Implications of non-commutativity of space-time : study of certain models

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DOCTOR OF PHILOSOPHY IN PHYSICS

BY

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DECLARATION

I hereby declare that, this thesis titled **"Implications of non-commutativity of space-time : study of certain models"** submitted by me, under the supervision of **Prof. E. Harikumar**, is a bonafide research work and is free from plagiarism. I also declare that it has not been submitted previously, in part or in full to this University or any other University or Institution, for the award of any degree or diploma. I hereby agree that my thesis can be deposited in Shodhganga/INFLIBNET.

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Further, the student has the following publications[1] before the submission of the thesis for adjudication.

1) $\star$ E. Harikumar, S. K. Panja and V. Rajagopal, *"Maximal acceleration in a Lorentz invariant non-commutative space-time"*, *Eur. Phys. J. Plus* **137** (2022) 966.

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5) E. Harikumar and V. Rajagopal, *"Quantization of κ-deformed Dirac equation"*, *Int. J. Mod. Phys.* **A 35** (2020) 2050147 (chapter 3).

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Abstract

In this thesis, we study some implications of the non-commutative space-times, by studying certain models in the κ-deformed space-time and Doplicher-Fredenhagen-Roberts (DFR) space-time.

We study the quantisation of scalar field (in κ-deformed as well as DFR space-times) and Dirac field (in κ-deformed space-time) using their deformed equations of motion. We obtain deformed (anti-) commutation relations between non-commutative field and its adjoint, by considering the usual form of the oscillator algebra. By demanding this (anti-) commutation relations between non-commutative field and its adjoint to be undeformed, we get the deformed oscillator algebra, which is a general feature of the non-commutative field theories. Further we have analysed the consequences of this deformed oscillator algebra by studying the Unruh effect in κ-deformed and DFR space-times and find that the Unruh profile is modified due to the non-commutativity. Using the global phase transformation symmetry, we have constructed the number operator corresponding to the κ-deformed Dirac field from its deformed equation of motion and we show that this number operator has a mass dependent correction term.

We also study the effects of the non-commutativity in the astrophysical objects such as superdense star. We analyse the superdense star in non-commutative space-time by generalising the core-envelope model having a perfect fluid distribution to the κ-deformed space-time. We construct the Einstein's equation in the κ-deformed space-time and its solutions give the expressions for the pressure and density of the superdense star in κ-deformed space-time. We further show that these equations admit physically acceptable solutions. In this study, we also obtain a bound on the κ-deformation parameter.

Finally, we analyse the notion of maximal acceleration in the non-commutative space-time. We derive the κ-deformed corrections associated with the maximal acceleration from the 8-dimensional κ-deformed line element and the κ-deformed uncertainty relations. From this we then obtain the expression for the maximal temperature associated with thermal radiation in the κ-deformed space-time and using this we obtain another bound on the κ-deformation parameter. We also show the emergence of maximal acceleration from the 4-dimensional κ-Minkowski space-time, which reduces to a finite value in the classical limit.

Dedicated to my parents

Contents

Chapter 1

Introduction

1.1 Motivation and overview

A complete quantum description of gravity is still an unsolved issue in theoretical physics. The quantum theory of gravity is to be understood as a theory that provides a microscopic description to gravity so that general relativity will become consistent with the principles of quantum mechanics. The quest for such quantum gravity theories has a long history; various approaches such as string theory [1], loop quantum gravity [2], causal dynamical triangulations [3], Horava-Lifschitz gravity [4], asymptotically safe gravity [5], non-commutative geometry [6], etc., have been developed and studied rigourosly. Most of these studies predict the existence of a minimal length scale [7–10], below which quantum gravity effects become important.

The non-commutative geometry [6] provides a geometrical framework to incorporate this fundamental length scale naturally and thus model quantum gravity effects. In the framework of non-commutative geometry, to capture the effects of quantum gravity, spectral action principle was developed [11]. The crucial ingredient in this approach is the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, which consists of symmetry algebra $\mathcal{A}$, Dirac operator $\mathcal{D}$, and a Hilbert space $\mathcal{H}$, on which $\mathcal{D}$ has a well defined action. By using this spectral action, gravity on the non-commutative space-time has been studied [11]. In [12], the standard model coupled to gravity has been studied in this framework.

Though recent activities on non-commutative geometry and construction and study of physical models on non-commutative space-time is due to its connection to quantum gravity, non-commutative space-time was originally introduced, way back in 40s. Heisenberg suggested to use non-commutative space-time as a possible way to remove the divergences that render quantum field theories ineffective. It was anticipated that the fundamental length scale naturally incorporated in the non-commutative space-time theories would regularise the UV divergences, by providing an upper cut-off in the momentum integration. This has been a major motivation for Snyder to replace the usual picture of the space-time continuum with the notion of a quantised space-time having a Lorentz invariant discrete space-time structure. With this in mind, in [13] a non-commutative space-time has been introduced and this is known as the Snyder space-time. The space-time coordinates of Snyder space-time satsify

$$[\hat{x}_\mu, \hat{x}_\nu] = i\lambda^2 M_{\mu\nu}, \tag{1.1}$$

where λ is a real parameter having the dimension of length and $M_{\mu\nu}$ is the Lorentz generator.

Following this, the Lorentz invariant equations of motion for the electromagnetic field and its solutions had also been derived in this quantized space-time [14]. However, it was shown that algebra associated with the non-commutative space-time in [13] lacked the translational invariance, and this translational invariance has then been recovered in [15] by interpreting the coordinate operators as the generators of the Lorenz transformations in the 5-dimensional de-Sitter space.

Non-commutativity has been found to arise in different string theory models. It has been shown that certain string theory and the M-theory models lead to non-commutative gauge theory [16–18]. Under compactification limit of certain string theory models Moyal space-time, a non-commutative space-time, has been shown to emerge [19–21]. The Moyal space-time coordinates obey

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \tag{1.2}$$

where $\theta_{\mu\nu}$ is a constant tensor (of length square dimension). Numerous features of the Moyal space-time has been studied extensively over the past decades [22, 23].

Using Weyl-Moyal correspondence, one can map the functions defined on the non-commutative space-time with its counterpart in the commutative space-time. As a result, Weyl-Moyal map induces a new multiplication rule, called the $\star$-star product, between the functions defined on the commutative space-time. The Moyal star product between two arbitrary functions $f(x)$ and $g(x)$ is defined as [20, 22–24]

$$f(x) \star g(x) = e^{\frac{i}{2}\theta_{\mu\nu}\partial_x^\mu \partial_y^\nu} f(x)g(y)\Big|_{x=y}. \tag{1.3}$$

In general one can study physical models on the non-commutative space-time by directly using the functions defined on-commutative space-times. Alternatively one can also analyse the physical models in the non-commutative space-time by working with the commutative functions, but with the above modified product rule. Different non-commutative field theories have been constructed and analysed in the Moyal space-time using this star product formalism.

Moyal star product, given in Eq.(1.3), contains infinite number of higher derivative terms and these terms result in bringing non-local and non-linear effects into non-commutative models. Therefore the presence of such non-local and non-linear terms in the non-commutative field theory models show characteristic features different from their commutative counter parts. The non-commutative field theory in Moyal space-time has been shown to exhibit effects such as mixing of ultraviolet and infrared divergences, known as the UV/IR mixing [22, 23, 25]. This UV/IR mixing in Moyal space-time has also been found to appear as a quantum anomaly of the twisted-Poincare symmetry [26]. Such UV/IV divergences have also been found to exist in other non-commutative space-times such as κ-deformed space-time and Snyder space-time [27, 28]. These non-local and non-linear terms has also been shown to introduce novel, stable soliton solutions in the non-commutative field theory [29].

Seiberg-Witten map [21] is another important result obtained from the studies of string theory models, which is extensively used for studying the non-commutative field theories. Seiberg-Witten map shows an equivalence between non-commutative gauge theories and the gauge theories on commutative space-time. This mapping has been used to analyse various properties associated with the non-commutative space-times [30–32]. Different aspects of the Cherns-Simons theories defined in the Moyal non-commutative space-time have been analysed with the help of Seiberg-Witten map [33–37]. The Seiberg-Witten map has also been used to study the quantum hall effect in the Moyal plane [38]. This map has also been used to study the renormalisation of the photon self-energy to all orders [39].

The algebra of symmetries corresponding to the non-commutative space-times is described by deforming the Poincare algebra, such that these deformed algebras reduce to the usual Poincare algebra in the commutative limit. In [40, 41], Hopf-algebra structure has been used for analysing the symmetries associated with the non-commutative geometry. Further, it has been shown that the deformed Poincare algebra, which preserves the Hopf algebra structures, can be obtained by twisting the classical Poincare algebra [42]. The symmetry algebra of the Moyal space-time has been shown to be twisted-Poincare algebra (which is a Hopf algebra), where the coproduct of the Lorentz generator has been deformed due to the twist element [43]. The symmetry of the Snyder space-time has also been described using the deformed Poincare symmetry [44].

The statistics of the particles lying on non-commutative space-time is connected with the symmetry algebra of the corresponding space-time. In [45], Weyl-moyal product has been used to study the spin-statistics and the CPT theorem in Moyal space-time. Using the deformed coproduct structure associated with the twisted-Poincare algebra, a twisted statistics has been obtained of the particles in the Moyal space-time [46, 47].

The concept of non-commutativity has been also extended to curved space-times and this has been used to describe the gravity in non-commutative space-time. Deformed diffeomorphism invariant Einstein-Hilbert's action has been constructed in the Moyal space-time from the twisted diffeomorphism algebra [48, 49]. This twisted diffemorphism has been used to show that the solution for two dimensional non-commutative gravity is the same as that for the commutative theory [50]. In [51], deformed Einstein's gravity has been obtained by applying Seiberg-Witten map to certain non-commutative gauge groups. Another remarkable prediction of the non-commutativity is the emergence of gravity from the non-commutative field theories [52–54]. Such predictions have been made using the Seiberg-Witten map. Various attempts to construct the non-commutative gravity theories have also lead to the study of the possibility of having a complex metric tensor [55–58]. In [59], the deformed gravitational equations have been derived by generalising the Moyal product to the curved space-time.

The non-commutativity of the space-time allows to model the space-time uncertainity relations, which is argued to emerge when gravity and quantum mechanics are brought together. When we probe the structure of the space-time at short distance, using high energy probes, the gravitational effects become extremely strong, and this results in the formation of black hole over that region of space-time. The horizon of this black hole restricts us from probing the space-time below the horizon created. Thus the study of quantum gravity seems to introduce space-time uncertainty relations. In [60, 61], a Lorentz invariant non-commutative space-time has been obtained by incorporating Einstein's theory of classical gravity into Heisenberg's uncertainty relation.

The κ space-time, a Lie-algebraic type non-commutative space-time, has been shown to appear in the low energy limit of certain loop quantum gravity models. Loop quantum gravity theories also use the discretised space-time structure to describe the quantum geometry of the space-time. Symmetry algebra of the background space-time associated with the low-energy limit of the loop quantum gravity is shown to be κ-Poincare algebra [62]. The κ-deformed space-time coordinates satisfy

$$[\hat{x}_\mu, \hat{x}_\nu] = i\hat{x}_\lambda C_{\mu\nu}{}^{\lambda}, \tag{1.4}$$

where $C_{\mu\nu}{}^{\lambda} = a_\mu \delta_\nu{}^\lambda - a_\nu \delta_\mu{}^\lambda$ and a_μ has the dimension of length.

κ-Minkowski space-time has also been found to be the space-time corresponding to the doubly/deformed special relativity (DSR) theories [63, 64]. The special theory of relativity (STR) cannot accomodate a frame independent description of the minimal length scale, due to the Lorentz-Fitzgerald contraction. Therefore the STR is modified to incorporate the minimal length scale as an additional fundamental constant, apart from the velocity of light c and this modified relativity principle is known as doubly/deformed special relativity [65]. One of the major consequence of the DSR theory is the modified dispersion relation. This is known to result in the velocity of the photon acquiring energy dependent corrections. Different aspects of the DSR theory have been disscussed in [65–68].

The effective theory that emerges when the gravitational degrees of freedom are removed from the $2+1$ dimensional gravity coupled to matter fields has been shown to be the non-commutative field theory on the κ-Minkowski space-time [69–71]. Further, the phase space of a point particle in $2+1$ dimensional gravity has been shown to be equivalent to the phase space of DSR anti de Sitter algebra [72]

The symmetry algebra of the κ-deformed space-time has been constructed it has been referred in the literature as (deformed) κ-Poincare algebra [63, 73–77]. The symmetry algebra of the κ-Minkowski space-time has also been defined alternatively using the usual Poincare algebra. This has been achieved by deforming the explicit form of the Poincare generators in a specific manner. This symmetry algebra is known as the undeformed κ-Poincare algebra [78–80]. Further it has been shown that the co-product sector of this undeformed κ-Poincare algebra is deformed [78–80]. The twisted statistics has also been obtained in the κ-deformed space-time from the twisted flip-operator that commutes with the deformed coproduct of the symmetry group [81].

Several characteristic features of the κ-deformed space-time and its consequences have been studied over past years. Different field theory models on the κ-Minkowski space-time and their properties have been analysed meticulously in recent times. The κ-star product compatible with the κ-Poincare algebra has been used to study the interaction vertex of the non-commutative scalar ϕ^4 theory [82]. In [83], the quantisation of the κ-deformed scalar theory has been studied and deformed oscillator algebra has been obtained, using the twisted flip operator. Dirac equation has been constructed in the κ-Minkowski space-time and using this, it has been shown that the charge conjugation is not the symmetry of the κ-deformed Dirac equation [84]. Gauge theory on the κ-Minkowski space-time has been obtained in [85, 86], using the notion of the κ-deformed star product and Seiberg-Witten formalism. κ-deformed Maxwell's equations, which are invariant under the undeformed κ-Poincare algebra, has been derived using the Feynman's approach [87, 88].

Recently some attempts have been made to understand the κ deformation of the curved space-time and gravity. The κ-deformed geodesic equation and its implications have been studied by extending the Feynman's approach to κ-deformed space-time[89]. In [90], κ-deformed modifications to the metric tensor has been analysed using the non-commutative version of the co-tetrads and the κ differential calculus.

One of the major problems associated with physical models defined on non-commutative space-times is the violation of Lorentz invariance. The break down of the Lorentz symmetry in non-commutative field theory has shown to exhibit certain effects like vacuum birefringence [91]. Lose of Lorentz/Poincare symmetry also makes the particle interpretation of field quanta ambigous. Hence it is important to construct and study the Lorentz invariant non-commutative space-time and field theories in such space-times. Doplicher-Fredenhagen-Roberts (DFR) space-time [60, 61] is one such non-commutative space-time, whose coordinates obey

$$[\hat{x}_\mu, \hat{x}_\nu] = i\hat{\theta}_{\mu\nu}, \; [\hat{x}_\mu, \hat{\theta}_{\nu\rho}] = 0, \; [\hat{\theta}_{\mu\lambda}, \hat{\theta}_{\nu\rho}] = 0. \tag{1.5}$$

The symmetry algebra of the DFR space-time has been described using DFR Poincare algebra [92]. Various field theory models in DFR space-time have been studied in recent time [93–95]. Aspects of general relativity has also been analysed in DFR space-time [96, 97].

Lagrangians associated with non-commutative field theories are constructed by requiring to get well known commutative result in the appropriate limit. But this guiding principle alone cannot lead to a unique Lagrangian for the non-commutative field theories. Absence of unique Lagrangian render the usual quantisation schemes ineffective for quantising these non-commutative field theories. But on the other hand one can direclty obtain the equations of motion corresponding to non-commutative field theories in an alternate manner. This is obtained from the quadratic Casimir of the corresponding deformed Poincare algebra. These non-commutative field theories can be quantised using their equations of motion alone by using the Takahashi-Umezawa quantisation procedure [98–100]. This procedure provide the quantisation rules just from the equations of motion, without requiring the Lagrangian. In this thesis we use this method to quantise κ-deformed scalar field, κ-deformed Dirac field and Doplicher-Fredenhagen-Roberts-Amorim scalar field and also obtain their deformed oscillator algebras of the corresponding creation and annihilation operators.

The effects of the non-commutativity are expected to be more strong in an extremely strong gravitational background. Astrophysical objects such as superdense star [101] is a suitable candidate for studying the non-commutative effects. In this thesis we study the superdense star in non-commutative space-time by generalising the anisotropic core-envelope model to the κ-deformed space-time.

Apart from the dynamical properties, one also need to understand the kinematical aspects of the non-commutative space-time. Different kinematical properties of non-commutative space-time can be analysed and in this thesis we study the notion of maximal proper acceleration [102, 103] in non-commutative space-time. We derive here the non-commutative corrections to the maximal acceleration, in κ-deformed and DFR space-times, and further study the implications.

In the next subsection, we will provide a summary of essential definitions and tools required for the studies taken up in this thesis.

1.2 κ-deformed space-time

In this subsection, we summarise essential results concerning the realisation of the κ-deformed space-time and its symmetry algebra [78]. The discussions in this section will set our notations for the later chapters.

κ-deformed space-time is a Lie-algebraic type non-commutative space-time, whose space-time coordinates satisfy Eq.(1.4). With the choice $a_\mu = (a, \vec{0})$, Eq.(1.4) becomes

$$[\hat{x}_0, \hat{x}_i] = ia\hat{x}_i, \ \ [\hat{x}_i, \hat{x}_j] = 0. \tag{1.6}$$

Here we observe that the spatial coordinates of the κ space-time commute among themselves, but space coordinates do not commute with time coordinate. Thus one finds that the κ space-time preserves the spatial isotropy. Note that a in Eq.(1.6), has the dimension of length and $\frac{1}{a} = \kappa$ - the deformation parameter used in literature, giving the name κ space-time.

Field theory models on the κ space-time were constructed using the star product formalism, where the usual notion of the pointwise product, between the coordinates (and their functions), is replaced with the star product, which is invariant under the κ-Poincare algebra [76, 77]. Alternatively, one can also construct and study the field theory models using the realisation method, where the non-commutative cvariable is represented in terms of the functions of commutative coordinates and their derivatives [78, 79]. It has been shown that the realisation approach is equivalent to the star product formalism in the κ space-time [80]. In this thesis, we will be using the realisation method [78–80] to study the various aspects of physics on κ-deformed space-time.

The non-commutative coordinate $\hat{x}_\mu$ is written in terms of the commutative coordinate x_μ and its derivatives ∂_μ as [78]

$$\begin{aligned}\hat{x}_0 =& x_0\psi(A) + iax_j\partial_j\gamma(A)\\ \hat{x}_i =& x_i\varphi(A),\end{aligned} \tag{1.7}$$

where $A = ia\partial_0$ and

$$\psi(0) = 1, \;\; \varphi(0) = 1. \tag{1.8}$$

Substituting Eq.(1.7) in Eq.(1.6), one gets

$$\frac{\varphi'(A)}{\varphi(A)}\psi(A) = \gamma(A) - 1. \tag{1.9}$$

Two possible realisations of $\psi(A)$ are $\psi(A) = 1$ and $\psi(A) = 1 + 2A$ [78]. Now onwards we choose $\psi(A) = 1$. Thus Eq.(1.7) and Eq.(1.9) becomes

$$\begin{aligned}\hat{x}_0 =& x_0 + iax_j\partial_j\gamma(A)\\ \hat{x}_i =& x_i\varphi(A),\end{aligned} \tag{1.10}$$

and

$$\frac{\varphi'(A)}{\varphi(A)} = \gamma(A) - 1. \tag{1.11}$$

Some of the allowed choices of φ are $e^{-A}, e^{-\frac{A}{2}}, 1, \frac{A}{e^A-1}$, etc., [78]. In [78–80], it was shown that different choices of φ corresponds to different realisations.

One can also realise $\hat{x}_\mu$ in an alternate way as

$$\hat{x}_\mu = x_\nu\varphi^\nu_\mu. \tag{1.12}$$

We choose a specific realisation for φ^ν_μ (which keeps only the linear terms in the deformation parameter a) as [88, 89],

$$\varphi^\nu_\mu = \delta^\nu_\mu - ia\alpha\delta^\nu_\mu\partial_0 - ia\beta\delta^\nu_0\partial_\mu - ia\gamma\delta^0_\mu\partial^\nu, \tag{1.13}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ are dimensionless parameter. Substituting Eq.(1.13) and Eq.(1.12) in Eq.(1.6), we get $\gamma = \alpha + 1$.

We will study both these realisations in latter chapters.

In general, the symmetry algebra of the κ-Minkowski space-time is described using the κ-Poincare algebra. As a result, the commutation relations of the Poincare algebra get deformed due to the a dependent correction terms. However, one can also realise the symmetry algebra using the usual Poincare algebra, but by deforming the explicit form of the generators in a particular way. This algebra is known as the undeformed κ-Poincare algebra [78].

The Lorentz generator of the undeformed κ-Poincare algebra satisfies [78]

$$[M_{\mu\nu}, M_{\lambda\rho}] = M_{\mu\rho}\eta_{\nu\lambda} - M_{\nu\rho}\eta_{\mu\lambda} - M_{\mu\lambda}\eta_{\nu\rho} + M_{\nu\lambda}\eta_{\mu\rho}. \tag{1.14}$$

By demanding the commutation relation between the Lorentz generator and the κ-deformed space-time coordinate to be linear in $M_{\mu\nu}$ and $\hat{x}_\mu$, i.e,

$$[M_{\mu\nu}, \hat{x}_\lambda] = \hat{x}_\mu\eta_{\nu\lambda} - \hat{x}_\nu\eta_{\mu\lambda} + ia(M_{0\mu}\eta_{\nu\lambda} - M_{0\nu}\eta_{\mu\lambda}), \tag{1.15}$$

and using the Jacobi's identities, we get the explicit form of the Lorentz generators of the undeformed κ-Poincare algebra as

$$\begin{aligned} M_{ij} =& x_i\partial_j - x_j\partial_i \\ M_{i0} =& x_i\partial_0\varphi\frac{e^{2A}-1}{2A} - x_0\partial_i\frac{1}{\varphi} + iax_i\partial_k^2\frac{1}{2\varphi} - iax_k\partial_k\partial_i\frac{\gamma}{\varphi}. \end{aligned} \tag{1.16}$$

But the commutative derivative, i.e, ∂_μ, do not transform as a 4-vector under the undeformed κ-Poincare algebra. To rectify this one uses Dirac deivative, D_μ [78] which transform as a 4-vector under this algebra. Thus we have

$$\begin{aligned} [M_{\mu\nu}, D_\lambda] =& D_\mu\eta_{\nu\lambda} - D_\nu\eta_{\mu\lambda} \\ [D_\mu, D_\nu] =& 0, \end{aligned} \tag{1.17}$$

where the components of the Dirac derivative are defined as

$$\begin{aligned} D_0 =& \partial_0\frac{\sinh A}{A} + ia\partial_k^2\frac{e^{-A}}{2\varphi^2} \\ D_i =& \partial_i\frac{e^{-A}}{\varphi} \end{aligned} \tag{1.18}$$

satisfying

$$[D_\mu, \hat{x}_\nu] = \eta_{\mu\nu}(iaD_0 + \sqrt{1 + a^2D_\alpha D^\alpha}) + ia\eta_{\mu 0}D_\nu. \tag{1.19}$$

The Casimir corresponding to the undeformed κ-Poincare algebra is defined using the Dirac derivatives as,

$$D_\mu D^\mu = \Box\left(1 + \frac{a^2}{4}\Box\right) \tag{1.20}$$

where $\Box$ represents the κ-deformed Laplacian,

$$\Box = \partial_k^2\frac{e^{-A}}{2\varphi^2} - \partial_0^2\frac{2(1-\cosh A)}{A^2}, \tag{1.21}$$

satisfying

$$\begin{aligned}[M_{\mu\nu},\Box] =&0, \\ [\Box,\hat{x}_\mu] =&2D_\mu.\end{aligned} \tag{1.22}$$

Note that the quadratic Casimir defined in Eq.(1.20) will be used for constructing the Klein-Gordon equation in the κ space-time, discussed in the later chapters. Further we use the Casimir, given in Eq.(1.20), to obtain the κ-deformed dispersion relation as

$$\frac{4}{a^2}\sinh^2\frac{A}{2}-p_i^2\frac{e^{-A}}{\varphi}+\frac{a^2}{4}\Big(\frac{4}{a^2}\sinh^2\frac{A}{2}-p_i^2\frac{e^{-A}}{\varphi}\Big)^2=m^2. \tag{1.23}$$

We find that in the $\lim a\to 0$, the above expression reduces to the usual dispersion relation for a massive particle in the flat space-time.

We have seen that the non-commutativity of space-time necessitates the modification of the generators of the undeformed κ-Poincare algebra. Action of the generator of the symmetry algebra is modified by the non-commutativity of the space-time. We obtain the deformed Leibnitz rule for the generators of the undeformed κ-Poincare algebra by evaluating $[M_{\mu\nu},f(\hat{x})]$ using Eq.(1.15) [78]. Thus the κ-deformed Leibnitz rule for $M_{\mu\nu}$ is given as

$$\begin{aligned}M_{i0}(\hat{f}\cdot\hat{g}) =&(M_{i0}\hat{f})\cdot\hat{g}+(e^A\hat{f})\cdot(M_{i0}\hat{g})+ia\Big(\frac{\partial_j}{\varphi}\hat{f}\Big)\cdot(M_{ij}\hat{g}), \\ M_{ij}(\hat{f}\cdot\hat{g}) =&(M_{ij}\hat{f})\cdot g+\hat{f}\cdot(M_{ij}\hat{g}),\end{aligned} \tag{1.24}$$

where f and g are arbitrary functions of $\hat{x}_\mu$.

Similarly the κ-deformed Leibnitz rule for the Dirac derivative is obtained (by evaluating $[D_\mu,f(\hat{x})]$ using Eq.(1.19)) to be

$$\begin{aligned}D_0(\hat{f}\cdot\hat{g}) =&(D_0\hat{f})\cdot(e^{-A}\hat{g})+\Big(\frac{iaD_0+\sqrt{1+a^2D_\alpha D^\alpha}}{1+a^2D_\alpha D^\alpha}\hat{f}\Big)\cdot(D_0\hat{g})+ \\ &\Big(iaD_i\frac{iaD_0+\sqrt{1+a^2D_\alpha D^\alpha}}{1+a^2D_\alpha D^\alpha}\hat{f}\Big)\cdot(D_i\hat{g}), \\ D_i(\hat{f}\cdot\hat{g}) =&(D_i\hat{f})\cdot(e^{-A}\hat{g})+\hat{f}\cdot(D_i\hat{g}).\end{aligned} \tag{1.25}$$

From the modified Leibnitz rule, the coproducts for the generators of the undeformed κ-Poincare algebra is written as

$$\begin{aligned}\Delta M_{i0} =&M_{i0}\otimes 1+e^A\otimes M_{i0}+iaD_je^A\otimes M_{ij}, \\ \Delta M_{ij} =&M_{ij}\otimes 1+1\otimes M_{ij},\end{aligned} \tag{1.26}$$

$$
\begin{aligned}
\Delta D_0 =& D_0 \otimes e^{-A} + \frac{iaD_0 + \sqrt{1+a^2 D_\alpha D^\alpha}}{1+a^2 D_\alpha D^\alpha} \otimes D_0 + iaD_i \frac{iaD_0 + \sqrt{1+a^2 D_\alpha D^\alpha}}{1+a^2 D_\alpha D^\alpha} \otimes D_i, \\
\Delta D_i =& D_i \otimes e^{-A} + 1 \otimes D_i.
\end{aligned}
\tag{1.27}
$$

From the above, we observe that the coproduct sector of the undeformed κ-Poincare algebra is deformed. The deformed coproduct can also be obtained from the κ-deformed twist element [104–106], instead of obtaining from the modified Lebinitz rule.

1.3 Doplicher-Fredenhagen-Roberts space-time

In this subsection, we provide a brief discussion of the construction of the Doplicher-Fredenhagen-Roberts (DFR) space-time [92], which is obtained by extending the Moyal space-time. We also discuss the symmetry algebra of DFR space-time and the Casimir operator associated with it.

The NC space-time coordinates satisfying the Moyal space-time algebra (see Eq.(1.2)) violates the Lorentz symmetry due to the presence of the constant $\theta_{\mu\nu}$ tensor. It has been shown in [60, 61] that one can obtain a Lorentz invariant NC space-time by assigning a Lorentz transformation for the NC parameter $\theta_{\mu\nu}$. This NC parameter has further been promoted to a coordinate operator $\hat{\theta}_{\mu\nu}$ [107]. The resulting NC space-time is known as the DFR space-time, whose space-time coordinate operators are $\hat{x}_\mu$ and $\hat{\theta}_{\mu\nu}$ respectively.

The DFR space-time algebra is given by

$$
[\hat{x}_\mu, \hat{x}_\nu] = i\hat{\theta}_{\mu\nu}, \; [\hat{x}_\mu, \hat{\theta}_{\nu\rho}] = 0, \; [\hat{\theta}_{\mu\lambda}, \hat{\theta}_{\nu\rho}] = 0. \tag{1.28}
$$

The DFR space-time algebra has further been extended by incorporating the canonical conjugate momenta operators $\hat{k}_{\mu\nu}$ corresponding to $\hat{\theta}_{\mu\nu}$ (apart from the conjugate momenta operator $\hat{p}_\mu$ associated with $\hat{x}_\mu$). This is called in the literatures as the extended DFR space-time [92] or DFRA space-time [108]. The DFR space-time coordinate operators and their conjugate momenta satisfy the following commutation relations,

$$
\begin{aligned}
[\hat{x}_\mu, \hat{p}_\nu] =& i\eta_{\mu\nu}, \; [\hat{x}_\mu, \hat{k}_{\nu\lambda}] = -\frac{i}{2}(\eta_{\mu\nu}\eta_{\rho\lambda} - \eta_{\mu\lambda}\eta_{\nu\rho})\hat{p}^\rho, \\
[\hat{p}_\mu, \hat{p}_\nu] =& 0, \; [\hat{\theta}_{\mu\nu}, \hat{k}_{\rho\lambda}] = i(\eta_{\mu\rho}\eta_{\nu\lambda} - \eta_{\mu\lambda}\eta_{\nu\rho}) \\
[\hat{p}_\mu, \hat{\theta}_{\nu\lambda}] =& 0, \; [\hat{p}_\mu, \hat{k}_{\nu\lambda}] = 0, \; [\hat{k}_{\mu\nu}, \hat{k}_{\rho\lambda}] = 0.
\end{aligned}
\tag{1.29}
$$

Eq.(1.28) and Eq.(1.29) forms the DFRA space-time algebra [108]. We see that the above algebra is closed (see [92] for the constistency conditions of the above algebra, using Jacobi identities).

The Poincare algebra associated with the DFR space-time is given as [92]

$$\begin{aligned}
[M_{\mu\nu}, M_{\lambda\rho}] =& i(\eta_{\mu\rho}M_{\nu\lambda} - \eta_{\nu\rho}M_{\lambda\mu} - \eta_{\mu\lambda}M_{\rho\nu} + \eta_{\nu\lambda}M_{\rho\mu}),\\
[M_{\mu\nu}, \hat{p}_\lambda] =& i(\eta_{\mu\lambda}\hat{p}_\nu - \eta_{\nu\lambda}\hat{p}_\mu),\\
[M_{\mu\nu}, \hat{k}_{\alpha\beta}] =& i(\eta_{\mu\beta}\hat{k}_{\alpha\nu} - \eta_{\mu\alpha}\hat{k}_{\nu\beta} + \eta_{\nu\alpha}\hat{k}_{\beta\mu} - \eta_{\nu\beta}\hat{k}_{\alpha\mu}),\\
[\hat{p}_\mu, \hat{p}_\nu] =& 0,\ [\hat{k}_{\mu\nu}, \hat{k}_{\rho\lambda}] = 0,\ [\hat{p}_\mu, \hat{k}_{\nu\lambda}] = 0.
\end{aligned} \tag{1.30}$$

The explicit form of the Lorentz generator associated with the DFRA-Poincare algebra is defined as [92]

$$M_{\mu\nu} = \hat{x}_\mu\hat{p}_\nu - \hat{x}_\nu\hat{p}_\mu + \frac{1}{2}\hat{\theta}_{\mu\alpha}\hat{p}^\alpha\hat{p}_\nu - \frac{1}{2}\hat{\theta}_{\nu\alpha}\hat{p}^\alpha\hat{p}_\mu - \hat{\theta}_{\mu\lambda}\hat{k}_\nu{}^\lambda + \hat{\theta}_{\nu\lambda}\hat{k}_\mu{}^\lambda. \tag{1.31}$$

The Casimir operator corresponding to the DFRA-Poincare algebra is given as [92]

$$\hat{P}^2 = \hat{p}_\mu\hat{p}^\mu + \frac{\lambda^2}{2}\hat{k}_{\mu\nu}\hat{k}^{\mu\nu}, \tag{1.32}$$

The dispersion relation in the DFR space-time can be written using the Casimir operator given in Eq.(1.32), as

$$\hat{p}_\mu\hat{p}^\mu + \frac{\lambda^2}{2}\hat{k}_{\mu\nu}\hat{k}^{\mu\nu} = m^2, \tag{1.33}$$

where λ is the non-commutative parameter having the dimension of length. We see that in the $\lim \lambda \to 0$, the DFR dispersion reduces to the usual dispersion relation in the Minkowski space-time.

1.4 Organisation of the thesis

This thesis focuses on the study of various aspects of physical models on non-commutative space-times such as κ-deformed space-time and DFR space-time. This thesis is divided into seven chapters.

In chapter 2, we study the quantisation of the Lorentz non-invariant non-commutative fields by using the Takahashi-Umezawa quantisation scheme [98–100]. This scheme do not require the explicit form of the Lagrangian. Instead, it uses the equations of motion alone for the quantisation. This method is particularly suited for quantisation of non-commutative field theories as Lagrangian of these theories are not unique while equations of motion are unique.

Starting from the κ-deformed Klein-Gordon equation, valid up to first order in a, we derive the deformed unequal time commutation relation between deformed field and its adjoint, using the undeformed oscillator algebra. By demanding an undeformed unequal time commutation relation between the deformed field and its adjoint, we obtain a deformed oscillator algebra (valid up to first order in a). Using the deformed equations of motion, we derive the energy-momentum tensor and Lorentz generator (corresponding to the undeformed κ-Poincare algebra) of the κ-deformed scalar field. The number operator, corresponding to the deformed scalar field, is also derived from the equations of motion. Implications of this deformed oscillator algebra to Unruh effect is analysed using the method of Bogoliubov coefficients [109].

In chapter 3, we study the quantisation of κ-deformed Dirac field. Using κ-deformed Dirac equation (valid up to first order in a), we derive the deformed unequal-time anti-commutation relation between deformed field and its adjoint. In this derivation we assume that the fermionic oscillator satisfy the usual algebra. Next, by imposing the unequal time anti-commutation relation between the κ-deformed Dirac field and its adjoint to be undeformed, we show that the fermionic creation and annihilation operators obey deformed oscillator algebra. The energy-momentum tensor and Lorentz generator for the κ-Dirac field are derived from the deformed equations of motion. We construct the conserved currents corresponding to parity and time-reversal symmetries of κ-deformed Dirac equation. Further, we show that it is impossible to construct a conserved current associated with charge conjugation symmetry, showing that the Dirac particle and its anti-particle satisfy different equations of motion in κ-deformed space-time[110].

In chapter 4, we derive the equal time commutation relation between the DFRA scalar field and its conjugate, where we assume that the corresponding creation and annihilation operators satisfy the usual oscillator algebra. We then show that imposing the condition that the commutation relation between the field and its conjugate is the same as that in the commutative space-time, leads to the deformation of usual oscillator algebra. Unlike the κ-deformed fields, here both these deformed commutation relations derived are valid to all orders in the non-commutative parameter. We also derive the conserved currents, corresponding to translational and Lorentz symmetry, for the DFRA scalar field. Further, we analyse the effects of non-commutativity on the Unruh effect by analysing a monopole detector coupled to the DFRA scalar field, showing that the Unruh temperature is not modified, but the thermal radiation seen by the accelerated observer gets correction due to the non-commutativity of space-time [111].

In chapter 5, we study the effects of the non-commutativity in an astrophysical object-namely superdense star by generalising the anisotropic core-envelope model of a superdense star [101] to κ-deformed space-time. The equations of state, connecting the pressure and density, are obtained by solving the κ-deformed Einstein's field equation, valid up to first order in a. From the κ-deformed law of density variation, we show that the non-commutativity enhances the density of the superdense star. Using the

κ-deformed law of density variation and the positivity condition on the tangential pressure, we obtain different bounds on the κ-deformation parameter. We also show that the κ-defomed strong energy condition takes the same form as that in the commutative space-time. By showing that the velocity of sound inside the star is less than the velocity of light, and from the positivity conditions of the pressures and density, we show that the super dense star model obtained in the κ-deformed space-time is stable [112].

In chapter 6, we analyse the maximal acceleration [102, 103] in the non-commutative space-time. We study the maximal acceleration in κ-deformed space-time and analyse its implications using two different approaches. In the first method we derive the κ-deformed corrections to the maximal acceleration, valid up to first order in a, using the 8-dimensional line element of the κ-deformed phase-space. Further we derive the first order κ-deformed corrections to the maximal acceleration from the κ-deformed uncertainty principle. By combining the expressions for the Unruh temperature and the deformed maximal acceleration, we obtain the maximum attainable temperature in the κ-deformed space-time. We then obtain a bound on the deformation parameter by comparing the expression of the maximum attainable temperature with the experimental data on the Unruh radiation. In the second method we show the emergence of maximal acceleration (valid up to first order in a) from the causally connected 4-dimensional line element in κ-Minkowski space-time. We also obtain the maximum attainable temperature corresponding to this deformed maximal acceleration. We then derive the κ-deformed geodesic equation and obtain its Newtonian limit. We show that κ-deformed Newton's force equation contains an equivalence principle violating term. By comparing this term with the experimental result on the violation of equivalence principle, we obtain a bound on the dimensionless non-commutative parameters present in the maximal acceleration expression, obtained using the second approach. [114].

In chapter 7, we summarise the results discussed in this thesis. We also present our concluding remarks and discuss possible future direction of research work in this area.

In this thesis, we work with $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$.

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Chapter 2

Quantisation of κ-deformed Klein-Gordon field

2.1 Introduction

Several studies have been reported in recent times, investigating field theory models defined in the κ space-time. Most of these studies have used the quadratic Casimir of the κ-deformed algebra [1–5] to derive the equation of motion, invariant under the symmetry algebra corresponding to the κ-deformed space-time. This deformed equation of motion contain higher-order derivative terms indicating the non-local nature of non-commutative field theory models. Therefore the Lagrangian associated with these deformed equations of motion also contain higher-order derivative terms, making it difficult to quantise the κ-deformed field theories using canonical scheme.

The complete information regarding the exact form of Lagrangian is indispensable for the canonical quantisation of field theories. But there exists another quantisation scheme which does not use the explicit form of the Lagrangian for quantisation. This method allows to derive the quantisation rules by starting with the equations of motion [6–8]. In this approach, the equations of motion corresponding to the free field theories are first transformed into Klein-Gordon equation with the help of an operator known as Klein-Gordon divisor [6–8]. This Klein-Gordon divisor is further utilised to define an unequal-time commutation relation between the field and its adjoint, which is compatible with Heisenberg's equations of motion. In this scheme, the usual commutation relation between creation and annihilation operators, present in the Fourier decomposition of the field operator is assumed. This method has been used to construct the covariant commutation relations for the fields with arbitrary spin [8, 9]. It has also been used to quantise massive spin-1 and massive spin-3 fields [10, 11]. Using this quantisation method one can construct the conserved currents associated with the symmetry transformations, from the equations of motion alone, without referring its Lagrangian [8, 12, 13]. The conserved currents associated with the discrete symmetries have also been constructed using this procedure [12].

The guiding principle used for constructing Lagrangians of say real/complex scalar theories, Dirac theories in non-commutative space-times is that they reduce to the exact commutative one in the limit where the deformation parameter vanishes. But this requirement alone is not sufficient to obtain unique Lagrangians. Since the equations of motion are constructed from the quadratic Casimir of the symmetry algebra, they are unique. The existence of unique equation of motion motivate us to use the approach discussed above for the quantisation of field theories on κ-deformed space-time. Equations of motion satisfied by different κ-deformed fields such as real and complex scalar fields, Dirac field, have been constructed in the recent times, without using the explicit form of Lagrangians. The κ-deformed Maxwell's equations and κ-deformed geodesic equations have been obtained in [14, 15] and [16] respectively, by generalising the Feynman's procedure to κ space-time. In [5] and [17], the κ-deformed scalar field equations and the deformed Dirac field equations are constructed from the deformed quadratic Casimir and the Dirac derivative, respectively. Duffin-Kemmer-Petiau (DKP) equation has been constructed in the κ-Minkowski space-time using Dirac derivatives [18].

In this chapter, we quantise the field obeying the κ-deformed Klein-Gordon equation, by generalising the quantisation procedure of [7, 8] to the κ-deformed space-time. Here we begin with κ-deformed scalar field equation, constructed from the quadratic Casimir of the undeformed κ-Poincare algebra [19–21]. We then obtain a deformed commutation relation (valid up to first order in a) between the deformed scalar field and its adjoint, at unequal times, by assuming that the creation and annihilation operators satisfy the usual oscillator algebra. We then start with the assumption that the commutator between the deformed field and its adjoint, at unequal times, to be undeformed and arrive at a deformed oscillator algebra (valid up to first order in a). These quantisation rules are different from that obtained in [5], where the twisted flip operator [22] has been used to derive the deformed commutation relations as well as the deformed oscillator algebra, with a modified product rule. We also analyse the translation and Lorentz symmetry in κ space-time and derive the energy-momentum tensor and Lorentz generator, corresponding to deformed scalar field. We show that this energy-momentum tensor for the κ-scalar field is no longer symmetric due to the κ deformation.

The κ-deformed corrections to the Unruh effect has been discussed extensively in the recent times. In [23] and [24], these corrections have been obtained by calculating the response of massless κ-deformed Klein-Gordon field coupled to a uniformly accelerating detector. Such an analysis has also been employed for studying fermionic fields, by determining the response function of a massless κ-Dirac field coupled with a monopole detector, moving along a uniformly accelerating trajectory [25]. Here we study the modifications of the Unruh effect due to κ-deformation of space-time. Using the Bogoliubov coefficients we first connect the frequency modes of the deformed massless scalar field in Rindler basis with that in the Minkowski basis and then calculate the vacuum expectation value of the number operator corresponding to the Rindler particles in the Minkowski vacuum. We show that, up to first order in a, the Unruh temperature associated with this thermal bath remains unaffected under the κ-deformations.

This chapter is organised in the following manner. In sec.2.2, we discuss the Takahashi-Umezawa quantisation procedure reported in [7, 8]. The subsec.2.2.3 illustrates the details regarding the construction of conserved currents from the equations of motion. In sec.2.3, we generalise this quantisation procedure, discussed in sec.2.2, to κ-deformed case and quantise the κ-deformed scalar field. We apply this scheme to the κ-deformed scalar field equation, valid up to first order in the deformation parameter a. In subsec.2.3.1, we derive the conserved currents corresponding to the symmetries associated with the κ-deformed Klein-Gordon field. In sec.2.4, we derive the a dependent corrections to Unruh effect. Finally in sec.2.5, we give the concluding remarks.

2.2 Takahashi-Umezawa quantisation procedure

In this section, we summarise the quantisation procedure discussed in [7, 8]. This method requires only the equations of motion for quantising a field. By using this procedure one can obtain the commutation relations between the field and its adjoint from its equations of motion. Further, this method also helps in constructing the conserved currents corresponding to continuous as well as discrete symmetry transformations from the respective equations of motion [8, 12].

We begin with the equations of motion satisfied by the field operator $\phi(x)$ and its adjoint $\bar{\phi}(x)$ given by

$$\Lambda(\partial)\phi(x) = 0 \tag{2.1}$$

and

$$\bar{\phi}(x)\Lambda(-\overleftarrow{\partial}) = 0, \tag{2.2}$$

respectively. In the above equation, $\Lambda(\partial)$ is a polynomial in ∂_μ and in general this is defined as [7, 8]

$$\Lambda(\partial) = \sum_{l=0}^{N} \Lambda_{\mu_1\mu_2....\mu_l}\partial^{\mu_1}\partial^{\mu_2}....\partial^{\mu_l}$$
$$= \Lambda_{(0)} + \Lambda_\mu\partial^\mu + \Lambda_{\mu\nu}\partial^\mu\partial^\nu + \Lambda_{\mu\nu\rho}\partial^\mu\partial^\nu\partial^\rho + ... + \Lambda_{\mu_1\mu_2\mu_3....\mu_N}\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}....\partial^{\mu_N}. \tag{2.3}$$

- Note that for a Klein-Gordon equation, the $\Lambda(\partial)$ operator is defined as

$$\Lambda(\partial) = \eta_{\mu\nu}\partial^\mu\partial^\nu - m^2. \tag{2.4}$$

  Comparing this Eq.(2.3), we get the components of $\Lambda(\partial)$ operator for Klein-Gordon field as

$$\Lambda_{\mu\nu} = \eta_{\mu\nu}, \quad \Lambda_\mu = 0, \quad \Lambda_{(0)} = -m^2, \tag{2.5}$$

  and all other terms vanish, i.e., $\Lambda_{\mu\nu\lambda} = 0, ..\Lambda_{\mu_1\mu_2....\mu_N} = 0$.

- For Dirac equation, the $\Lambda(\partial)$ operator is defined as

$$\Lambda(\partial) = i\gamma_\mu\partial^\mu + m. \tag{2.6}$$

  We obtain the non-vanishing components of $\Lambda(\partial)$ operator for Dirac field, by comparing this with Eq.(2.3), as

$$\Lambda_\mu = i\gamma_\mu, \quad \Lambda_{(0)} = m. \tag{2.7}$$

According to the Takahashi-Umezawa quantisation procedure, every free field equations of motion (represented as in Eq.(2.1)) can be reduced to the Klein-Gordon equations of motion, by acting the $\Lambda(\partial)$ operator with another operator called Klein-Gordan divisor, $d(\partial)$, such that

$$d(\partial)\Lambda(\partial) = \Box - m^2 = \Lambda(\partial)d(\partial). \tag{2.8}$$

Here it is to be noted that this Klein-Gordon divisor should commute with the Lambda operator, i.e, $[\Lambda(\partial), d(\partial)] = 0$ and Klein-Gordon divisor $d(\partial)$ should have non-zero eigen values [7, 8], so that $d(\partial)$ can be inverted. Using this $d(\partial)$, one can convert any free field equations of motion into Klein-Gordon equation. Therefore it is important to obtain the explicit form of Klein-Gordon divisor. It is easily found that for

- Klein-Gordon field,

$$d(\partial) = \mathbb{I}, \tag{2.9}$$

- Dirac field,

$$d(\partial) = i\gamma_\mu \partial^\mu - m. \tag{2.10}$$

Klein-Gordon divisor has been constructed for different field theories like Duffin-Kemmer field, Rarita-Schwinger field, etc. (see [8] for more details).

