

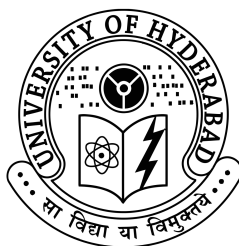
Ph. D. Thesis

Applications of Thermofield Dynamics to Quantum Entanglement and Correlations

A thesis submitted for the award of
DOCTOR OF PHILOSOPHY

by

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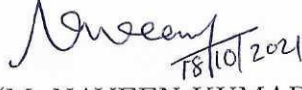
My LORD and Saviour Jesus Christ
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I here by declare that the work reported in this thesis entitled “**Applications of Thermofield dynamics to Quantum entanglement and Correlations**” has been carried out by me independently in the School of Physics, University of Hyderabad, under the supervision of **Prof. Bindu A. Bambah**. I also declare that this is my own work and effort, and it has not been submitted at any other University or Institution for any degree. Wherever contributions of others are involved, every effort is made to indicate that clearly with due reference to literature, and acknowledgement of collaborative research and discussions. I hereby agree that my thesis can be deposited in Shodganga/INFLIBNET.

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1. **Naveen Kumar Mogurampally**, K.V.S.S. Chaitanya and Bindu A. Bambah, “*Quantum Entanglement in coupled lossy waveguides using $SU(2)$ and $SU(1,1)$ Thermo-algebras*”, Journal of Modern Physics, (2015) 6, pp. 1554-1571. doi:10.4236/jmp.2015.611158, arXiv:1507.01539 [quant-ph].
2. T. Shreecharan and **M. Naveen Kumar**, “*Statistical Aspects of Coherent States of the Higgs Algebra*”, International Journal of Theoretical Physics, (2018) Vol.57, Issue 7, pp. 2133-2144. <https://doi.org/10.1007/s10773-018-3738-y>
3. Bindu A. Bambah and **Naveen Kumar Mogurampally**, (2016) “*Thermal effects in the hadronic and photonic multiplicity distributions and correlations: A Thermo-Field Dynamic approach*”, Proceedings of the sixth Asian Triangle Heavy Ion Conference, NewDelhi, 2016; https://inis.iaea.org/search/search.aspx?orig_q=RN:48075444.

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~Naveen

Abstract

The formalism of Thermo-field dynamics (TFD) is a powerful tool for exact studies of quantum physics problems at finite temperature. We apply it to the physical phenomenon of quantum entanglement (QE) which plays a fundamental role in quantum communication between parties separated by macroscopic distances. We study QE in two systems: the first one is the coupled waveguides in quantum optics. We solve the master equation for the coupled lossy waveguides by using the TFD formalism which allows the use of the underlying symmetry algebras $SU(2)$ and $SU(1,1)$, associated with the Hamiltonian of the system. We compute entanglement and decoherence as a function of time for various input states such as NOON states and thermal states of the system with and without damping. In the second application, the coupled two site, two species Bose-Hubbard model (in condensed matter physics) is studied. We calculate the entanglement properties and decoherence for the system considering the interaction of bosons in system with non-linear damping by solving the master equation using the TFD. We compare the role of the repulsive on-site interaction parameter and the hopping parameter on these features. We note the significance to optical models with ultracold atoms and show how we can tune the system for optimum quantum information processing. In yet another application of TFD to multiparticle production, we study the salient features of multiplicity distributions and correlations of pions which emerge in the collision of particles using quantum optical and statistical techniques. We introduce new distributions which incorporate finite temperature effects using TFD and study the resulting correlation functions. These results may be useful as tool to study the properties of the quark-gluon-plasma (QGP) believed to be produced in the collision of heavy ions at Relativistic heavy ion collider (RHIC) and Large Hadron Collider (LHC) (in future).

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Chapter 1

Introduction

Understanding the properties of many-body quantum systems is one of the most challenging problems in physics. Over the years many methods have been proposed in quantum mechanics to provide a precise description of the properties of these many body quantum systems. The aim of many-body physics theories is to compute the emergent thermodynamic, spectroscopic and response properties of a system of interacting particles. Although conventional quantum mechanics can be generalised to solve a few body problems, it is insufficient to deal with systems containing a large number of constituents and hence a new theory is called for. Important progress has been made through the development of quantum field theory(QFT)[1, 2, 3]. QFT, in its usual formulation, is a combination of classical field theory, quantum mechanics and special theory of relativity and does not involve temperature. Thus it is a zero temperature relativistic quantum field theory. When one goes to relativistic regime one encounters the possibility of creation of new particles from energy and annihilation of the intermediate particles, which can be handled only in QFT, by incorporating these quanta or particles as its essential feature allowing meaningful calculations of experimentally verifiable results. Although, initially, QFT was developed for dealing with the physics of elementary particles, it was adapted to the study of many-body problems in condensed matter, by the introduction of the fock representation, which could give a description of quantum systems whose state could be classified by a number series.

However, we need a much more extended and developed QFT for studying the properties of many-body quantum systems, in which, there is an entanglement and correlations among the particles of the quantum systems with thermal environments. For such systems it was necessary to reformulate QFT to take into account thermal effects. For that, there are two

exciting fields of studies very popular in current research of quantum physics, which are: thermo-field dynamics(TFD)[4, 5, 6, 7, 8, 9] and quantum entanglement(QE) dynamics[10, 11, 12, 13].

1.1 Thermo-field dynamics

TFD is an important formalism belonging to thermal quantum field theory(TQFT) (also called 'finite temperature field theory' (FTFT)), which is used for studying quantum physics problems at finite temperature. In TQFT, there are two finite temperature formalisms, those are: imaginary-time formalism (ITF) and real-time formalism (RTF). The Imaginary time formalism applicable for quantum systems at thermal equilibrium which is also called 'Matsubara formalism'. The most popular ITFs are: the path integral method and the operator method. Real time formalism is more applicable to quantum systems at non-thermal equilibrium and is used to study time dependent thermal systems. It is more complicated and it has not been as widely used as ITF. Not many things are standardly known about it. But this is a most suitable formalism, for, how a finite temperature system changes in time. The most popular RTFs are: **the “closed time path formalism”** and **the “thermo-field dynamics”**.

Thermo-field dynamics is a real time finite temperature formalism which is a very useful and powerful method of TQFT. It was developed as an extension of zero temperature formalism to finite temperature problem in which the thermal average of an operator in a statistical ensemble is replaced by an expectation value w.r.t. **“temperature dependent vacuum state”**. It was introduced by Takahashi & Umezawa[4, 5] in 1975. They constructed a representation by considering **a doubling of Hilbert space**, the “thermal vacuum state” is defined so that, the statistical average of an observable 'A' is given by “ $\langle A \rangle = Z^{-1}(\beta)Tr[Ae^{-\beta H}]$ ” is the expectation value of the operator A over a doubled Hilbert space spanned by $|n, n\rangle$, where, the partition function $Z(\beta) = Tr[Ae^{-\beta H}]$. It has been used to study the superconductors[14], the thermal oscillators[15], in magnetic systems [16], in quantum optics [17, 18, 19], in transport phenomena[20] and to study the properties of hadron source (quark gluon plasma(**QGP**))[21].

In this thesis, special attention is given towards applying the TFD formalism to study open quantum systems interacting with a thermal reservoir. “ $\langle A \rangle = Tr(\rho A) = Z^{-1}(\beta)Tr(Ae^{-\beta H})$ ”, is corresponds to Wick rotation “ $\beta \rightarrow -i\tau$ ”, can solve problems using, for example, perturbation and diagrammatic techniques. An alternative way is that the operator average $\langle A \rangle$

would be performed in a Hilbert space, with “temperature dependent state” “ $|0(\beta)\rangle$ ”, such that “ $\langle A \rangle = \langle 0(\beta)|A|0(\beta)\rangle$ ”. The details about TFD formalism are given in chapter 2: its construction, thermal vacuum state using density matrix method by considering the doubling of Hilbert space. Application of the formalism to bosonic and fermionic systems is given and TFD is used to convert the Liouville von Neumann equation of the evolution of the density operator into a Schrödinger type equation for the density matrix. This simplification allows us to study decoherence in open quantum systems. We also define various thermal states useful to study thermal correlations and multiplicity distributions in multiparticle production.

1.2 Quantum entanglement dynamics

The physical property responsible for the non-classical correlations between distinct quantum systems is called ‘quantum entanglement’(QE)[22]. That is, the entanglement is a physical phenomenon corresponding to interacting pairs of particles such that the quantum state of each particle of the group is not independent of the state of the others, even at large separations. The theory to study the QE of systems is called ‘quantum entanglement(QE) dynamics’, which seeks to understand the dynamics of quantum systems in various branches of physics. “It was Einstein, Podolsky and Rosen(EPR) and Schrödinger first recognized this ‘spooky’ feature of quantum machinery which lie down at center of interest of physics of 21st century”[10, 12]. “This feature implies the existence of global states of composite system which cannot be written as a product of the states of individual subsystems. This phenomenon, known as Entanglement, was originally called by Schrödinger ‘Verschränkung’, which underlines an intrinsic order of statistical relations between subsystems of compound quantum system”[11].

For a long time, QE was a topic discussed mostly in quantum optics[17, 23], but, in the last decades, it has seen a revival with input from very different areas. Quantum information relies on quantum entanglement.[24, 25] and communication, quantum computing [26, 27], quantum teleportation[28], quantum dense coding[22], and quantum cryptography[29]. Entanglement dynamics has been used to study information loss in the theory of black holes, the numerical investigation of many body quantum systems, the characterization of topological quantum states and quantum phase transitions(QPT). The study of many-body systems has been greatly aided by the study of exactly solvable spin systems. In fact, some of the experimental studies are carried out on systems which are well modeled by spin systems such

as the transverse Ising model that undergoes a QPT. The quantitative theory of entanglement may give a powerful unifying framework for the understanding of complex quantum systems exhibiting QPT. I have given great details of QE and its measures in chapter-3.

Combinedly, these two areas of research of quantum physics are now being studied in various fields ranging from condensed matter to quantum gravity. Most studies of quantum entanglement are for ideal systems at zero temperature. However interactions of many-body systems can cause systems to decohere and must be taken into account when carrying out experiments on quantum entanglement. Open quantum systems play a crucial role in quantum information. Decoherence is caused by systems-environment interactions. Using proper methods we can quantify the decoherence in open quantum systems by using master equation. The application of dynamics of the open quantum systems ranges from quantum cosmology to quantum optics and to quantum information. The control of entanglement against the dissipative effects of the environment is an important issue. TFD helps us in solving the master equation in a simple fashion and the decoherence effects can easily be calculated in a number of systems. We attempt to study the dynamics of entanglement and its evolution in open quantum systems, given their importance in quantum computation and quantum information.

In this thesis we focus on the techniques of TFD and its relation to the "Liouville-von Neumann equation" to study the quantum entanglement properties of a few many-body systems with thermal environments. These are generic many-body quantum systems in three different areas of physics, quantum optics, condensed matter physics and particle physics. We have studied the coupled lossy waveguides system in quantum optics, where we have shown how the damping effect is mimicked by the thermal effects in the system. In condensed matter physics, we studied the Bose-Hubbard model, where we compare the effect of three parameters (repulsive on-site interaction parameter, hopping parameter and damping parameter) on entanglement dynamics and the decoherence in the system. Again we show that the damping effects can be attributed to the system having thermal states. These features can be shown to tune the system for optimum quantum information processing. In particle physics we show how TFD can be used to define correlations and multiplicities of particles produced in thermal environments such as the quark gluon plasma in a heavy ion collision. We define different thermal states and calculate multiplicity distributions and correlations at finite temperature which gives the extension of BEC (Bose-Einstein Correlations) phenomenology, that gives new inputs in the experimental study of BEC.

