On the tautological algebra of the moduli space of semistable bundles over curve

A thesis submitted during 2020 to the
University of Hyderabad

In partial fulfillment of the award of
a Ph.D. degree in
School of Mathematics and Statistics

by

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CERTIFICATE

This is to certify that the thesis entitled "**On the tautological algebra of the moduli space of semistable bundles over curve**" by **Arijit Mukherjee** bearing Reg. No. 15MMPP01 in partial fulfillment of the requirements for the award of Doctor of Philosophy in **Mathematics** is a bonafide work carried out by him under my supervision and guidance.

The thesis has not been submitted previously in part or in full to this or any other University or Institution for the award of any degree or diploma.

Parts of this thesis have been:

A. published in the following publications :

Arijit Mukherjee, *Indian J. Pure Appl. Math.*, **51**(4) , 2020. (in press) (**Chapter 4**)

B. presented in following conferences :

1. International Conference on Analysis, Algebra, Combinatorics and their Applications (ICAACA - 2020), 20-22 January, Jadavpur University, Kolkata.

2. International Conference on Advances in Physical ,Chemical and Mathematical Sciences (ICAPCM - 2020), 13-16 February, RTMNU (Nagpur University), Nagpur.

Further, the student has passed the following courses towards fulfillment of course work requirement for Ph.D.

Sr. No.	Course code	Name of the course	Credits	Grade
1	MM801	Advanced Analysis	5	A^{+}
2	MM802	Advanced Algebra	5	B^{+}
3	MM810	Algebraic Geometry - II	5	A^{+}
4	MM830	Research Methodology and Ethics	4	A^{+}

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DATE: October 29, 2020.

DECLARATION

I, **Arijit Mukherjee**, hereby declare that this thesis entitled "**On the tautological algebra of the moduli space of semistable bundles over curve**" submitted by me under the guidance and supervision of **Dr. Archana S. Morye** is a bonafide research work. I also declare that it has not been submitted previously in part or in full to this University or any other University or Institution for the award of any degree or diploma.

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Date: 29.10.20

ACKNOWLEDGEMENTS

Undertaking this Ph.D. is a journey, which would not have been possible without direct and indirect involvement of many people. My sincere apology to those whom I forget to acknowledge.

At first I would like to express my deepest gratitude to my advisor Dr. Archana S. Morye for the continuous help, advise, unparalleled support and encouragements. The completion of this thesis would not have been possible without her constant guidance and support. She is always there for me for which I am deeply indebted to her. I extend my gratitude to her for allowing to visit IMSc, Chennai for my research work during my Ph.D.

I can not begin to express my thanks to Dr. Tathagata Sengupta, who was my supervisor before shifting to HBCSE, TIFR, Mumbai. I have had the opportunity to spend a couple of years under his guidance. He always encouraged me to learn a lot of things and not to be confined. Though I miss his presence, nevertheless he is always only a phone call away.

I am extremely grateful to Prof. Jaya. N. Iyer for suggesting and helping me in my research problems. I am deeply indebted to her for her generous help and valuable suggestions. I can never forget the way she supported and nurtured me. I extend my sincere gratitude to her for arranging my several visits to IMSc which was very beneficial to me.

I am very thankful to my doctoral committee members, Dr. S. Ilangovan and Dr. M. S. Datt, for their minute suggestions and continuous assessment. It was a real pleasure to attend a few courses taught by Prof. V. Kannan, Prof. R. Tandon and Prof. S. Kumaresan. I extend my sincere gratitude to Prof. V. Kannan, Dr. S. Ilangovan and Dr. Suman Kumar for always being there whenever I approached them for seeking suggestions. I would like to thank Prof. R. Radha, present dean of our school. I fondly recall the assistance I have received from my fellow research scholars, faculties and other members of our department and thank them all.

A major bonus of studying at University of Hyderabad is its charismatic campus. A countless memories of vagabond roaming amidst the jungles, meandering through the meadows and climbing the rocks will be eternal.

I am indebted to Kallol and Chandan as they made me feel at home when

I came to the university for the very first time to attend my Ph.D interview. I fondly recall the countless moments I spent with Apurba, Bappa, Sudipta da, Kuntal and Debika. Without these people my university life would not have been the same. My everyday locus intersected the most with the loci of these five persons - these bonds are that special. I like to thank Kaushik, Kohinoor da, Nivedita and Sugata da for their selfless support through thick and thin. I must not miss the opportunity to thank Sritam, Sazzad da, Suman da, Raja da, Rudra da, Suchana, Soutrick, Pabitra and Joydev for making my university days colourful. I want to thank all my friends, juniors and seniors of the university for their cheerful company.

I am deeply indebted to the professors of my alma mater St. Xavier's College (Kolkata) and Ballygunge Science College (Calcutta University) for encouraging me to pursue higher studies in Mathematics. Special thanks to Dr. Rabiul Islam (St. Xavier's College, Kolkata) for his invaluable insights into various branches of Mathematics. I gratefully acknowledge all my teachers from my childhood days who kept me motivated throughout to follow my dream.

I would like to recognize the assistance I have received from Chandra da and Suratno da academically during my stay at IMSc. Many of my friends from IMSc and IIT, Madras had made my days over there joyful and hence I thank them all.

I gratefully acknowledge UGC and 'Complex and Algebraic geometry' DAE-project for providing me financial support.

Finally and most importantly, I like to thank my parents, dadu, dida and thakuma for their sympathetic ear and unconditional support throughout. I miss my late grandfather Mr. Benukar Mukherjee who would have been very glad and proud for me like always. He always believed in my ability irrespective of the circumstances and his belief in me has made this journey possible. I can not express in words the importance of these persons in my life, literally.

ABSTRACT

This thesis consists of five chapters.

In Chapter 1, some basic algebraic geometry like Riemann-Roch theorem, Serre duality, Jacobian variety, symmetric product etc. that are relevant for our purpose have been discussed. We mainly recall Hodge structure, mixed Hodge structure, Chow groups, operational Chow groups and cycle class map in Chapter 2. Chapter 3 deals with the moduli spaces in general followed by the moduli spaces of stable and semistable vector bundles over curve.

In Chapter 4, the relations amongst the cohomology classes of the Brill-Noether subvarieties of the moduli space of semistable bundles over an elliptic curve have been found. We obtain results similar to the Poincaré relation on a Jacobian variety.

Chapter 5 is devoted to similar problems as in chapter 4, but for genus greater than one case. Here we determine the tautological algebra, the algebra generated by the cohomology classes of the Brill-Noether loci in the rational cohomology of the moduli space $\mathcal{M}_C(2,d)$ of semistable bundles of rank 2 and degree d. We show that when C is a general smooth projective curve of genus $g \geq 2$, $d = 2g-2$, the tautological algebra of $\mathcal{M}_C(2, 2g-2)$ (respectively $\mathcal{SU}_C(2, L)$, deg $L = 2g-2$) is generated by the divisor classes (respectively the class of the Theta divisor Θ). Here by $\mathcal{SU}_C(2, L)$ we mean the moduli space of semistable bundles over C of rank 2 and fixed determinant L with deg $L = 2g-2$. Also we prove some results about the non-emptiness of the loci.

SYNOPSIS

1 Introduction

In the nineteenth century, an abstract group was considered to be a subset of the general linear group GL_n for some $n \in \mathbb{N}$, which is closed under multiplication and inversion. In the modern language, this is a representation of a group G on a n-dimensional vector space. In the twentieth century, to classify or to understand groups, their representations were studied. The analogous transformation occurred in algebraic geometry during that time. Prior to that, algebraic curve simply meant the zero set of an irreducible polynomial in two variables. In the twentieth century, the notion of abstract curves changed and studying the classification of curves meant that one has to describe the moduli space $\mathcal{M}_g$ (of fixed genus g). Like the representation of groups, one can see how an algebraic curve can be embedded in the projective space $\mathbb{P}^r$, for some $r \in \mathbb{N}$. Brill-Noether theory is broadly related to the determinantal loci associated to the embeddings.

A study of the Brill-Noether loci was first carried out on the Jacobian of curves by Fulton, Lazarsfeld, Griffiths and Harris. They contributed in answering the natural questions on these loci, namely non-emptiness, irreducibility, dimension, cohomological relations and understanding the singular loci. One can refer to [Fu-La] and [Gf-Hr 1] to look into their work in this direction.

On the moduli space of higher rank semistable vector bundles of fixed degree on a smooth curve, C. S. Seshadri, N. Sundaram (cf. [Su]) and M. Teixidor i Bigas (cf. [Bg 1]) initiated a similar study of the Brill-Noether loci, answered some of the interesting questions, and posed further questions. Notable results were obtained in [Bg 1], [Bg 2], [Br-Gz-Ne], [Me 1] and [Me 2]. More recent developments on non-emptiness of the Brill-Noether loci can be found in [La-Ne-St], [La-Ne-Pr], [La-Ne 1], [La-Ne 2] and [La-Ne 3]. A compilation of the questions can be found in [Ne 2].

In this thesis we look at the questions of finding cohomological relations amongst the Brill-Noether loci for a general curve. On the Jacobian $J(C)$ of a smooth projective curve C, this is classical and is known as Poincaré formula, whereas the cohomological relations on the moduli space $J_d(C)$ of degree d line bundles on a general curve C are called Castelnuovo's formula (or Porteous' formula). Our aim in this thesis is to investigate similar cohomological relations on the higher rank moduli spaces of semistable vector bundles with fixed degree on a general smooth projective curve. The results are obtained in the rank two situation and when the degree is $2g-2$, g being the genus of the curve.

We now outline our proposed thesis. The thesis consists of five chapters. In the first chapter, some basic algebraic geometry like Riemann-Roch theorem, Serre duality etc. that are relevant for our purpose have been discussed. We mainly recall Hodge structure, mixed Hodge structure, Chow groups, operational Chow groups and cycle class map in the second chapter. The third chapter deals with the moduli spaces in general followed by the moduli spaces of vector bundles over curve. We also thoroughly go through the definition and some properties of the Brill-Noether subvarieties in this chapter which finally narrow down towards our work. In the fourth chapter, the relations amongst the cohomology classes of the Brill-Noether subvarieties of the moduli space of semistable bundles over an elliptic curve have been found. We obtain results similar to Poincaré formula on a Jacobian variety. The fifth chapter is devoted to similar problems as in chapter 4, but for genus greater than one case. Here we determine the tautological algebra, the algebra generated by the cohomology classes of the Brill-Noether loci in the rational cohomology of the moduli space $\mathcal{M}_C(2,d)$ of semistable bundles of rank 2 and degree d. We show that when C is a general smooth projective curve of genus $g \geq 2$, $d = 2g-2$, the tautological algebra of $\mathcal{M}_C(2, 2g-2)$ (respectively $\mathcal{SU}_C(2, L)$, deg $L = 2g-2$) is generated by the divisor classes (respectively the class of the Theta divisor Θ). Here by $\mathcal{SU}_C(2, L)$ we mean the moduli space of semistable bundles over C of rank 2 and fixed determinant L with deg $L = 2g-2$. Also we prove some results about the non-emptiness of the loci.

The summary of this thesis work is given in Section 2, 3 and 4. In Section 2, we recall Poincaré formula on the Jacobian variety $J(C)$ and Castelnuovo's formula on $J_d(C)$ and discuss how these are related to the line of our work. Section 3 is about a brief discussion on our first work (cf. [Mk]). Section 4 is based on our second work which has already been communicated (cf. [Ga-Iy-Mk]).

In following sections we take C to be a smooth projective curve of genus g over complex numbers. We denote the moduli space of S-equivalence classes of semistable bundles of rank r and degree d over C by $\mathcal{M}_C(r, d)$. By $\mathcal{SU}_C(r, L)$ we denote the moduli space of S-equivalence classes of semistable bundles of rank r

and fixed determinant L of degree d over C.

2 Poincaré formula on the Jacobian variety $J(C)$ and Castelnuovo's formula on $J_d(C)$

In this section we recall Poincaré's formula on the Jacobian variety $J(C)$ and Castelnuovo's formula on $J_d(C)$. We reinterpret these formulas and show that our problems in Section 3 and 4 arise quite naturally from them.

Let us denote the d-fold product of the curve C with itself by $C^{\times d}$. Here, $S^d(C)$, the *d-th symmetric product of* C, can be understood as the quotient space $\dfrac{C^{\times d}}{\sigma^d}$ of $C^{\times d}$ under the action of the permutation group σ^d of d symbols. Therefore the elements of $S^d(C)$ are unordered d-tuple $x_1 + x_2 + \cdots + x_d$ of points of C.

Let us denote by $\mathcal{O}(D)$ the line bundle corresponding to a divisor D on C. Consider the classical *Abel-Jacobi map* $\varphi_d \colon S^d(C) \to J_d(C)$, defined as follows:

$$\begin{aligned} \varphi_d \colon S^d(C) &\to J_d(C) \\ x_1 + x_2 + \cdots + x_d &\mapsto \mathcal{O}(x_1 + x_2 + \cdots + x_d). \end{aligned} \tag{1}$$

Here $x_1+x_2+\cdots+x_d$ is also thought as a degree d effective divisor on C and hence $\mathcal{O}(x_1 + x_2 + \cdots + x_d)$ makes sense. The image of $S^d(C)$ under the map φ_d are subvarieties of $J_d(C)$ and are denoted by W_d^0 for all $1 \leq d \leq g$. The subvariety W_d^0 parametrizes degree d line bundles over C having atleast one independent global section as this is the image of effective divisors of degree d.

If we want to compare the cohomology classes $[W_d^0]$ for all $1 \leq d \leq g$, it is not possible to do so at this stage as they sit inside different $J_d(C)$ with varying d. Thus, to compare their cohomology classes, it is natural to think those as subvarieties of one fixed variety. This can be obtained as follows. Let us choose a point $p \in C$ and fix it. Consider the map $\otimes\mathcal{O}(-dp) \colon J_d(C) \to J(C)$ defined as $L \mapsto L \otimes \mathcal{O}(-dp)$. Then the map $u \colon S^d(C) \longrightarrow J(C)$ is defined as $u = \otimes\mathcal{O}(-dp) \circ \varphi_d$ where φ_d is as in (1). Now define W_d, for all d, $1 \leq d \leq g$, called the *Brill-Noether subvarieties of* $J(C)$, as $W_d := u(S^d(C))$.

Let Θ be the Theta divisor in $J(C)$, the translate of the divisor W_{g-1}^0 of $J_{g-1}(C)$ via the map $\otimes\mathcal{O}(-(g-1)p) \colon J_{g-1}(C) \to J(C)$. Let $[W_d]$ be the cohomology class of W_d and $[\Theta]$ be the cohomology class of Θ in $H^*(J(C), \mathbb{Q})$. The classical *Poincaré's formula* expresses the cohomological classes of W_d, in terms of the Theta divisor on $J(C)$ (cf. [Ab-Cr-Gf-Hr, p. 25]). In $H^*(J(C), \mathbb{Q})$, for all

d, $1 \leq d \leq g$, we have

$$[W_d] = \frac{1}{(g-d)!} \cdot [\Theta]^{g-d}. \tag{2}$$

The subvariety W_d^0 of $J_d(C)$ which parametrizes degree d line bundles over C having atleast one independent global section can be further stratified by varying the number of global sections. Define $W_d^r := \{L \in J_d(C) \mid h^0(L) \geq r+1\}$. Then in $J_d(C)$, $1 \leq d \leq g$, we have the following stratification:

$$J_d(C) \supseteq W_d^0 \supseteq \cdots \supseteq W_d^r \supseteq \cdots .$$

Thus it is natural to compare the cohomology classes of these varieties in the cohomology ring $H^*(J_d(C), \mathbb{Q})$. Let us define the *Brill-Noether number*, denoted by ρ, as $\rho := g-(r+1)(g-d+r)$. When C is general, the Néron-Severi group of $J_d(C)$ is generated by a translate of the Θ divisor in $J(C)$. We denote this class as θ_d. We then have the following formula, known as *Castelnuovo's formula*, very much similar to (2) (cf. [Gf-Hr 1]). For a general curve C,

$$[W_d^r] = \prod_{\alpha=0}^{r} \frac{\alpha!}{(g-d+r+\alpha)!} \cdot \theta_d^{g-\rho}. \tag{3}$$

Poincaré's formula as in (2) can be interpreted as follows. Consider the subalgebra of $H^*(J(C), \mathbb{Q})$ a priori generated by the cohomology classes W_d, $1 \leq d \leq g$. Then this subalgebra is generated by $[\Theta]$ only. Similarly, (3) depicts that the subalgebra of $H^*(J_d(C), \mathbb{Q})$, a priori generated by the cohomology classes W_d^r for varying r, is actually generated by θ_d only. We consider similar problem in the cohomology ring of the moduli space of semistable bundles over an elliptic curve in Section 3 and over curve of genus greater than equal to two in Section 4.

3 Tautological algebra of the moduli space of semistable bundles over an elliptic curve

In this section, we describe the algebra generated by the cohomology classes of certain Brill-Noether subvarieties of the moduli space of semistable bundles over a curve C of genus 1, that is, over an elliptic curve. L. Tu proved that the Brill-Noether loci are trivial for positive degree vector bundles (either empty or the whole moduli space), and for line bundles of degree 0 (either empty or singleton) (cf. [Tu, Lemma 17 & p. 13 below Lemma 17]). Therefore, we consider the Brill-

Noether loci when the degree of a vector bundle is zero and the rank is more than one. Let L be a line bundle of degree 0 and let i be any non-negative integer. The following two definitions of the Brill-Noether loci are again due to [Tu, p. 4 & 5]. For a vector bundle F over C, we denote the S-equivalence class of F by f. Then the *Brill-Noether loci in* $\mathcal{SU}_C(r, L)$ are defined as follows:

$$W^i_{r,L}(\exists) := \left\{ f \in \mathcal{SU}_C(r, L) \mid h^0(F) \geq i+1 \text{ for some } F \in f \right\}.$$

We denote the cohomology class of $W^i_{r,L}(\exists)$ in $H^*(\mathcal{SU}_C(r, L), \mathbb{Z})$ by $[W^i_{r,L}(\exists)]$. This class is also called a *tautological class* and algebra generated by these classes $[W^i_{r,L}(\exists)]$ for varying i is called *tautological algebra of* $H^*(\mathcal{SU}_C(r, L), \mathbb{Z})$. We consider the analogous situation in the moduli space $\mathcal{M}_C(r, 0)$. The *Brill-Noether loci in* $\mathcal{M}_C(r, 0)$ are defined as follows:

$$W^i_{r,0}(\exists) := \left\{ f \in \mathcal{M}_C(r, 0) \mid h^0(F) \geq i+1 \ \text{ for some } F \in f \right\}.$$

We define *tautological class* and *tautological algebra of* $H^*(\mathcal{M}_C(r, 0), \mathbb{Z})$ similarly. In [Mk], we prove the main theorems on the relations amongst the tautological classes in $H^*(\mathcal{SU}_C(r, L), \mathbb{Z})$ and in $H^*(\mathcal{M}_C(r, 0), \mathbb{Z})$. We show the following:

Theorem 0.0.1 *Let r be any positive integer and let L be a degree 0 line bundle over C of genus 1. Then $W^0_{r,L}(\exists)$ is a divisor inside $\mathcal{SU}_C(r, L)$. Moreover, in $H^*(\mathcal{SU}_C(r, L), \mathbb{Z})$, we have*

$$[W^i_{r,L}(\exists)] = [W^0_{r,L}(\exists)]^{i+1},$$

for all $0 \leq i \leq r-2$ and the tautological algebra of $\mathcal{SU}_C(r, L)$ is $\mathbb{Z}[\zeta]/\langle \zeta^r \rangle$, where ζ is the cohomology class of $W^0_{r,L}(\exists)$ in $H^*(\mathcal{SU}_C(r, L), \mathbb{Z})$.

Moreover the determinant morphism $\det\colon \mathcal{M}_C(r, 0) \to J(C)$ is a projective bundle (cf. [Tu, p. 12]) and we use projective bundle formula to obtain the structure of the tautological algebra of the cohomology ring of $\mathcal{M}_C(r, 0)$. In particular, we prove the following:

Theorem 0.0.2 *The tautological algebra of $\mathcal{M}_C(r, 0)$ is*

$$H^*(C) \otimes \mathbb{Z}[\xi]/\langle \xi^r \rangle.$$

Here ξ is the cohomology class of the divisor $W^0_{r,0}(\exists)$ on $\mathcal{M}_C(r,0)$ in the cohomology ring $H^*(\mathcal{M}_C(r,0),\mathbb{Z})$.

4 Tautological algebra of the moduli space of semistable bundles of rank two on a general curve of genus greater than one

In this section, we discuss the outline of a joint work with C. Gangopadhyay and J. N. N Iyer. We consider the tautological algebra of the (rational) cohomology ring of the moduli space of semistable bundles over curve of genus greater than one. From now onwards, we are under rank 2 and degree $2g-2$ case. Let P be a Prym variety associated to a spectral curve $\pi\colon \widetilde{C} \to C$. We first prove the following theorem using [Bi, Corollary 5.3] as there exists an isogeny from $J(C)\times P$ to $J(\widetilde{C})$.

Theorem 0.0.3 *The cohomology class of a Brill-Noether locus on the Jacobian $J(\widetilde{C})$ of a general 2-sheeted spectral curve $\pi\colon \widetilde{C} \to C$ can be expressed as a sum of the powers of the divisor classes. In particular, the tautological algebra is generated by the divisor classes.*

The key idea is to relate the Brill-Noether loci on the moduli space with the Brill-Noether loci on the Jacobian variety of a general spectral curve. We utilise the rational map obtained in [Be-Na-Ra] from the Jacobian of a general spectral curve $\widetilde{C}$ to the moduli space $\mathcal{M}_C(2,2(g-1))$. Explicitly we get the following rational map:

$$\pi_*\colon J_{4(g-1)}(\widetilde{C}) \dashrightarrow \mathcal{M}_C(2,2(g-1)). \tag{4}$$

We use a finite regular dominant morphism corresponding to (4) and Theorem 0.0.3 to prove the following theorem (cf. [Ga-Iy-Mk]):

Theorem 0.0.4 *Suppose C is a general smooth projective curve of genus g, and $g\geq 2$. The cohomology class of a Brill-Noether locus on the moduli space $\mathcal{M}_C(2,2(g-1))$ can be expressed as a polynomial on the divisor classes.*

Similarly, in the moduli space $\mathcal{SU}_C(2,L)$ with $\deg L = 2(g-1)$, we obtain the following corollary:

Corollary 0.0.5 *1. The cohomology class of a Brill-Noether locus $\widetilde{W^{r,L}_{2,2(g-1)}}$ in the moduli space $\mathcal{SU}_C(2,L)$ is expressible in terms of a power of the class of the Theta divisor, with rational coefficients.*

2. If the Brill-Noether number is non-negative, then the cohomology classes are non-trivial and imply the non-emptiness of the corresponding loci.

It is likely that the Hodge conjecture holds for the Jacobian of a higher degree general spectral curve (cf. [Ar] for unramified coverings). The proofs employed in Theorem 0.0.4 will then be applicable also for higher rank moduli spaces. The proofs raise further questions whether a Castelnuovo type formula holds or not on the moduli space, for a general curve C.

List of Publications and Preprints:

1. A. Mukherjee, *Tautological algebra of the moduli space of semistable bundles on an elliptic curve*, accepted for publication in Indian Journal of Pure and Applied Mathematics.

2. C. Gangopadhyay, J. N. N. Iyer and A. Mukherjee, *Tautological algebra of the moduli space of semistable bundles of rank two on a general curve*, Preprint.

Dedicated to my parents

Mrs. Sipra Mukherjee & Mr. Tarun Kumar Mukherjee

Contents

Conventions and Notations

We denote by $\mathbb{C}$ (respectively $\mathbb{R}$) the field of complex numbers (respectively real numbers). The ring of integers is denoted by $\mathbb{Z}$ and a set of non-negative integers is denoted by $\mathbb{N}$.

Given a complex number z, by $\mathrm{Im}(z)$ we denote its imaginary part. A fixed square root of -1 is denoted by $\imath$. For a morphism f, we denote its image by $\mathrm{Im}(f)$. Given a space M, the identity morphism of M is denoted by Id_M or simply by Id whenever no confusion is likely to occur.

Let R be a commutative ring with unity. By $\mathrm{Spec}(R)$ we denote the spectrum of the ring R consisting of all prime ideals of R equipped with the structure of a locally ringed space. Given any two positive integers m and n, the set of all $m \times n$ matrices with entries from the ring R is denoted by $\mathrm{M}_{m\times n}(R)$. The same is denoted by $\mathrm{M}_n(R)$ whenever $m = n$. The subgroups of $\mathrm{M}_n(R)$ consisting of all invertible matrices is denoted by $\mathrm{GL}_n(R)$. By Id_n we denote the identity element of $\mathrm{M}_n(R)$. By $\mathrm{PGL}_n(R)$ we denote the quotient $\frac{\mathrm{GL}_n(R)}{\{\lambda\cdot\mathrm{Id}_n \mid \lambda\in R\}}$. As the spaces $\mathrm{GL}_n(R)$ and $\mathrm{PGL}_n(R)$ occur frequently, we use the notations GL_n and PGL_n respectively whenever there is no confusion.

All the varieties are taken over $\mathbb{C}$. The singular locus of a variety X is denoted by Sing X. We set the notation dim X to denote the dimension of a variety X. By C we denote a smooth projective curve of genus g. We denote an elliptic curve by E to differentiate it from higher genus curves, whenever needed. For a vector bundle V over a space M, its rank and degree are denoted by rank V and deg V respectively. For a non-negative integer n, by $\wedge^n V$ we denote the n-th exterior power of the vector bundle V.

If $\mathcal{C}$ is a (locally small) category, and if X and Y are objects in $\mathcal{C}$, then the set of morphisms from X into Y is denoted by $\mathrm{Hom}(X, Y)$. If $X = Y$, then the same set is denoted by $\mathrm{End}(X)$. By $\mathrm{Ext}^1(X, Y)$ we denote the collection of all extensions of Y by X.

We preserve the notation $\varinjlim_{i\in I} A_i$ to denote the direct limit of A_i's, where i varies over an directed indexing set I.

Introduction

In this thesis we study two problems on the tautological algebra, that is, the algebra generated by cohomology classes of Brill-Noether loci inside the cohomology ring of moduli space of semistable bundles over curve. The first problem is regarding the complete description of the tautological algebra over elliptic curve. The other problem concerns about a similar description in rank 2 case over curve having genus greater than equals to 2.

The thesis consists of five chapters. The first three chapters are devoted to a detailed study of the topics in algebraic geometry that are relevant for our purpose and hopefully reflects our attempt to make this thesis self-content as far as possible.

In the Chapter 1, some basic algebraic geometry like Riemann-Roch theorem, Serre duality, Jacobian variety, symmetric product etc. that are relevant for our purpose have been discussed. We mainly recall Hodge structure, mixed Hodge structure, Chow groups, operational Chow groups and cycle class map in Chapter 2. Chapter 3 deals with the moduli spaces in general followed by the moduli spaces of stable and semistable vector bundles over curve.

Tautological algebra of the moduli space of semistable bundles over an elliptic curve

In Chapter 4, we describe the algebra generated by the cohomology classes of certain Brill-Noether subvarieties of the moduli space of semistable bundles over a curve C of genus 1, that is, over an elliptic curve. L. Tu proved that the Brill-Noether loci are trivial for positive degree vector bundles (either empty or the whole moduli space), and for line bundles of degree 0 (either empty or singleton) (cf. [Tu, Lemma 17 & p. 13 below Lemma 17]). Therefore, we consider the Brill-Noether loci when degree of a vector bundle is 0 and the rank is more than 1. Let L be a line bundle of degree 0 and let i be any non-negative integer. L. Tu defined Brill-Noether loci $W^i_{r,L}(\exists)$ and $W^i_{r,0}(\exists)$ in the moduli spaces $\mathcal{SU}_C(r, L)$ and $\mathcal{M}_C(r, 0)$ respectively (cf. [Tu, p. 4 & 5]). We denote the cohomology class of $W^i_{r,L}(\exists)$ in $H^*(\mathcal{SU}_C(r, L), \mathbb{Z})$ by $[W^i_{r,L}(\exists)]$. This class is also called a *tautological class* and algebra generated by these classes $[W^i_{r,L}(\exists)]$ for varying i is called *tautological algebra of* $H^*(\mathcal{SU}_C(r, L), \mathbb{Z})$. We define *tautological class* and *tautological algebra of* $H^*(\mathcal{M}_C(r, 0), \mathbb{Z})$ similarly. In Subsection 4.3.2, we prove the main theorems on the relations amongst the tautological classes in $H^*(\mathcal{SU}_C(r, L), \mathbb{Z})$ and in $H^*(\mathcal{M}_C(r, 0), \mathbb{Z})$. We show that the Brill-Noether subvariety $W^0_{r,L}(\exists)$ is a divisor inside $\mathcal{SU}_C(r, L)$. Moreover, in $H^*(\mathcal{SU}_C(r, L), \mathbb{Z})$ we obtain Poincaré like relations $[W^i_{r,L}(\exists)] = [W^0_{r,L}(\exists)]^{i+1}$ for all $0 \leq i \leq r-2$. Moreover, denoting the cohomology class of $W^0_{r,L}(\exists)$ in $H^*(\mathcal{SU}_C(r, L), \mathbb{Z})$ by ζ, we show that the tautological algebra of $\mathcal{SU}_C(r, L)$ is $\mathbb{Z}[\zeta]/\langle \zeta^r \rangle$.

Furthermore as the determinant morphism $\det\colon \mathcal{M}_C(r, 0) \to J(C)$ is a projective bundle (cf. [Tu, p. 12]), we use projective bundle formula to obtain the structure of the tautological algebra of the cohomology ring of $\mathcal{M}_C(r, 0)$.

Tautological algebra of the moduli space of semistable bundles of rank two on a general curve of genus greater than one

In Chapter 5, we consider the tautological algebra of the (rational) cohomology ring of the moduli space of semistable bundles over curve of genus greater than one. In that sense, this problem is a natural successor of the problem described in previous chapter. In this chapter, we are under rank 2 and degree $d = 2g-2$ case. We use [Bi, Corollary 5.3] to prove that the cohomology class of a Brill-Noether locus on the Jacobian $J(\widetilde{C})$ of a general 2-sheeted spectral curve $\pi\colon \widetilde{C} \to C$ can be expressed as a sum of the powers of the divisor classes. In particular, the tautological algebra is generated by the divisor classes.

The key idea is to relate the Brill-Noether loci on the moduli space with the Brill-Noether loci on the Jacobian variety of a general spectral curve. We utilise the rational map obtained in [Be-Na-Ra] from the Jacobian of a general spectral curve $\widetilde{C}$ to the moduli space $\mathcal{M}_C(2, 2(g-1))$. We use a finite regular dominant morphism corresponding to this rational map and in Section 5.5 we show that when C is a general smooth projective curve of genus $g \geq 2$ then the cohomology class of a Brill-Noether locus on the moduli space $\mathcal{M}_C(2, 2(g-1))$ can be expressed as a polynomial on the divisor classes.

Similarly, in the moduli space $\mathcal{SU}_C(2, L)$ with $\deg L = 2(g-1)$, we obtain that the cohomology class of a Brill-Noether locus $\widetilde{W^{r,L}_{2,2(g-1)}}$ in the moduli space $\mathcal{SU}_C(2, L)$ is expressible in terms of a power of the class of the Theta divisor, with rational coefficients. Moreover, we show that if the Brill-Noether number is non-negative, then the cohomology classes are non-trivial and imply the non-emptiness of the corresponding loci. The proofs employed in these results will then also be applicable for higher rank moduli spaces. The proofs raise further questions whether a Castelnuovo type formula holds on the moduli space or not, for a general curve C.

A more detailed introduction to the thesis, including precise definitions, results and statements of the theorems, is given in the synopsis (p. **i**).

Chapter 1

Preliminaries

In this chapter, we go through a few basics of algebraic geometry that are relevant for our purpose.

We thoroughly discuss about the sheaf cohomology, relations between vector bundles and locally free sheaves and that between line bundles and divisors. The Riemann-Roch theorem, Serre duality have been discussed in case of curves. We also deal with abelian variety in general and Jacobian variety of a curve. In the process we raise the question whether all abelian varieties can be realised as Jacobian varieties or not. We answer this question affirmatively in case of an elliptic curve and mention that we get negative answer in general. We also spend some time on the symmetric product of curve and Abel-Jacobi map, both analytically and algebraically.

1.1 Sheaves and Cohomology

In this section, we quickly go through a few examples of sheaves. Followed by that, sheaf cohomology is described as derived functor cohomology and then compared with Čech cohomology as well.

1.1.1 Sheaves

Let M be a complex manifold. Then we define the sheaves $\mathcal{O}$, $\mathcal{O}^*$, $\mathcal{M}$, $\mathcal{M}^*$, Ω^p , $\mathbb{Z}$ by,

- $\mathcal{O}(U) :=$ the additive group of holomorphic functions on U,

- $\mathcal{O}^*(U)$:= the multiplicative group of non-zero holomorphic functions on U,
- $\mathcal{M}(U)$:= the additive group of meromorphic functions on U,
- $\mathcal{M}^*(U)$:= the multiplicative group of meromorphic functions on U that are not identically zero,
- $\Omega^p(U)$:= the group of holomorphic p-forms on U,
- $\mathbb{Z}(U)$:= the group of locally constant $\mathbb{Z}$-valued functions on U,

where U is an open subset of M. Sometimes in these notations of sheaves, M is used in the subscript especially when we need to emphasize about the base space involved. For example, often we use the notations $\mathcal{O}_M$ and $\mathcal{O}^*_M$ instead of $\mathcal{O}$ and $\mathcal{O}^*$ respectively. Also the notation $\mathcal{O}_M$ help us to differentiate the sheaf of additive group of holomorphic functions on M from the notation $\mathcal{O}$ which we use to relate a line bundle of a given divisor (cf. Section 1.3).

For a variety X, the ring of its regular functions is denoted by $k[X]$. Alternatively $k[X]$ is also denoted by $\mathcal{O}_X(X)$. Here $\mathcal{O}_X$ is a sheaf, called the *structure sheaf of* X and by $\mathcal{O}_X(U)$ we mean the ring of regular functions on an open subset U of X.

Let us define *skyscraper sheaf of* $x \in X$, denoted by $\mathbb{C}_x$, as follows:

$$\mathbb{C}_x(U) := \begin{cases} \mathbb{C} & \text{if } x \in U; \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1.1.1 For a given sheaf $\mathcal{F}$ over a topological space X, we denote the stalk of $\mathcal{F}$ at a point $x \in X$ by $\mathcal{F}_x$. By *support* Supp $\mathcal{F}$ of a sheaf $\mathcal{F}$ we mean the set $\{x \in X \mid \mathcal{F}_x \neq 0\}$. Therefore, Supp $\mathbb{C}_x = \{x\}$.

Now consider the following map on $\mathbb{C} - \{0\}$:

$$\begin{aligned} \exp\colon \mathcal{O} &\to \mathcal{O}^* \\ f &\mapsto e^{2\pi \imath f}. \end{aligned}$$

It can then be noted that when M is a complex manifold, the following sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0 \tag{1.1}$$

is an exact sequence of sheaves and is known as *exponential exact sequence*. We will again come across this while defining the Chern class of a line bundle.

It is easy to observe that any sheaf can be naturally restricted to an open subset to obtain a new sheaf. We denote by $\mathcal{F}|_U$ *the restriction sheaf of* $\mathcal{F}$ *to the open subset* U of X and is defined as follows:

$$\mathcal{F}|_U(V) := \mathcal{F}(V),$$

where $V \subseteq U$ is any open subset. The restriction sheaf $\mathcal{O}_X|_U$ of the structure sheaf of a variety X is often simply denoted by $\mathcal{O}_U$.

Definition 1.1.2 A sheaf $\mathcal{F}$ on a topological space X is said to be a *locally free sheaf of finite rank* n if for any $x \in X$, there exists an open neighbourhood U of X such that

$$\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus n},$$

as $\mathcal{O}_X$ modules.

Remark 1.1.3 From now onwards, by a locally free sheaf we mean a locally free sheaf of finite rank. A locally free sheaf of rank 1 is called an *invertible sheaf*.

1.1.2 The sheaf cohomology as a derived functor cohomology and Čech cohomology

The motivation for studying cohomology theory can be described in many ways. One of the ways arises from the observation that the global section functor is only left exact. Let us describe this precisely.

Let us denote by $\Gamma(X, \mathcal{F})$ the set $\mathcal{F}(X)$ of all global sections of the sheaf $\mathcal{F}$ over X. So, $\Gamma(X, \cdot)$ is a functor from the category of sheaves to the category of abelian groups, known as a *global section functor* and this covariant functor is not exact in general. In this context, let us give an example.