The field operator satisfying the equations of motion, given in Eq.(2.1), is decomposed into Fourier modes, using the creation and annihilation operators as

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \Big(u_p(x) a(p) + u_p^*(x) a^\dagger(p) \Big), \tag{2.11}$$

where $u_p(x)$ satisfies the equations of motion given in Eq.(2.1), i.e., $\Lambda(\partial)u_p(x) = 0$. The creation and annihilation operators present in Eq.(2.11) are assumed to satisfy the following commutation relations,

$$[a(p), a(p')] = [a^\dagger(p), a^\dagger(p')] = 0, \quad [a(p), a^\dagger(p')] = \delta^3(p - p'). \tag{2.12}$$

According to this quantisation procedure, we write down the (unequal-time) commutation relation between the field operator and its adjoint, using the Klein-Gordon divisor $d(\partial)$ as

$$[\phi(x), \bar{\phi}(x')] = i d(\partial) \Delta(x - x'), \tag{2.13}$$

where $\Delta(x - x')$ is defined as

$$\Delta(x - x') = \int \frac{d^3p}{(2\pi)^3 2E_p} \big(e^{-ip(x-x')} - e^{ip(x-x')} \big). \tag{2.14}$$

Note that if $\phi(x)$ is a fermionic field then the above commutation relations given in Eq.(2.12) and Eq.(2.13) are replaced with the corresponding anti-commutation relations as per the spin-statistics theorem.

From the above relation given in Eq.(2.13), one can get an equal-time commutation relation between the field, $\phi(x)$ and its time-derivative, $\partial_t \phi(x)$. This is obtained by acting Eq.(2.13) with $\partial_{t'}$, and then setting both the times to be equal, i.e, $t = t'$.

The consistency of this quantisation procedure can be verified using the compatibility of the field operator $\phi(x)$ with the Heisenberg's equation of motion, i.e.,

$$i\partial_t \phi(x) = [\phi(x), H] \tag{2.15}$$

where H is the Hamiltonian [7, 8].

2.2.1 Conserved currents

One of the major feature of this quantisation method is that it provides an alternate way for constructing the conserved currents (associated with the symmetry transformations) just from their equations of motion. Note that unlike Noether's method, here one does not requires the explicit form of the Lagrangian for constructing the conserved currents [8, 12, 13]. This method provides a unique way of constructing the conserved currents for discrete symmetry [12].

The conserved currents are constructed using an operator $\Gamma_\mu(\partial, -\overleftarrow{\partial})$, which is defined as

$$\begin{aligned}\Gamma_\mu(\partial, -\overleftarrow{\partial}) &= \sum_{l=0}^{N-1}\sum_{i=0}^{l}\Lambda_{\mu\mu_1.....\mu_l}\partial_{\mu_1}.....\partial_{\mu_i}(-\overleftarrow{\partial}_{\mu_{i+1}})......(-\overleftarrow{\partial}_{\mu_l}) \\ &= \Lambda_\mu + \Lambda_{\mu\nu}(\partial^\nu - \overleftarrow{\partial}^\nu) + \Lambda_{\mu\nu\rho}(\partial^\nu\partial^\rho - \partial^\nu\overleftarrow{\partial}^\rho + \overleftarrow{\partial}^\nu\overleftarrow{\partial}^\rho) + ...\end{aligned} \tag{2.16}$$

where $\Lambda_\mu,\ \Lambda_{\mu\nu}$,...etc.. appear as the coefficients of $\partial^\mu,\ \partial^\mu\partial^\nu$...etc in the explicit form of $\Lambda(\partial)$ operator defined in Eq.(2.3). We obtain these $\Lambda_\mu,\ \Lambda_{\mu\nu}$,...etc.. by comparing the equations of motion with the definition of $\Lambda(\partial)$ operator given in Eq.(2.3). Here it is to be noted that $\Lambda_{\mu\nu},\ \Lambda_{\mu\nu\lambda}$,...etc... should be symmetric in indices in order to calculate $\Gamma_\mu(\partial, -\overleftarrow{\partial})$ operator.

- Substituting Eq.(2.5) in Eq.(2.16), we obtain the $\Gamma_\mu(\partial, -\overleftarrow{\partial})$ operator for Klein-Gordon field as
$$\Gamma_\mu(\partial, -\overleftarrow{\partial}) = \partial_\mu - \overleftarrow{\partial}_\mu, \tag{2.17}$$
- Similarly by using Eq.(2.7) in Eq.(2.16), we get the $\Gamma_\mu(\partial, -\overleftarrow{\partial})$ operator for Dirac field as
$$\Gamma_\mu(\partial, -\overleftarrow{\partial}) = i\gamma_\mu. \tag{2.18}$$

Using Eq.(2.3) and Eq.(2.16), it has been shown that $\Gamma_\mu(\partial, -\overleftarrow{\partial})$ satisfies the identity [7, 8]

$$(\partial^\mu + \overleftarrow{\partial}^\mu)\Gamma_\mu(\partial, -\overleftarrow{\partial}) = \Lambda(\partial) - \Lambda(-\overleftarrow{\partial}). \tag{2.19}$$

In this formalism, the conserved current associated with a symmetry transformation is defined using this $\Gamma_\mu(\partial, -\overleftarrow{\partial})$ operator, as [7, 8]

$$J_\mu = \bar{\phi}(x)\Gamma_\mu(\partial, -\overleftarrow{\partial})\delta\phi(x), \tag{2.20}$$

where $\delta\phi(x)$ represents the variation of the field under the symmetry transformation. By using the identity Eq.(2.19) and the equations of motion, i.e., Eq.(2.1) and Eq.(2.2), it can be shown that the J_μ defined in Eq.(2.20) is a conserved quantity, i.e., $\partial_\mu J^\mu = 0$.

Note that this conserved current is used to fix the normalisation of the field (see [7, 8] for details).

Under the space-time translation, the space-time coordinate varies as $\delta x_\mu = \theta_\mu$ (where θ_μ is the constant parameter) and the scalar field transforms as $\delta\phi(x) = \theta^\mu \partial_\mu \phi(x)$. The energy-momentum tensor associated with this translational symmetry is written using Eq.(2.20) as

$$T_{\mu\nu} = \bar{\phi}(x)\Gamma_\mu(\partial, -\overleftarrow{\partial})\partial_\nu\phi(x) \tag{2.21}$$

and the generator corresponding to this translational symmetry is written as

$$P_\mu = \int d^3x \; T_{0\mu} = \int d^3x \; \bar{\phi}(x)\Gamma_0(\partial, -\overleftarrow{\partial})\partial_\mu\phi(x) \tag{2.22}$$

Similarly under the Lorentz transformation, the space-time coordinates transform as $\delta x_\mu = \omega_\mu{}^\nu x_\nu$ and the scalar field transforms as $\delta\phi(x) = \frac{1}{2}\big(x_\mu\partial_\nu\phi(x) - x_\nu\partial_\mu\phi(x)\big)\omega^{\mu\nu}$. The conserved quantity corresponding to this Lorentz transformation is given as

$$M_{\mu\nu} = \int d^3x \; \mathcal{M}_{0\mu\nu} \tag{2.23}$$

where

$$\mathcal{M}_{\mu\nu\lambda} = \frac{1}{2}\bar{\phi}(x)\Gamma_\mu(\partial, -\overleftarrow{\partial})\big(x_\nu\partial_\lambda\phi(x) - x_\lambda\partial_\nu\phi(x)\big). \tag{2.24}$$

The number operator corresponding to a field can be calculated by obtaining the conserved current corresponding to the global phase transformation symmetry. Under a global phase transformation, i.e, $\phi(x) \to \phi'(x) = e^{-i\alpha x}\phi(x)$, the infinitesimal change in the field and its adjoint are $\delta\phi(x) = -i\alpha\phi(x)$ and $\delta\bar{\phi}(x) = i\alpha\bar{\phi}(x)$ respectively. Thus using Eq.(2.20), we get the number operator corresponding to $\phi(x)$ as

$$N = -i\alpha \int d^3x \; \bar{\phi}(x)\Gamma_0(\partial, -\overleftarrow{\partial})\phi(x). \tag{2.25}$$

The Noether's prescription allows us to calculate the conserved currents for continuous symmetries only. However the above method provides a way for obtaining the conserved currents associated with the discrete symmetries also. Note that, unlike the Noether's method here one uses only the equations of motion to get conserved currents [12].

We now discuss the conserved currents corresponding to parity, charge conjugation and time reversal symmetry associated with the Dirac field $\psi(x)$, satisfying Eq.(2.1) for $\lambda(\partial)$ given in Eq.(2.6).

Under the parity transformation, the space-time coordinates and the corresponding derivatives change as $x_i \to -x_i$, $t \to t$, $\partial_i \to -\partial_i$ and $\partial_0 \to \partial_0$, so that the Dirac equation also changes as $\Lambda(-\partial_i, \partial_0)\psi(-x_i, t) = 0$. Now one can find a matrix $\mathcal{P}$

($\mathcal{P} = \gamma^0$ for Dirac field) such that $\mathcal{P}\psi(-x_i,t) = \psi_p(x_i,t)$ satisfies the Dirac equation, i.e, $\Lambda(\partial_i,\partial_0)\psi_p(x_i,t) = 0$. In order to construct the conserved currents, we take $\delta\psi(x) = \psi_p(x_i,t)$ [12]. By substituting this in Eq.(2.20), we obtain the expression for the conserved current corresponding to the parity transformation as

$$J_\mu = \bar{\psi}(x_i,t)\Gamma_\mu(\partial, -\overleftarrow{\partial})\psi_p(x_i,t). \tag{2.26}$$

Next we consider the time-reversal transformations. The space-time coordinates and their derivatives transform as $x_i \to x_i,\ t \to -t,\ \partial_i \to \partial_i$ and $\partial_0 \to -\partial_0$ under the time-reversal transformation. Thus the Dirac equation also transform as $\Lambda(\partial_i,-\partial_0)\psi(x_i,-t) = 0$. There exist a matrix $\mathcal{T}$ for which $\mathcal{T}\psi^*(x_i,-t) = \psi_T(x_i,t)$ satisfies the Dirac equation as $\Lambda(\partial_i,\partial_0)\psi_T(x_i,t) = 0$. Here we choose $\delta\psi(x) = \psi_T(x_i,t)$ [12] and using this in Eq.(2.20), we get the conserved currents for time-reversal symmetry as

$$J_\mu = \bar{\psi}(x_i,t)\Gamma_\mu(\partial, -\overleftarrow{\partial})\psi_T(x_i,t). \tag{2.27}$$

We study the charge conjugation symmetry of the Dirac equation by introducing a minimal coupling term, describing the interaction of electron with electromagnetic field A_μ, in the equations of motion. This is done by replacing $i\partial_\mu$ in Eq.(2.6) with $i\partial_\mu + eA_\mu$ and we denote this as $\Lambda_c(\partial,e) = i\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu + m$, satisfying $\Lambda_c(\partial,e)\psi(x_i,t) = 0$. Under the charge conjugation symmetry, $e \to -e$ and the complex conjugate of Dirac equation becomes $\Lambda_c^*(\partial,-e)\psi^*(x_i,t) = 0$. Here we can find a matrix $\mathcal{C}$ such that $\mathcal{C}\psi^*(x_i,t) = \psi_c(x_i,t)$ satisfies the Dirac equation for the anti-particle (i.e, $\Lambda_c(\partial,-e)\psi_c(x_i,t) = 0$). By choosing $\delta\psi(x_i,t) = \psi_c(x_i,t)$ [12] in Eq.(2.20), we obtain the conserved current corresponding to the charge conjugation symmetry as

$$J_c^\mu = \bar{\psi}(x_i,t)\Gamma_c^\mu(\partial, -\overleftarrow{\partial})\psi_c(x_i,t). \tag{2.28}$$

Thus one can construct the conserved currents corresponding to the discrete symmetries also from their equations of motion alone.

2.3 Quantisation of κ-deformed Klein-Gordon field

In this section we generalise the Takahashi-Umezawa quantisation scheme to the κ-deformed space-time. We will then use this method to quantise the field theory satisfying κ-deformed Klein-Gordon equation, valid up to first order in a. Further we derive the conserved currents corresponding to translational and Lorentz symmetry of the κ-deformed Klein-Gordon field. We also derive the number operator corresponding to deformed Klein-Gordon field.

The κ-deformed Klein-Gordon equation, invariant under the undeformed κ-Poincare algebra is written using the quadratic Casimir (see Eq.(1.20) of chapter 1), as [5]

$$\left(\Box\left(1+\frac{a^2}{4}\Box\right)-m^2\right)\hat{\phi}(x)=0, \tag{2.29}$$

where $\hat{\phi}(x)$ is the κ-deformed Klein-Gordon field and

$$\Box=\partial_k^2\frac{e^{-A}}{2\varphi^2}-\partial_0^2\frac{2(1-\cosh A)}{A^2}, \tag{2.30}$$

where $A=ia\partial_0$.

For a particular choice of realisation $\varphi=e^{-A}$, given in [19–21] we get the κ-deformed Klein-Gordon equation, valid up to first non-vanishing term in a, as

$$\left(\partial_i^2-\partial_0^2-m^2-ia\partial_0\partial_i^2\right)\hat{\phi}(x)=0. \tag{2.31}$$

The above equation reduces to the commutative Klein-Gordon equation in the limit $a\to 0$. Note that the additional term is a higher derivative term.

Now we generalise Eq.(2.8) to κ space-time by replacing Lambda operator $\Lambda(\partial)$ as well as the Klein-Gordon divisor $d(\partial)$ with their κ-deformed versions $\hat{\Lambda}(\partial)$ and $\hat{d}(\partial)$, respectively. We also re-write the RHS of Eq.(2.8) using the κ-deformed Klein-Gordon equation given in Eq.(2.31). Thus we have

$$\hat{\Lambda}(\partial)\hat{d}(\partial)=\hat{d}(\partial)\hat{\Lambda}(\partial)=\Box-m^2-ia\partial_0\partial_i^2. \tag{2.32}$$

Here Eq.(2.32) is the starting equation for the quantisation of κ-deformed scalar field theory using the Takahashi-Umezawa formalism. It is to be noted that Eq.(2.32) reduces to Eq.(2.8) in the commutative limit $a\to 0$.

From Eq.(2.31), we obtain the $\hat{\Lambda}(\partial)$ operator corresponding to the κ-deformed Klein-Gordon equation, valid up to first order in a, as

$$\hat{\Lambda}(\partial)=\Box-m^2-ia\partial_0\partial_i^2, \tag{2.33}$$

satisfying the equations of motion

$$\hat{\Lambda}(\partial)\hat{\phi}(x)=0. \tag{2.34}$$

As in the commutative case (see Eq.(2.9)) here also we take the κ-deformed Klein-Gordon divisor corresponding to the κ-deformed Klein-Gordon field as the identity operator, i.e., $\hat{d}(\partial)=\mathbb{I}$.

We decompose the κ-deformed Klein-Gordon field into positive and negative frequency modes using the deformed creation and annihilation operators as

$$\hat{\phi}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \Big(\hat{u}_p(x)\hat{a}(p) + \hat{u}_p^*(x)\hat{a}^\dagger(p) \Big), \tag{2.35}$$

where E_p, is the commutative energy which is defined using the usual dispersion relation as $E_p = \sqrt{\vec{p}^2 + m^2}$ [1]

Note that the $\hat{u}(x)$ appearing in Eq.(2.35) satisfy the κ-deformed Klein-Gordon equation, i.e.,

$$\hat{\Lambda}(\partial)\hat{u}(x) = 0. \tag{2.36}$$

We solve the above equation perturbatively by expanding $\hat{u}(x)$ up to first order in the deformation parameter as

$$\hat{u}(x) = u^{(0)}(x) + a\alpha u^{(1)}(x). \tag{2.37}$$

Here α is a real parameter having the dimension of $[L]^{-1}$. We also note that $\hat{\Lambda}(\partial)$ naturally split as

$$\hat{\Lambda}(\partial) = \Lambda^{(0)}(\partial) + a\Lambda^{(1)}(\partial). \tag{2.38}$$

Now we substitute Eq.(2.37) and Eq.(2.38) in Eq.(2.36) and keep the terms valid up to first order in a. Thus we get two equations corresponding to a independent and a dependent coefficient terms. They are given by

$$\begin{aligned} &\Lambda^{(0)}(\partial)u^{(0)}(x) = 0, \\ &\Lambda^{(1)}(\partial)u^{(0)}(x) + \alpha\Lambda^{(0)}(\partial)u^{(1)}(x) = 0. \end{aligned} \tag{2.39}$$

By solving the first equation of Eq.(2.39), we get $u^{(0)}(x) = e^{-ipx}$ and this represents the plane wave solution of the commutative Klein-Gordon equation. Substituting the commutative solution $u^{(0)}(x) = e^{-ipx}$ in the second equation of Eq.(2.39), we get

$$\Big(\Box - m^2\Big)u^{(1)}(x) = \frac{1}{\alpha}E_p p_i^2 e^{-ipx}, \tag{2.40}$$

We solve the above inhomogenous differential equation using the Green's function method and the solution is defined as

$$u^{(1)}(x) = u^{(0)}(x) + \int G(x - x')j(x')d^4x', \tag{2.41}$$

[1] In general one need to use the κ-deformed dispersion relation (i.e., $E_p^2 = p^2(1 + aE_p) + m^2$, valid up to first order in a). In order to simplify the calculations, we use the commutative dispersion relation. But here we have included the κ-deformed corrections through $\hat{u}(p)$ and through deformed creation and annihilation operators respectively.

where $j(x) = \frac{1}{\alpha} E_p p_i^2 e^{-ipx}$ and $G(x - x')$ is the Green's function that satisfy the commutative Klein-Gordon equation as

$$\Big(\Box - m^2\Big) G(x - x') = \delta^4(x - x'). \tag{2.42}$$

Using the Fourier transformation in the above equation, we obtain the form of the Green's function as

$$\begin{aligned} G(x - x') &= -\int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m^2} e^{-ip(x-x')} \\ &= \int \frac{d^4p}{(2\pi)^4} \frac{1}{p_0^2 - E_p^2} e^{-ip(x-x')} \end{aligned} \tag{2.43}$$

Now we solve the p_0 integral in Eq.(2.43) by shifting the poles, i.e, $p_0 = E_p$ and $p_0 = -E_p$ by $i\epsilon$. Thus we re-express Eq.(2.43) as

$$G(x - x') = \lim_{\epsilon \to 0} \int \frac{d^3p}{(2\pi)^3} \frac{e^{-i\vec{p}\cdot(\vec{x}-\vec{x}')}}{2E_p} \int \frac{dp_0}{2\pi} e^{ip_0(t-t')} \left(\frac{1}{(p_0 - E_p) + i\epsilon} - \frac{1}{(p_0 + E_p) - i\epsilon} \right). \tag{2.44}$$

Using the definition of the step function, i.e,

$$\lim_{\epsilon \to 0} \int dz \, \frac{e^{iz(t-t')}}{z + i\epsilon} = -2\pi i \theta(t - t'), \tag{2.45}$$

in the above equation and after re-arranging the terms we get the explicit form of the Green's function as

$$G(x - x') = -\int \frac{d^3p}{(2\pi)^3} \frac{i}{2E_p} \Big(\theta(t - t') e^{-ip(x-x')} + \theta(t' - t) e^{ip(x-x')} \Big). \tag{2.46}$$

Using this we calculate the second term on the RHS of Eq.(2.41) as

$$\begin{aligned} &\int d^4x' G(x - x') j(x') \\ &= \int d^4x' \int \frac{d^3p'}{(2\pi)^3} \frac{(-i)}{2E_{p'}} \left[\theta(t' - t) e^{ip'(x-x')} + \theta(t - t') e^{-ip'(x-x')} \right] \frac{E_p}{\alpha} (\vec{p})^2 e^{-ipx'} \\ &= -2\pi i \int \frac{d^3p'}{2E_{p'}} \left[\theta(t' - t) e^{ip'x} \int \frac{d^4x'}{(2\pi)^4} e^{-i(p'+p)x'} + \theta(t - t') e^{-ip'x} e^{-i(p-p')x'} \right] \frac{E_p}{\alpha} (\vec{p})^2 \\ &= -2\pi i \int \frac{d^3p'}{2E_{p'}} \left[\theta(t' - t) e^{ip'x} \delta^4(p + p') + \theta(t - t') e^{-ip'x} \delta^4(p - p') \right] \frac{E_p}{\alpha} (\vec{p})^2 \\ &= \frac{-i\pi}{\alpha} \left[\theta(t' - t) e^{-ip_0't} \delta(p_0 + p_0') + \theta(t - t') e^{ip_0't} \delta(p_0 - p_0') \right] e^{i\vec{p}\cdot\vec{x}} \vec{p}^2 \end{aligned} \tag{2.47}$$

Now using the identity $\delta(x - a) f(x) = \delta(x) f(a)$ and re-writing the delta and step functions using their integral representations, we find that the above expression vanishes. Substituting this and Eq.(2.41) in Eq.(2.37), we get the complete solution $\hat{u}(p)$, valid up to first order term in a, as

$$\hat{u}(x) = (1 + a\alpha)e^{-ipx} \tag{2.48}$$

We see that this $\hat{u}(x)$ (valid only up to first order in a) is proportional to the commutative solution e^{-ipx}. After substituting Eq.(2.48) in Eq.(2.35), we get the κ-deformed Klein-Gordon field operator, valid up to first order in a, as

$$\hat{\phi}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\Big((1 + a\alpha)e^{-ipx}\hat{a}(p) + (1 + a\alpha)e^{ipx}\hat{a}^\dagger(p)\Big). \tag{2.49}$$

From the above expression it can easily be seen that in the limit $a \to 0$, we get back the usual field operator satisfying the Klein-Gordon equation.

For a scalar theory we assume that the deformed creation and annihilation operators satisfy the usual commutation relations, i.e.,

$$[\hat{a}(p), \hat{a}(p')] = 0, \quad [\hat{a}^\dagger(p), \hat{a}^\dagger(p')] = 0, \quad [\hat{a}(p), \hat{a}^\dagger(p')] = \delta^3(p - p'). \tag{2.50}$$

Using Eq.(2.13) we write down the (unequal-time) commutation relations between κ-deformed real scalar field and its adjoint as

$$[\hat{\phi}(x), \bar{\hat{\phi}}(x')] = i\hat{\Delta}(x - x'). \tag{2.51}$$

We now assume $\hat{\Delta}(x - x')$ to have an a dependent correction term $f(x - x')$ (whose explicit form is unknown), so that we express $\hat{\Delta}(x-x') = \Delta(x-x') + af(x-x')$. Hence Eq.(2.51) becomes

$$[\hat{\phi}(x), \bar{\hat{\phi}}(x')] = i\Delta(x - x') + iaf(x - x'). \tag{2.52}$$

Using the explicit form of the κ-deformed field operator $\hat{\phi}(x)$, i.e, Eq.(2.35) in Eq.(2.52), we get the (unequal-time) commutation relation as

$$[\hat{\phi}(x), \bar{\hat{\phi}}(x')] = \int \frac{d^3p\, d^3p'}{\sqrt{(2\pi)^3 2E_p 2E_{p'}}}\Bigg(\Big(u_p^{(0)}(x)u_{p'}^{*(0)}(x') - u_p^{*(0)}(x)u_{p'}^{(0)}(x')\Big)[\hat{a}(p), \hat{a}^\dagger(p')] +$$
$$a\alpha\Big(u_p^{(1)}(x)u_{p'}^{*(0)}(x') + u_p^{(0)}(x)u_{p'}^{*(1)}(x') - u_p^{*(0)}(x)u_{p'}^{(1)}(x') - u_p^{*(1)}(x)u_{p'}^{(0)}(x')\Big)[\hat{a}(p), \hat{a}^\dagger(p')]\Bigg) \tag{2.53}$$

By using the explicit form of $u_p^{(0)}(x)$ and $u_p^{(1)}(x)$ from Eq.(2.48), we get

$$[\hat{\phi}(x), \bar{\hat{\phi}}(x')] = i(1 + 2a\alpha)\Delta(x - x'). \tag{2.54}$$

We identify $f(x - x') = 2\alpha\Delta(x - x')$ by comparing Eq.(2.54) with Eq.(2.52).

Thus we obtain a deformed (unequal-time) commutation relation between the deformed scalar field and its adjoint by assuming that the deformed creation and annihilation operators satisfy the usual oscillator algebra, as given in Eq.(2.50).

Now instead of using Eq.(2.50), let us assume that the oscillator algebra to be deformed (such that the deformation is valid up to first order in a)

$$[\hat{a}(k), \hat{a}(k')] = [\hat{a}^\dagger(k), \hat{a}^\dagger(k')] = 0, \quad [\hat{a}(k), \hat{a}^\dagger(k')] = g(a)\delta^3(k - k'). \tag{2.55}$$

where $g(a)$ is an arbitrary linear function in a, and in the limit $a \to 0, \ g(a) = 1$. Now we use this deformed commutation relation in Eq.(2.53) and we repeat the above steps to get the unequal-time commutation relation between κ-deformed scalar field and its adjoint as

$$[\hat{\phi}(x), \bar{\hat{\phi}}(x')] = ig(a)(1 + 2a\alpha)\Delta(x - x'). \tag{2.56}$$

The above commutation relation becomes undeformed (valid up to first order in a) for a particular choice $g(a) = 1 - 2a\alpha$. Thus we have

$$[\hat{\phi}(x), \bar{\hat{\phi}}(x')] = i\Delta(x - x'). \tag{2.57}$$

Now we substitute this $g(a) = 1 - 2a\alpha$ in Eq.(2.55) and we get the κ-deformed oscillator algebra, valid up to first order in a, as

$$[\hat{a}(k), \hat{a}(k')] = [\hat{a}^\dagger(k), \hat{a}^\dagger(k')] = 0, \quad [\hat{a}(k), \hat{a}^\dagger(k')] = (1 - 2a\alpha)\delta^3(k - k'). \tag{2.58}$$

Thus here we find that the commutation relation between κ-deformed scalar field and its adjoint becomes undeformed for the deformed oscillator algebra, given in Eq.(2.58).

2.3.1 Conserved currents for κ-deformed scalar field

In this subsection, we construct the conserved currents corresponding to translational and Lorentz symmetries, for the scalar field in the κ-deformed space-time, by reformulating the expression for conserved currents defined in Eq.(2.20) for the κ- space-time.

Comparing Eq.(2.3) and Eq.(2.33), we obtain the components $\hat{\Lambda}_\mu, \ \hat{\Lambda}_{\mu\nu}$, ..etc, corresponding to the κ-deformed Klein-Gordon equation as

$$\hat{\Lambda}_{\mu\nu\lambda} = -ia\delta_{\mu 0}\delta_{\nu i}\delta_{\lambda i}, \ \hat{\Lambda}_{\mu\nu} = \eta_{\mu\nu}, \ \hat{\Lambda}_\mu = 0, \ \hat{\Lambda}_{(0)} = -m^2. \tag{2.59}$$

Substituting Eq.(2.59) in Eq.(2.16) we get the expression for the $\hat{\Gamma}_\mu(\partial, -\overleftarrow{\partial})$ operator for κ-deformed Klein-Gordon field as

$$\begin{aligned}\hat{\Gamma}_\mu(\partial, -\overleftarrow{\partial}) =& \partial_\mu - \overleftarrow{\partial}_\mu - \frac{ia}{3}\Big(\delta_{\mu 0}\partial_i^2 - \delta_{\mu 0}\partial_i\overleftarrow{\partial}_i + \delta_{\mu 0}\overleftarrow{\partial}_i^2 + 2\delta_{\mu i}\partial_i\partial_0 + \\ & 2\delta_{\mu i}\overleftarrow{\partial}_i\overleftarrow{\partial}_0 - \delta_{\mu i}\partial_i\overleftarrow{\partial}_0 - \delta_{\mu i}\partial_0\overleftarrow{\partial}_i\Big).\end{aligned} \tag{2.60}$$

From the above, we see that a dependent terms in RHS of the equation are coming from the first order κ-deformation correction terms present in the equations of motion for scalar field. We find that the $\hat{\Gamma}_\mu(\partial, -\overleftarrow{\partial})$ operator defined above satisfies the identity, i.e., Eq.(2.19), in κ space-time also as

$$\Big(\partial_\mu + \overleftarrow{\partial}_\mu\Big)\hat{\Gamma}^\mu(\partial, -\overleftarrow{\partial}) = \hat{\Lambda}(\partial) - \hat{\Lambda}(-\overleftarrow{\partial}). \tag{2.61}$$

The general expression for the conserved currents associated with κ-deformed scalar field is obtained by substituting Eq.(2.60) in Eq.(2.20) as

$$\begin{aligned}\hat{J}_\mu =& \hat{\phi}(x)\Big(\partial_\mu - \overleftarrow{\partial}_\mu\Big)\delta\hat{\phi}(x) \\ & -\frac{ia}{3}\hat{\phi}(x)\Big(\delta_{\mu 0}\big(\partial_i^2 - \partial_i\overleftarrow{\partial}_i + \overleftarrow{\partial}_i^2\big) + \delta_{\mu i}\big(2\partial_i\partial_0 + 2\overleftarrow{\partial}_i\overleftarrow{\partial}_0 - \partial_i\overleftarrow{\partial}_0 - \partial_0\overleftarrow{\partial}_i\big)\Big)\delta\hat{\phi}(x).\end{aligned} \tag{2.62}$$

The conserved current for the deformed scalar field pick up a dependent correction terms from $\hat{\Gamma}_\mu(\partial, -\overleftarrow{\partial})$ and $\hat{\phi}(x)$.

Under the translation, the κ-deformed space-time coordinate transform as $\hat{x}_\mu \rightarrow \hat{x}'_\mu = \hat{x}_\mu + \delta\hat{x}_\mu$. We determine the infinitesimal change in the κ-deformed space-time coordinate $\delta\hat{x}_\mu$, using the relation $\delta\hat{x}_\mu = \theta^\nu[D_\mu, \hat{x}_\nu]$, where θ_ν is the parameter associated with translational symmetry. Thus by substituting the explict form of D_μ and $\hat{x}_\mu$ from Eq.(1.18) and Eq.(1.12) (of chapter 1), we obtain $\delta\hat{x}_\mu$, valid up to first order in a, as

$$\delta\hat{x}_\mu = \theta_\mu + ia\theta^\nu(\eta_{\mu\nu} + \delta_{\mu 0}\partial_\nu). \tag{2.63}$$

The infinitesimal change associated with the translation is given as $\delta\hat{\phi}(x) = \delta\hat{x}^\mu\partial_\mu\hat{\phi}(x)$. By substituting Eq.(2.63) in $\delta\hat{\phi}(x) = \delta\hat{x}^\mu\partial_\mu\hat{\phi}(x)$, we obtain the explicit form of $\delta\hat{\phi}(x)$, valid up to first order in a, as

$$\delta\hat{\phi}(x) = \theta^\mu\big(\partial_\mu\hat{\phi}(x) + ia(\partial_\mu\hat{\phi}(x) + \partial_0\partial_\mu\hat{\phi}(x))\big). \tag{2.64}$$

By substituting Eq.(2.64) in Eq.(2.62), we obtain the conserved current associated with the translational symmetry of the deformed scalar field as

$$\begin{aligned}\hat{J}_\mu =\ & \hat{\phi}(x)\big(\partial_\mu - \overleftarrow{\partial}_\mu\big)\partial_\nu\hat{\phi}(x)\theta^\nu - \frac{ia}{3}\delta_{\mu 0}\hat{\phi}(x)\Big(\partial_i - \partial_i\overleftarrow{\partial}_i + \overleftarrow{\partial}_i^2\Big)\partial_\nu\hat{\phi}(x)\theta^\nu \\ & - \frac{ia}{3}\delta_{\mu i}\hat{\phi}(x)\Big(2\partial_i\partial_0 - \partial_i\overleftarrow{\partial}_0 - \overleftarrow{\partial}_i\partial_0 + 2\overleftarrow{\partial}_i\overleftarrow{\partial}_0\Big)\partial_\nu\hat{\phi}(x)\theta^\nu \\ & + ia\hat{\phi}(x)\big(\partial_\mu - \overleftarrow{\partial}_\mu\big)\big(\partial_\nu + \partial_0\partial_\nu\big)\hat{\phi}(x)\theta^\nu\end{aligned} \tag{2.65}$$

From the above equation, we find that the last terms are contributed by the a dependent terms of $\delta\hat{x}_\mu$. Similarly the $\frac{1}{3}$ dependent terms of the conserved current expression are contributed by the a dependent terms present in $\hat{\Gamma}^\mu$, see Eq.(2.62). Using the relation $\hat{J}_\mu = \hat{T}_{\mu\nu}\theta^\nu$, we write down the expression for the energy-momentum for κ-deformed Klein-Gordon field, valid up to first order in a, as

$$\begin{aligned}\hat{T}_{\mu\nu} =\ & \hat{\phi}(x)\big(\partial_\mu - \overleftarrow{\partial}_\mu\big)\partial_\nu\hat{\phi}(x) - \frac{ia}{3}\delta_{\mu 0}\hat{\phi}(x)\Big(\partial_i - \partial_i\overleftarrow{\partial}_i + \overleftarrow{\partial}_i^2\Big)\partial_\nu\hat{\phi}(x) \\ & - \frac{ia}{3}\delta_{\mu i}\hat{\phi}(x)\Big(2\partial_i\partial_0 - \partial_i\overleftarrow{\partial}_0 - \overleftarrow{\partial}_i\partial_0 + 2\overleftarrow{\partial}_i\overleftarrow{\partial}_0\Big)\partial_\nu\hat{\phi}(x) \\ & + ia\hat{\phi}(x)\big(\partial_\mu - \overleftarrow{\partial}_\mu\big)\big(\partial_\nu + \partial_0\partial_\nu\big)\hat{\phi}(x).\end{aligned} \tag{2.66}$$

It is clear from the above expression that the energy-momentum tensor corresponding to the κ-deformed scalar field is not symmetric due to the a dependent terms of the κ deformation. From Eq.(2.66), we can obtain the momentum corresponding to κ-deformed scalar field as

$$\hat{P}_\mu = \int d^3x\ \hat{T}_{0\mu}. \tag{2.67}$$

The infinitesimal change in the κ-deformed space-time coordinate associated with the Lorentz symmetry is defined as $\delta\hat{x}_\mu = [M_{\mu\nu}, \hat{x}_\lambda]\omega^{\nu\lambda}$. By using the explicit form of $M_{\mu\nu}$ and $\hat{x}_\mu$, we get $\delta\hat{x}_\mu$ corresponding to Lorentz symmetry (valid up to first order in a) as

$$\begin{aligned}\delta\hat{x}_\mu =\ & x_\nu{\omega^\nu}_\mu + ia\Big(\delta_{\mu 0}x_\lambda\partial_\nu - \delta_{\nu 0}x_\lambda\partial_\mu + \eta_{\mu\lambda}x_\nu\partial_0 + \delta_{\mu 0}\delta_{\lambda 0}x_0\partial_\nu - \delta_{0\nu}\eta_{\mu\lambda}x_0\partial_0 \\ & -\delta_{0\nu}x_\mu\partial_\lambda + \delta_{0\mu}x_\nu\partial_\lambda\Big)\omega^{\nu\lambda}.\end{aligned} \tag{2.68}$$

Similarly the infinitesimal change in the deformed scalar field under the Lorentz transformation is obtained as $\delta\hat{\phi}(x) = \delta\hat{x}_\mu\partial^\mu\hat{\phi}(x)$. Substituting Eq.(2.68) in $\delta\hat{\phi}(x) = \delta\hat{x}_\mu\partial^\mu\hat{\phi}(x)$, we obtain the infinitesimal change in the deformed scalar field, valid up to first order in a, as

$$\begin{aligned}\delta\hat{\phi}(x) =\ & \frac{1}{2}\Big(x_\nu\partial_\lambda\hat{\phi}(x) - x_\lambda\partial_\nu\hat{\phi}(x)\Big)\omega^{\nu\lambda} + \frac{ia}{2}\Big(-\delta_{\nu 0}x_\lambda\partial_\alpha\partial^\alpha\hat{\phi}(x) + \delta_{\lambda 0}x_\nu\partial_\alpha\partial^\alpha\hat{\phi}(x) \\ & -\delta_{\nu 0}x_\alpha\partial_\lambda\partial^\alpha\hat{\phi}(x) + \delta_{\lambda 0}x_\alpha\partial_\nu\partial^\alpha\hat{\phi}(x) - x_\lambda\partial_\nu\partial_0\hat{\phi}(x) + x_\nu\partial_\lambda\partial_0\hat{\phi}(x)\Big)\omega^{\nu\lambda}\end{aligned} \tag{2.69}$$

After substituting Eq.(2.69) in Eq.(2.62), we get the explicit form of the conserved current corresponding to the undeformed κ-Lorentz symmetry of κ-deformed scalar field as

$$\begin{aligned}\hat{J}_\mu = &\frac{1}{2}\hat{\phi}(x)\big(\partial_\mu - \overleftarrow{\partial}_\mu\big)\big(x_\nu\partial_\lambda - x_\lambda\partial_\nu\big)\hat{\phi}(x)\omega^{\nu\lambda} \\ &-\frac{ia}{6}\delta_{\mu 0}\hat{\phi}(x)\Big(\partial_i - \partial_i\overleftarrow{\partial}_i + \overleftarrow{\partial}_i^2\Big)\big(x_\nu\partial_\lambda - x_\lambda\partial_\nu\big)\hat{\phi}(x)\omega^{\nu\lambda} \\ &-\frac{ia}{6}\delta_{\mu i}\hat{\phi}(x)\Big(2\partial_i\partial_0 - \partial_i\overleftarrow{\partial}_0 - \overleftarrow{\partial}_i\partial_0 + 2\overleftarrow{\partial}_i\overleftarrow{\partial}_0\Big)\big(x_\nu\partial_\lambda - x_\lambda\partial_\nu\big)\hat{\phi}(x)\omega^{\nu\lambda} \\ &-\frac{ia}{2}\hat{\phi}(x)\big(\partial_\mu - \overleftarrow{\partial}_\mu\big)\Big(\delta_{\nu 0}x_\lambda\partial_\alpha\partial^\alpha - \delta_{\lambda 0}x_\nu\partial_\alpha\partial^\alpha \\ &+\delta_{\nu 0}x_\alpha\partial_\lambda\partial^\alpha - \delta_{\lambda 0}x_\alpha\partial_\nu\partial^\alpha + x_\lambda\partial_\nu\partial_0 - x_\nu\partial_\lambda\partial_0\Big)\hat{\phi}(x)\omega^{\nu\lambda}\end{aligned} \tag{2.70}$$

In Eq.(2.70), we observe that $\frac{ia}{6}$ dependent correction terms come from the $\hat{\Gamma}_\mu(\partial, -\overleftarrow{\partial})$ operator and $\frac{ia}{2}$ dependent correction terms come from the infinitesimal change in the deformed scalar field associated with the undeformed κ-Lorentz symmetry.

Using the relation $\hat{J}_\mu = \hat{\mathcal{M}}_{\mu\nu\lambda}\omega^{\nu\lambda}$ in Eq.(2.70), we obtain $\hat{\mathcal{M}}_{\mu\nu\lambda}$ as

$$\begin{aligned}\hat{\mathcal{M}}_{\mu\nu\lambda} = &\frac{1}{2}\hat{\phi}(x)\big(\partial_\mu - \overleftarrow{\partial}_\mu\big)\big(x_\nu\partial_\lambda - x_\lambda\partial_\nu\big)\hat{\phi}(x) - \frac{ia}{6}\delta_{\mu 0}\hat{\phi}(x)\Big(\partial_i^2 - \partial_i\overleftarrow{\partial}_i + \overleftarrow{\partial}_i^2\Big)\big(x_\nu\partial_\lambda - x_\lambda\partial_\nu\big)\hat{\phi}(x) \\ &-\frac{ia}{6}\delta_{\mu i}\hat{\phi}(x)\Big(2\partial_i\partial_0 - \partial_i\overleftarrow{\partial}_0 - \overleftarrow{\partial}_i\partial_0 + 2\overleftarrow{\partial}_i\overleftarrow{\partial}_0\Big)\big(x_\nu\partial_\lambda - x_\lambda\partial_\nu\big)\hat{\phi}(x) \\ &-\frac{ia}{2}\hat{\phi}(x)\big(\partial_\mu - \overleftarrow{\partial}_\mu\big)\Big(\delta_{\nu 0}x_\lambda\partial_\alpha\partial^\alpha - \delta_{\lambda 0}x_\nu\partial_\alpha\partial^\alpha + \delta_{\nu 0}x_\alpha\partial_\lambda\partial^\alpha - \delta_{\lambda 0}x_\alpha\partial_\nu\partial^\alpha \\ &+x_\lambda\partial_\nu\partial_0 - x_\nu\partial_\lambda\partial_0\Big)\hat{\phi}(x).\end{aligned} \tag{2.71}$$

We obtain the expression for Lorentz generator (valid up to first order in a) corresponding to the deformed scalar field from Eq.(2.71) as

$$\hat{M}_{\mu\nu} = \int d^3x\ \hat{\mathcal{M}}_{0\mu\nu}. \tag{2.72}$$

2.4 Deformed Unruh effect

In this section we study the effects of the κ-deformation (valid up to first order in a) in the Unruh effect. This is studied using the deformed oscillator algebra corresponding to the κ-deformed scalar field derived in Eq.(2.58).

When a uniformly accelerating observer (with constant proper acceleration A) measures the vacuum expectation value of the number operator (corresponding to accelerating frame) in the Minkowski vacuum, particles are found to be in a thermal bath whose temperature is $T = \frac{\hbar A}{2\pi k}$. This is known as Unruh effect [26–29].

First order κ-deformed corrections to Unruh effect has been derived in [23], by calculating the response of uniformly accelerating monopole detector coupled to the massless κ-deformed complex scalar field (invariant under the κ-Poincare algebra). Similarly in [24], a^2 dependent corrections to the Unruh effect has been obtained from the response function of uniformly accelerating detector coupled to massless κ-deformed scalar field.

Here we derive the first order κ-deformed corrections to the Unruh effect corresponding to the deformed scalar field by using the method of Bogoliubov coefficients. For this, we first consider a massless κ-deformed Klein-Gordon field in the $1+1$ dimensional Minkowski space-time, such that the corresponding creation and annihilation operators satisfy deformed oscillator algebra. Next we consider the massless κ-deformed Klein-Gordon field in the $1+1$ dimensional Rindler space-time. We then use the method of Bogoliubov transformation to connect the frequency modes in the Minkowski basis with the frequency modes in the Rindler basis. Next we calculate the vacuum expectation value of the number operator defined in the Rindler basis over the Minkowski vacuum. In calculating this, we use the deformed oscillator algebra satisfied by the creation and annihilation operators given in Eq.(2.58).

Let us consider a $1+1$ dimensional Minkowski space-time[2] as

$$ds^2 = -dt^2 + dz^2. \tag{2.73}$$

Now we consider a massless κ-deformed Klein-Gordon field in this $1+1$ dimensional Minkowski space-time. The corresponding equation of motion (valid up to first order in a) is

$$(\partial_z^2 - \partial_t^2 - ia\partial_t\partial_z^2)\hat{\phi}(z,t) = 0. \tag{2.74}$$

The deformed field operator $\hat{\phi}(z,t)$ can be separated into left and right moving sectors and further by decomposing it into deformed positive and negative frequency modes, we get

$$\hat{\phi}(z,t) = (1+a\alpha)\int \frac{dk}{\sqrt{4\pi k}}\Big(\hat{b}_{+k}e^{ik(t+z)} + \hat{b}^\dagger_{+k}e^{-ik(t+z)} + \hat{b}_{-k}e^{-ik(z-t)} + \hat{b}^\dagger_{-k}e^{ik(z-t)}\Big). \tag{2.75}$$

We define the Minkowski vacuum state as $|0\rangle_M$, such that $\hat{b}_{+k}$ and $\hat{b}_{-k}$ annihilate the Minkowski vacuum (i.e., $\hat{b}_{+k}|0\rangle_M = 0$ and $\hat{b}_{-k}|0\rangle_M = 0$ respectively). These operators satisfy the deformed oscillator algebra as (see Eq.(2.58))

$$[\hat{b}_{\pm k}, \hat{b}^\dagger_{\pm k'}] = (1-2a\alpha)\delta(k-k'). \tag{2.76}$$

[2]In general one can use the κ-deformed metric to study the Unruh effect in κ space-time. But here we focuss on the κ-deformed corrections in Unruh effect due to the deformed oscillator algebra alone. Hence we consider the usual metric in this calculation. See chapter 5 to see the construction of κ-deformed metric.

Now we define the null coordinates as

$$U = t - z, \quad V = t + z. \tag{2.77}$$

In terms of these null coordinates Eq.(2.75) can be written as

$$\hat{\phi}(t,z) = \hat{\phi}_+(V) + \hat{\phi}_-(U), \tag{2.78}$$

where $\hat{\phi}_+(V)$ and $\hat{\phi}_-(U)$ represent the left and right moving sectors of the deformed scalar field in the 1 + 1 dimensional Minkowski space-time. Their explicit forms are given as

$$\hat{\phi}_+(V) = (1 + a\alpha) \int \frac{dk}{\sqrt{4\pi k}} \Big(\hat{b}_{+k} e^{ikV} + \hat{b}^\dagger_{+k} e^{-ikV} \Big), \tag{2.79}$$

and

$$\hat{\phi}_-(U) = (1 + a\alpha) \int \frac{dk}{\sqrt{4\pi k}} \Big(\hat{b}_{-k} e^{ikU} + \hat{b}^\dagger_{-k} e^{-ikU} \Big). \tag{2.80}$$

Here $\hat{\phi}_-(U)$ and $\hat{\phi}_+(V)$ are independent solutions of Eq.(2.74). So now onwards we consider only the left moving sector, i.e., $\hat{\phi}_+(V)$ for the remaining discussions [29].

Now we consider the Rindler space-time which is defined by the regions $|t| < z$ and $|t| < -z$. The region $|t| < z$ is known as the Right Rindler Wedge (RRW) and $|t| < -z$ is known as the Left Rindler Wedge (LRW). The coordinates in the RRW and LRW regions are defined as (τ, ζ) and $(\bar{\tau}, \bar{\zeta})$ respectively [29].

The coordinates of RRW are related to Minkowski coordinates by the following coordinate transformation [29]

$$t = \frac{e^{A\zeta}}{A} \sinh A\tau, \quad z = \frac{e^{A\zeta}}{A} \cosh A\tau, \tag{2.81}$$

where A is the constant proper acceleration. Under the above coordinate transformation the line element defined in Eq.(2.73) becomes

$$ds^2 = -e^{2A\zeta}(dt^2 - dz^2). \tag{2.82}$$

Similarly the coordinates of LRW are also related to Minkowski coordinates by the following coordinate transformation

$$t = \frac{e^{A\bar{\zeta}}}{A} \sinh A\bar{\tau}, \; z = -\frac{e^{A\bar{\zeta}}}{A} \cosh A\bar{\tau}. \tag{2.83}$$

In terms of LRW coordinates, the line element defined in Eq.(2.73) becomes

$$ds^2 = -e^{2A\bar{\zeta}}(dt^2 - dz^2). \tag{2.84}$$

Comparing Eq.(2.82) and Eq.(2.84) with Eq.(2.73), we observe that the Rindler metric is conformally related to the Minkowski metric in $1+1$ dimensions. Thus the deformed Klein-Gordon equation in $1+1$ dimensional Rindler space-time takes the same form as that in $1+1$ dimensional Minkowski space-time. Thus deformed scalar field equations in RRW and LRW are given as

$$(\partial_\zeta^2 - \partial_\tau^2 - ia\partial_\tau\partial_\zeta^2)\hat{\phi}(\zeta,\tau) = 0, \tag{2.85}$$

and

$$(\partial_{\bar{\zeta}}^2 - \partial_{\bar{\tau}}^2 - ia\partial_{\bar{\tau}}\partial_{\bar{\zeta}}^2)\hat{\phi}(\bar{\zeta},\bar{\tau}) = 0, \tag{2.86}$$

respectively.