This thesis is organised as follows. In this first chapter, I have introduced the basis and motivation towards my work in the area of quantum physics: the finite temperature field the-

ory, thermo-field dynamics, quantum entanglement and more concepts and results discussed in further chapters. In chapter two, we describe the formalism of thermo-field dynamics by considering a few exemplary thermal systems. In chapter three, we describe the quantum entanglement, density operator formalism and various measures of entanglement & an exemplary quantum system. In chapter four, we compute the entanglement and decoherence of the coupled waveguides system with and without damping by using TFD. The decoherence of various input states “NOON states” and “Thermal states” as function of time is studying. In chapter five, we compute the entanglement and decoherence for “the coupled two site, two species Bose-Hubbard model in the presence of a non-linear damping” with coupling strength, J and interaction term, U by using TFD. In chapter six, multiplicity distributions and correlations arise for different pion states and also derive and discuss how thermal effects can be incorporated in multiplicity distributions and correlations of pions by using the TFD. In the final chapter, we summarize and conclude with scope for future work.

Chapter 2

Thermo-field Dynamics formalism

“Thermo-field dynamics”(TFD) is a real time finite temperature formalism, which was developed initially by Umezawa and Takahashi to define finite temperature operators in the theory of superconductivity [4, 5]. Since then it has become a powerful finite temperature field theory to deal with non-equilibrium and real time evolution of thermal systems in many branches in physics[7, 8]. The central premise of the theory is to replace the “thermal average of a statistical ensemble” by an expectation value with respect to “temperature dependent vacuum”, by enlarging the Hilbert space. It basically consists of applying a general technique of enlarging the Hilbert Space of the theory, allowing thermal averages of quantities by a simple pure state vacuum expectation value of a local operator. In particular, density operator is represented as a vector in this “extended Hilbert space”. The von-Neumann-Liouville problem involves solving the master equation and is equivalent to solving a Schrödinger type equation. This is a powerful tool to study entanglement and decoherence effects in many body systems in a thermal environment. In this chapter we give the salient features of TFD relevant to the problems solved in the thesis. We give examples of the use of this formalism to describe thermal bosonic and fermionic systems and define thermal coherent and squeezed states and then used to study the entanglement and decoherence in a few many-body quantum systems in chapters -4, 5 and 6.

2.1 Construction of Thermal state -Doubling of Hilbert space

An essential quantity of a statistical system is, the ensemble average of an operator 'A' in thermal equilibrium at temperature 'T' is [7, 8, 30, 31],

$$\langle A \rangle = \frac{\text{Tr}(\rho A)}{\text{Tr}(\rho)} = \frac{1}{Z(\beta)} \text{Tr}(e^{-\beta H} A) \quad (2.1)$$

where, density operator " $\rho = e^{-\beta H}$ "; H is the Hamiltonian and partition function is $Z(\beta) = \text{Tr}(\rho)$ (similar to the vacuum to vacuum transition amplitude of QFT, but does not coincide with it).

Consider the set of eigenstates $|n\rangle$ of H , has discrete eigenvalues and they form a complete set of orthonormal states of a Hilbert space $\mathcal{H}$. Thus,

$$H|n\rangle = E_n|n\rangle; \quad \sum_n |n\rangle\langle n| = 1, \quad \langle m|n\rangle = \delta_{mn}, \quad (2.2)$$

and the density operator is written in terms of these states as

$$\rho = \sum \rho_{mn} |m\rangle\langle n|, \quad (2.3)$$

then,

$$\langle A \rangle = \frac{1}{Z(\beta)} \sum_n e^{-\beta E_n} \langle n|A|n\rangle \quad (2.4)$$

If we can define a new state, " $|0(\beta)\rangle$ " such that, we can write

$$\langle A \rangle = \langle 0(\beta)|A|0(\beta)\rangle = \frac{1}{Z(\beta)} \sum_n e^{-\beta E_n} \langle n|A|n\rangle \quad (2.5)$$

where, $|0(\beta)\rangle$ is the "**temperature dependent vacuum state**" in a new Hilbert space, which we will describe below.

Let us assume that the "thermal vacuum state", $|0(\beta)\rangle$ as a linear superposition of the states, $|n\rangle$, of Hilbert space, $\mathcal{H}$ as

$$|0(\beta)\rangle = \sum_n |n\rangle\langle n|0(\beta)\rangle = \sum_n f_n(\beta) |n\rangle \quad (2.6)$$

Then, we obtain (from eqns. 2.5, 2.6)

$$\langle 0(\beta) | A | 0(\beta) \rangle = \sum_{mn} f_m^*(\beta) f_n(\beta) \langle m | A | n \rangle = \frac{\sum e^{-\beta E_n} \langle n | A | n \rangle}{\sum e^{-\beta E_n}}. \quad (2.7)$$

which can be satisfied only if

$$f_m^*(\beta) f_n(\beta) = Z^{-1}(\beta) e^{-\beta E_n} \delta_{mn}, \quad (2.8)$$

then the δ_{mn} shows that $f_n(\beta)$ cannot be complex numbers. $f_n(\beta)$ are behave like orthogonal vector, so $|0(\beta)\rangle$ can not be an element of the original Hilbert space, but is a vector in a space spanned by state vectors, $|n\rangle$ and $f_n(\beta)$. To realize this representation we introduced a **“doubling of the Hilbert space”**.

Now we introduce a non-physical set of fields called the **“tilde fields”**. This new system needs to have vectors in a new space similar to $\mathcal{H}$, which has a unique states $|\tilde{n}\rangle$ corresponding to every state $|n\rangle$ in $\mathcal{H}$. Hence, the $|\tilde{n}\rangle$ are eigenstates of a Hamiltonian $\tilde{H}$, which form a complete set of orthonormal vectors in the new space $\tilde{\mathcal{H}}$. Thus,

$$\tilde{H}|\tilde{n}\rangle = E_n|\tilde{n}\rangle; \quad \sum_n |\tilde{n}\rangle \langle \tilde{n}| = 1, \quad \langle \tilde{m} | \tilde{n} \rangle = \delta_{mn} \quad (2.9)$$

The direct product $\mathcal{H} \otimes \tilde{\mathcal{H}}$ is a new Hilbert space, whose elements are $|m\rangle \langle n|$ which we denote as $|m, \tilde{n}\rangle$. These form a complete set of orthonormal vectors and any operator is a vector in that new Hilbert space.

The average of any operator 'A' is,

$$\langle m, \tilde{m} | A | n, \tilde{n} \rangle = \langle m | \otimes \langle \tilde{m} | A | n \rangle \otimes | \tilde{n} \rangle = \langle m | A | n \rangle \langle \tilde{m} | \tilde{n} \rangle = A_{mn} \delta_{mn} \text{ and that corresponding to '}\tilde{A}\text{' is, } \langle m, \tilde{m} | \tilde{A} | n, \tilde{n} \rangle = \langle \tilde{m} | \tilde{A} | \tilde{n} \rangle \langle m | n \rangle = \tilde{A}_{\tilde{m}\tilde{n}} \delta_{mn}.$$

So, we have written $\langle \tilde{m} | \tilde{n} \rangle = \delta_{mn}$, without reference to the tilde in the δ_{mn} . Now, we take

$$|f_n(\beta)\rangle = Z^{-1/2}(\beta) e^{-\beta E_n/2} |\tilde{n}\rangle, \quad (2.10)$$

which satisfies the properties of Hilbert space(may verify eq.[2.7]) as,

$$f_m^*(\beta) f_n(\beta) = z^{-1}(\beta) e^{-\beta E_m/2} \langle \tilde{m} | \tilde{n} \rangle e^{-\beta E_n/2} \text{ and then, } f_n^*(\beta) f_n(\beta) = z^{-1}(\beta) e^{-\beta E_n}.$$

Thus, **the thermal vacuum state** can be constructed as,

$$\begin{aligned} |0(\beta)\rangle &= Z^{-1/2}(\beta) \sum_n e^{-\beta E_n/2} |n, \tilde{n}\rangle \\ &= [Tr \rho]^{-1/2} \sum_n e^{-\beta E_n/2} |n, \tilde{n}\rangle \end{aligned} \quad (2.11)$$

An operator A acting on $\mathcal{H}$, we have

$$\begin{aligned}
\langle 0(\beta)|A|0(\beta) \rangle &= [Tr\rho]^{-1} \sum_m \sum_n e^{-\beta(E_m+E_n)/2} \langle m|A|n \rangle \langle \tilde{m}|\tilde{n} \rangle \\
&= [Tr\rho]^{-1} \sum_m \sum_n e^{-\beta(E_m+E_n)/2} A_{mn} \delta_{mn} \\
&= [Tr\rho]^{-1} Tr[\rho A] = \langle A \rangle.
\end{aligned} \tag{2.12}$$

2.2 Liouville space method: Construction of Density matrix

Another way we can show the doubling of the Hilbert space of a thermal system is the Liouville space concept. It is based on the observation that the operators acting on “the vector space $\mathcal{H}$ ” themselves form a “vector space $\tilde{\mathcal{H}}$ ”. Any operator, A which acts on $\mathcal{H}$ having the form

$$|A \rangle = \sum_m \sum_n A_{mn} |m \rangle \langle n| = \sum_m \sum_n A_{mn} |m, \tilde{n} \rangle, \tag{2.13}$$

can be treated as a vector $|A \rangle$ in the space spanned by the basis $|m \rangle \langle n|$. A scalar product of two such vectors $|A \rangle$ and $|B \rangle$ in this space is defined as

$$\langle A|B \rangle = Tr A^\dagger B = \sum_n \langle n|A^\dagger B|n \rangle. \tag{2.14}$$

A density operator is a state in Liouville space is given by

$$\rho = \sum_n e^{-\beta E_n} |n \rangle \langle n|, \tag{2.15}$$

the scalar product of states $|A \rangle$ and $|\rho \rangle$ is

$$\langle \rho|A \rangle = Tr \rho A = \langle A \rangle \tag{2.16}$$

Thus, the scalar product in the Liouville space is suitably defined in order that the thermal averages be represented as a scalar product. The orthonormality of the basis states $|m \rangle \langle n|$ of the Liouville space follows from the definition of the scalar product. When the states $|m \rangle \langle n|$ belonging to a new space are identified with the states $|m, \tilde{n} \rangle$ of the doubled Hilbert space of TFD, then two approaches-Liouville space formalism and TFD are essentially the same.

If we choose $|n\rangle$ to be the eigenstates of the number operators, then

$$|A\rangle = \sum_m \sum_n A_{mn} |m, \tilde{n}\rangle = \sum_m \sum_n A_{mn} \frac{(a^\dagger)^m (\tilde{a}^\dagger)^n}{\sqrt{m!n!}} |0, \tilde{0}\rangle. \quad (2.17)$$

with $[a, a^\dagger] = 1$, $[\tilde{a}, \tilde{a}^\dagger] = 1$ and $[a, \tilde{a}] = [a, \tilde{a}^\dagger] = 0$ and other commutators will also be zero.

