|^2=1$ and τ (a)=0.05, (b) =0.1, (c)=0.5 and (d)=1.

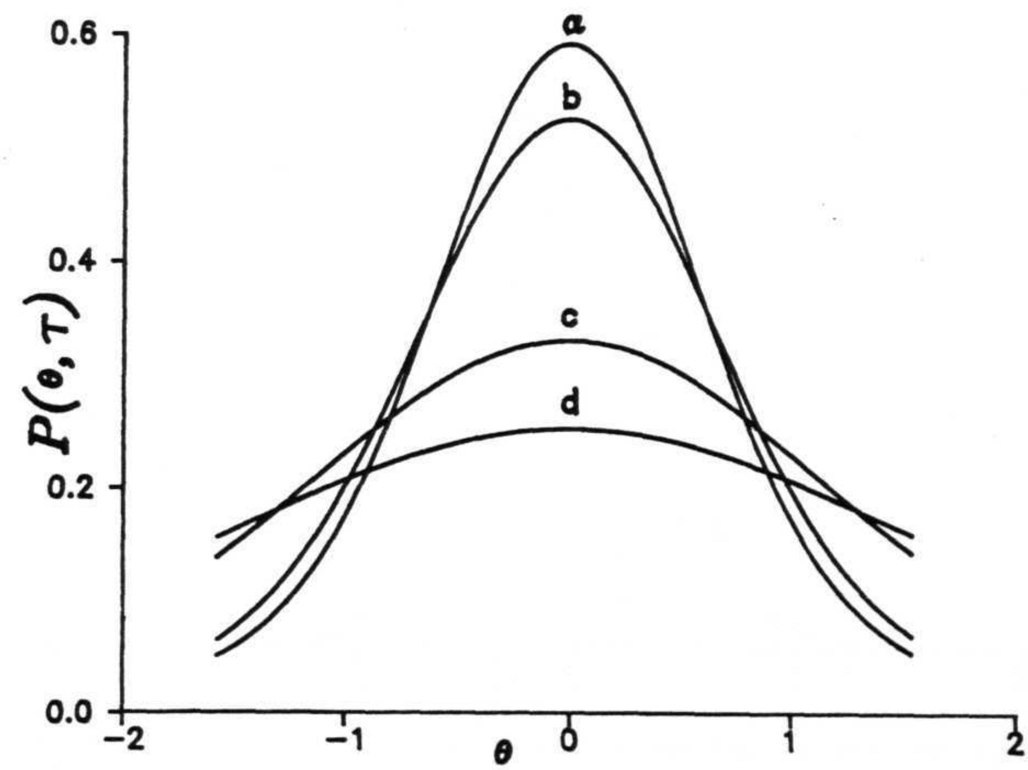

Fig. 8.2. Comparison of the classical (Eq.(8.8)) and the quantum mechanical (Eq.(1.18)) phase distribution for $|\alpha|^2=1$. Classical phase distribution corresponds to the curves τ (a)=0.05, (b)=0.1 and τ (c) =0.05, (d)=0.1 corresponds to the quantum phase distribution.