Let $\mathcal{O}_{\mathbb{P}^1}(n)$ denote the standard *twisting sheaf* over $\mathbb{P}^1$. Consider the following exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \xrightarrow{\cdot x_0 x_1} \mathcal{O}_{\mathbb{P}^1} \xrightarrow{(ev_{[1:0]}, ev_{[0:1]})} \mathbb{C}_{[1:0]} \oplus \mathbb{C}_{[0:1]} \longrightarrow 0. \qquad (1.2)$$

Here $\cdot x_0 x_1 \colon \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{O}_{\mathbb{P}^1}$ denotes the multiplication map by the monomial

x_0x_1. By $ev_{[1:0]}\colon \mathcal{O}_{\mathbb{P}^1} \to \mathbb{C}_{[1:0]}$ and $ev_{[0:1]}\colon \mathcal{O}_{\mathbb{P}^1} \to \mathbb{C}_{[0:1]}$ we denote the evaluation maps at the point $[1:0]$ and $[0:1]$ respectively. We have $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$ as the invertible sheaf $\mathcal{O}_{\mathbb{P}^1}(-2)$ has negative degree and $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$ as the only global holomorphic maps from the projective line $\mathbb{P}^1$ to $\mathbb{C}$ are constants. Applying global section functor to (1.2), we obtain the following:

$$0 \longrightarrow \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \longrightarrow \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \longrightarrow \mathbb{C} \oplus \mathbb{C}. \tag{1.3}$$

Therefore the map $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ to $\mathbb{C} \oplus \mathbb{C}$ in (1.3) is the standard map

$$\begin{aligned} \mathbb{C} &\to \mathbb{C} \oplus \mathbb{C} \\ z &\mapsto (z, z), \end{aligned}$$

which is not surjective. As a result, we loose exactness while passing from (1.2) to (1.3).

Remark 1.1.4 When X is an affine variety, the global section functor is exact. One can refer to [Ha, Chapter III, Theorem 3.5], [Gr 1] and [Gr 2] for more general results.

Let us denote the category of sheaves of $\mathcal{O}_X$ modules by $\mathcal{M}od(X)$ and category of abelian groups by $\mathcal{A}b$. By [Ha, Chapter III, Proposition 2.2], we note that $\mathcal{M}od(X)$ has enough injectives. Recall that we already noted that the global section functor $\Gamma(X, \cdot)$ from the category $\mathcal{M}od(X)$ to the category $\mathcal{A}b$ is only left exact. The *cohomology functors*, denoted by $H^i(X, \cdot)$, is defined as the right derived functors of $\Gamma(X, \cdot)$. For any $\mathcal{F} \in \mathcal{M}od(X)$, the groups $H^i(X, \mathcal{F})$ are called the *cohomology groups* of $\mathcal{F}$. Thus the cohomology of sheaves is defined as a derived functor cohomology.

Remark 1.1.5

1. By [Ha, Chapter III, Theorem 1.1A], it can be noted that $\Gamma(X, \mathcal{F}) = H^0(X, \mathcal{F})$. This also justifies the reason behind two notations of the collection of all global sections of a sheaf as in Remark 1.2.3.

2. Given a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0,$$

we obtain the following long exact sequence at the cohomology level as follows:

$$0 \longrightarrow H^0(X,\mathcal{E}) \longrightarrow H^0(X,\mathcal{F}) \longrightarrow H^0(X,\mathcal{G}) \longrightarrow H^1(X,\mathcal{E}) \longrightarrow \cdots$$

This phenomenon is an intrinsic property of right derived functor (cf. [Ha, Chapter III, Theorem 1.1A]) and is of utmost importance for our purpose.

Let us denote the *i-th Čech cohomology group of $\mathcal{F}$ with respect to the covering $\mathcal{U}$* by $\check{H}^i(\mathcal{U},\mathcal{F})$. The *$i$-th Čech cohomology group of $\mathcal{F}$* , denoted by $\check{H}^i(X,\mathcal{F})$, is then defined as follows:

$$\check{H}^i(X,\mathcal{F}) := \varinjlim_{\mathcal{U}} \check{H}^i(\mathcal{U},\mathcal{F}).$$

Moreover, at level one, we have the following isomorphism between Čech cohomology and the sheaf cohomology for any abelian sheaf $\mathcal{F}$ on X (cf. [Ha, p. 223]):

$$\check{H}^1(X,\mathcal{F}) \cong H^1(X,\mathcal{F}).$$

Remark 1.1.6 1. If we choose X to be a Noetherian and separated scheme, the cover $\mathcal{U}$ to be affine and the sheaf $\mathcal{F}$ to be quasi coherent, we have the equality of two cohomology theories (cf. [Ha, Chapter III, p. 225]). That is to say, a large number of spaces are there for which these two cohomology theories coincide.

2. Let M be a differentiable manifold. Let us denote the singular cohomology and de Rham cohomology with coefficients from the constant sheaf $\mathbb{R}$ by $H^*_{sing}(M,\mathbb{R})$ and $H^*_{DR}(M,\mathbb{R})$ respectively. Denoting $H^*_{DR}(M,\mathbb{R}) \otimes \mathbb{C}$ by $H^*_{DR}(M)$, we have the following isomorphisms (cf. [Gf-Hr 2, p. 43 & p. 44]):

$$H^*_{sing}(M,\mathbb{R}) \cong H^*_{DR}(M) \cong \check{H}^i(M,\mathbb{R}).$$

This allows us to use these cohomology theories interchangeably.

1.2 Locally free sheaves and vector bundles

In this section we recall the notion of holomorphic vector bundle. Also the connection between locally free sheaves and vector bundles has been discussed.

Let M be a given differentiable manifold and E a C^∞ complex vector bundle over M. Let $\pi\colon E \to M$ be the usual projection map. For every $x_0 \in M$, there exists an open set U_{x_0} in M, we have the diffeomorphisms

$$\varphi_{U_{x_0}} : \pi^{-1}(U_{x_0}) \to U_{x_0} \times \mathbb{C}^n. \tag{1.4}$$

The $\mathbb{C}$-vector spaces $E_x := \pi^{-1}\{x\}$ are called *fibers* of E over x.

Remark 1.2.1 1. The diffeomorphisms $\varphi_{U_{x_0}}$ as in (1.4) are called a *trivialisations* of E over U. It is easily noted from (1.4) that for any two such trivialisations φ_U and φ_V, the map

$$\begin{aligned} g_{UV}\colon U \cap V &\to \mathrm{GL}_n(\mathbb{C}) \\ x &\mapsto (\varphi_U \circ \varphi_V^{-1})|_{\{x\}\times\mathbb{C}^n} \end{aligned} \tag{1.5}$$

is C^∞. The maps g_{UV} are called *transition functions* for E corresponding to φ_U and φ_V.

2. The transition functions clearly satisfy the following two properties, known as *cocycle conditions*:

   (a) $g_{UV}(x) \cdot g_{VU}(x) = \mathrm{Id}_n$ for all $x \in U \cap V$.

   (b) $g_{UV}(x) \cdot g_{VW}(x) \cdot g_{WU}(x) = \mathrm{Id}_n$ for all $x \in U \cap V \cap W$.

Definition 1.2.2 Let $E \to M$ be a C^∞ complex vector bundle and U be an open set of M. A *section over* U is a C^∞ map $s\colon U \to E$ such that for all $x \in U$, $s(x) \in E_x$. A section over M is called a *global section*.

Remark 1.2.3 The space of global sections of a vector bundle E over M is denoted by $\Gamma(M, E)$ or by $H^0(M, E)$. In short, we use the notations $\Gamma(E)$ and $H^0(E)$ instead of $\Gamma(M, E)$ and $H^0(M, E)$ respectively, when the underlying space involved is clear from the context. Also, sometimes the notation $\Gamma(M)$ is used for the same, if there is no confusion regarding the bundle involved. So, for an open subset U of M, the space of all sections over U is denoted by $\Gamma(U)$. Then following this notation, Γ can be interpreted as a sheaf.

Clearly, the definitions of vector bundles and sections suggest that these can be defined in many other categories by taking the morphisms involved in these definitions suitable for that particular category. Throughout we are going to work on holomorphic category and so let's just precisely define vector bundles in this category.

Definition 1.2.4 A complex vector bundle $\pi\colon E \to M$ over a complex manifold M is said to be a *holomorphic vector bundle* if E gets equipped with the structure of a complex manifold such that for any $x \in M$, there exists a open set U_x in M with $x \in U_x$ and a trivialization

$$\varphi_{U_{x_0}}\colon \pi^{-1}(U_{x_0}) \to U_{x_0} \times \mathbb{C}^n$$

that is a biholomorphic map.

So transition functions, sections etc. involved with any such holomorphic vector bundle are holomorphic maps. From now onwards, by vector bundles we would mean a holomorphic vector bundle and everything related should be considered in holomorphic category only unless otherwise specified.

Let X be a smooth projective variety over $\mathbb{C}$. It can be noted that Γ is sheaf of modules over the sheaf of rings $\mathcal{O}_X$. Indeed, for any open subset U of X for which there exists a trivialization, we have:

$$\begin{aligned}\Gamma(U) &\to \pi^{-1}(U) \cong U \times \mathbb{C}^n\\ x &\mapsto (x, f_1(x), \ldots, f_n(x)),\end{aligned}$$

$f_i\colon U \to \mathbb{C}$ being regular functions. That is to say, on sufficiently smaller open set U,

$$\Gamma(U) \cong \mathcal{O}_X^{\oplus n}.$$

So, Γ is a locally free sheaf of $\mathcal{O}_X$ modules. In fact, the converse is also true. In this regard, let's state the following theorem.

Theorem 1.2.5 *[Sh, Theorem 6.2] Let $Vect_X(n)$ denote the set of all vector bundles over X of rank n modulo bundle isomorphism. Also, by $Loc_X(n)$ let us denote* *the set of all locally free sheaves of rank n over X upto sheaf isomorphism. Then we have the following one-one correspondence between $Vect_X(n)$ and $Loc_X(n)$:*

$$\begin{aligned} Vect_X(n) &\to Loc_X(n) \\ E &\mapsto \Gamma(E). \end{aligned}$$

1.3 Line bundles and divisors

In the preceding section we observed that upto isomorphism vector bundles and locally free sheaves of same rank can be identified. Therefore there is a one-one correspondence between invertible sheaves over X and line bundles on X upto isomorphism. In this section, we go through sheaf theoretic interpretation of line bundles and divisors and connection between them.

Let $\pi\colon L \to X$ be a holomorphic *line bundle*, that is a rank 1 vector bundle over X. Let $\{U_\alpha, \varphi_\alpha\}$ be a set of trivializations. Corresponding to these trivializations, we have transition functions $g_{U_\alpha U_\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*$ as defined in (1.5) of Remark 1.2.1, but in the category of holomorphic line bundles. Let us denote $g_{U_\alpha U_\beta}$ simply by $g_{\alpha\beta}$. They satisfy the following cocycle conditions as in Remark 1.2.1:

$$\begin{aligned} g_{\alpha\beta}(x) \cdot g_{\beta\alpha}(x) &= \mathrm{Id}_1 = 1 \text{ for all } x \in U_\alpha \cap U_\beta, \\ g_{\alpha\beta}(x) \cdot g_{\beta\gamma}(x) \cdot g_{\gamma\alpha}(x) &= \mathrm{Id}_1 = 1 \text{ for all } x \in U_\alpha \cap U_\beta \cap U_\gamma. \end{aligned} \tag{1.6}$$

Moreover, these transition functions $\{g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)\}$ defines the line bundle L uniquely.

This naturally leads us to a sheaf-theoretic description of line bundle. For a given line bundle $L \to X$, its transition functions $\{g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)\}$ can be regarded as the representation of a Čech 1-cochain on X having coefficients from the sheaf $\mathcal{O}^*$. Moreover, (1.6) depicts that $\{g_{\alpha\beta}\}$ is in fact a Čech cocycle.

For the given line bundle $L \to X$ and the same open cover $\{U_\alpha\}$, we can define another set of trivializations $\{\psi_\alpha\}$ as follows:

$$\psi_\alpha = f_\alpha \varphi_\alpha,$$

where f_α is any non-zero holomorphic function for all α. Corresponding to these newly given trivializations $\{\psi_\alpha\}$, we have a new set of transition functions $\{h_{\alpha\beta}\}$

as well and they are related to the old ones through the following relations:

$$h_{\alpha\beta} = \frac{f_\alpha}{f_\beta} \cdot g_{\alpha\beta}. \tag{1.7}$$

Therefore we can conclude similarly that $\{h_{\alpha\beta}\}$ is a Čech cocycle. Something more can be said in context of relating the Čech cocycles $\{g_{\alpha\beta}\}$ and $\{h_{\alpha\beta}\}$. That is to say, the cocycles $\{g_{\alpha\beta}\}$ and $\{h_{\alpha\beta}\}$ give same line bundle if and only if $\{g_{\alpha\beta} \cdot h_{\alpha\beta}^{-1}\}$ is a Čech coboundary. Hence $H^1(X, \mathcal{O}^*)$ is the set of all line bundles on X (cf. [We, Lemma 4.4]). Also for given two line bundles L_1 and L_2 with $\{g^1_{\alpha\beta}\}$ and $\{g^2_{\alpha\beta}\}$ respectively, $\{g^1_{\alpha\beta} \cdot g^2_{\alpha\beta}\}$ and $\{{g^1_{\alpha\beta}}^{-1}\}$ give the bundles $L_1 \otimes L_2$ and L_1^* respectively. As a result, $H^1(X, \mathcal{O}^*)$ naturally gets equipped with a group structure and is called the *Picard group of* X, denoted by $\mathrm{Pic}(X)$.

Definition 1.3.1 By an *analytic hypersurface* we mean an analytic subvariety V of X of codimension 1 in X, that is around any of its point, V is given by a single holomorphic function.

Definition 1.3.2 By a *divisor* D on X, we mean a locally finite formal linear combination of irreducible analytic hypersurfaces V_i of X of the form $D = \sum a_i V_i$.

Remark 1.3.3

1. From now on, we simply call an analytic hypersurface by hypersurface.

2. The sum in the expression $D = \sum a_i V_i$ of Definition 1.3.2 is finite whenever compactness of X is assumed. In that case, by *degree of a divisor* we simply mean the integer $\sum_i a_i$.

3. The set of all divisors on X is naturally an additive group and is denoted by $\mathrm{Div}(X)$. On compact X, $\mathrm{Div}(X)$ can therefore be interpreted as free abelian group generated by codimension 1 irreducible subvarieties of X. Moreover when X is a curve, that is of dimension 1, a divisor D on X looks like $D = \sum_{i=1}^n a_i p_i$ for some closed points p_i of X.

4. A divisor $D = \sum a_i V_i$ is said to be *effective* if $a_i \geq 0$ for all i. We use the notation $D \geq 0$ for such a divisor.

Let us go through the notion of order of a holomorphic function at a point of a hypersurface. Suppose V is an irreducible hypersurface of X. Let f be a local defining function for V near a point $x \in X$. For any other holomorphic function g that is defined near x, by the *order of* g *along* V *at* x we mean the largest integer a such that the equation $g = f^a \cdot h$ is satisfied in the local ring $\mathcal{O}_{X,x}$. A priori this definition is very much dependent on the point x, but [Gf-Hr 2, Proposition, p. 10] says that it is not so. Thus, the integer a can now be called the *order of* g *along* V and is denoted by $\mathrm{ord}_V(g)$.

This definition leads us to the sheaf theoretic description of divisors. A global section f of the quotient sheaf $\frac{\mathcal{M}^*}{\mathcal{O}^*}$ can be given by an open cover $\{U_\alpha\}$ and meromorphic functions f_α that are not identically zero on U_α with $\frac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ so that for any hypersurface V of X,

$$\mathrm{ord}_V(f_\alpha) = \mathrm{ord}_V(f_\beta).$$

Then the divisor $D = \sum_V \mathrm{ord}_V(f_\alpha) \cdot V$ can be associated to the given global section f. Here for each such V, α's are chosen with the property $V \cap U_\alpha \neq \emptyset$. Thus given a global section f of the sheaf $\frac{\mathcal{M}^*}{\mathcal{O}^*}$, we obtain a divisor. In fact, the converse is also true. Let $D = \sum_{V_i} a_i V_i$ be a divisor on X. a open cover $\{U_\alpha\}$ of X can be so chosen that in each U_α, every V_i appearing in D locally given by the functions $g_{i\alpha} \in \mathcal{O}(U_\alpha)$. Then $f_\alpha = \prod_i g_{i\alpha}^{a_i} \in \mathcal{M}^*(U_\alpha)$ gives us a global section of $\frac{\mathcal{M}^*}{\mathcal{O}^*}$. As a result, we obtain the following identification:

$$\mathrm{Div}(X) = H^0\left(X, \frac{\mathcal{M}^*}{\mathcal{O}^*}\right). \tag{1.8}$$

Remark 1.3.4 Any element of $H^0\left(X, \frac{\mathcal{M}^*}{\mathcal{O}^*}\right)$ is often called *Cartier divisor.* On the other hand, any element of $\mathrm{Div}(X)$ is called *Weil divisor.* Then (1.8) suggests that those two types of apparently different divisors are same in our case. It can be noted that they are not same in general. For more details about the conditions on X under which these two notions coincide, one can refer to [Ha, Chapter II, Proposition 6.11].

Now we are in a stage to relate $\mathrm{Div}(X)$ and $\mathrm{Pic}(X)$. Let D be a divisor on X. Let $\{f_\alpha\}$ be the local defining functions over some open cover $\{U_\alpha\}$ of X. Then on $U_\alpha \cap U_\beta$, the functions $g_{\alpha\beta}$ defined by $g_{\alpha\beta} := \frac{f_\alpha}{f_\beta}$ are holomorphic and non-zero.

Moreover they satisfy cocycle conditions, that is on $U_\alpha \cap U_\beta \cap U_\gamma$, we have:

$$g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = \frac{f_\alpha}{f_\beta} \cdot \frac{f_\beta}{f_\gamma} \cdot \frac{f_\gamma}{f_\alpha} = 1.$$

The unique line bundle given by these transition functions $\{g_{\alpha\beta}\}$ is called the *associated line bundle of* D and is denoted by $\mathcal{O}(D)$. This definition, though a priori depends upon the chosen local defining equation of D, is actually independent of local data and hence makes sense.

We now go through the reverse construction. Recall that, we have defined $\mathrm{ord}_V(f)$ is defined for a holomorphic function f on X. This definition can be extended to meromorphic functions as well. Let l be a meromorphic function on X, locally expressed as $\frac{g}{h}$, quotient of two holomorphic maps. For an irreducible hypersurface V of M, we define:

$$\mathrm{ord}_V(l) = \mathrm{ord}_V(g) - \mathrm{ord}_V(h). \tag{1.9}$$

We denote the *divisor of a meromorphic function* l by (l) and define by

$$(l) = \sum_V \mathrm{ord}_V(l) \cdot V. \tag{1.10}$$

Without loss of generality, if we assume that the g and h are relatively prime in the local expression of l, then Definition 1.10 can be checked to be well defined. Given a line bundle L and for any meromorphic section s of L, we have (cf. [Gf-Hr 2, p. 136]):

$$L = \mathcal{O}((s)), \tag{1.11}$$

(s) being defined similarly as (1.10). Thus we have the following maps:

$$\begin{aligned} \varphi\colon \mathrm{Div}(X) &\to \mathrm{Pic}(X) \\ D &\mapsto \mathcal{O}(D). \end{aligned} \tag{1.12}$$

This is the first instance of *Abel-Jacobi map*. We discuss this map in details in Section 1.6 and 1.7. Also for any global meromorphic section s of L which is not identically zero, we have:

$$\begin{aligned} \chi\colon \operatorname{Pic}(X) &\to \operatorname{Div}(X) \\ L &\mapsto (s). \end{aligned} \tag{1.13}$$

Then (1.11) basically says that $\varphi \circ \chi = \mathrm{Id}$. We immediately ask whether $\chi \circ \varphi = \mathrm{Id}$ holds or not. Unfortunately, it doesn't hold. That is to say the map φ has a non-trivial kernel. Let us find out the details about this kernel.

Suppose D be a divisor on X given by a meromorphic function f on X, that is, $D = (f)$. Then any open cover $\{U_\alpha\}$ of X and $f_\alpha := f|_{U_\alpha}$ can be considered as a local data for D and we therefore have $\frac{f_\alpha}{f_\beta} = 1$. As a result, these as transition functions gives us the trivial line bundle $\mathcal{O}(D)$. Conversely, if $\mathcal{O}(D)$ is the trivial line bundle with local data $\{f_\alpha\}$, then there exists $h_\alpha \in \mathcal{O}^*(U_\alpha)$ satisfying

$$\frac{f_\alpha}{f_\beta} = g_{\alpha\beta} = \frac{h_\alpha}{h_\beta}.$$

Then f defined by $f := f_\alpha h_\alpha^{-1} = f_\beta h_\beta^{-1}$ is a global meromorphic function on X with divisor D. In short, we have the following proposition.

Proposition 1.3.5 *The line bundle $\mathcal{O}(D)$, associated to a divisor D on X, is trivial if and only if $D = (f)$ for some meromorphic function f.*

Thus Proposition 1.3.5 leads us to the notion of linear equivalence of divisors very naturally.

Definition 1.3.6 Two divisors D_1 and D_2 on X are said to be *linearly equivalent*, denoted by $D_1 \sim D_2$, if for some $f \in \mathcal{M}^*(X)$, $D_1 = D_2 + (f)$.

Remark 1.3.7

1. Definition 1.3.6 says, two divisors D_1 and D_2 on X are linearly equivalent if $\mathcal{O}(D_1) = \mathcal{O}(D_1)$, that is, if they lie in the same fiber of the map $\varphi\colon \operatorname{Div}(X) \to \operatorname{Pic}(X)$ as defined in (1.12).

2. The map $\varphi\colon \operatorname{Div}(X) \to \operatorname{Pic}(X)$ as defined in (1.12) is a group homomorphism as we have the following:

$$\mathcal{O}(D_1 + D_2) = \mathcal{O}(D_1) \otimes \mathcal{O}(D_2),$$

for any two divisors D_1 and D_2 on X.

3. Let $\mathrm{PDiv}(X)$ denote the subgroup of $\mathrm{Div}(X)$ consisting of all *principal divisors*, that is, divisors of meromorphic functions. Then $\mathrm{Ker}(\varphi) = \mathrm{PDiv}(X)$. That is to say, φ is the inverse of χ when we go modulo $\mathrm{PDiv}(X)$. As a result,
$$\frac{\mathrm{Div}(X)}{\mathrm{PDiv}(X)} \cong \mathrm{Pic}(X). \tag{1.14}$$

So, we now can summarize by stating the following proposition (cf. [Sh, Theorem 6.3]).

Proposition 1.3.8 *The group of divisors on X upto linear equivalence is same as the group of line bundles over X upto isomorphism.*

This allows us to swap the notions of divisors, line bundles and invertible sheaves on X interchangeably. By this correspondence, we can immediately define the *degree of a line bundle* over a compact Riemann surface X as the order of any associated divisor. Therefore degree of a line bundle makes sense As degree of a principal divisor is zero (cf. [Ha, Chapter II, Corollary 6.10]), any two linearly equivalent divisors have same degree. As a result, the notion of degree of a line bundle is well defined. Moreover, by (1.14), we can conclude that two isomorphic line bundles have same degree as well. That is to say, though by definition degree can be thought of as a homomorphism $\deg\colon \mathrm{Div}(X) \to \mathbb{Z}$, it actually descends down to $\frac{\mathrm{Div}(X)}{\mathrm{PDiv}(X)}$ and $\mathrm{Pic}(X)$ as well.

Denoting by $\mathrm{Div}^d(X) \subseteq \mathrm{Div}(X)$ and $\mathrm{Pic}^d(X) \subseteq \mathrm{Pic}(X)$ the set of all degree d divisors and the set of all degree d line bundles (upto isomorphism) respectively, we have the following stratifications:

$$\mathrm{Div}(X) = \bigcup_{d\in\mathbb{Z}} \mathrm{Div}^d(X),$$
$$\mathrm{Pic}(X) = \bigcup_{d\in\mathbb{Z}} \mathrm{Pic}^d(X).$$

We come across this stratifications in in Section 1.6 and 1.7 and reconsider the maps as in (1.12) and (1.13) by restricting them to these stratifications. Also we prove that $\mathrm{Pic}^0(C)$ is isomorphic to the Jacobian variety $J(C)$ of a curve C (cf. Theorem 1.6.8). Therefore we are going to use these two notations interchangeably. Extending this notation, we also use the notation $J_d(C)$ instead of $\mathrm{Pic}^d(C)$ in coming sections.

Let's now find out that what extra property a line bundle must posses when it corresponds to an effective divisor. For any global meromorphic section s of L which is not identically zero, the divisor (s) is effective if s is actually a holomorphic section. Therefore, L is a line bundle associated to an effective divisor if and only if L has a non-trivial global holomorphic section, that is, $H^0(X, L) \neq 0$.

Given a divisor D on X, let $\mathcal{L}(D)$ be defined as follows:

$$\mathcal{L}(D) := \{f \mid f \text{ is meromorphic function on } X \text{ and } D + (f) \geq 0\}.$$

Then $\mathcal{L}(D)$ can be identified with a known space as follows. Let us define $|D| \subseteq \mathrm{Div}(M)$ as the set of all effective divisors that are linearly equivalent to D. By $|L|$ we mean $|D|$ for a line bundle L over X with $L = \mathcal{O}(D)$. Assume that $D = (s_0)$ for some global meromorphic function s_0 of the line bundle $\mathcal{O}(D)$. Then for an arbitrary global holomorphic section s of $\mathcal{O}(D)$, we have

$$\left(\frac{s}{s_0}\right) + D = (s) - (s_0) + D \geq 0.$$

That is to say, $\frac{s}{s_0} \in \mathcal{L}(D)$ and $(s) \in |D|$. Conversely, given any $f \in \mathcal{L}(D)$, $(f \cdot s_0) = (f) + (s_0) = (f) + D \geq 0$ and hence $f \cdot s_0$ is a global section of $\mathcal{O}(D)$. This leads us to the following natural identification:

$$\mathcal{L}(D) \cong H^0(X, \mathcal{O}(D)). \tag{1.15}$$

This identification therefore relates the space $\mathcal{L}(D)$ with the line bundle $\mathcal{O}(D)$. Moreover we relate both of them with $|D|$. It can be readily observed that given any $D_1 \in |D|$, there exists $f \in \mathcal{L}(D)$ satisfying

$$D_1 = D + (f).$$

. Also as divisors of two meromorphic functions differing by a non-zero scalar multiple are the same, we have:

$$|D| = \mathbb{P}(\mathcal{L}(D)). \tag{1.16}$$

We now have the following definition which is very useful for our purpose.

Definition 1.3.9 A *linear system* on X is a family of effective divisors corresponding to a linear subspace V of $\mathbb{P}(H^0(X, \mathcal{O}(D)))$. Moreover, it is called a *complete linear system* if $V = \mathbb{P}(H^0(X, \mathcal{O}(D)))$.

Remark 1.3.10 By (1.15) and (1.16) we can conclude that, given any divisor D on X, $|D|$ is a complete linear system.

We have described both $\mathrm{Pic}(X)$ and $\mathrm{Div}(X)$ sheaf theoretically. So it is natural to think whether the maps as in (1.12) and (1.13) also have an analogous sheaf-theoretic description or not. We end this section by answering this affirmatively.

Consider the following short exact sequence of sheaves on X:

$$0 \longrightarrow \mathcal{O}^* \xrightarrow{i} \mathcal{M}^* \xrightarrow{j} \frac{\mathcal{M}^*}{\mathcal{O}^*} \longrightarrow 0. \tag{1.17}$$

We then have the following exact sequence at cohomology level corresponding to (1.17):

$$H^0(X, \mathcal{M}^*) \xrightarrow{j_*} H^0\left(X, \tfrac{\mathcal{M}^*}{\mathcal{O}^*}\right) \xrightarrow{\delta} H^1(X, \mathcal{O}^*). \tag{1.18}$$

It can be checked that the map j_* and the connecting homomorphism δ are given as follows once we identify $\mathrm{Div}(X)$ as $H^0\left(X, \frac{\mathcal{M}^*}{\mathcal{O}^*}\right)$ and $\mathrm{Pic}(X)$ as $H^1(X, \mathcal{O}^*)$:

$$j_* : H^0(X, \mathcal{M}^*) \to H^0\left(X, \tfrac{\mathcal{M}^*}{\mathcal{O}^*}\right)$$
$$f \mapsto (f),$$

$$\delta : H^0\left(X, \tfrac{\mathcal{M}^*}{\mathcal{O}^*}\right) \to H^1(X, \mathcal{O}^*)$$
$$D \mapsto \mathcal{O}(D).$$

So the exactness of (1.18) simply depicts the isomorphism as in (1.14) in a sheaf theoretic approach.

1.4 Riemann-Roch theorem and Serre duality for curve

The Riemann-Roch theorem and Serre duality are one of the most significant results in algebraic geometry. Initially Riemann proved an inequality, called *Rie-*

mann's Inequality, in the year 1957. Then it got its present form after a work of Roch, a student of Riemann, in the year 1965. Then it was proved for Riemann surfaces. Later the result was proved for algebraic varieties. In this section we state the theorem over curve. We also state the Serre duality on the way.

Let X be a smooth projective curve over complex numbers and let E be a vector bundle over X. By abuse of notation, we denote the locally free sheaf corresponding to the vector bundle E again by E. Let us denote the dimension of the $\mathbb{C}$-vector space $H^i(X, E)$ by $h^i(X, E)$. The *Euler characteristic* of E, denoted by $\chi(E)$, is defined as $\chi(E) := h^0(X, E) - h^1(X, E)$.

Let us define a sheaf on X and genus of X as the dimension of the space of global sections of that sheaf followed by that. By *sheaf of differentials* Ω_X we mean the sheaf dual to the locally free sheaf $\mathcal{T}_X$ associated to the tangent bundle of X. As X is smooth the rank of the sheaf $\mathcal{T}_X$, also called as *tangent sheaf*, is same as the dimension of X. Also, Ω_X is a locally free sheaf of dimension n (cf. [Ha, Chapter II, Theorem 8.15]). By *canonical sheaf* we mean the sheaf $\wedge^n \Omega_X$, where n is the dimension of the variety X. The canonical sheaf is denoted by ω_X. Let $H^0(X, \omega_X)$ be the $\mathbb{C}$-vector space of global sections of the canonical sheaf of X, then *geometric genus* of X, denoted by g_{geo}, is defined as the complex dimension of $H^0(X, \omega_X)$. That is to say, we define:

$$g_{geo} := h^0(X, \omega_X).$$

When X is a curve, geometric genus is same as *genus* and is simply denoted by g. So, in this case, we have (cf. [Ha, Chapter IV, Proposition 1.1]):

$$g = g_{geo} = h^0(X, \omega_X). \tag{1.19}$$

Let us denote $\wedge^p \Omega_X$ by Ω^p_X or simply by Ω^p, when no confusion is likely to occur. It can be noted that Ω^p also can be thought of as the sheaf of holomorphic p-forms as mentioned in Subsection 1.1.1.

Also by [Ha, Chapter II, Theorem 8.15],

$$\omega_X = \wedge^1 \Omega_X = \Omega^1_X.$$

The line bundle associated to the invertible sheaf ω_X is called *canonical line bundle* and is denoted by K_X.

Serre duality theorem was first proved by Serre. He proved it for abstract algebraic geometry (cf. [Se 1]) and for a locally free sheaf on a compact complex manifold (cf. [Se 2]) followed by that. Let us now state Serre duality theorem without proof (cf. [Ha, Chapter III, Corollary 7.7]).

Theorem 1.4.1 *Let X be a smooth projective curve X over complex numbers and $\mathcal{E}$ is a locally free sheaf over X. Then the following isomorphism of $\mathbb{C}$-vector spaces holds:*

$$H^0(X, \mathcal{E}^* \otimes \omega_X) \cong H^1(X, \mathcal{E})^*.$$

Hence, the equality $h^0(X, \mathcal{E}^* \otimes \omega_X) = h^1(X, \mathcal{E})$ holds.

Finally we state the Riemann-Roch theorem for line bundles.

Theorem 1.4.2 *Let X be a smooth projective curve over complex numbers of genus g. Assume $\mathcal{L}$ be an invertible sheaf on X of degree d. Then*

$$h^0(X, \mathcal{L}) - h^0(X, \mathcal{L}^* \otimes \omega_X) = d + 1 - g.$$

Remark 1.4.3

1. Taking $\mathcal{L}$ to be the canonical sheaf ω_X in Theorem 1.4.2, we have the following equation:

$$h^0(X, \omega_X) - h^0(X, \omega_X^* \otimes \omega_X) = h^0(X, \omega_X) - h^1(X, \mathcal{O}_X) = d + 1 - g.$$

   As a result, we have $d = 2g - 2$. So, degree of the canonical sheaf over the curve X of genus g is $2g - 2$.

2. When $g = 1$, we immediately get that degree of ω_X is 0. Also from (1.19), we have $h^0(X, \omega_X) = g = 1$. Therefore, $\omega_X \cong \mathcal{O}_X$, that is, in case of elliptic curve canonical line bundle K_X corresponding to the canonical sheaf ω_X becomes trivial.

We end this section by stating Riemann-Roch theorem for any locally free sheaf on X. This can be proved by induction on the rank of the locally free sheaf involved, considering Theorem 1.4.2 as a base case for induction.

For a vector bundle E over X of rank n, the *determinant line bundle* $\det E$ of E is defined to be the line bundle $\wedge^n E$. The *degree* of the bundle E is denoted by deg E and is defined as the degree of $\det E$, that is, deg $E :=$ deg $\det E$. By degree of a locally free sheaf we mean the degree of the corresponding vector bundle. Then the Riemann-Roch theorem for any locally free sheaf on X can be stated as follows.

Corollary 1.4.4 *Let X be a smooth projective curve of genus g over complex numbers. Assume $\mathcal{E}$ be a locally free sheaf on X of rank n and degree d. Then*

$$\chi(\mathcal{E}) = d + n(1-g).$$

1.5 Projective bundle formula

In this part we mainly recall projective bundle formula for a projective bundle associated to a given vector bundle over any compact oriented C^∞ manifold. In the process we recall the definition of the Chern class of a line bundle and cohomology class of a variety and some relations between them as well.

Let M be a compact manifold over $\mathbb{C}$ of dimension m. Recall that $\mathcal{O}_M$ and $\mathcal{O}_M^*$ be the sheaf of holomorphic functions and non-vanishing holomorphic functions on M respectively. Also recall the exponential exact sequence as in (1.1) given as follows:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_M \xrightarrow{exp} \mathcal{O}_M^* \longrightarrow 0.$$

Corresponding to this short exact sequence we have a long exact sequence at cohomology level and therefore the following boundary homomorphism δ as follows.

$$H^1(M, \mathcal{O}_M^*) \xrightarrow{\quad\delta\quad} H^2(M, \mathbb{Z}) \ . \tag{1.20}$$

We now have the following definitions once we identify $\mathrm{Pic}(M)$ with $H^1(M, \mathcal{O}_M^*)$ (cf. Section 1.3).

Definition 1.5.1 Let $L \in \mathrm{Pic}(M)$. Then the first Chern class of the line bundle L, denoted by $c_1(L)$, is defined as

$$c_1(L) := \delta(L) \in H^2(M, \mathbb{Z}).$$

So the map in (1.20) is also denoted by

$$H^1(M, \mathcal{O}_M^*) \xrightarrow{\quad c_1\quad} H^2(M, \mathbb{Z}) \ . \tag{1.21}$$

Definition 1.5.2 The image of the homomorphism as in (1.21), a subgroup of $H^2(M, \mathbb{Z})$, is called Néron-Severi group of M and is denoted by $\mathrm{NS}(M)$.

Definition 1.5.3 Let Z be a smooth subvariety of M of codimension p. Then the *cohomology class* of Z, denoted by $[Z]$, is an element of $H^{2p}(X, \mathbb{Z})$ and is defined as

$$[Z] := i_* 1,$$

where $i_* \colon H^*(Z, \mathbb{Z}) \to H^{*+2p}(M, \mathbb{Z})$ is pushforward map associated to the embedding $i \colon Z \hookrightarrow M$.

Remark 1.5.4

1. When Z is not smooth then its cohomology class defined through a resolution of Z. In this case, let Z^s be the smooth locus of Z. Then by a theorem of Hironaka (cf. [Hi]), there exists a smooth variety $\widetilde{Z}$ and a morphism $f \colon \widetilde{Z} \to Z$ such that $f|_{f^{-1}(Z^s)} \colon f^{-1}(Z^s) \to Z^s$ is an isomorphism. Consider the morphism $i \circ f \colon \widetilde{Z} \to X$. Then the cohomology class of Z, denoted again by $[Z]$, is defined as

   $$[Z] := (i \circ f)_* 1 \in H^{2p}(X, \mathbb{Z}).$$

   Moreover, this is well-defined as it is independent of the choice of a resolution of Z (cf. [Be 2]).

2. Whenever we talk about $[Z]$ as an element of $H^{2p}(X, \mathbb{Q})$, we actually mean its image via the natural map $H^{2p}(X, \mathbb{Z}) \to H^{2p}(X, \mathbb{Q})$, irrespective of smoothness of Z.

Now let D be a divisor on M and $L = \mathcal{O}(D)$ be the corresponding line bundle. Then D being a $\mathbb{Z}$-linear combination of codimension 1 subvarieties of M, $[D] \in H^2(M, \mathbb{Z})$ by Definition 1.5.3. Also $c_1(L)$ is an element of $H^2(M, \mathbb{Z})$ by Definition 1.5.1 and (1.20). As L is the line bundle corresponding to the divisor D, it is natural to ask whether there is any relation between the cohomology class $[D]$ and the Chern class $c_1(L)$. In that regard we have the following Proposition.