Again the solutions can be separated into left and right moving sectors. But here we consider only the left moving sectors. For the calculational simplications we define the null coordinates in RRW and LRW and they are given by

$$u = \tau - \zeta,\ v = \tau + \zeta, \tag{2.87}$$

and

$$\bar{v} = \bar{\tau} - \bar{\zeta},\ \bar{u} = \bar{\tau} + \bar{\zeta}, \tag{2.88}$$

respectively.

Substituting Eq.(2.81) in Eq.(2.77) and using Eq.(2.87) we get the relation between the null coordinates in Minkowski space-time and RRW as

$$U = -\frac{e^{-Au}}{A},\ V = \frac{e^{Av}}{A} \tag{2.89}$$

Similarly by Substituting Eq.(2.83) in Eq.(2.77) and using Eq.(2.88) we get the relation between the null coordinates in Minkowski space-time and LRW as

$$U = \frac{e^{A\bar{u}}}{A},\quad V = -\frac{e^{-A\bar{v}}}{A}. \tag{2.90}$$

Here we consider only the left moving sectors of the field. Thus the solutions in RRW and LRW are given using the left moving sector alone as

$$\hat{\phi}_+(v) = (1+a\alpha)\int \frac{dw}{\sqrt{4\pi w}}\left(\hat{a}^R_{+w}e^{iwv} + \hat{a}^{\dagger R}_{+w}e^{-iwv}\right) \tag{2.91}$$

and

$$\hat{\phi}_+(\bar{v}) = (1+a\alpha)\int \frac{dw}{\sqrt{4\pi w}}\left(\hat{a}^L_{+w}e^{iw\bar{v}} + \hat{a}^{\dagger L}_{+w}e^{-iw\bar{v}}\right), \tag{2.92}$$

respectively.

The vacuum state in the Rindler space-time is defined as $|0\rangle_R$. The operators $\hat{a}^R_{+w}$ and $\hat{a}^L_{+w}$ annihilate the vacuum in RRW and LRW, i.e., $\hat{a}^R_{+w}\,|0\rangle_R = 0$ and $\hat{a}^L_{+w}\,|0\rangle_R = 0$, respectively. These operators also satisfy the deformed oscillator algebra (in RRW and LRW separately) as

$$[\hat{a}^R_{+w}(k), \hat{a}^{\dagger R}_{+w}(k')] = (1 - 2a\alpha)\delta(k - k'), \tag{2.93}$$

and

$$[\hat{a}^L_{+w}(k), \hat{a}^{\dagger L}_{+w}(k')] = (1 - 2a\alpha)\delta(k - k'). \tag{2.94}$$

We now use the method of Bogoliubov coefficients to connect the frequency modes of RRW and LRW with that of the Minkowski space-time. These Bogoliubov coefficients, $\alpha^R_{wk}, \beta^R_{wk}, \alpha^L_{wk}$ and β^L_{wk}, are introduced through [29]

$$\frac{e^{-iwv}}{\sqrt{4\pi w}} = \int \frac{dk}{\sqrt{4\pi k}}(\alpha^R_{wk}e^{-ikV} + \beta^R_{wk}e^{ikV}), \tag{2.95}$$

$$\frac{e^{-iw\bar{v}}}{\sqrt{4\pi w}} = \int \frac{dk}{\sqrt{4\pi k}}(\alpha^L_{wk}e^{-ikV} + \beta^L_{wk}e^{ikV}), \tag{2.96}$$

In order to obtain α^R_{wk}, we first multiply Eq.(2.95) by $\frac{e^{ikV}}{2\pi}$ for $k > 0$ and then by integrating over V, we obtain

$$\alpha^R_{wk} = \frac{1}{2\pi}\sqrt{\frac{k}{w}} \int dV e^{ikV} e^{-i\omega v} \tag{2.97}$$

We rewrite $e^{-i\omega v}$ using Eq.(2.89) and thus Eq.(2.97) becomes

$$\alpha^R_{wk} = \frac{1}{2\pi}\sqrt{\frac{k}{w}} \int dV (AV)^{\frac{-iw}{A}} e^{ikV} \tag{2.98}$$

Now we take $V = \frac{ix}{k}$ and using the integral representation of Gamma function, obtain

$$\begin{aligned} \alpha^R_{wk} &= \frac{1}{2\pi}\sqrt{\frac{k}{w}} \int \frac{idx}{k}\Big(A\frac{ix}{k}\Big)^{\frac{-iw}{A}} e^{-x} \\ &= \frac{ie^{\frac{\pi w}{2A}}}{2\pi\sqrt{wk}}\Big(\frac{A}{k}\Big)^{-\frac{iw}{A}}\Gamma\Big(1 - \frac{iw}{A}\Big). \end{aligned} \tag{2.99}$$

For finding β^R_{wk}, we first multiply Eq.(2.95) with $\frac{e^{-ikV}}{2\pi}$ and then integrate the resulting expression over V. We then do a change of variable by $V = -\frac{ix}{k}$ and using the definition of Gamma function, we get

$$\beta^R_{wk} = -\frac{ie^{-\frac{\pi w}{2A}}}{2\pi\sqrt{wk}}\Big(\frac{A}{k}\Big)^{-\frac{iw}{A}}\Gamma\Big(1 - \frac{iw}{A}\Big) \tag{2.100}$$

We now multiply Eq.(2.95) with $\frac{e^{-ikV}}{2\pi}$ and integrate over V. After doing a change of variable $V = \frac{ix}{k}$ and using the definition of Gamma function we get α^L_{wk} as

$$\alpha^L_{wk} = -\frac{ie^{\frac{\pi w}{2A}}}{2\pi\sqrt{wk}}\Big(\frac{A}{k}\Big)^{\frac{iw}{A}}\Gamma\Big(1+\frac{iw}{A}\Big), \tag{2.101}$$

Next we multiply Eq.(2.96) with $\frac{e^{ikV}}{2\pi}$ and integrate over V. After doing a change of variable $V = \frac{-ix}{k}$ and using the definition of Gamma function we get β^L_{wk} as

$$\beta^L_{wk} = \frac{ie^{-\frac{\pi w}{2A}}}{2\pi\sqrt{wk}}\Big(\frac{A}{k}\Big)^{\frac{iw}{A}}\Gamma\Big(1+\frac{iw}{A}\Big). \tag{2.102}$$

From Eq.(2.99), Eq.(2.100), Eq.(2.101) and Eq.(2.102), we find that the Bogoliubov coefficients obey the following relations

$$\begin{aligned}\beta^L_{wk} &= -\,e^{-\frac{\pi w}{A}}\alpha^{R*}_{wk}, \\ \beta^R_{wk} &= -\,e^{-\frac{\pi w}{A}}\alpha^{L*}_{wk}\end{aligned} \tag{2.103}$$

We notice here that the Bogoliubov coefficients satisfy the same relations as that in the commutative case [29].

Now we replace the positive and negative frequency modes of the deformed field (i.e, e^{iwv} and e^{-iwv}) in RRW using Eq.(2.95) and the relations given in Eq.(2.103), we get

$$\begin{aligned}\hat{\phi}_+(v) = (1+a\alpha)\int dw\Big[&\hat{a}^R_{+w}\Big(\int\frac{dk}{\sqrt{4\pi k}}\Big(\alpha^{R*}_{wk}e^{ikV} - e^{-\frac{\pi w}{A}}\alpha^L_{wk}e^{-ikV}\Big)\Big) + \\ &\hat{a}^{\dagger R}_{+w}\Big(\int\frac{dk}{\sqrt{4\pi k}}\Big(\alpha^R_{wk}e^{-ikV} - e^{-\frac{\pi w}{A}}\alpha^{L*}_{wk}e^{ikV}\Big)\Big)\Big].\end{aligned} \tag{2.104}$$

Similarly by replacing the positive and negative frequency modes of the deformed field (i.e, $e^{iw\bar{v}}$ and $e^{-iw\bar{v}}$) in LRW using Eq.(2.96) and using the relations given in Eq.(2.103), we get

$$\begin{aligned}\hat{\phi}_+(\bar{v}) = (1+a\alpha)\int dw\Big[&\hat{a}^L_{+w}\Big(\int\frac{dk}{\sqrt{4\pi k}}\Big(\alpha^{L*}_{wk}e^{ikV} - e^{-\frac{\pi w}{A}}\alpha^R_{wk}e^{-ikV}\Big)\Big) + \\ &\hat{a}^{\dagger L}_{+w}\Big(\int\frac{dk}{\sqrt{4\pi k}}\Big(\alpha^L_{wk}e^{-ikV} - e^{-\frac{\pi w}{A}}\alpha^{R*}_{wk}e^{ikV}\Big)\Big)\Big].\end{aligned} \tag{2.105}$$

The complete solution to deformed field equation in the Rindler space-time is given (in terms of left moving sector alone) as

$$\hat{\phi}_+(v,\bar{v}) = \hat{\phi}_+(v) + \hat{\phi}_+(\bar{v}). \tag{2.106}$$

By subtituting Eq.(2.104) and Eq.(2.105) in Eq.(2.106) and after some rearrangement, we obtain

$$\hat{\phi}_+(v,\bar{v}) = (1+a\alpha)\int \frac{dk}{\sqrt{4\pi k}}\Bigg[\int dw\,\left(\alpha^{R*}_{wk}\left(\hat{a}^R_{+w} - e^{-\frac{\pi w}{A}}\hat{a}^{\dagger L}_{+w}\right) + \alpha^{L*}_{wk}\left(\hat{a}^L_{+w} - e^{-\frac{\pi w}{A}}\hat{a}^{\dagger R}_{+w}\right)\right)e^{ikV} + \int dw\,\left(\alpha^{R}_{wk}\left(\hat{a}^{\dagger R}_{+w} - e^{-\frac{\pi w}{A}}\hat{a}^{L}_{+w}\right) + \alpha^{L}_{wk}\left(\hat{a}^{\dagger L}_{+w} - e^{-\frac{\pi w}{A}}\hat{a}^{R}_{+w}\right)\right)e^{-ikV}\Bigg]. \tag{2.107}$$

e^{ikV} and e^{-ikV} are the positive and negative frequency modes of the deformed scalar field in $1+1$ dimensional Minkowski space-time. By comparing Eq.(2.105) with Eq.(2.79), we observe that both represent the deformed scalar field, decomposed in terms of Minkowski modes in $1+1$ dimension. Thus from the above we find that the operators $(\hat{a}^R_{+w} - e^{-\frac{\pi w}{A}}\hat{a}^{\dagger L}_{+w})$ and $(\hat{a}^L_{+w} - e^{-\frac{\pi w}{A}}\hat{a}^{\dagger R}_{+w})$ annihilate the Minkowski vacuum. Thus we have

$$\left(\hat{a}^L_{+w} - e^{-\frac{\pi w}{A}}\hat{a}^{\dagger R}_{+w}\right)|0\rangle_M = 0, \tag{2.108}$$

$$\left(\hat{a}^R_{+w} - e^{-\frac{\pi w}{A}}\hat{a}^{\dagger L}_{+w}\right)|0\rangle_M = 0. \tag{2.109}$$

Now we multiply Eq.(2.108) and Eq.(2.109) with their hermitian conjugates and by using Eq.(2.93) and Eq.(2.94) in the resulting expression, get

$${}_M\langle 0|\,\hat{a}^{\dagger R}_{+w}\hat{a}^R_{+w}\,|0\rangle_M = e^{-\frac{2\pi w}{A}}(1-2a\alpha) + e^{-\frac{2\pi w}{A}}\,{}_M\langle 0|\,\hat{a}^{\dagger L}_{+w}\hat{a}^L_{+w}\,|0\rangle_M\,, \tag{2.110}$$

and

$${}_M\langle 0|\,\hat{a}^{\dagger L}_{+w}\hat{a}^L_{+w}\,|0\rangle_M = e^{-\frac{2\pi w}{A}}(1-2a\alpha) + e^{-\frac{2\pi w}{A}}\,{}_M\langle 0|\,\hat{a}^{\dagger R}_{+w}\hat{a}^R_{+w}\,|0\rangle_M\,. \tag{2.111}$$

By solving the above equations simultaneously we obtain the vacuum expectation value of the Rindler number operators in Minkowski vacuum, i.e., ${}_M\langle 0|\,\hat{\mathcal{N}}_R\,|0\rangle_M$, where $\hat{\mathcal{N}}_R = \hat{a}^{\dagger L}_{+w}\hat{a}^L_{+w} = \hat{a}^{\dagger R}_{+w}\hat{a}^R_{+w}$. Thus we get

$${}_M\langle 0|\,\hat{\mathcal{N}}_R\,|0\rangle_M = \frac{1-2a\alpha}{e^{\frac{2\pi w}{A}}-1}. \tag{2.112}$$

This shows that the Unruh temperature is $T_U = \frac{A}{2\pi}$. Due to κ-deformation, the vacuum expectation value of the number operator corresponding to Rindler particle gets modified by a $(1-2a\alpha)$ factor. This modification has been contributed by the deformed algebra associated with the creation and annihilation operators of the left and right Rindler wedges, i.e., Eq.(2.93) and Eq.(2.94). Here we find that the Unruh temperature associated with vacuum expectation value of the number operator is unaffected by the κ-deformation and this is in contrast with the results obtained in [23, 24]. It is to be noted that these observations are valid only up to first order in a.

2.5 Conclusions

In this chapter, we have generalised the quantisation procedure discussed in [7, 8] to the κ-deformed space-time. This method enable us to quantise the κ-deformed field theories from their equations of motion alone, without referring to its Lagrangian. This is desirable as only the equation of motion of the κ-deformed field is uniquely defined, but not the Lagrangian. The quantisation is achieved by constructing the operators such as $\hat{\Lambda}(\partial)$ and $\hat{d}(\partial)$ (Klein-Gordon divisor) in the κ-space-time. Further, this method is also used to analyse the symmetries associated with the field theories in the κ-space-time. These deformed conserved currents are constructed form their equations of motion with the help of $\hat{\Gamma}_\mu(\partial)$ operator in the κ-deformed space-time.

We have begun the quantisation of κ-deformed scalar field from its deformed equations of motion (valid up to first order in a) derived from the quadratic Casimir of the undeformed κ-Poincare algebra. From the solutions of the equations of motion and Klein-Gordon divisor, we have derived the deformed unequal-time commutation relation (valid up to first order in a) between the deformed field operator and its adjoint, by assuming the standard commutation relation for the oscillator algebra. By assuming the unequal-time commutation relation between deformed field and its adjoint to be undeformed, we obtain a deformed oscillator algebra, valid up to first order in a. This deformed oscillator algebra is different from the one derived in [22].

We have also constructed the conserved currents (valid up to first order in a) corresponding to the translational as well as the Lorentz symmetry for the κ-deformed scalar field. It has been shown that the energy-momentum tensor of the κ-deformed scalar field is no longer symmetric in its indices (even at the first order in a). Similar non-symmetric energy momentum tensor has also been obtained for the scalar fields in Moyal space-time [30].

Here we have studied the effects of the κ-deformation in the Unruh effect using the massless κ-deformed Klein-Gordon equation in the $1+1$ dimension and its associated deformed oscillator algebra. By using the method of Bogoliubov transformation we have calculated the expectation value of the (Rindler) number operator in the Minkowski vacuum and obtained the thermal distribution at $T_U = \frac{A}{2\pi}$. This distribution gets an overall modification and this deformation factor appearing in the modification of Unruh effect is exactly the same as that appearing in the deformed oscillator algebra (valid up to first order in a). An another observation is that the Unruh temperature of the thermal bath remains unaffected under the κ deformation (valid up to first order in a). It will be quite interesting to check whether this Unruh temperature remains the same even if we consider all the higher order terms in a.

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Chapter 3

Quantisation of κ-deformed Dirac field

3.1 Introduction

Various forms of deformed Dirac equation have been constructed in the κ-deformed space-time and different properties of the κ-deformed Dirac field satisfying such modified equations of motion have been studied extensively in the recent times [1–7]. The κ-deformed Dirac equation has been obtained in [1] such that its square would give the deformed Klein-Gordon equation. But this deformed Dirac equation is not invariant under the κ-Poincare algebra. In [2], κ-deformed Dirac equation, invariant under the κ-Poincare algebra has been constructed in such a way that its square gives the second Casimir of κ-Poincare algebra (which is the square of deformed Pauli-Lubanski vector). A modified Dirac equation consistent with the doubly special theory has been constructed in [3] and it has further been shown that this modified Dirac equation arises as the deformed Dirac equation in the κ-Minkowski space-time. In [4], Dirac equations compatible with doubly special relativity, describing particles and anti-particles, were constructed and these equations are shown to be different from each other. The κ-deformed Dirac equation has been constructed in [5] by replacing the usual derivative in Dirac equation with the Dirac derivative corresponding to the κ-Poincare algebra. In [6], deformed Dirac equation, invariant under the undeformed κ-Poincare algebra has been constructed using the Dirac derivative of the undeformed κ-Poincare algebra. In [7], κ-deformed Dirac equation (invariant under undeformed κ-Poincare algebra) has been constructed from the undeformed κ-Lorentz transformation. In all these works, the deformed Dirac equations have been derived without reference to the Lagrangian.

Therefore it is crucial to study the quantisation of the κ-Dirac field using its deformed equations of motion.

Here we quantise the κ-Dirac field from its deformed equation of motion, using the quantisation procedure discussed in the chapter 2 (see [8–11] for more details). We start with the κ-deformed Dirac equation [6], which is invariant under the undeformed κ-Poincare algebra and construct the corresponding Klein-Gordon divisor in the κ space-time. By assuming a usual fermionic oscillator algebra between the creation and annihilation operators, we derive a deformed anti-commutation relation (valid up to first order in the non-commutative parameter) between the deformed Dirac field and its adjoint, at unequal times. We also derive a deformed oscillator algebra (valid up to first order in a) by assuming the unequal-time anti-commutation relation between the κ-Dirac field and its adjoint to be undeformed. This deformed oscillator algebra differs from the one derived in [12], where it has been derived using the twisted flip operator compatible with the undeformed κ-Poincare algebra [13, 14]. By analysing the translational and Lorentz symmetries associated with the κ-Dirac field, we derive the energy-momentum tensor and κ-Lorentz generators corresponding to the deformed Dirac field. We also derive the number operator corresponding to the κ-deformed Dirac field and show that it contains a mass-dependent correction term.

Different aspects of discrete symmetry have been studied in the κ-Minkowski space-time. It has been shown in [6] that κ-Dirac field is not symmetric under the charge conjugation. The κ-deformed Dirac equation, obtained from the κ-Poincare Hopf algebra has been shown to break charge conjugation as well as time-reversal symmetry [15]. It has been shown in [16] that the κ-deformed Duffin-Kemmer-Petiau equation also violates the charge conjugation symmetry. In [17], the discrete symmetries associated with the κ-deformed complex scalar field have been discussed in detail and it has been shown that under the charge conjugation symmetry, a particle, associated with κ-complex scalar field, transforms into an antiparticle with different momenta. The quantisation procedure discussed in [11, 18, 19] provides a unique way of constructing the conserved currents associated with the discrete symmetries, from their equations of motion. We derive the conserved currents corresponding to the discrete symmetries of the κ-deformed Dirac field using the equations of motion itself. We show that charge conjugation is not a symmetry of the κ-deformed Dirac field, even at the first order in the deformation factor.

This chapter is organised in the following manner. In sec.3.2, we study the quantisation of κ-Dirac field (valid up to first order in a) using the quantisation procedure discussed in sec.2.2 (see chapter 2). We then derive the deformed anti-commutation relations between the κ-Dirac field and its adjoint by assuming the usual form of the fermionic oscillator algebra. We then obtain the deformed oscillator algebra for the κ-Dirac field, by demanding the anti-commutation relations between the κ-Dirac field and its adjoint to be undeformed. In sec.3.3, we analyse the continuous symmetries and construct the energy-momentum tensor and Lorentz generator of the κ-Dirac field. In subsec.3.3.1, we study the discrete symmetries associated with the κ-deformed Dirac field and obtain the corresponding conserved currents from the deformed equations of motion. By using the consistency condition, for the conserved current, of Takahashi-Umezawa formalism, we show that the deformed Dirac equation is not symmetric under the charge conjugation, even up to the first order in the deformation parameter. Finally, in sec.3.4, we give the concluding remarks.

3.2 Quantisation of κ-deformed Dirac field

In this section, we present the derivation of the κ-deformed Dirac equation, valid up to first order in the deformation parameter a, from the Dirac derivative defined in Eq.(1.18) (of chapter 1). We then quantise the deformed Dirac field satisfying the κ-deformed Dirac equation, by applying the Takahashi-Umezawa quantisation procedure, summarised in chapter 2 [10, 11]. We then demand that the anti-commutator between the field and its adjoint to be undeformed and show that this leads to a deformed oscillator algebra, valid up to first order in a.

The κ-deformed Dirac equation (invariant under the undeformed κ-Poincare algebra) is constructed by replacing the commutative derivative with Dirac derivative [6] and is given by

$$\Big(i\gamma^\mu D_\mu + m\Big)\hat{\psi}(x) = 0, \tag{3.1}$$

where $\hat{\psi}(x)$ is the κ-Dirac field. The explicit form of the Dirac derivative D_μ is given as (see chapter 1 for details)

$$D_0 = \partial_0 \frac{\sinh A}{A} + ia\partial_k^2 \frac{e^{-A}}{2\varphi^2},\ D_i = \partial_i \frac{e^{-A}}{\varphi}, \tag{3.2}$$

where $A = ia\partial_0$.

Similarly, the κ-deformed conjugate Dirac equation can be written as

$$\bar{\hat{\psi}}(x)\Big(i\overleftarrow{D}_\mu\gamma^\mu - m\Big) = 0. \tag{3.3}$$

The product of Eq.(3.1) and Eq.(3.3) gives the κ-deformed Klein-Gordon equation given in Eq.(2.29) (see chapter 2).

From now onwards we choose the realisation $\varphi = e^{-A}$ [20–22] as in chapter 2. Thus by substituting the explicit form of Dirac derivative, given above, in Eq.(3.1) and Eq.(3.3) and expanding it up to the first order in a, we get the κ-deformed Dirac equation and its conjugate equation, valid up to first order in a, as

$$\Big(i\gamma^\mu\partial_\mu + \frac{a}{2}\gamma^0\partial_i^2 + m\Big)\hat{\psi}(x) = 0, \tag{3.4}$$

and

$$\bar{\hat{\psi}}(x)\Big(i\overleftarrow{\partial}_\mu\gamma^\mu + \frac{a}{2}\gamma^0\overleftarrow{\partial}_i^2 - m\Big) = 0, \tag{3.5}$$

respectively. The product of Eq.(3.4) and Eq.(3.5) gives the deformed Klein-Gordon equation, valid up to first order in a, (see Eq.(2.31) (here we have used $[\gamma^\mu,\gamma^\nu]_+ = -2\eta^{\mu\nu}$), i.e.,

$$\Big(i\gamma^\mu\partial_\mu + \frac{a}{2}\gamma^0\partial_i^2 + m\Big)\Big(i\gamma^\nu\partial_\nu + \frac{a}{2}\gamma^0\partial_i^2 - m\Big) = \partial_\mu\partial^\mu - ia\partial_0\partial_i^2 - m^2. \tag{3.6}$$

In chapter 2, we have generalised the quantisation procedure discussed in [10, 11], to the κ-deformed space-time. Here we will use the same procedure to quantise the deformed Dirac field. From Eq.(2.32) (of chapter 2), we have

$$\hat{\Lambda}(\partial)\hat{d}(\partial) = \partial_\mu\partial^\mu - ia\partial_0\partial_i^2 - m^2. \tag{3.7}$$

Comparing Eq.(3.6) with Eq.(3.7), we get $\hat{\Lambda}(\partial)$ and $\hat{d}(\partial)$ operators corresponding to κ-deformed Dirac field as

$$\hat{\Lambda}(\partial) = i\gamma^\mu\partial_\mu + \frac{a}{2}\gamma^0\partial_i^2 + m, \tag{3.8}$$

and

$$\hat{d}(\partial) = i\gamma^\mu\partial_\mu + \frac{a}{2}\gamma^0\partial_i^2 - m. \tag{3.9}$$

Note that in the commutative limit $a \to 0$, Eq.(3.8) and Eq.(3.9) reduce to Eq.(2.6) and Eq.(2.10), respectively.

Using $\hat{\Lambda}(\partial)$ in Eq.(3.8), the κ-deformed Dirac equation can be written as

$$\hat{\Lambda}(\partial)\hat{\psi}(x) = 0. \tag{3.10}$$

The κ-deformed Dirac field operator and its adjoint are expressed in terms of Fourier components as

$$\hat{\psi}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \sum_{s=1,2}\Big(\hat{a}_s(p)\hat{u}_s(p)e^{-ipx} + \hat{b}_s^\dagger(p)\hat{v}_s(p)e^{ipx}\Big) \tag{3.11}$$

and

$$\bar{\hat{\psi}}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \sum_{s=1,2} \Big(\hat{a}_s^\dagger(p)\bar{\hat{u}}_s(p)e^{ipx} + \hat{b}_s(p)\bar{\hat{v}}_s(p)e^{-ipx}\Big), \tag{3.12}$$

where $E_p^2 = p^2 + m^2$ [1], $\hat{a}_s$, $\hat{b}_s$ and $\hat{a}_s^\dagger$, $\hat{a}_s^\dagger$ are annihilation and creation operators, respectively. Note that in the above relations, $\hat{u}_s(p)e^{-ipx}$ and $\hat{v}_s(p)e^{ipx}$ satisfy the κ-deformed Dirac equation, i.e., $\hat{\Lambda}(\partial)\big(\hat{u}_s(p)e^{-ipx}\big) = 0$ and $\hat{\Lambda}(\partial)\big(\hat{v}_s(p)e^{ipx}\big) = 0$. We solve these equations perturbatively by expanding $\hat{u}(p)$ and $\hat{v}(p)$ keeping terms up to first order in a, obtaining

$$\begin{aligned} \hat{u}(p) &= u^{(0)}(p) + a\alpha u^{(1)}(p), \\ \hat{v}(p) &= v^{(0)}(p) + a\alpha v^{(1)}(p), \end{aligned} \tag{3.13}$$

where the parameter α has the dimension of length inverse. Substituting Eq.(3.13) in the expressions $\hat{\Lambda}(\partial)\big(\hat{u}_s(p)e^{-ipx}\big) = 0$ and $\hat{\Lambda}(\partial)\big(\hat{v}_s(p)e^{ipx}\big) = 0$, we get

$$\Big(i\gamma^\mu\partial_\mu + \frac{a}{2}\gamma^0\partial_i^2 + m\Big)\big(u^{(0)}(p) + a\alpha u^{(1)}(p)\big)e^{-ipx} = 0, \tag{3.14}$$

and

$$\Big(i\gamma^\mu\partial_\mu + \frac{a}{2}\gamma^0\partial_i^2 + m\Big)\big(v^{(0)}(p) + a\alpha v^{(1)}(p)\big)e^{ipx} = 0. \tag{3.15}$$

By separating the a independent and a dependent terms of Eq.(3.14), we get two equations and they are

$$\Big(i\gamma^\mu\partial_\mu + m\Big)u^{(0)}(p)e^{-ipx} = 0, \tag{3.16}$$

$$\Big(i\alpha\gamma^\mu\partial_\mu + \alpha m\Big)u^{(1)}(p)e^{-ipx} = -\frac{1}{2}\gamma^0\partial_i^2 u^{(0)}e^{-ipx}. \tag{3.17}$$

Now we solve Eq.(3.17) using Green's function method and the solution is given as

$$u^{(1)}e^{-ipx} = u^{(0)}e^{-ipx} + \int d^4x' G(x-x')j(x'), \tag{3.18}$$

where $G(x-x')$ satisfies $\Big(i\gamma^\mu\partial_\mu + m\Big)G(x-x') = \delta^4(x-x')$ and the source term is defined as $j(x) = \frac{1}{2\alpha}\gamma^0 p_i^2 u^{(0)}e^{-ipx}$. The the explict form of the Green's function is given as

$$G(x-x') = -\int \frac{d^4p}{(2\pi)^4}\frac{\not{p} - m}{p^2 + m^2}e^{-ip(x-x')}. \tag{3.19}$$

We now rewrite the above Green's function using that for the scalar field, $G_{scalar}(x-x')$ (see Eq.(2.43 of chapter 2), as

$$G(x-x') = (i\not{\partial} - m)G_{scalar}(x-x'). \tag{3.20}$$

[1] In general one need to use the κ-deformed dispersion relation (i.e., $E_p^2 = p^2(1 + aE_p) + m^2$, valid up to first order in a). In order to simplify the calculations, we use the commutative dispersion relation. But here we have included the κ-deformed corrections through deformed spinors $\hat{u}(p)$, $\hat{v}(p)$ and through deformed creation and annihilation operators respectively

Using Eq.(3.20) in Eq.(3.18), we get the second term on RHS of Eq.(3.18) as

$$
\begin{aligned}
&\int d^4x' G(x-x')j(x') \\
=& -i\int \frac{d^4x'}{(2\pi)^3}\int \frac{d^3p'}{2E_{p'}}\Big((i\not{\partial}' - m)\Big[\theta(t-t')e^{-ip'(x-x')} - \theta(t'-t)e^{ip'(x-x')}\Big]\Big)\frac{p_i^2}{2\alpha}\gamma^0 u^{(0)}(p)e^{-ipx'} \\
=& -\frac{i}{2\alpha}\int \frac{d^4x'}{(2\pi)^3}\int \frac{d^3p'}{2E_{p'}}\Big[-i\delta(t-t')e^{-ip'(x-x')} - i\delta(t'-t)e^{ip'(x-x')}\Big]p_i^2 u^{(0)}(p)e^{-ipx'} \\
&+ \frac{i}{2\alpha}\int \frac{d^4x'}{(2\pi)^3}\int \frac{d^3p'}{2E_{p'}}\Big[(\not{p}' + m)\theta(t-t')e^{-ip'x}e^{-i(p-p')x'} \\
&+ (\not{p}' - m)\theta(t'-t)e^{ip'x}e^{-i(p+p')x'}\Big]\gamma^0 p_i^2 u^{(0)}(p) \\
=& -\frac{1}{2\alpha}\int \frac{d^3p'}{2E_{p'}}\Big[e^{-ip'\cdot x}e^{iE_p t}\int \frac{d^3x'}{(2\pi)^3}e^{i(p'-p)\cdot x'} + e^{ip'\cdot x}e^{iE_p t}\int \frac{d^3x'}{(2\pi)^3}e^{-i(p'+p)\cdot x'}\Big]p_i^2 u^{(0)}(p) \\
&+ \frac{i\pi}{\alpha}\int \frac{d^3p'}{2E_{p'}}\Big[(\not{p}' + m)\theta(t-t')e^{-ip'x}\int \frac{d^4x'}{(2\pi)^4}e^{-i(p-p')x'} \\
&+ (\not{p}' - m)\theta(t'-t)e^{ip'x}\int \frac{d^4x'}{(2\pi)^4}e^{-i(p+p')x'}\Big]\gamma^0 p_i^2 u^{(0)}(p).
\end{aligned}
\tag{3.21}
$$

Using the integral representation for step function, i.e., $\Theta(t-t') = \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int dk \frac{e^{ik(t-t')}}{k-i\epsilon}$ and the identity, $\delta(x-a)f(x) = \delta(x)f(a)$, in the last two terms on RHS of Eq.(3.21), we get

$$\int d^4x' G(x-x')j(x') = -\frac{1}{2\alpha}\frac{p_i^2}{E_p}u^{(0)}(p). \tag{3.22}$$

Substituting Eq.(3.22) in Eq.(3.18), we obtain

$$u^{(1)}(p) = u^{(0)}(p) - \frac{1}{2\alpha}\frac{p_i^2}{E_p}u^{(0)}(p). \tag{3.23}$$

Now by substituting Eq.(3.23) in Eq.(3.13), we get $\hat{u}(p)$ as

$$\hat{u}(p) = \left(1 + a\alpha - \frac{a}{2}\frac{p_i^2}{E_p}\right)u^{(0)}(p). \tag{3.24}$$

Next we consider Eq.(3.15) and we separate it into a independent and a dependent coefficient terms, we get

$$\Big(i\gamma^\mu\partial_\mu + m\Big)v^{(0)}(p)e^{ipx} = 0, \tag{3.25}$$

$$\Big(i\alpha\gamma^\mu\partial_\mu + \alpha m\Big)v^{(1)}(p)e^{ipx} + \frac{1}{2}\gamma^0\partial_i^2 v^{(0)}e^{ipx} = 0. \tag{3.26}$$

Now we follow the similar steps as we did to find out $\hat{u}(p)$. Thus following those steps, we obtain $\hat{v}(p)$ as

$$\hat{v}(p) = \left(1 + a\alpha - \frac{a}{2}\frac{p_i^2}{E_p}\right)v^{(0)}(p). \tag{3.27}$$

We now consider the κ-deformed version of the unequal time anti-commutation relation between κ-deformed Dirac field and its adjoint. Thus Eq.(2.13) becomes

$$[\hat{\psi}(x), \hat{\bar{\psi}}(x')]_+ = i\hat{d}(\partial)\hat{\Delta}(x-x'). \tag{3.28}$$

We assume that the deformed creation and annihilation operators satisfy the usual anti-commutation relations,

$$\begin{aligned}
&[\hat{a}_s(p), \hat{a}_{s'}(p')]_+ = [\hat{b}_s(p), \hat{b}_{s'}(p')]_+ = [\hat{a}^\dagger_s(p), \hat{a}^\dagger_{s'}(p')]_+ = [\hat{b}^\dagger_s(p), \hat{b}^\dagger_{s'}(p')]_+ = 0, \\
&[\hat{a}_s(p), \hat{a}^\dagger_{s'}(p')]_+ = [\hat{b}_s(p), \hat{b}^\dagger_{s'}(p')]_+ = \delta_{ss'}\delta^3(p-p').
\end{aligned} \tag{3.29}$$

Substituting Eq.(3.24) and Eq.(3.27) in Eq.(3.11) and Eq.(3.13) and using this in Eq.(3.28), we get

$$\begin{aligned}
\hat{\psi}(x), \bar{\hat{\psi}}(x')t_+ = \int \frac{d^3p\, d^3p'}{(2\pi)^3\sqrt{2E_p 2E_{p'}}} \sum_{s,s'=1,2} \Big[&[\hat{a}_s(p), \hat{a}^\dagger_{s'}(p')]_+ \Big(\tilde{u}^{(0)}_s(p,x)\bar{\tilde{u}}^{(0)}_{s'}(p',x') \\
&+ a\alpha\big(\tilde{u}^{(0)}_s(p,x)\bar{\tilde{u}}^{(1)}_{s'}(p',x') + \tilde{u}^{(1)}_s(p,x)\bar{\tilde{u}}^{(0)}_{s'}(p',x')\big)\Big) \\
&+ [\hat{b}_s(p), \hat{b}^\dagger_{s'}(p')]_+ \Big(\tilde{v}^{(0)}_s(p,x)\bar{\tilde{v}}^{(0)}_{s'}(p',x') + a\alpha\big(\tilde{v}^{(0)}_s(p,x)\bar{\tilde{v}}^{(1)}_{s'}(p',x') \\
&+ \tilde{v}^{(1)}_s(p,x)\bar{\tilde{v}}^{(0)}_{s'}(p',x')\big)\Big)\Big].
\end{aligned} \tag{3.30}$$

(Note that in the above, we have defined $\tilde{u}(p,x) = u(p)e^{-ipx}$ and $\tilde{v}(p,x) = v(p)e^{ipx}$). Substituting the expression for undeformed oscillator algebra, i.e, Eq.(3.29) in Eq.(3.30), we get

$$\begin{aligned}
\hat{\psi}(x), \bar{\hat{\psi}}(x')t_+ = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_{s=1,2} \Big(&u^{(0)}_s(p)\bar{u}^{(0)}_s(p)e^{-ip(x-x')} + v^{(0)}_s(p)\bar{v}^{(0)}_s(p)e^{ip(x-x')} \\
&+ a\alpha\Big(u^{(0)}_s(p)\bar{u}^{(1)}_s(p) + u^{(1)}_s(p)\bar{u}^{(0)}_s(p)\Big)e^{-ip(x-x')} \\
&+ a\alpha\Big(v^{(0)}_s(p)\bar{v}^{(1)}_s(p) + v^{(1)}_s(p)\bar{v}^{(0)}_s(p)\Big)e^{ip(x-x')}\Big).
\end{aligned} \tag{3.31}$$

Using Eq.(3.24) and Eq.(3.27), we evaluate $u^{(0)}_s(p)\bar{u}^{(1)}_s(p)$, $u^{(1)}_s(p)\bar{u}^{(0)}_s(p)$, $v^{(0)}_s(p)\bar{v}^{(1)}_s(p)$ and $v^{(1)}_s(p)\bar{v}^{(0)}_s(p)$ as

$$u^{(0)}_s(p)\bar{u}^{(1)}_s(p) = u^{(0)}_s(p)\bar{u}^{(0)}_s(p)\left(1 - \frac{1}{2\alpha}\frac{p_i^2}{E_p}\right), \tag{3.32}$$

$$u^{(1)}_s(p)\bar{u}^{(0)}_s(p) = u^{(0)}_s(p)\bar{u}^{(0)}_s(p)\left(1 - \frac{1}{2\alpha}\frac{p_i^2}{E_p}\right), \tag{3.33}$$

$$v^{(0)}_s(p)\bar{v}^{(1)}_s(p) = v^{(0)}_s(p)\bar{v}^{(0)}_s(p)\left(1 - \frac{1}{2\alpha}\frac{p_i^2}{E_p}\right), \tag{3.34}$$

$$v_s^{(1)}(p)\bar{v}_s^{(0)}(p) = v_s^{(0)}(p)\bar{v}_s^{(0)}(p)\bigg(1 - \frac{1}{2\alpha}\frac{p_i^2}{E_p}\bigg). \tag{3.35}$$

Substituting Eq.(3.32), Eq.(3.33), Eq.(3.34) and Eq.(3.35) in Eq.(3.31) and using the completeness relation as that in the commutative space-time, i.e., $\sum_{s=1,2} u_s^{(0)}\bar{u}_s^{(0)} = \not{p}-m$ and $\sum_{s=1,2} v_s^{(0)}\bar{v}_s^{(0)} = \not{p}+m$, we get

$$\begin{aligned}[\hat{\psi}(x), \bar{\hat{\psi}}(x')]_+ =& i\big(1+2a\alpha\big)(i\not{\partial} - m)\Delta(x-x') \\ &- 2a\big(i\not{\partial} - m\big)\int \frac{d^3p}{(2\pi)^3}\frac{p_i^2}{(2E_p)^2}\Big(e^{-ip(x-x')} - e^{ip(x-x')}\Big).\end{aligned} \tag{3.36}$$

We observe that the first terms of Eq.(3.32), Eq.(3.33), Eq.(3.34) and Eq.(3.35) add up to contribute the $i2a\alpha\big(i\not{\partial}-m\big)\Delta(x-x')$ term present in the RHS of Eq.(3.36). Similarly the second terms of the Eq.(3.32), Eq.(3.33), Eq.(3.34) and Eq.(3.35) add up to give the α independent correction term in Eq.(3.36). The last term in the RHS of Eq.(3.36) vanishes as the integrand is an odd function in p.

Now we explicitly calculate the RHS of Eq.(3.28) as

$$\begin{aligned} i\hat{d}(\partial)\hat{\Delta}(x-x') =& i\Big(i\not{\partial} + \frac{a}{2}\gamma^0\partial_i^2 - m\Big)\Big(\Delta(x-x') + a\Delta^{(1)}(x-x')\Big) \\ =& i(i\not{\partial} - m)\Delta(x-x') + ia(i\not{\partial} - m)\Delta^{(1)}(x-x') + \\ & i\frac{a}{2}\gamma^0\int \frac{d^3p}{(2\pi)^3 2E_p}p_i^2\Big(e^{-ip(x-x')} - e^{ip(x-x')}\Big).\end{aligned} \tag{3.37}$$

$\Delta^{(1)}(x-x')$ in the above equation represents the first order correction to $\hat{\Delta}(x-x')$ and by comparing the RHS of Eq.(3.36) with the RHS of Eq.(3.37) we find that $\Delta^{(1)}(x-x') = 2\alpha\Delta(x-x')$, for the undeformed oscillator algebra given in Eq.(3.29). Here we find that the last integral on the RHS of Eq.(3.37) vanishes as the integrand is an odd function in p. We observe that this $\hat{\Delta}(x-x')$ obtained is same as that of the κ-deformed Klein-Gordon equation given in Eq.(2.54) (of chapter 2). The anti-commutation relation between κ-deformed Dirac field operator and its adjoint, valid up to first order in a, becomes

$$[\hat{\psi}(x), \bar{\hat{\psi}}(x')]_+ = i\big(1+2a\alpha\big)(i\not{\partial} - m)\Delta(x-x'). \tag{3.38}$$

Now let us assume that the anti-commutation relation between creation and annihilation operators is deformed such that this deformation is valid up to first order in a. Thus we consider this deformed oscillator algebra as

$$\begin{aligned}&[\hat{a}_s(p), \hat{a}_{s'}(p')]_+ = [\hat{b}_s(p), \hat{b}_{s'}(p')]_+ = [\hat{a}_s^\dagger(p), \hat{a}_{s'}^\dagger(p')]_+ = [\hat{b}_s^\dagger(p), \hat{b}_{s'}^\dagger(p')]_+ = 0, \\ &[\hat{a}_s(p), \hat{a}_{s'}^\dagger(p')]_+ = [\hat{b}_s(p), \hat{b}_{s'}^\dagger(p')]_+ = h(a)\delta_{ss'}\delta^3(p-p').\end{aligned} \tag{3.39}$$

In the above equation note that $h(a)$ is an arbitrary function of a. Using this expression for the deformed oscillator algebra in Eq.(3.30) and repeating the above steps, we get an unequal time anti-commutation relation between κ-deformed Dirac field and its adjoint, valid up to first order in a, as

$$[\hat{\psi}(x), \bar{\hat{\psi}}(x')]_+ = ih(a)\big(1 + 2a\alpha\big)(i\partial\!\!\!/ - m)\Delta(x - x'). \tag{3.40}$$

Now we choose an explicit form for $h(a)$, such that the anti-commutation relation between κ-deformed Dirac field and its adjoint becomes undeformed (valid up to first order in a) and this gives $h(a) = 1-2a\alpha$. Therefore the undeformed anti-commutation relation between κ-deformed Dirac field and its adjoint becomes

$$[\hat{\psi}(x), \bar{\hat{\psi}}(x')]_+ = i(i\partial\!\!\!/ - m)\Delta(x - x'). \tag{3.41}$$

Thus the κ-deformed anti-commutation relations between deformed creation and annihilation operators are given by

$$\begin{aligned} &[\hat{a}_s(p), \hat{a}_{s'}(p')]_+ = [\hat{b}_s(p), \hat{b}_{s'}(p')]_+ = [\hat{a}^\dagger_s(p), \hat{a}^\dagger_{s'}(p')]_+ = [\hat{b}^\dagger_s(p), \hat{b}^\dagger_{s'}(p')]_+ = 0, \\ &[\hat{a}_s(p), \hat{a}^\dagger_{s'}(p')]_+ = [\hat{b}_s(p), \hat{b}^\dagger_{s'}(p')]_+ = (1 - 2a\alpha)\delta_{ss'}\delta^3(p - p'). \end{aligned} \tag{3.42}$$

We find that the deformation factor (valid up to first order in a) present in the deformed oscillator algebra of the κ-deformed Dirac field is exactly the same as that of the deformation factor present in the deformed oscillator algebra of κ-deformed Klein-Gordon field [23].

3.3 Conserved currents for κ-deformed Dirac field

In this section, we construct the conserved currents corresponding to translational and Lorentz symmetry of the κ-deformed Dirac field. We also derive the number operator corresponding to the deformed Dirac field from the global phase transformation symmetry. Further, we also obtain the conserved currents corresponding to discrete symmetries. All these conserved currents (valid up to first order in a) are obtained by constructing the deformed Gamma operator, $\hat{\Gamma}_\mu(\partial, -\overleftarrow{\partial})$ (see Eq.(2.16) of chapter 2) corresponding to the deformed Dirac equation.

We obtain the $\hat{\Gamma}_\mu(\partial, -\overleftarrow{\partial})$ operator (valid up to first order in a) corresponding to κ-deformed Dirac equation by substituting Eq.(3.8) in Eq.(2.16), as

$$\hat{\Gamma}^\mu(\partial, -\overleftarrow{\partial}) = i\gamma^\mu + \frac{a}{2}\gamma^0\delta^{\mu i}(\partial_i - \overleftarrow{\partial}_i). \tag{3.43}$$

From the above equation, we find that the second term in the RHS is contributed by a dependent term of the κ-deformed Dirac equation. The above equation reduces to the commutative expression (see Eq.(2.18)) in the limit $a \to 0$.

In order to check whether the $\hat{\Gamma}^{\mu}(\partial, -\overleftarrow{\partial})$ operator defined above satisfies the identity (see Eq.(2.19))

$$\left(\partial_{\mu} + \overleftarrow{\partial}_{\mu}\right)\hat{\Gamma}^{\mu}(\partial, -\overleftarrow{\partial}) = \hat{\Lambda}(\partial) - \hat{\Lambda}(-\overleftarrow{\partial}), \tag{3.44}$$

we calculate

$$\left(\partial_{\mu} + \overleftarrow{\partial}_{\mu}\right)\hat{\Gamma}^{\mu}(\partial, -\overleftarrow{\partial}) = i\gamma^{\mu}\left(\partial_{\mu} + \overleftarrow{\partial}_{\mu}\right) + \frac{a}{2}\gamma^{0}\delta^{\mu i}\left(\partial_{\mu} + \overleftarrow{\partial}_{\mu}\right)\left(\partial_{i} - \overleftarrow{\partial}_{i}\right). \tag{3.45}$$

and

$$\hat{\Lambda}(\partial) - \hat{\Lambda}(-\overleftarrow{\partial}) = i\gamma^{\mu}\left(\partial_{\mu} + \overleftarrow{\partial}_{\mu}\right) + \frac{a}{2}\gamma^{0}\delta^{\mu i}\left(\partial_{\mu} + \overleftarrow{\partial}_{\mu}\right)\left(\partial_{i} - \overleftarrow{\partial}_{i}\right). \tag{3.46}$$

respectively. From Eq.(3.45) and Eq.(3.46) we observe that the identity in Eq.(3.44) is satisfied.