For bosonic system we introduce the operators $a, \tilde{a}, a^\dagger$, and $\tilde{a}^\dagger$ as follows:

$$a|n, \tilde{m}\rangle = a|n\rangle\langle m| = \sqrt{n}|n-1\rangle\langle m| = \sqrt{n}|n-1, \tilde{m}\rangle, \quad (2.18a)$$

$$\tilde{a}|n, \tilde{m}\rangle = \tilde{a}|n\rangle\langle m| = \sqrt{m}|n\rangle\langle m-1| = \sqrt{m}|n, \tilde{m}-1\rangle, \quad (2.18b)$$

$$a^\dagger|n, \tilde{m}\rangle = a^\dagger|n\rangle\langle m| = \sqrt{n+1}|n+1\rangle\langle m| = \sqrt{n+1}|n+1, \tilde{m}\rangle, \quad (2.18c)$$

$$\tilde{a}^\dagger|n, \tilde{m}\rangle = \tilde{a}^\dagger|n\rangle\langle m| = \sqrt{m+1}|n\rangle\langle m+1| = \sqrt{m+1}|n, \tilde{m}+1\rangle. \quad (2.18d)$$

Because of the tilde operators commute with non-tilde operators, we can easily see that the operators $\tilde{a}^\dagger$ and $\tilde{a}$ respectively replicate the action of $a^\dagger$ and a on $|n\rangle\langle m|$ from the right. i.e.,

$$|n\rangle\langle m|a^\dagger = \sqrt{m}|n\rangle\langle m-1| = \sqrt{m}|n, \tilde{m}-1\rangle, \quad (2.19a)$$

$$|n\rangle\langle m|a = \sqrt{m+1}|n\rangle\langle m+1| = \sqrt{m+1}|n, \tilde{m}+1\rangle. \quad (2.19b)$$

The identity state, $|I\rangle$ in terms of the number states

$$|I\rangle = \sum |n, \tilde{n}\rangle, \quad (2.20)$$

it gives,

$$a|I\rangle = \tilde{a}^\dagger|I\rangle \quad \text{and} \quad a^\dagger|I\rangle = \tilde{a}|I\rangle, \quad (2.21)$$

which are called “**tilde conjugation rules**”.

From (2.11) $|0(\beta)\rangle$ in terms of $|\rho\rangle$ is

$$|0(\beta)\rangle = \sum_n e^{-\beta E_n/2} |n, \tilde{n}\rangle = \rho^{1/2} |I\rangle, \quad (2.22)$$

for a normalized density operator. Here, ρ is a diagonal matrix with elements $e^{-\beta E_n}$ representing the density operator in the basis of the eigenstates of H . The state $|I\rangle = \sum |n, \tilde{n}\rangle$ is to be noted that it does not have a finite norm. But this vector proves to be useful in defining most of the states $|\rho\rangle$. It has the property that it is independent of basis vectors.

Here it is possible to generalize the definition of the thermal state from eq.2.22 to

$$|\rho^\alpha \rangle = \sum_n e^{-\beta E_n \alpha} |n, \tilde{n}\rangle = \rho^\alpha |I\rangle \quad (2.23)$$

where, $1/2 \leq \alpha \leq 1$ correspondingly, we have

$$\langle \rho^{1-\alpha} | = \sum_n \langle n, \tilde{n} | e^{-\beta E_n (1-\alpha)} = \langle I | \rho^{1-\alpha}, \quad (2.24)$$

so that, the thermal expectation value is

$$\langle A \rangle = \langle \rho^{1-\alpha} | A | \rho^\alpha \rangle. \quad (2.25)$$

This generalization of the thermal state is easily justified by using the cyclic property of trace operation, since

$$\langle A \rangle = \text{Tr}(\rho A) = \text{Tr} \rho^{1-\alpha} A \rho^\alpha. \quad (2.26)$$

In the $\alpha = \frac{1}{2}$ representation, the state $|\rho^\alpha \rangle$ is called the thermal ground state[5]. Because of its symmetrical nature, ensemble averages seems like pure state averages. Although, the only representation available for non-equilibrium phenomena is $\alpha = 1$.

The Liouville von Neumann equation is

$$\frac{\partial}{\partial t} \rho^\alpha(t) = \frac{-i}{\hbar} [H, \rho^\alpha], \quad (2.27)$$

and by applying $|I\rangle$ on RHS of above equation, obtains Schrödinger type equation.

$$\frac{\partial}{\partial t} |\rho^\alpha(t)\rangle = -i\hat{H} |\rho^\alpha\rangle, \quad (2.28)$$

where,

$$-i\hat{H} = i(H - \tilde{H}). \quad (2.29)$$

Now the solution is,

$$|\rho^\alpha(t)\rangle = \exp[-iHt] \otimes \exp[-i\tilde{H}t] |\rho^\alpha(0)\rangle \quad (2.30)$$

here, $|\rho^\alpha(0)\rangle$ is an initial state in $\mathcal{H} \otimes \tilde{\mathcal{H}}$. We use this density operator to study the entanglement properties of various systems in my research work.

For dissipative systems, the density operator evolves as (for $\alpha = 1$) is,

$$\frac{\partial}{\partial t} \rho(t) = \frac{-i}{\hbar} (H\rho - \rho H) + L\rho, \quad (2.31)$$

here, L be the Liouville term. In similar way, we apply $|I\rangle$ on RHS of the above equation, one obtains the Schrödinger-type equation for this system is

$$\frac{\partial}{\partial t}|\rho(t)\rangle = -i\hat{H}|\rho\rangle, \quad (2.32)$$

where,

$$-i\hat{H} = i(H - \tilde{H}) + L. \quad (2.33)$$

Then, similar way like in general case (eq.2.30), we can construct a density operator of the system as the solution of the eqn.(2.32) and study its entanglement properties.

2.3 The construction of density operator for various thermal systems using TFD

2.3.1 For Bosonic oscillator

Now we construct the density matrix $|\rho\rangle$ for the bosonic oscillator using the rule in eq.2.23. Consider an ensemble of free bosons[7, 8] with frequency ' ω ' described by the Hamiltonian

$$H = \omega a^\dagger a; (\hbar = 1) \quad (2.34)$$

with $H|n\rangle = n\omega|n\rangle, \quad \text{here, } n = 0, 1, 2, 3, \dots\infty \quad (2.35)$

here, $|n\rangle$ be the fock states and $a, a^\dagger$ are annihilation and creation operators, which satisfy the commutation relations: $[a, a^\dagger] = 1; [a, a] = [a^\dagger, a^\dagger] = 0$.

The finite temperature of the Hamiltonian of the system is

$$\hat{H} = H - \tilde{H}, \quad (2.36)$$

here, we introducing the tilde field as, $\tilde{H} = \omega \tilde{a}^\dagger \tilde{a}$ with tilde creation and annihilation operators also satisfy the commutation relations. The product space is spanned by the set of orthonormal basis states, $\{|0, \tilde{0}\rangle, a^\dagger|0, \tilde{0}\rangle, \dots, \frac{1}{\sqrt{n!}}(a^\dagger)^n|0, \tilde{0}\rangle, \dots\}$ where " $|0, \tilde{0}\rangle$ " is the "thermal vacuum state". Thus, the thermal vacuum state for bosonic oscillator can be

written as,

$$\begin{aligned}
|0(\beta) > &= \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-n\beta\omega/2} |n, \tilde{n} > \\
&= \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-n\beta\omega/2} \frac{(a^\dagger)^n (\tilde{a}^\dagger)^n}{n!} |0, \tilde{0} > \\
&= \sqrt{(1 - e^{-\beta\omega})} \exp(e^{-\beta\omega/2} a^\dagger \tilde{a}^\dagger) |0, \tilde{0} >,
\end{aligned} \tag{2.37}$$

since, by the normalization, we have $Z(\beta) = \frac{1}{1 - e^{-\beta\omega}}$.

In general, the normalized canonical density operator for a bosonic oscillator is

$$\rho = (1 - e^{-\beta\omega}) \exp(-\beta\omega a^\dagger a), \tag{2.38}$$

with $\beta = \frac{1}{kT}$. The expectation value of the “number operator” is

$$\bar{n} = \langle a^\dagger a \rangle = \text{Tr} \rho a^\dagger a = \frac{e^{-\beta\omega}}{1 - e^{-\beta\omega}} = \frac{f}{1 - f}, \tag{2.39}$$

which is called the **boson distribution function** in thermal equilibrium. Here, $f = e^{-\beta\omega}$ by inverting it, we have $f = \frac{\bar{n}}{1 + \bar{n}}$, and we get

$$\rho = (1 - f) \exp(-\beta\omega a^\dagger a). \tag{2.40}$$

In TFD, the state $|\rho^\alpha >$ is given by

$$\begin{aligned}
|\rho^\alpha > &= \rho^\alpha |I > = (1 - f)^\alpha \exp(-\beta\omega \alpha a^\dagger a) |I > \\
&= (1 - f)^\alpha \sum_n e^{-\beta\omega n \alpha} |n, \tilde{n} > \\
&= (1 - f)^\alpha \sum_n e^{-\beta\omega n \alpha} \frac{(a^\dagger)^n (\tilde{a}^\dagger)^n}{n!} |0, \tilde{0} > \\
&= (1 - f)^\alpha \exp(f^\alpha a^\dagger \tilde{a}^\dagger) |0, \tilde{0} >
\end{aligned} \tag{2.41}$$

which can be written as

$$|\rho^\alpha > = \frac{(1 - f)^\alpha}{(1 - f^{2\alpha})^{1/2}} \exp(-iG_B) |0, \tilde{0} >, \tag{2.42}$$

$$\text{with} \quad G_B = -i\theta(\beta)(a\tilde{a} - a^\dagger \tilde{a}^\dagger) \quad \& \quad \tanh \theta = e^{-\beta\omega} = f. \tag{2.43}$$

Thus the unitary operator transforming “ $|0, \tilde{0} \rangle$ ” into “ $|0(\beta) \rangle$ ”, is given by “ $U_B(\beta) = e^{-iG_B}$ ” which is called a **Bogoliubov transformation**.

The equivalent of 2.41 and 2.42 can be shown using the “disentangling theorem” for $SU(1,1)$ and its realization in terms of $a, a^\dagger, \tilde{a}$ and $\tilde{a}^\dagger$ is given as [[32, 33]]

$$K_+ = a^\dagger \tilde{a}^\dagger, \quad K_- = a \tilde{a}, \quad \& \quad K_3 = \frac{1}{2}(a^\dagger a + \tilde{a}^\dagger \tilde{a} + 1), \quad (2.44)$$

satisfies “commutation relations of $SU(1,1)$ algebra”,

$$[K_-, K_+] = 2K_3; \quad [K_3, K_\pm] = \pm K_\pm \quad (2.45)$$

with “ $K_0 = (a^\dagger a - \tilde{a}^\dagger \tilde{a})$ ” is the “Casimir operator”.

The disentangling theorem enables operators like $\exp(\gamma_+ K_+ + \gamma_3 K_3 + \gamma_- K_-)$ to be written as products of exponentials as

$$\text{“} \exp(\gamma_+ K_+ + \gamma_3 K_3 + \gamma_- K_-) = \exp(\Gamma_+ K_+) \exp(\ln \Gamma_3 K_3) \exp(\Gamma_- K_-) \text{”} \quad (2.46)$$

where,

$$\Gamma_\pm = \frac{2\gamma_\pm \sinh \phi}{2\phi \cosh \phi - \gamma_3 \sinh \phi} \quad \& \quad \Gamma_3 = \left(\frac{2\phi}{2\phi \cosh \phi - \gamma_3 \sinh \phi} \right)^2 \quad (2.47)$$

with

$$\phi^2 = \frac{\gamma_3^2}{4} - \gamma_+ \gamma_-, \quad (2.48)$$

Only in the $\alpha = \frac{1}{2}$ representation are $|0, \tilde{0} \rangle$ and $|\rho^{\frac{1}{2}} \rangle$ related by a unitary transformation (from eqns. 2.37 & 2.42); for which,

$$|0(\beta) \rangle = |\rho^{1/2} \rangle = \exp(-iG_B) |0, \tilde{0} \rangle, \quad (2.49)$$

The Bogoliubov-Valatin(BV) transformation used for defining thermal counterparts of non-thermal operators.

$$a(\beta) = \exp[-iG_B] a \exp[iG_B] = (\cosh \theta) a - (\sinh \theta) \tilde{a}^\dagger. \quad (2.50)$$

We can show that

$$a(\beta) |\rho^{1/2} \rangle = 0, \quad (2.51)$$

where $|\rho^{1/2} \rangle$ is the “thermal vacuum” in the $\alpha = \frac{1}{2}$ representation. This is called the “thermal-state condition”.