Proposition 1.5.5 *[Gf-Hr 2, Proposition, p. 141] Let D be any divisor on M. Then in $H^2(M, \mathbb{Z})$ we have the following equality,*

$$[D] = c_1(\mathcal{O}(D)).$$

Let $V \to M$ be any vector bundle of rank n over M. Then one can construct an associated projective bundle whose fiber are projective space of the fiber of the bundle V. This bundle, denoted by $\mathbb{P}(V)$, is of rank $n-1$. Let $\pi\colon \mathbb{P}(V) \to M$ be the usual projection map. Then at cohomology level we have the following pullback map denoted by π^*.

$$\pi^*\colon H^*(M,\mathbb{Z}) \to H^*(\mathbb{P}(V),\mathbb{Z}). \tag{1.22}$$

Then by (1.22), $H^*(\mathbb{P}(V),\mathbb{Z})$ can be thought of as $H^*(M,\mathbb{Z})$ algebra. Following theorem depicts this with some more details.

Theorem 1.5.6 *[Gf-Hr 2, Proposition, p. 606] For any complex vector bundle V of rank n over a compact oriented C^∞ manifold M, $H^*(\mathbb{P}(V))$ is generated as an $H^*(M)$ algebra by the Chern class $\eta = c_1(T)$ satisfying the following equation,*

$$\eta^n - c_1(V)\eta^{n-1} + \cdots + (-1)^n c_n(V) = 0,$$

$T \to \mathbb{P}(V)$ being the tautological line bundle.

1.6 Abelian variety and Jacobian

In this section we recall the definitions and a few basic properties of an abelian variety and Jacobian variety. Then we discuss that study of a Jacobian variety is not at all very far away from studying an abelian variety.

By a *lattice* in a g-dimensional complex vector space V, one means a discrete subgroup Λ of V of rank $2g$, that is, a free abelian group of maximal rank. The quotient $\frac{V}{\Lambda}$ is called a *complex torus*. For notational simplicity, we denote such a complex torus by X. As quotienting by a discrete subgroup does not change the local structure, the complex torus X is a complex manifold of dimension g. Moreover, it is compact as Λ is a discrete subgroup of V of maximal rank.

Let us recall the notion of a period matrix associated to a complex torus. Let $e_1, e_2, \ldots, e_g$ be a basis of V. Let $\lambda_1, \lambda_2, \ldots, \lambda_{2g}$ be a set of generators of Λ. Then λ_j can be written in terms of e_j, $1 \leq j \leq g$, for all $1 \leq i \leq 2g$ as follows.

$$\lambda_i = \sum_{j=1}^{g} \lambda_{ji} e_j, \;\; \lambda_{ji} \in \mathbb{C} \text{ for all } 1 \leq j \leq g \text{ and } 1 \leq i \leq 2g.$$

The coefficients of these $2g$ many equations determine a matrix which we denote by Π.

$$\Pi = \begin{pmatrix} \lambda_{1,1} & \lambda_{1,2} & \cdots & \lambda_{1,2g} \\ \lambda_{2,1} & \lambda_{2,2} & \cdots & \lambda_{2,2g} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{g,1} & \lambda_{g,2} & \cdots & \lambda_{g,2g} \end{pmatrix} \tag{1.23}$$

This $g \times 2g$ matrix Π with complex entries is called a *period matrix* of the complex torus X. Clearly period matrix of any complex torus is not unique as its definition is based upon the choice of some bases. But definitely it determines a complex torus. It is natural to ask the following. Given any matrix in $\mathrm{M}_{g\times 2g}(\mathbb{C})$ whether one can determine that it is a period matrix of some complex torus or not. In this regard, we have the following proposition.

Proposition 1.6.1 *Let $\overline{\Pi}$ be the complex conjugate matrix of a given matrix $\Pi \in \mathrm{M}_{g\times 2g}(\mathbb{C})$ and $P \in \mathrm{M}_{2g}(\mathbb{C})$ be the matrix $\left(\begin{smallmatrix}\Pi \\ \overline{\Pi}\end{smallmatrix}\right)$. Then non-singularity of P implies that Π is a period matrix of a complex torus and vice versa.*

Proof.See [La-Bk, Proposition 1.1.2]. Indeed, $P \in \mathrm{M}_{2g}(\mathbb{C})$ is non-singular if and only if the columns of the matrix Π are independent over $\mathbb{R}$ if and only if if the columns of the matrix Π span a lattice. □

Remark 1.6.2 The columns $\Pi_i := \left(\begin{smallmatrix}\lambda_{1,i} & \lambda_{2,i} & \cdots & \lambda_{g,i}\end{smallmatrix}\right)^t$ of the matrix Π are called *periods*, for all $1 \leq i \leq 2g$. Proposition 1.6.1 says that a criteria for a given matrix $\Pi \in \mathrm{M}_{g\times 2g}(\mathbb{C})$ to be a period matrix of some complex torus X is that the the free abelian group Λ defined as $\Lambda := \{\sum_{i=1}^{2g} n_i\Pi_i \,|\, n_i \in \mathbb{Z}\}$ spanned by the periods needs to be of maximal rank $2g$. Moreover, in that case we have $X = \frac{\mathbb{C}^g}{\Lambda}$.

Let us now interpret the first Chern class of a holomorphic line bundle on a complex torus X in terms of real valued alternating forms and hermitian form on V. Combining [La-Bk, Proposition 2.1.6] and[La-Bk, Lemma 2.1.7] one can conclude that the Néron-Severi group $\mathrm{NS}(X)$ can be identified with the group of all hermitian forms $H : V \times V \to \mathbb{C}$ satisfying $\mathrm{Im}(H(\Lambda,\Lambda)) \subseteq \mathbb{Z}$ as well as with the group of all real valued alternating 2-forms $E : V \times V \to \mathbb{R}$ with $E(\Lambda,\Lambda) \subseteq \mathbb{Z}$ and $E(\imath v, \imath w) = E(v,w)$ $v,w \in V$.

We are now in a stage to define abelian variety. A line bundle L over a complex torus X is called a *positive definite line bundle* or simply a *positive line bundle* if the first Chern class $c_1(L)$ is a positive definite hermitian form. A complex torus that admits a positive line bundle is called an *abelian variety*. Such choice of a positive line bundle on an abelian variety is called a *polarisation*. So abelian varieties are also called *polarised abelian variety*. We have the following proposition due to Riemann which basically says that a complex torus is an abelian variety if and only if there exists a period matrix which looks simpler. More precisely,

Proposition 1.6.3 *[Gf-Hr 2, **Riemann Relations**, p. 306] A complex torus $X = \frac{V}{\Lambda}$ is an abelian variety if and only if there exists bases $e_1, e_2, \ldots, e_g$ of V and $\lambda_1, \lambda_2, \ldots, \lambda_{2g}$ of Λ, called simplectic or canonical bases, such that the period matrix Π takes the form $\begin{pmatrix} D & Z \end{pmatrix}$. Here Z is symmetric and $\mathrm{Im}(Z)$ is positive definite and the matrix $D \in \mathrm{M}_g(\mathbb{Z})$ is given by*

$$D = \begin{pmatrix} \delta_1 & & & \\ & \delta_2 & & \\ & & \ddots & \\ & & & \delta_g \end{pmatrix} \tag{1.24}$$

with $\delta_i > 0$ for all $1 \leq i \leq g$ and $\delta_i | \delta_{i+1}$ for all $1 \leq i \leq g-1$.

Also, the matrix of $\mathrm{Im}(c_1(L))$ takes the form

$$\mathrm{Im}(c_1(L)) = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

with respect to the canonical bases as in Proposition 1.6.3 and D is the matrix defined in (1.24). The integers δ_i are called the *elementary divisors of the polarisation* L. Moreover, L is called a *principal polarisation* if $\delta_i = 1$ for all $1 \leq i \leq g$.

As any line bundle on the g-dimensional complex vector space V is trivial, any line bundle L over $\frac{V}{\Lambda}$ can be thought of a quotient of the trivial line bundle $\pi^*(L)$ where $\pi : V \to \frac{V}{\Lambda}$ is the usual quotient map. Therefore when $L \to \frac{V}{\Lambda}$ is positive, one can realise global sections of L as entire functions on V satisfying some functional equations. Following proposition depicts that the elementary divisors of the polarisation L determines the size of the space of its global sections.

Proposition 1.6.4 *[Gf-Hr 2, Theorem, p. 317] Let $L \to \frac{V}{\Lambda}$ be a polarisation and $\delta_1, \ldots, \delta_g$ be the corresponding elementary divisors. Then*

1. *$h^0(L) = \prod_{i=1}^{g} \delta_i$.*

2. *$H^0(L^n)$ is base point free for $n \geq 2$ and gives an embedding of $\frac{V}{\Lambda}$ in some $\mathbb{P}^N$ for $n \geq 3$.*

Remark 1.6.5 Second assertion of the Proposition 1.6.4 is part of a characterisation of an abelian variety due to Lefschetz (cf. [Le 1] and [Le 2]) which says that for a complex torus being an abelian variety is same as being an algebraic variety.

Let C be a smooth projective curve of genus g over complex numbers. Then one can naturally associate to C an (principally polarised) abelian variety called the Jacobian variety $J(C)$. The study of abelian variety originated from the analysis of Jacobian variety though a Jacobian variety is a special case of an abelian variety. Now we recall the construction and a few properties of a Jacobian variety and discuss that Jacobian varieties are best known examples of abelian varieties.

Let $H^0(C, \Omega^1)$ be the complex vector space of holomorphic 1-forms on C and $H_1(C, \mathbb{Z})$ be the first homology group of topological 1-cycles with integer coefficients. As C is of genus g, $H^0(C, \Omega^1)$ is of complex dimension g and $H_1(C, \mathbb{Z})$ is a free abelian group of rank $2g$. The following proposition says that the 1-cycles of $H_1(C, \mathbb{Z})$ can actually be thought as linear forms on the space $H^0(C, \Omega^1)$.

Proposition 1.6.6 *The following canonical map*

$$\begin{aligned} H_1(C, \mathbb{Z}) &\to H^0(C, \Omega^1)^* \\ \lambda &\mapsto \left(\omega \mapsto \int_\lambda \omega\right). \end{aligned} \tag{1.25}$$

is injective.

Proof.See [La-Bk, Lemma 11.1.1]. Indeed, decomposing $H^1_{DR}(C)^*$ into the holomorphic part $H^0(C, \Omega^1)^*$ and the antiholomorphic part $\overline{H^0(C, \Omega^1)^*}$ according to Hodge decomposition we get that the map in (1.25) is the composition of following natural maps:

$$H_1(C, \mathbb{Z}) \hookrightarrow H_1(C, \mathbb{C}) = H^1_{DR}(C)^* = H^0(C, \Omega^1)^* \oplus \overline{H^0(C, \Omega^1)^*} \to H^0(C, \Omega^1)^*.$$

Then the proof follows from the fact that image of any element of $H_1(C,\mathbb{Z})$ in $H^1_{DR}(C)^*$ is invariant under complex conjugation. $\square$

As $H_1(C,\mathbb{Z}) \cong \mathbb{Z}^{2g}$, let us choose $2g$ canonical generators of $H_1(C,\mathbb{Z})$, say, $\lambda_1, \lambda_2, \dots, \lambda_{2g}$ such that

$$^{\#}(\lambda_i \cdot \lambda_j) = ^{\#}(\lambda_{g+i} \cdot \lambda_{g+j}) = 0 \text{ and } ^{\#}(\lambda_i \cdot \lambda_{g+j}) = \delta_{ij} \text{ for all } 1 \leq i,j \leq g,$$

where $^{\#}(\lambda_i \cdot \lambda_j)$ denotes the intersection number of the cycles λ_i and λ_j. Also as $H^0(C,\Omega^1) \cong \mathbb{C}^g$, we choose a basis $\omega_1, \omega_2, \dots, \omega_g$ of $H^0(C,\Omega^1)$. Now consider the following matrix $\Pi \in \mathrm{M}_{g\times 2g}(\mathbb{C})$:

$$\Pi = \begin{pmatrix} \int_{\lambda_1} \omega_1 & \int_{\lambda_2} \omega_1 & \cdots & \int_{\lambda_{2g}} \omega_1 \\ \int_{\lambda_1} \omega_2 & \int_{\lambda_2} \omega_2 & \cdots & \int_{\lambda_{2g}} \omega_2 \\ \vdots & \vdots & \ddots & \vdots \\ \int_{\lambda_1} \omega_g & \int_{\lambda_2} \omega_g & \cdots & \int_{\lambda_{2g}} \omega_g \end{pmatrix} \tag{1.26}$$

As the map in (1.25) is injective by Proposition 1.6.6, by abuse of notation we denote the image $(\omega \mapsto \int_\lambda \omega)$ of $\lambda \in H_1(C,\mathbb{Z})$ also by λ. So λ, thought as a linear form on $H^0(C,\Omega^1)$, is then defined as follows:

$$\begin{aligned} \lambda \colon H^0(C,\Omega^1) &\to \mathbb{C} \\ \omega &\mapsto \int_\lambda \omega. \end{aligned}$$

Therefore λ is completely known if its value on a basis is known, that is, if the values $\int_\lambda \omega_1, \int_\lambda \omega_2, \dots, \int_\lambda \omega_g$ are known. In fact, these values are coordinates of λ with respect to the basis $\omega_1^*, \omega_2^*, \dots, \omega_g^*$ of $H^0(C,\Omega^1)^*$, dual of the chosen basis of $H^0(C,\Omega^1)$, as the following equality holds:

$$\lambda = \int_\lambda = \sum_{j=1}^{g} \left(\int_\lambda \omega_i \right) \omega_j^*.$$

Therefore by Proposition 1.6.6, columns of the matrix π as in (1.26) span the lattice $\Lambda = \{\sum_{i=1}^{2g} n_i\lambda_i \,|\, n_i \in \mathbb{Z}\}$, which in fact is the lattice $H_1(C,\mathbb{Z})$. By Proposition 1.6.1, the matrix π as in (1.26) is therefore a period matrix of an abelian variety known as the *Jacobian variety of* C and is denoted by $J(C)$. Clearly the Jacobian variety $J(C)$ is the complex torus $\frac{H^0(C,\Omega^1)^*}{H_1(C,\mathbb{Z})}$ of dimension equal to genus

g of the curve C and is therefore an algebraic variety by Remark 1.6.5.

Remark 1.6.7 By construction, the Jacobian $J(\mathbb{P}^1)$ of the projective line $\mathbb{P}^1$ is trivial as genus of $\mathbb{P}^1$ is zero. So to exclude triviality, we will work on genus greater than zero case.

One can choose normalised basis $\gamma_1, \gamma_2, \ldots, \gamma_{2g}$ of $H_1(C, \mathbb{Z})$ so that

$$\begin{aligned} {}^{\#}(\gamma_i \cdot \gamma_j) &= {}^{\#}(\gamma_{g+i} \cdot \gamma_{g+j}) = 0, \\ {}^{\#}(\gamma_i \cdot \gamma_{g+j}) &= \delta_{ij}, \\ {}^{\#}(\gamma_{g+i} \cdot \gamma_j) &= -\delta_{ij} \text{ for all } 1 \leq i, j \leq g. \end{aligned} \tag{1.27}$$

By *intersection matrix* of this chosen basis of $H_1(C, \mathbb{Z})$, one means the matrix $\mathcal{P} \in \mathrm{M}_{2g}(\mathbb{Z})$ whose (i, j)-th element is the intersection number ${}^{\#}(\gamma_j \cdot \gamma_i)$. Therefore, from the relations as in (1.27), we obtain:

$$\mathcal{P} = \begin{pmatrix} 0 & -\mathrm{Id}_g \\ \mathrm{Id}_g & 0 \end{pmatrix}. \tag{1.28}$$

Moreover, one can choose a basis $\tau_1, \tau_2, \ldots, \tau_g$ of $H^0(C, \Omega^1)^*$ such that the following holds:

$$\int_{\gamma_i} \tau_j = \delta_{ij}, \ 1 \leq i, j \leq g. \tag{1.29}$$

But something more happens. Together with the chosen normalised basis of $H_1(C, \mathbb{Z})$ they form a symplectic basis, that is, the period matrix Π of $J(C)$ as defined in (1.26) takes a simpler form as in Proposition 1.6.3:

$$\mathcal{P} = \begin{pmatrix} \mathrm{Id}_g & Z \end{pmatrix}. \tag{1.30}$$

It can be checked that there exists a divisor on $J(C)$ known as *Theta divisor*, denoted by Θ, such that $\mathcal{O}(\Theta)$ is a canonical polarisation on $J(C)$ (cf. [La-Bk, Proposition 11.1.2]). As $\mathrm{Im}(c_1(\mathcal{O}(\Theta))) = \mathcal{P}^{-1}$ where $\mathcal{P}$ is the intersection matrix as in (1.28), $\mathcal{O}(\Theta)$ is then a principal polarisation on $J(C)$. Sometimes we denote by $(J(C), \Theta)$ the principally polarised Jacobian variety of C.

Recall the exponential exact sequence as in (1.1) for the curve C:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_C \xrightarrow{\exp} \mathcal{O}_C^* \longrightarrow 0. \tag{1.31}$$

Then corresponding to (1.31) we have the following long exact sequence:

$$\begin{tikzcd} H^0(C,\mathcal{O}_C) \arrow[r, "\exp"] & H^0(C,\mathcal{O}_C^*) \arrow[r] & H^1(C,\mathbb{Z}) \arrow[dll] & \\ H^1(C,\mathcal{O}_C) \arrow[r] & H^1(C,\mathcal{O}_C^*) \arrow[r, "\delta"] & H^2(C,\mathbb{Z}). \end{tikzcd} \tag{1.32}$$

As C is a compact and connected Riemann surface, only global holomorphic maps are constants, that is, $H^0(C,\mathcal{O}_C) \cong \mathbb{C}$. Also $H^0(C,\mathcal{O}_C)^* \cong \mathbb{C}^*$ and therefore the exponential map $\exp\colon H^0(C,\mathcal{O}_C) \to H^0(C,\mathcal{O}_C^*)$ as in (1.32) is surjective. From exactness of (1.32), the map $H^0(C,\mathcal{O}_C^*) \to H^1(C,\mathbb{Z})$ is therefore the zero map and hence the map $H^1(C,\mathbb{Z}) \to H^1(C,\mathcal{O}_C)$ is injective. So we obtain the following exact sequence from (1.32):

$$0 \longrightarrow \frac{H^1(C,\mathcal{O}_C)}{H^1(C,\mathbb{Z})} \longrightarrow H^1(C,\mathcal{O}_C^*) \xrightarrow{\delta} H^2(C,\mathbb{Z}). \tag{1.33}$$

As mentioned in the Section 1.3, the Picard group $\mathrm{Pic}(C)$ of all isomorphism classes of line bundles on C can be identified with the group $H^1(C,\mathcal{O}_C^*)$. Therefore $\mathrm{Pic}^0(C)$, the subgroup of $\mathrm{Pic}(C)$ consisting of line bundles with vanishing Chern class, is nothing but $\mathrm{Ker}(\delta)$ and therefore is isomorphic to $\frac{H^1(C,\mathcal{O}_C)}{H^1(C,\mathbb{Z})}$. Now by Serre duality, $H^1(C,\mathcal{O}_C) \cong H^0(C,\Omega^1)^*$ and by Poincaré duality, $H^1(C,\mathbb{Z}) \cong H_1(C,\mathbb{Z})$. Therefore we have the following interesting isomorphism:

$$\mathrm{Pic}^0(C) \cong \frac{H^1(C,\mathcal{O}_C)}{H^1(C,\mathbb{Z})} \cong \frac{H^0(C,\Omega^1)^*}{H_1(C,\mathbb{Z})} \cong J(C). \tag{1.34}$$

Hence, the Jacobian variety can also be interpreted as the space parametrising all isomorphism classes of line bundles with vanishing Chern classes, that is, degree zero line bundles on C. For any degree zero line bundle L, the dual bundle L^* is also of degree zero and $L \otimes L^* \cong \mathcal{O}_C$. As a result, $J(C)$ is immediately endowed with a group structure, the group operation being the tensor product of line bundles and $\mathcal{O}_C$ being the identity element of $J(C)$.

This alternative description of the Jacobian variety $J(C)$ can be used to compute its dimension in an alternative manner. As the trivial line bundle $\mathcal{O}_C$ has only one independent global section,that is $h^0(C,\mathcal{O}_C) = 1$, by Riemann-Roch theorem we have $h^1(C,\mathcal{O}_C) = g$. It can be proved that the tangent space of $J(C)$ at the point $\mathcal{O}_C$ is canonically isomorphic to $H^1(C, \mathcal{O}_C)$ (cf. [Mi, Proposition 2.1]). Now as $J(C)$ is smooth in our context, it readily follows that $\dim J(C) = g$.

Now we discuss the importance of the Jacobian variety $J(C)$ from the point of view of the curve C itself. For that, let's start with a map from the curve C to its Jacobian $J(C)$ which is defined very naturally once a base point x_0 of the curve C is fixed.

$$\begin{aligned} u\colon C &\to J(C) \cong \frac{H^0(C,\Omega^1)^*}{H_1(C,\mathbb{Z})} \\ x &\mapsto \left(\omega \mapsto \int_{x_0}^{x} \omega\right) \pmod{H_1(C,\mathbb{Z})}. \end{aligned} \tag{1.35}$$

Let us once again choose a basis $\omega_1, \omega_2, \ldots, \omega_g$ of $H^0(C, \Omega^1)$. Consider the following map from C to $J(C)$ which we again denote by u. While defining this we use the fact that $H^0(C, \Omega^1)^* \cong \mathbb{C}^g$.

$$\begin{aligned} u\colon C &\to J(C) \cong \frac{\mathbb{C}^g}{H_1(C,\mathbb{Z})} \\ x &\mapsto \left(\int_{x_0}^{x} \omega_1, \int_{x_0}^{x} \omega_2, \ldots, \int_{x_0}^{x} \omega_g\right) \pmod{H_1(C,\mathbb{Z})}. \end{aligned} \tag{1.36}$$

Note that the coordinates of the linear map $\omega \mapsto \int_{x_0}^{x} \omega$ are $\int_{x_0}^{x} \omega_1, \int_{x_0}^{x} \omega_2, \ldots, \int_{x_0}^{x} \omega_g$ with respect to the dual basis of the chosen basis of $H^0(C, \Omega^1)$, as mentioned earlier. Therefore two apparently different maps defined as in (1.35) and (1.36) basically are the same, the first one is coordinate free approach whether the other is not. Here, $\int_{x_0}^{x} \omega$ for any $\omega \in H^0(C, \Omega^1)$ means $\int_\gamma \omega$ for a fixed path γ from x_0 to x. So we need to check that the definitions are independent of the chosen path γ. Let us choose another path λ from x_0 to x. Then as $\eta(= \gamma - \lambda)$ is an element of $H_1(C, \mathbb{Z})$, therefore the difference $\int_\gamma \omega - \int_\lambda \omega = \int_\eta \omega$ is an element of $H_1(C, \mathbb{Z})$. Here we again identify $H_1(C, \mathbb{Z})$ with its image using Proposition 1.6.6. So the map $u\colon C \to J(C)$ is well defined and is known as *Abel-Jacobi map*.

Recall that by $\mathrm{Div}^n(C)$ we denote the set of divisors of degree n on C. Then one can extend the domain of definition of the map u to $\mathrm{Div}^n(C)$ linearly as follows:

$$\begin{aligned} u\colon \mathrm{Div}^n(C) &\to J(C) \\ \sum_i n_i x_i &\mapsto \sum_i n_i u(x_i). \end{aligned} \tag{1.37}$$

For $n = 0$, the map is more interesting. In this case, the Abel-Jacobi map is canonical, that is, it becomes independent of the chosen base point on the curve C. We therefore write down the definition for $n = 0$ case separately.

$$\begin{aligned} u\colon \mathrm{Div}^0(C) &\to J(C) \\ \sum_i (p_i - q_i) &\mapsto \left(\omega \mapsto \sum_i \int_{p_i}^{q_i} \omega\right) \pmod{H_1(C,\mathbb{Z})}. \end{aligned} \tag{1.38}$$

It can be checked that this definition is well defined as this is independent of the representation of a degree zero divisor. This is in fact a group homomorphism. Moreover the following theorem, known as *Abel's theorem* (cf. [Gf-Hr 2, p. 235]), says that this is an isomorphism.

Theorem 1.6.8 *Let* $\mathrm{PDiv}^0(C)$ *denote the subgroup of* $\mathrm{Div}^0(C)$ *consisting of degree zero principal divisors. Then the map* $u\colon \mathrm{Div}^0(C) \to J(C)$ *as in* (1.38) *fits into the following exact sequence:*

$$0 \longrightarrow \mathrm{PDiv}^0(C) \longrightarrow \mathrm{Div}^0(C) \xrightarrow{u} J(C) \longrightarrow 0\,.$$

Hence $\mathrm{Pic}^0(C)$ *is isomorphic to the Jacobian variety* $J(C)$.

Proof. See [La-Bk, Theorem 11.1.3]. Indeed, by Abel's theorem kernel of the map u is the subgroup $\mathrm{PDiv}^0(C)$ of $\mathrm{Div}^0(C)$. Surjectivity of the map u follows from Jacobi Inversion theorem (cf. [Gf-Hr 2, p. 235]). □

Remark 1.6.9

1. The isomorphism between $\mathrm{Pic}^0(C)$ and $J(C)$ as in Theorem 1.6.8 has already been discussed in (1.34).

2. It can be noted that restricting the map $\varphi\colon \mathrm{Div}(C) \to \mathrm{Pic}(C)$ as in (1.12) to $\mathrm{Div}^0(C)$, we obtained in Proposition 1.3.5 that the kernel is nothing but $\mathrm{PDiv}^0(C)$. This is same as Abel's theorem once we have Theorem 1.6.8.

We have defined the classical Abel-Jacobi map analytically. Let us define it algebraically now. Choosing a divisor D_n of degree n on C, we define:

$$\begin{aligned} u\colon \mathrm{Div}^n(C) &\to J(C) \\ D &\mapsto \mathcal{O}(D - D_n). \end{aligned}$$

As a particular case, if we choose $D_n = nx_0$, then the map becomes:

$$\begin{aligned} u\colon \mathrm{Div}^n(C) &\to J(C) \\ D &\mapsto \mathcal{O}(D - nx_0). \end{aligned} \tag{1.39}$$

We get the following map $u\colon C \to J(C)$ by restricting the map $u\colon \mathrm{Div}^1(C) \to J(C)$ as in (1.39) to the curve C as C can be thought of as a subset of $\mathrm{Div}^1(C)$.

$$\begin{aligned} u\colon C &\to J(C) \\ x &\mapsto \mathcal{O}(x - x_0). \end{aligned} \tag{1.40}$$

It can be easily verified that the map u defined in (1.40), (1.35) and (1.36) are the same and we denote this map also by u_{x_0} whenever the base point x_0 needs to be specified. The first one is algebraic approach where as the last two definitions are analytical in nature.

Theorem 1.6.10 *The map*

$$\begin{aligned} u\colon C &\to J(C) \\ x &\mapsto \mathcal{O}(x - x_0) \end{aligned}$$

is an embedding, when genus g of the curve C is greater than or equal to one.

Proof. See [La-Bk, Proposition 11.1.4 & Corollary 11.1.5]. Indeed, for $g \geq 1$, the projectivized differential of the map is nothing but the canonical map $\varphi_{\omega_C}\colon C \to \mathbb{P}^{g-1}$ which is injective at every point of C as ω_C is base point free. □

Remark 1.6.11

1. For $g = 0$, $J(C) = 0$ by Remark 1.6.7 and hence of dimension zero where as the dimension of the curve is one, so the map (1.40) can't be an embedding. So Theorem 1.6.10 does not hold for $g = 0$ case.

2. For $g \geq 2$, $\dim J(C) = g \geq 2 > 1 = \dim C$. Therefore, the map u as in (1.40) can't be an isomorphism. This embedding of the curve C in its Jacobian variety $J(C)$ can be an isomorphism only for $g = 1$ case.

Let us now discuss another importance of the Abel-Jacobi map u_{x_0} as in (1.40) which in turn will help us to define a Poincaré bundle over the curve C. To discuss that let us start with a very natural question. We have seen that the Jacobian variety $J(C)$ parametrizes all degree zero line bundles on C, a smooth projective variety of genus g and of dimension one. Now one may consider the Jacobian variety of $J(C)$ itself, that is, the space parametrizing all degree zero line bundles on the variety which in turn parametrizes all degree zero line bundles on C. Clearly this can be repeated infinitely many times. At this point one can ask whether this process, which a priori seems to be never ending, terminates or not. To answer this question we need to go through the concept of dual complex torus of a given torus.

Given a complex torus X of dimension g, one can define another complex torus $\widehat{X}$, known as *dual complex torus associated to* X, of same dimension. This torus $\widehat{X}$ parametrizes all degree zero line bundles on X (cf. [La-Bk, Proposition 2.4.1]). This description of $\widehat{X}$ gives rise to a line bundle on $X \times \widehat{X}$ known as Poincaré bundle for X. We recall this definition.

Definition 1.6.12 By a *Poincaré bundle* for X we mean a holomorphic line bundle $\mathcal{P}$ on $X \times \widehat{X}$ which is isomorphic to L when gets restricted to $X \times \{L\}$ for all $L \in J(X)$ and is trivial when gets restricted to $\{0\} \times \widehat{X}$.

Taking $X = J(C)$, we have

$$\widehat{J(C)} \cong \mathrm{Pic}^0(J(C)). \tag{1.41}$$

Let $u_x^*\colon \mathrm{Pic}^0(J(C)) \to \mathrm{Pic}^0(C)$ be the pullback of the Abel-Jacobi map $u_x\colon C \to J(C)$ with respect to base point $x \in C$ as defined in (1.40). The pullback is also a restriction map as u_x is an embedding by Theorem 1.6.10 for $g \geq 1$. For $g = 0$, $\mathrm{Pic}^0(C)$ and $\mathrm{Pic}^0(J(C))$ both are trivial by Remark 1.6.7. Altogether by [La-Bk, Lemma 11.3.1], we have the following isomorphism:

$$\mathrm{Pic}^0(J(C)) \xrightarrow[\cong]{u_x^*} \mathrm{Pic}^0(C)\,. \tag{1.42}$$

Therefore by (1.41) and (1.42) we can conclude that the process mentioned in the previous question, which seems to be never ending apparently, gets terminated at a very early stage.

Taking $X = J(C)$ in Definition 1.6.12, we get a Poincaré bundle for $J(C)$. Then the isomorphism as in (1.42) allows us to construct a Poincaré bundle for the curve C itself.

Definition 1.6.13 By a *Poincaré bundle of degree n for C normalised with respect to $x \in C$* we mean a holomorphic line bundle $\mathcal{P}_C^n$ on $C \times \mathrm{Pic}^n(C)$ which is isomorphic to L when gets restricted to $C \times \{L\}$ for all $L \in \mathrm{Pic}^n(C)$ and is trivial when gets restricted to $\{x\} \times \mathrm{Pic}^n(C)$.

Following proposition assures the existence of such a Poincaré bundle over C.

Proposition 1.6.14 *[La-Bk, Proposition 11.3.2] There exists a Poincaré bundle of degree n for C for all $n \in \mathbb{Z}$ uniquely determined by the base point $x \in C$.*

Remark 1.6.15 We will need this Poincaré bundle for C in a more general set up to provide variety structure on some special subsets in the space parametrizing semistable bundles of fixed rank and degree over C. See, for example, Subsection 5.2.2 for more details.

It can be noted that if two compact Riemann surfaces are isomorphic, then so is their Jacobians. In fact, converse is also true and popularly known as Torelli's Theorem (cf. [La-Bk, Theorem 11.1.7]). This emphasises the fact that the Jacobian variety of a curve is intrinsically related to that curve.

We end this section by mentioning the importance of a Jacobian variety from the viewpoint of an abelian variety. Let us start with a different but useful interpretation of Torelli's Theorem. Let us denote the moduli space of smooth projective curves of genus g by $\mathcal{M}_g$. By $\mathcal{A}_g^1$ we denote the moduli space of all (principally polarised) abelian variety of dimension g. Consider the following map:

$$\begin{aligned} J \colon \mathcal{M}_g &\to \mathcal{A}_g^1 \\ C &\mapsto J(C). \end{aligned} \tag{1.43}$$

Note that, Torelli's theorem says that the map as in (1.43) is injective. The image $J(\mathcal{M}_g)$ is therefore a $3g-3$ dimensional subvariety of $\mathcal{A}_g^1$. At this point, one can naturally ask a question: Is the map in (1.43) surjective for any g? In other words, given any principally polarised abelian variety $\mathcal{A}$ of dimension g, does there exist a smooth projective curve C of genus g such that $J(C) \cong \mathcal{A}$? The answer is negative in general for $g \geq 4$.

Though Jacobian varieties do not exhaust abelian varieties, possibly the next best thing happens. For a given curve C Jacobian variety is the abelian variety nearest to the curve C in some sense. That is to say, given a morphism from the curve C to an abelian variety X, it factors through $J(C)$ upto a translation of X (cf. [La-Bk, Universal Property of the Jacobian 11.4.1]).

1.7 Symmetric product and Abel-Jacobi map

In this section we discuss briefly on symmetric product of curve and Abel-Jacobi map. Throughout this section we take C to be a projective curve of genus g over complex numbers. Then we apply results from this section in the context of elliptic curve later on.

Let us denote $\overbrace{C \times C \times \cdots \times C}^{d}$ by $C^{\times d}$. Here, $S^d(C)$, the *d-th symmetric product of C* can be understood as the quotient space $\frac{C^{\times d}}{\sigma^d}$ of $C^{\times d}$ under the action of the permutation group σ^d of d symbols and $p_1 + p_2 + \cdots + p_d$ can be thought as $[(p_1, p_2, \cdots, p_d)]$, the image of $(p_1, p_2, \cdots, p_d)$ under σ^d action, that is, in $[(p_1, p_2, \cdots, p_d)]$ order of p_i's doesn't matter. Therefore the notation $p_1+p_2+\cdots+p_d$ instead of $[(p_1, p_2, \cdots, p_d)]$ makes more sense. Here $p_1+p_2+\cdots+p_d$ can also be thought as a degree d effective divisor on C and $S^d(C)$ is nothing but the set of all degree d effective divisors on C. So for $d \geq 0$, we have:

$$S^d(C) = \mathrm{Div}^d(C).$$

Consider the classical *Abel-Jacobi map* $\varphi_d \colon S^d(C) \to J_d(C)$, defined as follows:

$$\begin{aligned} \varphi_d \colon S^d(C) &\to J_d(C) \\ x_1 + x_2 + \cdots + x_d &\mapsto \mathcal{O}(x_1 + x_2 + \cdots + x_d). \end{aligned} \tag{1.44}$$

This map is also called as *Abel-Jacobi map*. In fact, in a way this map is defined more naturally as we don't need to fix any base point on the curve unlike (1.39). This map can also be thought of as the restriction of the map $\varphi \colon \mathrm{Div}(C) \to \mathrm{Pic}(C)$ as in (1.12) to $\mathrm{Div}^d(C)$.

Consider the following morphism.

$$\begin{aligned} \psi_d \colon C^{\times d} &\to S^d(C) \\ (p_1, p_2, \cdots, p_d) &\mapsto p_1 + p_2 + \cdots + p_d. \end{aligned} \tag{1.45}$$

This is clearly a quotient map. Moreover, $S^d(C)$ gets the structure of a topological space and also of a complex manifold from $C^{\times d}$ via the map ψ_d as in (1.45). In fact, for a given coordinate chart on an open set of $C^{\times d}$, one can get a coordinate chart on the image of that open set using elementary symmetric functions and the map ψ_d (cf. [Gf-Hr 2, p. 236]). By Chow's theorem, $S^d(C)$ therefore gets the structure of an algebraic variety as well.

Let us now discuss about the smoothness of the variety $S^d(C)$. Let $\mathrm{Quot}^d_{\mathcal{F}}$ denote the Quot scheme parametrizing all torsion quotients of $\mathcal{F}$ having degree d (cf. Subsection 3.3.2 for more details). Following theorem is about the smoothness of the Quot scheme $\mathrm{Quot}^n_{\mathcal{O}^r_C}$.

Theorem 1.7.1 *Let C be a non-singular projective curve and let n be any non-negative integer. Then $\mathrm{Quot}^n_{\mathcal{O}^r_C}$ is a smooth projective scheme.*

Proof.See [Hb, Theorem 4.3.3, p. 47]. In fact let $(\mathcal{F}, q) \in \mathrm{Quot}^n_{\mathcal{O}^r_C}$. Then we have the following exact sequence.

$$0 \longrightarrow \mathrm{Ker}(q) \longrightarrow \mathcal{O}^r_C \xrightarrow{q} \mathcal{F} \longrightarrow 0\,.$$

where $\mathcal{F}$ is supported on a zero dimensional subscheme of C and $\mathrm{Ker}(q)$ is locally free of rank r. We also have the following equality.

$$\mathrm{Ext}^1(\mathrm{Ker}(q), \mathcal{F}) = H^1(C, (\mathrm{Ker}(q))^* \otimes \mathcal{F}).$$

But again $(\mathrm{Ker}(q))^* \otimes \mathcal{F}$ being supported on a zero dimensional subscheme of C, $H^1(C, (\mathrm{Ker}(q))^* \otimes \mathcal{F}) = 0$. Hence $\mathrm{Ext}^1(\mathrm{Ker}(q), \mathcal{F}) = 0$. Hence the theorem follows. $\square$

Let P be a polynomial with rational coefficients and let Hilb^P_C, which will be mostly denoted by Hilb^P, be the Hilbert scheme parametrizing subschemes of C having Hilbert polynomial P. Let d be any given non-negative integer. Then considering d as a constant polynomial, we have the following isomorphism.

$$\mathrm{Quot}^d_{\mathcal{O}_C} \cong \mathrm{Hilb}^d \cong S^d(C). \tag{1.46}$$

Hence by Theorem 1.7.1 and (1.46), $S^d(C)$ is a smooth algebraic variety. Alternatively, it can be proved using fundamental theorem on symmetric functions (cf. [Mi, Proposition 3.2]). Interestingly something more is true. For a non-singular variety S, $S^d(S)$ is smooth only if S is of dimension one, that is a curve (cf. [Mi, Remark 3.3]).