Since $\hat{\Gamma}^{\mu}(\partial, -\overleftarrow{\partial})$ corresponding to the deformed Dirac equation satisfies the identity given in Eq.(3.44), we can now use this for constructing the conserved currents associated with the deformed Dirac field.

Substituting Eq.(3.43) in Eq.(2.20) (see chapter 2) we obtain the general expression for the conserved current corresponding to κ-deformed Dirac field (valid up to first order in a) as

$$\hat{J}^{\mu} = \bar{\hat{\psi}}(x)i\gamma^{\mu}\delta\hat{\psi}(x) + \frac{a}{2}\bar{\hat{\psi}}(x)\gamma^{0}\delta^{\mu i}(\partial_{i} - \overleftarrow{\partial}_{i})\delta\hat{\psi}(x). \tag{3.47}$$

The infinitesimal change in the deformed Dirac field under the translational symmetry is given as $\delta\hat{\psi}(x) = \delta\hat{x}_{\mu}\partial^{\mu}\hat{\psi}(x)$, where $\delta\hat{x}_{\mu}$ (see Eq.(2.63 of chapter 2) is

$$\delta\hat{x}_{\mu} = \theta_{\mu} + ia\theta^{\nu}(\eta_{\mu\nu} + \delta_{\mu 0}\partial_{\nu}). \tag{3.48}$$

Thus substituting Eq.(3.48) in $\delta\hat{\psi}(x) = \delta\hat{x}_{\mu}\partial^{\mu}\hat{\psi}(x)$, we get

$$\delta\hat{\psi}(x) = \big(\partial_{\mu}\hat{\psi}(x) + ia(\partial_{\mu}\hat{\psi}(x) + \partial_{0}\partial_{\mu}\hat{\psi}(x))\big)\theta^{\mu}. \tag{3.49}$$

Using Eq.(3.49) in Eq.(3.47), we get the conserved current (valid up to first order in a) corresponding to the translational symmetry of the κ-deformed Dirac field as

$$\hat{J}_{\mu} = i\bar{\hat{\psi}}(x)\gamma_{\mu}\partial_{\nu}\hat{\psi}(x)\theta^{\nu} - a\bar{\hat{\psi}}(x)\gamma_{\mu}(\partial_{\nu} + \partial_{0}\partial_{\nu})\hat{\psi}(x)\theta^{\nu} + \frac{a}{2}\delta^{\mu i}\bar{\hat{\psi}}(x)\gamma^{0}(\partial_{i} - \overleftarrow{\partial}_{i})\partial_{\nu}\hat{\psi}(x)\theta^{\nu}. \tag{3.50}$$

The second term in the RHS of the above equation is contributed by the a dependent term of $\delta\hat{x}_\mu$. The third term is contributed by the a dependent term of $\hat{\Gamma}_\mu(\partial, -\overleftarrow{\partial})$ (see Eq.(3.43)). Using the relation $\hat{J}_\mu = \hat{T}_{\mu\nu}\theta^\nu$ in Eq.(3.50), we obtain the energy-momentum tensor corresponding to κ-deformed Dirac field, valid up to first order in a, as

$$\begin{aligned}\hat{T}_{\mu\nu} = & \; i\bar{\hat{\psi}}(x)\gamma_\mu\partial_\nu\hat{\psi}(x) - a\bar{\hat{\psi}}(x)\gamma_\mu\partial_\nu\hat{\psi}(x) - a\bar{\hat{\psi}}(x)\gamma_\mu\partial_\nu\partial_0\hat{\psi}(x) \\ & + \frac{a}{2}\delta_{\mu i}\bar{\hat{\psi}}(x)\gamma^0\partial_i\partial_\nu\hat{\psi}(x) - \frac{a}{2}\delta_{\mu i}\partial_i\bar{\hat{\psi}}(x)\gamma^0\partial_\nu\hat{\psi}(x).\end{aligned} \tag{3.51}$$

Here we write down the conserved momenta corresponding to the translational symmetry of the deformed Dirac field from its energy-momentum tensor (in a similar manner as done for κ-deformed scalar field, i.e., Eq.(2.67) derived in chapter 2).

$$\hat{P}_\mu = \int d^3x \; \hat{T}_{0\mu}. \tag{3.52}$$

Under the Lorentz transformation, the infinitesimal change in the κ-deformed Dirac field is given as

$$\delta\hat{\psi}(x) = \delta\hat{x}_\mu\partial^\mu\hat{\psi}(x) + \frac{1}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\hat{\psi}(x), \tag{3.53}$$

where the infinitesimal change in the κ-deformed space-time coordinate ($\delta\hat{x}_\mu$) under the Lorentz transformation is given as (see Eq.(2.68))

$$\begin{aligned}\delta\hat{x}_\mu = & \; x_\nu\omega^\nu{}_\mu + ia\Big(\delta_{\mu 0}x_\lambda\partial_\nu - \delta_{\nu 0}x_\lambda\partial_\mu + \eta_{\mu\lambda}x_\nu\partial_0 + \delta_{\mu 0}\delta_{\lambda 0}x_0\partial_\nu - \delta_{0\nu}\eta_{\mu\lambda}x_0\partial_0 \\ & -\delta_{0\nu}x_\mu\partial_\lambda + \delta_{0\mu}x_\nu\partial_\lambda\Big)\omega^{\nu\lambda}\end{aligned} \tag{3.54}$$

and $\sigma_{\mu\nu}$ is defined as $\sigma_{\mu\nu} = [\gamma_\mu, \gamma_\nu]$.

Thus by substituting Eq.(3.53) and Eq.(3.54) in Eq.(3.47), we obtain the conserved current (valid up to first order in a) corresponding to the Lorentz transformation of the κ-deformed Dirac field as

$$\begin{aligned}\hat{J}_\mu = & \; \frac{i}{2}\bar{\hat{\psi}}(x)\gamma_\mu(x_\nu\partial_\lambda - x_\lambda\partial_\nu)\hat{\psi}(x)\omega^{\nu\lambda} + \frac{i}{4}\bar{\hat{\psi}}(x)\gamma_\mu\sigma_{\nu\lambda}\hat{\psi}(x)\omega^{\nu\lambda} + \\ & \frac{a}{8}\delta_{\mu i}\bar{\hat{\psi}}(x)\gamma^0(\partial_i - \overleftarrow{\partial}_i)\sigma_{\nu\lambda}\hat{\psi}(x)\omega^{\nu\lambda} + \frac{a}{4}\delta_{\mu i}\bar{\hat{\psi}}(x)\gamma^0(\partial_i - \overleftarrow{\partial}_i)(x_\nu\partial_\lambda - x_\lambda\partial_\nu)\hat{\psi}(x)\omega^{\nu\lambda} + \\ & \frac{a}{2}\bar{\hat{\psi}}(x)\gamma_\mu\Big(\delta_{\nu 0}x_\lambda\partial_\alpha\partial^\alpha - \delta_{\lambda 0}x_\nu\partial_\alpha\partial^\alpha + \delta_{\nu 0}x_\alpha\partial_\lambda\partial^\alpha - \delta_{\lambda 0}x_\alpha\partial_\nu\partial^\alpha + x_\lambda\partial_\nu\partial_0 - x_\nu\partial_\lambda\partial_0\Big)\hat{\psi}(x)\omega^{\nu\lambda}\end{aligned} \tag{3.55}$$

Using the relation $\hat{J}_\mu = \mathcal{M}_{\mu\nu\lambda}\omega^{\nu\lambda}$ in Eq.(3.55) we get $\mathcal{M}_{\mu\nu\lambda}$ as

$$\begin{aligned}\mathcal{M}_{\mu\nu\lambda} = & \; \frac{i}{2}\bar{\hat{\psi}}(x)\gamma_\mu(x_\nu\partial_\lambda - x_\lambda\partial_\nu)\hat{\psi}(x) + \frac{i}{4}\bar{\hat{\psi}}(x)\gamma_\mu\sigma_{\nu\lambda}\hat{\psi}(x) + \\ & \frac{a}{8}\delta_{\mu i}\bar{\hat{\psi}}(x)\gamma^0(\partial_i - \overleftarrow{\partial}_i)\sigma_{\nu\lambda}\hat{\psi}(x)\omega^{\nu\lambda} + \frac{a}{4}\delta_{\mu i}\bar{\hat{\psi}}(x)\gamma^0(\partial_i - \overleftarrow{\partial}_i)(x_\nu\partial_\lambda - x_\lambda\partial_\nu)\hat{\psi}(x) + \\ & \frac{a}{2}\bar{\hat{\psi}}(x)\gamma_\mu\Big(\delta_{\nu 0}x_\lambda\partial_\alpha\partial^\alpha - \delta_{\lambda 0}x_\nu\partial_\alpha\partial^\alpha + \delta_{\nu 0}x_\alpha\partial_\lambda\partial^\alpha - \delta_{\lambda 0}x_\alpha\partial_\nu\partial^\alpha + x_\lambda\partial_\nu\partial_0 - x_\nu\partial_\lambda\partial_0\Big)\hat{\psi}(x).\end{aligned} \tag{3.56}$$

The Lorentz generator corresponding to the κ-deformed Dirac field (valid up to first order in a) is written as

$$\hat{M}_{\mu\nu} = \int d^3x\ \mathcal{M}_{0\mu\nu}. \tag{3.57}$$

The κ-deformed Dirac field transform as $\hat{\psi}(x) \to \hat{\psi}'(x) = e^{-i\theta}\hat{\psi}(x)$ under the global phase transformation. Thus the infinitesimal change in the deformed Dirac field in this case is given as $\delta\hat{\psi}(x) = -i\theta\hat{\psi}(x)$. Substituting this in Eq.(3.47) we obtain the conserved current (valid up to first order in a) corresponding to the global phase transformation symmetry as

$$\hat{J}^\mu = \bar{\hat{\psi}}(x)\gamma^\mu\hat{\psi}(x)\theta - \frac{ia}{2}\delta^{\mu i}\bar{\hat{\psi}}(x)\gamma^0(\partial_i - \overleftarrow{\partial}_i)\hat{\psi}(x)\theta. \tag{3.58}$$

From the above conserved current, we obtain the number operator corresponding to the κ-deformed Dirac field as $\hat{N} = \int d^3x\ \hat{J}^0(x)$. Thus the explicit form of the number operator, valid up to first order in a, is given as

$$\begin{aligned}
\hat{N} &= \int d^3x \bar{\hat{\psi}}(x)\gamma^0\hat{\psi}(x) \\
&= \int \frac{d^3x\ d^3p\ d^3p'}{\sqrt{(2\pi)^6 2E_p 2E_{p'}}}\Big(\hat{a}^\dagger(p)\bar{\hat{u}}(p)e^{ipx} + \hat{b}(p)\bar{\hat{v}}(p)e^{-ipx}\Big)\gamma^0\Big(\hat{a}(p')\hat{u}(p')e^{-ip'x} + \hat{b}^\dagger(p')\hat{v}(p')e^{ip'x}\Big) \\
&= \int \frac{d^3x\ d^3p\ d^3p'}{\sqrt{(2\pi)^6 2E_p 2E_{p'}}}\Big(\hat{a}^\dagger(p)\hat{a}(p')\hat{u}^\dagger(p)\hat{u}(p')e^{i(p-p')x} + \hat{a}^\dagger(p)\hat{b}^\dagger(p')\hat{u}^\dagger(p)\hat{v}(p')e^{i(p+p')x} \\
&\quad + \hat{b}(p)\hat{a}(p')\hat{v}^\dagger(p)\hat{u}(p')e^{-i(p+p')x} + \hat{b}(p)\hat{b}^\dagger(p')\hat{v}^\dagger(p)\hat{v}(p')e^{-i(p-p')x}\Big).
\end{aligned} \tag{3.59}$$

Now we substitute the explicit form of $\hat{u}(p)$ and $\hat{v}(p)$ from Eq.(3.24) and Eq.(3.27) in Eq.(3.59). We then use the relations $u_s^{(0)\dagger}(p)v_{s'}^{(0)}(-p) = v_s^{(0)\dagger}(p)u_{s'}^{(0)}(-p) = 0$ and $u_s^{(0)\dagger}(p)u_{s'}^{(0)}(p') = v_s^{(0)\dagger}(p)v_{s'}^{(0)}(p') = \delta_{ss'}2E_p$ to get the number operator, valid up to first order in a, as

$$\hat{N} = \int \frac{d^3p}{(2\pi)^3}\Big(1 + 2a\alpha - a\frac{p^2}{E_p}\Big)\Big(\hat{a}^\dagger(p)\hat{a}(p) + \hat{b}(p)\hat{b}^\dagger(p)\Big) \tag{3.60}$$

In the above expression, we define $\hat{N}_a(p) = \hat{a}^\dagger(p)\hat{a}(p)$ and $\hat{N}_b(p) = \hat{b}^\dagger(p)\hat{b}(p)$. Thus we obtain the normal ordered number operator (valid up to first order in a) corresponding to the κ-deformed Dirac field as

$$:\hat{N}: = \int \frac{d^3p}{(2\pi)^3}\Big(1 + 2a\alpha - \frac{ap}{\sqrt{1 + m^2/p^2}}\Big)\Big(\hat{N}_a(p) - \hat{N}_b(p)\Big). \tag{3.61}$$

From the above expression, we find that the number operator picks up two a dependent correction terms. We obtain usual Dirac number operator, $: N := \int \frac{d^3p}{(2\pi)^3}\Big(N_a(p) - N_b(p)\Big)$ in the limit $a \to 0$.

3.3.1 Discrete symmetries

We now analyse the discrete symmetries associated with the κ-deformed Dirac field. We first check whether parity, time-reversal and charge conjugation are symmetries of the κ-deformed Dirac field or not. We then construct the conserved currents corresponding to discrete symmetries of κ-deformed Dirac equation.

We begin this discussion with the κ-deformed Dirac equation, valid up to first order in a, given by

$$\Big(i\gamma^0\partial_0 + i\gamma^i\partial_i + \frac{a}{2}\gamma^0\partial_i^2 + m\Big)\hat{\psi}(x_i,t) = 0. \tag{3.62}$$

We first analyse the parity symmetry associated with the deformed Dirac field. Under the parity, space-time coordinates and their derivatives transform as $x_i \to -x_i,\ t \to t,\ \partial_i \to -\partial_i$ and $\partial_0 \to \partial_0$. Thus Eq.(3.62) becomes

$$\Big(i\gamma^0\partial_0 - i\gamma^i\partial_i + \frac{a}{2}\gamma^0\partial_i^2 + m\Big)\hat{\psi}(-x_i,t) = 0. \tag{3.63}$$

Now we consider a matrix operator $\mathcal{P}$ satisfying

$$\mathcal{P}\Big(i\gamma^0\partial_0 - i\gamma^i\partial_i + \frac{a}{2}\gamma^0\partial_i^2 + m\Big)\mathcal{P}^{-1}\mathcal{P}\hat{\psi}(-x_i,t) = 0 \tag{3.64}$$

such that $\mathcal{P}\Big(i\gamma^0\partial_0 - i\gamma^i\partial_i + \frac{a}{2}\gamma^0\partial_i^2 + m\Big)\mathcal{P}^{-1}$ represents the $\hat{\Lambda}(\partial)$ operator given in Eq.(3.8) and thus $\mathcal{P}\hat{\psi}(-x_i,t)$ satisfies the κ-deformed Dirac equation. We now define this $\mathcal{P}\hat{\psi}(-x_i,t) = \hat{\psi}_p(x_i,t)$. By comparing Eq.(3.63) with Eq.(3.64) we obtain the matrix/operator $\mathcal{P}$ as $\mathcal{P} = \gamma^0$. Note that this parity operator is the same as that in the commutative case.

Now we obtain the expression for the conserved current corresponding to parity symmetry of the deformed Dirac equation by taking $\delta\hat{\psi}(x) = \hat{\psi}_p(x_i,t)$. Substituting this in Eq.(3.47) we get the conserved current, valid up to first order in a, as

$$\begin{aligned}\hat{J}^\mu =& \bar{\hat{\psi}}(x_i,t)\hat{\Gamma}^\mu(\partial,-\overleftarrow{\partial})\hat{\psi}_p(x_i,t)\\ =& i\bar{\hat{\psi}}(x_i,t)\gamma^\mu\gamma^0\hat{\psi}(-x_i,t) + \frac{a}{2}\delta^{\mu i}\bar{\hat{\psi}}(x_i,t)(\partial_i - \overleftarrow{\partial}_i)\hat{\psi}(-x_i,t).\end{aligned} \tag{3.65}$$

By substituting the explicit form of the deformed field operator and its adjoint in the above equation, we get the expression for the conserved charge (valid up to first order in a) associated with the parity symmetry of the deformed Dirac field as

$$\hat{Q}_p = \int \frac{dp^3}{(2\pi)^3}\Big(1 + 2a\alpha - \frac{ap}{\sqrt{1+m^2/p^2}}\Big)\Big(\hat{a}^\dagger(p)\hat{a}(-p) + \hat{b}^\dagger(p)\hat{b}(-p)\Big). \tag{3.66}$$

We observe that the deformed conserved charge associated with the parity symmetry picks up two a dependent correction terms. Note that the second correction term is a mass-dependent term. These correction terms are contributed by a dependent terms of the deformed Dirac field and its adjoint. The conserved charge for parity symmetry in κ space-time gets scaled by $\Big(1+2a\alpha-\frac{ap}{\sqrt{1+m^2/p^2}}\Big)$ factor. We also note that this deformation factor is exactly the same as that in the deformed number operator given in Eq.(3.61). In the limit $a\to 0$, the equation, Eq.(3.65) reduces to Eq.(2.26), which is the conserved current for the parity symmetry of the commutative Dirac field [18].

Now we analyse the time-reversal symmetry associated with the κ-deformed Dirac field. Under the time-reversal transformation, space-time coordinates and their derivatives transform as $x_i\to x_i,\ t\to -t,\ \partial_i\to\partial_i$ and $\partial_0\to-\partial_0$ respectively. Thus under the time-reversal transformation the κ-deformed Dirac equation given in Eq.(3.62) becomes

$$\Big(-i\gamma^0\partial_0+i\gamma^i\partial_i+\frac{a}{2}\gamma^0\partial_i^2+m\Big)\hat{\psi}(x_i,-t)=0. \tag{3.67}$$

By taking the complex conjugate of Eq.(3.67) we obtain

$$\Big(i\gamma^{0*}\partial_0-i\gamma^{i*}\partial_i+\frac{a}{2}\gamma^{0*}\partial_i^2+m\Big)\hat{\psi}^*(x_i,-t)=0. \tag{3.68}$$

We now consider a matrix/operator $\mathcal{T}$ satisfying

$$\mathcal{T}\Big(i\gamma^{0*}\partial_0-i\gamma^{i*}\partial_i+\frac{a}{2}\gamma^{0*}\partial_i^2+m\Big)\mathcal{T}^{-1}\mathcal{T}\hat{\psi}^*(x_i,-t)=0, \tag{3.69}$$

such that $\mathcal{T}\Big(i\gamma^{0*}\partial_0-i\gamma^{i*}\partial_i+\frac{a}{2}\gamma^{0*}\partial_i^2+m\Big)\mathcal{T}^{-1}$ corresponds to the $\hat{\Lambda}(\partial)$ operator given in Eq.(3.8). We find that $\mathcal{T}\hat{\psi}^*(x_i,-t)=\hat{\psi}_T(x_i,t)$ obeys the κ-deformed Dirac equation, valid up to first order in a. From the conditions $\mathcal{T}\gamma^{0*}\mathcal{T}^{-1}=\gamma^0$ and $\mathcal{T}\gamma^{i*}\mathcal{T}^{-1}=-\gamma^i$, obtained by comparing Eq.(3.62) with Eq.(3.69), we find that $\mathcal{T}=i\gamma^1\gamma^3$. Note that this time-reversal operator is exactly the same as that in the commutative case.

Now we write down the expression for the conserved current associated with the time-reversal symmetry of the κ-deformed Dirac field using Eq.(3.47). By taking $\delta\hat{\psi}(x_i,t)=\hat{\psi}_T(x_i,t)$ and substituting this in Eq.(3.47), we obtain the conserved current valid up to first order in a as

$$\begin{aligned}\hat{J}^\mu=&\bar{\hat{\psi}}(x_i,t)\hat{\Gamma}^\mu(\partial,-\overleftarrow{\partial})\mathcal{T}\hat{\psi}_T(x_i,t)\\=&-\bar{\hat{\psi}}(x_i,t)\gamma^\mu\gamma^1\gamma^3\hat{\psi}^*(x_i,-t)+\frac{ia}{2}\delta^{\mu i}\bar{\hat{\psi}}(x_i,t)\gamma^0(\partial_i-\overleftarrow{\partial}_i)\gamma^1\gamma^3\hat{\psi}^*(x_i,-t).\end{aligned} \tag{3.70}$$

From the above equation we obtain the expression for the conserved current by substituting Eq.(3.24) and Eq.(3.27) in Eq.(3.70). Thus we obtain the conserved charge (valid up to first order in a) corresponding to the time-reversal symmetry of the κ-deformed

Dirac field as

$$\hat{Q}_t = \int \frac{d^3p}{(2\pi)^3}\Big(1 + 2a\alpha - \frac{ap}{\sqrt{1+m^2/p^2}}\Big)\Big(\hat{a}^\dagger(p)\hat{a}^*(-p) + \hat{b}(p)\hat{b}^T(-p)\Big). \tag{3.71}$$

Here also we observe that the deformed conserved charge for the time-reversal symmetry possesses a mass-dependent correction term. This deformation factor is exactly the same as that seen in Eq.(3.61) and Eq.(3.66). In the limit $a \to 0$, Eq.(3.70) reduces to Eq.(2.27), which represents the conserved current for the time-reversal symmetry of the commutative Dirac field [18].

Next, we analyse whether the charge conjugation is a symmetry of the κ-deformed Dirac field or not. For this, we first introduce an interaction term in the κ-deformed Dirac equation. This term is introduced as the minimal coupling of electron with electromagnetic field A_μ, by replacing $i\partial_\mu$ with $i\partial_\mu + eA_\mu$ in the κ-deformed Dirac equation given in Eq.(3.62). Thus the $\hat{\Lambda}(\partial)$ operator in Eq.(3.8) becomes

$$\hat{\Lambda}_c(\partial) = \gamma^\mu\Big(i\partial_\mu + eA_\mu\Big) - \frac{a}{2}\gamma^0\Big(i\partial_i + eA_i\Big)^2 + m \tag{3.72}$$

and the corresponding Dirac equation becomes

$$\hat{\Lambda}_c(\partial)\hat{\psi}(x_i,t) = 0. \tag{3.73}$$

Under the charge conjugation, the charge of the electron changes as $e \to -e$ and thus we get the κ-deformed equation of motion (valid up to first order in a) for the anti-particle as

$$\Big[\gamma^\mu\Big(i\partial_\mu - eA_\mu\Big) - \frac{a}{2}\gamma^0\Big(i\partial_i - eA_i\Big)^2 + m\Big]\hat{\psi}_c(x_i,t) = 0. \tag{3.74}$$

By taking the complex conjugate of Eq.(3.73) we obtain

$$\Big[\gamma^{*\mu}\Big(-i\partial_\mu + eA_\mu\Big) - \frac{a}{2}\gamma^{*0}\Big(-i\partial_i + eA_i\Big)^2 + m\Big]\hat{\psi}^*(x_i,t) = 0. \tag{3.75}$$

The derivative terms in the above equation pick up negative sign after taking the complex conjugate of $\hat{\Lambda}_c(\partial)$.

Now we consider a matrix $\mathcal{C}$ satisfying

$$\mathcal{C}\Big[-\gamma^{*\mu}\Big(i\partial_\mu - eA_\mu\Big) - \frac{a}{2}\gamma^{*0}\Big(i\partial_i - eA_i\Big)^2 + m\Big]\mathcal{C}^{-1}\mathcal{C}\hat{\psi}^*(x_i,t) = 0, \tag{3.76}$$

such that Eq.(3.76) is equivalent to Eq.(3.74) and thus we have $\hat{\psi}_c(x_i,t) = \mathcal{C}\hat{\psi}^*(x_i,t)$.

Using Eq.(2.16), we construct the $\hat{\Gamma}^{\mu}_{c}(\partial, -\overleftarrow{\partial})$ corresponding to the $\hat{\Lambda}_c(\partial)$ given in Eq.(3.72). This $\hat{\Gamma}^{\mu}_{c}(\partial, -\overleftarrow{\partial})$ operator, valid up to first order in a, is given as

$$\hat{\Gamma}^{\mu}_{c}(\partial, -\overleftarrow{\partial}) = i\left(\gamma^{\mu} - a\gamma^0 eA_i\delta^{\mu i}\right) + \frac{a}{2}\gamma^0\delta^{\mu i}\left(\partial_i - \overleftarrow{\partial}_i\right). \tag{3.77}$$

Now we check whether the $\hat{\Gamma}^{\mu}_{c}(\partial, -\overleftarrow{\partial})$ obtained above satisfy the identity given in Eq.(3.44) or not. By acting Eq.(3.77) with $\partial_\mu + \overleftarrow{\partial}_\mu$, we obtain

$$\left(\partial_\mu + \overleftarrow{\partial}_\mu\right)\hat{\Gamma}^{\mu}_{c}(\partial, -\overleftarrow{\partial}) = i\gamma^{\mu}\left(\partial_\mu + \overleftarrow{\partial}_\mu\right) - ia\gamma^0 eA_i\left(\partial_i + \overleftarrow{\partial}_i\right) + \frac{a}{2}\gamma^0\left(\partial_i^2 - \overleftarrow{\partial}_i^2\right). \tag{3.78}$$

Similarly from Eq.(3.72) we obtain

$$\begin{aligned}\hat{\Lambda}_c(\partial) - \hat{\Lambda}_c(-\overleftarrow{\partial}) = i\gamma^{\mu}\left(\partial_\mu + \overleftarrow{\partial}_\mu\right) - ia\gamma^0 eA_i\left(\partial_i + \overleftarrow{\partial}_i\right) \\ + \frac{a}{2}\gamma^0\left(\partial_i^2 - \overleftarrow{\partial}_i^2\right) + i\gamma^0 e\left(\overleftarrow{\partial}_i A_i\right) - i\gamma^0 e\left(\partial_i A_i\right).\end{aligned} \tag{3.79}$$

By comparing Eq.(3.78) with Eq.(3.79) we find that, $\left(\partial_\mu + \overleftarrow{\partial}_\mu\right)\hat{\Gamma}^{\mu}_{c}(\partial, -\overleftarrow{\partial}) \neq \hat{\Lambda}_c(\partial) - \hat{\Lambda}_c(-\overleftarrow{\partial})$. Hence $\hat{\Gamma}^{\mu}_{c}(\partial, -\overleftarrow{\partial})$ operator does not satisfy the identity given in Eq.(3.44). Therefore we cannot construct a consistent $\hat{\Gamma}^{\mu}(\partial)$ operator and conserved current corresponding to the charge conjugation symmetry of the deformed Dirac field. Thus it is clear that charge conjugation is not a symmetry for the particles obeying κ-deformed Dirac equation (valid up to first order in a). This result is consistent with that obtained in [6], where it has been shown that one cannot obtain a unique charge conjugation matrix $\mathcal{C}$ for the κ-deformed Dirac equation (to all orders in a).

3.4 Conclusions

In this chapter, we have started with the κ-deformed Dirac field, obeying the κ-deformed Dirac equation, valid up to first order in a. This undeformed κ-Poincare invariant Dirac equation has been constructed by replacing the usual derivative in the Dirac equation with the Dirac derivative [6]. Using the method discussed in chapter 2, we quantised this deformed Dirac field from its equation of motion alone without referring to Lagrangian.

We solved the equations of motion perturbatively and obtained the explicit form of the κ-Dirac field, valid up to first order in a. From the deformed equations of motion, we have obtained $\hat{\Lambda}(\partial)$ operator and Klein-Gordon divisor, $\hat{d}(\partial)$, valid up to first order in a, corresponding to the κ-deformed Dirac field. We have derived a deformed unequal-time anti-commutation relation between the deformed Dirac field and its adjoint (where $(1 + 2a\alpha)$ is the deformation factor, valid up to first order in a), by assuming the usual form of the oscillator algebra. This $(1 + 2a\alpha)$ factor has also been seen in the unequal-time deformed commutation relation between κ-scalar field and its adjoint (see chapter 2). Thus we observe that the undeformed oscillator algebra gave rise to a deformed unequal-time (anti-) commutator between the deformed field and its adjoint.

We have obtained a deformed oscillator algebra by demanding that the unequal-time anti-commutation relation between the deformed Dirac field and its adjoint to be undeformed (valid up to first order in a). This lead to a deformed oscillator algebra given in Eq.(3.42). The factor $(1 - 2a\alpha)$ present in the κ-deformed fermionic oscillator algebra has also been seen to be present in the deformed oscillator algebra of the κ-deformed scalar field [23]. This deformed oscillator algebra is a novel feature of the κ-deformed field theories. Such modified oscillator algebras have shown to modify Unruh effect [23] and Hawking radiation [24].

We have also studied the symmetries associated with the κ-deformed Dirac field and derived the deformed conserved currents (valid up to first order in a) from the equation of motion alone by using the $\hat{\Gamma}_\mu(\partial, -\overleftarrow{\partial})$ operator. We have constructed the energy-momentum tensor and the Lorentz generator (valid up to first order in a) corresponding to the κ-deformed Dirac field from the translational as well as Lorentz symmetry in the κ-space-time. By deriving the conserved current associated with the global phase transformation symmetry, we have obtained the number operator (valid up to first order in a) for the κ-Dirac field. We have seen that the number operator of κ-Dirac field picks up a mass-dependent correction term.

We have analysed the discrete symmetries associated with the κ-deformed Dirac field and have shown that parity as well as time-reversal are the symmetries of κ-Dirac field. Further, we have derived the conserved currents (valid up to first order in a) corresponding to parity and time-reversal symmetry from the $\hat{\Gamma}_\mu(\partial, -\overleftarrow{\partial})$ operator. In the limit $a \to 0$, the corresponding conserved charges reduce to those obtained in [18]. We have also observed that κ-deformed Dirac field violates the charge conjugation symmetry, by showing that the $\hat{\Gamma}^\mu_c(\partial, -\overleftarrow{\partial})$ operator (corresponding to the deformed Dirac equation minimally coupled with electromagnetic field) is not consistent with the requirement [10, 11] imposed by Takahashi-Umezawa formalism. This result agrees with that obtained in [6].

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Chapter 4

Quantisation of Doplicher-Fredenhagen-Roberts-Amorim scalar field

4.1 Introduction

The non-commutative space-times such as Moyal space-time and κ-deformed space-time violate the Lorentz symmetry. However, in [1–3], Lorentz invariant non-commutative space-time, known as Doplicher-Fredenhagen-Roberts (DFR) space-time, has been constructed and its coordinates satisfy

$$[\hat{x}_\mu, \hat{x}_\nu] = i\hat{\theta}_{\mu\nu},\ [\hat{x}_\mu, \hat{\theta}_{\nu\lambda}] = 0,\ [\hat{\theta}_{\mu\nu}, \hat{\theta}_{\rho\lambda}] = 0. \tag{4.1}$$

This DFR algebra has further been extended by incorporating the canonical conjugate momenta operators $\hat{p}_\mu$ (corresponding to $\hat{x}_\mu$) and $\hat{k}_\mu$ (corresponding to $\hat{\theta}_{\mu\nu}$) and this forms the Doplicher-Fredenhagen-Roberts-Amorim (DFRA) algebra [4, 5] (see Eq.(1.29) of chapter 1).

In recent times various features of field theory models on the DFR space-time have been investigated. The canonical quantisation of the DFRA complex scalar field was examined and its symmetries were investigated in [6]. In this study, [6] Green's function technique was used to obtain the general solution for the DFRA complex scalar field. It has been demonstrated that UV/IR mixing was absent at the one-loop level of the DFRA scalar field with the ϕ^4 interaction [7]. The covariant Dirac equation in DFR space-time has been constructed in[8], such that its square gives the DFR Klein-Gordon equation. Introduction of a Lorentz invariant weight function $W(\theta)$ (which depends on

θ^2) in the action was shown to be necessary for consistent perturbative calculation in DFR theories [3, 9–12]. Different phenomenological aspects have been examined in DFR space-time [3, 9–12].

The quadratic Casimir of the DFRA algebra (see Eq.(1.32) of chapter 1) was used to obtain the equations of motion associated with the scalar field in DFR space-time [4, 5]. It was demonstrated that the requirement of obtaining these equations of motion by variation principle is satisfied by two different scalar field theory actions in DFR space-time (even when the weight function is fixed to unity); one is a model whose action is [1]

$$S = \int d^3x \; d^6\theta \; W(\theta)\Big(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 - \Lambda\Big(\partial_{\mu\nu}\partial^{\mu\nu} - \mu\Big)\phi\Big) \tag{4.2}$$

where Λ is a Lagrangian multiplier and the second action is

$$S = \int d^3x \; d^6\theta \; W(\theta)\Big(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 - \frac{\lambda^4}{4}\partial_{\mu\nu}\phi\partial^{\mu\nu}\phi\Big). \tag{4.3}$$

Note that these two actions are not equal up to total derivative terms. Since the definition of conjugate momenta of a field depends on the form of the Lagrangian, the canonical quantisation of the DFRA scalar theory described by the above two Lagrangians are expected to be different. The actions corresponding to these non-commutative scalar field theory models are distinct. Therefore these non-commutative generalisations are not unique. Thus the non-uniqueness of the action/Lagrangian makes it important to study the quantisation from the equations of motion.

The DFRA scalar field equation constructed in [4] includes contribution from the weight function. By setting this weight function to be a constant, the equation of motion becomes exactly the same as the one coming from the quadratic Casimir of the corresponding symmetry algebra. Here we quantise the non-commutative DFRA scalar field using the quantisation procedure discussed in [13–15]. We obtain the deformed commutation relation between the DFRA scalar field and its conjugate (at equal times) by considering a conventional form for the DFRA oscillator algebra. By demanding the commutation relation (at equal times) between the DFRA scalar field and its conjugate to be undeformed, we then derive the deformed DFRA oscillator algebra. We also show that the first non-vanishing corrections of the deformed oscillator algebra depend on the non-commutative length scale as $\frac{1}{\lambda^4}$. We analyse the translation and Lorentz symmetry of scalar theory in DFR space-time and then construct the corresponding conserved currents without any reference to the action of the DFRA scalar field [16, 17]. We also show that the energy-momentum tensor of the DFRA scalar field is asymmetric in its indices when the weight function is included.

[1]Here $\partial_{\mu\nu} = \frac{\partial}{\partial\theta^{\mu\nu}}$ and μ has the dimension of $(mass)^4$

The deformed oscillator algebra of the non-commutative field theories play an important role in constructing Fock space and it is important to study the modifications brought by the deformed oscillator algebra in different phenomenon such as Hawking radiation, Unruh effect etc. Non-commutative corrections to the Unruh effect [18–22] have been studied in different non-commutative space-time models such as Moyal space-time [23] and κ-space-time [24–26]. Here we derive the non-commutative corrections to Unruh effect in DFR space-time. We study this by examining the response of a monopole detector coupled with the massless DFRA scalar field in $1+3+3$ dimensional DFR space-time. These three extra spatial dimensions associated with the DFR space-time vanish in the compactification limit and we obtain the usual result of $1+3$ dimensional commutative space-time. We obtain the response function by evaluating the positive Wightman function calculated from the vacuum expectation value of the massless DFRA scalar field. We show that the thermal distribution is either Bose-Einstein or Fermi-Dirac, depending on the dimension of the DFR space-time. Though the profile of the thermal distribution is modified by the non-commutativity, we show that the temperature associated with this thermal distribution is unaffected by the non-commutativity of space-time.

This chapter is organised in the following manner. In sec.4.2, we set up the non-commutative action corresponding to the DFRA scalar field and get the equations of motion by fixing the weight function to be Gaussian. In sec.4.3, we obtain the deformed equal-time commutation relation between the DFRA scalar field and its conjugate by assuming corresponding creation and annihilation operators to follow the usual harmonic oscillator algebra. We then demand the commutation relation between the DFRA scalar field and its conjugate to be undeformed and this gives a particular form of deformed DFRA oscillator algebra. In sec.4.4, we construct energy-momentum tensor and Lorentz generator of the DFRA scalar field from its equation of motion. In sec.4.5, we examine the Unruh effect in DFR space-time by evaluating the transition probability rate of a massless DFRA scalar field coupled with a uniformly accelerating detector. Finally, in sec.4.5, we give our concluding remarks.

4.2 DFRA scalar field theory

In this section, we discuss the construction of action for field theory defined in DFR space-time using the star product formalism. We obtain the equations of motion corresponding to the scalar field in DFR space-time by varying the action. We also discuss the importance of introducing the θ-dependent weight function [3, 9–12].

The Moyal-star product [28] associated with the DFR space-time is given as

$$f(x,\theta)\star g(x,\theta) = e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu\partial_{\nu'}} f(x,\theta)g(x',\theta)\Big|_{x=x'}, \tag{4.4}$$

where f and g are functions of the DFR space-time coordinates. The star product reduces to the usual pointwise multiplication in the limit $\theta \to 0$. Here we note that the $\star$ product introduces non-local terms, which depend on the derivatives with respect to the coordinate x_μ alone. In [27], it has been shown that the unitarity requirement of field theories on DFR space-time imposes the condition $\theta_{0i} = 0$. We thus set $\theta_{0i} = 0$ in the remaining part of this chapter.

The action for the scalar field theory in DFR space-time is obtained by replacing the usual product with the star product (given in Eq.(4.4)) and by introducing a θ dependent weight function in the measure. Thus the action for the scalar field theory in DFR space-time is given as [6]

$$S = \int d^4x\ d^3\theta\ W(\theta)\Big(\partial_\mu\phi\star\partial^\mu\phi + \lambda^2\partial_{\theta_i}\phi\star\partial_{\theta_i}\phi + m^2\phi\star\phi\Big). \tag{4.5}$$

Note in the above expression, θ_i is defined as $\theta_i = \frac{1}{2}\epsilon_{ijk}\theta^{jk}$.

$W(\theta)$ in the above equation represents the weight function, which depends on the θ coordinates and not on x_μ. This has been introduced in the action to regulate the divergences that appear in the perturbative calculations corresponding to the field theories in DFR space-time [3, 9–12]. In order to have a Lorentz invariant field theory, the weight function has to be an even function in θ, i.e., $W(-\theta) = W(\theta)$ [3, 9–12]. Hence one chooses Gaussian form for the weight function, i.e.,

$$W(\theta) = \Big(\frac{1}{4\pi^2\lambda^4}\Big)^{3/2} e^{-\frac{\theta^2}{4\lambda^4}}, \tag{4.6}$$

$(\frac{1}{4\pi^2\lambda^4})^{3/2}$ in the above expression corresponds to the normalisation factor. Other choices for the weight function satisfying the above conditions are considered in [11], showing that the weight function is not unique.

The Moyal product, defined in Eq.(4.4), satisfies

$$\int d^4x\ d^3\theta\ W(\theta)\ f(x,\theta)\star g(x,\theta) = \int d^4x\ d^3\theta\ W(\theta)\ f(x,\theta)g(x,\theta). \tag{4.7}$$

By using the above identity, i.e., Eq.(4.7), in Eq.(4.5), the action for scalar field in DFR space-time becomes

$$S = \int d^4x\ d^3\theta\ W(\theta)\Big(\partial_\mu\phi\partial^\mu\phi + \lambda^2\partial_{\theta_i}\phi\partial_{\theta_i}\phi + m^2\phi^2\Big). \tag{4.8}$$

The λ^2 dependent term in the Lagrangian comes from the θ coordinates of the non-commutative space-time. We obtain the commutative scalar field action in the limit $\lambda \to 0$ (note $\lim_{\lambda\to 0} \frac{e^{-\frac{\theta_i^2}{4\lambda^4}}}{\lambda^2} = \delta(\theta_i)$ [29, 30].

From Eq.(4.8), we obtain equations of motion for the scalar field in DFR space-time as

$$\Big(\Box + \lambda^2 \Box_\theta - m^2\Big)\phi(x,\theta) + \lambda^2 \partial_{\theta_i}\Big(\ln W(\theta)\Big)\partial_{\theta_i}\phi(x,\theta) = 0, \tag{4.9}$$

where $\Box = \partial_\mu \partial^\mu$ and $\Box_\theta = \partial_{\theta_i}\partial^{\theta_i}$.

The last term on the LHS of Eq.(4.9) depends on the derivative of the weight function and it vanishes when the weight function becomes a constant, i.e., $W(\theta) = 1$ as in [6]. When $W(\theta) = 1$, the equation of motion in Eq.(4.9) reduces to the equation of motion obtained from the quadratic Casimir of DFRA algebra (see Eq.(1.32) of chapter 1) [6], i.e.,

$$\Big(\Box + \lambda^2 \Box_\theta - m^2\Big)\phi(x,\theta) = 0. \tag{4.10}$$

Note that in the limit $\lambda \to 0$, Eq.(4.9) and Eq.(4.10) reduce to the well known commutative scalar field equation of motion.

By substituting Eq.(4.6) in Eq.(4.9), we get

$$\Big(\Box + \lambda^2 \Box_\theta - m^2\Big)\phi(x,\theta) - \frac{\theta_i}{2\lambda^2}\partial_{\theta_i}\phi(x,\theta) = 0. \tag{4.11}$$

In the limit $\lambda \to 0$ with $\theta = 0$, the above expression reduces to the commutative scalar field equation of motion. The first three terms on the LHS of Eq.(4.11) are the equation of motion associated with the quadratic Casimir of the DFRA algebra [4, 29, 30]. Canonical quantisation of the non-commutative scalar field obeying the equation of motion coming Casimir of DFRA algebra has been studied thoroughly in [6], from its Lagrangian.

We observe that the equations of motion coming from the DFRA scalar field action (see Eq.(4.9)) and that coming from the quadratic Casimir of the DFRA algebra (see Eq.(1.32) of chapter 1) are different due to the presence of the weight function in action. Therefore the equations of motion coming from the action depend on the choice of the weight function (which is not unique). This results in the non-uniqueness of the equations of motion associated with the DFRA scalar field (note that all these equations of motion reduce to the same commutative limit). Thus it is necessary to quantise the field theories in DFR space-time from their equations of motion itself instead of following the canonical procedure.

4.3 Quantisation of DFRA scalar field

We have observed that the Lagrangian associated with the DFRA scalar field is not unique. Hence it is important to study the quantisation of DFRA field theories from their equation of motion, instead of starting from the Lagrangian. In this section, we apply the quantisation method discussed in [13–15] (see chapter 2) to quantise the DFRA scalar field by starting from its equations of motion, i.e., Eq.(4.11).

We start by generalising the definition for $\Lambda(\partial)$ operator (see Eq.(2.3), chapter 2) to the DFR space-time as $\Lambda(\partial, \partial_\theta)$. Thus $\Lambda(\partial, \partial_\theta)$ operator in DFR space-time is defined as

$$\begin{aligned}\Lambda(\partial, \partial_\theta) &= \sum_{l=0}^{N} \Lambda_{A_1 A_2 \dots . A_l} \partial^{A_1} \partial^{A_2} \dots . \partial^{A_l} \\ &= \Lambda_0 + \Lambda_A \partial^A + \Lambda_{AB} \partial^A \partial^B + \Lambda_{ABC} \partial^A \partial^B \partial^C + \dots\dots\dots + \Lambda_{A_1 A_2 A_3 \dots . A_N} \partial^{A_1} \partial^{A_2} \partial^{A_3} \dots . \partial^{A_N},\end{aligned} \tag{4.12}$$

where the index $A = (\mu, \theta_i)$ and $\partial_A = (\partial_\mu, \lambda \partial_{\theta_i})$.

In the DFR space-time, the Klein-Gordon divisor is defined as $d(\partial, \partial_\theta)$. Thus Eq.(2.8) (see chapter 2) in the commutative space-time can be extended to the DFR space-time by replacing $\Lambda(\partial)$ operator, $d(\partial)$ and Klein-Gordon equation with the corresponding definitions in the DFR space-time as $\Lambda(\partial, \partial_\theta)$, $d(\partial, \partial_\theta)$ and the DFRA scalar field equation, i.e., Eq.(4.11). Thus we have

$$d(\partial, \partial_\theta)\Lambda(\partial, \partial_\theta) = \Box + \lambda^2 \Box_\theta - m^2 - \frac{\theta}{2\lambda^2}\partial_\theta. \tag{4.13}$$

For the DFRA scalar field, we have

$$\Lambda(\partial, \partial_\theta) = \Box + \lambda^2 \Box_\theta - m^2 - \frac{\theta}{2\lambda^2}\partial_\theta \text{ and } d(\partial, \partial_\theta) = \mathbb{I}. \tag{4.14}$$

By comparing Eq.(4.14) with Eq.(4.12), we obtain the components of $\Lambda(\partial, \partial_\theta)$ operator as

$$\Lambda_0 = -m^2, \ \Lambda_A = \Big(0, -\frac{\theta_i}{2\lambda^3}\Big), \ \Lambda_{AB} = diag(-1, 1, 1, 1, 1, 1, 1). \tag{4.15}$$

The DFRA scalar field, $\phi(x, \theta)$ is decomposed in terms of creation and annihilation operators as

$$\phi(x, \theta) = \int \frac{d^3p}{\sqrt{(2\pi)^3}} \frac{d^3\tilde{k}}{\sqrt{(2\pi)^3}} \frac{1}{\sqrt{2\omega(p, \tilde{k})}} \Big(u(x, \theta)a(p, k) + u^*(x, \theta)a^\dagger(p, \tilde{k})\Big). \tag{4.16}$$

Now we need to obtain the explicit form of $u(x,\theta)$ satisfying $\Lambda(\partial,\partial_\theta)u(x,\theta)=0$. We derive this solution by taking an ansatz for $u(x,\theta)$ as

$$u(x,\theta)=e^{-ipx}f(\theta). \tag{4.17}$$

Fixing $f(0)=1$, we have $u(x,0)=u(x)$ and it satisfies the commutative equation of motion, i.e., $\Lambda(\partial)u(x)=0$.

Substituting Eq.(4.17) in the equation $\Lambda(\partial,\partial_\theta)u(x,\theta)=0$, we obtain the equation for $f(\theta)$ as

$$\Big(\partial_\theta^2-\frac{\theta}{2\lambda^4}\partial_\theta-p^2-m^2\Big)f(\theta)=0. \tag{4.18}$$

In the above equation, the second term is contributed by the weight function dependent term. We solve the above equation using the power series method. Thus $f(\theta)$ is expanded as

$$f(\theta)=\sum_{n=0}^{\infty}a_n\theta^n, \tag{4.19}$$

where $a_0=1$, using the condition $f(0)=1$.