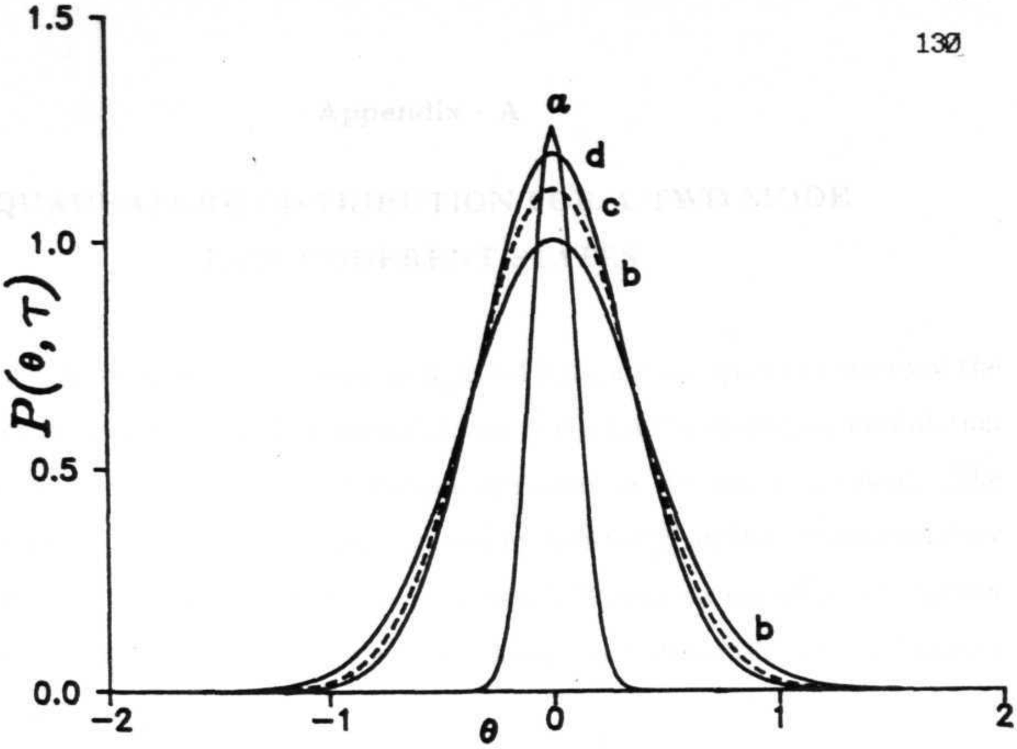

Fig. 8.3. Comparison of classical and quantum phase distribution for τ=0.05, (a) corresponds to the classical phase distribution. It is independent of $|\alpha|^2$. Curves (b), (c) and (d) corresponds to the quantum phase distribution for $|\alpha|^2$=5, 10 and 30 respectively.

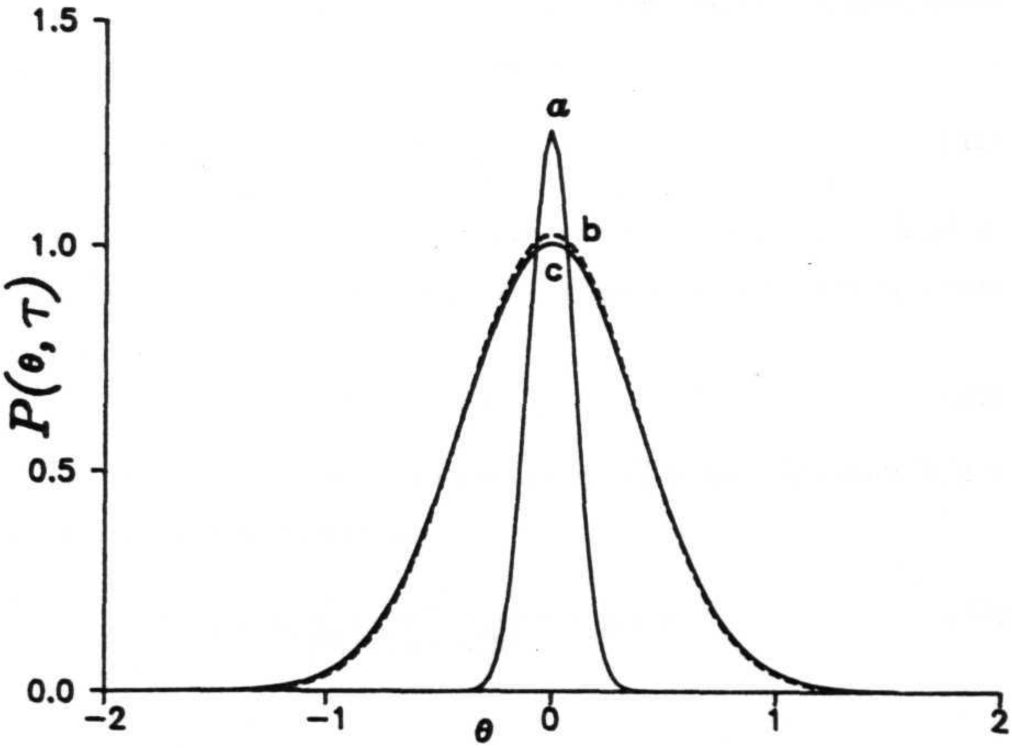

Fig. 8.4. Plot of classical phase distribution, (a); Gaussian appoximation (Eq.(8.19)) to the quantum phase distribution, (b); and quantum phase distribution, (c); for $|\alpha|^2$=5 and τ=0.05.

Appendix - A

QUADRATURE DISTRIBUTION FOR A TWO MODE PAIR COHERENT STATES

Pair coherent states introduced recently by Agarwal [1,2], are the quantum states of the two mode electromagnetic field which are simultaneous eigenstates of the pair annihilation operator and of the difference in the number operators of the two mode field. The quantum features of such states has been discussed by Agarwal[2] and has shown that they exhibit remarkable nonclassical properties such as sub-Poissonian statistics, correlation in the photon number fluctuations in the two modes, violations of the Cauchy-Schwarz inequalities and squeezing.