Thermal operators for Bosonic system

The “**temperature dependent operators**” are defined in terms of the zero temperature annihilation ($a, \tilde{a}$) and creation ($a^\dagger, \tilde{a}^\dagger$) operators. as follows,

$$a(\beta) = u(\beta)a - v(\beta)\tilde{a}^\dagger; \quad a^\dagger(\beta) = u(\beta)a^\dagger - v(\beta)\tilde{a}; \quad (2.52a)$$

$$\tilde{a}(\beta) = u(\beta)\tilde{a} - v(\beta)a^\dagger; \quad \tilde{a}^\dagger(\beta) = u(\beta)\tilde{a}^\dagger - v(\beta)a, \quad (2.52b)$$

and they can be represented in matrix form as,

$$\begin{bmatrix} a(\beta) \\ \tilde{a}^\dagger(\beta) \end{bmatrix} = \bar{U}(\theta) \begin{bmatrix} a \\ \tilde{a}^\dagger \end{bmatrix} = \begin{bmatrix} \cosh\theta(\beta) & -\sinh\theta(\beta) \\ -\sinh\theta(\beta) & \cosh\theta(\beta) \end{bmatrix} \begin{bmatrix} a \\ \tilde{a}^\dagger \end{bmatrix} \quad (2.53)$$

.

2.3.2 For Fermionic oscillator

In similar way, we construct the density operator $|\rho \rangle$ for the fermionic oscillator[7, 8] with the hamiltonian, $H = \omega a^\dagger a$ where, the annihilation and creation operators obey anti-commutation relations $\{a, a^\dagger\} = 1$, and $\{a, a\} = 0 = \{a^\dagger, a^\dagger\} = 0$. Here this system has two dimensional Hilbert space with the states given by $|0 \rangle$ and $|1 \rangle = a^\dagger|0 \rangle$.

Considering TFD, we define $\tilde{H} = \omega \tilde{a}^\dagger \tilde{a}$; here, $\tilde{a}^\dagger$ and $\tilde{a}$ operators also follows the anti-commutation relations. The product space of states for two systems, becomes four dimensional with the basis states, $\{|0, \tilde{0} \rangle, \tilde{a}^\dagger|0, \tilde{0} \rangle (= |0, \tilde{1} \rangle), a^\dagger|0, \tilde{0} \rangle (= |1, \tilde{0} \rangle), a^\dagger \tilde{a}^\dagger|0, \tilde{0} \rangle (= |1, \tilde{1} \rangle)\}$.

Then the “thermal vacuum state” for fermionic oscillator can be written as,

$$\begin{aligned}
|0(\beta) \rangle &= \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\beta E_n/2} |n, \tilde{n} \rangle = \frac{1}{\sqrt{Z(\beta)}} (|0, \tilde{0} \rangle + e^{-\beta \epsilon_1/2} |1, \tilde{1} \rangle \\
&= \frac{1}{\sqrt{Z(\beta)}} (1 + e^{(-\beta\omega/2)a^\dagger \tilde{a}^\dagger}) |0, \tilde{0} \rangle \\
&= \frac{1}{\sqrt{(1 + e^{-\beta\omega})}} (1 + e^{(-\beta\omega/2)a^\dagger \tilde{a}^\dagger}) |0, \tilde{0} \rangle, \quad (2.54)
\end{aligned}$$

since, by the normalization, we have “ $Z(\beta) = 1 + e^{-\beta\omega}$ ”.

In general, the normalized canonical density operator for a fermionic oscillator is

$$\rho = (1 + e^{-\beta\omega})^{-1} \sum_{n=0}^1 e^{-\beta\omega a^\dagger a} |n \rangle \langle n| \quad (2.55)$$

The “number operator” is

$$\bar{n} = \langle a^\dagger a \rangle = \text{Tr} \rho a^\dagger a = \frac{1}{1 + e^{\beta\omega}} = \frac{f}{1 + f}, \quad (2.56)$$

which is called the fermionic distribution function in thermal equilibrium. Here, $f = e^{-\beta\omega}$ by inverting it, we have $f = \frac{\bar{n}}{1-\bar{n}}$, and we get

$$\rho = (1 + f)^{-1} \exp(-\beta\omega a^\dagger a) |n, \tilde{n} \rangle. \quad (2.57)$$

In TFD, the general state $|\rho^\alpha \rangle$ is given by

$$\begin{aligned}
|\rho^\alpha \rangle &= \rho^\alpha |I \rangle = (1 + f)^{-\alpha} \sum_n e^{-\beta\omega n \alpha} \frac{a^{\dagger n} \tilde{a}^{\dagger n}}{n!} |0, \tilde{0} \rangle \\
&= (1 + f)^\alpha \exp(f^\alpha a^\dagger \tilde{a}^\dagger) |0, \tilde{0} \rangle \quad (2.58)
\end{aligned}$$

We can write above expression in the form

$$|\rho^\alpha \rangle = \frac{(1 + f^{2\alpha})^{1/2}}{(1 + f)^\alpha} \exp(-iG_F) |0, \tilde{0} \rangle \quad (2.59)$$

with

$$G_F = i\theta(\beta)(a^\dagger \tilde{a}^\dagger + a \tilde{a}) \quad \& \quad \tan \theta = e^{-\beta\omega} = f. \quad (2.60)$$

In terms of fermionic operators we have the realization of SU(1,1) in terms of $a, a^\dagger, \tilde{a}$ and

$\tilde{a}^\dagger$ is given as

$$K_+ = a^\dagger \tilde{a}^\dagger, \quad K_- = a \tilde{a}, \quad \& \quad K_3 = \frac{1}{2}(a^\dagger a + \tilde{a}^\dagger \tilde{a} + 1), \quad (2.61)$$

satisfies “commutation relations of SU(1,1) algebra”,

$$[K_-, K_+] = 2K_3; \quad [K_3, K_\pm] = \pm K_\pm \quad (2.62)$$

with “ $K_0 = (a^\dagger a + \tilde{a}^\dagger \tilde{a})$ ” as the “Casimir operator”.

The disentangling theorem enables operators like $\exp(\gamma_+ K_+ + \gamma_3 K_3 + \gamma_- K_-)$ to be written as products of exponentials as

$$“\exp(\gamma_+ K_+ + \gamma_3 K_3 + \gamma_- K_-) = \exp(\Gamma_+ K_+) \exp(\ln \Gamma_3 K_3) \exp(\Gamma_- K_-)” \quad (2.63)$$

where,

$$\Gamma_\pm = \frac{2\gamma_\pm \sinh \phi}{2\phi \cosh \phi - \gamma_3 \sinh \phi} \quad \& \quad \Gamma_3 = \left(\frac{2\phi}{2\phi \cosh \phi - \gamma_3 \sinh \phi} \right)^2 \quad (2.64)$$

with

$$\phi^2 = \frac{\gamma_3^2}{4} - \gamma_+ \gamma_-, \quad (2.65)$$

Only in the $\alpha = \frac{1}{2}$ representation are $|0, \tilde{0} \rangle$ and $|\rho^{\frac{1}{2}} \rangle$ related by a unitary transformation (from eqns.2.54 & 2.59); for which,

$$|0(\beta) \rangle = |\rho^{1/2} \rangle = \exp(-iG_F)|0, \tilde{0} \rangle, \quad (2.66)$$

This is a Bogoliubov-Valatin(BV) transformation. Now

$$a(\beta) = \exp[-iG_F] a \exp[iG_F] = (\cos \theta) a - (\sin \theta) \tilde{a}^\dagger. \quad (2.67)$$

It can be easily verified that

$$a(\beta) |\rho^{1/2} \rangle = 0, \quad (2.68)$$

where $|\rho^{1/2} \rangle$ is the thermal vacuum in the $\alpha = \frac{1}{2}$ representation. This is the thermal-state condition.

Thermal operators for fermionic system

The “**temperature dependent operators**” are defined in terms of non-thermal operators as follows, (by using the identity, $A(\theta) = U(\theta)AU^\dagger(\theta)$)

$$a(\beta) = u(\beta)a - v(\beta)\tilde{a}^\dagger; \quad a^\dagger(\beta) = u(\beta)a^\dagger - v(\beta)\tilde{a} \quad (2.69a)$$

$$\tilde{a}(\beta) = u(\beta)\tilde{a} + v(\beta)a^\dagger; \quad \tilde{a}^\dagger(\beta) = u(\beta)\tilde{a}^\dagger + v(\beta)a \quad (2.69b)$$

The thermal operators can be represented in matrix form as,

$$\begin{bmatrix} a(\beta) \\ \tilde{a}^\dagger(\beta) \end{bmatrix} = \bar{U}(\theta) \begin{bmatrix} a \\ \tilde{a}^\dagger \end{bmatrix} = \begin{bmatrix} \cos\theta(\beta) & -\sin\theta(\beta) \\ \sin\theta(\beta) & \cos\theta(\beta) \end{bmatrix} \begin{bmatrix} a \\ \tilde{a}^\dagger \end{bmatrix} \quad (2.70)$$

2.4 Construction of various thermal states using TFD

Since we have seen earlier that temperature effects can be incorporated by doubling the Hilbert space($|0 > \otimes |\tilde{0} >$). We can define special thermal states which will be useful for our applications. A general thermal state can be defined as as,

$$|\alpha, r, \beta > = S(r)D(\alpha)|0(\beta) > \quad (2.71)$$

where, $S(r) = \text{Exp}[\frac{1}{2}(ra^2 - r\tilde{a}^{\dagger 2})]$, $D(\alpha) = \text{Exp}[\alpha a^\dagger - \bar{\alpha}a]$ and $|0(\beta) >$ given in eq.(2.11); $G(\theta) = \text{Exp}[\theta(a\tilde{a} - a^\dagger\tilde{a}^\dagger)]$ are the squeezing, coherence and thermal operators respectively which re defined for complex r , α and real θ as,[34]. Here $(a, a^\dagger)$ and $(\tilde{a}, \tilde{a}^\dagger)$ from commuting sets of boson operators. The parameter θ related to temperature T through the expressions $f(\beta) = [\exp(\beta\omega) - 1]^{-1}$; $f^{1/2}(\beta) = \sinh(\theta)$ and $\beta = \hbar\omega/k_B T$.

2.4.1 Thermal coherent state

A “thermal coherent state” (TCS) defined as,

$$|TCS > = e^{-iG}|CS > = e^{\alpha a^\dagger - \alpha^* a} e^{\phi \tilde{a}^\dagger - \phi^* \tilde{a}} |0, \tilde{0} > = D(\alpha)|0(\beta) > \quad (2.72)$$

The density operator for TCS can be defined as,

$$|\rho_{TCS} \rangle = |TCS \rangle \langle TCS| = N_{\rho_{TCS}} |I \rangle \quad (2.73)$$

where, $N_{\rho_{TCS}} = D^\dagger(\alpha) e^{-\beta\omega a^\dagger a} D(\alpha)$

2.4.2 Thermal squeezed coherent state

A “thermal squeezed coherent state” (TSCS) defined as,

$$|TSCS \rangle = e^{-iG} |SCS \rangle \quad (2.74)$$

. In TFD, the density operator for this state written as,

$$|\rho_{TSCS} \rangle = |TSCS \rangle \langle TSCS| = N_{\rho_{TSCS}} |I \rangle \quad (2.75)$$

where, $N_{\rho_{TSCS}} = D^\dagger(\alpha) S^\dagger(r) e^{-\beta\omega a^\dagger a} S(r) D(\alpha)$ and $|I \rangle = \sum |n, \tilde{n} \rangle$.