Substituting Eq.(4.19) in Eq.(4.18) and re-writing the p^2 term using the DFRA dispersion relation (see Eq.(4.11)), i.e., $p^2+m^2+\lambda^2k^2+\frac{ik\theta}{2\lambda^2}=0$, we obtain the solution for $f(\theta)$ as

$$f(\theta)=a_0\Big(\cos k\theta+\sum_{n=1}^{\infty}A_n(k\theta)^{2n}\Big)+a_1\Big(\sin k\theta+\sum_{n=1}^{\infty}B_n(k\theta)^{2n+1}\Big), \tag{4.20}$$

where

$$\begin{aligned}A_n(\lambda,k,\theta)&=\frac{(-1)^n}{(2n)!}\Big[\Pi_{j=0}^{n-1}\Big(1-\frac{j}{\lambda^4k^2}+\frac{ik\theta}{2\lambda^2}\Big)-1\Big],\\ B_n(\lambda,k,\theta)&=\frac{(-1)^n}{(2n+1)!}\Big[\Pi_{j=0}^{n-1}\Big(1-\frac{(2j+1)}{2\lambda^4k^2}+\frac{ik\theta}{2\lambda^2}\Big)-1\Big].\end{aligned} \tag{4.21}$$

Here $(k\theta)$ and $k\lambda^2$ are dimensionless quantities. A_n and B_n in Eq.(4.20) are contributed by the weight function dependent terms of Eq.(4.18). Thus A_n and B_n vanish when the weight function becomes unity. Note that A_n and B_n vanishes in the limit $\lambda\to 0$ with $\theta=0$.

Using the transformation laws given in [4], it can be shown that the solution given in Eq.(4.20) is Lorentz invariant. Therefore we can decompose the DFRA scalar field operator $\phi(x,\tilde{\theta})$ [2] into positive and negative modes using the creation and annihilation operators. Substituting Eq.(4.17) and Eq.(4.20) in Eq.(4.16), we obtain the DFRA scalar

[2]We define $\tilde{\theta}=\theta/\lambda$ and $\tilde{k}=\lambda k$. In general we denote DFRA scalar field as $\phi(x,\theta)$. However when we decompose the field into momentum space, we use $\phi(x,\tilde{\theta})$

field as

$$\phi(x,\tilde{\theta}) = \int \frac{d^3p}{\sqrt{(2\pi)^3}} \frac{d^3\tilde{k}}{\sqrt{(2\pi)^3}} \frac{1}{\sqrt{2\omega(p,\tilde{k})}} \Bigg[e^{-ipx}\bigg(a_0\Big(\cos\Big(\tilde{k}\tilde{\theta}\Big) + \sum_{n=1}^{\infty} A_n(\lambda,\tilde{k},\tilde{\theta})(\tilde{k}\tilde{\theta})^{2n}\Big) + a_1\Big(\sin\Big(\tilde{k}\tilde{\theta}\Big) + \sum_{n=1}^{\infty} B_n(\lambda,\tilde{k},\tilde{\theta})(\tilde{k}\tilde{\theta})^{2n+1}\Big)\bigg) a(p,\tilde{k}) + h.c \Bigg], \tag{4.22}$$

where $k_i = \frac{1}{2}\epsilon_{ijk}k^{jk}$. We now re-express the cosine and sine functions in the above expression using Euler's identity and after some re-arrangement, we obtain

$$\phi(x,\tilde{\theta}) = \int \frac{d^3p}{\sqrt{(2\pi)^3}} \frac{d^3\tilde{k}}{\sqrt{(2\pi)^3}} \frac{1}{\sqrt{2\omega(p,\tilde{k})}} \Bigg[e^{-ipx}\bigg(\frac{1}{2}(a_0 + ia_1)e^{-ik\theta} + \frac{1}{2}(a_0 - ia_1)e^{ik\theta} + a_0 \sum_{n=1}^{\infty} A_n(\lambda,\tilde{k},\tilde{\theta})(\tilde{k}\tilde{\theta})^{2n} + a_1 \sum_{n=1}^{\infty} B_n(\lambda,\tilde{k},\tilde{\theta})(\tilde{k}\tilde{\theta})^{2n+1}\bigg) a(p,\tilde{k}) + h.c \Bigg] \tag{4.23}$$

By changing $\tilde{k} \to -\tilde{k}$ in the second term of the above equation and performing the integration, we get the solution for the scalar field in DFR space-time as

$$\phi(x,\tilde{\theta}) = \int \frac{d^3p}{\sqrt{(2\pi)^3}} \frac{d^3\tilde{k}}{\sqrt{(2\pi)^3}} \frac{1}{\sqrt{2\omega(p,\tilde{k})}} \Bigg(e^{-ipx}\bigg(a_0 e^{-i\tilde{k}\tilde{\theta}} + a_0 \sum_{n=1}^{\infty} A_n(\lambda,\tilde{k},\tilde{\theta})(\tilde{k}\tilde{\theta})^{2n} + a_1 \sum_{n=1}^{\infty} B_n(\lambda,\tilde{k},\tilde{\theta})(\tilde{k}\tilde{\theta})^{2n+1}\bigg) a(p,\tilde{k}) + h.c \Bigg). \tag{4.24}$$

When the weight function becomes unity, the above equation, i.e., Eq.(4.24), reduces to the 7 dimensional plane wave solution as in [6, 29]. Thus for $a_0 = 1$ (see discussion after Eq.(4.19)), the above expression for the DFRA scalar field becomes

$$\phi(x,\tilde{\theta}) = \int \frac{d^3p}{\sqrt{(2\pi)^3}} \frac{d^3\tilde{k}}{\sqrt{(2\pi)^3}} \frac{1}{\sqrt{2\omega(p,\tilde{k})}} \bigg(\Big(e^{-i(px+\tilde{k}\tilde{\theta})} + e^{-ipx}\mathcal{G}(\tilde{k},\tilde{\theta}) \Big) a(p,\tilde{k}) + h.c \bigg), \tag{4.25}$$

where

$$\mathcal{G}(\tilde{k},\tilde{\theta}) = \sum_{n=1}^{\infty} \Big(A_n(\lambda,\tilde{k},\tilde{\theta})(\tilde{k}\tilde{\theta})^{2n} + a_1 B_n(\lambda,\tilde{k},\tilde{\theta})(\tilde{k}\tilde{\theta})^{2n+1} \Big). \tag{4.26}$$

When the weight function becomes one and in the limit $\lambda \to 0$ with $\theta = 0$, the above equation, i.e, Eq.(4.26), vanishes. Thus we observe that the DFRA scalar field reduces to the commutative one in the limit $\lambda \to 0$.

The unequal-time commutation relation between the non-commutative field and its adjoint in DFR space-time can be obtained by generalising Eq.(2.13) (see chapter 2) to DFR space-time. This is done by replacing $\Delta(x - x')$ with $\Delta(x, x'; \theta, \theta')$ as well as $d(\partial)$

with $d(\partial, \partial_\theta)$. Thus we get

$$\Big[\phi(x,\tilde{\theta}), \bar{\phi}(x',\tilde{\theta}')\Big] = id(\partial,\partial_\theta)\Delta(x,x';\tilde{\theta},\tilde{\theta}'). \tag{4.27}$$

For the DFRA scalar field we have seen that $d(\partial, \partial_\theta) = \mathbb{I}$ and therefore Eq.(4.27) becomes

$$\Big[\phi(x,\tilde{\theta}), \bar{\phi}(x',\tilde{\theta}')\Big] = i\Delta(x,x';\tilde{\theta},\tilde{\theta}'). \tag{4.28}$$

The unknown function, $\Delta(x,x';\tilde{\theta},\tilde{\theta}')$, in the above equation, Eq.(4.28), can be obtained by evaluating the LHS of Eq.(4.28). Thus by using Eq.(4.25), we obtain the LHS of Eq.(4.28) as

$$\Big[\phi(x,\tilde{\theta}), \phi(x',\tilde{\theta}')\Big] = \int \frac{d^3p}{\sqrt{(2\pi)^3}}\frac{d^3\tilde{k}}{\sqrt{(2\pi)^3}}\frac{d^3p'}{\sqrt{(2\pi)^3}}\frac{d^3\tilde{k}'}{\sqrt{(2\pi)^3}}\frac{1}{\sqrt{2\omega(p,\tilde{k})}}\frac{1}{\sqrt{2\omega(p',\tilde{k}')}}$$
$$\bigg(u_{p,\tilde{k}}(x,\tilde{\theta})u^*_{p',\tilde{k}'}(x',\tilde{\theta}')[a(p,\tilde{k}),a^\dagger(p',\tilde{k}')] - u^*_{p,\tilde{k}}(x,\tilde{\theta})u_{p',\tilde{k}'}(x,\tilde{\theta}')[a(p',\tilde{k}'),a^\dagger(p,\tilde{k})]\bigg). \tag{4.29}$$

Now we assume the creation and annihilation operators of the DFRA scalar field to obey the following commutation relation

$$[a(p,\tilde{k}),a^\dagger(p',\tilde{k}')] = \delta^3(p-p')\delta^3(\tilde{k}-\tilde{k}'). \tag{4.30}$$

We derive the unequal time commutation relation between the DFRA scalar field and its adjoint by substituting the above undeformed oscillator algebra, Eq.(4.30), in Eq.(4.29). Thus we have

$$\Big[\phi(x,\tilde{\theta}), \phi(x',\tilde{\theta}')\Big] = \int \frac{d^3p}{(2\pi)^3}\frac{d^3\tilde{k}}{(2\pi)^3}\frac{1}{2\omega(p,\tilde{k})}\bigg(e^{-ip(x-x')}e^{-i\tilde{k}(\tilde{\theta}-\tilde{\theta}')} - e^{ip(x-x')}e^{i\tilde{k}(\tilde{\theta}-\tilde{\theta}')} +$$
$$e^{-ip(x-x')}\Big(e^{-i\tilde{k}\tilde{\theta}}\mathcal{G}^*(\tilde{k},\tilde{\theta}') + e^{i\tilde{k}\tilde{\theta}'}\mathcal{G}(\tilde{k},\tilde{\theta}) + \mathcal{G}(\tilde{k},\tilde{\theta})\mathcal{G}^*(\tilde{k},\tilde{\theta}')\Big) -$$
$$e^{ip(x-x')}\Big(e^{i\tilde{k}\tilde{\theta}}\mathcal{G}(\tilde{k},\tilde{\theta}') + e^{-i\tilde{k}\tilde{\theta}'}\mathcal{G}^*(\tilde{k},\tilde{\theta}) + \mathcal{G}^*(\tilde{k},\tilde{\theta})\mathcal{G}(\tilde{k},\tilde{\theta}')\Big)\bigg). \tag{4.31}$$

We obtain the explicit form of $i\Delta(x,x';\tilde{\theta},\tilde{\theta}')$ by comparing Eq.(4.31) with Eq.(4.28). We find that the last six terms on the RHS of Eq.(4.31) are the weight function dependent terms.

By taking the time derivative of Eq.(4.31) and setting $t = t'$, we obtain the commutation relation between the DFRA scalar field and its time derivative at equal times as

$$\Big[\phi(x,\tilde{\theta}), \frac{d\phi(x',\tilde{\theta}')}{dt'}\Big]\Big|_{t=t'} = i\delta^3(x-x')\delta^3(\tilde{\theta}-\tilde{\theta}') + i\delta^3(x-x')\int \frac{d^3\tilde{k}}{(2\pi)^3}\Big(\frac{e^{-i\tilde{k}\tilde{\theta}}\mathcal{G}^*(\tilde{k},\tilde{\theta}')}{2} + \frac{e^{i\tilde{k}\tilde{\theta}'}\mathcal{G}(\tilde{k},\tilde{\theta})}{2} + \frac{e^{i\tilde{k}\tilde{\theta}}\mathcal{G}(\tilde{k},\tilde{\theta}')}{2} + \frac{e^{-i\tilde{k}\tilde{\theta}'}\mathcal{G}^*(\tilde{k},\tilde{\theta})}{2} + \frac{\mathcal{G}(\tilde{k},\tilde{\theta})\mathcal{G}^*(\tilde{k},\tilde{\theta}')}{2} + \frac{\mathcal{G}(\tilde{k},\tilde{\theta}')\mathcal{G}^*(\tilde{k},\tilde{\theta})}{2}\Big). \tag{4.32}$$

The above equation gives the deformed (equal-time) commutation relation between the DFRA scalar field and its (time) derivative, valid to all orders in θ/λ. Due to the presence of the weight function dependent term $\mathcal{G}(\tilde{k},\tilde{\theta})$, the commutation relation given in Eq.(4.32) is deformed. The last six terms in the RHS of Eq.(4.32) change with the choice of weight function. It is to be noted that the above expression reduces to the corresponding commutation relation between the DFRA scalar field and its conjugate obtained in [6] when the weight function becomes unity.

In order to show the non-commutative corrections more clearly, we write down the deformed commutation relation given in Eq.(4.32) by keeping the expression valid up to the first non-vanishing terms in $\tilde{\theta}$. Thus we get

$$\Big[\phi(x,\tilde{\theta}), \frac{d\phi(x',\tilde{\theta}')}{dt'}\Big]\Big|_{t=t'} = i\delta^3(x-x')\delta^3(\tilde{\theta}-\tilde{\theta}') + i\delta^3(x-x')\int \frac{d^3\tilde{k}}{(2\pi)^3}\Big(\frac{(\tilde{k}\tilde{\theta}')^3}{12\tilde{k}^2\lambda^2} + \frac{(\tilde{k}\tilde{\theta})^3}{12\tilde{k}^2\lambda^2}\Big). \tag{4.33}$$

Recalling $\tilde{\theta} = \theta/\lambda$, $\tilde{k} = k\lambda$, we find that the first non-vanishing correction terms associated with the deformed commutation relation depends on θ^3 terms. Further, we also observe that the dependency of this correction term on the non-commutative length scale is of the form $1/\lambda^4$.

In the above calculations, we have considered an undeformed DFRA oscillator algebra as given in Eq.(4.30). Now let us consider the DFRA oscillator algebra to be deformed such that the commutation relation between the DFRA field and its adjoint is undeformed. Thus we assume the deformed oscillator algebra to be

$$[a(p,\tilde{k}), a^\dagger(p',\tilde{k}')] = \delta^3(p-p')\delta^3(\tilde{k}-\tilde{k}')g(\tilde{k}), \tag{4.34}$$

where $g(\tilde{k})$ is an unknown function whose exact form has to be evaluated.

By substituting the above deformed oscillator algebra, i.e., Eq.(4.34), in Eq.(4.29) and repeating the above steps, we obtain the unequal-time commutation relation between

DFRA scalar field and its adjoint as

$$\Big[\phi(x,\tilde{\theta}),\phi(x',\tilde{\theta}')\Big] = \int \frac{d^3p}{(2\pi)^3}\frac{d^3\tilde{k}}{(2\pi)^3}\frac{1}{2\omega(p,\tilde{k})}g(\tilde{k})\Big(e^{-ip(x-x')}e^{-i\tilde{k}(\tilde{\theta}-\tilde{\theta}')} - e^{ip(x-x')}e^{i\tilde{k}(\tilde{\theta}-\tilde{\theta}')}+ \\ e^{-ip(x-x')}\Big(e^{-i\tilde{k}\tilde{\theta}}\mathcal{G}^*(\tilde{k},\tilde{\theta}') + e^{i\tilde{k}\tilde{\theta}'}\mathcal{G}(\tilde{k},\tilde{\theta}) + \mathcal{G}(\tilde{k},\tilde{\theta})\mathcal{G}^*(\tilde{k},\tilde{\theta}')\Big) - \\ e^{ip(x-x')}\Big(e^{i\tilde{k}\tilde{\theta}}\mathcal{G}(\tilde{k},\tilde{\theta}') + e^{-i\tilde{k}\tilde{\theta}'}\mathcal{G}^*(\tilde{k},\tilde{\theta}) + \mathcal{G}^*(\tilde{k},\tilde{\theta})\mathcal{G}(\tilde{k},\tilde{\theta}')\Big)\Big). \tag{4.35}$$

From Eq.(4.35), we get the equal-time commutation relation between the DFRA scalar field and its time derivative as

$$\Big[\phi(x,\tilde{\theta}),\frac{d\phi(x',\tilde{\theta}')}{dt'}\Big]\Big|_{t=t'} = i\delta^3(x-x')\int\frac{d^3\tilde{k}}{(2\pi)^3}\,g(\tilde{k})\,\Big(\frac{e^{-i\tilde{k}(\tilde{\theta}-\tilde{\theta}')}}{2} + \frac{e^{i\tilde{k}(\tilde{\theta}-\tilde{\theta}')}}{2} + \frac{e^{-i\tilde{k}\tilde{\theta}}\mathcal{G}^*(\tilde{k},\tilde{\theta}')}{2} + \\ \frac{e^{i\tilde{k}\tilde{\theta}'}\mathcal{G}(\tilde{k},\tilde{\theta})}{2} + \frac{e^{i\tilde{k}\tilde{\theta}}\mathcal{G}(\tilde{k},\tilde{\theta}')}{2} + \frac{e^{-i\tilde{k}\tilde{\theta}'}\mathcal{G}^*(\tilde{k},\tilde{\theta})}{2} + \frac{\mathcal{G}(\tilde{k},\tilde{\theta})\mathcal{G}^*(\tilde{k},\tilde{\theta}')}{2} + \\ \frac{\mathcal{G}(\tilde{k},\tilde{\theta}')\mathcal{G}^*(\tilde{k},\tilde{\theta})}{2}\Big). \tag{4.36}$$

Now we demand the equal-time commutation relation between the DFRA scalar field and its time derivative to be undeformed, i.e.,

$$\Big[\phi(x,\tilde{\theta}),\frac{d\phi(x',\tilde{\theta}')}{dt'}\Big]\Big|_{t=t'} = i\delta^3(x-x')\delta^3(\tilde{\theta}-\tilde{\theta}'), \tag{4.37}$$

Thus by comparing the RHS of Eq.(4.36) with that RHS of Eq.(4.37), we get

$$\delta^3(\tilde{\theta}-\tilde{\theta}') = \int\frac{d^3\tilde{k}}{(2\pi)^3}g(\tilde{k})\Big(\frac{e^{-i\tilde{k}(\tilde{\theta}-\tilde{\theta}')}}{2} + \frac{e^{i\tilde{k}(\tilde{\theta}-\tilde{\theta}')}}{2} + \frac{e^{-i\tilde{k}\tilde{\theta}}\mathcal{G}^*(\tilde{k},\tilde{\theta}')}{2} + \frac{e^{i\tilde{k}\tilde{\theta}'}\mathcal{G}(\tilde{k},\tilde{\theta})}{2} + \\ \frac{e^{i\tilde{k}\tilde{\theta}}\mathcal{G}(\tilde{k},\tilde{\theta}')}{2} + \frac{e^{-i\tilde{k}\tilde{\theta}'}\mathcal{G}^*(\tilde{k},\tilde{\theta})}{2} + \frac{\mathcal{G}(\tilde{k},\tilde{\theta})\mathcal{G}^*(\tilde{k},\tilde{\theta}')}{2} + \frac{\mathcal{G}(\tilde{k},\tilde{\theta}')\mathcal{G}^*(\tilde{k},\tilde{\theta})}{2}\Big). \tag{4.38}$$

By utilising the definition for $\delta^3(\tilde{\theta}-\tilde{\theta}')$ on the LHS of Eq.(4.38) and after doing some re-arrangement, we obtain the explicit form of $g(\tilde{k})$ as

$$g(\tilde{k}) = \frac{1}{1+\frac{e^{i\tilde{k}\tilde{\theta}'}\mathcal{G}(\tilde{k},\tilde{\theta})+e^{i\tilde{k}\tilde{\theta}}\mathcal{G}(\tilde{k},\tilde{\theta}')+e^{-i\tilde{k}\tilde{\theta}'}\mathcal{G}^*(\tilde{k},\tilde{\theta})+e^{-i\tilde{k}\tilde{\theta}}\mathcal{G}^*(\tilde{k},\tilde{\theta}')+\mathcal{G}(\tilde{k},\tilde{\theta})\mathcal{G}^*(\tilde{k},\tilde{\theta}')+\mathcal{G}(\tilde{k},\tilde{\theta}')\mathcal{G}^*(\tilde{k},\tilde{\theta})}{2\cos\tilde{k}(\tilde{\theta}-\tilde{\theta}')}}. \tag{4.39}$$

Thus by using the above choice of $g(\tilde{k})$, i.e., Eq.(4.39), in Eq.(4.34), we get the equal-time commutation relation between the DFRA scalar field and its time derivative to be undeformed. For this specific choice of $g(\tilde{k})$, the deformed oscillator algebra given in

Eq.(4.34) becomes

$$[a(p,\tilde{k}),a^\dagger(p',\tilde{k}')]=\frac{\delta^3(p-p')\delta^3(\tilde{k}-\tilde{k}')}{1+\frac{e^{i\tilde{k}\tilde{\theta}'}\mathcal{G}(\tilde{k},\tilde{\theta})+e^{i\tilde{k}\tilde{\theta}}\mathcal{G}(\tilde{k},\tilde{\theta}')+e^{-i\tilde{k}\tilde{\theta}'}\mathcal{G}^*(\tilde{k},\tilde{\theta})+e^{-i\tilde{k}\tilde{\theta}}\mathcal{G}^*(\tilde{k},\tilde{\theta}')+\mathcal{G}(\tilde{k},\tilde{\theta})\mathcal{G}^*(\tilde{k},\tilde{\theta}')+\mathcal{G}(\tilde{k},\tilde{\theta}')\mathcal{G}^*(\tilde{k},\tilde{\theta})}{2\cos\tilde{k}(\tilde{\theta}-\tilde{\theta}')}}. \tag{4.40}$$

In the above equation, we find that the deformation factor present in the DFRA oscillator algebra depends on the non-commutative coordinates $\tilde{\theta}$ and $\tilde{\theta}'$ respectively. This deformation factor is non-unique due to non-uniqueness associated with the weight function $W(\theta)$ and this deformation factor becomes one when the weight function reduces to unity. For the particular choice $\theta=\theta'$, the deformation factor becomes, $g(\tilde{k})=\frac{1}{1+e^{-i\tilde{k}\tilde{\theta}}\mathcal{G}^*(\tilde{k},\tilde{\theta})+e^{i\tilde{k}\tilde{\theta}}\mathcal{G}(\tilde{k},\tilde{\theta})+\mathcal{G}(\tilde{k},\tilde{\theta})\mathcal{G}^*(\tilde{k},\tilde{\theta})}$. From Eq.(4.40), we obtain the deformed DFRA oscillator algebra, valid up to first non-vanishing term in θ, as

$$[a(p,\tilde{k}),a^\dagger(p',\tilde{k}')]=\delta^3(p-p')\delta^3(\tilde{k}-\tilde{k}')\bigg(1-\frac{(\tilde{k}\tilde{\theta}')^3}{12\tilde{k}^2\lambda^2}-\frac{(\tilde{k}\tilde{\theta})^3}{12\tilde{k}^2\lambda^2}\bigg). \tag{4.41}$$

In the above Eq.(4.41), we see that the first non-vanishing correction associated with the deformed DFRA oscillator algebra depends on θ^3. This is in contrast with the results obtained in the κ-deformed space-time, where the correction term depends linearly on the non-commutative parameter [31, 32]. Here the correction terms of the deformed DFRA oscillator depend on $1/\lambda^4$ also.

We observe the appearance of weight function dependent term in the deformed (equal-time) commutation relation between the DFRA scalar field and its (time) derivative in Eq.(4.32). These terms reduce to zero when the weight function becomes one and the resulting commutation relations are in agreement with that derived in [6].

4.4 Conserved currents

In this section, we use quantisation method discussed in chapter 2 to construct the conserved currents corresponding to the DFRA scalar field from its equation of motion alone [16, 17]. Using this procedure, we obtain the conserved currents associated with the translation and Lorentz symmetry of the DFRA scalar field.

In order to derive the conserved currents, we first generalise the definition of $\Gamma_\mu(\partial,-\overleftarrow{\partial})$ operator to the DFR space-time as $\Gamma_A(\partial,-\overleftarrow{\partial})$. By using Eq.(2.16) (see chapter 2), we obtain the definition for $\Gamma_A(\partial,-\overleftarrow{\partial})$ as

$$\begin{aligned}\Gamma_A(\partial,-\overleftarrow{\partial})&=\sum_{l=0}^{N-1}\sum_{i=0}^{l}\Lambda_{AA_1.....A_l}\partial_{A_1}.....\partial_{A_i}(-\overleftarrow{\partial}_{A_{i+1}})......(-\overleftarrow{\partial}_{A_l})\\&=\Lambda_A+\Lambda_{AB}(\partial^B-\overleftarrow{\partial}^B)+\Lambda_{ABC}(\partial^B\partial^C-\partial^B\overleftarrow{\partial}^C+\overleftarrow{\partial}^B\overleftarrow{\partial}^C)+......\end{aligned} \tag{4.42}$$

Substituting the components of the $\Lambda(\partial, \partial_\theta)$ operator, i.e., Eq.(4.15), in Eq.(4.42) we get the explicit form of $\Gamma_A(\partial, -\overleftarrow{\partial})$ as

$$\Gamma_A(\partial, -\overleftarrow{\partial}) = \Big(\partial_\mu - \overleftarrow{\partial}_\mu, \lambda(\partial_{\theta_i} - \overleftarrow{\partial}_{\theta_i}) - \frac{\theta_i}{2\lambda^3}\Big). \tag{4.43}$$

Using Eq.(2.20) (of chapter 2), we write down the general expression for conserved current associated with the DFRA scalar field as

$$J_A = \phi(x,\theta)\Gamma_A(\partial, -\overleftarrow{\partial})\delta\phi(x,\theta). \tag{4.44}$$

Under the translation symmetry, the DFR space-time coordinates transform as

$$x_\mu \to x_\mu + a_\mu, \;\; \theta_i \to \theta_i + b_i, \tag{4.45}$$

where a_μ and b_i are the translation parameters in the DFR space-time. The corresponding infinitesimal change in the DFRA scalar field under the above transformation is given as

$$\delta\phi(x,\theta) = -a^\mu \partial_\mu \phi(x,\theta) - b^i \partial_{\theta_i}\phi(x,\theta) \equiv -C^B \partial_B \phi(x,\theta), \tag{4.46}$$

where $C^B = (a^\mu, b^i/\lambda)$. Substituting Eq.(4.46) in Eq.(4.44), the expression for the conserved current corresponding to the translation symmetry in DFR space-time becomes

$$J_A = -C^B \phi(x,\theta)\Gamma_A(\partial, -\overleftarrow{\partial})\partial_B\phi(x,\theta). \tag{4.47}$$

By using Eq.(4.46) and Eq.(4.43) in Eq.(4.47), we obtain the explicit form of the components of the conserved current associated with the translation symmetry in DFR space-time as

$$\begin{aligned} J_\mu =& a^\nu \partial_\mu\phi(x,\theta)\partial_\nu\phi(x,\theta) - a^\nu\phi(x,\theta)\partial_\mu\partial_\nu\phi(x,\theta) - b^i\phi(x,\theta)\partial_\mu\partial_{\theta_i}\phi(x,\theta) + b^i\partial_\mu\phi(x,\theta)\partial_{\theta_i}\phi(x,\theta), \\ J_{\theta_j} =& a^\nu \partial_{\theta_j}\phi(x,\theta)\partial_\nu\phi(x,\theta) - a^\nu\phi(x,\theta)\partial_{\theta_j}\partial_\nu\phi(x,\theta) - b^i\phi(x,\theta)\partial_{\theta_j}\partial_{\theta_i}\phi(x,\theta) + b^i\partial_{\theta_j}\phi(x,\theta)\partial_{\theta_i}\phi(x,\theta) + \\ & a^\nu\frac{\theta_j}{2\lambda^3}\phi(x,\theta)\partial_\nu\phi(x,\theta) + b^i\frac{\theta_j}{2\lambda^3}\phi(x,\theta)\partial_{\theta_i}\phi(x,\theta). \end{aligned} \tag{4.48}$$

The Minkowskian part of the conserved current in the above expression does not have weight function dependent terms. Whereas the θ_i components of the conserved current possess two weight function dependent terms. Using the relation $J_A \equiv T_{AB}C^B$, we get the expression for the energy-momentum tensor T_{AB} of the DFRA scalar field as

$$T_{AB} = \phi(x,\theta)\Gamma_A(\partial, -\overleftarrow{\partial})\partial_B\phi(x,\theta). \tag{4.49}$$

The explicit form of the components of the energy-momentum tensor for the DFRA scalar field are given as

$$\begin{aligned}
T_{\mu\nu} =&\phi(x,\theta)\partial_\mu\partial_\nu\phi(x,\theta) - \partial_\mu\phi(x,\theta)\partial_\nu\phi(x,\theta),\\
T_{\mu\theta_i} =&\lambda\phi(x,\theta)\partial_\mu\partial_{\theta_i}\phi(x,\theta) - \lambda\partial_\mu\phi(x,\theta)\partial_{\theta_i}\phi(x,\theta),\\
T_{\theta_i\mu} =&\lambda\phi(x,\theta)\partial_{\theta_i}\partial_\mu\phi(x,\theta) - \lambda\partial_{\theta_i}\phi(x,\theta)\partial_\mu\phi(x,\theta) - \frac{\theta_i}{2\lambda^3}\phi(x,\theta)\partial_\mu\phi(x,\theta),\\
T_{\theta_i\theta_j} =&\lambda^2\phi(x,\theta)\partial_{\theta_i}\partial_{\theta_j}\phi(x,\theta) - \lambda^2\partial_{\theta_i}\phi(x,\theta)\partial_{\theta_j}\phi(x,\theta) - \frac{\theta_i}{2\lambda^2}\phi(x,\theta)\partial_{\theta_j}\phi(x,\theta).
\end{aligned} \tag{4.50}$$

The last three components of the energy-momentum tensor in the above expression are contributed by the θ coordinate of the DFR space-time. The $T_{\mu\nu}$ tensor is symmetric in its indices, as in the commutative case. Whereas $T_{\mu\theta_i} \neq T_{\theta_i\mu}$ and $T_{\theta_i\theta_j}$ is not symmetric. It is to be noted that the $T_{\mu\theta_i}$ tensor becomes symmetric when the weight function becomes one. As the weight function reduces to unity, the $T_{\theta_i\theta_j}$ tensor also becomes symmetric. Thus we observe that the energy-momentum tensor of the DFRA scalar field is no longer symmetric in its indices due to θ dependency of the weight function $W(\theta)$.

Using the expression for the energy-momentum tensor T_{AB}, we define the conserved momenta associated with the DFRA scalar field as

$$P_B = \int d^3x\ d^3\theta\ W(\theta)T_{0B}. \tag{4.51}$$

Under the Lorentz transformation, the coordinates of the DFR space-time transform as

$$x_\mu \to x_\mu + \omega_\mu{}^\nu x_\nu,\ \ \theta_i \to \theta_i + \omega_i{}^j\theta_j. \tag{4.52}$$

Using the above given Lorentz transformation rule, the infinitesimal change in the DFRA scalar field is given as

$$\begin{aligned}
\delta\phi(x,\theta) =& -\frac{1}{2}\omega^{\mu\nu}(x_\nu\partial_\mu - x_\mu\partial_\nu)\phi(x,\theta) - \frac{1}{2}\omega^{ij}(\theta_j\partial_{\theta_i} - \theta_i\partial_{\theta_j})\phi(x,\theta)\\
\delta\phi(x,\theta) \equiv& -\frac{1}{2}C^{AB}\phi(x,\theta)\big(X_B\partial_A - X_A\partial_B\big)\phi(x,\theta),
\end{aligned} \tag{4.53}$$

where $C^{AB} = diag(\omega^{\mu\nu},\omega^{ij})$ and $X_B = (x_\mu, \theta_i/\lambda)$.

Therefore the definition for the conserved current associated with the Lorentz symmetry of the DFR scalar field is given as

$$J_A = -\frac{1}{2}C^{CB}\phi(x,\theta)\Gamma_A(\partial,-\overleftarrow{\partial})\big(X_B\partial_C - X_C\partial_B\big)\phi(x,\theta) \equiv C^{BC}M_{ABC}, \tag{4.54}$$

where,

$$M_{ABC} = \frac{1}{2}\phi(x,\theta)\Gamma_A(\partial, -\overleftarrow{\partial})\big(X_B\partial_C - X_C\partial_B\big)\phi(x,\theta). \tag{4.55}$$

Substituting Eq.(4.53) and Eq.(4.43) in Eq.(4.44), we get the explicit form of the components of the conserved currents associated with the Lorentz symmetry of DFRA scalar field as

$$\begin{aligned}
J_\mu =& \frac{1}{2}\omega^{\nu\lambda}\phi(x,\theta)\big(\partial_\mu - \overleftarrow{\partial}_\mu\big)\big(x_\nu\partial_\lambda - x_\lambda\partial_\nu\big)\phi(x,\theta) + \frac{1}{2}\omega^{ij}\phi(x,\theta)\big(\partial_\mu - \overleftarrow{\partial}_\mu\big)\big(\theta_i\partial_{\theta_j} - \theta_j\partial_{\theta_i}\big)\phi(x,\theta)\\
J_{\theta_i} =& \frac{\lambda}{2}\omega^{\mu\nu}\phi(x,\theta)\big(\partial_{\theta_i} - \overleftarrow{\partial}_{\theta_i}\big)\big(x_\mu\partial_\nu - x_\nu\partial_\mu\big)\phi(x,\theta) + \frac{1}{2}\omega^{\mu\nu}\phi(x,\theta)\frac{\theta_i}{2\lambda^3}\big(x_\mu\partial_\nu - x_\nu\partial_\mu\big)\phi(x,\theta)+\\
& \frac{\lambda}{2}\omega^{jk}\phi(x,\theta)\big(\partial_{\theta_i} - \overleftarrow{\partial}_{\theta_i}\big)\big(\theta_j\partial_{\theta_k} - \theta_k\partial_{\theta_j}\big)\phi(x,\theta) + \frac{1}{2}\omega^{jk}\phi(x,\theta)\frac{\theta_i}{2\lambda^3}\big(\theta_j\partial_{\theta_k} - \theta_k\partial_{\theta_j}\big)\phi(x,\theta)
\end{aligned} \tag{4.56}$$

Here also, we notice that only the θ_i components of the conserved current contain the weight function dependent terms.

We write down the Lorentz symmetry generator in DFR space-time, using the tensor M_{ABC} (given in Eq.(4.55)) as,

$$\begin{aligned}
M_{BC} &= \int d^3x\ d^3\theta\ W(\theta)M_{0BC}\\
&= \int d^3x\ d^3\theta\ W(\theta)\Big(\frac{1}{2}\phi(x,\theta)\Gamma_0(\partial, -\overleftarrow{\partial})\big(X_B\partial_C - X_C\partial_B\big)\phi(x,\theta)\Big).
\end{aligned} \tag{4.57}$$

By substituting Eq.(4.43) in Eq.(4.57), we get the explicit form of the components of the Lorentz generator associated with the DFRA scalar field as

$$\begin{aligned}
M_{\mu\nu} =& \frac{1}{2}\int d^3x\ d^3\theta\ W(\theta)\bigg(\phi(x,\theta)\delta_{\mu 0}\partial_\nu\phi(x,\theta) + \phi(x,\theta)x_\mu\partial_0\partial_\nu\phi(x,\theta) - \partial_0\phi(x,\theta)x_\mu\partial_\nu\phi(x,\theta)-\\
& \phi(x,\theta)\delta_{\nu 0}\partial_\mu\phi(x,\theta) - \phi(x,\theta)x_\nu\partial_0\partial_\mu\phi(x,\theta) + \partial_0\phi(x,\theta)x_\nu\partial_\mu\phi(x,\theta)\bigg),\\
M_{\mu\theta_i} =& \frac{1}{2}\int d^3x\ d^3\theta\ W(\theta)\bigg(\lambda\phi(x,\theta)\delta_{\mu 0}\partial_{\theta_i}\phi(x,\theta) + \lambda\phi(x,\theta)x_\mu\partial_0\partial_{\theta_i}\phi(x,\theta)-\\
& \lambda\partial_0\phi(x,\theta)x_\mu\partial_{\theta_i}\phi(x,\theta) - \frac{\theta_i}{\lambda}\phi(x,\theta)\partial_0\partial_\mu\phi(x,\theta) + \frac{\theta_i}{\lambda}\partial_0\phi(x,\theta)\partial_\mu\phi(x,\theta)\bigg),\\
M_{\theta_i\mu} =& \frac{1}{2}\int d^3x\ d^3\theta\ W(\theta)\bigg(-\lambda\phi(x,\theta)\delta_{\mu 0}\partial_{\theta_i}\phi(x,\theta) - \lambda\phi(x,\theta)x_\mu\partial_0\partial_{\theta_i}\phi(x,\theta)+\\
& \lambda\partial_0\phi(x,\theta)x_\mu\partial_{\theta_i}\phi(x,\theta) + \frac{\theta_i}{\lambda}\phi(x,\theta)\partial_0\partial_\mu\phi(x,\theta) - \frac{\theta_i}{\lambda}\partial_0\phi(x,\theta)\partial_\mu\phi(x,\theta)\bigg),\\
M_{\theta_i\theta_j} =& \frac{1}{2}\int d^3x\ d^3\theta\ W(\theta)\bigg(\phi(x,\theta)\theta_i\partial_0\partial_{\theta_j}\phi(x,\theta) - \partial_0\phi(x,\theta)\theta_i\partial_{\theta_j}\phi(x,\theta)-\\
& \phi(x,\theta)\theta_j\partial_0\partial_{\theta_i}\phi(x,\theta) + \partial_0\theta_j\partial_{\theta_i}\phi(x,\theta)\bigg).
\end{aligned} \tag{4.58}$$

We see that all the components of the Lorentz generator of the DFRA scalar field depend on the weight function as an overall multiplication factor $W(\theta)$ coming through the measure. In the limit $\lambda \to 0$ with $\theta = 0$, the components such as $M_{\mu\theta_i}$, $M_{\theta_i\nu}$, $M_{\theta_i\theta_j}$ vanishes and $M_{\mu\nu}$ becomes the Lorentz generator of the commutative scalar field.

4.5 Response of detector coupled to DFRA scalar field and Unruh effect

In this section, we analyse the response of a massless DFRA scalar field coupled to a uniformly accelerating Unruh-DeWitt detector [18–20]. Here we study the Unruh effect in the presence of extra θ dimensions associated with the DFR space-time by evaluating the response function in both even and odd dimensions of DFR space-time.

We consider a monopole detector $m(\tau)$ whose trajectory is parametrised using the proper time τ and this is coupled to the massless DFRA scalar field. We define this interacting Lagrangian by replacing the usual field in the commutative interaction term with the massless DFRA scalar field. Thus we have

$$\mathcal{L}_{int} = m(\tau)\phi\big(x(\tau), \tilde{\theta}(\tau)\big). \tag{4.59}$$

We assume the field to be initially in the vacuum state $|0\rangle$ and the detector to be in its ground state $|E_0\rangle$. As the detector moves along the uniformly accelerating path, the detector will make a transition from its ground state $|E_0\rangle$ to its excited state $|E\rangle$, where $E > E_0$. As a result, the massless DFRA field makes a transition to its excited state $|\psi\rangle$. By using the time-independent, first-order perturbation theory, we write down the transition amplitude as

$$\mathcal{M}_{i\to f} = i\,\langle E, \psi| \int_{-\infty}^{\tau_0} d\tau\; m(\tau)\phi\big(x(\tau), \tilde{\theta}(\tau)\big)\,|0, E_0\rangle\,. \tag{4.60}$$

The monopole moment associated with the detector evolve as $m(\tau) = e^{iH_0\tau}m(0)e^{-iH_0\tau}$. Here H_0 represents the Hamiltonian of the monopole detector, satisfying $H_0\,|E\rangle = E\,|E\rangle$ and $H_0\,|E_0\rangle = E_0\,|E_0\rangle$, respectively. Using Eq.(4.60), we write down the transition probability as

$$\big|\mathcal{M}_{i\to f}\big|^2 = \sum_E \Big|\,\langle E|\, m(0)\,|E_0\rangle\,\Big|^2 \mathcal{F}(\Delta E) \tag{4.61}$$

where $\mathcal{F}(\Delta E)$ corresponds to the response function, which is defined as

$$\mathcal{F}(\Delta E) = \int_{-\infty}^{\tau_0}\int_{-\infty}^{\tau_0} d\tau d\tau' e^{-i\Delta E(\tau-\tau')}G^{+}\big(x(\tau), \tilde{\theta}(\tau); x(\tau'), \tilde{\theta}(\tau')\big). \tag{4.62}$$

Here $\Delta E = E - E_0$ and $G^+\big(x(\tau), \tilde{\theta}(\tau); x(\tau'), \tilde{\theta}(\tau')\big)$ is known as the positive Wightman function which is defined as

$$G^+(x, \tilde{\theta}; x', \tilde{\theta}') = \langle 0| \, \phi(x, \tilde{\theta})\phi(x', \tilde{\theta}') \, |0\rangle \, . \tag{4.63}$$

It is straightforward to calculate the Wightman function from the equation of motion alone. But here, we use the vacuum expectation value of the quantised massless DFRA scalar fields to evaluate the Wightman function. We utilise this method as one can easily see the effects of the quantised massless DFRA fields on the response function in this approach.

We get the explicit form of the positive Wightman function in $(1 + 3 + d_\theta)$ dimension [3] DFR space-time by calculating the vacuum expectation value of the massless DFRA scalar field given in Eq.(4.25). Substituting Eq.(4.25) in Eq.(4.63), we evaluate the Wightman function valid up to first order in $1/\lambda^2$ as

$$\begin{aligned} G^+(x, \tilde{\theta}; x', \tilde{\theta}') = & \frac{N(d_\theta)}{\Big[(x-x')^2 + (\tilde{\theta}-\tilde{\theta}')^2 - (t-t')^2\Big]^{(2+d_\theta)/2}} - \frac{(\tilde{\theta}')^4}{4\lambda^2} \frac{N(d_\theta)(2+d_\theta)}{\Big[(x-x')^2 + \tilde{\theta}^2 - (t-t')^2\Big]^{(4+d_\theta)/2}} \\ & - \frac{(\tilde{\theta})^4}{4\lambda^2} \frac{N(d_\theta)(2+d_\theta)}{\Big[(x-x')^2 + \tilde{\theta}'^2 - (t-t')^2\Big]^{(4+d_\theta)/2}} + \frac{ia_1(\tilde{\theta})^4}{6\lambda^2} \frac{N(d_\theta)(2+d_\theta)}{\Big[(x-x')^2 + \tilde{\theta}'^2 - (t-t')^2\Big]^{(4+d_\theta)/2}} \\ & - \frac{ia_1(\tilde{\theta}')^4}{6\lambda^2} \frac{N(d_\theta)(2+d_\theta)}{\Big[(x-x')^2 + \tilde{\theta}^2 - (t-t')^2\Big]^{(4+d_\theta)/2}} - \frac{ia_1(\tilde{\theta})^5}{12\lambda^2} \frac{N(d_\theta)(2+d_\theta)(4+d_\theta)}{\Big[(x-x')^2 + \tilde{\theta}'^2 - (t-t')^2\Big]^{(6+d_\theta)/2}} \\ & + \frac{ia_1(\tilde{\theta})^5}{12\lambda^2} \frac{N(d_\theta)(2+d_\theta)(4+d_\theta)}{\Big[(x-x')^2 + \tilde{\theta}'^2 - (t-t')^2\Big]^{(6+d_\theta)/2}} \end{aligned} \tag{4.64}$$

where, $N(d_\theta) = \frac{\Gamma\left(\frac{2+d_\theta}{2}\right)}{4\pi^{(4+d_\theta)/2}}$. All the $1/\lambda^2$ dependent terms in the above expression are due to the $\mathcal{G}(\tilde{k}, \tilde{\theta})$, $\mathcal{G}^*(\tilde{k}, \tilde{\theta})$ dependent terms of Eq.(4.25). The last six terms of the above equation are not unique due to the non-unique nature of the weight function and these terms are absent when the weight function becomes one. From the above expression, we get the commutative positive Wightman function in the limit $\theta \to 0$ and $d_\theta \to 0$.

Now we assume the detector to be moving along a uniformly accelerating trajectory, whose constant proper acceleration is denoted as A. The coordinates in this uniformly accelerating trajectory are given as

$$t(\tau) = \frac{1}{A} \sinh A\tau, \; x(\tau) = \frac{1}{A} \cosh A\tau, \; y = \text{constant}, \; z = \text{constant}, \; \tilde{\theta}_i = \text{constant (say } \theta), \tag{4.65}$$

[3] Here d_θ is the number of extra spatial dimension associated with the non-commutativity of DFR space-time.

Here we have considered the detector to be accelerating in the commutative $t-x$ plane. These non-commutative coordinates are just additional spatial coordinates in the extra dimensions and they act in the same way as y and z coordinates. Therefore the detector will continue to move along the uniformly accelerating path even in the commutative limit $\theta \to 0$. The explicit form of the positive Wightman function in this uniformly accelerating trajectory (Eq.(4.65)) is given as

$$G^{+}(\tau-\tau') = \left(\frac{A}{2}\right)^{2+d_\theta} \frac{N(d_\theta)}{\left(\sinh^2\frac{A(\tau-\tau')}{2}\right)^{(2+d_\theta)/2}} - \frac{\theta^4}{2\lambda^2}\left(\frac{A}{2}\right)^{4+d_\theta} \frac{N(d_\theta)(2+d_\theta)}{\left(\sinh^2\frac{A(\tau-\tau')}{2} + \frac{A^2\theta^2}{4}\right)^{(4+d_\theta)/2}}. \tag{4.66}$$

In the above expression, i.e., Eq.(4.66), we find that the positive Wightman function in terms of the uniformly accelerating coordinates contain θ dependent correction terms. These correction terms are not unique, as it changes with the choice of the weight function. From Eq.(4.66), we get the commutative result in the limit $\theta = 0$ and $d_\theta \to 0$.