We now count the dimension of the smooth variety $S^d(C)$. let us recall the map ψ_d as in (1.45). As σ^d acts on $C^{\times g}$ by permuting the coordinates, for any $p_1 + p_2 + \cdots + p_g \in S^g(C)$ we have,

$$\psi_d^{-1}(p_1+\cdots+p_d) = \left\{(x_1, \cdots, x_d) \in C^{\times d} \,|\, \rho(x_i) = p_i \text{ for all } i, \text{and for all } \rho \in \sigma^d\right\}.$$

Therefore cardinality of any fiber is $d!$ and hence ψ_d is a finite morphism of degree $d!$. Therefore dimension of $S^d(C)$ is d as dimension of $C^{\times d}$ is so. Also as dimension of $\mathrm{Quot}^n_{\mathcal{O}^r_C}$ is $r \cdot n$, dimension of $S^d(C)$ can be calculated from (1.46) as well (cf. [Bi-Si]).

We have the following commutative diagram which consists of some natural subvarieties and depicts the scenario we are under quite nicely.

$$\begin{tikzcd}
C^{\times g} \arrow[r, "\psi_g"] & S^g(C) \arrow[r, "\varphi_g"] & J_g(C) \arrow[d, "\simeq"', "\otimes\mathcal{O}(-p)"] \\
C^{\times(g-1)} \arrow[u, "\text{inclusion}"] \arrow[r, "\psi_{g-1}"] & S^{g-1}(C) \arrow[u, "\text{inclusion}"'] \arrow[r, "\varphi_{g-1}"] & J_{g-1}(C) \arrow[d, "\simeq"', "\otimes\mathcal{O}(-p)"] \\
C^{\times(g-2)} \arrow[u, "\text{inclusion}"] \arrow[r, "\psi_{g-2}"] & S^{g-2}(C) \arrow[u, "\text{inclusion}"'] \arrow[r, "\varphi_{g-2}"] & J_{g-2}(C) \arrow[d, "\simeq"', "\otimes\mathcal{O}(-p)"] \\
\vdots \arrow[u, "\text{inclusion}"] & \vdots \arrow[u, "\text{inclusion}"'] & \vdots \arrow[d, "\simeq"', "\otimes\mathcal{O}(-p)"] \\
C \arrow[u, "\text{inclusion}"] \arrow[r, "\psi_1", "\simeq"'] & S^1(C) \arrow[u, "\text{inclusion}"'] \arrow[r, "\varphi_1"] & J_1(C)
\end{tikzcd}$$

The maps $\varphi_d \colon S^d(C) \to J_d(C)$, for all d with $1 \leq d \leq g$, are birational morphisms. Moreover, the image of $S^d(C)$ under the map φ_d are subvarieties of $J_d(C)$ and are denoted by W^0_d for all $1 \leq d \leq g$. The subvariety W^0_d parametrizes degree d line bundles over C having atleast one independent global section as this is the image of effective divisors of degree d.

If we want to compare the cohomology classes $[W^0_d]$'s, it is not possible to do so at this stage as W^0_d's sit inside different $J_d(C)$'s. So, to compare their cohomology classes it is natural to think them as subvarieties of one fixed variety. This can be obtained as follows. Let us consider the point $p \in C$ which we have chosen and fixed already. Consider the classical Abel-Jacobi map map $\varphi_d\colon S^d(C) \to J_d(C)$ as defined in (1.44). Also consider the map $\otimes\mathcal{O}(-dp)\colon J_d(C) \to J(C)$ defined as follows.

$$\begin{aligned} \otimes\mathcal{O}(-dp)\colon J_d(C) &\to J(C) \\ L &\mapsto L \otimes \mathcal{O}(-dp). \end{aligned}$$

Then the map $u\colon S^d(C) \longrightarrow J(C)$ is defined as $u = \otimes\mathcal{O}(-dp) \circ \varphi_d$ which is same as the map defined in (1.39).

$$\begin{array}{ccccc} S^d(C) & \xrightarrow{\varphi_d} & J_d(C) & \xrightarrow{\otimes\mathcal{O}(-dp)} & J(C) \\ & & & & \\ x_1 + \cdots + x_d & \longmapsto & \mathcal{O}(x_1 + \cdots + x_d) & \longmapsto & \mathcal{O}(x_1 + \cdots + x_d - dp). \end{array}$$

Now define W_d, for all d, $1 \leq d \leq g$, called the *Brill-Noether subvarieties of* $J(C)$, as follows:

$$W_d := u(S^d(C)). \tag{1.47}$$

Let Θ be the Theta divisor in $J(C)$, the translate of the divisor W^0_{g-1} of $J_{g-1}(C)$ via the map $\otimes\mathcal{O}(-(g-1)p)\colon J_{g-1}(C) \to J(C)$. Let $[W_d]$ be the cohomology class of W_d and $[\Theta]$ be the cohomology class of Θ in $H^*(J(C), \mathbb{Q})$. The classical *Poincaré relation* then expresses the cohomological classes of W_d, in terms of the Theta divisor on $J(C)$.

Lemma 1.7.2 *[Ab-Cr-Gf-Hr, chapter 1, §5, p-25] In $H^*(J(C), \mathbb{Q})$, we have*

$$[W_d] = \frac{1}{(g-d)!} \cdot [\Theta]^{g-d} \tag{1.48}$$

for all d, $1 \leq d \leq g$.

This Lemma 1.7.2 can be interpreted as follows. Consider the subalgebra of $H^*(J(C), \mathbb{Z})$ a priori generated by the cohomology classes W_d, $1 \leq d \leq g$. Then this subalgebra is generated by $[\Theta]$ only. Moreover, the relation (1.48) holds. We consider similar problem in the cohomology ring of the moduli space of semistable bundles over elliptic curve.

Now we have the following theorem due to Abel by which one can describe the fiber of the Abel-Jacobi map explicitly.

Theorem 1.7.3 *Two effective divisors D and D_1 of degree d on C are linearly equivalent if and only if $\varphi_d(D) = \varphi_d(D_1)$.*

Proof. See [Ab-Cr-Gf-Hr, p. 18]. Also, follows from Proposition 1.3.5 and restricting the map $\varphi\colon \mathrm{Div}(C) \to \mathrm{Pic}^(C)$ as in (1.12) to $\mathrm{Div}^d(C)$. $\square$

We observe that $S^d(C)$ can thought of as collection of all effective divisors on C of degree d. The variety $\mathrm{Pic}^d(C)$ can be interpreted as $\frac{S^d(C)}{\sim}$, where ' $\sim$' is the linear equivalence of divisors. For any $D \in S^d(C)$, let us denote its linear equivalence class in $\mathrm{Pic}^d(C)$ by $[D]$. Then Theorem 1.7.3 can be restated as follows. "Only if" part says that the Abel-Jacobi map factors through $\mathrm{Pic}^d(C)$, that is we have the following commutative diagram,

$$\begin{tikzcd} S^d(C) \arrow[rr, "{[\ ]}"] \arrow[dr, "\varphi_d"'] & & \mathrm{Pic}^d(C) \arrow[dl, "{[\varphi_d]}"] \\ & J_d(C) & \end{tikzcd}$$

where $[\ ]\colon S^d(C) \to \mathrm{Pic}^d(C)$ is given by $D \mapsto [D]$ and $[\varphi_d]\colon \mathrm{Pic}^d(C) \to J_d(C)$ is defined by $[D] \mapsto \varphi_d(D)$. Moreover "if" part of Theorem 1.7.3 depicts that the map $[\varphi_d]\colon \mathrm{Pic}^d(C) \to J_d(C)$ is injective.

Remark 1.7.4 By Theorem 1.7.3, fiber of the map φ_d over any line bundle $L \in J_d(C)$ is the complete linear system $|D|$ of a divisor D on C with $\mathcal{O}(D) = L$. Now if $d > 0$, then by Serre duality $h^1(C, \mathcal{O}(D)) = 0$ and by Riemann-Roch theorem $h^0(C, \mathcal{O}(D)) = d$. Therefore each fiber of the map $\varphi_d\colon S^d(C) \to J_d(C)$ is isomorphic to $\mathbb{P}^{d-1}$ if $d > 0$.

1.8 Elliptic curve

Let us begin this section with a question which we have raised already : Is the map in (1.43) surjective? We have mentioned that answer to this question is negative in general for $g \geq 4$. We now look at this map for $g = 1$ case and investigate its surjectivity for this special case. Towards that let us introduce the definition of elliptic curve and a few of its properties.

Definition 1.8.1 A smooth projective curve of genus 1 over complex numbers is called an elliptic curve (over $\mathbb{C}$).

For a smooth projective curve C of genus g, we define degree of the curve to be the degree d of its defining polynomial. Then using Riemann-Hurwitz formula genus of the curve can be given in terms of its degree as follows (cf. [Gf-Hr 2, p. 220]).

$$g = \frac{d(d-1)}{2}. \tag{1.49}$$

This is called *degree genus formula* for a smooth plane curve. As an elliptic curve is smooth by definition, using (1.49) it can be defined alternatively as follows.

Definition 1.8.2 A smooth cubic projective curve is called an elliptic curve.

So an elliptic curve can be thought of as projective plane curve given by a cubic polynomial. We denote an elliptic curve by E. For any smooth projective curve C over complex numbers, for all integer d, $J(C) \cong J_d(C)$, where the isomorphism can be naturally obtained by tensoring with a line bundle of appropriate degree. But for an elliptic curve E, something more happens to be true. Towards that, we have the following proposition.

Proposition 1.8.3 *The Picard variety $J_1(E)$ can be identified with E.*

Proof.Consider the map

$$\begin{aligned} E &\to J_1(E) \\ p &\mapsto \mathcal{O}(p). \end{aligned} \tag{1.50}$$

We want to verify that two distinct points p_1 and p_2 of E are not linearly equivalent, that is, $\mathcal{O}(p_1) \ncong \mathcal{O}(p_2)$. By Riemann-Roch theorem we have:

$$h^0(E, \mathcal{O}(p)) - h^0(E, \omega_E \otimes \mathcal{O}(p)^*) = 1 + 1 - 1.$$

As by (2) of Remark 1.4.3 ω_E is trivial, deg $\mathcal{O}(p)^* = -1$ and hence $h^0(E, \omega_E \otimes \mathcal{O}(p)^*) = 0$. Therefore, $h^0(E, \mathcal{O}(p)) = 1$. Hence by (1.15), (1.16) and Definition 1.3.9, we observe that only effective divisor linearly equivalent to p is p itself. As a result the map as in (1.50) is injective.

Now let L be a degree one line bundle over E and s be any non-zero section of L. Let (s) be the divisor corresponding to the section s. Then the map

$$\begin{aligned} J_1(E) &\to E \\ L &\mapsto (s) \end{aligned}$$

is the inverse of the map in (1.50) (cf. §1.3). Hence $J_1(E) \cong E$. $\square$

Proposition 1.8.4 *For an elliptic curve E, its Jacobian variety $J(E)$ is isomorphic to the curve E itself.*

Proof. Let C be any smooth projective curve over complex numbers and $x_0 \in C$ be a chosen point. Let us define the following map:

$$\begin{aligned} J_1(C) &\to J(C) \\ L &\mapsto L \otimes \mathcal{O}(-x_0). \end{aligned}$$

Clearly this is an isomorphism. Therefore the proposition now follows from Proposition 1.8.3. $\square$

Remark 1.8.5 In Remark 1.6.11, we observed that the map u as in (1.40) can only be an isomorphism when the curve C is of genus one. Proposition 1.8.4 answers that possibility affirmatively.

Let us end this section by answering the question we have mentioned in the beginning of this section. Let us denote the set containing all non-isomorphic one dimensional complex torus by $\mathscr{T}_1$. Then Uniformization Theorem (cf. [Di-Sh, §1.4]) says that the inclusion map $i\colon \mathcal{M}_1 \hookrightarrow \mathscr{T}_1$ is actually an isomorphism. Moreover it factors through the map $J\colon \mathcal{M}_1 \to \mathcal{A}_1^1$ defined by $C \mapsto J(C)$ as in (1.43). We therefore have the following commutative diagram.

$$\begin{tikzcd} \mathcal{M}_1 \arrow[rr, "J"] \arrow[dr, hook, "i"'] & & \mathcal{A}_1^1 \arrow[dl, hook] \\ & \mathscr{T}_1 & \end{tikzcd}$$

Therefore the map

$$\begin{aligned} J\colon \mathcal{M}_1 &\to \mathcal{A}_1^1 \\ C &\mapsto J(C) \end{aligned}$$

is surjective and hence an isomorphism.

Chapter 2

On the cycle class map and Hodge structure - smooth and non-smooth cases

In this chapter we discuss the Hodge decomposition of a complex submanifold sitting inside a projective space. We thoroughly go through few properties of the Hodge decomposition like functoriality, Hodge symmetry etc. that are compatible with the corresponding properties of the cohomology ring. After that the Chow groups and the cycle class map have been defined for smooth cases. Then we come across one of the seven "Millennium Problems" of Clay Mathematics Institute (CMI), Cambridge, namely, the Hodge conjecture. It was first formulated by Hodge in 1941 and is now known as the Integral Hodge conjecture. Then Atiyah and Hirzebruch proved that integral Hodge conjecture can't hold (cf. [At-Hz]). We mention another example by Kollár in this context. The Hodge conjecture then gets modified and it asserts that Hodge cycles are (rational linear) combinations of some geometric pieces called algebraic cycles for some particularly nice spaces. One can refer to [Hg 1] and [Hg 2] for details regarding the Hodge conjecture.

Finally we give an instance where the Hodge conjecture holds, namely, for a general polarised Jacobian variety and go through mixed Hodge structure, operational Chow groups and the cycle class map on the singular varieties followed by that.

2.1 Hodge decomposition

Let X be a complex manifold of dimension n. Then by definition we can cover X with open sets U such that there exists an open set Δ of $\mathbb{C}^n$ satisfying following isomorphism:

$$(z_1, z_2, \ldots, z_n)\colon U \to \Delta.$$

Here z_i's are called complex coordinates for X on U. Moreover we can pass from one coordinate to the another via holomorphic functions. Writing $z_k = x_k + \imath y_k$, $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ are real coordinates of X on U, treating X as a real manifold of dimension $2n$. Hence every differential form on U can be expressed in terms of these coordinates, that is to say, with respect to dx_k and dy_k for all $1 \leq k \leq n$. Let us define dz_k and $d\bar{z_k}$ as follows:

$$\begin{aligned} dz_k &= dx_k + \imath dy_k, \\ d\bar{z_k} &= dx_k - \imath dy_k. \end{aligned} \tag{2.1}$$

Then the differential forms can be expressed in terms of dz_k and $d\bar{z_k}$ for all $1 \leq k \leq n$. If a form on X can be expressed as a sum of the terms like $a(z_k, \bar{z_k}) dz_{i_1} \wedge dz_{i_2} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge d\bar{z}_{j_2} \wedge \cdots \wedge d\bar{z}_{j_q}$ in any system of coordinates, then we say that the form is *of type* (p, q). Here $1 \leq i_1 < \cdots < i_p \leq n$ and $1 \leq j_1 < \cdots < j_q \leq n$. Let $H^m_{DR}(X, \mathbb{C})$, or simply $H^m(X, \mathbb{C})$ when no confusion is likely to occur, be the m-th de Rahm cohomology group of X consisting of closed m-forms modulo the exact ones. By $H^{p,q}(X)$, or simply by $H^{p,q}$ when the space involved is clear from the context, we denote the subspace of $H^{p+q}(X, \mathbb{C})$ consisting of closed $(p+q)$-forms of type (p, q). Let us assume that X is a complex submanifold of some projective space, then for any non-negative integer r, the space $H^r(X, \mathbb{C})$ can be decomposed as follows:

Hodge decomposition:

$$H^r(X, \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}. \tag{2.2}$$

Remark 2.1.1 This result does not need the projectivity of X, it uses some coarser condition instead. That is to say, it uses existence of a Kähler metric. Any projective space carries such metric and moreover restriction of that to any complex submanifold is again a Kähler metric. So the decomposition holds in that case also. For the proof of the Hodge decomposition for a complex Kähler manifold, one can refer to [Gf-Hr 2, p. 116].

Let us now recall a few properties of the de Rahm cohomology groups. Let M and N be two oriented, complex manifolds of dimension m and n respectively. Then we have the following:

1. The de Rahm cohomology $H^*(M, \mathbb{Z})$ is a graded, skew-commutative ring and as an abelian group it is finitely generated. We denote the product operation of this ring by '$\cdot$'.

2. When $p < 0$ or $p > m$, $H^p(M, \mathbb{Z}) = 0$. Moreover, $H^0(M, \mathbb{Z}) = \mathbb{Z} \cdot 1$ and $H^m(M, \mathbb{Z}) \cong \mathbb{Z}$ via the canonical isomorphism $\int_M$.

3. Let $f \colon M \to N$ be a smooth map. We denote the associated pull back morphism by $f^* \colon H^*(N, \mathbb{Z}) \to H^*(M, \mathbb{Z})$ and push forward morphism by $f_* \colon H^*(M, \mathbb{Z}) \to H^*(N, \mathbb{Z})$. Then

   (a) The morphism f^* is a morphism of graded rings, that is, $f^*(H^r(N, \mathbb{Z})) \subseteq H^r(M, \mathbb{Z})$.

   (b) The morphism f_* is a group homomorphism and not a ring homomorphism in general. Also, degree of this morphism is $n - m$ and so $f_*(H^p(M, \mathbb{Z})) \subseteq H^{p+n-m}(N, \mathbb{Z})$.

   (c) For all $\alpha \in H^*(M, \mathbb{Z})$ and $\beta \in H^*(N, \mathbb{Z})$, $f_*(\alpha \cdot f^*\beta) = f_*\alpha \cdot \beta$.

   (d) For all $\alpha \in H^m(M, \mathbb{Z})$, $\int_N f_*\alpha = \int_M \alpha$.

4. **Poincaré duality:**

   The following bilinear form is non-degenerate :

$$H^p(M, \mathbb{Z}) \otimes H^{m-p}(M, \mathbb{Z}) \xrightarrow{\ \cdot\ } H^m(M, \mathbb{Z}) \xrightarrow{\int_M} \mathbb{Z}.$$

We now list down some properties of the Hodge decomposition that are nicely compatible with the properties given above. Let us assume that X and Y be two projective varieties of dimension m and n respectively. We need the variety structure here for existence of the Hodge decomposition as in (2.2). We now have the following:

1. The equalities $H^0(X,\mathbb{C}) = H^{0,0}$ and $H^{2m}(X,\mathbb{C}) = H^{m,m}$ hold. Furthermore, the operation '$\cdot$' of $H^*(X,\mathbb{C})$ gets restricted as follows:

$$H^{p_1,q_1} \times H^{p_2,q_2} \to H^{p_1+p_2,q_1+q_2}.$$

2. Let $f\colon X \to Y$ be a morphism between two projective varieties. Then we have:

$$\begin{aligned} &f^*(H^{p,q}(Y)) \subseteq H^{p,q}(X),\\ &f_*(H^{p,q}(X)) \subseteq H^{p+n-m,q+n-m}(Y). \end{aligned} \tag{2.3}$$

3. Poincaré duality gets restricted to the following perfect pairing:

$$H^{p,q} \times H^{m-p,m-q} \longrightarrow H^{m,m} \xrightarrow{\cong} H^{2m}(X,\mathbb{C}) \xrightarrow{\cong} \mathbb{C}. \tag{2.4}$$

4. For any r, $H^r(X,\mathbb{C})$ is equipped with a natural involution corresponding to the conjugation of differential forms, that is, dz_k and $d\bar{z}_k$,defined in (2.1), get interchanged under this involution. Hence we have the following.

   Hodge symmetry:

$$H^{p,q} = \overline{H^{q,p}}.$$

We use aforementioned properties heavily to prove the following proposition.

Proposition 2.1.2 *Let X be a complex projective manifold of dimension n and Z be an irreducible, codimension p subvariety. Then the cohomology class $[Z]$ in $H^{2p}(X,\mathbb{Z})$ is of type (p,p).*

Proof. See [Be 2, Proposition 3.3]. Indeed, we are now in the situation of Definition 1.5.3 and Remark 1.5.4. Let us quickly recall the notations. Let $\widetilde{Z}$ be a resolution of Z and $f \circ i\colon \widetilde{Z} \to X$ be the map as defined in Remark 1.5.4. So, whenever Z is smooth, we have $\widetilde{Z} = Z$ and $f \circ i = i$. For any $\alpha \in H^{2n-2p}(X,\mathbb{Z})$, we have:

$$\begin{aligned} [Z]\cdot\alpha &= i_*1\cdot\alpha \quad \text{(By Definition 1.5.3)}\\ &= i_*i^*\alpha. \quad \text{(By property 3(c) of de Rahm cohomology)} \end{aligned} \tag{2.5}$$

Therefore we obtain:

$$\begin{aligned}\int_X [Z]\cdot\alpha &= \int_X i_* i^*\alpha \quad \text{(By (2.5))} \\ &= \int_{\widetilde{Z}} i^*\alpha. \quad \text{(By property 3(d) of de Rahm cohomology)}\end{aligned} \tag{2.6}$$

As (2.4) holds, it is enough to show that $[Z]\cdot\alpha = 0$ hold for all α in $H^{n-a,n-b}(X)$ with a being different from b and $a+b=2p$. Without loss of generality, assume that $a > p$. So, $n-b > n-p$ and as a result $H^{n-a,n-b}(\widetilde{Z}) = 0$. As by (2.3)

$$\alpha \in H^{n-a,n-b}(X) \Rightarrow i^*\alpha \in H^{n-a,n-b}(\widetilde{Z}),$$

we have $i^*\alpha = 0$. Therefore, we are done by (2.6). □

2.2 The cycle class map on smooth varieties and the Hodge (p,p)-conjecture

Let X be a smooth projective variety over complex numbers. Let $Z^p(X)$ be the free abelian group generated by the codimension p subvarieties of X. The elements of $Z^p(X)$ are called *algebraic cycles* of codimension p.

Let us now recall the notion of rational equivalence. A codimension p algebraic cycle Z is said to be *rationally equivalent* to 0, denoted by $Z \sim 0$, if there exists a finite number of codimension $p-1$ subvarieties V_i of X and non-zero elements r_i of the field of rational functions of V_i satisfying

$$Z = \sum_i (r_i).$$

Here (r_i) denote the divisor of the rational function r_i as defined in (1.10). As $(r^{-1}) = -(r)$ for any non-zero element of the field of rational functions, the collection of all codimension p algebraic cycles rationally equivalent to 0 is a subgroup of the group $Z^p(X)$ and is denoted by $Rat^p(X)$. Rational equivalence can be alternatively interpreted in a more geometric way. Informally, two cycles Z_0 and Z_1 in $Z^k(X)$ are rationally equivalent if there is a rationally parametrized family of cycles interpolating between them, that is, Z_0 and Z_1 are obtained as restrictions of a cycle on $\mathbb{P}^1 \times X$ to the fibers $\{t_0\} \times X$ and $\{t_1\} \times X$ for two distinct points t_0 and t_1 of $\mathbb{P}^1$ (cf. [Fu, §1.6]).

Let $CH^p(X)$ be the *p-th graded piece of the Chow group* $CH^*(X)$. Here $CH^p(X)$ is defined as

$$CH^p(X) := Z^p(X)/Rat^p(X).$$

Moreover the Chow group $CH^*(X)$ can be given the structure of a ring with *intersection of cycles* as the product operation (cf. [Fu, §8.3]). The ring is then called the *Chow ring* of X.

Remark 2.2.1

1. It can be noted that $\mathrm{Div}(X) = Z^1(X)$. Furthermore we have, $CH^1(X) = \mathrm{Pic}(X)$, that is, the notion of rational equivalence generalises the notion of linear equivalence of divisors as defined in Definition 1.3.6.

2. Defining $A_p(X) := CH^{n-p}(X)$, n being the dimension of the variety X, we get the group $A_*(X) = \oplus_p A_p(X)$ graded by dimension instead of codimension. This group does not have a ring structure in general. But for smooth X, $A_*(X) \cong CH^*(X)$ and hence $A_*(X)$ is a ring too.

Consider the *cycle class map* defined as follows.

$$\begin{aligned} cl\colon CH^p(X) &\to H^{2p}(X,\mathbb{Z}) \\ Z &\mapsto [Z], \end{aligned} \tag{2.7}$$

where Z is an irreducible subvariety of codimension p in X and extend it linearly to the whole of $CH^p(X)$. The image of this map is denoted by $H^{2p}(X,\mathbb{Z})_{alg}$ and the elements in $H^{2p}(X,\mathbb{Z})_{alg}$ are called *integral algebraic classes*. Let us consider the natural inclusion $i\colon H^{2p}(X,\mathbb{Z}) \to H^{2p}(X,\mathbb{C})$. Then Proposition 2.1.2 can be restated as follows:

$$H^{2p}(X,\mathbb{Z})_{alg} \subseteq i^{-1}(H^{p,p}). \tag{2.8}$$

The Hodge conjecture initially meant that converse of (2.8) holds. Now this is known as the *integral Hodge conjecture* which is as follows :

$$H^{2p}(X,\mathbb{Z})_{alg} = i^{-1}(H^{p,p}).$$

Remark 2.2.2 Let $i\colon H^{2p}(X,\mathbb{Z}) \to H^{2p}(X,\mathbb{C})$ be the inclusion map. Then $i^{-1}(H^{p,p})$ is often denoted by $H^{p,p} \cap H^{2p}(X,\mathbb{Z})$.

Many mathematicians observed that the integral Hodge conjecture doesn't hold. In [At-Hz], Atiyah and Hirzebruch proved that there are some torsion elements that can't be represented as algebraic classes. Here is an example due to Kollár. Let us take a general hypersurface X of $\mathbb{P}^4$ having degree p^3, p being a prime greater than 5. Let h be the class of a hyperplane section of X generating $H^2(X,\mathbb{Z})$. Let l be the Poincaré dual of h that generates $H^4(X,\mathbb{Z})$. Then l is not algebraic as any algebraic class is a multiple of $p \cdot l$ (cf. [Be 2, Proposition 4.3 & Corollary 4.4]).

As the integral Hodge conjecture failed to be true, people started looking at analogous result in rational cases. Let $CH^*(X) \otimes \mathbb{Q}$ be denoted by $CH^*(X)_{\mathbb{Q}}$ and $j_{\mathbb{Q}}\colon H^{2p}(X,\mathbb{Q}) \to H^{2p}(X,\mathbb{C})$ be the obvious map. Then the subspace of the *Hodge classes* of $H^{2p}(X,\mathbb{Q})$, denoted by $H^{2p}_{Hodge}(X)$, is defined as:

$$H^{2p}_{Hodge}(X) := H^{2p}(X,\mathbb{Q}) \cap j_{\mathbb{Q}}^{-1}(H^{p,p}(X)). \tag{2.9}$$

Consider the cycle class map $cl\colon CH^p(X)_{\mathbb{Q}} \to H^{2p}(X,\mathbb{Q})$ defined similarly as in (2.7). The image of this map is denoted by $H^{2p}(X,\mathbb{Q})_{alg}$ and the elements in $H^{2p}(X,\mathbb{Q})_{alg}$ are called *rational algebraic classes*. The *Hodge (p,p)-conjecture* asserts the following:

Hodge conjecture:

$$H^{2p}(X,\mathbb{Q})_{alg} = H^{2p}_{Hodge}(X)$$

that is, any rational algebraic class is a Hodge class and vice versa.

For $p = 1$, Hodge (p,p)-conjecture is true even in integral case. This was proved by Lefschetz and is known as *Lefschetz theorem on $(1,1)$ classes*. Lefschetz proved this using a tool introduced by Poincaré called normal functions (cf. [Le 3]). Here we state the theorem and give an outline of the proof using a modern approach.

Theorem 2.2.3 *Let $X \subseteq \mathbb{P}^n$ be a complex submanifold. Then given any $\gamma \in H^{1,1} \cap H^2(X,\mathbb{Z})$, there exists a divisor D on X such that $\gamma = [D]$.*

Proof. See [Gf-Hr 2, p. 163]. Indeed, recall the exponential exact sequence as in (1.1)

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathcal{O}_M \xrightarrow{exp} \mathcal{O}_M^* \longrightarrow 0.$$

Then in corresponding cohomology sequence we obtain:

$$H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \xrightarrow{i_*} H^2(X, \mathcal{O}_X) \xrightarrow{\cong} H^{0,2}.$$

Here c_1 is as defined in (1.21) and the isomorphism $H^2(X, \mathcal{O}_X) \cong H^{0,2}$ follows from [Gf-Hr 2, Dolbeault Theorem, p. 45]. It can be checked that the map $i_*\colon H^2(X, \mathbb{Z}) \to H^{0,2}$ actually factors through $H^2(X, \mathbb{C})$, that is, we have the following commutative diagram:

$$\begin{tikzcd}
H^2(X,\mathbb{Z}) \arrow[r] \arrow[dr, "i_*"'] & H^2(X,\mathbb{C}) \arrow[r, "\cong"] & H^{2,0} \oplus H^{1,1} \oplus H^{0,2} \arrow[dl, "\pi^{0,2}"] \\
& H^{0,2} &
\end{tikzcd}$$

Here $\pi^{0,2}$ denotes the usual projection. Theorem now follows from Proposition 1.5.5. □

Remark 2.2.4 The Hodge (p, p)-conjecture is trivially true for $p = 0$. Proposition 2.2.3 can be proved to be true for rational case and hence Hodge (p, p)-conjecture is true for $p = 1$ too.

Surjectivity is very intrinsically related to a whole lot of conjectures. The Hodge conjecture is no different. Proposition 2.1.2 depicts the fact that Hodge conjecture can be stated in a more refined way as follows:

$$H^{2p}_{Hodge}(X) \subseteq H^{2p}(X, \mathbb{Q})_{alg}.$$

That is to say the cycle class map

$$cl\colon CH^p(X)_{\mathbb{Q}} \to H^{2p}_{Hodge}(X)$$

is surjective. The Hodge conjecture is one of the seven millennium problems of Clay Mathematics Institute of Cambridge. It was formulated by Hodge in 1941 (cf. [Hg 1]). Many more conjectures can be interpreted as surjectivity of certain maps. Let us just mention one more instance. Poincaré formulated a conjecture in 1904 known as the Poincaré conjecture. It is a theorem now as it was proved in November, 2002 by Perelman. Poincaré asked if the three dimensional sphere is characterized as the unique simply connected closed three manifold upto homeomorphism. Let S^3 denote the three sphere. Let us denote the set of all simply connected closed three manifolds by M^3. Let '$\sim$' denote the homeomorphism of topological spaces. Then obviously the following map

$$\{S^3\} \to \frac{M^3}{\sim}$$

is injective. Moreover, the Poincaré conjecture asserts that the map is surjective too.

2.3 The Hodge conjecture for a general Jacobian

Let X be an abelian variety. The subring of $H^{2*}_{Hodge}(X)$ generated by $H^0_{Hodge}(X)$ and $H^2_{Hodge}(X)$ is denoted by $D^*(X)$. The cycle classes in $D^*(X)$ are all algebraic by Remark 2.2.4. Let $D^p(X)$ be the p-th graded piece of $D^*(X)$. In particular, the Hodge (p,p)-conjecture is true if

$$D^p(X) = H^{2p}_{Hodge}(X).$$

Mattuck proved that Hodge conjecture is true for a general polarised abelian variety (cf. [Ma]). Tate proved the Hodge conjecture for self product of an elliptic curve (cf. [Ta] and [Gr 3, §3]) and Murasaki did some explicit computations for the same (cf. [Mr]). Then using degeneration technique one can prove that Hodge conjecture holds for a general polarised Jacobian variety with Theta divisor Θ as a polarisation.

Theorem 2.3.1 *For a general polarised Jacobian* $(J(C), \Theta)$ *of dimension* g

$$H^{2p}_{Hodge}(J(C)) = D^p(J(C)) \cong \mathbb{Q}$$

for all $p = 0, \cdots, g$.

Proof. See [La-Bk, Theorem 17.5.1; p. 561]. □

2.4 Pure and mixed Hodge structure

In this section we recall the notion of pure and mixed Hodge structures. We have already discussed pure Hodge structure without mentioning explicitly. Let's begin with that.

Let us take a complex manifold X of dimension m sitting inside some projective space. Then recall that the space $H^r(X,\mathbb{C})$ has a decomposition, called Hodge decomposition, as follows (cf. 2.1):

$$H^r(X,\mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}.$$

Furthermore this decomposition satisfies the condition $H^{p,q} = \overline{H^{q,p}}$, known as Hodge symmetry. Such a direct sum decomposition of $H^r(X,\mathbb{C})$ is known as a *pure Hodge structure of weight* r. The space $H^r(X,\mathbb{C})$ has a finite decreasing filtration by the subspaces $F^pH^r(X,\mathbb{C})$, with $p \in \mathbb{Z}$, defined as follows:

$$F^pH^r(X,\mathbb{C}) := \bigoplus_{i\geq p} H^{i,r-i},$$

that is, an element of $F^pH^r(X,\mathbb{C})$ has atleast p many dz's when expressed in terms of local coordinates. This filtration is known as *Hodge filtration.* Moreover the subspaces $H^{p,q}$ can be recovered from these new subspaces as we have the following equality:

$$H^{p,q} = F^pH^r(X,\mathbb{C}) \cap \overline{F^qH^r(X,\mathbb{C})}.$$

We now briefly recall the notion of mixed Hodge structure. This was introduced by P. Deligne around 1970 as a generalisation of pure Hodge structure which is applicable for singular and non-complete varieties as well. We then have the notion of an additional filtration, finite and increasing, known as weight filtration which is trivial over a smooth compact variety over complex numbers. One can refer to [De 1], [De 2], [Ca-Ze-Gf-Tg, Chapter 3] and [Du].

Let's be more precise. Assume that X is a complex projective variety possibly singular. A *mixed Hodge structure* on $H^r(X,\mathbb{Z})$ consists of the following data:

1. a Hodge filtration F^p of $H^r(X,\mathbb{C})$,

2. a finite increasing filtration W_j of $H^r(X,\mathbb{Q})$ called *weight filtration*,

such that the *i-th associated graded quotient* $gr_i^W H^r(X,\mathbb{Q})$ of $H^r(X,\mathbb{Q})$ with respect to the weight filtration, defined as $gr_i^W H^r(X,\mathbb{Q}) := \frac{W_i}{W_{i-1}}$, along with the filtration induced by the given Hodge filtration on its complexification is a pure Hodge structure of weight i, for all $i \in \mathbb{Z}$. Here, the complexification $gr_i^W H^r(X,\mathbb{Q}) \otimes \mathbb{C}$ of $gr_i^W H^r(X,\mathbb{Q})$ is given by $\frac{W_i \otimes \mathbb{C}}{W_{i-1} \otimes \mathbb{C}}$ and the filtration induced by F^p on this is denoted by $F^p(gr_i^W H^r(X,\mathbb{Q}) \otimes \mathbb{C})$ and is defined as

$$F^p(gr_i^W H^r(X,\mathbb{Q}) \otimes \mathbb{C}) := \frac{(F^p \cap W_i \otimes \mathbb{C}) + W_{i-1} \otimes \mathbb{C}}{W_{i-1} \otimes \mathbb{C}}.$$

Remark 2.4.1

1. Both pure and mixed Hodge structure can be defined in a more general set up. In this context, one can refer to [Ca-Ze-Gf-Tg, Definition 3.1.1, 3.1.2, 3.2.11 & 3.2.15]. Then their existence can be proved in our situations (cf. [Ca-Ze-Gf-Tg, Chapter 3, §3.1 & §3.4]).

2. We use the i-th associated graded quotient $gr_i^W H^r(X,\mathbb{Q})$ of $H^r(X,\mathbb{Q})$ to define cycle class map for singular varieties in the next section.

2.5 The cycle class map on the singular varieties

Let X be any scheme of finite type over $\mathbb{C}$. Fulton defined the operational Chow groups $A^*(X)$ for any scheme X (cf. [Fu, Chapter 17, §17.3]). These are the same as the Chow groups when X is smooth.