If $\hat{a}$ and $\hat{b}$ are the two annihilation operators associated with the two modes, the operator $\hat{a}\hat{b}$, acting on a Fock state, simultaneously annihilates photons of modes a and b. Thus it can be referred to as the pair annihilation operator. The pair coherent states are defined as eigenstates of the pair annihilation operator

$$\hat{a}\hat{b}|\zeta, q>= |\zeta, q> \quad , \tag{A1}$$

where ζ is a complex number and q is the degeneracy parameter, which can be fixed by the requirement that $|\zeta, q>$ is an eigenstate of the difference in the number operators for the two modes.

$$(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})|\zeta, q>= q|\zeta, q> \quad . \tag{A2}$$

If the pair creation starts from vacuum, the paramater q will be zero. The state $|\zeta, q>$ can be expanded in terms of Fock states as

$$|\zeta, q>= N_q \sum_{n=o}^{\infty} \frac{\zeta^n}{[n!(n+q)!]^{1/2}}|n+q, n> \quad , \tag{A3}$$

where N_q is the normalisation constant

$$N_q = \left[\sum_{n=o}^{\infty} \frac{|\zeta|^{2n}}{n!(n+q)!}\right]^{-1/2} = |\zeta|^{-q}[I_q(2|\zeta|)]^{-1/2} \quad . \tag{A4}$$

$I_q(x)$ is modified Bessel function [3]. Eqs. (A1-A4) define the pair coherent states. In this Appendix we derive the quadrature distribution for the pair coherent states.

We consider the case when the parameter $q = 0$. Then the state (A3) becomes

$$|\zeta, 0> \equiv |\psi> = N_o \sum_{n=o}^{\infty} \frac{\zeta^n}{n!}|n, n> \quad . \tag{A5}$$

We calculate the coordinate space wave function

$$< x, y|\psi> = N_o \sum_{n=o}^{\infty} \frac{\zeta^n}{n!} < x|n>< y|n>$$

$$= N_o \sum_{n=o}^{\infty} \frac{s^n}{n!} \frac{1}{\sqrt{\pi}} \frac{H_n(x)H_n(y)}{2^n n!} \exp\left[-\frac{(x^2+y^2)}{2}\right] \quad , \tag{A6}$$

where, $< x|n >$ is a harmonic oscillator wave function given in terms of Hermite polynomials as

$$< x|n >= \left(2^n n!\sqrt{\pi}\right)^{-1/2} H_n(x) e^{-x^2/2} \quad . \tag{A7}$$

Here $\hat{x}$ and $\hat{y}$ correspond to

$$\hat{x} = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}} \ ; \qquad \hat{y} = \frac{\hat{b} + \hat{b}^\dagger}{\sqrt{2}} \quad . \tag{A8}$$

The distribution $P_{pc}(x, y)$ for the field in the pair coherent is then obtained from (A6) as

$$P_{pc}(x, y) = |\langle x, y|\psi\rangle|^2 \quad . \tag{A9}$$

A two mode quadrature operator is defined by [4]

$$\hat{X} = \frac{(\hat{a} + \hat{b} + \hat{a}^\dagger + \hat{b}^\dagger)}{2\sqrt{2}} = \frac{\hat{x} + \hat{y}}{2} \quad , \tag{A10}$$

$$\hat{Y} = \frac{(\hat{a} - \hat{b} + \hat{a}^\dagger - \hat{b}^\dagger)}{2\sqrt{2}} = \frac{\hat{x} - \hat{y}}{2} \quad . \tag{A11}$$

So we make a transformation

$$\hat{x} = \hat{X} + \hat{Y} \ , \qquad \hat{y} = \hat{X} - \hat{Y} \ , \tag{A12}$$

and substitute in (A9), The quadrature distribution $P_{pc}(X, Y)$ is then

$$P_{pc}(X,Y) = \left[\sum_{m=o}^{\infty} \frac{|\zeta|^{2m}}{m!^2}\right]^{-1} \frac{e^{-(X^2+Y^2)}}{\pi} \left|\sum_{n=o}^{\infty} \frac{H_n(X+Y)H_n(X-Y)\zeta^n}{2^n n!^2}\right|^2 \ . \tag{A13}$$

In the Figs. (A1) and (A2) we plot the quadrature distribution $P_{pc}(X, Y)$ for different values of ζ. Clearly these figures show that the quadrature X is squeezed. This squeezing occurs because of the strong correlation between the a and b modes. This can be seen analytically as follows. It is obvious from (A5) and (A10) that the mean value of X is zero.