The transition probability rate is defined using Eq.(4.61) as

$$\mathcal{T}(\Delta E) = \frac{d\left|\mathcal{M}_{i\to f}\right|^2}{d\tau_0} = \sum_E \left| \langle E|\, m(0)\, |E_0\rangle \right|^2 \frac{d\mathcal{F}(\Delta E)}{d\tau_0}. \tag{4.67}$$

By calculating $\frac{d\mathcal{F}(\Delta E)}{d\tau_0}$ (where $\tau_0 = \tau - \tau'$) using the response function defined in Eq.(4.62) and simplifying it, we obtain the rate of transition probability as

$$\mathcal{T}(\Delta E) = \sum_E \left| \langle E|\, m(0)\, |E_0\rangle \right|^2 \int_{-\infty}^{\infty} d\tau\; e^{-i\Delta E\tau} G^{+}(\tau). \tag{4.68}$$

Substituting the explicit form of the positive Wightman function, i.e., Eq.(4.66), in Eq.(4.68), we get the rate of transition probability in DFR space-time as

$$\begin{aligned}\mathcal{T}(\Delta E) =& \sum_E \left| \langle E|\, m(0)\, |E_0\rangle \right|^2 \left(\frac{A}{2}\right)^{2+d_\theta} N(d_\theta) \int_{-\infty}^{\infty} d\tau\; \frac{e^{-i\Delta E\tau}}{\left(\sinh^2\frac{A\tau}{2}\right)^{(2+d_\theta)/2}} - \\ & \sum_E \left| \langle E|\, m(0)\, |E_0\rangle \right|^2 \frac{\theta^4}{2\lambda^2}\left(\frac{A}{2}\right)^{4+d_\theta} N(d_\theta)(2+d_\theta) \int_{-\infty}^{\infty} d\tau\; \frac{e^{-i\Delta E\tau}}{\left(\sinh^2\frac{A\tau}{2} + \frac{A^2\theta^2}{4}\right)^{(4+d_\theta)/2}}.\end{aligned} \tag{4.69}$$

By solving the integrals (using [33]) and after doing some simplifications, we obtain the explicit form of the transition probability rate in $(1+3+d_\theta)$ DFR space-time as

$$\mathcal{T}(\Delta E) = \sum_E \left| \langle E|\, m(0)\, |E_0\rangle \right|^2 \left(\xi_0^{(d_\theta)} - \frac{(A\theta)^4}{32\lambda^2}\xi_1^{(d_\theta)}\right) \frac{1}{e^{2\pi\Delta E/A} + (-1)^{d_\theta+1}}, \tag{4.70}$$

where,

$$\xi_0^{(d_\theta)}(\Delta E, A) = \frac{\Gamma\left(\frac{d_\theta+2}{2}\right) A^{(1+d_\theta)}}{2\pi^{(2+d_\theta)/2}} \begin{cases} \prod_{k=0}^{d_\theta/2} \left(k^2 + \frac{(\Delta E)^2}{A^2}\right) \frac{A}{\Delta E}, & \text{if } d_\theta \text{ is even} \\ \prod_{k=0}^{(d_\theta-1)/2} \left(\frac{(2k+1)^2}{4} + \frac{(\Delta E)^2}{A^2}\right), & \text{if } d_\theta \text{ is odd} \end{cases} \tag{4.71}$$

and

$$\begin{aligned} \xi_1^{(d_\theta)}(\Delta E, A, \theta) =& \frac{2^{(4+d_\theta)}\pi e^{\pi\Delta E/A}}{\Gamma(4+d_\theta)A} \frac{\left(\frac{A^2\theta^2}{2} - 1 + A\theta\sqrt{\frac{A^2\theta^2}{4} - 1}\right)^{i\Delta E/A}}{\left(\frac{A^2\theta^2}{2} - 1 - A\theta\sqrt{\frac{A^2\theta^2}{4} - 1}\right)^{(4+d_\theta)/2}} \times \\ & {}_2F_1\left(\frac{4+d_\theta}{2}; \frac{i\Delta E}{A} + \frac{4+d_\theta}{2}; 4+d_\theta; 1 - \frac{e^{(4+d_\theta)/2}}{\left(\frac{A^2\theta^2}{2} - 1 + A\theta\sqrt{\frac{A^2\theta^2}{4} - 1}\right)}\right) \times \\ & \begin{cases} \prod_{k=1}^{d_\theta} \left(k^2 + \frac{(\Delta E)^2}{A^2}\right), & \text{if } d_\theta \text{ is even} \\ \prod_{k=1}^{d_\theta} \left(\frac{(2k-1)^2}{4} + \frac{(\Delta E)^2}{A^2}\right), & \text{if } d_\theta \text{ is odd} \end{cases} \end{aligned} \tag{4.72}$$

In Eq.(4.70), we see that the transition rate has θ dependent term and the distribution function, $(e^{2\pi\Delta E/A} + (-1)^{d_\theta+1})^{-1}$ is same as that in the commutative situation. For $d_\theta = 1$, we obtain the explicit form of the transition probability rate as

$$\begin{aligned} \mathcal{T}(\Delta E) =& \sum_E \Big| \langle E| \, m(0) \, |E_0\rangle \Big|^2 \frac{A^2}{4\pi}\left(\frac{1}{4} + \frac{(\Delta E)^2}{A^2}\right) \\ & \left[1 - \frac{A\theta^4\pi^2 e^{\pi\Delta E/A}}{6\lambda^2} \frac{\left(\frac{A^2\theta^2}{2} - 1 + A\theta\sqrt{\frac{A^2\theta^2}{4} - 1}\right)^{i\Delta E/A}}{\left(\frac{A^2\theta^2}{2} - 1 - A\theta\sqrt{\frac{A^2\theta^2}{4} - 1}\right)^{5/2}} \times \right. \\ & \left. {}_2F_1\left(\frac{5}{2}; \frac{i\Delta E}{A} + \frac{5}{2}; 5; 1 - \frac{e^{5/2}}{\left(\frac{A^2\theta^2}{2} - 1 + A\theta\sqrt{\frac{A^2\theta^2}{4} - 1}\right)}\right)\right] \frac{1}{e^{2\pi\Delta E/A} + 1}. \end{aligned} \tag{4.73}$$

From Eq.(4.73), we see that $\mathcal{T}(\Delta E)$ gets a Fermi-Dirac (FD) distribution factor, for $d_\theta = 1$ (odd-dimensional DFR space-time, i.e., total space-time dimension is (1+3)+1=5).

Similarly, for $d_\theta = 2$, we get the explicit form of the transition probability rate as

$$\mathcal{T}(\Delta E) = \sum_E \Big| \langle E|\, m(0)\, |E_0\rangle \Big|^2 \frac{A^2}{2\pi^2}\Big(1 + \frac{(\Delta E)^2}{A^2}\Big) \times \\ \Bigg[\Delta E - \frac{A\theta^4\pi^2 e^{\pi\Delta E/A}}{30\lambda^2} \frac{\left(\frac{A^2\theta^2}{2} - 1 + A\theta\sqrt{\frac{A^2\theta^2}{4} - 1}\right)^{i\Delta E/A}}{\left(\frac{A^2\theta^2}{2} - 1 - A\theta\sqrt{\frac{A^2\theta^2}{4} - 1}\right)^3} \times \\ \Big(4 + \frac{(\Delta E)^2}{A^2}\Big)\, {}_2F_1\left(3; \frac{i\Delta E}{A} + 3; 6; 1 - \frac{e^3}{\left(\frac{A^2\theta^2}{2} - 1 + A\theta\sqrt{\frac{A^2\theta^2}{4} - 1}\right)}\right)\Bigg] \frac{1}{e^{2\pi\Delta E/A} - 1}. \tag{4.74}$$

Here $\mathcal{T}(\Delta E)$ gets a Bose-Einstein (BE) distribution factor for $d_\theta = 2$ (even-dimensional DFR space-time, i.e., total space-time dimension is (1+3)+2=6). Thus we conclude that the transition probability rate has a BE distribution when the DFR space-time is even dimensional and has a FD distribution when the DFR space-time is odd dimensional. Thus these results are in terms with the results in the commutative space-time [21, 22].

The non-commutative correction terms of the transition probability rate depend on the choice of the weight function. We get the commutative results from Eq.(4.70), by setting $\theta = 0$ and $d_\theta = 0$.

From Eq.(4.70), we see that the temperature, $T = \frac{2\pi}{A}$ (Unruh temperature) associated with the thermal distribution of the transition probability rate is exactly the same as that in the commutative case. Similar results were also obtained in the κ-Minkowski space-time [24, 25].

4.6 Conclusion

The Lagrangian for the scalar theory in non-commutative space-time (particularly DFR space-time) is not unique. But the equation of motion constructed from the quadratic Casimir is unique. Thus it is imperative to study the quantisation of field theory in non-commutative space-time from its equation of motion rather than Lagrangian. In this chapter, we have quantised the DFRA scalar field from its equation of motion alone.

We have generalised the quantisation procedure [14, 15], discussed in chapter 2, to DFR space-time and derived the deformed commutation relation between the DFRA scalar field and its time-derivative (at equal-times), valid to all orders in the non-commutative parameter, by considering the usual form for the DFRA oscillator algebra. We have shown that the requirement of the commutation relation between the DFRA scalar field and its time derivative being undeformed gives rise to a deformed DFRA oscillator algebra. Furthermore, the first non-vanishing correction terms of this deformed oscillator algebra depends on the non-commutative parameter as $1/\lambda^4$. It is shown that our results are in agreement with that obtained in [6] when the weight function reduces to unity, where the commutation relation between the DFRA scalar field and its conjugate is undeformed. Deformed oscillator algebras have also been shown to occur in the κ-deformed space-time by quantising the κ-scalar field [31] and κ-Dirac field [32], from their equations of motion. Therefore we find that the deformed oscillator is a generic feature associated with the non-commutative field theories.

We have obtained the energy-momentum tensor as well as the Lorentz generator of the DFRA scalar field, by deriving the conserved currents associated with the translation and Lorentz symmetry of the scalar field, in DFR space-time. We observe that $T_{\theta_i\mu}$ components of the energy-momentum tensor are asymmetric, in its indices, due to the θ dependent weight function. Similar asymmetric energy-momentum tensor for the scalar field has also been obtained in Moyal [34] and κ-Minkowski (see chapter 2) space-times.

We have derived the non-commutative correction to the Unruh effect by examining the interaction between the uniformly accelerating monopole detector and the massless DFRA scalar field. We find that the transition probability rate picks up an extra non-commutative parameter dependent multiplication factor. The thermal distribution factor in the transition probability rate is found to be either Bose-Einstein or Fermi-Dirac, depending on whether the dimension of the (extra spatial non-commutative coordinate θ of the) DFR space-time is even or odd. We also show that the non-commutativity of the DFR space-time does not modify the temperature of the thermal distribution.

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Chapter 5

Superdense star in κ-space-time

5.1 Introduction

Superdense stars contain tightly bound matter concentrated over a small region of space. It has a central density of about $\sim 10^{16}$ kg/m^3 and radius of about $\sim 10^4 m$ [1, 2]. The gravitational force near the superdense star is very high, but it does not create an event horizon. Thus it provides an ideal laboratory to test models to study the effects of high gravitational field. The non-commutativity is expected to modify the space-time structure when the gravitational field is very strong and thus it is natural to study effects of non-commutativity on superdense star. In this chapter, we analyse superdense star in κ-deformed space-time.

For a stable star, the attractive gravitational pull towards its massive core is balanced by the outward pressure produced by nuclear fusion. When this fusion ceases, the outward pressure drops and the massive star undergo a gravitational collapse, producing a supernova explosion. The remnants of this supernova explosion contain superdense matter whose matter density is comparable to the nuclear density. This superdense matter then gets cooled and finally attain a neutron rich equilibrium state and this results in the formation of a superdense star [1, 2]. White dwarfs [3] and neutron stars [4] are some typical examples for the superdense stars.

The physical structure of these compact stars is studied using their equations of state. These equations of state are obtained by solving Einstein's equation, which is constructed from the spherically symmetric metric and the energy-momentum tensor of the matter content shich is modelled as that of the perfect fluid distribution. When the matter density of the compact star is greater than or comparable to the nuclear density, it becomes difficult to describe the dynamics of such compact stars from a single equation of state. In such situations, one studies compact stars using an approach known as the core-envelope model [5, 6]. The core-envelope model assumes a compact star to be composed of two regions - a central core and a surrounding envelope.

It has been shown in [7, 8] that the superdense stars have an anisotropic fluid distribution; as a result, the pressure exerted by the fluid in the radial and tangential directions is not uniform. This anisotropy in the pressure can happen due to various reasons like the presence of superfluids [9], phase transition [10], pion condensation [11], etc. Different implications of the anisotropy associated with compact stars have been studied in recent times. The anisotropic pressure has been shown to have significant effects on the mass to radius ratio and surface redshift of the relativistic, spherically symmetric stars [12–14]. Studies on the influence of anisotropy on the adiabatic contraction of certain spherically symmetric, non-static compact stars has been shown that the anisotropic compact stars are more stable than the isotropic ones [15]. Variety of relativistic stars have been studied using the anisotropic core-envelope model [16–24].

The density (ρ) and pressures (core pressure p, radial pressure p_r and tangential pressure p_t), obtained from the solutions of the Einstein's equation, of the core-envelope models, are physically acceptable if they satisfy the following requirements [5, 6]

(i) The density and (isotropic or anisotropic) pressures should be positive quantities throughout the star, i.e., $\rho > 0,\ p > 0,\ p_r > 0,\ p_t > 0$.

(ii) The density and pressures should decrease monotonically from a maximum value at the centre to a minimum value on the outer boundary of the star, i.e., $\frac{d\rho}{dr} < 0,\ \frac{dp}{dr} < 0,\ \frac{dp_r}{dr} < 0,\ \frac{dp_t}{dr} < 0$.

(iii) The speed of the sound[1] inside the superdense star should be less than the speed of light, i.e., $\frac{dp}{d\rho} < 1,\ \frac{dp_r}{d\rho} < 1,\ \frac{dp_t}{d\rho} < 1$.

We will generalise the above three conditions to the κ space-time and check whether the solutions to the superdense star in κ-deformed space-time are physically acceptable or not.

Recently, several studies to analyse the effects of the κ-deformed non-commutativity in cosmological objects have been reported. In [25], κ-deformed Schwarzschild metric has been constructed and this has further been used to derive κ-deformed corrections to the Hawking radiation, using the method of Bogoliubov coefficients. Compact stars in κ-deformed space-time have been investigated by deriving the κ-deformed degenerate

[1]The speed of sound associated with a medium is defined as the variation of pressure with respect to the variation of density in that medium

pressure from the partition function and also from the generalised uncertainty principle [26]. κ-deformed corrections to the entropy of the BTZ black hole have been calculated using the brick wall method as well as using the quasinormal mode frequency of the κ-scalar field (in the background of BTZ black hole) [27, 28].

In this chapter, we consider a superdense star in the κ-deformed space-time whose inner core is assumed to have an isotropic pressure distribution and an anisotropic pressure distribution in the outer envelope. We construct the κ-deformed Einstein's equation by replacing the quantities in the commutative Einstein's equation with the κ-deformed ones. We then derive the κ-deformed equations of state for the superdense star by solving deformed Einstein's equation, valid up to first order in a. We show that the κ-deformation enhances the density of the superdense star in κ-background. The core and envelope pressures also pick κ-deformed correction terms. We also show that deformed density and pressures satisfy the (above three) requirements of the physically acceptable solutions. We also obtain a bound on the κ-deformation parameter from the κ-deformed law of density variation and from the positivity condition on the deformed tangential pressure. We also calculate the κ-deformed redshift by using the compactness factor, which is defined as the ratio of the mass to the radius.

The organisation of this chapter is as follows. In section 5.2, we give a brief summary of superdense star. One starts with the 4-dimensional flat space metric to which a 3-spheroid space is embedded. This metric is parameterised by defining two geometric parameters, which play a crucial role in developing the core-envelope model for the superdense star and obtain the static, spherically symmetric metric. Then one constructs the energy-momentum tensor for superdense matter distribution by considering it as a perfect fluid with an additional anisotropic term. Using this metric and energy-momentum tensor, the Einstein's field equation describing superdense stars is derived. In section 5.3, we derive the κ-deformed metric corresponding to the superdense star, from the generalised commutation relation between the κ-deformed phase-space coordinates. Next, we construct the κ-deformed energy-momentum tensor. Using this and the deformed metric, we formulate the κ-deformed Einstein's field equation. In subsection 5.3.1, we derive the κ-deformed law of density variation, valid up to first order in a, by solving the temporal component of the deformed Einstein's equation. From this κ-deformed law of density variation, we also obtain a bound on the κ-deformation parameter. In section 5.4, we solve the deformed Einstein's field equation, explicitly in the isotropic core and the anisotropic envelope, separately, valid up to first order in a. In subsection 5.4.1, we derive the κ-deformed isotropic pressure in the core. In subsection 5.4.2, we derive the expression for the κ-deformed radial as well as tangential pressures in the envelope. We also derive a bound on the deformation parameter from the positivity condition on the deformed tangential pressure. In subsection 5.4.3, we discuss the boundary conditions associated with the κ-deformed metric and calculate the surface redshift by using the expression for compactness factor. Finally, in section 5.5, we present our results and concluding remarks.

5.2 Static spherically symmetric metric

In this subsection, we briefly summarise the derivation of the metric appropriate for representing superdense stars. We also discuss the construction of energy-momentum tensor for the same. This will enable one to write down Einstein's field equations for the superdense star, which is used for obtaining the equations of state.

We begin the discussion with the expression for the line element in the 4-dimensional Euclidean flat space.

$$d\sigma^2 = dx^2 + dy^2 + dz^2 + dw^2. \tag{5.1}$$

Now we consider a 3-spheroid whose equation is given in terms of the coordinates (x, y, z, w) as [29, 30]

$$\frac{x^2 + y^2 + z^2}{R^2} + \frac{w^2}{b^2} = 1, \tag{5.2}$$

where R and b are the semi-major and semi-minor axes associated with the 3-spheroid. The coordinates of this 3-spheroid are parametrised as [29, 30]

$$\begin{aligned} x &= R \sin\psi \sin\theta \cos\phi, \\ y &= R \sin\psi \sin\theta \sin\phi, \\ z &= R \sin\psi \cos\theta, \\ w &= b \cos\psi, \end{aligned} \tag{5.3}$$

where $0 \leq \psi \leq \pi$ and $0 \leq \phi < 2\pi$. Substituting this parametrisation, i.e., Eq.(5.3), in Eq.(5.1), the line element becomes

$$d\sigma^2 = \left(R^2 \cos^2\psi + b^2 \sin^2\psi\right) d\psi^2 + R^2 \sin^2\psi \left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{5.4}$$

It is to be noted that R and b are geometric parameters having the dimension of length. We re-express Eq.(5.4) using the transformation given by [21]

$$r = R \sin\psi \text{ and } K = 1 + \frac{b^2}{R^2}. \tag{5.5}$$

Here K represents another geometric parameter, which is dimensionless, such that $K \geq 1$. Using the transformation in Eq.(5.5), we re-write the line element given in Eq.(5.4), as

$$d\sigma^2 = \frac{\left(1 + K\frac{r^2}{R^2}\right)}{\left(1 + \frac{r^2}{R^2}\right)} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{5.6}$$

From Eq.(5.6), we find that $K = 1$ corresponds to the case of 3-dim spherically symmetric metric. After obtaining this metric, we next define the 4-dimensional space-time metric as,

$$ds^2 = e^{f(r)}dt^2 - d\sigma^2. \tag{5.7}$$

The function $f(r)$ in the above is a spherically symmetric function that depends only on the radial distance r. We use the spherical coordinates to write the spatial components of the 4-dimensional space-time metric. Substituting Eq.(5.6) in Eq.(5.7), we get the 4-dimension space-time line element as [21]

$$ds^2 = e^{f(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5.8}$$

where $e^{\lambda(r)}$ is defined as

$$e^{\lambda(r)} = \frac{1 + K\frac{r^2}{R^2}}{1 + \frac{r^2}{R^2}}. \tag{5.9}$$

From the line element given in Eq.(5.8), we obtain the non-vanishing components of corresponding metric, $g_{\mu\nu}$, as

$$g_{00}(x) = e^{f(r)},\ g_{11}(x) = -\frac{1 + K\frac{r^2}{R^2}}{1 + \frac{r^2}{R^2}},\ g_{22}(x) = -r^2,\ g_{33}(x) = -r^2\sin^2\theta. \tag{5.10}$$

We will use Eq.(5.10) to construct the non-commutative metric for studying the superdense star in κ-space-time.

The superdense matter distribution can be modelled in different ways [16–24]. Here we study the superdense matter distribution using the core-envelope model. Many compact objects, like neutron stars, white dwarfs etc, have been studied extensively using the core-envelope model [5, 6]. In this study, we assume the inner core to have an isotropic fluid distribution and the outer envelope to have anisotropic fluid distribution. Thus we express the energy-momentum tensor for the superdense star, using the energy-momentum tensor corresponding to the perfect fluid distribution, as [21, 29, 30]

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu} + \Pi_{\mu\nu}. \tag{5.11}$$

Here ρ is the density, u_μ is the unit 4-velocity of fluid distribution and p is the pressure of superdense matter and $\Pi_{\mu\nu}$ characterises the anisotropy. We assume that superdense stars possess spherical symmetry and therefore, the dynamical quantities have an explicit dependence on the radial distance r alone.

The anisotropy of the perfect fluid distribution is represented using the tensor $\Pi_{\mu\nu}$ (note that $\Pi_{\mu\nu} = 0$ for isotropic fluid distribution and therefore $\Pi_{\mu\nu} = 0$ inside the core). $\Pi_{\mu\nu}$ is defined as

$$\Pi_{\mu\nu} = \sqrt{3}S(r)\left[C_\mu C_\nu - \frac{1}{3}(u_\mu u_\nu - g_{\mu\nu})\right] \tag{5.12}$$

In the above definition, $S(r)$ denotes the magnitude of the anisotropic stress parameter (which vanishes for an isotropic fluid distribution) and C_μ denotes the unit radial vector. We choose $u_\mu = (e^{f(r)/2}, 0, 0, 0)$ and $C_\mu = (0, e^{\lambda(r)/2}, 0, 0)$, satisfying the conditions $u_\mu u_\nu g^{\mu\nu} = 1$, $C_\mu C_\nu g^{\mu\nu} = -1$ and $u_\mu C_\nu g^{\mu\nu} = 0$.

The superdense matter has uniform pressure (due to isotropy) throughout the core. However the situation inside the envelope is different. Inside the envelope, radial and tangential pressures are different due to the anisotropy. The anisotropic stress parameter S is defined using the radial pressure p_r and tangential pressure p_t as

$$S(r) = \frac{p_r(r) - p_t(r)}{\sqrt{3}}. \tag{5.13}$$

Thus now we define the radial as well as tangential pressures in terms of the core pressure $p(r)$ and anisotropic stress parameter $S(r)$ as

$$p_r(r) = p(r) + \frac{2S(r)}{\sqrt{3}} \tag{5.14}$$

and

$$p_t(r) = p(r) - \frac{S(r)}{\sqrt{3}}, \tag{5.15}$$

respectively. From above relation we clearly see that for an isotropic fluid distribution (note $S(r) = 0$ inside the core), radial and tangential pressures are equal, i.e., $p_r = p_t$.

Einstein's equation is given (in natural units) as

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{5.16}$$

where $G_{\mu\nu}$ is Einstein's tensor and it is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \tag{5.17}$$

In the above expression, $R_{\mu\nu}$ and R represent Ricci tensor and Ricci scalar, respectively. Thus by evaluating $R_{\mu\nu}$ as well as R from the metric given in Eq.(5.8) and using the energy-momentum tensor given in Eq.(5.11), one writes down the Einstein's equations explicitly. The solution to these equations give the relation between various quantities of interest relevant for the superdense star.

5.3 κ-deformed Einstein field equations

In this section, we first construct the metric in the κ-deformed space-time using the generalised commutation relation between the κ-deformed phase-space coordinates [25]. We then formulate the κ-deformed version of the energy-momentum tensor, defined in Eq.(5.11), suitable for studying the non-commutative superdense star. Einstein's field equation is generalised to the κ-deformed space-time by promoting the commutative quantities (such as energy-momentum tensor, space-time metric, Ricci tensor and Ricci scalar) to the corresponding κ-deformed quantities. We then obtain the solution corresponding to the κ-deformed Einstein's equations, valid up to the first order in the deformation parameter a. Following this, we derive the κ-deformed law of density variation for the superdense star and obtain a bound on the κ deformation parameter from this.

The generalised commutation relation for the κ-deformed phase space coordinates is [25, 31],

$$[\hat{x}_\mu, \hat{P}_\nu] = i\hat{g}_{\mu\nu}, \tag{5.18}$$

where $\hat{g}_{\mu\nu}$ is the κ-deformed metric and it is a function of the κ-deformed space-time coordinate $\hat{x}_\mu$.

We choose a specific realisation for the κ-deformed phase-space coordinates as [25],

$$\hat{x}_\mu = x_\alpha \varphi^\alpha_\mu,\ \hat{P}_\mu = g_{\alpha\beta}(\hat{y}) k^\beta \varphi^\alpha_\mu, \tag{5.19}$$

where $\hat{P}_\mu$ is the κ-deformed generalised momenta and k_μ is the conjugate momenta corresponding to the commutative coordinate x_μ. In the commutative limit, i.e., $a \to 0$, we obtain $\hat{x}_\mu \to x_\mu$ and $\hat{P}_\mu \to k_\mu$.

Note that we have introduced another set of κ-deformed space-time coordinates $\hat{y}_\mu$ in Eq.(5.19). This $\hat{y}_\mu$ is assumed to commute with $\hat{x}_\mu$, i.e., $[\hat{y}_\mu, \hat{x}_\nu] = 0$. These new coordinates are introduced only for calculational simplification [25]. The $g_{\alpha\beta}(\hat{y})$ appearing in Eq.(5.19) has same functional form as the metric in the commutative coordinate, but x_μ replaced with non-commutative coordinate $\hat{y}_\mu$.

Substituting Eq.(5.19) in the κ-deformed space-time commutation relation, i.e., $[\hat{x}_0, \hat{x}_i] = ia\hat{x}_i,\ [\hat{x}_i, \hat{x}_j] = 0$, we find a particular realisation for φ^α_μ as

$$\varphi^0_0 = 1,\ \varphi^0_i = 0,\ \varphi^i_0 = 0,\ \varphi^i_j = \delta^i_j e^{-ak^0}. \tag{5.20}$$

The coordinates $\hat{y}_\mu$ are also assumed to satisfy the κ-deformed space-time commutation relation as $[\hat{y}_0, \hat{y}_i] = ia\hat{y}_i,\ [\hat{y}_i, \hat{y}_j] = 0$. We now express $\hat{y}_\mu$ in terms of the commutative coordinate and its conjugate momenta as

$$\hat{y}_\mu = x_\alpha \phi^\alpha_\mu. \tag{5.21}$$

Using $[\hat{y}_0, \hat{y}_i] = ia\hat{y}_i,\ [\hat{y}_i, \hat{y}_j] = 0$ and $[\hat{x}_\mu, \hat{y}_\nu] = 0$, one obtains ϕ^α_μ as (see [25, 31] for details)

$$\phi^0_0 = 1,\ \phi^0_i = -ak^i,\ \phi^i_0 = 0,\ \phi^j_i = \delta^j_i. \tag{5.22}$$

Thus the explicit form of $\hat{y}_\mu$ are

$$\hat{y}_0 = x_0 - ax_jk^j, \quad \hat{y}_i = x_i. \tag{5.23}$$

Using the above in Eq.(5.19) and substituting $\hat{x}_\mu$ and $\hat{P}_\mu$ in Eq.(5.18), the κ-deformed metric is obtained as [25]

$$[\hat{x}_\mu, \hat{P}_\nu] \equiv i\hat{g}_{\mu\nu} = ig_{\alpha\beta}(\hat{y})\Big(k^\beta \frac{\partial \varphi^\alpha_\nu}{\partial k^\sigma}\varphi^\sigma_\mu + \varphi^\alpha_\mu \varphi^\beta_\nu\Big). \tag{5.24}$$

Note that $g_{\mu\nu}(\hat{y})$ in the above can be obtained by replacing the commutative coordinates with the κ-deformed coordinates in the commutative metric given in Eq.(5.10).

Substituting Eq.(5.20) in Eq.(5.24), we find the RHS of Eq.(5.24), valid up to first order in a as

$$\begin{aligned}
&[\hat{x}_0, \hat{P}_0] = ig_{00}(\hat{y}),\\
&[\hat{x}_0, \hat{P}_i] = ig_{0i}(\hat{y})\big(1 - 2ak^0\big) - ag_{ik}(\hat{y})k^k,\\
&[\hat{x}_i, \hat{P}_0] = ig_{i0}(\hat{y})\big(1 - ak^0\big),\\
&[\hat{x}_i, \hat{P}_j] = ig_{ij}(\hat{y})\big(1 - 2ak^0\big).
\end{aligned} \tag{5.25}$$

Thus we get the explicit form of the components of $\hat{g}_{\mu\nu}$ as

$$\begin{aligned}
&\hat{g}_{00} = g_{00}(\hat{y}),\\
&\hat{g}_{0i} = g_{0i}(\hat{y})\big(1 - 2ak^0\big) - ag_{im}(\hat{y})k^m,\\
&\hat{g}_{i0} = g_{i0}(\hat{y})\big(1 - ak^0\big),\\
&\hat{g}_{ij} = g_{ij}(\hat{y})\big(1 - 2ak^0\big).
\end{aligned} \tag{5.26}$$

The line element in κ-deformed space-time is now defined by replacing the commutative metric as well as the differential of the space-time coordinates with their κ-deformed versions [25], i.e.,

$$d\hat{s}^2 = \hat{g}_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu. \tag{5.27}$$

Thus the explicit form of the κ-deformed line element, valid up to first order in a, is given by

$$\begin{aligned} d\hat{s}^2 &= g_{00}(\hat{y})dx^0dx^0 + \Big(g_{0i}(\hat{y})\big(1-3ak^0\big) - ag_{im}(\hat{y})k^m\Big)dx^0dx^i \\ &+ g_{i0}(\hat{y})\big(1-2ak^0\big)dx^idx^0 + g_{ij}(\hat{y})\big(1-4ak^0\big)dx^idx^j. \end{aligned} \tag{5.28}$$

From Eq.(5.23), we see that $g_{\mu\nu}(\hat{y}_i) = g_{\mu\nu}(x_i)$. Since the cross terms involving time and space indices of the metric tensor given in Eq.(5.8) are zero (i.e., $g_{0i} = 0$), the κ-deformed metric given in Eq.(5.28) becomes [2]

$$d\hat{s}^2 = g_{00}(\hat{y})dx^0dx^0 + g_{ij}(\hat{y})\big(1-4ak^0\big)dx^idx^j. \tag{5.29}$$

From Eq.(5.8) and Eq.(5.23), we read off the metric components $g_{00}(\hat{y})$ and $g_{ij}(\hat{y})$, respectively. Using this in Eq.(5.29), we obtain κ-deformed space-time metric corresponding to the superdense star as

$$d\hat{s}^2 = e^{f(r)}dt^2 - \Big(e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\Big)\big(1-4ak^0\big). \tag{5.30}$$

From the above equation, we note that the κ-deformed metric continues to be spherically symmetric. Here we observe that only the spatial components of the space-time metric get modified under the κ-deformation and this change is through an overall multiplication factor of $(1-4ak^0)$. In the limit $a \to 0$, we recover the commutative line element given in Eq.(5.8).

Now we write the expression for the κ-deformed energy-momentum tensor, by promoting the commutative quantities, present in Eq.(5.11), to their κ-deformed versions. Thus we get

$$\hat{T}_{\mu\nu} = (\hat{\rho} + \hat{p})\hat{u}_\mu\hat{u}_\nu - \hat{p}\hat{g}_{\mu\nu} + \hat{\Pi}_{\mu\nu}, \tag{5.31}$$

where the κ-deformed unit 4-velocity is defined as $\hat{u}_\mu = u_\alpha\varphi^\alpha_\mu$ and the κ-deformed unit radial vector is defined as $\hat{C}_\mu = C_\alpha\varphi^\alpha_\mu$ (see Eq.(5.12)). Using the realisation given in Eq.(5.20), we get $\hat{u}_\mu = (e^{f(r)/2}, 0, 0, 0)$ and $\hat{C}_\mu = (0, e^{\lambda(r)/2}(1-ak^0), 0, 0)$. Substituting these quantities, $\hat{C}_\mu$ and $\hat{u}_\mu$ in Eq.(5.31), we find the components of the κ-deformed

[2] Since κ-deformed space-time is rotational invariant, the κ-deformed metric is taken to be symmetric in its indices.

energy-momentum tensor, valid up to first order in a, as

$$\begin{aligned}
\hat{T}_{00} &= \hat{\rho} e^{f(r)}, \\
\hat{T}_{11} &= \Big(\hat{p}\big(1-2ak^0\big) + \frac{2\hat{S}}{\sqrt{3}}\big(1-ak^0\big)\Big)e^{\lambda(r)}, \\
\hat{T}_{22} &= \Big(\hat{p} - \frac{\hat{S}}{\sqrt{3}}\Big)r^2\big(1-2ak^0\big), \\
\hat{T}_{33} &= \Big(\hat{p} - \frac{\hat{S}}{\sqrt{3}}\Big)r^2\sin^2\theta\big(1-2ak^0\big).
\end{aligned} \tag{5.32}$$

Next, we write down the κ-deformed Einstein's tensor by replacing the commutative quantities present in Eq.(5.17) with their κ-deformed counterparts. Therefore the explicit form of the κ-deformed Einstein's tensor is given as

$$\hat{G}_{\mu\nu} = \hat{R}_{\mu\nu} - \frac{1}{2}\hat{R}\hat{g}_{\mu\nu}. \tag{5.33}$$

Note that in the above equation, $\hat{R}_{\mu\nu}$ and $\hat{R}$ represent the κ-deformed Ricci tensor and Ricci scalar, respectively. The non-vanishing components of the κ-deformed Ricci tensor and κ-deformed Ricci scalar are calculated explicitly using the κ-deformed metric given in Eq.(5.30) and they are

$$\begin{aligned}
\hat{R}_{00} = &\frac{\Big(5R^2f'^2(r) + 2R^2rf''(r) + Kr^3f'^2(r) + r^3f'^2(r)\Big)R^2e^{f(r)}(1+2ak^0)}{4r(R^4+2R^2Kr^2+K^2r^4)} + \\
&\frac{\Big(6R^2r^2f'(r) + Kr^5f'^2(r) + 2R^2Kr^3f''(r) + 2R^2r^3f''(r)\Big)e^{f(r)}(1+2ak^0)}{4r(R^4+2R^2Kr^2+K^2r^4)} + \\
&\frac{\Big(2Kr^5f''(r) + 2R^2Kr^2f'(r) + 4Kr^4f'(r)\Big)e^{f(r)}(1+2ak^0)}{4r(R^4+2R^2Kr^2+K^2r^4)}
\end{aligned} \tag{5.34}$$

$$\begin{aligned}
\hat{R}_{11} = &\frac{2R^2r(R^2+r^2)(K-1)f'(r) - (R^2+Kr^2)(R^4+2R^2r^2+r^4)f'^2(r)}{4(R^2+Kr^2)(R^4+2R^2r^2+r^4)} + \\
&\frac{8R^2(R^2+r^2)(K-1) - 2(R^2+Kr^2)(R^4+2R^2r^2+r^4)f''(r)}{4(R^2+Kr^2)(R^4+2R^2r^2+r^4)},
\end{aligned} \tag{5.35}$$

$$\begin{aligned}
\hat{R}_{22} &= \frac{-rR^2\Big(R^2f(r) + Kr^2f(r) - 4Kr + r^2f'(r)\Big) - r^2\Big(4R^2 - 2K^2r^2 + Kr^3f'(r) + 2Kr^2\Big)}{2R^4+4R^2Kr^2+2K^2r^4}, \\
\hat{R}_{33} &= \hat{R}_{22}\sin^2\theta,
\end{aligned} \tag{5.36}$$

$$\hat{R} = \frac{\Big(5R^2f'^2(r) + 2R^2rf''(r) + Kr^3f'^2(r) + r^3f'^2(r) + 10r^2f'(r) + 2Kr^3f''(r)\Big)R^2(1+2ak^0)}{4r(R^4+2R^2Kr^2+K^2r^4)} +$$
$$\frac{\Big(Kr^5f'^2(r) + 2R^2r^3f''(r) + 2Kr^5f''(r) + 2R^2Kr^2f'(r) + 12Kr^4f'(r)\Big)(1+2ak^0)}{4r(R^4+2R^2Kr^2+K^2r^4)} -$$
$$\frac{\Big(2R^2r(R^2+r^2)^2(K-1)f'(r) - (R^2+Kr^2)(R^2+r^2)(R^4+2R^2r^2+r^4)f'^2(r)\Big)(1+2ak^0)}{4(R^2+Kr^2)^2(R^4+2R^2r^2+r^4)} -$$
$$\frac{\Big(8R^2(R^2+r^2)^2(K-1) - 2(R^2+Kr^2)(R^2+r^2)(R^4+2R^2r^2+r^4)f''(r)\Big)(1+2ak^0)}{4(R^2+Kr^2)^2(R^4+2R^2r^2+r^4)} +$$
$$\frac{\Big(R^2f(R^2+Kr^2) - 2r^2(K-1)(4R^2+K)\Big)(1+2ak^0)}{r\Big(2R^4+4R^2Kr^2+2K^2r^4\Big)} \tag{5.37}$$

In above equations, K is a dimensionless parameter whose values are bounded as $K \geq 1$ (see Eq.(5.5)).

Next, we determine the components of κ-deformed Einstein's tensor using above $\hat{R}_{\mu\nu}$ and $\hat{R}$ in Eq.(5.33) and obtain

$$\begin{aligned}
\hat{G}_{00} &= \frac{\Big(3KR^2 - 3R^2 + K^2r^2 - Kr^2\Big)e^{f(r)}(1-2ak^0)}{R^4+2KR^2r^2+K^2r^4}, \\
\hat{G}_{11} &= \frac{f'(r)R^2 - Kr + r^2f'(r) + r}{r(R^2+r^2)}, \\
\hat{G}_{22} &= \frac{r\Big(R^4rf'^2(r) + 2r^4f''(r) + 2R^4f'(r) + R^2Kr^3f^2(r) + R^2r^3f'^2(r) + 2R^2r^3f''(r)\Big)}{4(R^4+2R^2Kr^2+K^2r^4)} + \\
&\quad \frac{r\Big(4R^2r^2f'(r) + 4R^2r2R^2Kr^3f''(r) - 4R^2Kr + Kr^5f'^2(r) + 2Kr^4f'(r)\Big)}{4(R^4+2R^2Kr^2+K^2r^4)}, \\
\hat{G}_{33} &= \sin^2\theta\hat{G}_{22}.
\end{aligned} \tag{5.38}$$

From the above equation, we find that only the temporal component of Einstein's tensor gets modified under the κ-deformation, whereas the spatial components remain unchanged. Also, the modification of $\hat{G}_{00}$ is by an overall multiplication factor of $(1-2ak^0)$.

In the κ-space-time, the Einstein field equation, given in Eq.(5.16) takes the form

$$\hat{G}_{\mu\nu} = 8\pi\hat{T}_{\mu\nu}. \tag{5.39}$$

The explicit form of the κ-deformed Einstein's field equations are obtained by substituting the components of the deformed Einstein's tensor, i.e., Eq.(5.38) and deformed energy-momentum tensor, i.e., Eq.(5.32) in Eq.(5.39). From this, we obtain the three

κ-deformed equations, valid up to first order in a, as

$$8\pi\hat{\rho}\Big(1-2ak^0\Big) = \frac{(K-1)\Big(3+K\frac{r^2}{R^2}\Big)}{R^2\Big(1+K\frac{r^2}{R^2}\Big)}, \tag{5.40}$$

$$8\pi\Big(\hat{p_r}-2ak^0-\frac{2\hat{S}}{\sqrt{3}}ak^0\Big) = \frac{\Big(1+\frac{r^2}{R^2}\Big)\frac{f'(r)}{r}-\frac{(K-1)}{R^2}}{1+K\frac{r^2}{R^2}}, \tag{5.41}$$

$$\begin{aligned} 8\pi\hat{S}\sqrt{3}\Big(1-ak^0\Big) &= \frac{r(K-1)}{R^2}\left(\frac{1}{r}+\frac{f'(r)}{2}\right)\left(1+K\frac{r^2}{R^2}\right)^{-2} - \frac{(K-1)}{R^2}\left(1+K\frac{r^2}{R^2}\right)^{-1} \\ &- \left(\frac{f''(r)}{2}+\frac{f'^2(r)}{4}-\frac{f'(r)}{2r}\right)\left(1+\frac{r^2}{R^2}\right)\left(1+K\frac{r^2}{R^2}\right)^{-1}. \end{aligned} \tag{5.42}$$

Note that all these equations given above are modified under the κ-deformation. These correction terms appear due to the contributions from $\hat{G}_{00}$ and $\hat{T}_{\mu\nu}$ components, respectively.

5.3.1 κ-deformed law of density variation

In this subsection, we derive the law of density variation for the superdense star in κ-deformed space-time, valid up to first order in a, from the κ-deformed Einstein's equation given in Eq.(5.40).

According to the law of density variation [30], the density of the superdense star should be a positive quantity and its value should decrease monotonically from a maximum value at the centre to a minimum value on the outer boundary. We will check the validity of this condition in the κ-deformed space-time.

We obtain the expression for the density of superdense star in κ-space-time, by bringing the $(1-2ak^0)$ term of Eq.(5.40) to the right-hand side and keeping the terms valid up to first order in a. Thus the density of the superdense star in κ space-time becomes

$$\hat{\rho} = \frac{(K-1)\Big(3+K\frac{r^2}{R^2}\Big)}{8\pi R^2\Big(1+K\frac{r^2}{R^2}\Big)^2}\Big(1+2ak^0\Big). \tag{5.43}$$

Under the κ-deformation, density (of the super-dense star) gets modified by an overall multiplication factor $(1+2ak^0)$. The K in the above equation is a dimensionless parameter. Thus we find that the non-commutative correction enhances the density of the super-dense star (if $a>0,\ K>1$). The commutative expression for the density is recovered in the limit $a\to 0$.

The density of the superdense star has to be a positive quantity. In the κ-deformed setting, we need to choose $K > 1$ and $ak^0 > -0.50$ for the $\hat{\rho}$ to be a positive quantity. If we choose k^0 as Planck energy (i.e, $k^0 \approx 10^{19}$ GeV), we obtain a lower bound on the deformation parameter (in SI units) as $|a| \geq 10^{-36}$m.

From the expression for the density of the superdense star in κ-deformed space-time given in Eq.(5.43), we obtain the explicit form of the central density (denoted by $\hat{\rho}_0 = \hat{\rho}(r = 0)$) of the superdense star in κ-deformed space-time to be

$$\hat{\rho}_0 = \frac{3(K-1)}{8\pi R^2}\Big(1 + 2ak^0\Big). \tag{5.44}$$

Similarly we obtain the expression for the density (of super-dense star in κ-deformed space-time) at the boundary, $r = r_2$ (i.e., at outer boundary of the envelope) as,

$$\hat{\rho}(r_2) = \frac{(K-1)\Big(3 + K\frac{r_2^2}{R^2}\Big)}{8\pi R^2\Big(1 + K\frac{r_2^2}{R^2}\Big)^2}\Big(1 + 2ak^0\Big). \tag{5.45}$$

The expression for the κ-deformed density gradient is calculated from Eq.(5.43), by taking the derivative of $\hat{\rho}(r)$ with respect to r and we get

$$\frac{d\hat{\rho}}{dr} = \frac{-2K(K-1)r}{8\pi R^4}\frac{\big(5 + K\frac{r^2}{R^2}\big)}{\big(1 + K\frac{r^2}{R^2}\big)^3}\Big(1 + 2ak^0\Big). \tag{5.46}$$

By inspecting the above equation, we find that the density gradient in the κ space-time decreases monotonically throughout the super-dense star if the conditions $K > 1$ and $ak^0 > -0.50$ are maintained. Thus we observe that as r increases, $\hat{\rho}$ decreases from a maximum value $\hat{\rho}_0$ at the centre to a minimum value $\hat{\rho}(r_2)$ on the boundary and this has been shown graphically in Fig.(5.1). This behaviour is exactly similar to the behaviour of the density in the commutative case and therefore, we can say that the general form of the law of density variation is preserved in the κ-deformed space-time.

Note that from now onwards, we will use a particular choice of the geometric parameter $K = 2$ for the simplification in the remaining calculations. For $K = 2$, Eq.(5.43), Eq.(5.44) and Eq.(5.46) become

$$\hat{\rho} = \frac{\big(3 + 2\frac{r^2}{R^2}\big)}{8\pi R^2\big(1 + 2\frac{r^2}{R^2}\big)}\Big(1 + 2ak^0\Big), \tag{5.47}$$

$$\hat{\rho}_0 = \frac{3}{8\pi R^2}\Big(1 + 2ak^0\Big), \tag{5.48}$$

$$\frac{d\hat{\rho}}{dr} = \frac{-r}{2\pi R^4}\frac{\big(5 + 2\frac{r^2}{R^2}\big)}{\big(1 + 2\frac{r^2}{R^2}\big)^3}\Big(1 + 2ak^0\Big). \tag{5.49}$$

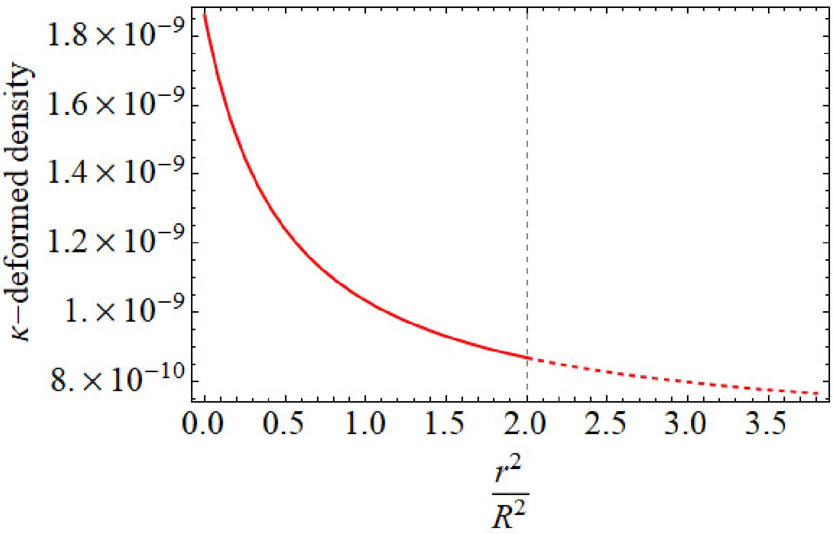

FIGURE 5.1: Variation of $\hat{\rho}$ against $\frac{r^2}{R^2}$ throughout the star ($0 \leq \frac{r^2}{R^2} \leq 3.8$)

5.4 Superdense star

In this section, we analyse the superdense star in a space-time with a minimal length (introduced through non-commutativity). We use the results of previous sections in setting up the equations of state relevant to handle this scenario. We use the core-envelope model, generalised to κ-deformed space-time to study the superdense star in the non-commutative setting. Thus we first set up the κ-deformed Einstein field equations for the isotropic core. This is followed by the same analysis for the anisotropic envelope.

5.4.1 Isotropic Core

In this subsection, we begin our analysis with κ-deformed Einstein's field equations for the isotropic core and then solve these equations, valid up to first order in a. These solutions will give the expression for the κ-deformed pressure (valid up to first order in a) inside the isotropic core of the super-dense star. We then derive the κ-deformed strong energy condition inside the isotropic core using the expression for the density and isotropic core pressure in κ space-time.