In this chapter we have given a description of Thermofield dynamics, a finite temperature formalism which allows statistical averages of thermal operators to be written and quantum mechanical expectation values on a doubled Hilbert space. We have defined the thermal vacuum and shown how to find thermal counterparts of fermionic and bosonic systems. We have also introduced the thermal coherent state and the thermal squeezed state. This formalism will be applied to solve the master equation of a variety of many bodies in physics in the later chapters.

Chapter 3

Quantum entanglement and its measures

Quantum entanglement(QE) occurs when a group of interacting quantum particles are produced such that the quantum state of each particle of the group is not independent of the other states, even for large spatial separation. Quantum entanglement is a purely quantum feature associated with composite quantum system and used extensively as a powerful computational resource. There exist different criterion to quantify entanglement. To study the entanglement properties, we use the density matrix formalism. In this chapter we define the various measures of quantum entanglement for a bipartite system: entropy, logarithmic negativity and decoherence. The influence of the environment can cause open systems to lose their key quantum features like decoherence and entanglement. Since in this thesis we are using thermo-field dynamics to study systems interacting with thermal baths and with noise, we must give a description of the master equation. In standard quantum treatments, the dynamics of the reduced quantum states is given by a master equation(ME) after tracing out the environmental degrees of freedom. This will be explained in detail in this chapter.

3.1 Density matrix formalism- Master Equation

For an interacting quantum system, the density matrix formalism is used to study its interaction with the environment. The evolution of density matrix is given by its associated master equation. The density operator $\rho(t)$ evolves according to a (first-order) differential

equation in time t , then that means the $\rho(t + dt)$ is completely determined by $\rho(t)$. This can fail in an open system because dissipation leads to information loss to the reservoir, which can flow back causing non-Markovian fluctuations of the system.

In quantum mechanics, in general, we define a density operator of a quantum system with the complete set of orthonormal states $|\psi_i\rangle$ in a Hilbert space H as

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|. \quad (3.1)$$

The statistical distribution of these states is governed by the probability, p_i : $0 \leq p_i \leq 1$. $\langle\psi_i|\rho|\psi_i\rangle = p_i$ and $\sum_i \langle\psi_i|\rho|\psi_i\rangle = \sum_i p_i = 1$.

The density operator in the number basis is

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_i p_i \sum_{nm} c_{ni} c_{mi}^* |n\rangle\langle m|. \quad (3.2)$$

The matrix elements of ρ in this representation are $\langle n|\rho|m\rangle = \sum_i p_i c_{ni} c_{mi}^*$ with the diagonal matrix elements, $\langle n|\rho|n\rangle = \sum_i p_i |c_{ni}|^2$ and the density matrix is the Hermitian.

For two level systems described by a single wavefunction, the density matrix is given by

$$\rho = \sum_{nm} c_{ni} c_{mi}^* |n\rangle\langle m|, \quad (3.3)$$

here, $\langle n|\rho|m\rangle = c_{ni} c_{mi}^*$ and $\langle n|\rho|n\rangle = |c_{ni}|^2$.

The trace of the density matrix is equal to unity, because $Tr(\rho) = \sum_{ni} p_i \langle\psi_i|n\rangle\langle n|\psi_i\rangle = \sum_i p_i \langle\psi_i|\psi_i\rangle = 1$; and is independent of the basis used for the representation.

The equation of motion of the density matrix follows normally from the definition of ρ and the time-dependent Schrödinger equation. Using

$$\frac{\partial}{\partial t} |\psi(t)\rangle = -\frac{i}{\hbar} H |\psi(t)\rangle; \quad \frac{\partial}{\partial t} \langle\psi(t)| = \frac{i}{\hbar} \langle\psi(t)| H. \quad (3.4)$$

$$\begin{aligned} \frac{\partial \rho(t)}{\partial t} &= \frac{\partial}{\partial t} [|\psi\rangle\langle\psi|] = \left[\frac{\partial}{\partial t} |\psi(t)\rangle \right] \langle\psi(t)| + |\psi(t)\rangle \left[\frac{\partial}{\partial t} \langle\psi(t)| \right] \\ &= -\frac{i}{\hbar} H |\psi(t)\rangle\langle\psi(t)| + \frac{i}{\hbar} |\psi(t)\rangle\langle\psi(t)| H \end{aligned}$$

$$= -\frac{i}{\hbar}[H, \rho(t)], \quad (3.5)$$

here, H is the “Hamiltonian of system-environment interaction”. The above equation is called the ‘Liouville von-Neumann equation’.

The Lindblad or quantum Liouvillian equation [35] is a general Markovian master equation which describes the non-unitary evolution of the density operator, ρ and is given by

$$\frac{\partial \rho(t)}{\partial t} = -\frac{i}{\hbar}[H, \rho(t)] + \gamma \sum_j [L_j \rho L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_j, \rho\}]. \quad (3.6)$$

here, L_j are the Lindbladian operators. The first term on the RHS is the Liouville von Neumann equation. The second term is the Lindbladian and is non-unitary and it is used for dissipation and decoherence to the quantum measurement process.

In subsequent chapters we will construct the density matrix for some open quantum systems, we use it to quantify their entanglement properties.

3.2 Quantum entanglement measures in bipartite system

We present tools to quantify quantum entanglement. We define the pure and mixed states as:

Pure states

For a quantum system the density operator is said to be in a pure state if $\rho = |\psi_i\rangle\langle\psi_i|$, which is obtained by fixing $p_i = 1$ in eq.3.1. For a pure state one has

$$Tr(\rho^2) = Tr(\rho) = 1. \quad (3.7)$$

Mixed states

The density operator of a mixed state is,

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \quad (3.8)$$

with $p_i \geq 0$ and $\sum_i p_i = 1$. In this case, $Tr(\rho) = 1$ still holds but, $Tr(\rho^2) < 1$. In general, for bipartite system the density operator of a mixed state can be written as

$$\rho = \sum_i p_i |\psi_A\rangle\langle\psi_A| \otimes |\psi_B\rangle\langle\psi_B| = \sum_i p_i \rho_A \otimes \rho_B \quad (3.9)$$

.

Let us take A, B are two quantum systems in Hilbert spaces H_A, H_B respectively. Then the composite quantum system of A and B in Hilbert space $H_A \otimes H_B$. Suppose $|e\rangle$ is a basis in H_A and $|f\rangle$ is a basis in H_B , then $|e_i\rangle \otimes |f_j\rangle$ be the basis in $H_A \otimes H_B$.

A general state $|\psi\rangle$ in $H_A \otimes H_B$ can be written in the Schmidt form as

$$|\psi_{ij}\rangle = \sum_{i,j} p_{ij} |e_i\rangle \otimes |f_j\rangle, \quad (3.10)$$

here, the number of non-zero elements in p_{ij} 's is called the 'Schmidt rank'. If the Schmidt rank is equal to 1, $|\psi\rangle$ is a separable or product state and if it is greater than 1, $|\psi\rangle$ is a non-separable or entangled state.

Another way of checking, whether the state $|\psi\rangle$ in $H_A \otimes H_B$ is entangled through **the reduced density matrix**. Given the density matrix,

$$\rho = |\psi\rangle\langle\psi|, \quad (3.11)$$

and tracing over the subsystem B

$$\rho_A = Tr_B(\rho) \quad (3.12)$$

gives the “reduced density matrix” of ρ on subsystem A . If ρ_A is such that $\rho_A^2 = \rho_A$, then the state $|\psi\rangle$ is a “separable state”, otherwise $|\psi\rangle$ is an “entangled state”.

3.2.1 Entropy of entanglement

Entanglement for a pure state bipartite system is defined as[36] “the von-Neumann entropy of the reduced density matrix traced with respect to one of the systems” (either of its two parts when considered separately). Thus, the entanglement of a state with density operator ρ is

$$S(\rho) = -Tr(\rho_A \log_2 \rho_A) = -Tr(\rho_B \log_2 \rho_B) = -\sum_{i=0}^N \lambda_i \log \lambda_i, \quad (3.13)$$

here, $S(\rho)$ is called ‘the entropy of entanglement’ or ‘the von-Neumann entropy of the system with density operator ρ ’; $\rho_A = Tr_B(\rho)$ and $\rho_B = Tr_A(\rho)$ are the reduced density operators for A and B systems respectively defined in eq.3.12 and the $\{\lambda_i\}$ are the eigenvalues of the reduced density operator. $S(\rho)$ varies from ‘0’ to ‘1’ (‘0’ for product states & ‘1’ for maximally entangled states). For the rest of the thesis we will call $S(\rho)$, simply S .

While for closed systems, in accordance with the second law of thermodynamics, entropy is always positive, for a damped system in interaction with, say, a heat bath system, we can see the negative entropy of entanglement when the damping effect is increasing. The significance of negative entropy in a many-body quantum system is that it describes classical correlations from quantum entanglement. Because quantum systems are non-separable quantum entropies can be negative is called von-Neumann entropy. This is because of virtual information carry by entangle particles.[37]. Therefore quantum mechanical extension of classical information theory can take into an account entanglement by negative conditional entropies. This provides an insight into the information flow problem in quantum communication processes.

3.2.2 Logarithmic negativity

Another useful entanglement measure is the “logarithmic negativity”[38]. For a bipartite system with density matrix, ρ the log-negativity is,

$$E_N(T) = \log_2 \|\rho^T\|, \quad \text{here, } \|\rho^T\| = (2N(\rho) + 1) \quad (3.14)$$

where “ ρ^T is the partial transpose of ρ ”, the symbol $\|\cdot\|$ denotes the trace norm and “ $N(\rho)$ is the absolute value of the sum of all the negative eigenvalues of ρ^T ”. Entanglement is implied by a non-zero value of E_N .

Logarithmic negativity using Covariance matrix(CM) method:

For a bipartite system with a local thermal environment which can be described by the Lindblad master equation (in eq.3.6) for the density operator $|\rho(t)\rangle$ in the Schrödinger representation and with the initial state, $|\rho(0)\rangle$ and the Hamiltonian of this open system in phase space is $H(x, y, p_x, p_y)$. In phase space one can define the “**Covariance matrix**”(CM) [39, 40](which is a real, symmetric and positive 4x4 matrix) as

$$\sigma = \begin{pmatrix} \sigma_{xx}(t) & \sigma_{xp_x}(t) & \sigma_{xy}(t) & \sigma_{xp_y}(t) \\ \sigma_{xp_x}(t) & \sigma_{p_x p_x}(t) & \sigma_{yp_x}(t) & \sigma_{p_x p_y}(t) \\ \sigma_{xy}(t) & \sigma_{yp_x}(t) & \sigma_{yy}(t) & \sigma_{yp_y}(t) \\ \sigma_{xp_y}(t) & \sigma_{p_x p_y}(t) & \sigma_{yp_y}(t) & \sigma_{p_y p_y}(t) \end{pmatrix} \quad (3.15)$$

here the matrix elements are defined as:

$$\sigma_{ij} = Tr[(\xi_i \xi_j + \xi_j \xi_i) \rho] - Tr(\xi_i \rho) Tr(\xi_j \rho), \quad (3.16)$$

where $\xi = (x, p_x, y, p_y)$.