$$\langle\psi|\hat{X}|\psi\rangle = 0 \ , \tag{A14}$$

and

$$\langle\psi|\hat{X}^2|\psi\rangle = \frac{1}{4}\langle\psi|\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b} + 1 + \hat{a}\hat{b} + a^\dagger b^\dagger|\psi\rangle \ . \tag{A15}$$

Since the terms like $\langle\hat{a}^2\rangle, \langle\hat{a}^{\dagger 2}\rangle, \langle\hat{a}^\dagger\hat{b}\rangle, \cdots$ etc. goes to zero and

$$< \hat{a}^\dagger\hat{a} > = < \hat{b}^\dagger\hat{b} > = \frac{N_o^2|\zeta|^2}{N_1^2} = \frac{|\zeta|^2 I_1(2|\zeta|)}{I_o(2|\zeta|)} \ , \text{(see ref.2, Eq.(2.8))}.$$

Using the above result in (A15), $< \hat{X}^2 >$ becomes,

$$< \hat{X}^2 > = \frac{1}{4}\left[2|\zeta|\frac{I_1(2|\zeta|)}{I_o(2|\zeta|)} + 2|\zeta|\cos\theta + 1\right] \ , \tag{A16}$$

with $\zeta = |\zeta|e^{i\theta}$. When the asymptotic expansion [3] for $I_\nu(z)$,(for large values of $|\zeta|$)

$$I_\nu(z) \cong \frac{e^z}{(2\pi z)^{1/2}}\left\{1 - \frac{4\nu^2 - 1}{8z}\right\} \ , \tag{A17}$$

is used, for the first term in (A16), i.e.,

$$\frac{|\zeta| I_1(2|\zeta|)}{I_o(2|\zeta|)} \sim |\zeta| - \frac{1}{4} + o(1/|\zeta|) \ . \tag{A18}$$

We get the asymptotic value for $<\hat{X}^2>$ as

$$<\hat{X}^2>=\frac{1}{4}\left[2|\zeta|(1+\cos\theta)+\frac{1}{2}\right] \quad . \tag{A19}$$

Then from (A14)-(A19), the fluctuations in quadrature $\hat{X}$ in the asympotic limit is found to be

$$\begin{aligned} S_x \equiv 4(\Delta X)^2 &= <\hat{X}^2> - <\hat{X}>^2 \\ &= \frac{1}{2} + 2|\zeta|(1+\cos\theta) \quad (A20) \\ &= \frac{1}{2} \quad \text{for } \theta=\pi \quad . \quad (A21) \end{aligned}$$

Thus asymptotically 50% squeezing is obtained. It is interesting to note that if the correlation between the a and b modes reduces the fluctuations in $\hat{X}$ then at the same time the fluctuations in $\hat{Y}$ are enhanced.

$$(\Delta X)^2 = (\Delta x)^2 + (\Delta y)^2 + 2\left[<\hat{x}\hat{y}> - <\hat{x}><\hat{y}>\right] \quad , \tag{A22}$$

$$(\Delta Y)^2 = (\Delta x)^2 + (\Delta y)^2 - 2\left[<\hat{x}\hat{y}> - <\hat{x}><\hat{y}>\right] \quad . \tag{A23}$$

We now compare the quadrature distribution $P_{pc}(X,Y)$ (A13) obtained for pair coherent state with that associated with the two mode squeezed vacuum state for the parameter values which give rise to same amount of squeezing in quadrature X for both the states. The two mode squeezed vacuum state [5] is defined by

$$|r,\varphi>\equiv \hat{S}(r,\varphi)|0,0> \quad . \tag{A24}$$

The two mode squeeze operator

$$\hat{S}(r,\varphi)\equiv \exp\left[r(\hat{a}\hat{b}e^{-i\varphi} - \hat{a}^\dagger\hat{b}^\dagger e^{i\varphi})\right] \quad , \tag{A25}$$

is characterised by a real squeeze parameter r and an angle φ that determines the phase of the squeezing. The fluctuations in the X quadrature (A10) for this state is found to be

$$S_x \equiv 4(\Delta X^2) = \cosh 2r - \sinh 2r \cos\varphi \quad , \tag{A26}$$

$$= e^{-2r} \quad , \quad \text{for } \varphi = 0 \quad . \tag{A27}$$

The coordinate space wave function for this states given by

$$< x, y|r, \varphi >= \frac{\sec hr}{\sqrt{\pi}} \frac{1}{\sqrt{1-t^2}} \exp\left\{\frac{4xyt - (x^2 + y^2)(1 + t^2)}{2(1 - t^2)}\right\} \quad , \tag{A28}$$

where $t = -e^{i\varphi} \tanh r$.