An element of the operational Chow group $A^p(X)$ is a collection of homomorphisms $A_k(X') \to A_{k-p}(X')$, for all $X' \to X$, compatible with proper push-forward, flat pullback and intersections. Here X' is a fiber product $X \times_X Y$ with respect to the morphism $\mathrm{Id}\colon X \to X$ and a given map $Y \to X$. The map $X' \to X$ is the usual projection map which fits into the commutative diagram of fiber product. So an element of $A^p(X)$ can be thought of as a special bivariant class (cf. [Fu, Definition 17.1]). Moreover we have the following:

1. There is a product, such that $A^*(X) = \oplus_p A^p(X)$ is an associative, graded ring with 1.

2. For any $f\colon Y \to X$, the pullback

$$f^*\colon A^p(X) \to A^p(Y)$$

is a ring homomorphism. This is functorial in f.

3. There is a projection formula:

$$f_*(f^*\beta \cap \alpha) = \beta \cap f_*(\alpha).$$

4. The Chern classes of vector bundles are defined in this theory.

5. Bloch, Gille, and Soulé (cf. [Bl-Gi-Sl]) defined the cycle class maps, which we again denote by cl, on the rational operational Chow group of X:

$$cl\colon A^i(X)_{\mathbb{Q}} \to gr^W_{2i} H^{2i}(X, \mathbb{Q}). \tag{2.10}$$

(cf. [To, §8, p. 22] also).

In particular, if Y is a smooth projective variety and $f\colon Y \to X$ is a generically finite morphism, then there are pushforward and pullback maps:

$$f_*\colon CH^p(Y) \to A^p(X),\ f^*\colon A^p(X) \to CH^p(Y).$$

Lemma 2.5.1 *Consider a smooth projective variety Y, and $f\colon Y \to X$ is generically finite and flat. There is a commutative diagram:*

$$\begin{tikzcd} A^p(X)_{\mathbb{Q}} \arrow[r, "f^*"] \arrow[d, "cl"'] & CH^p(Y)_{\mathbb{Q}} \arrow[d, "cl"] \\ gr^W_{2p} H^{2p}(X, \mathbb{Q}) \arrow[r, "f^*_{coh}"'] & H^{2p}(Y, \mathbb{Q}) \end{tikzcd}$$

Furthermore, f^*_{coh} is injective.

Proof. The second assertion on injectivity of f^*_{coh} follows from [Pe-Sb, Chapter 5, Corollary 5.42]. Suppose $W \subset Y$ is a codimension p closed subvariety. By definition, $[W]$ corresponds to a class in $A^p(X)$ and $f^*[W] \in CH^p(Y)$ is the same as the class $[W]$. The commutativity follows from the functoriality of the cycle class maps. □

We will utilise this map to define the cohomology classes of the Brill-Noether loci on the singular moduli space $\mathcal{M}_C(2, 2g-2)$, in the graded pieces of its singular cohomology group (cf. Theorem 5.5.7).

Chapter 3

On the moduli spaces

In mathematics, among many other interesting problems classification of objects in a given category is one. The concept of moduli spaces arise in order to deal this problem in algebraic geometry. Though moduli problems use many fancy techniques, the basics of these problems are naturally embedded in all branches of mathematics. For example, consider the problem of classifying all finite dimensional vector space over a given field k upto vector space isomorphism. As upto isomorphism there is only one vector space of dimension n for all non-negative integer n, the space $\mathbb{N} \cup \{0\}$ can be considered as a space classifying all finite dimensional k-vector spaces. The set of natural numbers $\mathbb{N}$ can be thought as the space parametrizing all non-zero finite dimensional k-vector spaces. But the same set can also be interpreted as the space classifying all cyclic groups of finite order. So by classifying problem one means a collection of objects $\mathcal{A}$, an equivalence relation '$\sim$' on $\mathcal{A}$. By solving this problem one means to describe $\frac{\mathcal{A}}{\sim}$, the set of equivalence classes of $\mathcal{A}$ under the given equivalence relation '$\sim$'. To do so, one usually find some discrete invariant. For example, *dimension* of a vector space and *order* of a group served as the discrete invariants in the problems discussed above.

Let us go through a few more examples of classifying spaces which are more relevant for our purpose. Any elliptic curve serves as a classifying space. To be more specific, a point on an elliptic curve E represents the isomorphism class of a degree zero line bundle on E (cf. Proposition 1.8.4). This classification problem can be immediately seen as a particular case of a more general moduli problem, namely the problem of classifying all degree zero line bundles upto isomorphism on a smooth, projective curve C over complex numbers. We have discussed that the g-dimensional variety $J(C)$, known as the Jacobian variety, parametrizes all non-isomorphic degree zero line bundles on the curve C of genus g (cf. Theorem 1.6.8). Let $\mathcal{A}$ be the set of all effective divisors of a fixed degree d on a given curve C. Suppose '$\sim_1$' be taken as equality of two such divisors and '$\sim_2$' be taken as linear equivalence. Then $\frac{\mathcal{A}}{\sim_1}$ is nothing but the d-th symmetric product $S^d(C)$ whereas $\frac{\mathcal{A}}{\sim_2}$ is nothing but $\mathrm{Pic}^d(C)$ (cf. Section 1.7).

3.1 Fine and coarse moduli spaces

In this chapter our main target is to go through the problem of classifying all stable bundles over C upto isomorphism and all semistable bundles over C upto S-equivalence. For that let us explain what we mean by a moduli problem more rigorously and two types of moduli spaces, namely fine and coarse moduli space, followed by that.

Let us recall the notion of functor of points. For that we introduce the following notations. By *Sch* we denote the category of schemes of finite type over a given field k. In this section by a scheme we refer to a scheme of finite type over the field k. We denote the category of sets by *Set*. we then have the following definition.

Definition 3.1.1 For a scheme X, the contravariant functor $\mathrm{Hom}(-, X)\colon \mathit{Sch} \to \mathit{Set}$, denoted by h_X, is called *functor of points of the scheme X*.

The functor $\mathrm{Hom}(-, X)$ is defined naturally. Given a scheme Y, it sends to the set $\mathrm{Hom}(Y, X)$. Note that, here we are assuming that our chosen categories are locally small categories and therefore the definition of functor of points makes sense. Also, given a morphism $f\colon Y \to Z$ of schemes, the functor h_X sends it to $h_X(f)$ defined as follows:

$$\begin{aligned} h_X(f)\colon h_X(Z) &\to h_X(Y) \\ g &\mapsto g \circ f. \end{aligned}$$

Remark 3.1.2 1. Any morphism $f\colon X \to Y$ of schemes gives rise to a natural transformation h_f of functors h_X and h_Y defined as follows:

$$\begin{aligned} {h_f}_Z \colon h_X(Z) &\to h_Y(Z) \\ g &\mapsto f \circ g. \end{aligned}$$

2. The category of contravariant functors from any category $\mathcal{C}$ are called *presheaves on* $\mathcal{C}$ and is denoted by $Psh(\mathcal{C})$. Using this notation we summarize what we have discussed as follows: There exists a functor $h \colon \mathcal{C} \to Psh(\mathcal{C})$ defined as

$$\begin{aligned} h \colon \mathcal{C} &\to Psh(\mathcal{C}) \\ X &\mapsto h_X && \text{[At object level]}, \\ (f \colon X \to Y) &\mapsto (h_f \colon h_X \to h_Y) && \text{[At morphism level]}. \end{aligned} \tag{3.1}$$

Let us now state a very important lemma in category theory which is very useful in our context too.

Proposition 3.1.3 *[Ho, **Yoneda lemma**, Lemma 2.4] Let $\mathcal{C}$ be any given category. Suppose that C be an object in $\mathcal{C}$ and F be an object in $Psh(\mathcal{C})$. Then there is a one to one correspondence between the set of all natural transformations from h_C to F and $F(C)$ given by,*

$$(\eta \colon h_C \to F) \mapsto \eta_C(\mathrm{Id}_C).$$

Definition 3.1.4 Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F \colon \mathcal{C} \to \mathcal{D}$ be a functor between them. Then the functor F is said to be *fully faithful* if the morphism $F_{X,Y} \colon \mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$ induced by F is bijective for all objects X, Y in the category $\mathcal{C}$.

Corollary 3.1.5 *The functor $h \colon \mathcal{C} \to Psh(\mathcal{C})$ as in (3.1), known as Yoneda embedding, is fully faithful.*

Proof. See [Ho, Corollary 2.5]. Indeed, let C, C_1 be two arbitrary objects of the category $\mathcal{C}$. To show that the functor $h \colon \mathcal{C} \to Psh(\mathcal{C})$ is fully faithful, we need to show that the morphism $h_{C,C_1} \colon \mathrm{Hom}(C, C_1) \to \mathrm{Hom}(h_C, h_{C_1})$ is bijective. That follows immediately from Proposition 3.1.3 by taking $F = h_{C_1}$. $\square$

This leads to the following definition which is very much crucial to define certain kind of moduli spaces.

Definition 3.1.6 A presheaf F in $Psh(\mathcal{C})$ is said to be *representable* if it lies in the image of the Yoneda emdedding, that is, if there exists an object C in $\mathcal{C}$ and a natural isomorphism between the functors F and h_C. We also say that the scheme C *represents* the functor F.

We already have discussed a few examples of moduli spaces without defining it precisely. Now we are in position to do so and towards that we have the following definitions.

Definition 3.1.7 By a *naive moduli problem in algebraic geometry* one means a collection $\mathcal{A}$ of objects in algebraic geometry together with an equivalence relation '$\sim$' on $\mathcal{A}$. A naive moduli problem is denoted by $(\mathcal{A}, \sim)$.

The next most important concept is the concept of a family of objects of $\mathcal{A}$ parametrized by a variety S. Precise definition of family is very intrinsically related to a given problem. One needs to mould the definition of a family depending on context to obtain best possible results. Intuitively, such a family $\mathcal{F}$ should consist of a collection of objects $\mathcal{F}_s$ of $\mathcal{A}$ for each $s \in S$, which vary in such a way that somehow reflects the structure of the variety S. Moreover, we demand these families to satisfy some natural conditions which should remain valid irrespective of which context we are under. These conditions are precisely incorporated in the following definition.

Definition 3.1.8 Let $(\mathcal{A}, \sim)$ be a naive moduli problem. Then an *extended moduli problem* is given by sets $\mathcal{A}_S$ of families over S for all schemes S, an equivalence relation '$\sim_S$' on $\mathcal{A}_S$ and pull back morphisms $f^* \colon \mathcal{A}_S \to \mathcal{A}_T$ for any morphism $f \colon T \to S$ of schemes satisfying following functorial properties:

1. $(\mathcal{A}_{\mathrm{Spec}(k)}, \sim_{\mathrm{Spec}(k)}) = (\mathcal{A}, \sim)$,
2. $\mathrm{Id}^*(\mathcal{F}) = \mathcal{F}$ for any family $\mathcal{F}$ over S and for the identity morphism $\mathrm{Id} \colon S \to S$,
3. $\mathcal{F} \sim_S \mathcal{G} \Rightarrow f^*(\mathcal{F}) \sim_T f^*(\mathcal{G})$ for any morphism $f \colon T \to S$,
4. $(g \circ f)^*(\mathcal{F}) \sim_T f^* g^*(\mathcal{F})$ for any given morphisms of schemes $f \colon T \to S$, $g \colon S \to R$ and any family $\mathcal{F}$ over the scheme R.

Here given a family $\mathcal{F}$ over S, by $\mathcal{F}_s$ we mean pull back $s^*(\mathcal{F})$ over the point $s\colon \mathrm{Spec}(k) \to S$.

Definition 3.1.8 in turn leads to the following definition.

Definition 3.1.9 An extended moduli problem defines a presheaf functor called *moduli functor* corresponding to the moduli problem, denoted by $\mathcal{M}$, is defined by

$$\begin{aligned} \mathcal{M}\colon Sch &\to Set \\ S &\mapsto \frac{\{\text{families over S}\}}{\sim_S} \qquad \text{[At object level]}, \\ (f\colon T \to S) &\mapsto f^*\colon \mathcal{M}(S) \to \mathcal{M}(T) \quad \text{[At morphism level]}. \end{aligned} \tag{3.2}$$

We now can define the best possible example of a moduli space known as the fine moduli space.

Definition 3.1.10 Let $\mathcal{M}$ be a moduli functor as in (3.2). A scheme M is said to be a *fine moduli space* if it represents $\mathcal{M}$.

Let us unwind Definition 3.1.10 a bit to understand the reasons for calling it best possible example of a moduli space. A priori it is not clear how the scheme M is related to the given (naive) moduli problem $(\mathcal{A}, \sim)$. Definition 3.1.10 says that there exists a natural transformation , say η between the moduli functor $\mathcal{M}$ and the functor h_M. Therefore we have the following bijections η_S for any scheme S:

$$\eta_S : \mathcal{M}_S := \frac{\{\text{families over S}\}}{\sim_S} \leftrightarrow \{\text{morphisms } S \to M\} := h_M(S). \tag{3.3}$$

In particular, taking $S = \mathrm{Spec}(k)$ in (3.3), we have:

$$\begin{tikzcd} \mathcal{M}_{\mathrm{Spec}(k)} \arrow[r, "\eta_{\mathrm{Spec}(k)}", "\cong"'] \arrow[d, "||"'] & h_M(\mathrm{Spec}(k)) \arrow[d, "||"'] \\ \frac{\{\text{families over } \mathrm{Spec}(k)\}}{\sim_{\mathrm{Spec}(k)}} \arrow[r, "\eta_{\mathrm{Spec}(k)}", "\cong"'] & \{\text{morphisms } \mathrm{Spec}(k) \to M\} \end{tikzcd} \tag{3.4}$$

Therefore by (1) of Definition 3.1.8 and (3.4) we can conclude that the set $\frac{\mathcal{A}}{\sim}$ is in bijection with the k-points of the scheme M representing the moduli functor $\mathcal{M}$.

Let us go through another important property possessed only by fine moduli spaces. Again consider (3.3) for $S = M$.

$$\eta_M : \mathcal{M}_M := \frac{\{\text{families over M}\}}{\sim_M} \leftrightarrow \{\text{morphisms } M \to M\} := h_M(M). \tag{3.5}$$

Immediately, we have a family $\mathcal{U}$ upto equivalence over M corresponding to the morphism $\mathrm{Id}_M\colon M \to M$, that is $\mathcal{U} := \eta_M^{-1}(\mathrm{Id}_M) \in \mathcal{M}$. Consider an arbitrary family $\mathcal{F}$ upto equivalence over a scheme S corresponding to a morphism $f\colon S \to M$. Consider the following diagram:

$$\begin{tikzcd} \{\mathcal{F}\} \arrow[r, hook] & \mathcal{M}(S) \arrow[r, "\eta_S"] \arrow[d, "||"'] & h_M(S) \arrow[d, "||"] & \{f\colon S \to M\} \arrow[l, hook'] \arrow[d, "||"] \\ \{f^*(\mathcal{U})\} \arrow[r, hook] & \mathcal{M}(S) \arrow[r, "\eta_S"'] & h_M(S) & \{\mathrm{Id}_M \circ f\colon S \to M\} \arrow[l, hook'] \end{tikzcd} \tag{3.6}$$

Diagram (3.6) depicts the fact that both the families $\mathcal{F}$ and $f^*(\mathcal{U})$ over the scheme S correspond to the same morphism $f\colon S \to M$. Therefore from (3.5) we have:

$$\mathcal{F} \sim_S f^*(\mathcal{U}),$$

that is, any family over any scheme can be obtained upto equivalence by pulling back the family $\mathcal{U}$, called the *universal family.*

So the situation is nice in all possible sense in case of a fine moduli space and hence such moduli spaces are very rare. So it is natural to obtain a weaker notion of such moduli space.

Definition 3.1.11 Given a moduli functor $\mathcal{M}$, a *coarse moduli space* is defined to be a scheme M along with a natural transformation $\eta\colon \mathcal{M} \to h_M$ satisfying following properties:

1. $\eta_{\mathrm{Spec}(k)}\colon \mathcal{M}(\mathrm{Spec}(k)) \to h_M(\mathrm{Spec}(k))$ is bijective,

2. Given any scheme N and any natural transformation $\mu\colon \mathcal{M} \to h_N$, there exists a unique morphism of schemes satisfying the following commutative diagram:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\eta"] \arrow[dr, "\mu"'] & h_M \arrow[d, "h_f"] \\ & h_N \end{tikzcd}$$

Here h_f is as defined in Remark 3.1.2.

Remark 3.1.12 1. It can be easily checked that both fine and coarse moduli spaces are unique upto unique isomorphism, if at all exists.

2. Though the notion of a coarse moduli space is weaker than that of a fine moduli space, still it is nearest to the moduli functor in the sense of (2) in Definition 3.1.11.

3.2 Stable and Semistable bundles

In this section, we recall some definitions and a few properties of semistable bundles over a smooth projective curve C over $\mathbb{C}$ of any genus.

Recall that given a vector bundle V over a smooth projective curve C, the degree deg V of V is defined to be the degree of the determinant line bundle $\det V$. Now one can associate a rational number to the given bundle V. This rational number, denoted by μ_V or μ in short if the bundle involved is clear from the context, called the *slope* of V and is defined by

$$\mu_V := \frac{\deg\, V}{\text{rank}\ V}.$$

Definition 3.2.1 A vector bundle V over C is called *semistable* if for any non-zero proper subbundle W,

$$\mu_W \leq \mu_V. \tag{3.7}$$

The bundle V is called *stable* if the inequality in (3.7) is strict.

Example 3.2.2 1. Any line bundle over a curve is stable and hence semistable.

2. The bundle $\mathcal{O}(1)^{\oplus 2}$ over $\mathbb{P}^1$ is semistable but not stable. It can be noted that the slope $\mu_{\mathcal{O}(1)}$ of the subbundle $\mathcal{O}(1)$ is 1 which is equal to the slope $\mu_{\mathcal{O}(1)^{\oplus 2}}$ of the bundle $\mathcal{O}(1)^{\oplus 2}$.

3. The bundle $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ over $\mathbb{P}^1$ is of slope 0 whereas the subbundle $\mathcal{O}(1)$ has slope 1. Hence the bundle $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ is not even semistable.

Given any rational number μ, let us denote the category of semistable bundles of slope μ by $SS_C(\mu)$ or simply by $SS(\mu)$ if the underlying curve is understood.

Consider the following exact sequence of vector bundles

$$0 \to V_1 \to V_2 \to V_3 \to 0. \tag{3.8}$$

Assume that $\mu_{V_1} = \mu_{V_3} = \mu$(say). Since degree and rank of vector bundles are additive, we have

$$\begin{aligned}\mu_{V_2} &= \frac{\deg V_2}{\text{rank } V_2} \\ &= \frac{\deg V_1 + \deg V_3}{\text{rank } V_1 + \text{rank } V_3} \\ &= \frac{\text{rank } V_1 \cdot \mu_{V_1} + \text{rank } V_3 \cdot \mu_{V_3}}{\text{rank } V_1 + \text{rank } V_3} \\ &= \frac{\text{rank } V_1 \cdot \mu + \text{rank } V_3 \cdot \mu}{\text{rank } V_1 + \text{rank } V_3} = \mu.\end{aligned}$$

Further we assume that the bundles V_1 and V_3 are semistable. Let F_2 be a subbundle of V_2. Let $F_1 := F_2 \cap V_1$ and F_3 be the image of F_2 in V_3 under the map $V_2 \to V_3$ as in the exact sequence (3.8). Then the vector bundles F_1 and F_3 are subsheaves of V_1 and V_3 respectively. We can get a subbundle $\tilde{F_1}$ of V_1 with $\mu_{F_1} \leq \mu_{\tilde{F_1}} \leq \mu$ and a subbundle $\tilde{F_3}$ of V_3 with $\mu_{F_3} \leq \mu_{\tilde{F_3}} \leq \mu$ as V_1 and V_3 are semistable bundles. Moreover, F_i, for $1 \leq i \leq 3$, satisfy the following exact sequence:

$$0 \to F_1 \to F_2 \to F_3 \to 0.$$

Therefore $\mu_{F_2} \leq \mu$ and hence V_2 is semistable. We can conclude that if V_1 and V_3 are semistable of slope μ, then so is V_2. In particular, direct sum of two semistable bundles of slope μ is again so. Therefore, a priori $SS_C(\mu)$ is only an additive category.

For any $V_1, V_2 \in SS_C(\mu)$ and for any non-zero map $\pi \colon V_1 \to V_2$, we have the inequality

$$\mu = \mu_{V_1} \leq \mu_{\text{Im}(\pi)} \leq \mu_{V_2} = \mu.$$

Therefore $\mu_{\text{Im}(\pi)} = \mu$ and $\text{Im}(\pi)$ is a subbundle of V_2. Hence π is of constant rank and $\text{Ker}(\pi)$, $\text{Coker}(\pi)$ are vector bundles. Let E be a subbundle of $\text{Ker}(\pi)$. So E is a subbundle of V_1 too and $\mu_E \leq \mu_{V_1} = \mu$ as V_1 is semistable. Hence $\text{Ker}(\pi)$ is semistable of slope μ. We also claim that $\text{Coker}(\pi)$ is semistable of slope μ. If not, then $\text{Coker}(\pi)$ has a non-zero locally free quotient bundle F with $\mu_F < \mu$. This

F is also a quotient bundle of V_2 of slope strictly less than μ. Dually, we get a subbundle of V_2 with slope strictly greater than μ contradicting the semistability of V_2. Hence the claim follows.

Therefore, for any $V_1, V_2 \in SS_C(\mu)$ and for any non-zero map $\pi\colon V_1 \to V_2$, $\mathrm{Ker}(\pi)$ and $\mathrm{Coker}(\pi)$ are also members of $SS_C(\mu)$. Hence $SS_C(\mu)$ is an abelian category. Therefore we have the notion of Jordan-Hölder filtration in this category.

Every semistable bundle V of slope μ has a *Jordan-Hölder filtration*,

$$0 = W_0 \subseteq W_1 \cdots \subseteq W_t = V,$$

where the successive quotients $\frac{W_i}{W_{i-1}}$ are stable bundles of slope μ for all $i = 1, \cdots, t$. First of all, the integer t is independent of the filtration and is called *the length of the filtration*. Moreover, for two such filtration of V, the successive quotients are uniquely determined up to a permutation, and so the *associated graded bundle*

$$\mathrm{Gr}(V) := \bigoplus_{i=1}^{t} \frac{W_i}{W_{i-1}}$$

is well defined.

Definition 3.2.3 Two semistable bundles V and V_1 are said to be *S-equivalent*, written as '$V \sim V_1$', if $\mathrm{Gr}(V) \cong \mathrm{Gr}(V_1)$.

Definition 3.2.4 A semistable bundle is called a *polystable* bundle if it is a direct sum of stable bundles.

Remark 3.2.5 From Definition 3.2.3, it is obvious that S-equivalence class of a semistable bundle contains exactly one polystable bundle upto isomorphism. In particular for a semistable bundle V, we have $V \sim \mathrm{Gr}(V)$.

Now we look into a few more type of bundles over curve and relations between them. Let V be a vector bundle over a curve C. Recall that by $\mathrm{End}(V)$ we denote the collection of all morphisms from the vector bundle V to itself. We recall that a bundle V is called *simple* if $\mathrm{End}(V) \cong \mathbb{C}$. Assume that V_1 and V_2 be two stable vector bundle of same slope. Then any non-zero morphism between them is an isomorphism (cf. [Ne 1, Lemma 5.3]). Let V be a stable vector bundle and $h\colon V \to V$ be a morphism. Then looking at an arbitrary fiber V_x of V over x, the linear map $h_x - \lambda \cdot \mathrm{Id}_{V_x} \colon V_x \to V_x$ is zero for any eigenvalue λ of h_x, the restriction of the morphism h to the fiber V_x. Therefore the morphism $h - \lambda \cdot \mathrm{Id}_V$ is not an isomorphism and hence equals to 0. So $h = \lambda \cdot \mathrm{Id}_V$ and $\mathrm{End}(V) \cong \mathbb{C}$, that is, the bundle V is simple.

Consider a decomposable bundle V with a decomposition $V = V_1 \oplus V_2$. Then two different homotheties on two summands V_1 and V_2 give rise to a non-trivial endomorphism of the bundle V. Hence the bundle V is not simple. Contrapositively, we can conclude that any simple bundle is indecomposable.

In short, we have the following implications:

$$\text{Stable} \Longrightarrow \text{Simple} \Longrightarrow \text{Indecomposable}. \tag{3.9}$$

By Definition 3.2.1, it is obvious that any stable bundle is semistable. Converse is also true when the rank and the degree of the bundle are coprime. It is a well-known fact. We still provide a proof for the sake of continuity. Let V be a semistable bundle of rank n and degree d with $\gcd(n, d) = 1$. Let W be a proper non-zero subbundle of V. Then

$$1 \leq \text{rank } W < \text{rank } V. \tag{3.10}$$

As V is semistable, we have:

$$\mu_W \leq \mu_V. \tag{3.11}$$

Therefore from (3.10) and (3.11) we get,

$$\deg W < \deg V. \tag{3.12}$$

Now if equality occurs in (3.11), then deg W and rank W both have to be integer mulltiples of deg V and rank V respectively as $\gcd(n, d) = 1$. But that contradicts (3.10) and (3.12). Therefore we have strict inequality in (3.11) and hence V is stable.

This can summarised by the following implications:

$$\text{Stable} \Longrightarrow \text{Semistable}, \tag{3.13}$$

$$\text{Semistable} \Longrightarrow \text{Stable}, \quad \text{if rank and degree are coprime}. \tag{3.14}$$

These implications as in (3.9), (3.13) and (3.14) combine together in case of elliptic curve as we will see later.

3.3 Moduli space of stable and semistable bundles over curve

In this section we go through the construction of the moduli space of stable and semistable bundles of rank n and degree d over curve. We recall different types of quotients under the action of an affine algebraic group on a scheme and discuss Quot scheme followed by that. Finally we outline the construction of the moduli spaces we are interested in through the PGL_N action over a Quot scheme.

3.3.1 Quotients and moduli spaces

Let G be an affine algebraic group and X be a scheme upon which G acts via an action '$\cdot$'. Then G has a natural induced action on the k-algebra $\mathcal{O}(X)$ of regular functions on X, which we again denote by '$\cdot$' by abuse of notation, is given by

$$\begin{aligned} G \times \mathcal{O}(X) &\to \mathcal{O}(X) \\ (g, f) &\mapsto g \cdot f, \text{ where} \\ (g \cdot f)(x) &:= f(g^{-1}x). \end{aligned} \tag{3.15}$$

By $\mathcal{O}(X)^G$ we denote the subalgebra of $\mathcal{O}(X)$ consisting of invariant functions under the action as in (3.15) and is therefore given by

$$\mathcal{O}(X)^G := \{f \in \mathcal{O}(X) \,|\, g.f = f \text{ for all } g \in G\}.$$

For any open subset U of X, $\mathcal{O}_X(U)^G$ is similarly defined as the subalgebra of $\mathcal{O}_X(U)$ consisting of all G-invariant functions.

Following these notations let us now define three types of quotients which are very much essential for constructing the moduli space of stable and semistable bundles over curve.

Definition 3.3.1 A G-invariant morphism of schemes $\varphi\colon X \to Y$ is said to be a *categorical quotient* for the G-action on X if it is universal, that is, given any other G-invariant morphism $\pi\colon X \to Z$, there exist a unique morphism $\theta\colon Y \to Z$

which fits into following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "\pi"'] & Y \arrow[d, "!\ \theta"] \\ & Z \end{tikzcd}$$

Definition 3.3.2 A morphism of schemes $\varphi\colon X \to Y$ is said to be a *good quotient* for the G-action on X if the following are satisfied.

1. The morphism φ is G-invariant.

2. The morphism φ is onto.

3. For any open subset U of Y, the morphism $\mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$ is an isomorphism onto $\mathcal{O}_X(\varphi^{-1}(U))^G$.

4. The image $\varphi(W)$ is closed in Y for every G-invariant closed subset W of X.

5. For any disjoint G-invariant closed subsets W_1 and W_2 of X, $\varphi(W_1) \cap \varphi(W_2) = \emptyset$.

6. The morphism φ is affine, that is, $\varphi^{-1}(U)$ is affine for every affine open subset U of Y.

   We denote this by $X//G$.

   If moreover, preimage $\varphi^{-1}y$ of any point $y \in Y$ is a single orbit, then the morphism $\varphi\colon X \to Y$ is said to be a *geometric quotient* for the G-action on X. We denote this by X/G.

Remark 3.3.3 1. Definition 3.3.2 immediately implies that any geometric quotient is a good quotient.

2. As (2) holds in Definition 3.3.2, (4) and (5) together can be stated as follows: For any disjoint G-invariant closed subsets W_1 and W_2 of X, the closures of $\varphi(W_1)$ and $\varphi(W_2)$ are disjoint.

3. Any good quotient is a categorical quotient (cf. [Ho, Proposition 3.30]) and therefore we have the following chain of implications:

$$\text{Geometric quotient} \Longrightarrow \text{Good quotient} \Longrightarrow \text{Categorical quotient.}$$

Given a moduli problem, a family $\mathcal{F}$ over a scheme S is said to have *local universal property* if for any family $\mathcal{G}$ over a scheme T and for any k-point $t \in T$, there exists a neighbourhood U_t of t in T and a morphism $f\colon U_t \to S$ satisfying

$$\mathcal{G}|_{U_t} \sim_{U_t} f^*(\mathcal{F}).$$

It can be noted that local universality doesn't demand the uniqueness of the morphism $f\colon U_t \to S$. The following proposition relates the three quotients just defined with moduli space.

Proposition 3.3.4 *[Ho, Proposition 3.35] Given a moduli problem $\mathcal{M}$, let $\mathcal{F}$ a family over a scheme S satisfying local universal property. Assume that there is an algebraic group G acting on S such that any two k-points s and t lie in the same G-orbit if and only if $\mathcal{F}_s \sim \mathcal{F}_t$. Then a categorical quotient of the G-action on S is a coarse moduli space if and only if preimage of every k-point under the quotient is a single orbit.*

3.3.2 Towards Quot Scheme

One of the major pathological behaviours of a moduli problem is *unboundedness*, which is essentially the non-existence of any family $\mathcal{F}$ over a scheme S parametrizing all objects in that moduli problem. To make the moduli problem of semistable bundles of rank n and degree d over a curve bounded, we need to impose few more conditions. We come to that shortly.

Let us now recall the definition of global generation of a sheaf.

Definition 3.3.5 Let $\mathcal{F}$ be a given sheaf over a space X. Then $\mathcal{F}$ is said to be *generated by its global sections* if the evaluation map ev is surjective, that is, we have the following:

$$H^0(X,\mathcal{F}) \otimes \mathcal{O}_X \xrightarrow{ev} \mathcal{F} \longrightarrow 0.$$

Let us now go through the notion of a generically generated bundle. Let X be a non-singular projective curve. As every torsion free module over a regular local ring of dimension 1 is free, every torsion free sheaf over X is locally free. Let F be a vector bundle and $\mathcal{F}$ be the corresponding sheaf of sections. Consider a subsheaf $\mathcal{G}$ of $\mathcal{F}$. Denoting the inverse image in $\mathcal{F}$ of the torsion subsheaf of $\mathcal{F}/\mathcal{G}$ by $\mathcal{G}_1$, we have $\mathcal{G}_1$ is torsion free and hence locally free as well. Let us denote the vector bundle corresponding to the sheaf $\mathcal{G}_1$ by G_1. Now as $\mathcal{F}/\mathcal{G}_1$ is locally free, the bundle homomorphism $G_1 \to F$ corresponding to the inclusion $\mathcal{G}_1 \subseteq \mathcal{F}$ is injective. As a result, G_1 can be treated as a subbundle of F and is called *generically generated by* G. From the construction described above, we have:

$$\text{rank } G_1 = \text{rank } G,$$
$$\deg G_1 \geq \deg G.$$

The following proposition provides some extra conditions on semistable bundles of higher degrees.

Proposition 3.3.6 *Let $\mathcal{F}$ be a locally free sheaf of rank n and degree d over X. Assume $d > n(2g-1)$. If the associated vector bundle F to $\mathcal{F}$ is semistable, then the following conditions hold:*

1. $H^1(X, \mathcal{F}) = 0$,

2. $\mathcal{F}$ is generated by global sections.

Proof.

1. If not, then by Serre duality, there exists a homomorphism $0 \neq f : \mathcal{F} \to K_X$. Let G be a subbundle of F generically generated by $\mathrm{Ker}(f)$. Then G is of rank $n-1$ and degree d_1 such that $d_1 \geq \deg \mathrm{Ker}(f) \geq \deg \mathcal{F} - \deg K_X$. By semistability of F, we have:

$$\frac{d-(2g-2)}{n-1} \leq \mu(G) \leq \mu(F) = \frac{d}{n}.$$

Hence we have $d \leq n(2g-2)$, contradicting the hypothesis.

2. Let us denote the fiber of the bundle F at the point $x \in X$ by F_x. Then F_x can be regarded as a torsion sheaf having support $\{x\}$. Let us denote by $\mathcal{F}(-x)$ the sheaf $\mathcal{O}(-x) \otimes \mathcal{F}$. Then we have the following short exact sequence:

$$0 \longrightarrow \mathcal{F}(-x) \longrightarrow \mathcal{F} \longrightarrow F_x \longrightarrow 0. \tag{3.16}$$

We need to show that the map $H^0(\mathcal{F}) \to H^0(F_x)$ is surjective. For that it is enough to show that $H^1(\mathcal{F}(-x)) = 0$ following the long exact sequence at cohomology level corresponding to (3.16). That follows from part (1) as

$$\deg \mathcal{F}(-x) = \deg \mathcal{O}(-x) \otimes \mathcal{F} = d - n > n(2g-2).$$

$\square$

Remark 3.3.7 1. For now, if we assume the existence of the moduli space of semistable bundles of rank n and degree d over a curve C, then denoting it by $\mathcal{M}_C(n,d)$ we have the following isomorphism:

$$\mathcal{M}_C(n,d) \xrightarrow[\cong]{\otimes L} \mathcal{M}_C(n, d+ne). \tag{3.17}$$

Here L is a line bundle of degree e. Hence (3.17) makes sense as tensoring by a line bundle preserves both stability and semistability. Therefore the condition on the degree of the sheaf $\mathcal{F}$ in Proposition 3.3.6 can be imposed without loss of any generality.

2. Properties (1) and (2) are essential for the boundedness of the family of semistable bundles of rank n and degree d over a curve. In fact, a strictly larger family of vector bundles of rank n and degree d is bounded, namely, the family satisfying properties (1) and (2) of Proposition 3.3.6.

Proposition 3.3.6 naturally leads to another example of a fine moduli space, known as Quot scheme, which in turn is very much essential to construct the moduli space of our concern.

Let $\mathcal{F}$ be a locally free sheaf of rank n and degree d over a curve C of genus g satisfying

1. $H^1(X, \mathcal{F}) = 0$,

2. The natural evaluation map $ev\colon H^0(X,\mathcal{F}) \otimes \mathcal{O}_X \to \mathcal{F}$ is surjective.

Then by Riemann-Roch theorem we have:

$$\chi(\mathcal{F}) = d + n(1-g) = h^0(X,\mathcal{F}) - h^1(X,\mathcal{F}) = h^0(X,\mathcal{F}).$$

Let $N := d + mn + n(1-g)$, then evaluation map satisfies the following exact sequence:

$$k^N \otimes \mathcal{O}_X \xrightarrow{ev} \mathcal{F} \longrightarrow 0. \tag{3.18}$$

Surjective morphisms as in (3.18) from a fixed coherent sheaf are parametrized by a scheme called Quot scheme. We recall the definition of this Quot Scheme.

Let $\mathcal{F}$ be a given coherent sheaf on C. Consider the moduli problem of classifying quotients of $\mathcal{F}$ having rank r and degree d. For that, we consider surjective morphism between sheaves

$$f\colon \mathcal{F} \twoheadrightarrow \mathcal{G}$$

up to the following equivalence relation

$$(f\colon \mathcal{F} \twoheadrightarrow \mathcal{G}) \sim (f_1\colon \mathcal{F} \twoheadrightarrow \mathcal{G}_1) \Leftrightarrow \mathrm{Ker}(f) = \mathrm{Ker}(f_1),$$

or equivalently by five lemma, if there exists a sheaf isomorphism $\eta\colon \mathcal{G} \to \mathcal{G}_1$ such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "f"] \arrow[d, "\mathrm{Id}"'] & \mathcal{G} \arrow[d, "\eta"] \\
\mathcal{F} \arrow[r, "f_1"] & \mathcal{G}_1
\end{tikzcd}$$

where $\mathcal{G}$ and $\mathcal{G}_1$ have rank r and degree d. The scheme that parametrizes all quotients of $\mathcal{F}$ of rank r and degree d upto the above equivalence is known as *Quot Scheme* and is denoted by $\mathrm{Quot}^{r,d}_{\mathcal{F}}$.

Let $\mathrm{Quot}^{d}_{\mathcal{F}}$ denote the Quot scheme parametrizing all the torsion quotients of $\mathcal{F}$ having degree d. Therefore we have

$$\mathrm{Quot}^{d}_{\mathcal{F}} = \mathrm{Quot}^{0,d}_{\mathcal{F}}.$$

Let us recall the notion of a polynomial, called Hilbert polynomial, of a given coherent sheaf.