The matrix σ is a bonafide “covariance matrix” iff it satisfies the uncertainty relation[41]

$$\det(\sigma + \frac{i}{2}\Omega) \geq 0; \quad \text{here, } \Omega = \bigoplus_{i=1}^2 J; \quad \text{with } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (3.17)$$

The evolution of the initial two-mode 4x4 CM $\sigma(0)$ of the system(in general Gaussian channel) is

$$\sigma(t) = (X_1(t) \oplus X_2(t))\sigma(0)(X_1(t) \oplus X_2(t))T + (Y_1(t) \oplus Y_2(t)), \quad (3.18)$$

where, $X_{1,2}(t)$ and $Y_{1,2}(t)$ are the system-environment interaction functions. The CM has a block form

$$\sigma(t) = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & -c \\ c & 0 & b & 0 \\ 0 & -c & 0 & b \end{pmatrix} \quad (3.19)$$

where, the symmetric covariance matrices corresponding to reduced one-mode states are A and B and C is the cross-correlation matrix.

Then the CM satisfies the separability condition (“**Simon’s criterion**”)[41] for any two

mode state is

$$Det(A)Det(B) + \left(\frac{1}{4} - |Det(C)|\right)^2 - tr(AJCJB JC^T J) \geq \frac{1}{4}(Det(A) + Det(B)) \quad (3.20)$$

If the condition is satisfied, the state $|\rho(t)\rangle$ is entangled.

For the two mode Gaussian state the smallest symplectic eigenvalues of the partial transposed CM are $v_{\pm} = \sqrt{ab} \pm c$. Then the log-negativity is,

$$E_N(t) = \max[0, -\ln 2v_{<}(t)]; \quad (3.21)$$

here, $v_{<}(t) = \min[v_+, v_-]$ smallest symplectic eigenvalue.

For $E_N > 0$, the state is entangled, otherwise separable. This form of logarithmic-negativity (using covariance matrix method) will be used for complex systems, we will use this method in the next chapter.

3.2.3 Decoherence

Entanglement of pure states is easy to quantify, however, for mixed states, quantum correlations are weakened, hence entanglement is hard to quantify for the composite quantum system. So, we need to define a new entanglement measure for mixed states. Here is an entanglement measure to quantify the mixed state entanglement, which is mainly because of the inevitable interactions of the system with the environment is called 'decoherence'. For a single environmental degree of freedom $\rho(t)$ is the density operator that describes the quantum state of the system on the Hilbert space H , for which decoherence is defined as

$$D(\rho) = Tr(\rho^2). \quad (3.22)$$

For open systems the study of decoherence, describes the effects of environment on entanglement evolution[42]. It helps us to study frictional and thermalization effects. Likewise, the predictions of decoherence theory in a number of experiments observed the gradual emergence of classical properties in a quantum system. The loss of coherence is due to environmental interaction, and is jointly due to both the environmental state and the interaction. The quantum behaviour loss due to the interaction with environmental degree of freedom that causes decoherence.

Chapter 4

Application of TFD to Coupled waveguides system in Quantum optics

Coupled waveguides have become useful tool for studying entanglement[43, 44]. Recently coherent phenomenon have been analysed by specially designed photonic waveguides[44]. These have shown promise of application to quantum computation and quantum metrology. In particular, superposition states of N-photons known as NOON-states have been generated directly on chip by Politi et. al.[45] and been shown to exhibit quantum interference. NOON states are special because they represent two orthogonal states in maximum superposition making them candidates for use in quantum information processing[46]. For using waveguides to generate entanglement we must take care of decoherence effects as they can destroy entanglement and decrease the efficiency[47]. Rai et. al.[48] have shown that entanglement between waveguide modes can be affected by losses by using the “quantum Liouville equation”. Here, we use thermo-field dynamics to calculate the effects of noise and decoherence in the coupled two mode waveguide system. By using different types of input states, we show the efficiency of the states for quantum information theory.

4.1 Coupled waveguides system

Coupled waveguides are used as a transmission medium in quantum communication. To transfer power between two waveguides and to study their entanglement properties the model of Rai and Agarwal[48] is used.

The Hamiltonian described by,

$$H = \hbar\omega(a^\dagger a + b^\dagger b) + \hbar J(a^\dagger b + b^\dagger a) \quad (4.1)$$

where modes a and b corresponds to the first and second waveguides respectively as shown in Figure(4.1), these modes obey bosonic commutation relations. The evanescent coupling in terms of distance between the two waveguides is given by ' J '. The density operator has a time evolution given by

$$\frac{\partial}{\partial t}\rho = -\frac{i}{\hbar}[H, \rho] \quad (4.2)$$

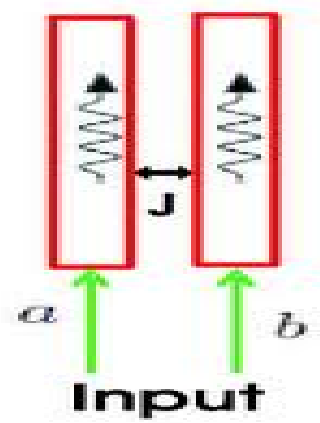


Figure 4.1: A two coupled waveguide system

The Liouville equation in the presence of damping is given by

$$\frac{\partial}{\partial t}\rho = -\frac{i}{\hbar}[H, \rho] + \mathcal{L}\rho, \quad (4.3)$$

where,

$$\mathcal{L}\rho = -\gamma(a^\dagger a \rho - \rho a^\dagger + a^\dagger \rho a + \rho a^\dagger a + b^\dagger b \rho - \rho b^\dagger + b^\dagger \rho b + \rho b^\dagger b) \quad (4.4)$$

where γ is dissipation in the material of the waveguide.

In absence of loss, the Heisenberg equations for the field operators give their evolution as,

$$a(t) = a(0) \cos(Jt) + ib(0) \sin(Jt); \quad (a \leftrightarrow b)$$

We solve the master equations(4.2, 4.3) exactly using TFD. This allows us to study the response to the coupling of different input states such as number, NOON and thermal states, in the coupled waveguide system. We show that in the absence of damping, an input vacuum state evolves into a two mode SU(2) coherent state. In the presence of damping, the vacuum state evolves into a two mode squeezed state and a thermal state into a thermal squeezed state.

4.2 Two coupled waveguides without damping($\gamma = 0$)

Without damping, the master equation is determined by,

$$\dot{\rho} = -i\omega(a^\dagger a \rho + b^\dagger b \rho - \rho a^\dagger a - \rho b^\dagger b) - iJ(a^\dagger b \rho + b^\dagger a \rho - \rho a^\dagger b - \rho b^\dagger a) \quad (4.5)$$

Applying the TFD-formalism to the master equation, we get the Schrodinger type wave equation

$$|\dot{\rho}\rangle = -i\hat{H}|\rho\rangle \quad (4.6)$$

here, the Hamiltonian $\hat{H}$ is given by

$$\hat{H} = \omega(a^\dagger a + b^\dagger b - \tilde{a}^\dagger \tilde{a} - \tilde{b}^\dagger \tilde{b}) + J(a^\dagger b + b^\dagger a - \tilde{a}^\dagger \tilde{b} - \tilde{b}^\dagger \tilde{a}) \quad (4.7)$$

which can be written as non-tilden and tilden parts: $\hat{H} = H - \tilde{H}$,

where,

$$H = \omega(a^\dagger a + b^\dagger b) + J(a^\dagger b + b^\dagger a) \quad (4.8)$$

$$\tilde{H} = \omega(\tilde{a}^\dagger \tilde{a} + \tilde{b}^\dagger \tilde{b}) + J(\tilde{a}^\dagger \tilde{b} + \tilde{b}^\dagger \tilde{a}) \quad (4.9)$$

Then the solution of eq.(4.6) is given by

$$|\rho(t)\rangle = \exp[-iHt] \otimes \exp[-i\tilde{H}t] |\rho(0)\rangle \quad (4.10)$$

where, $|\rho(0)\rangle$ is an initial state in $\mathcal{H} \otimes \tilde{\mathcal{H}}$. The two Hamiltonians H and $\tilde{H}$ are independent. Therefore, we work with one of the Hamiltonians which describes physical states by tracing over the tilde states.

The underlying symmetries associated with the Hamiltonians (eqns(5.41, 5.42)) are used to solve for the density matrix.

To see this symmetry explicitly we define the following operators.

$$L_+ = a^\dagger b, \quad L_- = b^\dagger a \quad \& \quad L_3 = \frac{1}{2}(a^\dagger a - b^\dagger b) \quad (4.11)$$

which satisfy the $SU(2)$ algebra,

$$[L_3, L_\pm] = \pm L_\pm \quad \& \quad [L_+, L_-] = 2L_3 \quad (4.12)$$

with number operator, $\mathcal{N} = a^\dagger a + b^\dagger b$.

The Hamiltonian from eq(5.41) in terms of the $SU(2)$ generators is,

$$H = \omega\mathcal{N} + J(L_+ + L_-) \quad (4.13)$$

Hence, the underlying symmetry of the Schrodinger like eq(4.6) is $SU(2) \otimes SU(2)$ and $|\rho(t)\rangle$ is given by

$$|\rho(t)\rangle = e^{\alpha(t)L_+ - \alpha^*(t)L_-} |\rho(0)\rangle; \quad \text{here, } \alpha(t) = iJt, . \quad (4.14)$$

Using the disentanglement formula[49, 50], and taking the initial state $|\rho(0)\rangle$ as the vacuum state, the solution(5.47) reduces to,

$$|\rho(t)\rangle = e^{\xi L_+} e^{\log(1+|\xi|^2)L_3} e^{-\xi^* L_-} |\rho(0)\rangle \quad (4.15)$$

here, $\xi = \xi(\alpha(t)) = \frac{\alpha(t) \tan(|\alpha(t)|)}{|\alpha(t)|}$

The density matrix in the number state basis (for $n_a \neq n_b$) is given by

$$|\rho(t)\rangle = \sum_{n_a, n_b=0}^N C_{n_a, n_a} C_{n_b, n_b}^* |n_a, N - n_a\rangle \langle n_b, N - n_b|$$

where,

$$C_{n_a, n_a} = \frac{\xi^{n_a}}{(1 + |\xi|^2)^{N/2}} \binom{N}{n_a}^{\frac{1}{2}}; \quad C_{n_b, n_b} = \frac{\xi^{n_b}}{(1 + |\xi|^2)^{N/2}} \binom{N}{n_b}^{\frac{1}{2}}; \quad \& N = n_a + n_b. \quad (4.16)$$

The entanglement properties are calculated by taking the partial transpose of ρ and calculating the eigenvalues of the resulting matrix(ρ_{PT}) [51] is,

$$|\rho(t)_{PT}\rangle = \sum_{n_a, n_b=0}^N C_{n_a, n_a} C_{n_b, n_b}^* |n_a, N - n_b\rangle \langle n_b, N - n_a| \quad (4.17)$$

Let us start with the following set of two Hermitian conjugate terms for $n_a \neq n_b$ in above equation :

$$C_{n_a, n_a} C_{n_b, n_b}^* |n_a, N - n_b\rangle \langle n_b, N - n_a| + C_{n_b, n_b} C_{n_a, n_a}^* |n_b, N - n_a\rangle \langle n_a, N - n_b|. \quad (4.18)$$