At $\varphi = 0$ it is found that the quadrature X (A10) is squeezed. Again making the same transformation as given by Eq.A12 to the above equation, we obtain the quadrature distribution $P_s(X, Y)$ for the two mode squeezed vacuum state.

$$P_s(X, Y) = \frac{1}{\pi} \exp\left\{-2(X^2 e^{2r} + Y^2 e^{-2r})\right\}, \text{ at } \varphi = 0 \tag{A29}$$

We plot this distribution in Figs.(A3) and (A4) for the values of r for which the amount of squeezing obtained in quadrature X is same as that obtained for pair coherent state whose quadrature distribution is plotted in Figs.(A1) and (A2).

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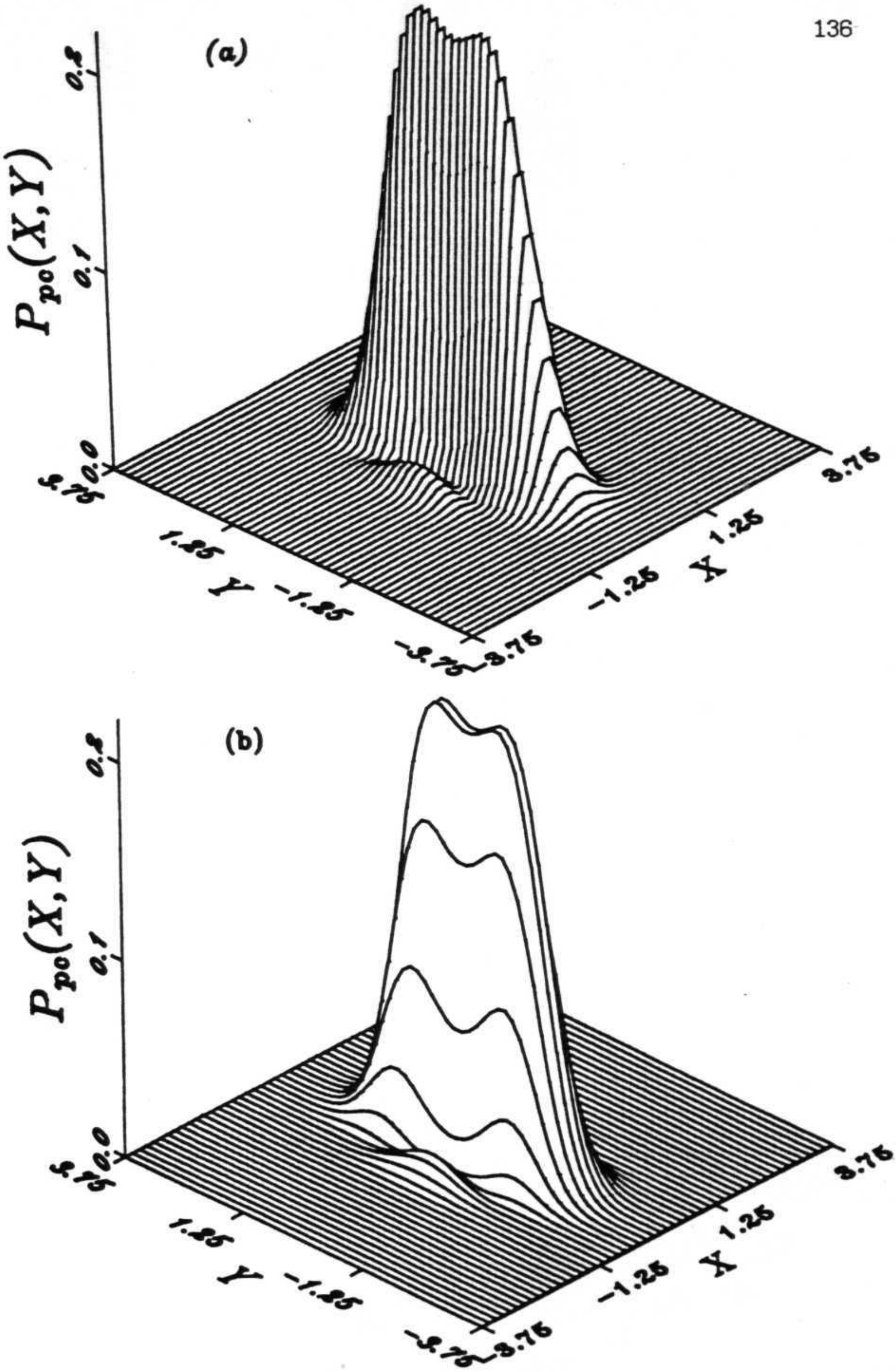


Fig. A.1. The quadrature distribution $P_{pc}(X,Y)$ (Eq.(A13)) for ζ=-2 which corresponds to $4(\Delta x)^2$=0.4541. The distribution is plotted for (a) constant Y, (b) constant X.