The isotropic core of the super-dense star is in the region $0 \leq r \leq r_1$. This core possess an isotropic fluid distribution and hence the anisotropic stress parameter in Eq.(5.31) vanishes, i.e., $S(r) = 0$. As a result, the radial as well as tangential pressures become equal inside the core. For the choice $K = 2$, κ-deformed Einstein's equation given in Eq.(5.42) becomes

$$\frac{r}{R^2}\left(\frac{1}{r} + \frac{f'(r)}{2}\right)\left(1 + 2\frac{r^2}{R^2}\right)^{-1} - \frac{1}{R^2} - \left(\frac{f''(r)}{2} + \frac{f'^2(r)}{4} - \frac{f'(r)}{2r}\right) = 0. \tag{5.50}$$

Now we re-express the above equation in a simpler form by choosing $z = \sqrt{1 + \frac{r^2}{R^2}}$ and $F(r) = e^{f(r)/2}$. Using these Eq.(5.50) becomes

$$(2z^2 - 1)\frac{d^2F}{dz^2} + 2z\frac{dF}{dz} - 2F = 0. \tag{5.51}$$

Solution to the above second-order differential equation is given as [21]

$$F(r) = A\sqrt{1 + \frac{r^2}{R^2}} + B\left(\sqrt{1 + \frac{r^2}{R^2}}L(r) - \frac{1}{\sqrt{2}}\sqrt{1 + 2\frac{r^2}{R^2}}\right). \tag{5.52}$$

Here A and B in the above expression represent the constants of integration (whose values will be determined later using the boundary conditions) and $L(r)$ is a function (that depends on r alone) whose explicit form is

$$L(r) = \ln\left(\sqrt{2}\sqrt{1 + \frac{r^2}{R^2}} + \sqrt{1 + 2\frac{r^2}{R^2}}\right). \tag{5.53}$$

Using the definition $F(r) = e^{f(r)/2}$ in the expression for the κ-deformed metric of the superdense star, given in Eq.(5.30), we get the explicit form of the κ-deformed metric inside the isotropic core as

$$\begin{aligned} d\hat{s}^2 = &\left[A\sqrt{1 + \frac{r^2}{R^2}} + B\left(\sqrt{1 + \frac{r^2}{R^2}}\ln\left(\sqrt{2}\sqrt{1 + \frac{r^2}{R^2}} + \sqrt{1 + 2\frac{r^2}{R^2}}\right) - \frac{1}{\sqrt{2}}\sqrt{1 + 2\frac{r^2}{R^2}}\right)\right]^2 dt^2 \\ &- \left(\frac{1 + 2\frac{r^2}{R^2}}{1 + \frac{r^2}{R^2}}\right)dr^2\left(1 - 4ak^0\right) - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\left(1 - 4ak^0\right). \end{aligned} \tag{5.54}$$

Using $f(r) = 2\ln F$ in the κ-deformed Einstein's equation, given in Eq.(5.41), we find the expression for the isotropic core pressure, in κ-deformed space-time, valid up to first order in a to be

$$\hat{p}(r) = \frac{\left[A\sqrt{1 + \frac{r^2}{R^2}} + B\left[\sqrt{1 + \frac{r^2}{R^2}}L(r) + \frac{1}{\sqrt{2}}\sqrt{1 + 2\frac{r^2}{R^2}}\right]\right]}{8\pi R^2(1 + 2\frac{r^2}{R^2})\left[A\sqrt{1 + \frac{r^2}{R^2}} + B\left[\sqrt{1 + \frac{r^2}{R^2}}L(r) - \frac{1}{\sqrt{2}}\sqrt{1 + 2\frac{r^2}{R^2}}\right]\right]}\left(1 + 2ak^0\right). \tag{5.55}$$

From the above expression, we observe that the pressure inside the isotropic core gets scaled by a $(1 + 2ak^0)$ factor. This suggests that under the κ-deformation, the isotropic core pressure of the superdense star scales in the same manner as the density gets scaled (for $a > 0$). From Fig.(5.2), we observe that the core pressure decreases monotonically from a maximum central pressure value (i.e., $\hat{p}(0)$) to a minimum value. This suggests that the gradient of the core pressure is negative, i.e., $\frac{d\hat{p}}{dr} < 0$ in $0 < r \leq r_1$.

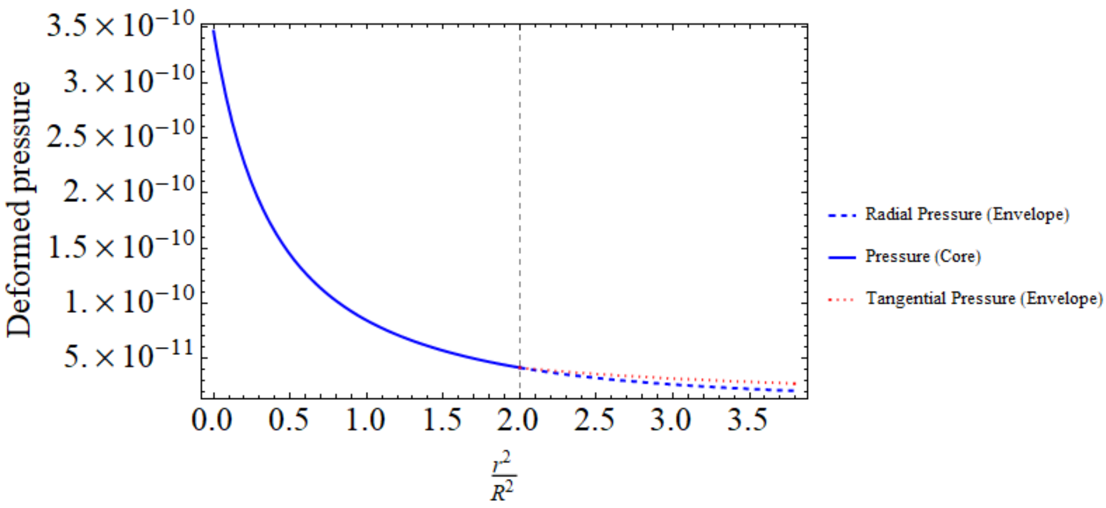

FIGURE 5.2: Variation of $\hat{p}$ against $\frac{r^2}{R^2}$ in the core ($0 \leq \frac{r^2}{R^2} \leq 2$) and variation of $\hat{p}_r$ and $\hat{p}_t$ against $\frac{r^2}{R^2}$ in the envelope ($2 \leq \frac{r^2}{R^2} \leq 3.8$).

In the commutative space-time, the speed of sound v_s associated with the fluid is defined in terms of the variation of pressure with respect to that of density, as

$$v_s^2 = \frac{dp}{d\rho} \tag{5.56}$$

This speed of sound should be less than the speed of light [3], i.e., $v_s^2 < 1$ (note that from above definition, $\frac{dp}{d\rho}$ will be a positive quantity). We obtain the expression for the speed of sound in κ-space-time, by replacing the pressure and density in Eq.(5.56) with their κ-deformed versions.

The speed of the sound inside the isotropic core is estimated by taking the derivative of core pressure with respect to the density. By exploiting the same definition, we calculate the speed of sound inside the isotropic core by taking the derivative of κ-deformed core pressure (given in Eq.(5.55)) with respect to κ-deformed density (given in Eq.(5.43)). Thus we have

$$\frac{d\hat{p}}{d\hat{\rho}} = \frac{\left(1+2\frac{r^2}{R^2}\right)}{\left(5+2\frac{r^2}{R^2}\right)} + \frac{\sqrt{2}BR^2F'(r)\left(1+2\frac{r^2}{R^2}\right)^{5/2}}{4rF^2(r)\left(5+2\frac{r^2}{R^2}\right)} + \frac{\sqrt{2}B(1+2\frac{r^2}{R^2})^{3/2}}{F(r)\left(5+2\frac{r^2}{R^2}\right)}. \tag{5.57}$$

In Fig.(5.3), we have plotted $\frac{d\hat{p}}{d\hat{\rho}}$ against $\frac{r^2}{R^2}$, using the above expression. From this graph we observe that $\frac{d\hat{p}}{d\hat{\rho}} < 1$ (inside the core). Thus we find that the speed of the sound inside the isotropic core of the κ-deformed super-dense star is less than the speed of light (which is in agreement with the commutative result of [30]). Thus the causality condition is satisfied inside the core of super-dense star in κ-deformed space-time. From Fig(5.2), we

[3] In natural units, we choose $c = 1$, whereas in SI units, this inequality becomes $v_s^2 < c^2$

also found that $\hat{p} > 0$ and $\frac{d\hat{p}}{dr} < 0$. Therefore the superdense matter distribution inside the core is physically acceptable model under κ-deformation, as in the commutative space-time [33].

5.4.2 Anisotropic Envelope

In this subsection, we analyse the κ-deformed Einstein's equation defined in the envelope region of the superdense star. The solutions to these equations yield the expression for the quantities such as κ-deformed anisotropic parameter, κ-deformed radial as well as tangential pressures, valid up to first order in a, in the envelope region.

The anisotropic envelope of the superdense star, in κ-deformed space-time, is defined as the region $r_1 < r \leq r_2$. Due to the anisotropy (i.e., $S \neq 0$), the κ-deformed envelope pressure splits into radial as well as tangential components.

We re-express the κ-deformed Einstein's field equation given in Eq.(5.41) by using the transformation (see [21]),

$$\psi = \frac{e^{f(r)/2}}{(2z-1)^{1/4}}, z = \sqrt{1 + \frac{r^2}{R^2}}, \tag{5.58}$$

as

$$\frac{d^2\psi}{dz^2} + \left(\frac{3(2z^2-1) - 5z^2}{(2z^2-1)^2} + \frac{8\sqrt{3}\pi R^2 S(1-ak^0)(2z^2-1)}{z^2-1}\right)\psi = 0. \tag{5.59}$$

Now we solve the above second-order differential equation to get the explicit form of $\psi(r)$. In order to simplify the calculation, we demand the terms in the bracket to be zero as in the commutative space-time [21]. This gives expression for the κ-deformed anisotropic stress parameter, valid up to first order in a, as

$$\hat{S} = \frac{\frac{r^2}{R^2}\left(2 - \frac{r^2}{R^2}\right)\left(1 + ak^0\right)}{8\pi\sqrt{3}R^2\left(1 + \frac{r^2}{R^2}\right)^3}. \tag{5.60}$$

From the above equation, we observe that the anisotropic parameter gets scaled by $(1 + ak^0)$ factor under the κ-deformation. Thus we find that the anisotropy associated with super-dense matter increases by non-commutativity (for $ak^0 > 0$). This increase depends on the fundamental length scale (i.e., a) and the deformation energy (k^0) entering through the non-commutative metric. We recover the commutative anisotropic stress parameter in the limit $a \to 0$. Note that, unlike the κ-deformed density and core pressure, the anisotropic stress parameter gets modified by a factor $1 + ak^0$.

After substituting Eq.(5.60) in Eq.(5.59) we get

$$\frac{d^2\psi}{dz^2} = 0. \tag{5.61}$$

The solution to the above ODE is given by

$$\psi(z) = Cz + D. \tag{5.62}$$

Here C and D are the constants of integration (whose explicit values will be determined later using boundary conditions). From Eq.(5.58) and Eq.(5.62), we get the explicit form of $f(r)$ in the envelope as

$$f(r) = 2\ln\bigg(\Big(1+2\frac{r^2}{R^2}\Big)^{1/4}\Big(C\sqrt{1+\frac{r^2}{R^2}}+D\Big)\bigg). \tag{5.63}$$

Using this explicit form of $f(r)$ given above in Eq(5.63), we obtain the expression for κ-deformed metric in the envelope of the superdense star as

$$\begin{aligned} d\hat{s}^2 = {} & \sqrt{1+2\frac{r^2}{R^2}}\Big(C\sqrt{1+\frac{r^2}{R^2}}+D\Big)^2 dt^2 - \left(\frac{1+2\frac{r^2}{R^2}}{1+\frac{r^2}{R^2}}\right)dr^2\Big(1-4ak^0\Big) \\ & - r^2\Big(d\theta^2+\sin^2\theta d\phi^2\Big)\Big(1-4ak^0\Big). \end{aligned} \tag{5.64}$$

Now we obtain the expression for the κ-deformed radial pressure (valid up to first order in a) by substituting the expression for the deformed anisotropic parameter (i.e., Eq.(5.60) in the κ-deformed Einstein's equation Eq.(5.41))

$$\hat{p}_r = \frac{\left[C\sqrt{1+\frac{r^2}{R^2}}\Big(3+4\frac{r^2}{R^2}\Big)+D\right]\Big(1+2ak^0\Big)}{8\pi R^2\Big(1+2\frac{r^2}{R^2}\Big)^2\Big(C\sqrt{1+\frac{r^2}{R^2}}+D\Big)} - \frac{\frac{r^2}{R^2}\Big(2-\frac{r^2}{R^2}\Big)ak^0}{8\pi R^2\Big(1+2\frac{r^2}{R^2}\Big)^3}. \tag{5.65}$$

Thus from the above equation, we find that the radial pressure picks up two correction terms under the κ-deformation. Here the first correction term, i.e., the $2ak^0$ dependent one is contributed by the κ-deformed core pressure and the second correction term, i.e., ak^0 dependent one is contributed by the κ-deformed anisotropic parameter (see the definition given in Eq.(5.14) for clarity). Note that the κ-deformed radial pressure has to be a positive quantity throughout the core, i.e., $\hat{p}_r > 0$, and this happens when the conditions $r \geq \sqrt{2}R$ and $ak^0 \geq 0$ are satisfied.

The expression for the κ-deformed tangential pressure (valid up to first order in a) is obtained by substituting the equations for κ-deformed radial pressure, i.e., Eq.(5.65) and anisotropic parameter, i.e., Eq.(5.60) in Eq.(5.15). Thus we have

$$\hat{p}_t = \frac{\left[C\sqrt{1+\frac{r^2}{R^2}}\left(3+4\frac{r^2}{R^2}\right)+D\right]\left(1+2ak^0\right)}{8\pi R^2\left(1+2\frac{r^2}{R^2}\right)^2\left(C\sqrt{1+\frac{r^2}{R^2}}+D\right)} - \frac{\frac{r^2}{R^2}\left(2-\frac{r^2}{R^2}\right)\left(1+2ak^0\right)}{8\pi R^2\left(1+2\frac{r^2}{R^2}\right)^3}. \tag{5.66}$$

We find that the tangential pressure also gets modified by the factor $1+2ak^0$ under the κ-deformation. This modification factor is exactly the same as the modification factor present in the density expression (see Eq.(5.47)). In order to have a physically acceptable model, the κ-deformed tangential pressure has to a positive quantity, i.e., $\hat{p}_t > 0$ and this happens when the conditions $r \geq \sqrt{2}R$ and $1+2ak^0 \geq 0$ are obeyed.

Now we consider a typical superdense neutron star whose central density, core radius and outer radius are given as $\rho_0 = 11.1145 \times 10^{17}\text{kg/m}^3$, $r_1 = 11.330 \times 10^3\text{m}$ and $r_2 = 12.527 \times 10^3\text{m}$, respectively [32]. Using the relation $r_2^2 > 2R^2$ (obtained from the positivity condition of $\hat{p}_t$) and the expression for the κ-deformed central density, $8\pi\rho_0(1-4ak^0) = \frac{3}{R^2}$, we get a bound on the deformation parameter a as $|a| > 10^{-16}\text{m}$.

From Eq.(5.65) and Eq.(5.66) and using the condition $r > \sqrt{2}R$, we see that κ-deformed tangential pressure is greater than the κ-deformed radial pressure (as in the commutative case) throughout the envelope. From Fig.(5.2), we observe that both the radial and tangential pressures decrease monotonically from the core-envelope boundary to the outer boundary of the star. Hence we can infer that the pressure (both radial and tangential) gradients are negative throughout the envelope of star, i.e, $\frac{d\hat{p}_r}{dr} < 0$, $\frac{d\hat{p}_t}{dr} < 0$ for $r_1 < r \leq r_2$.

The speed (both radial and tangential parts) of the sound inside the anisotropic envelope of the κ-deformed superdense star is estimated by taking the derivative of κ-deformed radial pressure (given in Eq.(5.55)) as well as that of the tangential pressure (given in Eq.(5.66)) with respect to κ-deformed density (given in Eq.(5.47)) as

$$\begin{aligned}\frac{d\hat{p}_r}{d\hat{\rho}} &= \frac{2\left(1+2\frac{r^2}{R^2}\right)+2\left(1-5\frac{r^2}{R^2}+\frac{r^4}{R^4}\right)ak^0}{\left(1+2\frac{r^2}{R^2}\right)\left(5+2\frac{r^2}{R^2}\right)} - \frac{C\left(1+2\frac{r^2}{R^2}\right)^2}{2\sqrt{1+\frac{r^2}{R^2}}\left(5+2\frac{r^2}{R^2}\right)\psi(r)} \\ &+ \frac{CR^2\sqrt{1+\frac{r^2}{R^2}}\left(1+2\frac{r^2}{R^2}\right)\left[\left(1+2\frac{r^2}{R^2}\right)\psi'(r)+4\frac{r^2}{R^2}\psi(r)\right]}{2r\left(5+2\frac{r^2}{R^2}\right)\psi'(r)},\end{aligned} \tag{5.67}$$

$$\frac{d\hat{p}_t}{d\hat{\rho}} = \frac{2\frac{r^2}{R^2}\left(1-\frac{r^2}{R^2}\right)}{\left(5+2\frac{r^2}{R^2}\right)\left(1+2\frac{r^2}{R^2}\right)} - \frac{C\left(1+2\frac{r^2}{R^2}\right)^2}{2\sqrt{1+\frac{r^2}{R^2}}\left(5+2\frac{r^2}{R^2}\right)\psi(r)} + \frac{CR^2\sqrt{1+\frac{r^2}{R^2}}\left(1+2\frac{r^2}{R^2}\right)\left[\left(1+2\frac{r^2}{R^2}\right)\psi'(r)+4\frac{r^2}{R^2}\psi(r)\right]}{2r\left(5+2\frac{r^2}{R^2}\right)\psi'(r)}. \tag{5.68}$$

From Eq.(5.67), we find that the speed of the sound associated with the radial pressure inside the envelope of κ-deformed superdense star picks up an ak^0 dependent correction term under the κ-deformation. Similarly, from Eq.(5.68), we find that the speed of the sound associated with the tangential pressure inside the envelope of κ-deformed superdense star remains unchanged under the κ-deformation. In Fig.(5.3), we have plotted $\frac{d\hat{p}_r}{d\hat{\rho}}$ as well as $\frac{d\hat{p}_t}{d\hat{\rho}}$ against $\frac{r^2}{R^2}$ using the above expressions. From this graph, we find that $\frac{d\hat{p}_r}{d\hat{\rho}} < 1$ and $\frac{d\hat{p}_t}{d\hat{\rho}} < 1$. Thus we infer that the speed of sound (due to radial and tangential pressures) inside the anisotropic envelope of the κ-deformed superdense star is less than the speed of light (which is in agreement with the commutative case [30, 33]). Therefore we find that the causality condition is followed in the envelope region of the superdense star in the κ-space-time. From Fig(5.2), we also find that $\hat{p}_t > 0$, $\hat{p}_r > 0$ and $\frac{d\hat{p}_t}{dr} < 0$, $\frac{d\hat{p}_r}{dt} < 0$. Hence we say that the superdense matter distribution inside the envelope is a physically acceptable model [33] under κ-deformation.

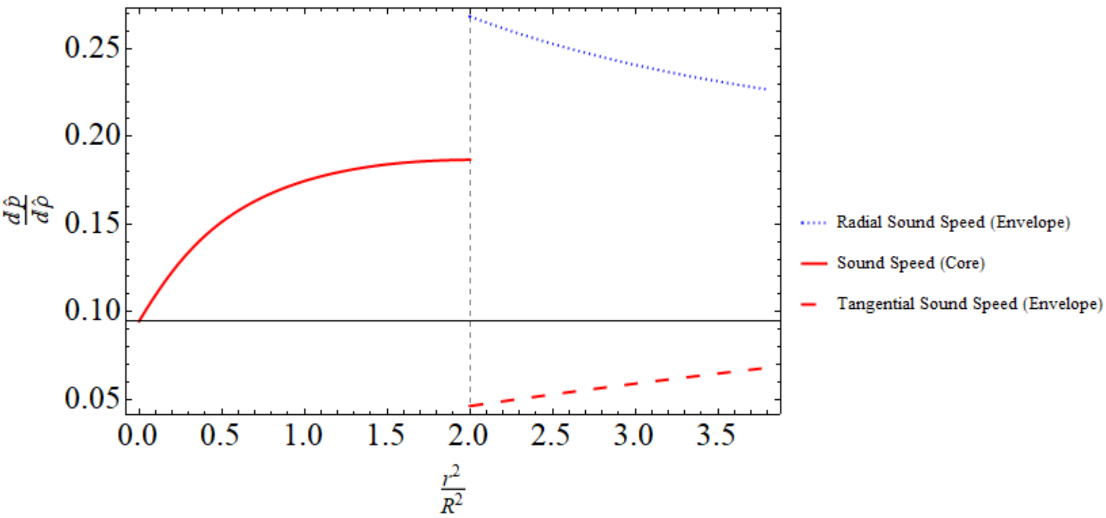

FIGURE 5.3: Variation of $\frac{d\hat{p}}{d\hat{\rho}}$ against $\frac{r^2}{R^2}$ in the core ($0 \le \frac{r^2}{R^2} \le 2$) and variation of $\frac{d\hat{p}_r}{d\hat{\rho}}$ and $\frac{d\hat{p}_t}{d\hat{\rho}}$ against $\frac{r^2}{R^2}$ in the envelope ($2 \le \frac{r^2}{R^2} \le 3.8$).

5.4.3 Matching conditions

In the commutative case, the constants A, B, C and D appearing in Eq.(5.55), Eq.(5.65), Eq.(5.66) are determined using the four matching conditions. Here we generalise these four matching conditions to the κ-deformed space-time. The following are the four matching conditions.

(i) We match the κ-deformed metric of the superdense star with κ-deformed Schwarzschild metric at the outer boundary of the envelope. The κ-deformed Schwarzschild metric is given as [25]

$$d\hat{s}^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \frac{1}{\left(1 - \frac{2m}{r}\right)}\left(1 - 4ak^0\right)dr^2 - r^2\left(1 - 4ak^0\right)d\Omega^2. \tag{5.69}$$

Now we equate the coefficients of the metric in Eq.(5.64) with that in Eq.(5.69) at $r = r_2$ and we get

$$m = \frac{r_2^3}{2R^2\left(1 + 2\frac{r_2^2}{R^2}\right)}, \tag{5.70}$$

$$C\sqrt{1 + \frac{r_2^2}{R^2}} + D = \frac{\sqrt{1 + \frac{r_2^2}{R^2}}}{\left(1 + 2\frac{r_2^2}{R^2}\right)^{3/4}}. \tag{5.71}$$

(ii) The (κ-deformed) radial pressure vanishes at the outer boundary (at $r = r_2$) of superdense star, i.e., $\hat{p}_r(r_2) = 0$ [30]. Thus we have

$$8\pi\hat{p}_r(r_2) = \frac{\left[C\sqrt{1 + \frac{r_2^2}{R^2}}\left(3 + 4\frac{r_2^2}{R^2}\right) + D\right]\left(1 + 2ak^0\right)}{R^2\left(1 + 2\frac{r_2^2}{R^2}\right)^2\left(C\sqrt{1 + \frac{r_2^2}{R^2}} + D\right)} - \frac{\frac{r_2^2}{R^2}\left(2 - \frac{r_2^2}{R^2}\right)ak^0}{R^2\left(1 + 2\frac{r_2^2}{R^2}\right)^3} = 0. \tag{5.72}$$

We obtain C and D, by solving the linear equations in Eq.(5.71) and Eq.(5.72), as

$$C = \frac{-\left(\left(1 + 2\frac{r_2^2}{R^2}\right)\left(1 + 2ak^0\right) - 2\frac{r_2^2}{R^2}\left(2 - \frac{r_2^2}{R^2}\right)ak^0\right)}{2\left(1 + 2\frac{r_2^2}{R^2}\right)^2\left(1 + 2ak^0\right)\left(1 + 2\frac{r_2^2}{R^2}\right)^{3/4}}, \tag{5.73}$$

$$D = \sqrt{1 + \frac{r_2^2}{R^2}}\frac{\left(\left(3 + 4\frac{r_2^2}{R^2}\right)\left(1 + 2\frac{r_2^2}{R^2}\right)\left(1 + 2ak^0\right) - 2\frac{r_2^2}{R^2}\left(2 - \frac{r_2^2}{R^2}\right)ak^0\right)}{2\left(1 + 2\frac{r_2^2}{R^2}\right)^2\left(1 + 2ak^0\right)\left(1 + 2\frac{r_2^2}{R^2}\right)^{3/4}}. \tag{5.74}$$

Thus we find that the κ-deformation modifies the constants C and D. Note that even though C and D get modified, the quantity $\left(C\sqrt{1 + \frac{r_2^2}{R^2}} + D\right)$ remains unchanged and therefore $\hat{g}_{00}$ component of the metric on the boundary of superdense star remains unchanged under the κ-deformation (see Eq.(5.25)).

(iii) The metric is continuous throughout the superdense star in the κ-deformed space-time. Hence the metric coefficients in the core and that in the envelope are equal at the core-envelope boundary (at $r = r_1$). Therefore we get from Eq.(5.54) and Eq.(5.64)

$$\sqrt{3}A + B[\sqrt{3}L(r_1) - \sqrt{2.5}] = \frac{\sqrt{3}C + D}{5^{3/4}}. \tag{5.75}$$

(iv) The (deformed) pressures are continuous throughout the superdense star. At the core-envelope boundary (i.e., at $r = r_1$), the anisotropy vanishes and all these pressures become equal. Hence we have

$$\hat{p}(r_1) = \hat{p}_r(r_1) = \hat{p}_t(r_1). \tag{5.76}$$

The above condition gives us another equation involving $A,\ B,\ C$ and D as

$$\sqrt{3}A + B[\sqrt{3}L(r_1) + \sqrt{2.5}] = \frac{11\sqrt{3}C + D}{5^{3/4}}. \tag{5.77}$$

Now we obtain A as well as B in terms of C and D by solving the above linear equations Eq.(5.75) and Eq.(5.77) simultaneously. Thus we get

$$\begin{aligned} A &= \frac{[5\sqrt{5} - 3\sqrt{2}(\sqrt{3}L(r_2) - \sqrt{2.5})]C + \frac{1}{\sqrt{3}}[5\sqrt{5} + 2\sqrt{2}(\sqrt{3}L(r_1) - \sqrt{2.5})]D}{5^{3/4}}, \\ B &= \frac{\sqrt{2}[3\sqrt{3}C - 2D]}{5^{3/4}}. \end{aligned} \tag{5.78}$$

By substituting Eq.(5.73) and Eq.(5.74) in Eq.(5.78), we obtain A and B.

An observer at infinity finds the wavelength of light coming out of the gravitational field (at r_2) to be shifted and this is known as gravitational redshift. We calculate the gravitational redshift using the gravitational redshift parameter, $\mathcal{Z}$, which is defined as

$$\mathcal{Z}_R = \sqrt{\frac{g_{00}(\infty)}{g_{00}(r_2)}} - 1. \tag{5.79}$$

$\mathcal{Z}$ plays an important role in the study of astrophysical objects. Therefore it is important to understand the influence of minimal length in the gravitational redshift for the analysis of the effects of non-commutativity on superdense star. In our case, we look for the effect on the κ-deformation on the redshift. We obtain this expression by replacing the commutative metric (in redshift expression i.e., Eq.(5.79)) with the corresponding κ-deformed metric and therefore we get

$$\mathcal{Z}_R = \sqrt{\frac{\hat{g}_{00}(\infty)}{\hat{g}_{00}(r_2)}} - 1 = \frac{1}{\sqrt{1 - \frac{2m}{r_2}}} - 1. \tag{5.80}$$

Now we define the compactness factor, u, which gives the mass-to-radius ratio of the superdense star. It is defined as [34, 35]

$$u = \frac{m(r_2)}{r_2}. \tag{5.81}$$

From the matching condition, i.e, Eq.(5.70), the expression for compactness factor is found to be

$$u = \frac{m(r_2)}{r_2} = \frac{r_2^2}{2R^2\left(1 + 2\frac{r_2^2}{R^2}\right)}. \tag{5.82}$$

Substituting the expression for compactness factor, i.e., Eq.(5.82) in Eq.(5.80), we obtain the expression for surface redshift [36] in κ-deformed space-time as

$$\mathcal{Z}_R = \frac{1}{\sqrt{1-2u}} - 1 = \sqrt{\frac{1 + 2\frac{r_2^2}{R^2}}{1 + \frac{r_2^2}{R^2}}} - 1. \tag{5.83}$$

From the above expression, we notice that the surface redshift (associated with the super-dense star) does not get modified under the κ-deformation. This is because of the fact that the $\hat{g}_{00}$ component of κ-deformed metric remains unchanged under κ-deformation, up to first order in a. Therefore, in this choice of realisation, we see that the gravitational redshift does not pick up a non-commutative correction term. This result is in contrast to that in [37]. There the authors have obtained a non-commutative correction to gravitational redshift by using a modified metric whose temporal part contains minimal length parameters (obtained from the GUP).

5.5 Conclusions

In this chapter, we have studied the effects on the non-commutativity in the superdense star, by generalising the anisotropic core-envelope model to the κ-deformed space-time. We have derived the κ-deformed energy-momentum tensor as well as the deformed metric (valid up to first order in a) for the superdense star. Using this, we have constructed κ-deformed Einstein's equation for the superdense star. These deformed field equations are solved separately inside the core and the envelope of the superdense star (valid up to first order approximation in a).

We observe that the density for the super-dense star in the κ-deformed space-time, Eq.(5.43), gets scaled by a $(1 + 2ak^0)$ factor. Here it is to be emphasised that the non-commutativity enhances the density of the star. Eq.(5.43) and Eq.(5.46) give the expressions for κ-deformed law of density variation, which is similar to the law of density variation in the commutative space. From the conditions $\hat{\rho} > 0$ and $\frac{d\hat{\rho}}{dr} < 0$ (obtained from the κ-deformed law of density variation), we have obtained a bound on the deformation parameter given by $|a| > 10^{-36}$m. In the Fig.(5.1), we see that the deformed density decreases from a maximum value at centre ($\hat{\rho}_0$) to a minimum value on the boundary of star and this is similar to the behaviour of the density in commutative case [21]. This similarity in the behaviour of density is due to the fact that the non-commutative correction is encoded only through scaling by the $(1 + 2ak^0)$ factor.

The isotropic core pressure, Eq.(5.55), of the superdense star also gets scaled by the factor $(1 + 2ak^0)$. Similarly, the anisotropic parameter also gets modified by a $(1 + ak^0)$ factor under the κ-deformation. The κ-deformed radial pressure inside the anisotropic envelope of the superdense star possesses two correction terms; $2ak^0$ dependent term (which is contributed by the deformed isotropic core pressure) and an ak^0 dependent term (which is contributed by the deformed anisotropic parameter). Similarly, we observe that the tangential pressure, Eq.(5.66), inside the anisotropic envelope, gets scaled by the $(1 + 2ak^0)$ factor. The positivity condition on the tangential pressure yields another limit on the deformation parameter given by $|a| > 10^{-16}$m. It is to be noted that this bound is in agreement with that obtained in [26]. It has been seen in Fig.(5.2) that the deformed pressure (inside the core as well as in the envelope) is positive throughout the star, i.e., $\hat{p} > 0$, $\hat{p}_r > 0$, $\hat{p}_t > 0$ throughout the star. From the Fig.(5.2), we observe that the deformed tangential pressure is greater than the deformed radial pressure throughout the envelope, i.e, $\hat{p}_t > \hat{p}_r$ for $r_1 < r \leq r_2$. Thus we observe that the tangential pressure is greater than radial pressure even under the κ-deformation and this is in agreement with the behaviour of radial and tangential pressures in the commutative case [21]. From the Fig.(5.3), we observe that $\frac{d\hat{p}}{d\hat{\rho}} < 1$ (speed of sound due to core pressure), $\frac{d\hat{p}_r}{d\hat{\rho}} < 1$ (speed of sound due to radial pressure) and $\frac{d\hat{p}_t}{d\hat{\rho}} < 1$ (speed of sound due to tangential pressure) are satisfied throughout the κ-deformed superdense star. Therefore we conclude that the speed of sound does not exceed the speed of light inside the κ-deformed superdense star (as that in commutative case [30]). We also find that $\hat{\rho} > 0$, $\hat{p} > 0$, $\hat{p}_r > 0$, $\hat{p}_t > 0$ and $\frac{d\hat{p}}{dt} < 1$, $\frac{d\hat{p}_t}{dt} < 1$, $\frac{d\hat{p}_r}{dt} < 1$. Thus we find that the solutions obtained satisfy the physical requirements and hence the κ-deformed superdense star model is physically acceptable (up to first order in a).

We have further observed that the surface gravitational redshift for superdense star does not get modified under the κ-deformation. This is due to the fact that the $\hat{g}_{00}$ component of the deformed metric does not pick a κ-deformation factor (valid up to first order in a).

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Chapter 6

Maximal acceleration in non-commutative space-time

6.1 Introduction

The presence of minimal length scale in certain quantum gravity models have been shown to be related to the occurrence of an upper bound on the proper acceleration [1], known as maximal acceleration. The existence of a critical acceleration associated with the string length has been shown in [2]. Covariant loop quantum gravity models have also been shown to have maximal acceleration, which is consistent with the local Lorentz symmetry [3]. The maximal acceleration relative to the vacuum has been defined using the fundamental constants c, $\hbar$ and G as [4]

$$\mathcal{A}_0 = \sqrt{\frac{c^7}{\hbar G}} \simeq 5.6 \times 10^{52}\ \mathrm{ms}^{-2} \tag{6.1}$$

In [1–3], the maximal acceleration has been studied using the notion of minimal length scale associated with the quantum gravity models. But the concept of maximal acceleration has been introduced and analysed much earlier in [4–11]. The maximal acceleration, corresponding to a massive particle, has been derived from the line element on 8-dimensional phase-space, which is constructed from the 4-dimensional Minkowski space-time and the energy-momentum dispersion relation by geometrising the quantum mechanics [5, 6]. Similar expression for the maximal acceleration has also been derived using Heisenberg's uncertainty relation [7–11]. Maximal acceleration has been obtained in [4], by comparing the Sakharov's absolute maximum temperature [12] with the Unruh temperature [13] of a thermal radiation. In [14], the invariance of the 8-dimensional line element under the reciprocity transformation has been discussed and the existence of maximal acceleration has been shown using the reciprocity principle [15].

In order to analyse the relativistic kinematics of a particle having maximal acceleration, the usual 4-dimensional Minkowski space-time has been extended to 8-dimensional space, whose extra four coordinates are the 4-momenta [16]. Further a new transformation law which leaves this 8-dimensional line element invariant has been constructed in [16]. This 8-dimensional space has been shown to induce a mass-dependent curvature, leading to the violation of equivalence principle [17]. The appearance of scalar curvature has also been obtained while extending the usual 4-dimensional Rindler metric to the 8-dimensional Rindler metric in phase space, which contains the maximal acceleration term [18]. This modified Rindler metric has further been used to obtain modified Unruh temperature, having maximal acceleration dependent correction term [19]. Modified expressions for the Unruh temperature as well as Hawking temperature, containing maximal acceleration dependent correction term has also been derived heuristically in [20]. The dynamics of particles (with maximal acceleration) has been analysed in the curved space-times such as Schwarzschild [21], Reissner-Nordstrom [22] and Kerr [23].

These line elements constructed in the above works were not general covariant. As a result different line elements were constructed to cure the problem of general covariance. One way of constructing the covariant line element is by introducing a non-linear connection in the velocity dependent terms of the line element. In [24], Christoffel symbol has been choosen as the non-linear connection, whereas in [25–28], the non-linear connection has been obtained as a combination of Christoffel symbol and Cartan tensor. The covariant line element associated with the maximal acceleration has also been constructed by the pseudo-complexification of Minkowski space [29].

The implications of the maximal acceleration has been studied in various contexts. The maximal acceleration has been shown to regularise the UV divergences of the quantum field theories [30]. It has been shown that the initial singularity associated with standard model of cosmology can be avoided using the maximal acceleration [31]. The maximal acceleration has also been used to obtain a finite entropy for a black hole [32]. In [33, 34], the maximal acceleration has been used to estimate the upper bound on the mass of the Higgs boson. The expression for maximal acceleration has been used to obtain Sakharov's absolute maximum temperature [35]. The maximal acceleration also helps in determining the parameters present in the generalised uncertainty principle [36].

It must be clear from the above summary that the analysis of maximal acceleration and its implications in models of quantum gravity and quantum geometry of intrinsic interest. Thus it is of importance to study how the non-commutativity of the space-time, which is expected at minimum length scales, affects the maximal acceleration and its consequences. We derive the first order corrections to the maximal acceleration in the κ-deformed space-time. This is derived from the κ-deformed 8-dimensional phase-space metric, which is constructed from the 4-dimensional κ-Minkowski space and the κ-deformed dispersion relation. We also obtain the first order corrections of maximal acceleration using the κ-deformed uncertainty principle. Further we show that the first order correction obtained in these two methods depend on the rest mass of the particle, but they differ by a numerical factor.

Apart from this, we also show that the non-commutative geometry associated with the κ-Minkowski space-time induces an upper limit on the proper acceleration of the massive particle. We then show the emergence of this maximal acceleration from the 4-dimensional κ-Minkowski space-time itself, instead of using the 8-dimensional phase-space metric. Unlike the former expression of the maximal acceleration, the maximal acceleration obtained here is found to depend on the metric deformation energy. In addition to this, it also depends on the dimensionless non-commutative parameters α and β, coming from the realisation (see Eq.(1.13) of chapter 1) associated with the κ-deformed space-time coordinate. We further derive the Newtonian force equation in the κ-deformed space-time, valid up to first order in a. This correction term is shown to violate the equivalence principle and by comparing this with the experimental results on the violation of equivalence principle, we set bounds on the dimensionless non-commutative parameters α and β.

This chapter is organised in the following way. In sec.6.2, we construct the 8-dimensional κ-deformed phase-space metric, valid up to first order in a, from 4-dimensional κ-Minkowski metric and κ-dispersion relation (which is obtained from the Casimir of the undeformed κ-Poincare algebra). We then derive the expression for maximal acceleration, valid up to first order in a, from the 8-dimensional κ-deformed space-time. We then use the Unruh temperature, which is the temperature of the thermal radiation seen by a uniformly accelerating observer. By relating this with the maximal acceleration of the observer, we set an upper cut off on the Unruh temperature. Further, by matching the first order correction term of the deformed maximal temperature with the experimental results on the Unruh radiation, we get a bound on the κ-deformation parameter a. In sec.6.3, we construct the maximal acceleration in κ space-time using the κ-deformed uncertainty relation. In sec.6.4, we construct 4-dimensional κ-Minkowski space-time using the α, β realisation (see Eq.(1.13) of chapter 1). We then show the emergence of maximal acceleration from this 4-dimensional κ-Minkowski space-time. Further, we also obtain the corresponding maximal temperature. In subsec.6.4.1, we derive the κ-geodesic equation and obtain the corresponding Newtonian limit, valid up to first order in a. We show that this κ-deformed Newton's force equation contains an equivalence principle violating term and by comparing this with the experimental data, we get bounds on α and β. Finally in sec.6.5, we summarise our results and give the concluding remarks.

6.2 κ-deformed maximal acceleration

In this section we first construct the 8-dimensional κ-deformed line element using the 4-dim κ-deformed Minkowski metric and the κ-deformed dispersion relation, valid up to first order in a. We then obtain the κ-deformed corrections to the maximal acceleration of a massive particle and analyse its implications.

In [5, 6], the maximal acceleration of a massive particle has been derived from the causally connected events in the 8-dimensional phase-space. The line element associated with this 8-dim phase-space has been constructed from the 4-dimensional Minkowski space-time and the energy-momentum dispersion relation. Here we follow the same approach for calculating the κ-deformed correction to the maximal acceleration.

The κ-deformed dispersion relation is given as (see Eq.(1.23) of chapter 1) [37]

$$\frac{4}{a^2}\sinh^2\frac{ap^0}{2} - p_i^2\frac{e^{-ap^0}}{\varphi} + \frac{a^2}{4}\Big(\frac{4}{a^2}\sinh^2\frac{ap^0}{2} - p_i^2\frac{e^{-ap^0}}{\varphi}\Big)^2 = m^2. \tag{6.2}$$

We now choose a realisation $\varphi = e^{-ap^0}$ [38–40] and using this realisation in Eq.(6.2), we get the κ-deformed dispersion relation, valid up to first order in a, as (with $p^0 = E$)

$$E^2 - p^2(1 + aE) - m^2 = 0. \tag{6.3}$$

By denoting $\hat{p} = p\sqrt{1 + aE}$, we re-write the above dispersion relation as

$$E^2 - \hat{p}^2 - m^2 = 0. \tag{6.4}$$

The line element in the κ-deformed space-time is defined as [41]

$$d\hat{s}^2 = \hat{g}_{\mu\nu}d\hat{x}^\mu d\hat{x}^\nu. \tag{6.5}$$

Up to first order in a, the above line element takes the form (see Eq.(5.28) of chapter 5 for details)

$$\begin{aligned} d\hat{s}^2 = {} & g_{00}(\hat{y})dx^0dx^0 + g_{i0}(\hat{y})(1 - 3ak^0)dx^idx^0 - ag_{im}(\hat{y})k^mdx^0dx^i \\ & + g_{i0}(\hat{y})(1 - 2ak^0)dx^idx^0 + g_{ij}(\hat{y})(1 - 4ak^0)dx^idx^j. \end{aligned} \tag{6.6}$$

where

$$g_{\mu\nu}(\hat{y}_0) = g_{\mu\nu}(x_0) - ax_jk^j\frac{\partial g_{\mu\nu}(x_0)}{\partial x_0}, \quad g_{\mu\nu}(\hat{y}_i) = g_{\mu\nu}(x_i), \tag{6.7}$$

and $\hat{y}_\mu$ is the auxiliary κ-deformed space-time coordinate (see Eq.(5.23) of chapter 5). Note that in the above equation k^0 represents the κ-deformation energy scale [41] associated with the non-commutative metric.

From Eq.(6.6), we obtain the 4-dimensional κ-deformed Minkowski line element, valid up to first order in a, as

$$d\hat{s}^2 = -dt^2 + (1 - 4ak^0)dx^2. \tag{6.8}$$

From the κ-deformed dispersion relation given in Eq.(6.4) and the 4-dimensonal κ-deformed Minkowski line element given in Eq.(6.8), we construct the 8-dimensional κ-deformed flat phase-space metric, valid up to first order in a, as (see [5] for the construction of 8-dimensional phase-space in commutative case)

$$d\hat{s}^2 = -dt^2 + (1 - 4ak^0)dx^2 + \frac{1}{\mu^4}\big(- dE^2 + d\hat{p}^2\big), \tag{6.9}$$

where the parameter μ has the dimension of mass.

The κ-deformed dispersion relation, in Eq.(6.2), has been constructed from the quadratic Casimir of the undeformed κ-Poincare algebra. Therefore the κ-deformed 8-dimensional line element is invariant under the undeformed κ-Poincare algebra (valid up to first order in a). Note that line element given in Eq.(6.9) reduces to the commutative line element given in [5] in the limit $a \to 0$.

Re-writing $\hat{p} = p\sqrt{1 + aE}$ in Eq.(6.9), we get the 8-dimensional metric valid up to first order in a, as

$$d\hat{s}^2 = -dt^2 + (1 - 4ak^0)dx^2 + \frac{1}{\mu^4}\Big(- dE^2 + dp^2(1 + aE) + ap_j dp^j dE\Big). \tag{6.10}$$

We know that line element of the time-like event obeys $d\hat{s}^2 \leq 0$ [1]. We now implement this to the κ-deformed case, i.e.,

$$- dt^2 + (1 - 4ak^0)dx^2 - \frac{1}{\mu^4}\Big(dE^2 - (1 + aE)dp^2 - ap_j dp^j dE\Big) \leq 0. \tag{6.11}$$

Let us divide the above inequality throughout by dt^2 and we denote $v = \frac{dx}{dt}$, as the velocity of the particle. Thus we get

$$1 - (1 - 4ak^0)v^2 + \frac{1}{\mu^4}\left[\Big(\frac{dE}{dt}\Big)^2 - (1 + aE)\Big(\frac{dp}{dt}\Big)^2 - ap_j \frac{dp^j}{dt}\frac{dE}{dt}\right] \geq 0 \tag{6.12}$$

Using the κ-deformed dispersion relation, Eq.(6.3), we get $\frac{dE}{dt} = (1 + aE + \frac{ap^2}{2E})\frac{p}{E}\frac{dp}{dt}$. Denoting the proper acceleration of the particle as $\mathcal{A}$, where $\frac{dp}{dt} = \frac{m\mathcal{A}}{(1-v^2)^{3/2}}$, Eq.(6.12)

[1] Note that for a metric with signature $\eta_{\mu\nu} = diag(1, -1, -1, -1)$, the time-like events follow $d\hat{s}^2 \geq 0$, i.e., $ds^2 = dt^2 - dx^2 \geq 0$

becomes

$$(1-v^2)\left[1+\frac{4ak^0v^2}{1-v^2}+\frac{1}{\mu^4}\frac{m^2A^2}{(1-v^2)^4}\left(\frac{p^2}{E^2}\left(1+2aE+\frac{ap^2}{E}\right)-(1+aE)-\frac{ap^2}{E}\right)\right]\geq 0. \quad (6.13)$$

As in the commutative flat space-time [5] here also the velocity obeys the condition $v \leq c$ and thus the acceleration becomes maximum when $v << c$. In order to obtain the maximal acceleration, we consider the κ-deformed instantaneous rest frame of the particle where the velocity of the particle vanishes, i.e., $v = 0$ and hence its momentum also reduce to zero, i.e., $p = 0$. Hence Eq.(6.13) becomes

$$1-\frac{m^2\mathcal{A}_{max}^2}{\mu^4}(1+am)\geq 0 \quad (6.14)$$

By simplifying Eq.(6.14), we get the explicit form of the maximal acceleration in the κ-deformed space-time, valid up to first order in a, as

$$\mathcal{A}_{max}\leq m\Big(1-\frac{am}{2}\Big). \quad (6.15)$$

Now we have choosen $\mu = m$, where m is the rest mass of the particle. In the SI units Eq.(6.15) takes the form

$$\mathcal{A}_{max}\leq \frac{mc^3}{\hbar}\Big(1-\frac{amc}{2\hbar}\Big) \quad (6.16)$$

We find that the maximal acceleration given in Eq.(6.16) is independent of the κ-deformation energy k^0 appearing in Eq.(6.12) to Eq.(6.13). Note that the first order correction term in the κ-deformed maximal acceleration depends on the rest mass of the particle. Here $\mathcal{A}_{max}$ is the magnitude of the maximal acceleration and thus $\mathcal{A}_{max}$ should be a positive quantity and this happens when $\frac{amc}{2\hbar} < 1$ (this is true when a is positive). When we go to the commutative limit, i.e., $a \to 0$, the maximal acceleration in Eq.(6.16) becomes $A_{\text{max}} = \frac{mc^3}{\hbar}$, as in [5, 6]. In the classical limit, i.e., $\hbar \to 0$, the maximal acceleration, in Eq.(6.16), becomes infinity, as in [5, 6]. We also observe that the κ-deformed correction to the maximal acceleration reduces to zero for a massless particle.

By substituting the definition of the reduced Compton wavelength, i.e., $\lambda\!\!\!^{-}_c = \frac{\hbar}{mc}$ in Eq.(6.16), we obtain the expression for the $\mathcal{A}_{max}$ as

$$\mathcal{A}_{max}\leq \frac{c^2}{\lambda\!\!\!^{-}_c}\Big(1-\frac{a}{2\lambda\!\!\!^{-}_c}\Big). \quad (6.17)$$

The magnitude of the maximal acceleration should be a positive quantity and this gives bound on the κ-deformation parameter as $a < 2\lambda\!\!\!^{-}_c$.