The eigenvalues are

$$\lambda_{n_a, n_a} = \frac{N!}{(N - n_a)! n_a!} \sin^{2n_a}(Jt) \cos^{2N-2n_a}(Jt); (for, n_a = n_b) \quad (4.19)$$

$$\lambda_{n_a, n_b} = \pm \frac{N!}{\sqrt{(N - n_a)! (N - n_b)! n_a! n_b!}} \sin^{n_a+n_b}(Jt) \cos^{2N-n_a-n_b}(Jt); (for, n_a \neq n_b). \quad (4.20)$$

4.2.1 Entanglement properties of the system

For a bipartite system, the entropy as defined in the eq.(3.13) (for the general number state, with the eigenvalues eqns(4.19, 4.20)) is given by

$$S = - \sum_{i=0}^N \lambda_i \log \lambda_i = - \sum_{n=0}^N \left[\frac{N!}{(N-n)!n!} \sin^{2n}(Jt) \cos^{2N-2n}(Jt) \right] \times \{ \log \left[\frac{N!}{(N-n)!n!} \right] + (2n) \log[\sin(Jt)] + (2N-2n) \log[\cos(Jt)] \} \quad (4.21)$$

We can also quantify the entanglement of the system by studying the logarithmic negativity[38], Recall from chapter 3 that the log negativity is

$$E_N(T) = \log_2 \|\rho^T\|, \quad \text{here, } \|\rho^T\| = (2N(\rho) + 1) \quad (4.22)$$

where ρ^T is the partial transpose of ρ and the symbol $\|\cdot\|$ denotes the trace norm. Also $N(\rho)$ is the absolute value of the sum of all the negative eigenvalues of the partial transpose of ρ . The log negativity is a non-negative quantity and a non-zero value of E_N would mean that the state is entangled. Now we consider various cases of optical input states,

Case-1: For two photon system as an input(i.e., $N = 2$):

The entropy of entanglement of the two photon input state is

$$S = -4 \cos^4(Jt) \log[\cos(Jt)] - 2 \sin^2(Jt) \cos^2(Jt) \log[2 \sin^2(Jt) \cos^2(Jt)] - 4 \sin^4(Jt) \log[\sin(Jt)] \quad (4.23)$$

from the dot dashed curve of figure(6.1(d)) we see that, if we begin with separable input state at time $t = 0$, the entanglement entropy increases to the maximum value of 1.5 at $Jt = 0.785212$ and then decreases to become zero at $Jt = 1.57061$. This shows a periodic behaviour due to the coupling J which makes a system gets entangled and disentangled periodically.

The logarithmic negativity is now considered as this is different for various two photon states, unlike the entropy.

Case-1(a.) If we take the input state as $|\psi_{in}\rangle = |1, 1\rangle$, the possible output state will be $|\psi_{out}\rangle = \alpha_1|0, 2\rangle + \alpha_2|1, 1\rangle + \alpha_3|2, 0\rangle$; where, $\alpha_1 = -i \sin(2Jt)/\sqrt{2}$, $\alpha_2 = \cos(2Jt)$, $\alpha_3 = -i \sin(2Jt)/\sqrt{2}$. Then we can write the log-negativity entanglement of this state is

$$E_N = \log_2[1 + 2(-i\sqrt{2}\sin(2Jt)\cos(2Jt) - \sin^2(2Jt)/2)] \quad (4.24)$$

from the thick curve of figure(6.1(a)) we see that, if we begin with separable input state at time $t = 0$, the log negativity is zero. E_N increases to a maximum value of 1.32875 for $Jt = 0.42879$, this is the maximally entangled state. Two-photon interference effects, such as the “Hong-Ou-Mandel” effect, can be used to characterize to what extent two photons are identical. In this case for $Jt = 0.785212$ we get the dips at $E_N = 1$ (If the identical input photons overlap perfectly in time, then the coincidence rate of the output modes of the beam splitter will drop to zero), due to Hong-Ou-Mandel interference[52] and for $Jt = 1.57061$, E_N vanishes. After, we see a periodic behavior, attributed to the inter-waveguide coupling(J).

Case-1(b.) Now we take $|\psi_{in}\rangle = |2, 0\rangle$, the possible output state will be

$|\psi_{out}\rangle = \beta_1|0, 2\rangle + \beta_2|1, 1\rangle + \beta_3|2, 0\rangle$; here, $\beta_1 = -\sin^2(Jt)$, $\beta_2 = -i\sin(2Jt)/\sqrt{2}$, $\beta_3 = \cos^2(Jt)$. The log-negativity of this state is,

$$E_N = \log_2[1 + 2(-i\sqrt{2}\sin(Jt)\cos^3(Jt) + i\sqrt{2}\sin^3(Jt)\cos(Jt) - \sin^2(Jt)\cos^2(Jt))] \quad (4.25)$$

from the dotted curve of figure(6.1(a)) we see that, E_N increases with time and attains a maximum value of 1.32193 at $Jt = 0.785212$, then decreases and eventually becomes equal to zero at $Jt = 1.57061$. Thus the state becomes disentangled at this point of time. After we see periodic behavior and the system gets entangled and disentangled periodically. Unlike the earlier case (1(a)), here, we do not see any interference effects. Clearly the entanglement dynamics of the states $|1, 1\rangle$ and $|2, 0\rangle$ are different. The first has “Hong-Ou-Mandel effect” and the second does not, this means that the probability of detecting two photons in the same cavity differs from having them detected in two separate cavities, and the log negativity can be the two different two photon cases.

NOON states: A NOON state is a non-classical state of many-body system, which is a superposition of N particles in one mode, with no particles in another mode, and vice versa and can be written as

$$|NOON\rangle_{a,b} = \frac{1}{\sqrt{2}}(|N\rangle_a |0\rangle_b + |0\rangle_a |N\rangle_b) \quad (4.26)$$

NOON states are special because they can be used to obtain high-precision phase measurements, more efficient as the number of photons grows. NOON states are mostly produced optically, thus we consider the effect of the coupled cavities on input NOON states of various photon numbers.

Case-1(c.) For two photon input NOON state: $|\psi_{in}\rangle = (|2, 0\rangle + |0, 2\rangle)/\sqrt{2}$,

the output state will be, $|\psi_{out}\rangle = a_1|0, 2\rangle + a_2|1, 1\rangle + a_3|2, 0\rangle$;

where, $a_1 = \cos(2Jt)/\sqrt{2}$, $a_2 = -i\sin(2Jt)$, $a_3 = \cos(2Jt)/\sqrt{2}$.

Then the logarithmic negativity of this state is

$$E_N = \log_2[1 + i\sqrt{2}\sin(4Jt) + \cos^2(2Jt)], \quad (4.27)$$

from thick curve of figure(6.1(c)) we see that, for a $|2002\rangle$ state as input there are periodic dips due to the Hong-Ou-Mandel interference, which is characteristic of NOON states, and the photons are identical. E_N increases with time and attains a maximum value of 1.32875 for $Jt = 0.356988$, which is a point of maximal entanglement. Further, at $Jt = 0.785212$, E_N vanishes. After, we see a periodic behavior, attributed to the inter-waveguide coupling(J). Increase the number of photons in the NOON state does not make the entanglement vanish. Thus for applications to quantum computing NOON states are highly beneficial as their entanglement survives over long periods.

Case-2: For four photon system as an input (i.e., $N = 4$), the entanglement entropy(S) from the dotted curve of the figure(6.1(d)), we see that, if we begin with a separable input state at $t = 0$, the S is zero. After which the value of ' S ' increases and attains a maximum value of 2.03064 at $Jt = 0.785212$, it then decreases and eventually becomes equal to zero at $Jt = 1.57061$. Thus the state becomes disentangled at this point of time. At later times we see periodic behavior, so the system gets entangled and disentangled periodically.

Now we can deduce the logarithmic negativity for each of the four photon states

$$\{|\psi_{in}\rangle\} = \{|2, 2\rangle, |3, 1\rangle, |1, 3\rangle, |4, 0\rangle, |0, 4\rangle, \text{ and } (|4, 0\rangle + |0, 4\rangle)/\sqrt{2} \text{ (N00N state)}\}$$

Case-2(a.) For the input state, $|\psi_{in}\rangle = |2, 2\rangle$, the log-negativity(E_N) shown in the thick curve of figure(6.1(b)). Because of the involvement of four photons, we can see the double the interference effect of the two photon system.

Case-2(b.) For the input state, $|\psi_{in}\rangle = |3, 1\rangle$, the log-negativity is shown in the dotted curve of figure(6.1(b)). Here, we can see the small difference in interference pattern with the $|2, 2\rangle$ state. The ‘‘Hong-Ou-Mandel effect’’ and the maximally entangled state occur at lower values.

Case-2(c.) Now we take $|\psi_{in}\rangle = |4, 0\rangle$, the log-negativity is shown in the thin curve of figure(6.1(b)).

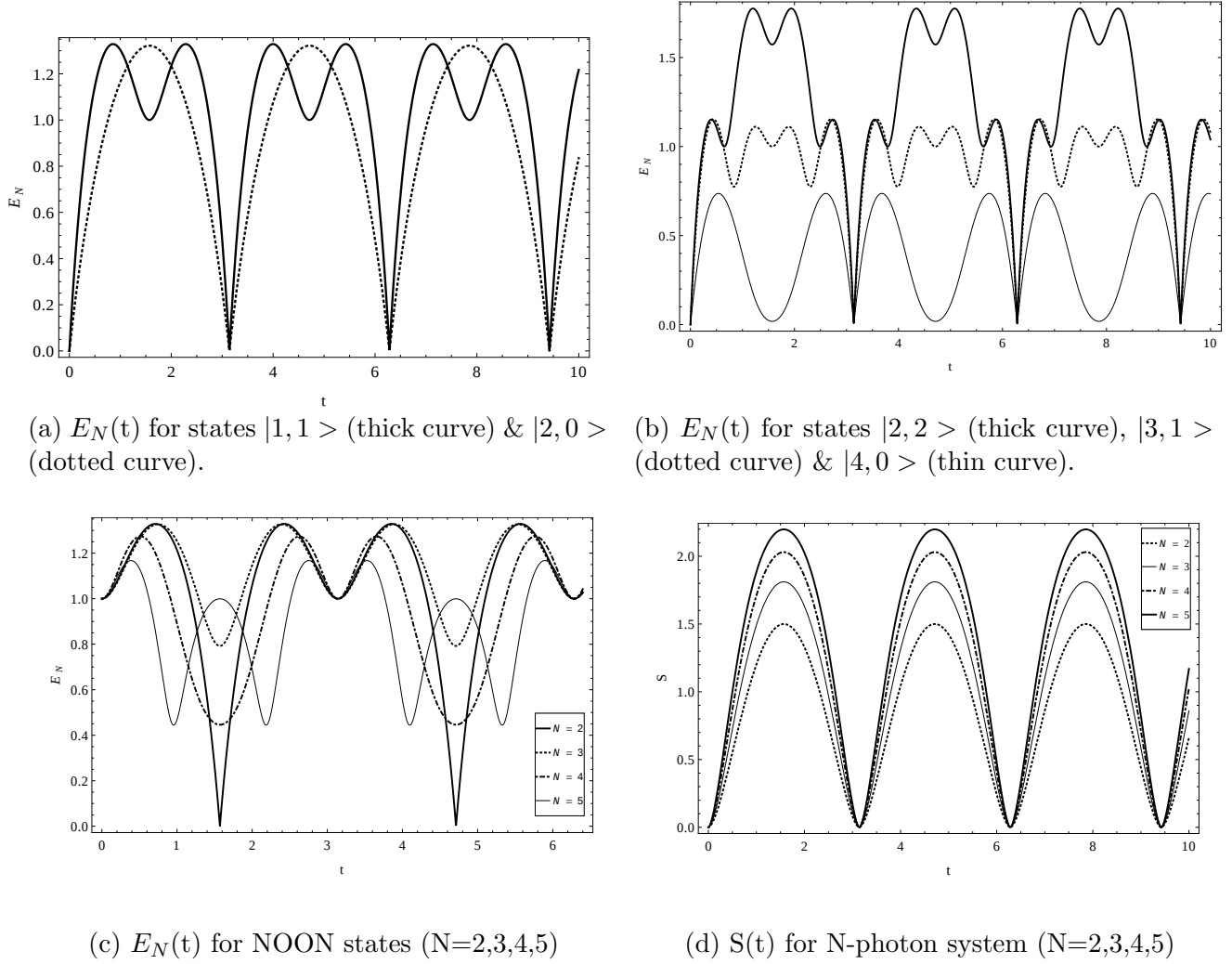


Figure 4.2: Shows the time evolution of entanglement of logarithmic negativity [$E_N(t)$] and time evolution of entropy entanglement [$S(t)$]: for various input photon number states with $J = 0.5$.