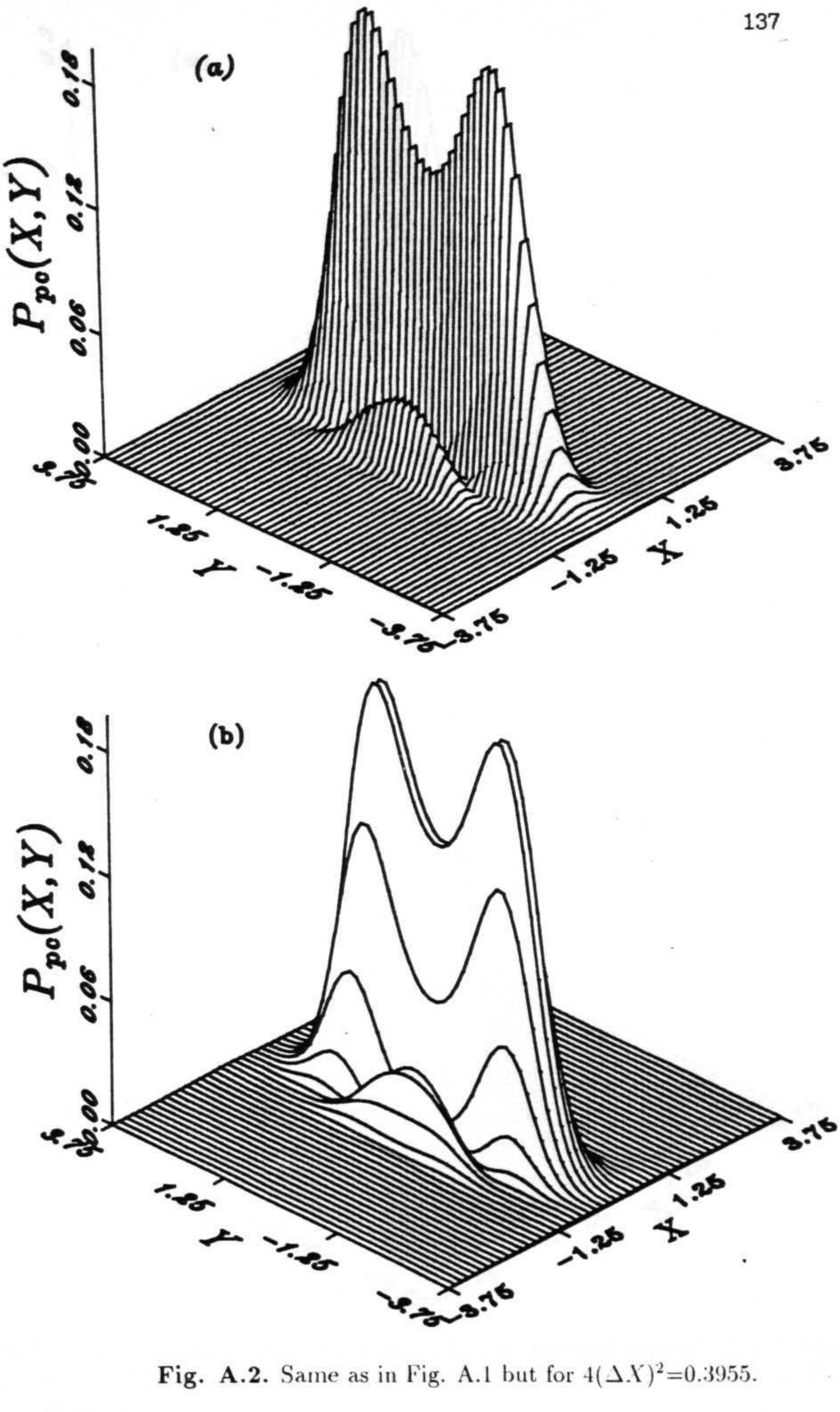
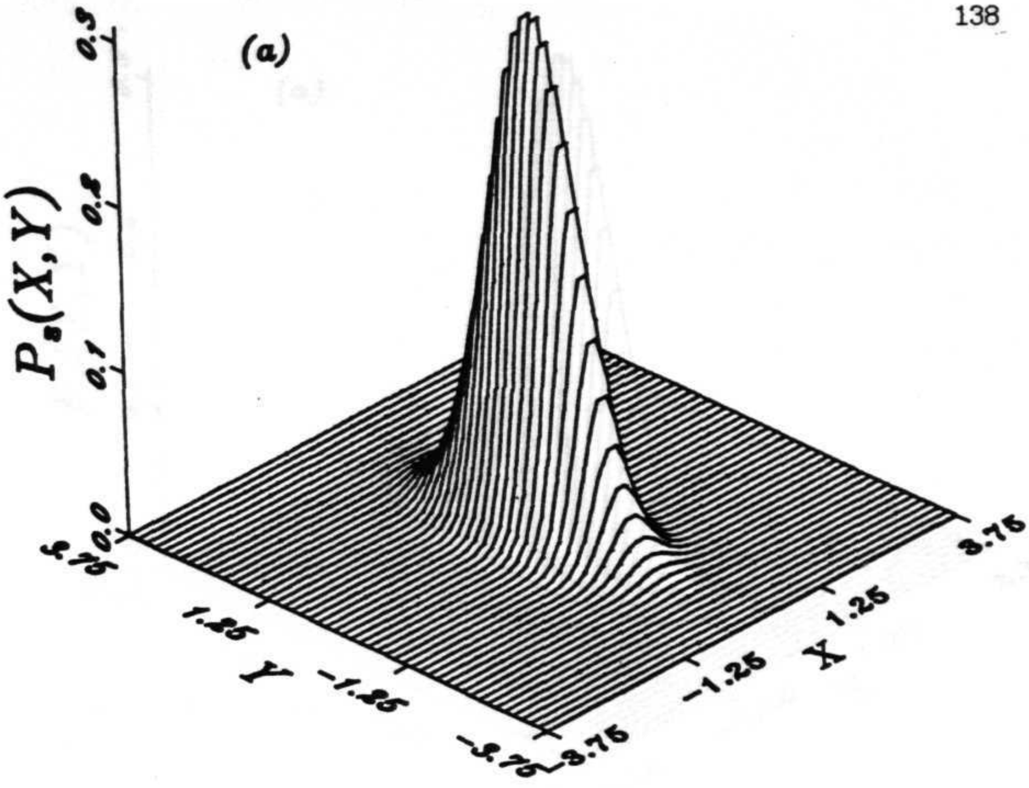