Definition 3.3.8 Let X be projective curve equipped with an ample invertible sheaf $\mathcal{L}$ and $\mathcal{E}$ be a coherent sheaf over X. The *Hilbert polynomial* of $\mathcal{E}$ with respect to $\mathcal{L}$ is a polynomial $P(\mathcal{E},\mathcal{L}) \in \mathbb{Q}[t]$ such that for sufficiently large $l \in \mathbb{N}$,

$$P(\mathcal{E},\mathcal{L},l) = \chi(\mathcal{E}\otimes\mathcal{L}^{\otimes l}) = h^0(X,\mathcal{E}\otimes\mathcal{L}^{\otimes l}) - h^1(X,\mathcal{E}\otimes\mathcal{L}^{\otimes l}).$$

Remark 3.3.9 1. As by Serre's vanishing theorem, higher cohomology group

$H^1(X, \mathcal{E} \otimes \mathcal{L}^{\otimes l})$ vanishes for sufficiently large l (cf. [Ha, Chapter III, Theorem 5.2]), therefore we have:

$$P(\mathcal{E}, \mathcal{L}, l) = h^0(X, \mathcal{E} \otimes \mathcal{L}^{\otimes l}).$$

2. We simply denote $P(\mathcal{E}, \mathcal{L}, l)$ by $P(l)$ when the sheaf and the line bundle involved is clear from the context.

Let $\mathcal{G}$ be any quotient of $\mathcal{F}$ having rank r and degree d. Then $\mathcal{G}$ has the Hilbert polynomial $P(t)$ with respect to a degree 1 line bundle $\mathcal{O}(1)$ which is given by,

$$P(t) = rt + d + r \cdot (1 - g). \tag{3.19}$$

As the curve C is given beforehand, its genus g is fixed and therefore $P(t)$ is dependent on r and d only. Hence the polynomial $P(t)$ as in (3.19) is denoted by $[r, d]$. Let $\mathrm{Quot}_{\mathcal{F}}^{P}$ or $\mathrm{Quot}_{\mathcal{F}}^{[r,d]}$ denote the Quot scheme parametrizing the quotients of $\mathcal{F}$ having Hilbert polynomial $P(t) = [r, d]$, then we have

$$\mathrm{Quot}_{\mathcal{F}}^{P(t)} = \mathrm{Quot}_{\mathcal{F}}^{[r,d]} = \mathrm{Quot}_{\mathcal{F}}^{r,d}.$$

Moreover, for $r = 0$ we have,

$$\mathrm{Quot}_{\mathcal{F}}^{[0,d]} = \mathrm{Quot}_{\mathcal{F}}^{0,d} = \mathrm{Quot}_{\mathcal{F}}^{d}. \tag{3.20}$$

Remark 3.3.10 The above equalities in (3.20) give an interpretation of $\mathrm{Quot}_{\mathcal{F}}^{d}$ in two apparently different ways. One can think of d as an integer and then the Quot scheme $\mathrm{Quot}_{\mathcal{F}}^{d}$ can be interpreted as the scheme parametrizing all the torsion quotients of $\mathcal{F}$ having degree d. Also considering d as the constant Hilbert polynomial, $\mathrm{Quot}_{\mathcal{F}}^{d}$ can be interpreted as the scheme parametrizing all the quotients of $\mathcal{F}$ having constant Hilbert polynomial d. Both are essentially the same. This is not at all surprising as the degree of the Hilbert polynomial of a sheaf $\mathcal{G}$ equals dim Supp $\mathcal{G}$. Here the sheaf $\mathcal{G}$ being a torsion sheaf is supported on finitely many closed points and hence degree of the Hilbert polynomial of $\mathcal{G}$ is zero. Therefore it is a constant polynomial and from (3.19) we get that this is exactly equal to d.

The following theorem is about the smoothness of a Quot scheme in case of constant Hilbert polynomial. We have discussed the proof of this theorem already in Section 1.7. We mention the statement again for the sake of continuity.

Theorem 3.3.11 *Let C be a non-singular projective curve and let n be any non-negative integer. Then* $\mathrm{Quot}^n_{\mathcal{O}_C^r}$ *is a smooth projective scheme.*

Proof. See Theorem 1.7.1. □

3.3.3 Moduli space construction: Stable and semistable bundles

Fix a line bundle $\mathcal{O}(1)$ of degree 1 over C. Choose an $m >> 0$ such that any semistable vector bundle E over C of rank n and degree d is *m-regular*, that is, $H^i(C, E(m-i)) = 0$ for all $i > 0$.

In particular, we have (cf. [Hu-Ln, Chapter 1, §1.7, Lemma 1.7.2]):

1. $h^1(C, E(m)) = 0$.
2. $h^0(C, E(m)) = d + mn + n(1-g) =: N$.
3. The natural map $H^0(C, E(m)) \otimes \mathcal{O} \to E(m)$ is surjective.

Let us denote the Quot Scheme $\mathrm{Quot}^{n,d+mn}_{\mathcal{O}^N}$ simply by $\mathcal{Q}$. Let

$$\mathcal{O}^N_{C\times\mathcal{Q}} \to \mathcal{F}$$

be the universal quotient.

Note that the group scheme GL_N acts on $\mathcal{Q}$ in the following manner:

Let T be an algebraic scheme over $\mathbb{C}$.

Let $g \in \mathrm{GL}_N(T)$ be an automorphism $\mathcal{O}^N_{C\times T} \xrightarrow{g} \mathcal{O}^N_{C\times T}$. Let $[\mathcal{O}^N_{C\times T} \to F_T] \in \mathcal{Q}(T)$.

Then, define

$$g.[\mathcal{O}^N_{C\times T} \to F_T] := [\mathcal{O}^N_{C\times T} \xrightarrow{g} \mathcal{O}^N_{C\times T} \to F_T]$$

It is clear that this action in fact factors through an action of the group scheme PGL_N.

We now have the main theorem of this chapter about the spaces parametrizing stable and semistable bundles over a curve X.

Proposition 3.3.12 *Let C be any smooth irreducible projective curve over complex numbers. Then there exists a coarse moduli space $\mathcal{M}^s_C(n,d)$, also denoted by $\mathcal{M}^s(n,d)$, for stable bundles of rank n and degree d. Moreover, the natural compactification of $\mathcal{M}^s(n,d)$ is the moduli space $\mathcal{M}_C(n,d)$ of semistable bundles of rank n and degree d, also denoted by $\mathcal{M}(n,d)$ and is in fact a projective variety.*

Proof.See [Ne 1, Chapter 5, §4] or [Hu-Ln, §4.3]. Let us give a brief outline of the proof. Let $\mathcal{R} \subseteq \mathcal{Q}$ be the open subset such that for all $x \in \mathcal{R}$,

1. $\mathcal{F}|_{C\times x}$ is a semistable bundle,
2. $H^0(C, \mathcal{O}^N) \to H^0(C, \mathcal{F}|_{C\times x})$ is an isomorphism.

It is immediate that $\mathcal{R}$ is PGL_N-equivariant. Then, for $d > r(2g-1)$, we construct $\mathcal{M}_C(n,d)$ as the following good quotient:

$$\mathcal{M}_C(n,d) := \mathcal{R}//\mathrm{PGL}_N.$$

and we have the quotient map

$$\mu\colon \mathcal{R} \to \mathcal{M}_C(n,d). \tag{3.21}$$

Let $\mathcal{R}^s \subseteq \mathcal{R}$ be subset such that for all $x \in \mathcal{R}^s$,

1. $\mathcal{F}|_{C\times x}$ is a stable bundle,
2. $H^0(C, \mathcal{O}^N) \to H^0(C, \mathcal{F}|_{C\times x})$ is an isomorphism.

Restricting μ as in (3.21) to $\mathcal{R}^s$, we obtain the geometric quotient

$$\mu|_{\mathcal{R}^s}\colon \mathcal{R}^s \to \mathcal{M}^s_C(n,d),$$

that is,

$$\mathcal{M}^s_C(n,d) := \mathcal{R}^s//\mathrm{PGL}_N.$$

Moreover, $\mathcal{R}^s$ parametrizes a family of stable vector bundles over C of rank n and degree d having local universal property. Also two k-points of $\mathcal{R}^s$ lie in the same orbit if and only if the vector bundles parametrized by these points are isomorphic. Therefore $\mathcal{M}_C^s(n,d)$ is a coarse moduli space by Proposition 3.3.4. For $d \leq r(2g-1)$, this holds too as tensoring by a line bundle of fixed degree is an isomorphism and does not affect semistability or stability. □

Remark 3.3.13 Putting $n = 1$ and $d = 0$ in Proposition 3.3.12, we immediately get that the Jacobian variety $J(X)$ is a coarse moduli space.

We end this chapter by mentioning the importance of S-equivalence in the following proposition.

Proposition 3.3.14 *Two semistable bundle F and F_1 determine the same point of the moduli space $\mathcal{M}(n,d)$ if and only if $\mathrm{Gr}(F) \cong \mathrm{Gr}(F_1)$. Hence the space $\mathcal{M}(n,d)$ can also be interpreted as the moduli space of polystable bundles of rank n and degree d.*

Proof. See [Ne 1, Complement 5.8.1]. Indeed, the first part follows from the fact that the orbit of $\mathrm{Gr}(F)$ is contained in the closure of the orbit of the semistable bundle F of rank n and degree d. The second part follows from Definition 3.2.3 and Remark 3.2.5. □

Chapter 4

Brill-Noether loci and tautological algebra of semistable bundles over elliptic curve

The moduli space of semistable bundles over an elliptic curve is identified with the symmetric product of the curve itself. On the other hand, the corresponding fixed determinant moduli space is isomorphic to projective space. Both these facts are well known due to Tu (cf. [Tu]). Moreover, the Brill-Noether loci inside these moduli spaces are thoroughly described (cf. [Tu, Section 4]). In this chapter, we study the algebra generated by the cohomology classes of Brill-Noether subvarieties and relations between them. Our problem is motivated by Poincaré relation on a Jacobian variety of a smooth projective curve of genus g over complex numbers.

Atiyah (cf. [At]) classified the indecomposable bundles over an elliptic curve completely. In Section 4.1 we recall indecomposable bundles over an elliptic curve and the moduli space of rank n degree d semistable bundles followed by that. Given a fixed line bundle L over an elliptic curve, we also describe the moduli space of semistable bundles of rank n and degree d whose determinant is L. In Section 4.2 we discuss Brill-Noether subvarieties inside the moduli of semistable bundles and show that in degree 0 we get an interesting stratification of those special subvarieties. Finally in Section 4.3 we define the tautological algebra as the algebra generated by some Brill-Noether loci and find the relations amongst the cohomology classes of Brill-Noether subvarieties of the moduli space of semistable bundles over an elliptic curve. We obtain results similar to the

Poincaré relation on a Jacobian variety.

4.1 Structure of moduli space of semistable bundles over elliptic curve

In this section we first describe the indecomposable bundles over elliptic curve, given by Atiyah in [At]. Then we go through the structure of an arbitrary semistable bundle of rank n and degree d modulo S-equivalence. We describe the moduli space of semistable bundles over elliptic curve followed by that. Finally, we explain the corresponding fixed determinant moduli space as a consequence of that. Results and proofs in this section are mainly taken from [Tu] and [At].

Let us now recall that by E we denote an elliptic curve, that is, a smooth projective curve over complex numbers of genus 1 (cf. Section 1.8). Throughout this chapter we use this notation for an elliptic curve to differentiate it from other higher genus curve C.

4.1.1 Indecomposable bundles on elliptic curve

Here we discuss indecomposable bundles over elliptic curve completely which in turn will be required to describe the moduli space of semistable bundles over elliptic curves. We recall that *indecomposable bundles* are the bundles which can not be written as direct sum of two proper subbundles. The set of isomorphism classes of indecomposable bundles of rank n and degree d is denoted by $\mathrm{Ind}_E(n,d)$.

Atiyah described $\mathrm{Ind}_E(n,d)$ completely. In the process he constructed F_n, the unique line bundle in $\mathrm{Ind}_E(n,0)$ with $\Gamma(E,F_n) \neq 0$. The construction of F_n and a few properties are listed below as these are very important for our work.

Theorem 4.1.1 *[At, Theorem 5] For any $n \geq 1$, there exists a degree 0 vector bundle $F_n \in \mathrm{Ind}_E(n,0)$, unique upto isomorphism. Also F_1 is chosen to be the trivial line bundle $\mathcal{O}_E$ and F_n is defined inductively such that they satisfy the following exact sequence:*

$$0 \to \mathcal{O}_E \to F_n \to F_{n-1} \to 0.$$

Moreover, $h^0(F_n) = 1$.

Remark 4.1.2 1. By Theorem 4.1.1, F_1 and F_2 satisfy the following exact sequence:

$$0 \to \mathcal{O}_E \to F_2 \to F_1 \to 0.$$

As $F_1 \cong \mathcal{O}_E$, F_2 therefore satisfies

$$0 \to \mathcal{O}_E \to F_2 \to \mathcal{O}_E \to 0.$$

Hence $F_2 \cong I_2$, where I_2 is the trivial bundle of rank 2.

2. These indecomposable bundles of type rank n and degree 0 provide a plenty of examples of the fact that two isomorphic bundles may have different number of independent global sections. For example, $F_2 \cong I_2$, but $h^0(F_2) = 1 \neq 2 = h^0(I_2)$.

For all $n, d \in \mathbb{Z}$, $n > 0$, Atiyah defined some special indecomposable bundles of rank n and degree d such that any indecomposable bundles can be written in terms of those canonical bundles. In this regard we have the following theorem.

Theorem 4.1.3 *[At, Theorem 7 and 10] Considering the elliptic curve E as an abelian variety with the chosen base point $p \in E$ as the zero element, we have the following.*

1. *Given any $n, d \in \mathbb{Z}$, $n > 0$, there exists a bundle $F_{\mathcal{O}(p)}(n, d)$ of rank n and degree d, such that any element F of $\mathrm{Ind}_E(n, d)$ is of the form*

$$F \cong F_{\mathcal{O}(p)}(n, d) \otimes L,$$

where L is a line bundle of degree 0.

2. *Let M be a degree 0 line bundle over E and $n_1 = \frac{n}{\gcd(n,d)}$. Then*

$$F_{\mathcal{O}(p)}(n, d) \otimes L \cong F_{\mathcal{O}(p)}(n, d) \otimes M \iff (L \otimes M^{-1})^{n_1} = \mathcal{O}_E.$$

3. $\det F_{\mathcal{O}(p)}(n, d) = \mathcal{O}(p)^d$.

4. $F_{\mathcal{O}(p)}(n, 0) \cong F_n$.

The bundle $F_{\mathcal{O}(p)}(n, d)$ as in Theorem 4.1.3 are often called *canonical indecomposable bundle.*

Definition 4.1.4 A line bundle L of degree zero is called an *n-torsion point of* $J(E)$ if $L^{\otimes n} = \mathcal{O}_E$. By Tor_n we denote the subgroup of $J(E)$ of all n- torsion points.

The following proposition follows directly from Theorem 4.1.3.

Proposition 4.1.5 *Consider the following map from the Jacobian of E to the moduli space* $\mathrm{Ind}_E(n,d)$.

$$\begin{aligned} \varphi\colon J(E) &\to \mathrm{Ind}_E(n,d) \\ L &\mapsto F_{\mathcal{O}(p)}(n,d) \otimes L. \end{aligned}$$

Then the following holds.

1. *φ is onto.*
2. *Fiber of φ is isomorphic to* Tor_{n_1}.
3. $\mathrm{Ind}_E(n,d) \cong \frac{J(E)}{\mathrm{Tor}_{n_1}}$.

Now we can describe $\mathrm{Ind}_E(n,d)$ modulo the following proposition.

Proposition 4.1.6 *The Jacobian variety $J(E)$ of E is isomorphic to* $\frac{J(E)}{\mathrm{Tor}_{n_1}}$.

Proof. Consider the map

$$\begin{aligned} \otimes n_1\colon J(E) &\to J(E) \\ L &\mapsto L^{\otimes n_1}. \end{aligned}$$

Then the following exact sequence

$$0 \longrightarrow \mathrm{Tor}_{n_1} \longrightarrow J(E) \xrightarrow{\otimes n_1} J(E) \longrightarrow 0$$

gives the required isomorphism. □

Theorem 4.1.7 *The moduli space $\mathrm{Ind}_E(n,d)$ can be identified with the curve E.*

Proof. See [At, Theorem 7]. Indeed, as $J(E) \cong E$ by Proposition 1.8.4, this follows from Proposition 4.1.5 and 4.1.6. □

4.1.2 Semistable bundles over an elliptic curve

In this section we describe the structure of the moduli space of semistable bundles over an elliptic curve. Following proposition asserts that in case of elliptic curve, stability of a bundle implies semistability which factors through indecomposability. Also all the notions of stability, semistability, indecomposability and being simple coincide when rank and degree of a bundle are relatively prime.

Proposition 4.1.8 *Any indecomposable bundle on an elliptic curve is semistable. Moreover, it is stable if and only if its degree and rank are coprime.*

Proof.See [Tu, Appendix A]. Indeed by Theorem 4.1.3, any indecomposable bundle F of rank n and degree d over E can be written as a product of a canonical indecomposable bundle $F_{\mathcal{O}(p)}(n,d)$ and a line bundle of degree zero. Then by [Tu, Lemma 29] , the bundle $F_{\mathcal{O}(p)}(n,d)$ is semistable. Therefore F is also semistable as tensor product by a line bundle does not effect the semistability. Rest follows from the discussion before (3.13). $\square$

Therefore we obtain the following diagram of some implications part of which we got earlier in (3.9), (3.13) and (3.14).

$$\begin{tikzcd}
\text{Stable} \arrow[r, Rightarrow] & \text{Simple} \arrow[r, Rightarrow] & \text{Indecomposable} \arrow[dl, Rightarrow, "\text{in genus one}"] \\
& \text{Semistable} \arrow[ul, Rightarrow, "\text{gcd (rank,deg)=1}"] &
\end{tikzcd}$$

The following theorem describes the structure of an arbitrary semistable bundle over elliptic curve in terms of canonical indecomposable bundles.

Theorem 4.1.9 *Let $n \geq 1, d$ be any two integer. Let $n = hn_1$ and $d = hd_1$, where n_1 and d_1 are coprime. Then, any semistable bundle over E of rank n and degree d is S-equivalent to a bundle of the form*

$$F_{\mathcal{O}(p)}(n_1, d_1) \otimes \bigoplus_{i=1}^{h} L_i, \tag{4.1}$$

L_i being degree zero line bundles, determined upto multiplication by an element of Tor_{n_1} of $J(E)$.

Proof. See [Tu, Theorem 16]. Indeed, let F be any semistable bundle over E and

$$0 = W_0 \subseteq W_1 \cdots \subseteq W_t = F$$

be a Jordan-Hölder filtration of F. Then denoting $\frac{W_i}{W_{i-1}}$ by F_i for all $i = 1, \cdots, t$; by Remark 3.2.5, we have

$$F \sim \bigoplus_{i=1}^{t} F_i. \tag{4.2}$$

Also for all i, we have,

$$\mu_{F_i} = \mu_F = \frac{n}{d} = \frac{n_1}{d_1}.$$

Now as all F_i's are stable, by Proposition 4.1.8, we get that F_i's are of rank n_1 and degree d_1. Now equating rank and degree in (4.2), we get that there are h many terms in the that equation, that is (4.2) can be rewritten as

$$F \sim \bigoplus_{i=1}^{h} F_i. \tag{4.3}$$

As stable bundles are indecomposable, by Theorem 4.1.3 and (4.3) we obtain

$$F \sim F_{\mathcal{O}(p)}(n_1, d_1) \otimes \bigoplus_{i=1}^{h} L_i,$$

L_i being degree zero line bundles, determined upto multiplication by an element of Tor_{n_1} of $J(E)$. $\square$

Using Theorem 4.1.9, we can get the structure of $\mathcal{M}_E(n, d)$, the moduli space of S-equivalence classes of semistable bundles over elliptic curve E which we discuss in the next section.

4.1.3 Moduli space of semistable bundles on elliptic curve

In this section we give the structure of $\mathcal{M}_E(n, d)$. In the process we describe $\mathcal{SU}_E(n, L)$, the moduli space of S-equivalence classes of semistable bundles of rank n and fixed determinant L of degree d over E. This is due to [Tu].

Theorem 4.1.10 *Let $n \geq 1, d$ be any two integer and $h = \gcd(n, d)$. Then the moduli space $\mathcal{M}_E(n, d)$ is isomorphic to $S^h(E)$.*

Proof.See [Tu, Theorem 16]. Indeed, let $n = hn_1$ and $d = hd_1$. By (4.1) an arbitrary element F of $\mathcal{M}_E(n,d)$ is S-equivalent to $F_{\mathcal{O}(p)}(n_1,d_1) \otimes \bigoplus_{i=1}^h L_i$. Observe that as L_i is a bundle of rank 1 and degree 0 and $\mathcal{O}(p)$ is a bundle of rank 1 and degree 1, $L_i^{n_1} \otimes \mathcal{O}(p)$ is a bundle of rank 1 and degree 1. Therefore $\sum_{i=1}^h L_i^{n_1} \otimes \mathcal{O}(p) \in S^h J_1(E)$. Consider the map

$$\begin{aligned} f \colon \mathcal{M}_E(n,d) &\to S^h(J_1(E)) \\ F_{\mathcal{O}(p)}(n_1,d_1) \otimes \bigoplus_{i=1}^h L_i &\mapsto \sum_{i=1}^h L_i^{n_1} \otimes \mathcal{O}(p). \end{aligned}$$

This map is an isomorphism. Therefore, the theorem follows from Proposition 1.8.3. □

Remark 4.1.11 Previously, we have observed the importance of S-equivalence in Proposition 3.3.14. Now, Theorem 4.1.10 depicts the reason for considering the moduli space $\mathcal{M}_E(n,d)$ as the collection of all semistable bundles of rank n and degree d over E modulo the S-equivalence instead of that collection only.

Now fixed determinant moduli space $\mathcal{SU}_E(n,L)$ can be described immediately modulo the following proposition.

Proposition 4.1.12 *Let $n \geq 1, d$ be any two integer and $h = \gcd(n,d)$. Then the Abel-Jacobi map $\varphi_h \colon S^h(E) \to J_h(E)$ can be identified with the determinant map $\det \colon \mathcal{M}_E(n,d) \to J_d(E)$.*

Proof.See [Tu, Theorem 2]. Indeed, using part (3) of Theorem 4.1.3 and by Theorem 4.1.10 we get the following commutative diagram.

$$\begin{tikzcd} \mathcal{M}_E(n,d) \arrow[r, "\cong"] \arrow[d, "\det"'] & S^h(E) \arrow[d, "\varphi_h"] \\ J_d(E) \arrow[r, "\cong"'] & J_h(E) \end{tikzcd} \tag{4.4}$$

Hence the proposition follows. □

Theorem 4.1.13 *Let $n \geq 1, d$ be any two integer and $h = \gcd(n,d)$. Let L be a line bundle of degree d over E. Then the fixed determinant moduli space $\mathcal{SU}_E(n,L)$ is isomorphic to $\mathbb{P}^{h-1}$.*

Proof. See [Tu, Theorem 3]. Indeed, the fiber of the map $\det\colon \mathcal{M}_E(n,d) \to J_d(E)$ over $L \in J_d(E)$ is nothing but $\mathcal{SU}_E(n,L)$. Theorem now follows from the commutative diagram 4.4 in Proposition 4.1.12 and Remark 1.7.4. □

4.2 Brill-Noether loci over elliptic curve

In this section, we discuss about some special subvarieties of the moduli space of semistable bundles over elliptic curve called *Brill-Noether* subvarieties. These subvarieties came into picture while studying semistable bundles over any curve whose space of global sections is varying. Here we work over elliptic curve. This section is also taken from [Tu], where he described all possible Brill-Noether subvarieties in a systematic manner.

We begin with two definitions.

Definition 4.2.1 A vector bundle F over E is said to be special if $h^1(F) \neq 0$. Otherwise it is said to be a non-special bundle.

It is easy to observe that for non-special bundles, $h^0(E)$ is fixed by Riemann-Roch theorem. Therefore, we are concerned about special bundles. Broadly, Brill-Noether subvarieties lie inside the locus of special bundles and possess a given number of independent global sections.

4.2.1 Bundles with positive degree

Here we work inside semistable bundles over elliptic curves of positive degree.

Lemma 4.2.2 *Any semistable bundle F of positive degree over E is non-special, that is, $h^1(F) = 0$.*

Proof. See [Tu, Lemma 17, p. 13]. Indeed, let F be a semistable vector bundle of degree $d > 0$ and K_E be the canonical line bundle over E. Then $K_E \cong \mathcal{O}_E$ by Remark 1.4.3. By Serre duality we have

$$h^1(F) = h^0(K_E \otimes F^*) = h^0(F^*).$$

As F^* is also a semistable bundle and of negative degree, $h^0(F^*) = 0$. Therefore, $h^1(F) = 0$. Moreover by Riemann-Roch theorem, $h^0(F) = d$. □

Remark 4.2.3 A consequence of Lemma 4.2.2 is that the map

$$h^0 \colon \mathcal{M}_E(n,d) \longrightarrow \mathbb{Z}_+ \cup \{0\}$$
$$f \mapsto h^0(F)$$

is well defined for $d > 0$, and is the constant function d.

Definition 4.2.4 Let $d > 0$ and $i \geq 0$ be any two integer. The *Brill-Noether loci* are defined by

$$W^i_{n,d} := \left\{ f \in \mathcal{M}_E(n,d) \,|\, h^0(F) \geq i+1 \right\}.$$

This definition is well defined by Remark 4.2.3.

The following lemma is a direct consequence of Lemma 4.2.2 (cf. [Tu, p. 13]).

Lemma 4.2.5 *Let $d > 0$. Then*

$$W^i_{n,d} \cong \begin{cases} \emptyset & \text{if } 1 \leq d \leq i; \\ \mathcal{M}_E(n,d) & \text{if } d \geq i+1. \end{cases}$$

Therefore Brill-Noether loci inside $\mathcal{M}_E(n,d)$ are not of much interest when $d > 0$.

4.2.2 Degree zero bundles

For degree 0 line bundles over E we have the following result.

Lemma 4.2.6 *The Brill-Noether loci for $d = 0$, $r = 1$ are*

$$W^i_{1,0} \cong \begin{cases} \emptyset & \text{if } 1 \leq i; \\ \{\mathcal{O}_E\} & \text{if } i = 0. \end{cases}$$

Proof. See [Tu, p. 13]. As $h^0(L) = 0$ or 1 for a line bundle L of degree zero over E and moreover $h^0(L) = 1$ if and only if $L \cong \mathcal{O}_E$. □

Therefore in this case also stratifications of Brill-Noether subvarieties is nothing non-trivial. So, only case remains to check is for bundles of rank n and degree d with $n \geq 2$ and $d = 0$.

Remark 4.2.7 Unlike $d > 0$ (cf. Remark 4.2.3), $h^0 \colon \mathcal{M}_E(n,0) \longrightarrow \mathbb{Z}_+ \cup \{0\}$ is not well defined when $d = 0$. For example, let F_2 be the *Atiyah's indecomposable* *bundle* of rank 2 and I_2 be the trivial bundle of rank 2. Then $F_2 \cong I_2$, but $h^0(F_2) = 1 \neq 2 = h^0(I_2)$. See Remark 4.1.2 for more details.

In this case, because of Remark 4.2.7, two types of Brill-Noether loci are defined inside $\mathcal{SU}_E(n, L)$ and $\mathcal{M}_E(n, 0)$. We denote by f the S-equivalence class of a semistable bundle F over E.

Definition 4.2.8 Let $n \geq 2$ be any integer. The *Brill-Noether loci in* $\mathcal{M}_E(n, 0)$ are denoted by $W^i_{n,0}(\forall)$ and $W^i_{n,0}(\exists)$, and are defined as

$$\begin{aligned} W^i_{n,0}(\forall) &:= \left\{ f \in \mathcal{M}_E(n,0) \,|\, h^0(F) \geq i+1 \ \text{ for all } F \in f \right\}, \\ W^i_{n,0}(\exists) &:= \left\{ f \in \mathcal{M}_E(n,0) \,|\, h^0(F) \geq i+1 \ \text{ for some } F \in f \right\}. \end{aligned}$$

Similarly we also need to define two types of Brill-Noether loci inside the moduli space $\mathcal{SU}_E(n, L)$, where L is line bundle over E of degree 0.

Definition 4.2.9 Let $n \geq 2$ be any integer. Let L be a line bundle over E of degree 0. Then, the *Brill-Noether loci in* $\mathcal{SU}_E(n, L)$ are denoted by $W^i_{n,L}(\forall)$ and $W^i_{n,L}(\exists)$, and are defined as

$$\begin{aligned} W^i_{n,L}(\forall) &:= \left\{ f \in \mathcal{SU}_E(n,L) \,|\, h^0(F) \geq i+1 \text{ for all } F \in f \right\}, \\ W^i_{n,L}(\exists) &:= \left\{ f \in \mathcal{SU}_E(n,L) \,|\, h^0(F) \geq i+1 \text{ for some } F \in f \right\}. \end{aligned}$$

We have,

$$W^i_{n,L}(\forall) = W^i_{n,0}(\forall) \cap \mathcal{SU}_E(n, L) \text{ and } W^i_{n,L}(\exists) = W^i_{n,0}(\exists) \cap \mathcal{SU}_E(n, L).$$

We now describe the Brill-Noether loci just defined. For that we require the following propositions. Let $SS^0(E)$ denote the collection of all semistable bundles over E of degree 0 and $F \in SS^0(E)$. Then we define

$$\widetilde{F} := \left\{ F' \in SS^0(E) \,|\, F \sim F' \right\}.$$

Then we have:

Proposition 4.2.10 *Consider the map*

$$\begin{aligned} h^0 \colon SS^0(E) &\to \mathbb{Z}^+ \cup \{0\} \\ F &\mapsto h^0(F). \end{aligned}$$

Then for any F, $\widetilde{F}$ is either contained in $(h^0)^{-1}(0)$ or ${h^0}^{-1}(\mathbb{Z}^+)$ depending on whether $h^0(F) = 0$ or not. Moreover, for any F with $h^0(F) \neq 0$, there exists $F' \in \widetilde{F} \cap {h^0}^{-1}(\mathbb{Z}^+)$ satisfying $h^0(F') = 1$.

Proof.See [Tu, Lemma 18]. Indeed, the indecomposable bundle F_n as in Theorem 4.1.1 plays an important role in the proof. For proving the first part, we need the fact that all Jordan-Hölder factors of F_n are isomorphic to the trivial line bundle. To prove the second part, we need $h^0(F_n) = 1$. □

Proposition 4.2.11 *Let F be a semistable bundle of degree 0 over E. Then most number of independent global sections are possessed by the direct sum of line bundles among all the elements of $\widetilde{F}$.*

Proof.See [Tu, Lemma 19]. Indeed, firstly we decompose F into its indecomposable factors and collect the factors having global sections together. That is, we write

$$F \cong F_{k_1} \oplus \cdots F_{k_t} \oplus H,$$

with $h^0(H) = 0$. Then clearly $h^0(F) = t$. If $\sum_{i=1}^t k_i = n$, then by (4.1) and following the same argument as in (1) of 4.1.2 we can write

$$F \sim I_n \oplus \sum L_j, \tag{4.5}$$

where L_j are some non-trivial line bundles over E. From that we obtain

$$h^0\left(I_n \oplus \sum L_j\right) = k \geq h^0(F).$$

Moreover if $F' \sim F$, then by transitivity of S-equivalence and by (4.5) we have $F' \sim I_n \oplus \sum L_j$ and therefore $k \geq h^0(F')$ as before. Hence the proposition follows. □

Finally we end this section by describing the structure of the Brill-Noether loci.

Theorem 4.2.12 *[Tu, Theorem 4 and 5] Let $n \geq 2$ be any integer and L be a degree zero line bundle over E. Then the structure of the Brill-Noether Loci $W^i_{n,0}(\forall)$ and $W^i_{n,L}(\forall)$ as in Definition 4.2.8 and 4.2.9 are given by*

$$W^i_{n,0}(\forall) \cong \begin{cases} \emptyset & \text{if } 1 \leq i; \\ S^{n-1}(E) & \text{if } i = 0. \end{cases}$$

$$W^i_{n,L}(\forall) \cong \begin{cases} \emptyset & \textit{if } 1 \le i; \\ \mathbb{P}^{n-2} & \textit{if } i = 0. \end{cases}$$

Remark 4.2.13 We have observed that inside the moduli space of semistable bundles of rank n and degree d with $d > 0$ or $(n, d) = (1, 0)$, Brill-Noether loci do not produce any non-trivial stratification. Now Theorem 4.2.12 says that among two types of Brill-Noether loci in the moduli space of semistable bundles of rank n and degree d with $d = 0$ and $n \ge 2$, the first type again fails to do so.

So let us move to the last possible case in search of some non-trivial stratification of Brill-Noether loci and hence an interesting algebra generated by cohomology classes of them.

Theorem 4.2.14 *Let $n \ge 2$ be any integer and L be a degree zero line bundle over E. Then the structure of the Brill-Noether Loci $W^i_{n,0}(\exists)$ and $W^i_{n,L}(\exists)$ as in Definition 4.2.8 and 4.2.9 are given by*

$$W^i_{n,0}(\exists) \cong S^{n-i-1}(E),$$
$$W^i_{n,L}(\exists) \cong \mathbb{P}^{n-i-2}(E).$$

Proof. See [Tu, Theorem 4 and 5]. Indeed, note that $W^i_{n,0}(\exists) \subseteq \mathcal{M}_E(n, 0) \cong S^n(E)$ by Theorem 4.1.10. Moreover by Proposition 4.2.11, $f \in W^i_{n,0}(\exists)$ if and only if $i+1$ terms in $\sum L_j$ as in (4.5) are copies of trivial line bundles and the remaining $n-i-1$ terms are any degree 0 line bundles. Hence $W^i_{n,0}(\exists) \cong S^{n-i-1}(E)$. Now we have the following commutative diagram which is the restriction of the commutative diagram (4.4) to the Brill-Noether loci.

$$\begin{tikzcd} W^i_{n,0}(\exists) \arrow[r, "\cong"] \arrow[d, "\det"'] & S^{n-i-1}(E) \arrow[d, "\varphi_{n-i-1}"] \\ J(E) \arrow[r, "\cong"'] & J_{n-i-1}(E) \end{tikzcd} \tag{4.6}$$

Therefore by Remark 1.7.4,

$$W^i_{n,L}(\exists) \cong \mathbb{P}^{n-i-2}.$$

Hence the theorem. □

Theorem 4.2.14 demonstrates that finally we obtain some interesting stratifications consisting of Brill-Noether subvarieties in $\mathcal{M}_E(n,0)$ and $\mathcal{SU}_E(n,L)$, where $n \geq 2$ and L is a line bundle of degree 0 over E.

4.3 Tautological algebra and main theorems

The classical Poincaré formula opened up a whole new direction of problems in algebraic geometry. Later on, many mathematicians have worked on the problems of finding similar formula in many different contexts. In [Co], A. Collino proved that Poincaré formula holds in Jacobian of a hyperelliptic curve under algebraic equivalence. In 2003, P. E. Newstead proposed a project called *the Brill-Noether project* in his web page (cf. [Ne 2]) and suggested for similar problems in more general context. He mentioned that to deal with the detailed geometry of the Brill-Noether loci, one can study their classes in the cohomology ring of the moduli space. We work in similar kind of problem in the moduli space $\mathcal{SU}_E(n,L)$ and prove that Poincaré like formula holds over there.

In this section we define tautological algebra and tautological classes in our situation and explain the reason for choosing the name tautological algebra. Finally we close this chapter after proving the main theorems.

4.3.1 Why the name tautological algebra

Mathematicians like R. Vakil, C. Faber, R. Pandharipande, A. Pixton, T. Graber and many others have used the terminology *Tautological Algebra*. By tautological algebra they meant a subalgebra of either cohomology ring or Chow ring of some moduli space generated by classes of some naturally defined subvarieties. In [Mo], Morita studied the tautological algebra of $\mathcal{M}_g$, the moduli space of smooth projective curves of genus g, generated by some tautological classes inside the Chow ring of $\mathcal{M}_g$ defined by Mumford (cf. [Mu 2]). As Brill-Noether subvarieties are very natural by definition, it is quite natural to refer the subalgebra generated by their cohomology classes as tautological algebra. In [Be 1], by (rational) tautological ring Beauville meant the smallest subring stable under pullback maps and pushforward maps $(\otimes n)^*$ and $(\otimes n)_*$ respectively, induced by $\otimes n \colon J(X) \to J(X)$, closed under Pontryagin product of the rational Chow ring of $J(X)$ and contains classes of W_i, $1 \leq i \leq g$ as in (1.47). Here X denotes a connected, smooth, projective curve of genus g over $\mathbb{C}$. As we are concerned for Brill-Noether subvarieties inside moduli space of semistable bundles over elliptic curve, we define the algebra generated by their cohomology classes as tautological algebra and then prove our main theorems.

4.3.2 Main theorems

Now we define tautological algebra properly in our context and then prove our main theorems regarding the structure of those algebras and a few relations in those algebras.

Definition 4.3.1 The cohomology classes $[W^i_{n,0}(\exists)] \in H^*(\mathcal{M}_E(n,0),\mathbb{Z})$ are called the *tautological classes*. The subalgebra of $H^*(\mathcal{M}_E(n,0),\mathbb{Z})$ generated by these tautological classes is called the *tautological subalgebra* of $H^*(\mathcal{M}_E(n,0),\mathbb{Z})$.