The temperature of thermal radiation, seen by an observer with uniform acceleration A, is given using the Unruh temperature [13], i.e., $T = \frac{\hbar A}{2\pi k_B c}$. Therefore the maximum temperature associated with a thermal radiation can also be expressed using the Unruh temperature as $T_{max} = \frac{\hbar \mathcal{A}_{max}}{2\pi k_B c}$. Substituting Eq.(6.16) in the expression for T_{max}, we get the maximum temperature as

$$T_{max} = \frac{mc^2}{2\pi k_B}\Big(1 - \frac{amc}{2\hbar}\Big). \tag{6.18}$$

In the limit $a \to 0$, the above expression reduces to that obtained in [4, 35]. In [42] it is reported that the experimentally measured value of the Unruh temperature associated with positron radiation as $T = 1.80 \pm 0.51\ PeV$. By matching the error bar of T, i.e., $\Delta T = 0.51\ PeV$ with the a dependent correction term of T_{max}, we find the bound on the κ-deformation parameter as $a \leq 10^{-26}m$.

6.3 Maximal acceleration from κ-deformed uncertainty principle

In this section we derive the κ-deformed correction (valid up to first order in a) to the maximal acceleration using the κ-deformed uncertainty relation.

The uncertainty relation between energy and an arbitrary function of time [7–11] is given as

$$\Delta E \Delta g(t) \geq \frac{1}{2}\frac{dg}{dt}. \tag{6.19}$$

Let us choose $g(t) = v$, where v is the velocity of the particle. Using this in Eq.(6.19), we obtain the uncertainty relation between energy and velocity as

$$\Delta E \Delta v(t) \geq \frac{1}{2}\frac{dv}{dt} \tag{6.20}$$

Next let us choose $g(t) = x$, where x is the position coordinate. Using this in Eq.(6.19), we get the uncertainty relation between energy and position coordinate as

$$\Delta E \Delta x(t) \geq \frac{1}{2}\frac{dx}{dt} \tag{6.21}$$

Multiplying Eq.(6.20) and Eq.(6.21), we get

$$\big(\Delta E \Delta x\big)\big(\Delta E \Delta v\big) \geq \frac{1}{4} v \mathcal{A} \tag{6.22}$$

where we define $\frac{dx}{dt} = v$ and $\frac{dv}{dt} = \mathcal{A}$.

$$\big(\Delta E\big)^2 \Delta x \frac{\Delta v}{v} \geq \frac{1}{4}\mathcal{A}. \tag{6.23}$$

Using the relation $\Delta E = v\Delta p$, we re-write ΔE in the above equation as

$$\big(\Delta p\big)^2 \Delta x \big(\Delta v\big) v \geq \frac{1}{4}\mathcal{A}. \tag{6.24}$$

We know that the uncertainty in the velocity of the particle should be less than the maximum attainable velocity and according to the special theory of relativity this maximum attainable velocity should be less than the velocity of light, i.e., $(\Delta v) = \sqrt{< v^2 > - < v >^2} \leq v_{max} \leq 1$ (note that here we take $c = 1$). Thus we have $\Delta v \leq 1$ and $v \leq 1$. So we take $(\Delta v)v \leq 1$ and hence Eq.(6.24) becomes

$$\big(\Delta p \Delta x\big)^2 \geq \frac{1}{4}\mathcal{A}\Delta x \tag{6.25}$$

Since the κ-deformed uncertainty relation (valid up to first order in a) between Δx and Δp has been shown to be [49]

$$\Delta x \Delta p \Big(1 + \frac{a}{2\Delta x}\Big) \geq \frac{1}{2}, \tag{6.26}$$

we find $\Delta x \Delta p \geq \frac{1}{2}\Big(1 - \frac{a}{2\Delta x}\Big)$. Substituting this in Eq.(6.25), we obtain

$$\mathcal{A}\Delta x \leq \Big(1 - \frac{a}{\Delta x}\Big) \tag{6.27}$$

Now we choose Δx as the reduced Compton's wavelength, i.e., $\Delta x = \lambda_c$ and it is defined as $\lambda_c = \frac{1}{m}$. Using this in above equation, we obtain the κ-deformed maximal acceleration, valid up to first order in a, as

$$\mathcal{A}_{max} \leq m\Big(1 - am\Big) \tag{6.28}$$

In the SI units, the expression for the κ-deformed maximal acceleration becomes

$$\mathcal{A}_{max} \leq \frac{mc^3}{\hbar}\Big(1 - \frac{amc}{\hbar}\Big) \tag{6.29}$$

By comparing Eq.(6.16) and Eq.(6.29), we find that the first order κ-deformed correction terms of the maximal acceleration differs only by a numerical factor 1/2. Thus the commutative limit and classical limit of both these expressions are same as in [5–7].

6.4 Emergence of maximal acceleration from κ-deformed space-time

In this section, we first construct the line element in the 4-dimensional κ-Minkowski space-time (valid up to first order in a) by considering the realisation given in Eq.(1.13) (of chapter 1). We then show the emergence of the maximal acceleration from this deformed line element.

The κ-deformed space-time coordinate is realised in terms of the commutative coordinate and momenta as

$$\hat{x}_\mu = x_\alpha \varphi^\alpha{}_\mu(p). \tag{6.30}$$

Here we choose the realisation (see Eq.(1.13) of chapter 1) [43] where

$$\varphi^\alpha{}_\mu(p) = \delta^\alpha{}_\mu\Big(1 + \alpha\frac{a\cdot p}{\hbar}\Big) + \beta\frac{a^\alpha p_\mu}{\hbar} + \gamma\frac{p^\alpha a_\mu}{\hbar}, \tag{6.31}$$

$\alpha, \beta, \gamma \in \mathbb{R}$ in the above expression are dimensionless parameters. Using Eq.(6.30) and Eq.(6.31) in $[\hat{x}_0, \hat{x}_i] = ia\hat{x}_i$ and $[\hat{x}_i, \hat{x}_j] = 0$, we get $\gamma = \alpha + 1$ [43]. We see that in the limit $a \to 0$, $\varphi^\alpha{}_\mu(p)$ reduces to $\delta^\alpha{}_\mu$. But $\varphi^\alpha{}_\mu(p)$ diverges in limit $\hbar \to 0$. Thus we obtain the commutative result in the limit $a \to 0$ (when there is no $\hbar$ dependence). From Eq.(6.31), we see that limit $\frac{a}{\hbar} \to 0$ is well defined and we obtain the commutative coordinates. Substituting Eq.(6.31) in Eq.(6.30), we get

$$\hat{x}_\mu = x_\mu\Big(1 + \alpha\frac{a\cdot p}{\hbar}\Big) + \beta\Big(x\cdot\frac{a}{\hbar}\Big)p_\mu + (\alpha+1)(x\cdot p)\frac{a_\mu}{\hbar}. \tag{6.32}$$

We obtain the commutative coordinate, from the above expression, in the limit $a \to 0$. Also, $\hat{x}_\mu = 0$, when $x_\mu = 0$. But the non-commutativity associated with the space-time restrict us from localising a particle below the minimal length scale set by the non-commutative parameter. Therefore taking the limit $x_\mu \to 0$, when $a \neq 0$ is not valid.

Using Eq.(5.18) (of chapter 5), we write the κ-deformed generalised commutation relation as

$$[\hat{x}_\mu, \hat{P}_\nu] = i\hbar\hat{g}_{\mu\nu}. \tag{6.33}$$

$\hat{P}_\mu$ is expressed in terms of the commutative coordinate and momenta as

$$\hat{P}_\mu = g_{\alpha\beta}(\hat{y})p^\alpha\varphi^\beta{}_\mu(p). \tag{6.34}$$

Here $\hat{y}$, introduced for simplifying the calculations [43] is another κ-deformed space-time coordinate obeying $[\hat{y}_\nu, \hat{x}_\mu] = 0$ and therefore $\hat{x}_\mu$ commutes with any function of $\hat{y}_\mu$, i.e., $[\hat{x}_\mu, f(\hat{y})] = 0$ [43]. Thus using Eq.(6.31), we write $\hat{y}_\mu$ and $f(\hat{y})$ as

$$\hat{y}_\mu = x_\mu + \alpha\frac{x\cdot p a_\mu}{\hbar} + \beta\frac{x\cdot a p_\mu}{\hbar} + (\alpha+1)\frac{x_\mu a\cdot p}{\hbar}, \tag{6.35}$$

$$f(\hat{y}) = f(x) + \alpha\Big(\frac{a}{\hbar}\cdot\frac{\partial f}{\partial x}\Big)(x\cdot p) + \beta\Big(\frac{a}{\hbar}\cdot x\Big)\Big(\frac{\partial f}{\partial x}\cdot p\Big) + (\alpha+1)\Big(x\cdot\frac{\partial f}{\partial x}\Big)\Big(\frac{a}{\hbar}\cdot p\Big). \tag{6.36}$$

Substituting Eq.(6.32) and Eq.(6.34) in Eq.(6.33), we get

$$\hat{g}_{\mu\nu} = g_{\alpha\beta}(\hat{y})\Big(p^{\beta}\frac{\partial\varphi^{\alpha}_{\nu}}{\partial p^{\sigma}}\varphi^{\sigma}_{\mu} + \varphi^{\alpha}_{\mu}\varphi^{\beta}_{\nu}\Big). \tag{6.37}$$

Replacing f in Eq.(6.36) with $g_{\mu\nu}$, we obtain the expression for $g_{\mu\nu}(\hat{y})$ in terms of the commutative coordinate and its conjugate momenta as

$$g_{\mu\nu}(\hat{y}) = g_{\mu\nu}(x)+\alpha\Big(\frac{a}{\hbar}\cdot\frac{\partial g_{\mu\nu}}{\partial x}\Big)(x\cdot p)+\beta\Big(\frac{a}{\hbar}\cdot x\Big)\Big(\frac{\partial g_{\mu\nu}}{\partial x}\cdot p\Big)+(\alpha+1)\Big(x\cdot\frac{\partial g_{\mu\nu}}{\partial x}\Big)\Big(\frac{a}{\hbar}\cdot p\Big). \tag{6.38}$$

Substituting Eq.(6.31) and Eq.(6.38) in Eq.(6.37), we obtain the explict form of the κ-deformed metric (valid up to first order in a) in terms of the commutative coordinate and momenta as

$$\begin{aligned}\hat{g}_{\mu\nu} = g_{\mu\nu} + \alpha\bigg(p^{\beta}g_{\nu\beta}\frac{a_{\mu}}{\hbar} + 2g_{\mu\nu}\Big(\frac{a}{\hbar}\cdot p\Big) + \Big(\frac{a}{\hbar}\cdot\frac{g_{\mu\nu}}{\partial x}\Big)x\cdot p\bigg) + \beta\bigg(p^{\beta}\frac{a^{\alpha}}{\hbar}g_{\alpha\beta}\eta_{\mu\nu} + \frac{a^{\beta}}{\hbar}p_{\nu}g_{\mu\beta}+ \\ \frac{a^{\beta}}{\hbar}p_{\mu}g_{\nu\beta} + \Big(\frac{a}{\hbar}\cdot x\Big)\Big(\frac{\partial g_{\mu\nu}}{\partial x}\cdot p\Big)\bigg) + (\alpha+1)\bigg(2g_{\mu\beta}p^{\beta}\frac{a_{\nu}}{\hbar} + p^{\alpha}\frac{a_{\mu}}{\hbar}g_{\alpha\nu} + \Big(x\cdot\frac{\partial g_{\mu\nu}}{\partial x}\Big)\Big(\frac{a}{\hbar}\cdot p\Big)\bigg).\end{aligned} \tag{6.39}$$

From the above, we obtain the commutative metric in the limit $a \to 0$ (also in the limit $\frac{a}{\hbar} \to 0$). We also find that the metric $\hat{g}_{\mu\nu}$ diverges in the limit $\hbar \to 0$ with $a \neq 0$.

The κ-deformed metric constructed in Eq.(6.39) depends on the commutative coordinate, momenta and the deformation parameter (This should be contrasted with the κ-deformed metric derived in Eq.(5.26) (of chapter 5) which depends on the commutative coordinate, deformation parameter as well as the deformation energy scale as the realisation ($\varphi = e^{-ak^0}$) depends on the deformation parameter and deformation energy scale). In Eq.(6.10), the momentum dependent terms are all due to the κ-deformed dispersion relation whereas in this case, i.e., Eq.(6.39) the momentum dependent term comes from the realisation given in Eq.(6.31).

Now we write down the definition for line element in the 4-dimensional κ-Minkowski space-time as

$$d\hat{s}^2 = \hat{\eta}_{\mu\nu}d\hat{x}^{\mu}d\hat{x}^{\nu}. \tag{6.40}$$

After replacing $g_{\mu\nu}$ in Eq.(6.39) with $\eta_{\mu\nu}$ and substituting this and also using Eq.(6.30) and Eq.(6.31) in Eq.(6.40), we get the explicit form of the 4-dimensional κ-Minkowskian line element, valid up to first order in a, as

$$d\hat{s}^2 = \eta_{\mu\nu}dx^\mu dx^\nu + \alpha\bigg(\Big(2p_\nu\frac{a_\mu}{\hbar} + 4\eta_{\mu\nu}\frac{a}{\hbar}\frac{p^0}{c} + 2p_\mu\frac{a_\nu}{\hbar}\Big)dx^\mu dx^\nu + \frac{a}{\hbar}\eta_{\mu\nu}\frac{dp^0}{c}dx^\mu x^\nu +$$
$$(x\cdot dx)\frac{a}{\hbar}\frac{dp^0}{c} + 2\frac{a}{\hbar}cdt(dx\cdot p) + 2(x\cdot dp)cdt\frac{a}{\hbar}\bigg) + \beta\bigg(\Big(\eta_{\mu\nu}\frac{a}{\hbar}\frac{p^0}{c} + \frac{a_\mu}{\hbar}p_\nu + \frac{a_\nu}{\hbar}p_\mu\Big)dx^\mu dx^\nu +$$
$$\frac{2a}{\hbar}(dx\cdot p)cdt + \frac{2a}{\hbar}(dx\cdot dp)ct\bigg) + \bigg(\frac{3}{2}\Big(p_\mu\frac{a_\nu}{\hbar} + p_\nu\frac{a_\mu}{\hbar}\Big)dx^\mu dx^\nu + \frac{2a}{\hbar}cdt(dx\cdot p + x\cdot dp))\bigg). \tag{6.41}$$

Using $\eta_{\mu\nu} = (1,-1,-1,-1)$, above line element becomes,

$$d\hat{s}^2 = c^2dt^2 - dx^2 + 5\frac{a}{\hbar}cdt\Big(p^0dt - p\cdot dx\Big) + 2\frac{a}{\hbar}cdt\Big(tdp^0 - x\cdot dp\Big) + \alpha\bigg(6\frac{a}{\hbar}cdt\Big(p^0dt - p\cdot dx\Big)$$
$$+ 4\frac{a}{\hbar}\frac{p^0}{c}\Big(c^2dt^2 - dx^2\Big) + 2\frac{a}{\hbar}\frac{dp^0}{c}\Big(c^2tdt - x\cdot dx\Big) + 2\frac{a}{\hbar}cdt\Big(tdp^0 - x\cdot dp\Big)\bigg)$$
$$+ \beta\bigg(\frac{a}{\hbar}\frac{p^0}{c}\Big(c^2dt^2 - dx^2\Big) + 4\frac{a}{\hbar}cdt\Big(p^0dt - p\cdot dx\Big) + 2\frac{a}{\hbar}ct\Big(dp^0dt - dp\cdot dx\Big)\bigg). \tag{6.42}$$

We now see that for a time-like event the corresponding line element satisfy $d\hat{s}^2 \geq 0$. We then divide Eq.(6.42) throughout by dt^2 and denote $\frac{dx}{dt} = v$, where v is the velocity and $\frac{dp}{dt} = \frac{mA}{(1-v^2/c^2)^{3/2}}$, where A is the proper acceleration of a particle of rest mass m. Thus the above expression becomes

$$c^2 - v^2 + \alpha\bigg(6\frac{a}{\hbar}c\Big(p^0 - p\cdot v\Big) + 4\frac{a}{\hbar}\frac{p^0}{c}\Big(c^2 - v^2\Big) + 2\frac{a}{\hbar}\frac{dp^0}{dt}\frac{1}{c}\Big(c^2t - x\cdot v\Big) +$$
$$2\frac{a}{\hbar}c\Big(t\frac{dp^0}{dt} - x\cdot\frac{mA}{(1-v^2/c^2)^{3/2}}\Big)\bigg) + \beta\bigg(\frac{a}{\hbar}\frac{p^0}{c}\Big(c^2 - v^2\Big) + 4\frac{a}{\hbar}c\Big(p^0 - p\cdot v\Big) +$$
$$2\frac{a}{\hbar}ct\Big(\frac{dp^0}{dt} - v\cdot\frac{mA}{(1-v^2/c^2)^{3/2}}\Big)\bigg) + \bigg(5\frac{a}{\hbar}c\Big(p^0 - p\cdot v\Big) + 2\frac{a}{\hbar}c\Big(t\frac{dp^0}{dt} - x\cdot\frac{mA}{(1-v^2/c^2)^{3/2}}\Big)\bigg) \geq 0 \tag{6.43}$$

The $x\cdot A$ and $v\cdot A$ dependent terms in the above expression depends on A. These terms are contributed by the $x\cdot dx = |\vec{x}||\vec{dx}|cos\theta$ and $dx\cdot dp = |\vec{dx}||\vec{dp}|\cos\theta$ terms of the κ-Minkowski line element given in Eq.(6.41). In order to evaluate the maximum acceleration we take $\cos\theta = 1$ and we also write $|x|$ in place of x in the below equations.

By dividing Eq.(6.43) throughout by $c^2 - v^2$ and using the commutative dispersion relation (because we are considering only the first order terms in a. So we need to use only commutative dispersion relation), we re-write $\frac{dp^0}{dt}$ as $\frac{dp^0}{dt} = \frac{pc^2}{p^0}\frac{dp}{dt} = \frac{pc^2}{p^0}\frac{mA}{(1-v^2/c^2)^{3/2}}$. Thus we obtain

$$1 + \alpha\bigg(\frac{6ac(p^0 - pv)}{\hbar(c^2 - v^2)} + \frac{4ap^0}{c\hbar} + \Big(\frac{2apc}{\hbar p^0}\frac{(c^2t - |x|v)}{c^2 - v^2} + \frac{2ac}{\hbar(c^2 - v^2)}\Big(t\frac{pc^2}{p^0} - |x|\Big)\Big)\frac{mA}{(1-v^2/c^2)^{3/2}}\bigg) +$$
$$\beta\bigg(\frac{ap^0}{\hbar c} + \frac{4ac(p^0 - pv)}{\hbar(c^2 - v^2)} + \frac{2act}{\hbar(c^2 - v^2)}\Big(\frac{pc^2}{p^0} - v\Big)\frac{mA}{(1-v^2/c^2)^{3/2}}\bigg) +$$
$$\bigg(\frac{5a(p^0 - pv)}{\hbar(c^2 - v^2)} + \frac{2ac}{\hbar(c^2 - v^2)}\Big(t\frac{pc^2}{p^0} - |x|\Big)\frac{mA}{(1-v^2/c^2)^{3/2}}\bigg) \geq 0, \tag{6.44}$$

As in the commutative flat space-time here also the maximum allowed velocity is c and thus the acceleration becomes maximum when $v << c$. In order to obtain the maximal acceleration, we consider the κ-deformed instantaneous rest frame of the particle where the velocity of the particle vanishes, i.e., $v = 0$ and hence its momentum also reduce to zero, i.e., $p = 0$. As a result all the t dependent terms of Eq.(6.44) also vanishes. Therefore we get the expression for the maximal acceleration $\mathcal{A}_{max}$, valid up to first order in a, as

$$\mathcal{A}_{max} \leq \frac{c}{2(\frac{a}{\hbar})|x|m} \frac{1}{(1+\alpha)} \left[1 + \frac{5ap^0}{\hbar c}\Big(1+\beta+2\alpha\Big)\right]. \tag{6.45}$$

The p^0 in Eq.(6.45) is interpreted as the deformation energy associated with the metric. The above expression contains $\hbar$ dependent as well as $\hbar$ independent terms, unlike the maximal acceleration obtained in Eq.(6.16). We observe that in the limit $\hbar \to 0$ with $a \neq 0$, $\mathcal{A}_{max} \leq \frac{5p^0(1+\beta+2\alpha)}{2|x|m(1+\alpha)}$, which in contrast with (the $\hbar \to 0$ limit of the) Eq.(6.16), where the maximal acceleration becomes infinity in the classical limit, i.e., $\hbar \to 0$. Further we also see that in the limit $a \to 0$, $\mathcal{A}_{max} \to \infty$ as seen in the commutative case.

Here $\mathcal{A}_{max}$ represents the magnitude of the maximal acceleration and therefore it is a positive quantity. This happens either when $\alpha > -1,\ \beta > -(1 + 2\alpha + \frac{c\hbar}{5ap^0})$ or when $\alpha < -1,\ \beta < -(1 + 2\alpha + \frac{c\hbar}{5ap^0})$. As stated above, with $a \neq 0$, in the limit $\hbar \to 0$, we find $\mathcal{A} \leq \frac{5p^0(1+\beta+2\alpha)}{2|z|m(1+\alpha)}$. In limit $\hbar \to 0$, the bounds on α and β are either $\alpha > -1,\ \beta > -(1+2\alpha)$ or $\alpha < -1,\ \beta < -(1+2\alpha)$. The parameters α and β come from the realisation given in Eq.(6.31). Their numerical values can be fixed only from experimental data.

From Eq.(6.45) we see that the maximal accleration induced by the κ-deformed space-time depends on the inverse of magnitude of the spatial coordinate. This $|x|$ term in the $\mathcal{A}_{max}$ expression is contributed by the x_μ dependent term of the realisation given in Eq.(6.31). Due to the non-commutativity we can not localise a particle below the minimal length scale, i.e., $|x| \leq a$ is not allowed. Therefore one cannot take the limit $x \to 0$ with $a \neq 0$ in the above expression. Note that this $|x|$ becomes the minimum distance of separation between the particles which moves under the influence of a force exerted between them.

Using Eq.(6.45) and expression for the Unruh temperature, here also we can calculate the maximum temperature as in sec.6.2. Thus by choosing $2|x| = \lambda_c$, we obtain the expression for the maximum temperature as

$$T_{max} = \frac{\hbar c}{2\pi k a} \frac{1}{(1+\alpha)} \bigg(1 + \frac{5ap^0}{\hbar c}(1+\beta+2\alpha)\bigg). \tag{6.46}$$

Unlike Eq.(6.18), here we find that the maximal temperature diverges in the commutative limit, $a \to 0$. In the classical limit $\hbar \to 0$, $T_{max} \to \frac{5p^0(1+\beta+2\alpha)}{2\pi k(1+\alpha)}$, where the non-commutative parameters α and β obey either $\alpha > -1,\ \beta > -(1+2\alpha+\frac{c\hbar}{ap^0})$ or $\alpha < -1,\ \beta < -(1+2\alpha+\frac{c\hbar}{ap^0})$.

6.4.1 Bounds on α and β using κ-deformed Newtonian limit

Here we first derive the κ-deformed geodesic equation, valid up to first order in a and then obtain the corresponding Newtonian limit. We use this Newtonian limit to compute the bound on α and β.

The κ-deformed Christoffel symbol is defined as

$$\hat{\Gamma}^{\mu}_{\nu\lambda} = \frac{1}{2}\hat{g}^{\mu\rho}\big(\partial_\nu \hat{g}_{\rho\lambda} + \partial_\lambda \hat{g}_{\nu\rho} - \partial_\rho \hat{g}_{\nu\lambda}\big) \tag{6.47}$$

Substituting Eq.(6.39) in Eq.(6.47), we obtain the Christoffel symbol in κ-deformed space-time, valid up to first order in a, as

$$\hat{\Gamma}^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\nu\lambda} + \frac{1}{2}\frac{a}{\hbar}\Big(m_2 \mathcal{B}^{\mu}_{\nu\lambda\sigma}\frac{dx^\sigma}{d\tau} + \frac{p^0}{c}\mathcal{C}^{\mu}_{\nu\lambda}\Big), \tag{6.48}$$

where,

$$\begin{aligned}
\mathcal{B}^{\mu}_{\nu\lambda\sigma} &= \alpha g^{\mu\rho}\bigg(2\partial_\nu g_{\sigma[\rho}\delta_{\lambda]0} + 2\partial_\lambda g_{\sigma[\rho}\delta_{\nu]0} - 2\partial_\rho g_{\sigma[\nu}\delta_{\lambda]0} + \partial_\nu\Big(\frac{\partial g_{\rho\lambda}}{\partial t}\frac{x_\sigma}{c}\Big) + \partial_\lambda\Big(\frac{\partial g_{\nu\rho}}{\partial t}\frac{x_\sigma}{c}\Big) - \partial_\rho\Big(\frac{\partial g_{\nu\lambda}}{\partial t}\frac{x_\sigma}{c}\Big)\bigg) \\
&+ \frac{3}{2}g^{\mu\rho}\bigg(\partial_\nu g_{\sigma[\rho}\delta_{\lambda]0} + \partial_\lambda g_{\sigma[\rho}\delta_{\nu]0} - \partial_\rho g_{\sigma[\nu}\delta_{\lambda]0}\bigg) + \frac{3}{2}\Big(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\nu\lambda}\Big)g^{[\mu}_{\sigma}\delta^{\rho]0} \\
&+ \alpha\Big(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\nu\lambda}\Big)\Big(2\delta^{0[\rho}g^{\nu]\sigma} + \frac{1}{c}\frac{\partial g^{\mu\rho}}{\partial t}x^\sigma\Big) + \beta g^{\mu\rho}\Big(\eta_{\rho[\lambda}\partial_{\nu]}g_{0\sigma} - \eta_{\nu\lambda}\partial_\rho g_{0\sigma} \\
&+ \delta_{\sigma[\lambda}\partial_{\nu]}g_{\rho 0} - \partial_\rho g_{0[\lambda}\delta_{\nu]\sigma} + \delta_{\rho 0}\partial_{[\nu}g_{\lambda]0} + \partial_\sigma g_{\rho[\lambda}\delta\nu]0 - \delta_{\rho 0}\partial_\sigma g_{\nu\lambda} + x_0\partial_\rho\partial_{[\nu}g_{\lambda]\rho} - x_0\partial_\rho\partial_\sigma g_{\nu\lambda}\Big) \\
&+ \beta\Big(g_{0\sigma}\eta^{\mu\rho} + \delta^{[\rho}_{\sigma}g^{\mu]}_{0} + x_0\partial_\sigma g^{\mu\rho}\Big)\Big(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\nu\lambda}\Big),
\end{aligned} \tag{6.49}$$

$$\begin{aligned}
\mathcal{C}^{\mu}_{\nu\lambda} =& \bigg(4\alpha g^{\mu\rho} + (\alpha+1)\Big(x\cdot\frac{\partial g^{\mu\rho}}{\partial x}\Big)\bigg)\Big(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\nu\lambda}\Big) \\
&+ (\alpha+1)g^{\mu\rho}\bigg(\partial_\nu\Big(x\cdot\frac{\partial g_{\rho\lambda}}{\partial x}\Big) + \partial_\lambda\Big(x\cdot\frac{\partial g_{\nu\rho}}{\partial x}\Big) - \partial_\rho\Big(x\cdot\frac{\partial g_{\nu\lambda}}{\partial x}\Big)\bigg).
\end{aligned} \tag{6.50}$$

The geodesic equation in the κ-deformed space-time is defined as

$$\frac{d^2\hat{x}^\mu}{d\tau^2} + \hat{\Gamma}^{\mu}_{\nu\lambda}\frac{d\hat{x}^\nu}{d\tau}\frac{d\hat{x}^\lambda}{d\tau} = 0. \tag{6.51}$$

Now we calculate $\frac{d\hat{x}^\mu}{d\tau}$ and $\frac{d^2\hat{x}^\mu}{d\tau^2}$ (valid up to first order in a) from Eq.(6.30) and Eq.(6.51) becomes

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}+\frac{a}{\hbar}\mathcal{D}^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}+\frac{a}{\hbar}\mathcal{E}^\mu_{\nu\lambda\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}+\frac{a}{\hbar}\mathcal{F}^\mu_{\nu\lambda}\frac{d^2x^\nu}{d\tau^2}\frac{dx^\lambda}{d\tau}=0, \quad (6.52)$$

where,

$$\begin{aligned}
\mathcal{D}^\mu_{\nu\lambda}=&2\alpha\frac{q^0}{c}\Gamma^\mu_{\nu\lambda}+\frac{1}{2}\frac{p^0}{c}\mathcal{C}^\mu_{\nu\lambda},\\
\mathcal{E}^\mu_{\nu\lambda\sigma}=&m(\alpha+1)\Gamma^\mu_{\rho\gamma}\left(\delta^\rho_\sigma\delta^{\gamma0}+\delta^\gamma_\sigma\delta^{\rho0}\right)\eta_{\nu\lambda}+\frac{1}{2}m_2\mathcal{B}^\mu_{\nu\lambda\sigma}+m\beta\delta_{\sigma0}\Gamma^\mu_{\nu\lambda},\\
\mathcal{F}^\mu_{\nu\lambda}=&3m(\alpha+1)\eta_{\nu\lambda}\delta^{\mu0}+m(\alpha+1)x_\nu\left(\delta^\rho_\lambda\delta^{\gamma0}+\delta^\gamma_\lambda\delta^{\rho0}\right)\Gamma^\mu_{\rho\gamma}+\\
&\beta m\left(\delta_{\lambda0}\delta^\mu_\nu+\delta_{\nu o}\delta^\mu_\lambda\right)+\beta m x_0\Gamma^\mu_{\nu\lambda}
\end{aligned} \quad (6.53)$$

In the above expressions m and q^0 are the mass and energy coming from the a dependent realisation of the κ-deformed coordinate $\hat{x}_\mu$. Similarly m_2 and p^0 are the mass and energy coming from realisation associated with the metric $\hat{g}_{\mu\nu}$.

Now we obtain the Newtonian limit of the κ-deformed geodesic equation given in Eq.(6.52), using the following conditions

- particles are moving slowly, i.e.,

$$\frac{dx_i}{d\tau}<<\frac{dx_0}{d\tau}, \quad (6.54)$$

- gravitational field is static, i.e.,

$$\frac{\partial g_{\mu\nu}}{\partial t}=0, \quad (6.55)$$

- gravitational field is weak and we linearise the metric as

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}, \quad |h_{\mu\nu}|<<1. \quad (6.56)$$

Thus using these condition we get from Eq.(6.52)

$$\frac{d^2x^0}{d\tau^2}\left(1+\frac{a}{\hbar}m(3\alpha+\beta+3)\frac{dx^0}{d\tau}\right)-\frac{1}{2}\frac{a}{\hbar}\beta m\partial_i h_{00}\frac{d^2x^i}{d\tau^2}\frac{dx^0}{d\tau}=0. \quad (6.57)$$

Multiplying Eq.(6.57) throughout by $\left(1-\frac{a}{\hbar}m(3\alpha+\beta+3)\frac{dx^0}{d\tau}\right)$ and considering the terms valid up to first order in a, we get

$$\frac{d^2x^0}{d\tau^2}-\frac{1}{2}\frac{a}{\hbar}\beta m\partial_i h_{00}\frac{d^2x^i}{d\tau^2}\frac{dx^0}{d\tau}=0. \quad (6.58)$$

In the commutative space-time, $\frac{1}{c}\frac{dx^0}{d\tau}$ can be written in terms of the linearised metric h_{00} as $\frac{1}{c}\frac{dx^0}{d\tau} = 1 - \frac{1}{2}h_{00}$. By substituting $\frac{1}{c}\frac{dx^0}{d\tau} = 1 - \frac{1}{2}h_{00}$ in Eq.(6.58), we get $\frac{1}{2}\frac{a}{\hbar}\beta m x_0 \partial_i h_{00}\frac{d^2x^i}{d\tau^2}\Big(1 - \frac{1}{2}h_{00}\Big) = 0$. This is possible only when, $\frac{d^2x^i}{d\tau^2} = 0$ or $h_{00} = 2$ or $\beta = 0$. But for an accelerating particle $\frac{d^2x^i}{d\tau^2} \neq 0$. Similarly h_{00} defines the commutative Newton's potential and hence $h_{00} \neq 2$. Thus in order to obtain a consistent result we have to choose $\beta = 0$. Thus now onwards we do the remaining calculations by choosing $\beta = 0$. Thus Eq.(6.58) becomes

$$\frac{d^2x^0}{d\tau^2} = 0. \tag{6.59}$$

From the deformed geodesic equation, we also get another equation for the spatial components as

$$\frac{d^2x^j}{d\tau^2}\left(\delta^{ij} + \frac{a}{\hbar}m(\alpha+1)x_j\partial_i h_{00}\frac{dx^0}{d\tau}\right) = -\frac{1}{2}\partial_i h_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} - \frac{1}{2}\frac{a}{\hbar}\bigg(\alpha\Big(2\frac{q_0}{c} + 4\frac{p_0}{c}\Big)\partial_i h_{00} +$$
$$(\alpha+1)\frac{p_0}{c}\partial_i\Big(x\cdot\frac{\partial h_{00}}{\partial x}\Big)\bigg)\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} - \frac{1}{2}\frac{a}{\hbar}\Big(2m\,(\alpha+1) + m_2\,(4\alpha+3)\Big)\partial_i h_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau}. \tag{6.60}$$

Now we multiply Eq.(6.60) throughout by $\Big(\delta^{ij} - \frac{a}{\hbar}m(\alpha+1)x_j\partial_i h_{00}\frac{dx^0}{d\tau}\Big)$ and by keeping the first order terms in a and h_{00}, we get

$$\frac{d^2x^i}{d\tau^2} = -\frac{1}{2}\partial_i h_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} - \frac{1}{2}\frac{a}{\hbar}\bigg(\alpha\Big(2\frac{q^0}{c} + 4\frac{p^0}{c}\Big)\partial_i h_{00} + (\alpha+1)\frac{p^0}{c}\partial_i\Big(x\cdot\frac{\partial h_{00}}{\partial x}\Big)\bigg)\frac{dx^0}{d\tau}\frac{dx^0}{d\tau}$$
$$- \frac{1}{2}\frac{a}{\hbar}\Big(2m\,(\alpha+1) + m_2\,(4\alpha+3)\Big)\partial_i h_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau}. \tag{6.61}$$

In order to get the Newtonian limit we use $\frac{1}{c}\frac{dx^0}{d\tau} = 1 - \frac{1}{2}h_{00}$ in the above equation and keep the terms valid up to h_{00}. We also take $q_0 = m$ and $p_0 = m_2$ as the particle is moving very slowly. Using these arguments and the commutative expression for Newton's potential, i.e., $h_{00} = -\frac{2GM}{c^2 r}$, we get expression for the Newton's force equation, in the κ-deformed space-time as

$$\hat{F}^i = F^i\bigg(1 + \frac{a}{\hbar}\Big(\frac{p^0}{c}\big(7\alpha+2\big) + mc\big(4\alpha+2\big)\Big)\bigg), \tag{6.62}$$

where $F^i = -\frac{mMGx^i}{r^3}$. From the above we observe that the first order κ-deformed correction of the Newton's force equation, contains two a dependent terms. The first correction term of Eq.(6.62) depends on the deformation energy scale of the metric p^0 and the second correction of Eq.(6.62) depends on the rest mass of the particle. Here the mass dependent term in the κ-deformed Newton's force equation is contributed by the momentum dependent term of the realisation given in Eq.(6.31).

We observe that the mass dependent term present in the first order correction term of the κ-deformed Newton's force equation violates the principle of equivalence. The change in the ratio of gravitational mass to the inertial mass is obtained as $\delta(\frac{m_g}{m_i}) = \frac{amc(4\alpha+2)}{\hbar}$. The violation of equivalence principle has been constrained experimentally as $\delta(\frac{m_g}{m_i}) < 10^{-13}$ [50] and thus we have $\frac{amc(4\alpha+2)}{\hbar} < 10^{-13}$. Therefore we get the bound on α as $\alpha < -0.5$, for a unit massive test particle and the deformation parameter within the range $a = 10^{-49}m$ to $a = 10^{-21}m$. From the positivity condition on the magnitude of the maximal acceleration we get $\alpha > -1$. By combining these two bounds on α, we obtain the range of allowed values for α as $-1 < \alpha < -0.5$ for $\beta = 0$.

6.5 Conclusions

In this chapter we have studied the maximal acceleration in the κ-deformed space-time and analysed its implications. We have evaluated the corrections to the maximal acceleration of a massive particle, valid up to the first order in the deformation parameter. This is derived using two different approaches.

This is first derived from the time-like events in 8-dimensional κ-deformed phase-space, which is constructed using the 4-dimensional κ-Minkowski line element and the κ-deformed dispersion relation. It was found that the first order correction to the maximal acceleration (see Eq.(6.16)) depends on the rest mass of particle and the maximal acceleration vanishes for a massless particle. But this correction does not depends on the metric deformation energy k^0, even though the realisation $\varphi = e^{-ak^0}$ depends on the deformation energy. In the commutative limit $a \to 0$, we recover the Caianiello's maximal acceleration [5, 6]. As in the commutative case here also the maximal acceleration diverges in the classical limit $\hbar \to 0$. We have also calculated the maximal temperature associated with thermal radiation in κ-deformed space-time, by using the expression for maximal acceleration in Unruh temperature. In the commutative limit $a \to 0$, this reduces to one obtained in [35]. Further by comparing this deformed maximal temperature with the experimental data on the Unruh temperature, we obtain a bound on the length scale associated with the κ-deformed space-time as $a \leq 10^{-26}$ m. The first order κ-deformed corrections of the maximal acceleration has also been derived in an alternate way by using the κ-deformed uncertainty principle between the spatial coordinate and its conjugate momenta. The first order correction terms of these expressions (i.e., Eq.(6.17) and Eq.(6.29)) differ only by a numerical factor 1/2.

We then show that κ-deformed space-time geometry induces an upper bound on the acceleration of a massive particle. This maximal acceleration (see Eq.(6.45)), valid up to first order in a, has been obtained from the time-like events associated with 4-dimensional κ-Minkowski space-time. This maximal acceleration has emerged soley due to the non-commutative geometry of the κ-Minkowski space-time. The maximal acceleration derived in Eq.(6.17) and Eq.(6.45) different as the first one is derived using the generalisation of Caianiello's approach to non-commutative space-time whereas the second one is derived by considering the line element in the 4-dimensional κ-Minkowski space-time.

The maximal acceleration (see Eq.(6.45)) in the κ-Minkowski space-time contains two terms. Both these terms depend on the rest mass of the particle, minimum distance of approach and the (dimensionless) non-commutative parameter α. In addition to this the first term also depends on κ-deformed length scale a. Note that the second term depends on metric deformation energy p^0 and the (dimensionless) non-commutative parameter β. Here these parameters α, β and the deformation energy p^0 come from the realisation given in Eq.(6.31)). By imposing the positivity condition on the magnitude of the maximal acceleration we get the bounds on α and β as either $\alpha > -1,\ \beta > -(1+2\alpha+\frac{c\hbar}{ap^0})$ or $\alpha < -1,\ \beta < -(1+2\alpha+\frac{c\hbar}{ap^0})$ respectively.

The maximal acceleration approaches infinity when we take the commutative limit $a \rightarrow 0$. Another novel feature associated with this maximal acceleration (see Eq.(6.45)) is the existence of an $\hbar$ independent term. Therefore in the classical limit $\hbar \rightarrow 0$, the maximal acceleration (induced by the 4-dimensional κ-Minkowski space-time) attains a finite value unlike the maximal acceleration in Eq.(6.17) (where it becomes infinity in the classical limit).

We also show that the consistent Newtonian limit of the κ-geodesic equation (corresponding to realisation given in Eq.(6.31)) can be obtained only when $\beta = 0$. Further we construct the κ-deformed Newton's force equation, valid up to first order in a. Here the first order correction term contains deformation energy p^0 dependent term as well as the mass dependent term. This mass dependent term violates the principle of equivalence. The violation of equivalence principle in the κ-deformed space-time has also been reported in [48], while studying the Kepler problem in κ space-time. By comparing this equivalence principle violating term with the experimental result, we find a bound on non-commutative parameter as $\alpha < -0.5$. Thus by combining this with earlier bounds we get possible allowed range of values for α as $-1 < \alpha < -0.5$ with $\beta = 0$.

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Chapter 7

Conclusions

In this thesis, we have investigated some implications of two non-commutative space-times, viz; κ-deformed and DFR space-times. We have analysed issues such as quantisation of certain non-commutative fields (such as κ-scalar, κ-Dirac and DFRA-scalar fields), core envelope model of κ-deformed superdense star and the maximal acceleration in κ-deformed space-time.

In chapter 1, we have presented motivations for studying non-commutative space-times. We have also given a brief summary of the non-commutative space-times such as κ space-time and DFR space-time in this introductory chapter.

The non-uniqueness associated with the Lagrangians of non-commutative fields makes the usual quantisation schemes ambiguous. To overcome this difficulty, we have studied the quantisation of non-commutative field theory using only their equation of motions, which are uniquely given by the quadratic Casimir of the symmetry algebra. This is done by generalising the Takahashi-Umezawa approach to non-commutative space-time. We have applied this approach and quantised the κ-scalar field (in chapter 2), κ-Dirac field (in chapter 3) and DFRA-scalar field (in chapter 4).

In chapter 2, we have derived the deformed unequal time commutation relation between the deformed scalar field and its adjoint by assuming the usual form of the oscillator algebra. Further, we obtained a deformed oscillator algebra (valid up to first order in a) by demanding the unequal time commutation relation between the deformed scalar field and its adjoint to be undeformed. We have constructed the energy-momentum tensor and the Lorentz generator (valid up to first order in a) for the κ-deformed scalar field. We have also studied the implications of the deformed oscillator algebra to Unruh effect using the method of Bogoliubov coefficients. We showed that the vacuum expectation value of the (Rindler) number operator (associated with the κ-deformed scalar field) in Minkowski vacuum is modified and this modification is caused due to the deformed oscillator algebra.

In chapter 3, we have derived the deformed unequal-time anti-commutation relation between the deformed Dirac field and its adjoint by using the usual form for the fermionic oscillator algebra. Next, by imposing the unequal time anti-commutation relation between the Dirac field and its adjoint to be undeformed, we show that the fermionic creation and annihilation operators obey deformed oscillator algebra (valid up to first order in a). This deformation factor is exactly the same as the deformation factor appearing in the deformed oscillator of the κ-scalar field. The number operator, energy-momentum tensor and Lorentz generator for the κ-Dirac field are also derived without referring to its Lagrangian. The a dependent term of the deformed number operator has a mass-dependence. Such mass-dependent term is expected to have phenomenological implications and this has to be studied separately. We have also studied the discrete symmetries associated with the κ-Dirac field and have constructed the conserved currents corresponding to parity and time-reversal symmetries of the κ-deformed Dirac equation. Further, by generalising the consistency condition (of the conserved quantities constructed in the Takahashi-Umezawa scheme) to the κ-deformed space-time, we have shown that charge conjugation is not symmetry for the κ-deformed Dirac equation (even up to first order in a).

In chapter 4, we have studied the quantisation of the DFRA scalar field. The action associated with the DFRA scalar field is not unique due to the non-uniqueness in the choice of weight function, which is introduced in action to control the divergences associated with the θ integration. We have obtained the deformed (equal-time) commutation relation between the DFRA scalar field and its conjugate by assuming the creation and annihilation operators to satisfy the usual oscillator algebra. We then showed that imposing the commutation relation between the DFRA scalar field and its conjugate to be undeformed, leads to a deformed oscillator algebra. We find that this deformation factor depends on the choice of the weight function and this is coming from the equation of motion, which has the weight function dependent term. We have also constructed the energy-momentum tensor and Lorentz generator for the DFRA scalar field. Further, we have studied the Unruh effect in DFR space-time by analysing a uniformly accelerating monopole detector coupled to a massless DFRA scalar field. We showed that this thermal distribution is either Bose-Einstein or Fermi-Dirac, depending upon the dimension of DFR space-time. As in the case of κ-deformed scalar field, here also we showed that the Unruh temperature is unaffected by the non-commutativity of space-time.

In chapter 5, we have studied the effects of the non-commutativity on superdense star by generalising the anisotropic core-envelope model of the superdense star to κ-deformed space-time. We have constructed the κ-deformed Einstein's equation by replacing the commutative quantities with the κ-deformed ones. We then solved this deformed Einstein's equation (valid up to first order in a) in the core and envelope separately. We observe that density is scaled under the κ-deformation. The deformed density decreases from a maximum value at the centre to a minimum value on the outer boundary and thus, we showed that the law of density variation is preserved under the κ-deformation. We also obtained the expressions for the deformed isotropic core pressure, deformed radial pressure and deformed tangential pressure (valid up to first order in a). Further, we showed that their values are positive and these values decrease monotonically as we move from the central core to the outer envelope of the superdense star. We observe that the speed of sound inside the superdense star is less than the speed of light. The density and pressures of the κ-deformed superdense star is found to satisfy the conditions for a physically acceptable model, i.e., $\hat{\rho} > 0,\ \hat{p} > 0,\ \hat{p}_t > 0,\ \hat{p}_r > 0,\ \frac{d\hat{\rho}}{dr} < 0,\ \frac{d\hat{p}}{dr} < 0,\ \frac{d\hat{p}_t}{dr} < 0,\ \frac{d\hat{p}_r}{dr} < 0,\ \frac{d\hat{p}}{d\hat{\rho}} < 1,\ \frac{d\hat{p}_t}{d\hat{\rho}},\ \frac{d\hat{p}_r}{d\hat{\rho}} < 1.$

In chapter 6, we have studied the maximal acceleration in the κ-deformed space-time using two different approaches. In the first method, we have derived the κ-deformed corrections to the maximal acceleration, valid up to the first order in a, using the 8-dimensional line element defined on the κ-deformed phase-space. Further, we derived the first order κ-deformed corrections to the maximal acceleration from the κ-deformed uncertainty principle. In the second method, we have shown the emergence of maximal acceleration (valid up to first order in a) using the 4-dimensional line element on κ-Minkowski space-time. We then derived the κ-deformed geodesic equation and obtained its Newtonian limit. We showed that κ-deformed Newton's force equation contains an equivalence principle violating term. By comparing this violation with the experimental result on the violation of the equivalence principle, we obtain a bound on the dimensionless non-commutative parameters present in the non-commutative correction to the maximal acceleration obtained using the second approach.

It will be quite interesting to use the Takahashi-Umezawa quantisation procedure to study the quantisation of κ-deformed gauge fields and see the implications of the deformed oscillator algebra. Further, we plan to study the quantisation of DFRA-Dirac field and obtain its deformed oscillator algebra. We also plan to apply this method to study the quantisation of high spin fields. We are also interested in studying the behaviour of quark star in non-commutative space-time.

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To Whom It May Concern

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