Fig. A.2. Same as in Fig. A.1 but for $4(\Delta X)^2=0.3955$.



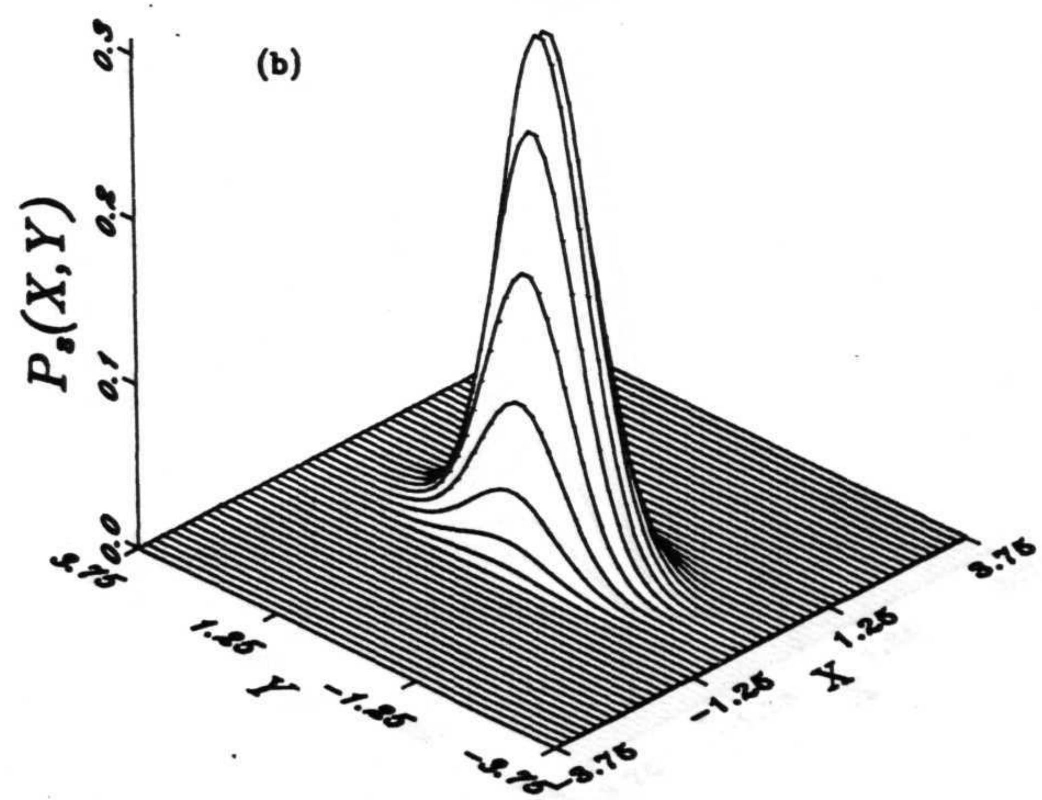


Fig. A.3. The quadrature distribution $P_s(X, Y)$ for squeezed vacuum (Eq.(A29)) for $4(\Delta X)^2$=0.4541. The squeezing parameter r=0.395 and $\varphi = \pi$, (a) for constant Y and (b) for constant X.