Definition 4.3.2 Let L be a degree zero line bundle over E. Then the cohomology classes $[W^i_{n,L}(\exists)] \in H^*(\mathcal{SU}_E(n,L),\mathbb{Z})$ are called the *tautological classes*. The subalgebra of $H^*(\mathcal{SU}_E(n,L),\mathbb{Z})$ generated by these tautological classes is called the *tautological subalgebra* of $H^*(\mathcal{SU}_E(n,L),\mathbb{Z})$.

Following theorem shows that the tautological class $\zeta := [W^0_{n,L}(\exists)]$ is the generator of the tautological subalgebra of $H^*(\mathcal{SU}_E(n,L),\mathbb{Z})$.

Theorem 4.3.3 *Let r be any positive integer and let L be a degree 0 line bundle over E. Then $W^0_{n,L}(\exists)$ is a divisor inside $\mathcal{SU}_E(n,L)$. Furthermore, in $H^*(\mathcal{SU}_E(n,L),\mathbb{Z})$, we have*

$$[W^i_{n,L}(\exists)] = [W^0_{n,L}(\exists)]^{i+1},$$

for all $0 \leq i \leq n-2$ and the tautological algebra of $\mathcal{SU}_E(n,L)$ is $\mathbb{Z}[\zeta]/\langle \zeta^n \rangle$, where ζ is the cohomology class of $W^0_{n,L}(\exists)$ in $H^*(\mathcal{SU}_E(n,L),\mathbb{Z})$.

Proof. We have the following stratification inside $\mathcal{SU}_E(n,L)$ by Theorem 4.1.13

and 4.2.14.

$$
\begin{array}{ccccc}
\mathcal{SU}_E(n,L) & \cong & \mathbb{P}^{n-1} & & \\
\cup\vert & & \cup\vert & & \\
W^0_{n,L}(\exists) & \cong & \mathbb{P}^{n-2} & & \\
\cup\vert & & \cup\vert & & \\
W^1_{n,L}(\exists) & \cong & \mathbb{P}^{n-3} & & \\
\cup\vert & & \cup\vert & & \\
\cdot & & \cdot & & \\
\cdot & & \cdot & & \\
\cdot & & \cdot & & \\
\cup\vert & & \cup\vert & & \\
W^{n-3}_{n,L}(\exists) & \cong & \mathbb{P}^{1} & & \\
\cup\vert & & \cup\vert & & \\
W^{n-2}_{n,L}(\exists) & \cong & \mathbb{P}^{0} & \cong & \{\cdot\}
\end{array}
$$

So, $W^0_{n,L}(\exists)$ is a subvariety of $\mathcal{SU}_E(n,L)$ of codimension 1 and hence a divisor. We can calculate relations between $[\mathbb{P}^i]$'s as follows. Inside $\mathbb{P}^{n-1}$ we have the following stratification:

$$\{\cdot\} \subseteq \mathbb{P}^1 \subseteq \mathbb{P}^2 \subseteq \cdots \subseteq \mathbb{P}^{n-2} \subseteq \mathbb{P}^{n-1}.$$

Then we have:

$$H^*(\mathbb{P}^{n-1},\mathbb{Z}) = \frac{\mathbb{Z}[\zeta]}{\langle \zeta^n \rangle}, \tag{4.7}$$

where ζ is the cohomology class of $\mathbb{P}^{n-2}$, that is, $\zeta = [\mathbb{P}^{n-2}] = c_1(\mathcal{O}(1))$ by Proposition 1.5.5, $c_1(\mathcal{O}(1))$ being the first Chern class of $\mathcal{O}(1)$ over $\mathbb{P}^{n-1}$. Moreover in $H^*(\mathbb{P}^{n-1},\mathbb{Z})$, we have:

$$[\mathbb{P}^{n-1-k}] = \zeta^k. \tag{4.8}$$

Therefore, by (4.7) and Theorem 4.1.13 we get:

$$H^*(\mathcal{SU}_E(n,L),\mathbb{Z}) = \frac{\mathbb{Z}[\zeta]}{\langle \zeta^n \rangle}.$$

Furthermore in $H^*(\mathcal{SU}_E(n,L),\mathbb{Z})$, we get the following equality by (4.8) and Theorem 4.2.14 :

$$[W^i_{n,L}(\exists)] = [\mathbb{P}^{n-i-2}] = [\mathbb{P}^{n-1-(i+1)}] = \zeta^{i+1} = [\mathbb{P}^{n-2}]^{i+1} = [W^0_{n,L}(\exists)]^{i+1}.$$

Hence the theorem follows. □

The next theorem is about some relations between the generators of the tautological subalgebra of $H^*(\mathcal{SU}_E(n,L),\mathbb{Z})$ and $H^*(\mathcal{M}_E(n,0),\mathbb{Z})$. Consider the map (cf. [Bg-Tu, p. 338] and [Do-Tu, p. 348]):

$$\begin{aligned} \pi\colon J(E)\times \mathcal{SU}_E(n,L) &\to \mathcal{M}_E(n,d) \\ (l,F) &\mapsto l\otimes F. \end{aligned} \tag{4.9}$$

This map as in (4.9) suggests that the structure of the cohomology subalgebra generated by $[W^i_{n,0}(\exists)]$'s is similar to the algebra generated by $[W^i_{n,L}(\exists)]$'s with coefficients lying in $H^*(J(E))$. We make this precise in the next theorem. The theorem says that tautological algebra of $\mathcal{M}_E(n,0)$ is generated by the cohomology class of the Brill-Noether subvariety $W^0_{n,0}(\exists)$ as $H^*(E)$-algebra.

Theorem 4.3.4 *The tautological algebra of $\mathcal{M}_E(n,0)$ is*

$$H^*(E)\otimes\mathbb{Z}[\xi]/\langle\xi^n\rangle.$$

Here ξ is the cohomology class in $H^*(\mathcal{M}_E(n,0),\mathbb{Z})$ of the divisor $W^0_{n,0}(\exists)$ on $\mathcal{M}_E(n,0)$.

Proof. We have the following stratification inside $\mathcal{M}_E(n,0)$ by Theorem 4.1.10 and 4.2.14.

$$\begin{array}{ccccc}
\mathcal{M}_E(n,0) & \cong & S^n(E) & & \\
\cup\!\mid & & \cup\!\mid & & \\
W^0_{n,0}(\exists) & \cong & S^{n-1}(E) & & \\
\cup\!\mid & & \cup\!\mid & & \\
W^1_{n,0}(\exists) & \cong & S^{n-2}(E) & & \\
\cup\!\mid & & \cup\!\mid & & \\
\cdot & & \cdot & & \\
\cdot & & \cdot & & \\
\cdot & & \cdot & & \\
\cup\!\mid & & \cup\!\mid & & \\
W^{n-2}_{n,0}(\exists) & \cong & S^1(E) & \cong & E
\end{array}$$

So, $W^0_{n,0}(\exists)$ is a subvariety of $\mathcal{M}_E(n,0)$ of codimension 1 and hence a divisor. By

Proposition 4.1.12 and by Remark 1.7.4, the determinant morphism

$$\mathcal{M}_E(n,0) \to J(E)$$

is a projective bundle $\mathbb{P}^{n-1}_{J(E)} \to J(E)$.

Hence, by projective bundle formula as in Theorem 1.5.6,

$$H^*(\mathcal{M}_E(n,0),\mathbb{Z}) \ = \ H^*(J(E)) \otimes \mathbb{Z}[\xi]/\langle \xi^n \rangle. \tag{4.10}$$

Here ξ is the first Chern class of $\mathcal{O}(1)$ on $\mathbb{P}^{n-1}_{J(E)}$.

Therefore, by Proposition 1.8.4 and by (4.10) we get,

$$H^*(\mathcal{M}_E(n,0),\mathbb{Z}) \ = \ H^*(E) \otimes \mathbb{Z}[\xi]/\langle \xi^n \rangle.$$

However, by (4.6), we have the equality of the cohomology classes:

$$[W^i_{n,0}(\exists)] = \xi^{i+1},$$

for all $0 \le i \le n-2$. This gives the assertion.

$\square$

Remark 4.3.5 The cycle class map as defined in (2.7) is an isomorphism in case of only two curves, namely $\mathbb{A}^1$ and $\mathbb{P}^1$. The same is true for $\mathbb{P}^n$ for $n > 1$ as well. This happens because of vanishing of all odd dimensional cohomologies (cf. [Fu, Example 19.1.11]). So, using [Ei-Hr, Theorem 2.1] very much similar results can be obtained in the Chow ring $CH^*(\mathcal{SU}_E(n,L))$ and hence in $CH^*(\mathcal{M}_E(n,0))$ like we obtained in Theorem 4.3.3 and 4.3.4.

Chapter 5

Brill-Noether loci and tautological algebra for higher genus curve

In this chapter, we deal with a similar problem as in the previous chapter. We consider the tautological algebra of the (rational) cohomology ring of the moduli space of semistable bundles over a curve of genus g greater than one. We deal with the case when rank is 2 and degree is $2g-2$. We obtain relations in the tautological algebra of $H^*(\mathcal{M}_C(2, 2(g-1)), \mathbb{Q})$ as well as $H^*(\mathcal{SU}_C(2, L), \mathbb{Q})$, L being a line bundle of degree $2g-2$ over C.

A study of the Brill-Noether loci was first carried out on the Jacobian of curves by Fulton, Lazarsfeld, Griffiths and Harris. They contributed in answering the natural questions on these loci, namely non-emptiness, irreducibility, dimension, cohomological relations and understanding the singular loci. One can refer to [Fu-La] and [Gf-Hr 1] to look into their work in this direction.

On the moduli space of higher rank semistable vector bundles of fixed degree on a smooth curve, N. Sundaram (cf. [Su]) and M. Teixidor i Bigas (cf. [Bg 1]) initiated a similar study of the Brill-Noether loci, answered some of the interesting questions, and posed further questions. Notable results were obtained in [Bg 1], [Bg 2], [Br-Gz-Ne], [Me 1] and [Me 2]. More recent developments on non-emptiness of the Brill-Noether loci can be found in [La-Ne-St], [La-Ne-Pr], [La-Ne 1], [La-Ne 2] and [La-Ne 3]. We mention a few of those in this chapter that are relevant for our problem.

In this problem, the key idea is to relate the Brill-Noether loci on the moduli space $\mathcal{M}_C(2, 2(g-1))$ with the Brill-Noether loci on the Jacobian variety of a general spectral curve. We utilise a finite regular dominant morphism corresponding to the rational map obtained in [Be-Na-Ra] from the Jacobian of a general spectral curve $\widetilde{C}$ to the moduli space $\mathcal{M}_C(2, 2(g-1))$. The constructions used to prove the main results may give an insight for studying similar problems in other moduli spaces as well.

5.1 Spectral curves and Moduli Spaces

In this section we recall the construction of spectral curve from [Be-Na-Ra] as this is very much essential for our purpose.

5.1.1 Spectral curve

Let C be a smooth projective curve of genus $g \geq 2$ defined over complex numbers. Let L be a line bundle on C and $s = (s_k)$ be sections of L^k for $k = 1, 2, \cdots, n$. Let $\pi \colon \mathbb{P}(\mathcal{O} \oplus L^*) \to C$ be the natural projection map and $\mathcal{O}(1)$ be the relatively ample bundle. Then $\pi_*(\mathcal{O}(1))$ is naturally isomorphic to $\mathcal{O} \oplus L^*$ and therefore has a canonical section. This provides a section of $\mathcal{O}(1)$ denoted by y. By projection formula we have:

$$\pi_*(\pi^*L \otimes \mathcal{O}(1)) \cong L \otimes \pi_*(\mathcal{O}(1)) \cong L \otimes (\mathcal{O} \oplus L^*) = L \oplus \mathcal{O}.$$

Therefore $\pi_*(\pi^*L \otimes \mathcal{O}(1))$ also has a canonical section and we denote the corresponding section of $\pi^*L \otimes \mathcal{O}(1)$ by x. Consider the section

$$x^n + (\pi^*s_1)yx^{n-1} + \cdots + (\pi^*s_n)y^n \tag{5.1}$$

of $\pi^*L^n \otimes \mathcal{O}(n)$. Zero scheme of this section is a subscheme of $\mathbb{P}(\mathcal{O} \oplus L^*)$ and is called a *spectral curve* of the given curve C and is denoted by $\widetilde{C}_s$ or $\widetilde{C}$ in short. Let $\pi \colon \widetilde{C} \to C$ be the restriction of the natural projection $\pi \colon \mathbb{P}(\mathcal{O} \oplus L^*) \to C$. It can be checked that $\pi \colon \widetilde{C} \to C$ is finite and its fiber over any point $c \in C$ is a subscheme of $\mathbb{P}^1$ given by

$$x^n + a_1yx^{n-1} + \cdots + a_ny^n = 0,$$

where (x, y) is a homogeneous coordinate system and a_i is the value of s_i at c.

Let $\widetilde{g}$ be the genus of $\widetilde{C}$. As $\pi_*(\mathcal{O}) \cong \mathcal{O} \oplus L^{-1} \oplus \cdots \oplus L^{-(n-1)}$, we have the following relation between genus $\widetilde{g}$ of the spectral curve $\widetilde{C}$ and genus g of C using Riemann-Roch theorem.

$$1-\widetilde{g} = \chi(\widetilde{C}, \mathcal{O}) = \chi(C, \pi_*(\mathcal{O})) = \sum_{i=0}^{n-1} \chi(C, L^{-i}) = -(\deg L) \cdot \frac{n(n-1)}{2} + n(1-g).$$

Hence we have:

$$\widetilde{g} = (\deg L) \cdot \frac{n(n-1)}{2} + n(g-1) + 1. \tag{5.2}$$

Moreover if we take the line bundle L to be of degree $2g-2$, say the canonical line bundle K_C for example, then from (5.2) the genus $\widetilde{g}$ of the corresponding spectral curve $\widetilde{C}$ is given by:

$$\widetilde{g} = n^2(g-1) + 1 = \dim \mathcal{M}_C(n, d). \tag{5.3}$$

5.1.2 Spectral curve and moduli space of semistable bundles

Here we relate the spectral curve $\widetilde{C}$ with the moduli space of semistable bundles of fixed rank and degree over C. Consider the following theorem.

Theorem 5.1.1 *[Be-Na-Ra, Proposition 3.6, Remark 3.1, 3.5 & 3.8, p. 172-174] Let C be any curve and L any line bundle on C. Let $(s) = ((s_i)) \in \Gamma(L) \oplus \Gamma(L^2) \oplus \cdots \oplus \Gamma(L^n)$ be so chosen such that the corresponding spectral curve $\widetilde{C}_s$ is integral, smooth and non-empty. Then there is a bijective correspondence between isomorphism classes of line bundles on $\widetilde{C}_s$ and isomorphism classes of pairs (E, φ) where E is a vector bundle of rank n and $\varphi\colon E \to L \otimes E$ a homomorphism with characteristic coefficients s_i.*

Let n be any positive integer. Then following the construction of spectral curve, by Theorem 5.1.1 we get a smooth, irreducible curve $\widetilde{C}$ and an n-sheeted branched covering $\pi\colon \widetilde{C} \to C$ such that a general $E \in \mathcal{M}_C(n, d)$ is the direct image $\pi_* l$ of a $l \in J_\delta(\widetilde{C})$. The relation between δ and d can be calculated as follows (cf. [Bg-Tu, p. 332]). By the Leray spectral sequence we have:

$$H^i(\widetilde{C}, l) = H^i(C, \pi_* l) \tag{5.4}$$

for all i. Hence we have:

$$\chi(\widetilde{C}, l) = \chi(C, \pi_* l) = \chi(C, E).$$

So by Riemann-Roch theorem we get,

$$\begin{aligned} &\chi(\widetilde{C}, l) = \chi(C, E) \\ \Rightarrow &\delta - (\widetilde{g} - 1) = d - n(g-1) \\ \Rightarrow &\delta = d + (\widetilde{g} - 1) - n(g-1). \end{aligned}$$

Therefore by (5.3) we get the following relation between $\delta(=\deg\ l)$ and $d(=\deg\ E)$.

$$\delta = d + (n^2 - n)(g-1). \tag{5.5}$$

As direct image of a line bundle is not necessarily semistable, the map

$$\pi_* \colon J_\delta(\widetilde{C}) \dashrightarrow \mathcal{M}_C(n, d)$$

is only a rational map. Let us denote by J^{ss} the semistable locus of $J_\delta(\widetilde{C})$ defined as:

$$J^{ss} := \left\{ l \in J_\delta(\widetilde{C}) \,|\, \pi_* l \in \mathcal{M}_C(n, d) \right\}.$$

Then J^{ss} is a Zariski open subset of $J_\delta(\widetilde{C})$ and the map

$$\pi_* \colon J^{ss} \to \mathcal{M}_C(n, d) \tag{5.6}$$

is a regular dominant map (cf. [Be-Na-Ra, Theorem 1, p. 169]). Moreover, the following theorem shows that the map π_* is a finite map.

Theorem 5.1.2 *[Be-Na-Ra, Remark 5.4, p. 177] The map $\pi_* \colon J^{ss} \to \mathcal{M}_C(n, d)$, as in (5.6), is of degree $2^{3g-3} \cdot 3^{5g-5} \cdots n^{(2n-1)(g-1)}$.*

5.1.3 Prym variety associated to a spectral curve

In this section we consider the moduli space $\mathcal{M}_C(2, d)$. For a general $E \in \mathcal{M}_C(2, d)$, we get a spectral curve $\pi \colon \widetilde{C} \to C$ where the map π is a 2-sheeted branched covering. Let n be the number of branch points. Then by Riemann-

Hurwitz formula we get (cf. [Gf-Hr 2, p. 219]):

$$\widetilde{g} = \frac{n}{2} + 2g - 1. \tag{5.7}$$

Also we have from (5.3):

$$\widetilde{g} = 4g - 3. \tag{5.8}$$

Therefore from (5.7) and (5.8) we get,

$$n = 4g - 4 \neq 0 \text{ as } g \geq 2,$$

that is, $\pi \colon \widetilde{C} \to C$ is ramified with $4g-4$ branch points. Now we have the following lemma.

Lemma 5.1.3 *[Mu 1, Lemma, p. 332] The map $\pi^* \colon J(C) \to J(\widetilde{C})$ is injective.*

Consider the following *Norm map*, denoted by $\mathrm{Nm}(\pi)$, associated to the map $\pi \colon \widetilde{C} \to C$.

$$\begin{aligned} \mathrm{Nm}(\pi) \colon J(\widetilde{C}) &\to J(C) \\ \sum_{i=1}^{m} n_i \widetilde{x}_i &\mapsto \sum_{i=1}^{m} n_i \pi(\widetilde{x}_i). \end{aligned}$$

Identity component of $\mathrm{Ker}(\mathrm{Nm}(\pi))$ is defined to be the *Prym variety* associated to the covering $\pi \colon \widetilde{C} \to C$. But in our context the definition of Prym variety can be further improved. For that consider the following lemma.

Lemma 5.1.4 *[Ka, Lemma 1.1, p. 337] The following conditions are equivalent.*

1. *The map $\pi^* \colon J(C) \to J(\widetilde{C})$ is injective.*
2. *$\mathrm{Ker}(\mathrm{Nm}(\pi))$ is connected.*

So by Lemma 5.1.3 and 5.1.4, Prym variety associated to the covering $\pi \colon \widetilde{C} \to C$ is nothing but $\mathrm{Ker}(\mathrm{Nm}(\pi))$. Moreover we have, $J(\widetilde{C}) \cong J(C) + P$, where P is the Prym variety associated to the covering $\pi \colon \widetilde{C} \to C$. But this sum is not a direct sum as cardinality of $J(C) \cap P$ is a non-zero finite number. Let H be the kernel of the map

$$\begin{aligned} J(C) \times P &\to J(\widetilde{C}) \cong J(C) + P \\ (x, y) &\mapsto x + y. \end{aligned}$$

Then we have:

Theorem 5.1.5 *[Mu 1, Corollary 1, p. 332] The Jacobian $J(\widetilde{C})$ is isomorphic to $\frac{J(C)\oplus P}{H}$, where P is a Prym variety and H is a finite group. In other words, there exists an isogeny from $J(C) \times P$ to $J(\widetilde{C})$.*

5.2 Tautological algebra generated by the Brill-Noether loci on $J_d(C)$

In this section, we investigate the cohomology algebra generated by Brill-Noether subvarieties of $J(C)$ and $J_d(C)$. This problem is motivated by the classical Poincaré formula on $J(C)$.

5.2.1 Brill-Noether loci on $J(C)$ and $J_d(C)$

In this subsection we recall the Poincaré formula on $J(C)$ once again for sake of completeness.

Let us fix a point $P \in C$. Recall the classical Abel-Jacobi map $u\colon S^d(C) \longrightarrow J(C)$ as defined in Section 1.7.

Recall that W_d^0, for all d, $1 \leq d \leq g$, called Brill-Noether subvarieties of $J(C)$, was defined as follows:

$$W_d^0 := u(S^d(C)).$$

Let $\Theta := u(S^{g-1}(C))$. The classical Poincaré relation determines the relations between the cohomological classes of W_i^0 on $J(C)$ (cf. [Ab-Cr-Gf-Hr, Chapter 1, §5, p. 25]):

$$[W_i^0] = \frac{1}{(g-i)!}[\Theta]^{g-i} \in H^*(J(C), \mathbb{Q}).$$

5.2.2 Brill-Noether loci in $J_d(C)$

For a fixed d, we recall the Brill-Noether loci W_d^r, which are defined to be certain natural closed subschemes of $J_d(C)$ and discuss some of its properties relevant to us.

Definition 5.2.1 As a set, for $r \geq 0$, we define

$$W_d^r := \left\{ L \in J_d(C) \,|\, h^0(L) \geq r+1 \right\} \subseteq J_d(C).$$

It is clear from semicontinuity theorem (cf. [Ha, Chapter III, Theorem 12.8]) that W_d^r is closed. In fact, W_d^r has a natural scheme structure as determinantal loci (cf. [Ab-Cr-Gf-Hr, §4, Chapter II, p. 83]) of certain morphisms of vector bundles over $J_d(C)$. We define these morphisms as follows:

Let us fix a Poincaré bundle $\mathcal{L}$ over $C \times J_d(C)$. It can be noted that existence of such a Poincaré bundle assured by Proposition 1.6.14. Let E be an effective divisor on C with

$$\deg E = m \geq 2g - d - 1.$$

Let $\Gamma := E \times J_d(C)$. Then, over $C \times J_d(C)$ we have the exact sequence:

$$0 \to \mathcal{L} \to \mathcal{L}(\Gamma) \to \mathcal{L}(\Gamma)|_{\Gamma} \to 0. \tag{5.9}$$

Let v be the projection from $C \times J_d(C) \to J_d(C)$. Now, applying the functor v_* to the morphism $\mathcal{L}(\Gamma) \to \mathcal{L}(\Gamma)|_{\Gamma}$ as in (5.9), we get a morphism

$$\gamma := v_*(\mathcal{L}(\Gamma)) \to v_*(\mathcal{L}(\Gamma)|_{\Gamma}).$$

Note that, by the choice of the degree of E and Grauert's theorem (cf. [Ha, Chapter III, Corollary 12.9]), we get that both $v_*(\mathcal{L}(\Gamma))$ and $v_*(\mathcal{L}(\Gamma)|_{\Gamma})$ are vector bundles of rank $d + m - g + 1$ and m respectively.

Definition 5.2.2 The Brill-Noether loci W_d^r is defined to the $(m + d - g - r)$-th determinantal loci associated to the morphism γ.

To see that Definition 5.2.2 indeed agrees with Definition 5.2.1, in the sense that the set theoretic support of 5.2.2 is exactly 5.2.1, we refer to [Ab-Cr-Gf-Hr, Lemma 3.1, p. 178].

From general properties of determinantal loci, we have the following lemma:

Lemma 5.2.3 *[Ab-Cr-Gf-Hr, Lemma 3.3, p. 181] Suppose $r \geq d - g$. Then every component of W_d^r has dimension greater or equal to the Brill-Noether number*

$$\rho := g - (r + 1)(g - d + r).$$

Remark 5.2.4 Note that if $r \leq d - g - 1$, then by Riemann-Roch theorem $W_d^r = J_d(C)$. So, from here onwards, we will assume that $r \geq d - g$.

In general, the above inequality can be strict (cf. [Ab-Cr-Gf-Hr, Theorem 5.1, p. 191]). Even, in the case when equality holds, W^r_d can have more than one components (cf. [Ab-Cr-Gf-Hr, Chapter V, p. 208]).

We recall the following theorem due to Griffith and Harris.

Theorem 5.2.5 *[Gf-Hr 1, Main Theorem, p. 235] Let C be a smooth projective curve of genus g and let ρ be the Brill-Noether number. Then*

(a) $\dim W^r_d \geq \rho$.

(b) For a general curve C,

$$\dim W^r_d = \rho.$$

Furthermore,

$$[W^r_d] = \prod_{\alpha=0}^{r} \frac{\alpha!}{(g-d+r+\alpha)!}.\theta_d^{g-\rho}. \tag{5.10}$$

The formula as in (5.10) is called the Castelnuovo formula. Regarding the irreducibility, we have:

Theorem 5.2.6 *[Fu-La, Corollary 2.4, p. 280] If C is general and $\rho > 0$, then W^r_d is irreducible.*

Now, recall that in the case when C is general, by Theorem 2.3.1 we have that the Néron-Severi group $\mathrm{NS}(J_d(C))$ of $J_d(C)$ is generated by a translate of the Θ divisor in $J(C)$. We denote this class as θ_d. In particular, this implies that the class of W^r_d can be written in terms of powers of θ_d.

5.2.3 Tautological algebra generated by the Brill-Noether loci in $J(\widetilde{C})$

In this section we investigate the subalgebra of $H^*(J(\widetilde{C}), \mathbb{Q})$ generated by the Brill-Noether loci on $J(\widetilde{C})$. Towards this, we consider the case when we have a ramified double cover $\pi\colon \widetilde{C} \to C$.

Let $\mathcal{R}^r_g$ denote the moduli space of ramified two sheeted covering of a connected smooth projective curves of genus g with fixed ramification r. Then we have the following theorem.

Theorem 5.2.7 *[Bi, Corollary 5.3, p. 634] The Néron-Severi group of the Jacobian of a general element of $\mathcal{R}_g^r$ is generated by two elements; the two elements are obtained from the decomposition (up to isogeny) of the Jacobian of a covering curve (cf. Theorem 5.1.5). Furthermore, the Néron-Severi group generates the algebra of Hodge cycles (of positive degree) on the Jacobian of the general double cover.*

Note that even if C is general, $\widetilde{C}$ may not be general. However, in our situation, we will check that the above theorem still holds.

Theorem 5.2.8 *The cohomology class of a Brill -Noether locus on a Jacobian $J(\widetilde{C})$ of a general 2-sheeted spectral curve $\pi\colon \widetilde{C} \to C$ can be expressed as a sum of powers of divisor classes. In particular the tautological algebra is generated by the divisor classes.*

Proof. We only need to check that Theorem 5.2.7 can be applied to the Jacobian of a general spectral curve. Fix a degree $d > 0$. Denote $\mathcal{S}_{g,s}$ the moduli space of tuples

$$\{(C, L, s = (s_0, s_1)\},$$

where C is a curve of genus g, L is a line bundle on C of degree d, and $s_0 \in H^0(C, L)$, $s_1 \in H^0(C, L^2)$. This moduli space can be interpreted as the moduli space of spectral curves, as in § 5.1. There is a dominant rational map (on the component where $(s_0 = 0)$)

$$\mathcal{S}^0_{g,s} \to \mathcal{R}^r_g \to \mathcal{M}_g.$$

Here r is the ramification type corresponding to a general section s equivalently the zeroes of the equation (5.1) (cf. [Ba-Ci-Ve] also, for a similar moduli space). The maps are given by

$$(C, L, s) \mapsto (C, L, B(s)) \mapsto C,$$

where B is the branch divisor of the spectral curve $\tilde{C}_s \to C$, such that $L^2 = \mathcal{O}(B)$. Since $J(\tilde{C}_s)$ depends only the ramification type B and L, Theorem 5.2.7 can be applied to the Jacobian of a general spectral curve. □

5.3 Brill-Noether loci on $\mathcal{M}_C(n,d)$

To define the Brill-Noether loci for $\mathcal{M}_C(n,d)$, we start with a more general set up. Let S be an algebraic scheme over $\mathbb{C}$. Let $\mathcal{E}$ be a vector bundle over $C \times S$ such that for all $s \in S$, $\mathcal{E}_s := \mathcal{E}|_{C\times s}$ is a vector bundle of rank n and degree d over C.

Just as in 5.2.1, we have the following definition of Brill-Noether loci as a closed set.

Definition 5.3.1 We define Brill-Noether loci $W^r_{S,\mathcal{E}}$ associated to pair $(S,\mathcal{E})$ to be the set
$$W^r_{S,\mathcal{E}} := \left\{ s \in S \,|\, h^0(C,\mathcal{E}_s) \geq r+1 \right\}.$$

By [Hu-Ln, Lemma 1.7.6, p. 28], since the family $\mathcal{E}$ is a bounded family, we can choose a divisor D in C of sufficiently high degree such that $H^1(C,\mathcal{E}_s(D)) = 0$ for all $s \in S$. For notational convenience, we continue to denote the pullback of D to $C \times S$ by D. Then, over $C \times S$ we have the exact sequence:

$$0 \to \mathcal{E} \to \mathcal{E}(D) \to \mathcal{E}(D)|_D \to 0.$$

Let $v\colon C \times S \to S$ be the projection.

Then, we have the morphism

$$f\colon v_*(\mathcal{E}(D)) \to v_*(\mathcal{E}(D)|_D).$$

Now for any $s \in S$ we have $h^1(C,\mathcal{E}(D)_s) = h^1(C,\mathcal{E}_s(D)) = 0$. By Riemann-Roch theorem we get

$$\begin{aligned} h^0(C,\mathcal{E}(D)_s) =& d + n \deg D + n(1-g), \\ h^0(C,(\mathcal{E}(D)|_D)_s) =& \deg D. \end{aligned}$$

Hence, by [Ha, Chapter III, Theorem 12.11], we get that both $v_*(\mathcal{E}(D))$ and $v_*(\mathcal{E}(D)|_D)$ are vector bundles and for any $s \in S$, we have isomorphisms:

$$\begin{aligned} v_*(\mathcal{E}(D))|_s &\xrightarrow{\cong} H^0(C,\mathcal{E}|_{C\times s}(D)), \\ v_*(\mathcal{E}(D)|_D)|_s &\xrightarrow{\cong} H^0(C,\mathcal{E}|_{C\times s}(D)|_D) \end{aligned} \tag{5.11}$$

Using Riemann-Roch theorem, we get that

$$\begin{aligned} \text{rank } v_*(\mathcal{E}(D)) =& d + n \deg D + n(1-g), \\ \text{rank } v_*(\mathcal{E}(D)|_D) =& n \deg D. \end{aligned}$$

Definition 5.3.2 We define $W^r_{S,\mathcal{E}}$ to be the $(d+n \deg D+n(1-g)-(r+1))$-th determinantal loci associated to the morphism f.

Remark 5.3.3 To see that the set-theoretic support of 5.3.2 is indeed 5.3.1, note that we have the following commutative diagram:

$$\begin{tikzcd} & & v_*(\mathcal{E}(D))|_s \arrow[r, "f|_s"] \arrow[d, "\cong"] & v_*(\mathcal{E}(D)|_D)|_s \arrow[d, "\cong"] \\ 0 \arrow[r] & H^0(C,\mathcal{E}_s) \arrow[r] & H^0(C,\mathcal{E}_s(D)) \arrow[r] & H^0(C,\mathcal{E}_s(D)|_D) \end{tikzcd}$$

Hence,

$$\text{rank } f|_s \leq d + n \deg D + n(1-g) - (r+1) \iff h^0(C,\mathcal{E}_s) \geq r+1.$$

From this, it follows that definition 5.3.2 agrees with definition 5.3.1.

Lemma 5.3.4 *If $W^r_{S,\mathcal{E}} \neq \emptyset$, then, codimension of each component of $W^r_{S,\mathcal{E}} \leq (r+1)(r+1-d+n(g-1))$.*

Proof. This follows from [Ab-Cr-Gf-Hr, §4, Chapter II, p. 83]. □

Lemma 5.3.5 *Let S_1, S_2 be two algebraic schemes over $\mathbb{C}$ and let $\mathcal{E}$ be a bundle on $C \times S_2$ such that for all $s \in S_2$, $\mathcal{E}_s$ is a vector bundle of rank n and degree d. If $g\colon S_1 \to S_2$ be a morphism, then*

$$g^{-1}W^r_{S_2,\mathcal{E}} = W^r_{S_1,(\mathrm{Id}_C \times g)^*\mathcal{E}}.$$

Proof. Let $v_1\colon C \times S_1 \to S_1$ and $v_2\colon C \times S_2 \to S_2$ be the projections. Let $G := \mathrm{Id}_C \times g\colon C \times S_1 \to C \times S_2$. Then we have the following commutative diagram:

$$\begin{tikzcd} C \times S_1 \arrow[r, "G"] \arrow[d, "v_1"] & C \times S_2 \arrow[d, "v_2"] \\ S_1 \arrow[r, "g"] & S_2 \end{tikzcd}$$

This induces the following commutative diagram:

$$\begin{tikzcd} g^*(v_2)_*(\mathcal{E}(D)) \arrow[r] \arrow[d] & g^*(v_2)_*(\mathcal{E}(D)|_D) \arrow[d] \\ (v_1)_*G^*(\mathcal{E}(D)) \arrow[r] & (v_1)_*G^*(\mathcal{E}(D)|_D) \end{tikzcd}$$

By (5.11), we get that the vertical arrows in the above diagram are isomorphisms. Now, the lemma follows from general properties of determinantal loci. □

Now suppose $\widetilde{C}$ be a smooth projective curve of genus $\widetilde{g}$ and $\pi\colon \widetilde{C} \to C$ be a finite morphism. Let $\mathcal{E}$ be a vector bundle over $\widetilde{C} \times S$ such that $\mathcal{E}_s$ is of rank n and degree d for all $s \in S$. Since the map $\pi \times \mathrm{Id}\colon \widetilde{C} \times S \to C \times S$ is a finite flat morphism, we get that $(\pi \times \mathrm{Id})_*\mathcal{E}$ is a vector bundle over $C \times S$ and in fact,

$$((\pi \times \mathrm{Id})_*\mathcal{E})_s = \pi_*(\mathcal{E}_s) \text{ for all } s \in S.$$

We will denote this bundle $(\pi \times \mathrm{Id})_*\mathcal{E}$ by $\mathcal{E}'$. Note that rank of $\mathcal{E}'$ is

$$n' := n(\deg \pi) \text{ for all } s \in S,$$

and degree of $\mathcal{E}'_s$ is

$$d' := d + n(1 - \widetilde{g}) - n(\deg \pi)(1 - g).$$

Then we have the following lemma:

Lemma 5.3.6 *The following equality of Brill-Noether loci holds:*

$$W^r_{S,\mathcal{E}} = W^r_{S,\mathcal{E}'}$$

.

Proof.We have the commutative diagram:

$$\begin{tikzcd} \widetilde{C} \times S \arrow[r, "\pi\times\mathrm{Id}"] \arrow[dr, "\widetilde{v}"'] & C \times S \arrow[d, "v"] \\ & S \end{tikzcd}$$

Fix D a divisor on C such that $h^1(\mathcal{E}'_s(D)) = 0$ for all $s \in S$. Then

$$h^1(\widetilde{C}, \mathcal{E}_s(\pi^*D)) = h^1(C, \pi_*(\mathcal{E}_s(\pi^*D))) = h^1(C, \mathcal{E}'_s(D)) = 0.$$

Therefore we can use the divisor π^*D for the construction of $W^r_{S,\mathcal{E}}$.

Let us consider the morphism

$$f\colon \mathcal{E}(\pi^*D) \to (\mathcal{E}(\pi^*D))|_{\pi^*D}.$$

Then $W^r_{S,\mathcal{E}}$ is defined to be the $(d+n \deg \pi^*D+n(1-\widetilde{g})-(r+1))$-th determinantal loci of the morphism $\widetilde{v}_*f$. Now $\widetilde{v} = v\circ(\pi\times\mathrm{Id})$. It follows from projection formula that $(\pi \times \mathrm{Id})_*f$ is nothing but the morphism

$$\mathcal{E}'(D) \to \mathcal{E}'(D)|_D,$$

and therefore, $W^r_{S,\mathcal{E}'}$ is the $(d' + n' \deg D + n'(1-g) - (r+1))$-th determinantal loci of $v_*(\pi \times \mathrm{Id})_*f = \widetilde{v}_*f$. It can be checked easily that

$$d' + n' \deg D + n'(1-g) - (r+1) = d + n \deg \pi^*D + n(1-\widetilde{g}) - (r+1).$$

□

Next, we will define Brill-Noether Loci for $\mathcal{M}_C(n,d)$. Note that if $(n,d) = 1$, we have a universal bundle over $C \times \mathcal{M}_C(n,d)$ and hence, we can apply the previous construction to get the notion of Brill-Noether loci in this case. However, in general we don't have a universal bundle.

Recall the construction of the moduli space of semistable bundles of fixed rank and degree over C (cf. Subsection 3.3.3 for more details).