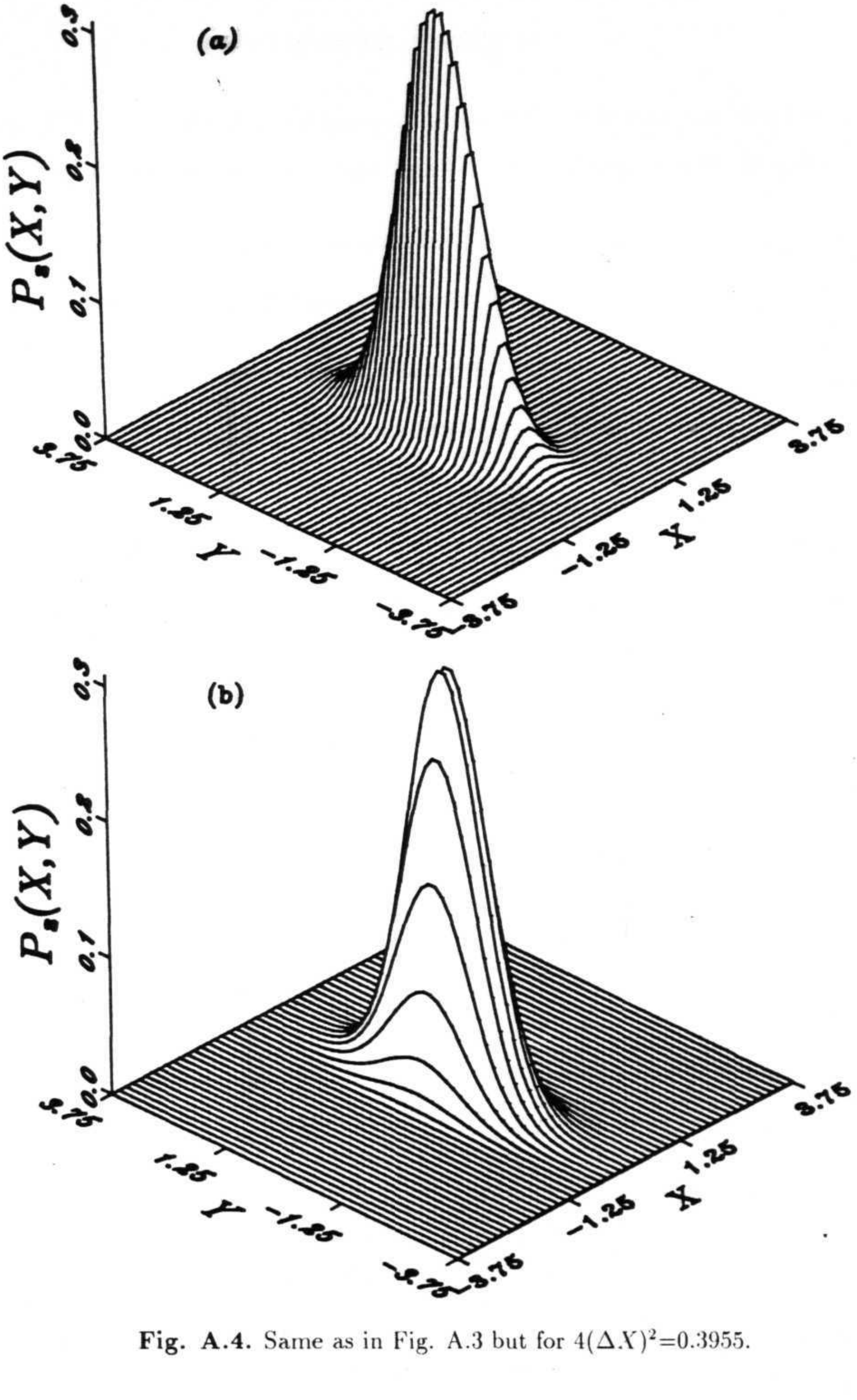


Fig. A.4. Same as in Fig. A.3 but for $4(\Delta X)^2=0.3955$.

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