Let $\mathcal{R} \subseteq \mathcal{Q}$ be the open subset such that for all $x \in \mathcal{R}$, $\mathcal{F}|_{C\times x}$ is a semistable bundle and $H^0(C, \mathcal{O}^N) \to H^0(C, \mathcal{F}|_{C\times x})$ is an isomorphism. It is immediate that $\mathcal{R}$ is PGL_N-equivariant. Then, we define

$$\mathcal{M}_C(n,d) := \mathcal{R}//\mathrm{PGL}_N.$$

and we have the quotient map

$$\mu\colon \mathcal{R} \to \mathcal{M}_C(n,d).$$

Let us denote $\mathcal{F}|_{C\times\mathcal{R}}$ by $\mathcal{F}'$. Then, over $C\times\mathcal{R}$, $\mathcal{F}'(-m)$ is a vector bundle satisfying (A). Hence, we have the closed subscheme $W^r_{\mathcal{R},\mathcal{F}'(-m)} \subseteq \mathcal{R}$.

Next, we will show that $W^r_{\mathcal{R},\mathcal{F}'(-m)}$ is GL_N-equivariant (and consequently, PGL_N-equivariant).

Let $q\colon T \to W^r_{\mathcal{R},\mathcal{F}'(-m)}$ be a T-valued point of $W^r_{\mathcal{R},\mathcal{F}'(-m)}$. Let $\mathcal{O}^N_{C\times T} \to F_T$ be the pullback of the universal quotient under q.

By Lemma 5.3.5, we get that $W^r_{T,F_T(-m)} = q^{-1}W^r_{\mathcal{R},\mathcal{F}'(-m)} = T$.

Let $g \in \mathrm{GL}_N(T)$. Then, by definition, the quotient corresponding to $g.q\colon T \to \mathcal{R}$ is given by

$$\mathcal{O}^N_{C\times T} \xrightarrow{g} \mathcal{O}^N_{C\times T} \to F_T.$$

Again, by Lemma 5.3.5,

$$(g.q)^{-1}W^r_{\mathcal{R},\mathcal{F}'(-m)} = W^r_{T,F_T(-m)} = T.$$

In other words, we get that $g.q\colon T \to \mathcal{R}$ factors through $W^r_{\mathcal{R},\mathcal{F}'(-m)}$.

Hence, the closed subscheme $W^r_{\mathcal{R},\mathcal{F}'(-m)}$ is GL_N-equivariant.

Definition 5.3.7 We define Brill-Noether loci $\widetilde{W^r_{n,d}}(C)$ to be the scheme theoretic image of $W^r_{\mathcal{R},\mathcal{F}'(-m)}$ under the morphism μ.

Notation 5.3.8 We will denote $\widetilde{W^r_{n,d}}(C)$ by $\widetilde{W^r_{n,d}}$ when there is no chance of confusion.

Remark 5.3.9 Note that since the morphism

$$\mu\colon \mathcal{R} \to \mathcal{M}_C(n,d)$$

is a good quotient and $W^r_{\mathcal{R},\mathcal{F}'(-m)}$ is PGL_N-equivariant, we get that $\mu(W^r_{\mathcal{R},\mathcal{F}'(-m)})$ is a closed subset of $\mathcal{M}_C(n,d)$. Hence, as sets $\widetilde{W^r_{n,d}} = \mu(W^r_{\mathcal{R},\mathcal{F}'(-m)})$. That is to say, denoting the strong equivalence class of a semistable bundle E over C by e

as before, we get

$$\widetilde{W^r_{n,d}} = \left\{ e \in \mathcal{M}_C(n,d) \,|\, \text{there exists } E \in e \text{ such that } h^0(C,E) \geq r+1 \right\}. \quad (5.12)$$

Recall that $\mathcal{M}^s_C(n,d)$, the moduli space of stable bundles on C of rank n and degree d, is an open subset of $\mathcal{M}_C(n,d)$.

Definition 5.3.10 We define Brill-Noether loci $W^r_{n,d}$ of $\mathcal{M}^s_C(n,d)$ to be the closed subscheme

$$W^r_{n,d} := \widetilde{W^r_{n,d}} \cap \mathcal{M}^s_C(n,d) \subset \mathcal{M}^s_C(n,d).$$

Remark 5.3.11 Let $\mathcal{R}^s \subseteq \mathcal{R}$ be the set of all $x \in \mathcal{R}$ such that $\mathcal{F}'|_{C\times x}$ is stable. Let $\mathcal{F}'' := \mathcal{F}'|_{C\times\mathcal{R}^s}$. Let $\mu_s\colon \mathcal{R}^s \to \mathcal{M}^s_C(n,d)$ be the restriction of μ to $\mathcal{R}^s$. Then, $W^r_{n,d}$ is the scheme-theoretic image of $W^r_{\mathcal{R}^s,\mathcal{F}''(-m)}$ under the map μ_s.

Now, using the fact that $\mu_s\colon \mathcal{R}^s \to \mathcal{M}^s_C(n,d)$ is a principal PGL_N-bundle (cf. [Hu-Ln, Corollary 4.3.5, p. 91]), and Lemma 5.3.4 we have that

Lemma 5.3.12 *If $W^r_{n,d} \neq \emptyset$, then dimension of each component of $W^r_{n,d}$ is at least*

$$n^2(g-1)+1-(r+1)(r+1-d+n(g-1)).$$

Definition 5.3.13 We define

$$\rho^r_{n,d} := n^2(g-1)+1-(r+1)(r+1-d+n(g-1))$$

to be the expected dimension of $W^r_{n,d}$.

Remark 5.3.14 The above lemma is not true in the case of $\widetilde{W^r_{n,d}}$. It may have components whose dimensions are less than $\rho^r_{n,d}$ (cf.[Br-Gz-Ne, §7] for example).

Lemma 5.3.15 *Let S be an algebraic scheme and $\mathcal{E}$ be a vector bundle over $C\times S$ such that for all $s \in S, \mathcal{E}_s$ is stable of rank n and degree d. If $f\colon S \to \mathcal{M}^s_C(n,d)$ is the induced map, then*

$$f^{-1}W^r_{n,d} = W^r_{S,\mathcal{E}}.$$

Proof. First we show that the statement is true in the case when $S = \mathcal{R}^s$ and $\mathcal{E} = \mathcal{F}''(-m)$. As we saw earlier, $W^r_{\mathcal{R}^s,\mathcal{F}''(-m)}$ is a PGL_N-equivariant subscheme and since $\mathcal{R}^s \to \mathcal{M}_C^s(n,d)$ is a principal PGL_N-bundle, $W^r_{\mathcal{R}^s,\mathcal{F}''(-m)}$ descends to a closed subscheme Z in $\mathcal{M}_C^s(n,d)$, i.e.

$$\mu_s^{-1}Z = W^r_{\mathcal{R}^s,\mathcal{F}''(-m)}.$$

Since $W^r_{n,d} = \mu_s(W^r_{\mathcal{R}^s,\mathcal{F}''(-m)})$, it is clear that $Z = W^r_{n,d}$. Hence

$$\mu_s^{-1}W^r_{n,d} = W^r_{\mathcal{R},\mathcal{F}''(-m)}.$$

Now let $(S,\mathcal{E})$ be as in the hypothesis. Since $F''(-m)$ is a locally universal family, for any $x \in S$ there exists $U_x \subset \mathcal{R}$ which is open and a map $g\colon U_x \to R^s$ such that

$$(\mathrm{Id} \times g)^*\mathcal{F}''(-m) = \mathcal{E}|_{C\times U_x}.$$

By Lemma 5.3.5 we have

$$g^{-1}W^r_{\mathcal{R},\mathcal{F}''(-m)} = W^r_{S,\mathcal{E}} \cap U_x.$$

Since $\mu_s \circ g = f|_{U_x}$, we have

$$(f|_{U_x})^{-1}W^r_{n,d} = W^r_{S,\mathcal{E}} \cap U_x.$$

The lemma now follows from this. □

Now we are going to recall a few properties like non-emptiness and irreducibility of Brill-Noether loci in the moduli spaces $\mathcal{M}_C^s(n,d)$ and $\mathcal{M}_C(n,d)$. These properties are quite different in higher rank cases in comparison with rank one case, as we'll see below.

Let us now fix different notations of Brill-Noether subvarieties in different spaces to avoid confusion as we deal with all the spaces together after some point. For a given scheme and for a given sheaf $\mathcal{E}$ over $C \times S$, we denote the Brill-Noether loci by $W^r_{S,\mathcal{E}}$ as in Definition 5.3.1 or in Definition 5.3.2. We also denote this by W^r_S when the sheaf involved is clear from the context. In $\mathcal{M}_C(n,d)$ the Brill-Noether locus is denoted by $\widetilde{W^r_{n,d}}$ as in (5.12). The same is denoted by $W^r_{n,d}$ in $\mathcal{M}_C^s(n,d)$ as in Definition 5.3.10. Inside $J_d(C)$, that is inside $\mathcal{M}_C(1,d)$, the Brill-Noether locus $W^r_{1,d}$ is denoted by W^r_d as in Definition 5.2.1 or in Definition 5.2.2. Inside $J_d(\widetilde{C})$ the same is denoted by $W^r_d(\widetilde{C})$.

5.3.1 Brill-Noether Loci in rank one case

The following properties of Brill-Noether loci in rank one case is already mentioned in §6. We recall those results in a bit more detail for the sake of completeness.

Theorem 5.3.16 *[Gz-Bg, Theorem 3.3, 3.4 & 3.5, p. 6] Let $d \geq 1$, $r \geq 0$. Then*

1. *$\rho^r_{1,d} \geq 1 \Rightarrow W^r_d$ is connected for any curve C and irreducible on the generic curve C.*

2. *Let C be a generic curve. Then*

   (a) *$\rho^r_{1,d} < 0 \Rightarrow W^r_d$ is empty.*

   (b) *$\rho^r_{1,d} \geq 0 \Rightarrow W^r_d$ is non-empty, reduced , of pure dimension $\rho^r_{1,d}$ and* $\operatorname{Sing} W^r_d = W^{r+1}_d$.

5.3.2 Brill-Noether loci in higher rank case

Now we assume $n \geq 2$ and $0 \leq d \leq n$, that is $\mu(E) \leq 1$ for any $E \in \mathcal{M}^s_C(n,d)$ or $E \in \mathcal{M}_C(n,d)$. Then we have the following result due to Brambila-Paz, Grzegorczyk and Newstead (cf. [Br-Gz-Ne]).

Theorem 5.3.17 *Let $n \geq 2$ and $0 \leq d \leq n$. Then*

1. *$W^r_{n,d}$ is non-empty if and only if $d > 0$, $n \leq d+(n-r-1)g$ and $(n,d,r+1) \neq (n,n,n)$.*

2. *$W^r_{n,d}$ is non-empty $\Rightarrow W^r_{n,d}$ is irreducible, of dimension $\rho^r_{n,d}$ and* $\operatorname{Sing} W^r_{n,d} = W^{r+1}_{n,d}$.

3. *$\widetilde{W^r_{n,d}}$ is non-empty if and only if either $d = 0$ and $r+1 \leq n$ or $d > 0$ and $n \leq d + (n-r-1)g$.*

4. *$\widetilde{W^r_{n,d}}$ is non-empty $\Rightarrow \widetilde{W^r_{n,d}}$ is irreducible.*

This result was later extended by Mercat (cf. [Me 1] and [Me 2]) for $\mu(E) < 2$.

5.3.3 Brill-Noether loci for large number of sections

Let us denote $\rho^r_{n,d}$ as in Definition 5.3.13 by $\rho(n,d,r,g)$. Then we have the following theorem.

Theorem 5.3.18 *[Bg 1, Theorem 1, p. 386] Let $d = mn+d_1$, $r+1 = tn+r_1$ with $0 \leq r_1 < n$ and $0 \leq d_1 < n$. Also let C be a generic curve. Then $W^r_{n,d}$ is non-empty and has a component of right dimension, namely minimum of $\rho(n,d,r,g)$ and $n^2(g-1)+1$, if the following hold.*

1. *$\rho(1, m-1, t, g-1) \geq 0$ if $r_1 > d_1$.*
2. *$\rho(1, m, t, g-1) \geq 0$ if $0 \neq r_1 \leq d_1$.*
3. *$\rho(1, m-1, t-1, g-1) \geq 0$ if $r_1 = 0$.*

The above result is also true for $\widetilde{W^r_{n,d}}$ if either $d_1 \neq 0$ or the number ρ is strictly positive in (1), (2) or (3).

Remark 5.3.19 When the Brill-Noether loci have a large number of sections, that is, $r+1 > n$, the conditions of the above theorem are probably close to being the best possible for existence of a component of right dimension.

5.4 Cohomology class of the Brill-Noether locus

Consider the semistable locus $\mathcal{R}$ of the Quot scheme, together with the classifying morphism as in (3.21):

$$\mu \colon \mathcal{R} \to \mathcal{M}_C(n,d).$$

By [Dr-Na], $\mathcal{R}$ is a smooth variety. Furthermore, the quotient map μ is a flat morphism. Recall that the Brill-Noether locus is defined as (cf. Definition 5.3.7)

$$\widetilde{W^r_{n,d}} = \mu(W^r_{\mathcal{R},\mathcal{F}'(-m)}).$$

and it corresponds to a cohomology class

$$[\widetilde{W^r_{n,d}}] \in \oplus_i gr^W_{2i} H^{2i}(\mathcal{M}_C(n,d), \mathbb{Q}).$$

To see this, consider the cycle class map into the cohomology:

$$CH^i(\mathcal{R}) \to H^{2i}(\mathcal{R}, \mathbb{Z}).$$

Since we do not know if the Brill-Noether locus is of pure dimension, we will use the cycle class map on the Chow ring:

$$CH^*(\mathcal{R}) \to \oplus_i H^{2i}(\mathcal{R}, \mathbb{Z}).$$

The Chow class

$$[W^r_{\mathcal{R}}] \in CH^*(\mathcal{R}) = \oplus_i CH^i(\mathcal{R})$$

defines the Brill-Noether cohomology class

$$[W^r_{\mathcal{R}}] \in \oplus_i H^{2i}(\mathcal{R}, \mathbb{Z}).$$

Since $\mathcal{R}$ is an open variety, there is a weight filtration on the rational cohomology and we obtain a cycle class in $\oplus_i gr^W_{2i} H^{2i}(\mathcal{R}, \mathbb{Q})$.

Recall the cycle class map from (2.10), on the operational Chow ring, for any projective variety X:

$$cl\colon A^i(X)_{\mathbb{Q}} \to gr^W_{2i} H^{2i}(X, \mathbb{Q}).$$

Since $\mathcal{M}_C(2, 2(g-1))$ is a singular variety, we will consider $A^*(\mathcal{M}_C(r,d))$ instead of Chow groups.

Furthermore, due to universal property of the Brill-Noether locus on $\mathcal{R}$, the Chow class $[W^r_{\mathcal{R}}] \in CH^*(\mathcal{R})$ defines a class:

$$[\widetilde{W^r_{n,d}}] \in A^*(\mathcal{M}_C(n,d))_{\mathbb{Q}}.$$

In particular, we have the following lemma.

Lemma 5.4.1 *The Brill-Noether class $[W^r_{\mathcal{R}}]$ is non-zero if and only if $[\widetilde{W^r_{n,d}}]$ is non-zero, in Chow cohomology (respectively in weighted graded cohomology ring).*

Proof.The quotient map μ is a flat morphism. Hence there is a pullback map:

$$A^*(\mathcal{M}_C(r,d)) \to CH^*(\mathcal{R})$$

and compatible with the cycle class map (cf. Lemma 2.5.1):

$$\begin{tikzcd} A^*(\mathcal{M}_C(n,d))_{\mathbb{Q}} \arrow[r, "\mu^*"] \arrow[d, "cl"'] & CH^*(\mathcal{R})_{\mathbb{Q}} \arrow[d, "cl"] \\ \oplus_i gr^W_{2i} H^{2i}(\mathcal{M}_C(n,d),\mathbb{Q}) \arrow[r, "\mu^*_{coh}"'] & \oplus_i gr^W_{2i} H^{2i}(\mathcal{R},\mathbb{Q}). \end{tikzcd}$$

Since the subvariety $\widetilde{W^r_{n,d}}$ is the quotient of the GL_n-invariant subvariety $W^r_{\mathcal{R}}$ under μ, the pullback $\mu^{-1}(\widetilde{W^r_{n,d}})$ is the same as $W^r_{\mathcal{R}}$. Hence the lemma is clear. □

5.5 Main theorems, when the rank is two

We want to give some relations amongst the Brill-Noether loci in $\mathcal{M}_C(2,d)$. In our context we fix degree d to be $2(g-1)$.

In this case, Sundaram proved that $\widetilde{W^0_{2,2(g-1)}}$ is a divisor in $\mathcal{M}_C(2,2(g-1))$ (cf. [Su]). We give some relations between the cohomology classes of the Brill-Noether loci in terms of cohomology class of $\widetilde{W^0_{2,2(g-1)}}$.

Since the moduli spaces $\mathcal{M}_C(2,2(g-1))$ and $\mathcal{SU}_C(2,\mathcal{L})$ are singular varieties, recall from §2.5, that the cohomology classes are taken in the graded piece for the weight filtration on the singular cohomology group $H^*(\mathcal{M}_C(2,2(g-1)),\mathbb{Q})$ (respectively $H^*(\mathcal{SU}_C(2,\mathcal{L}),\mathbb{Q})$).

Consider the map $\pi_*\colon J^{ss} \subseteq J_{4(g-1)}(\widetilde{C}) \to \mathcal{M}_C(2,2(g-1))$ as in (5.6). Note that as we have taken $d = 2(g-1)$, therefore it follows from (5.5) that $\delta = 4(g-1)$. Also from (5.3), we have

$$\delta = 4(g-1) = \{4(g-1)+1\} - 1 = \widetilde{g} - 1.$$

Hence we have the Theta divisor $\Theta := W^0_{4(g-1)}(\widetilde{C})$ in $J_{4(g-1)}(\widetilde{C})$. Following theorem says that the Theta divisor of $\widetilde{C}$ intersects both J^{ss} and its complement in $J_{4(g-1)}(\widetilde{C})$.

Theorem 5.5.1 *[Be-Na-Ra, Proposition 5.1, p. 176] The Theta divisor of the moduli space $J_{4(g-1)}(\widetilde{C})$, denoted by Θ, does not lie inside the complement of J^{ss} in $J_{4(g-1)}(\widetilde{C})$. More precisely,*

1. *For any point $l \in J_{4(g-1)}(\widetilde{C}) - \Theta$, $\pi_*(l)$ is semistable.*

2. *There is a point $\xi \in \Theta$ such that $\pi_*(\xi)$ is semistable.*

Moreover we have that pullback of the divisor $\widetilde{W^0_{2,2(g-1)}}$ of $\mathcal{M}_C(2, 2(g-1))$ is the restriction of Θ to J^{ss}.

Theorem 5.5.2 *Let us denote the restriction of Θ to J^{ss} by $\Theta|_{J^{ss}}$. Then*

$$\pi_*^{-1}(\widetilde{W^0_{2,2(g-1)}}) = \Theta|_{J^{ss}}.$$

Proof. See [Bg-Tu, Lemma 6, p. 335]. Also follows directly from the fact that $H^0(\widetilde{C}, l) = H^0(C, \pi_* l)$. □

We now revisit Theorem 5.3.18 in this context. Then the following theorem gives some sufficient conditions for the Brill-Noether loci in $\mathcal{M}_C(2, 2(g-1))$ to be non-empty. For the rest of the section, we will assume that any one of these conditions holds.

Theorem 5.5.3 *Let $r + 1 = tn + r_1$ with $0 \leq r_1 < n$. Then $\widetilde{W}^r_{n,d}$ is non-empty and has a component of right dimension, namely minimum of $\rho(2, 2(g-1), r, g)$ and $4(g-1) + 1$, if the following hold.*

1. *$\rho(1, g-2, t, g-1) > 0$ if $r_1 > 0$.*

2. *$\rho(1, g-1, t, g-1) > 0$ if $r_1 < 0$.*

3. *$\rho(1, g-2, t-1, g-1) > 0$ if $r_1 = 0$.*

Proof. Follows directly from Theorem 5.3.18. □

Now we want to check whether Theorem 5.5.1 and 5.5.2 hold for other Brill-Noether subvarieties of higher codimension. By (5.4), we have

$$\pi_*^{-1}(\widetilde{W^r_{2,2(g-1)}}) = W^r_{4(g-1)}(\widetilde{C})|_{J^{ss}}.$$

A priori, it is not clear whether $W^r_{4(g-1)}(\widetilde{C})$ lies inside the complement of J^{ss} or not, that is even if $\widetilde{W^r_{2,2(g-1)}}$ is non-empty, its inverse image $W^r_{4(g-1)}(\widetilde{C})|_{J^{ss}}$ could be empty when $r > 0$. (This question will be treated in the next subsection). However for our purpose, it will suffice to consider a scheme S to give relations between Brill-Noether subvarieties of $\mathcal{M}_C(2, 2(g-1))$.
We can construct a scheme S with the following properties.

1. S is a smooth projective variety.

2. There exists a birational morphism $\varphi\colon S \to J_{4(g-1)}(\widetilde{C})$ and a generically finite morphism $\psi\colon S \to \mathcal{M}_C(2, 2(g-1))$.

3. There is a morphism $q\colon S \to \mathcal{Q}$ such that $\psi = \mu \circ q$.

4. The morphism $\varphi\colon \varphi^{-1}(J^{ss}) \to J^{ss}$ is an isomorphism.

5. The following diagram is commutative.

$$\begin{tikzcd} & S \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ J_{4(g-1)}(\widetilde{C}) \arrow[rr, dashed, "\pi_*"'] & & \mathcal{M}_C(2, 2(g-1)) \end{tikzcd} \tag{5.13}$$

Moreover this diagram is commutative whenever the domains of the involved rational maps are chosen properly. In particular, we have the following commutative diagram.

$$\begin{tikzcd} & \varphi^{-1}(J^{ss}) \arrow[dl, "\varphi"', "\cong"] \arrow[dr, "\psi"] & \\ J^{ss} \arrow[rr, "\pi_*"'] & & \mathcal{M}_C(2, 2(g-1)) \end{tikzcd}$$

Then we have the following diagram.

$$\begin{tikzcd} \tilde{C} \times S \arrow[r, "\pi \times \mathrm{Id}"] \arrow[d, "\mathrm{Id} \times \varphi"'] & C \times S \arrow[r] \arrow[d, "\mathrm{Id} \times \psi"'] & S \arrow[d, "\psi"] \\ \widetilde{C} \times J_{4(g-1)}(\widetilde{C}) \arrow[r, dashed, "\pi \times \pi_*"'] & C \times \mathcal{M}_C(2, 2(g-1)) \arrow[r] & \mathcal{M}_C(2, 2(g-1)) \end{tikzcd}$$

Let us denote by J^s the following set.

$$J^s := \left\{ l \in J_\delta(\widetilde{C}) \mid \pi_* l \in \mathcal{M}^s_C(2, 2(g-1)) \right\}.$$

Define $S_0 := \varphi^{-1}(J^s)$. Then we have the following lemma:

Lemma 5.5.4 $\psi^{-1}(\widetilde{W^r_{2,2(g-1)}}) \cap S_0 = \varphi^{-1}(W^r_{4(g-1)}(\widetilde{C})) \cap S_0$.

Proof. If $\mathcal{P}$ is a Poincaré bundle over $\widetilde{C} \times J_{4(g-1)}(\widetilde{C})$, then the morphism $S_0 \to \mathcal{M}_C(2, 2(g-1))$ is induced by the family $(\pi \times \mathrm{Id})_*((\mathrm{Id} \times \varphi)^*\mathcal{P})|_{C \times S_0}$. Now by Lemma 5.3.6 we get that

$$W^r_{S,(\mathrm{Id}\times\varphi)^*\mathcal{P}} \cap S_0 = W^r_{S,(\pi\times\mathrm{Id})_*((\mathrm{Id}\times\varphi)^*\mathcal{P})} \cap S_0.$$

The Lemma 5.3.15 then implies

$$W^r_{S,(\pi\times\mathrm{Id})_*((\mathrm{Id}\times\varphi)^*\mathcal{P})} \cap S_0 = \psi^{-1}\widetilde{W^r_{2,2(g-1)}},$$

and Lemma 5.3.5 implies

$$W^r_{S,(\mathrm{Id}\times\varphi)^*\mathcal{P}} \cap S_0 = \varphi^{-1}W^r_{4(g-1)}(\widetilde{C}) \cap S_0.$$

Hence we get

$$\psi^{-1}(\widetilde{W^r_{2,2(g-1)}}) \cap S_0 = \varphi^{-1}(W^r_{4(g-1)}(\widetilde{C})) \cap S_0.$$

□

Hence we obtain the following:

Lemma 5.5.5 *We have the equality of the closures*

$$\overline{\psi^{-1}(\widetilde{W^r_{2,2(g-1)}}) \cap S_0} = \overline{\varphi^{-1}(W^r_{4(g-1)}(\widetilde{C})) \cap S_0}$$

of a component of Brill-Noether loci on S. In particular, of the corresponding cohomology classes in $H^*(S,\mathbb{Z})$.

Denote this component $\mathcal{W}^r_S$, in S.

5.5.1 Poincaré type relations on moduli spaces

Assume that C is a general smooth projective curve and $\widetilde{C} \to C$ is a general smooth spectral curve, which is a double ramified covering of C. We denote

$$gr^W_* H^*(X,\mathbb{Q}) = \oplus_i gr^W_{2i} H^{2i}(X,\mathbb{Q}).$$

the associated graded ring (for the weight filtration) of the even degree cohomology of the singular moduli spaces $X = \mathcal{M}_C(2, 2(g-1))$ and $\mathcal{SU}_C(2,\mathcal{L})$, where $\mathcal{L} \in J_{2(g-1)}(C)$, (cf. §2.5).

We start with the following lemma.

Lemma 5.5.6 *The divisor classes on $J_{4(g-1)}(\widetilde{C})$ descend to the moduli space $\mathcal{M}_C(2, 2(g-1))$ via the above diagram (5.13).*

Proof. Recall from [Be-Na-Ra, Proposition 5.7] a commutative diagram:

$$\begin{tikzcd} P' \times J_{g-1}(C) \arrow[r] \arrow[d] & J_{4(g-1)}(\widetilde{C}) \arrow[d, "\pi_*"] \\ \mathcal{SU}_C(2) \times J_{g-1}(C) \arrow[r] & \mathcal{M}_C(2, 2(g-1)). \end{tikzcd}$$

Here P' is the Prym variety associate to the covering $\widetilde{C} \to C$ and $\mathcal{SU}_C(2)$ is a fixed determinant (of degree $2(g-1)$) moduli space. Furthermore, it is shown that the indiscrepancy loci of the dominant rational map π_* has codimension at least two and the same is true when restricted to P'. The proof of loc.cit. implies that the polarisations on P' and $J_{g-1}(C)$ descend on the moduli space $\mathcal{M}_C(2, 2(g-1))$. By functoriality, via the diagram (5.13), the divisor classes descend on $\mathcal{M}_C(2, 2(g-1))$. $\square$

We now show the following.

Theorem 5.5.7 *The cohomology class of a Brill-Noether locus on the moduli space $\mathcal{M}_C(2, 2(g-1))$ can be expressed as a polynomial on divisor classes. In particular, the tautological algebra generated by the Brill-Noether loci is generated by the divisor classes.*

Proof. Recall the morphisms

$$\varphi \colon S \to J_{4(g-1)}(\tilde{C})$$

and

$$\psi \colon S \to \mathcal{M}_C(2, 2(g-1)).$$

Now φ is a birational morphism and let $E \subset S$ be the exceptional loci, and ψ is a generically finite morphism. Hence, we have the following equalities of cohomology rings:

$$H^*(S, \mathbb{Q}) = H^*(J_{4(g-1)}(\widetilde{C}), \mathbb{Q}) \oplus H^*(E, \mathbb{Q}),$$

and an inclusion of rings (cf. Lemma 2.5.1):

$$\psi^*_{coh} \colon gr^W_* H^*(\mathcal{M}_C(2, 2(g-1), \mathbb{Q}) \hookrightarrow H^*(S, \mathbb{Q}).$$

By Theorem 5.2.8, the cohomology class of the Brill-Noether loci $W^r_{1,d}(\widetilde{C}) \subset J_{4(g-1)}(\widetilde{C})$ is a polynomial expression on the divisor classes in $H^*(J_{4(g-1)}(\widetilde{C}))$. This implies that in $H^*(S, \mathbb{Q})$, the pullback of the cohomology class $[W^r_{1,d}(\widetilde{C})]$ is the cohomology class of the Brill-Noether loci $W^r_S \subset S$ and it is a polynomial expression on the pullback of the divisor classes on $J_{4(g-1)}(\widetilde{C})$.

Recall that S was constructed such that $q \colon S \to \mathcal{Q}$ and $\psi = \mu \circ q$, wherever μ is defined.

Denote $S' := q^{-1}(\mathcal{R})$. Since μ is flat and $\mathcal{R}$ is a smooth variety there are pullback maps on the Chow cohomologies:

$$A^*(\mathcal{M}_C(2, 2(g-1))) \xrightarrow{\mu^*} CH^*(\mathcal{R}) \xrightarrow{q^*} CH^*(S').$$

By Lemma 5.4.1,

$$q^*\mu^*[\widetilde{W^r_{2,2(g-1)}}] = q^*[W^r_{\mathcal{R}}] = [W^r_{S'}]. \tag{5.14}$$

Since $\mathcal{R} \subset \mathcal{Q}$ is an open subvariety of $\mathcal{Q}$, using the localization sequence

$$CH^*(S) \to CH^*(S') \to 0,$$

we deduce that $[W^r_S] \mapsto [W^r_{S'}]$.

The above Chow cohomology diagram is compatible, via cycle class maps, with the weighted graded cohomology rings:

$$gr^W_* H^*(\mathcal{M}_C(2, 2(g-1)), \mathbb{Q}) \xrightarrow{\mu^*} gr^W_* H^*(\mathcal{R}, \mathbb{Q}) \xrightarrow{q^*} gr^W_* H^*(S', \mathbb{Q})$$

together with a restriction

$$H^*(S, \mathbb{Q}) \xrightarrow{t} gr^W_* H^*(S', \mathbb{Q}).$$

Here t is a surjection. In particular, $[W^r_S] \mapsto [W^r_{S'}]$, in cohomology. Furthermore, by (5.14), we deduce that

$$\psi^*_{coh}[\widetilde{W^r_{2,2(g-1)}}] = [W^r_S]. \tag{5.15}$$

By Lemma 5.5.5, Lemma 5.5.6 we know that the divisor classes descend, and (5.15) imply that the cohomology class of the Brill-Noether locus in the graded cohomology $gr_*^W H^*(\mathcal{M}_C(2,2(g-1)),\mathbb{Q})$ is expressible as a polynomial on the divisor classes. $\square$

Consider the determinant morphism

$$\det\colon \mathcal{M}_C(2,2(g-1)) \to J_{2(g-1)}(C).$$

The inverse image $\det^{-1}(\mathcal{L})$ is the moduli space $\mathcal{SU}_C(2,\mathcal{L})$, for a line bundle $\mathcal{L}$ on C of degree $2(g-1)$. Denote the Brill-Noether loci

$$\widetilde{W^{r,\mathcal{L}}_{2,2(g-1)}} := \widetilde{W^{r}_{2,2(g-1)}} \cap \mathcal{SU}_C(2,\mathcal{L}).$$

Corollary 5.5.8 *1. The cohomology class of a Brill-Noether locus $\widetilde{W^{r,\mathcal{L}}_{2,2(g-1)}}$ in the moduli space $\mathcal{SU}_C(2,\mathcal{L})$ is expressible in terms of a power of the class of the Theta divisor, with rational coefficients. In particular the tautological algebra is generated by the class of the Theta divisor Θ.*

2. If the Brill-Noether number is non-negative, then the cohomology classes are non-trivial and imply the non-emptiness of the corresponding loci.

Proof.

1. Consider the inclusion:

$$j\colon \mathcal{SU}_C(2,\mathcal{L}) \hookrightarrow \mathcal{M}_C(2,2(g-1)).$$

The pullback map on the cohomology ring

$$j^*\colon gr_*^W H^*(\mathcal{M}_C(2,2(g-1),\mathbb{Q}) \to gr_*^W H^*(\mathcal{SU}_C(2,\mathcal{L}),\mathbb{Q})$$

is a ring homomorphism. By Theorem 5.5.7, the cohomology class of the Brill-Noether loci is a polynomial expression on the divisor classes on $\mathcal{SU}_C(2,2(g-1))$. The Picard group of $\mathcal{SU}_C(2,2(g-1))$ is generated by the Theta divisor Θ. This gives the relation, for any irreducible component:

$$[\widetilde{W^{r,\mathcal{L}}_{2,2(g-1)}}] = \alpha.[\Theta]^{t(r)} \in gr_*^W H^*(\mathcal{SU}_C(2,\mathcal{L}),\mathbb{Q}),$$

for some $\alpha \in \mathbb{Q}$ and $t(r)$ is the codimension of an irreducible component of Brill-Noether loci.

2. Using Theorem 5.2.5 a), we obtain that the dimension of every component of the Brill-Noether loci on $J_{4(g-1)}(\widetilde{C})$ is at least the expected dimension. This implies that the corresponding cohomology class is non-zero in $H^*(J_{4(g-1)}(\widetilde{C}), \mathbb{Z})$. In turn, the same is true for the pullback class on S, which further descends as a non-zero class on $\mathcal{M}_C(2, 2(g-1))$ (see proof of Theorem 5.5.7). This implies the non-emptiness of the Brill-Noether loci on the moduli spaces, whenever the expected dimension is non-negative.

$\square$

Let us end this chapter with the following remark.

Remark 5.5.9 It is likely that the Hodge conjecture holds for the Jacobian of a higher degree general spectral curve (cf. [Ar] for unramified coverings). The proofs employed in Theorem 5.5.7 will then be applicable also for higher rank moduli spaces. The proofs raise further questions whether a Castelnuovo type formula holds or not on the moduli space, for a general curve C.

Conclusion

A few days back I was going through the celebrated thesis of Piper Harron (cf. [Hr]). The conclusion of her thesis goes as follows:

"You're still here? Oh, I guess I should tell you math papers generally don't have what you or I might call a "conclusion". They just sort of stop.

So, yeah, you can, um, go now. But, cheers!

Seriously, it's over."

After going through her thesis a bit, I would like to believe that she was being sarcastic while writing the mentioned conclusion. Hence I am going to conclude by mentioning very briefly about what we have tried to investigate in our thesis and what are the problems that can be studied along the same line.

We have investigated the tautological algebra, the algebra generated by the cohomology classes of the Brill-Noether subvarieties, inside the cohomology ring of the moduli space of semistable bundles over curves. In our first work, the relations amongst the cohomology classes of the Brill-Noether subvarieties of the moduli space of semistable bundles over an elliptic curve have been found (cf.

Chapter 4). We have obtained results similar to Poincaré's formula on a Jacobian variety. In our second work (cf. Chapter 5), we have showed that when C is a general smooth projective curve of genus $g \geq 2$, $d = 2g-2$, the tautological algebra of $\mathcal{M}_C(2, 2g-2)$ (respectively $\mathcal{SU}_C(2, L)$, deg $L = 2g-2$) is generated by the divisor classes (respectively the class of the Theta divisor Θ). Also we have proved some results about the non-emptiness of the loci.

Let us first quickly justify the two different scenario we are under for the two mentioned work. In our first work, we have obtained the main results (cf. Theorem 4.3.3 & 4.3.4) for degree 0 bundles having rank greater than 1. Apparently, this might lead to the fact that we have looked over rest of the cases. But we actually have not missed those cases as then the stratifications of Brill-Noether subvarieties are trivial, which is clear from Lemma 4.2.5 and Lemma 4.2.6. In the second problem, we have worked under rank 2 and degree $2g-2$ case. In this thesis we have considered Theta divisor on the moduli space $J_{g-1}(C)$. Now it can be noted that for any $L \in J_{g-1}(C)$, $\chi(L) = 0$. In fact, this is a necessary criteria to define Theta divisor without "twist" (cf. [Bg-Tu, Section 2.3]). To do so in the moduli space $\mathcal{M}_C(n, d)$, we need to take $d = n(g-1)$. As the first main result (cf. Theorem 5.2.8) of our second work was obtained in rank 2 case, we had no choice but to work on $\mathcal{M}_C(2, 2g-2)$.

We now mention a key difference in the approach of the two work we have dealt with in this thesis. Following the techniques of the second problem (cf. proof of part 1 of Corollary 5.5.8), it is quite clear that if we can obtain the tautological algebra of $\mathcal{M}_C(n, d)$ first, then the same can be easily obtained for the corresponding fixed determinant moduli space. But we have investigated the problem other way round for the first work, that is in genus one case. That we were forced to do only because we were unable to obtain the tautological algebra of the semistable moduli $\mathcal{M}_C(n, 0)$ directly in that case.

It can be easily noted that problems similar to what we have dealt with can be considered in some other suitable moduli spaces as well because of the basic nature of the problem. For example, the problem can be studied in the moduli space $\mathcal{M}_C(n, d)$, where $(n, d) \neq (2, 2g-2)$, for any curve C with genus greater than equal to 2. The investigation of similar algebra over some particular Quot scheme is presently under way. As the results obtained so far are not convincing enough, we abstain from including them in the thesis.

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