Case-2(d.) For four photon input NOON states, the log-negativity is shown in the dot dashed curve of the figure(6.1(c)). In this case, unlike for the two photon NOON state, the entanglement never goes to zero. This means that increasing the number of photons in a NOON state gives a more robust entanglement which is sustained at large times.

To show this we calculate the entanglement entropy(S) and logarithmic negativity(E_N) for a three photon and a five photon NOON states. The S shown in fig.(6.1(d)), the thin curve for three photon and thick curve for five photon states and E_N shown in fig.(6.1(c)), the dotted curve for three photon and thin curve for five photon states.

We see that as the photon number increases the NOON state gets more and more robust and shows that “high-NOON states” can be used for more precision measurements. These results are relevant in light of the recent experimental detection of entangled 5-photon “NOON states” [53].

4.2.2 Entanglement for input thermal states

Now we consider the initial state $|\rho(0)\rangle$ to be the two mode thermal state. This introduces a thermal bath in the system with which the photons interact. Then, in the TFD formalism, one can define the time evolved state $\rho(t)$ as

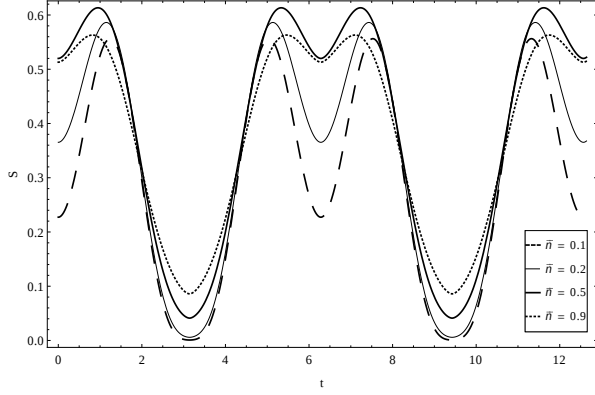
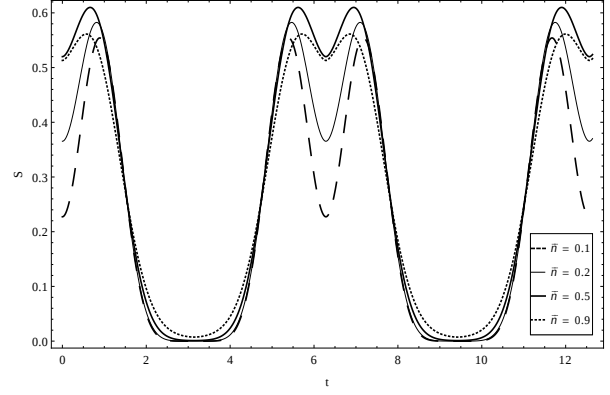
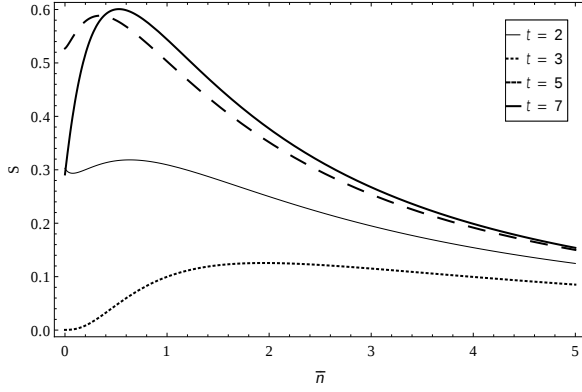
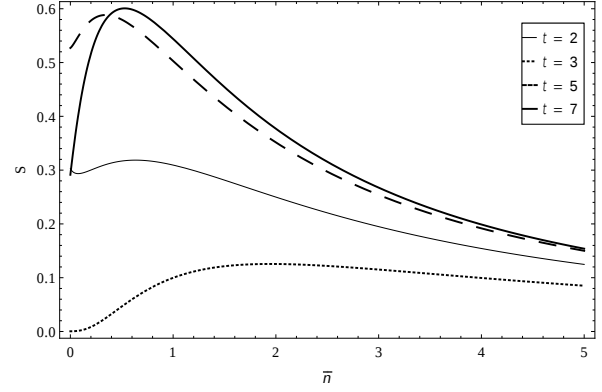
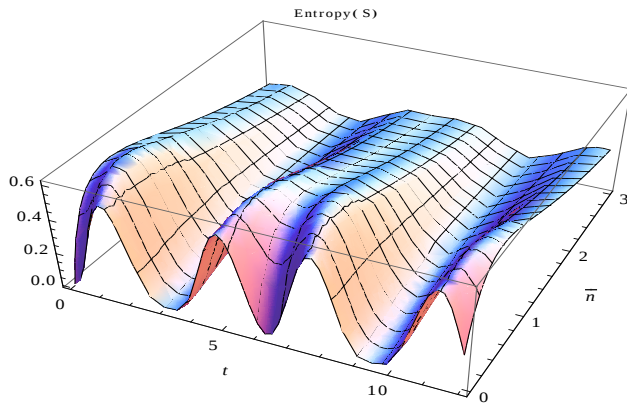
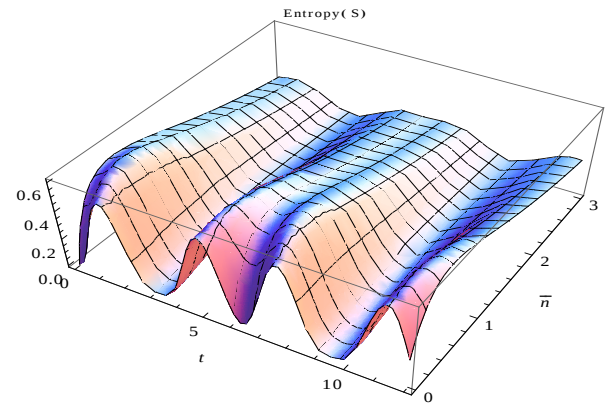
$$\begin{aligned} |\rho(t)\rangle &= e^{-iG(\theta)} e^{-iHt} \otimes e^{-i\tilde{H}t} |\rho(0)\rangle \\ &= \sum_{n_a=0}^N \frac{\bar{n}_a^{n_a}}{(\bar{n}_a + 1)^{n_a+1}} \sum_{n_b=0}^N \frac{\bar{n}_b^{n_b}}{(\bar{n}_b + 1)^{n_b+1}} e^{-iHt} |n_a, n_b\rangle \langle n_a, n_b| e^{iHt} \end{aligned} \quad (4.28)$$

$$= \sum_{n_a=0}^N C_{n_a, n_a} \frac{\bar{n}_a^{n_a}}{(\bar{n}_a + 1)^{n_a+1}} \sum_{n_b=0}^N C_{n_b, n_b} \frac{\bar{n}_b^{n_b}}{(\bar{n}_b + 1)^{n_b+1}} |n_a, N - n_a\rangle \langle n_b, N - n_b| \quad (4.29)$$

where, $G(\theta) = -i\theta(\tilde{a}a - \tilde{a}^\dagger a^\dagger + \tilde{b}b - \tilde{b}^\dagger b^\dagger)$, and $\bar{n}_a$ and $\bar{n}_b$ are thermal distribution functions
 $C_{n_a, n_a} = \frac{\xi^{n_a}}{(1+|\xi|^2)^{N/2}} \binom{N}{n_a}^{\frac{1}{2}} ; \quad C_{n_b, n_b} = \frac{\xi^{n_b}}{(1+|\xi|^2)^{N/2}} \binom{N}{n_b}^{\frac{1}{2}} ; \quad \& \quad N = n_a + n_b.$

The entanglement properties are calculated by taking the partial transpose of ρ and finding its eigenvalues

$$\lambda_{n_a, n_a} = \frac{(\bar{n}_a)^{2n_a}}{(\bar{n}_a + 1)^{2(n_a+1)}} \left\{ \frac{N!}{(N - n_a)! n_a!} \sin^{2n_a}(Jt) \cos^{2N-2n_a}(Jt) \right\}; (for, n_a = n_b) \quad (4.30)$$

(a) for $N = 2$, time evolution of entropy(S) for different $\bar{n}$ -values(b) for $N = 4$, time evolution of entropy(S) for different $\bar{n}$ -values(c) for $N = 2$, entropy(S) in function of ($\bar{n}$) for discrete time values(d) for $N = 4$, entropy(S) in function of ($\bar{n}$) for discrete time values(e) for $N = 2$, entropy(S) in $\bar{n}$ and t(f) for $N = 4$, entropy(S) in $\bar{n}$ and tFigure 4.3: Shows the time evolution of entropy of entanglement $[S(t)]$ in thermal distribution function($\bar{n}$): for input 2-, 4-photon systems.

$$\lambda_{n_a, n_b} = \pm \left[\frac{\bar{n}_a^{n_a}}{(\bar{n}_a + 1)^{n_a+1}} \frac{\bar{n}_b^{n_b}}{(\bar{n}_b + 1)^{n_b+1}} \left\{ \frac{N!}{\sqrt{(N - n_a)!(N - n_b)!n_a!n_b!}} \sin^{n_a+n_b}(Jt) \cos^{2N-n_a-n_b}(Jt) \right\} \right]; (n_a \neq n_b) \quad (4.31)$$

The von Neumann entropy of the reduced density matrix in terms of $\{\lambda_i\}$ the eigenvalues of the reduced density operator (as eq.3.13 is:

$$S = - \sum_{i=0}^N \lambda_i \log \lambda_i = - \sum_{i=0}^N \lambda_i \log \lambda_i = - \sum_{n_a=0}^N \frac{(\bar{n}_a)^{2n_a}}{(\bar{n}_a + 1)^{2(n_a+1)}} \left[\frac{N!}{(N - n_a)!n_a!} \sin^{2n_a}(Jt) \cos^{2N-2n_a}(Jt) \right] \\ \times \left\{ \log \left[\frac{(\bar{n}_a)^{2n_a}}{(\bar{n}_a + 1)^{2(n_a+1)}} \left[\frac{N!}{(N - n_a)!n_a!} \sin^{2n_a}(Jt) \cos^{2N-2n_a}(Jt) \right] \right\} \right. \quad (4.32)$$

Case-3(a.) For two photon input state (i.e., $N = 2$), the entanglement entropy is

$$S = - \frac{1}{(\bar{n} + 1)^2} \cos^4(Jt) \log \left[\frac{1}{(\bar{n} + 1)^2} \cos^4(Jt) \right] - \frac{2\bar{n}^2}{(\bar{n} + 1)^4} \sin^2(Jt) \cos^2(Jt) \log \left[\frac{2\bar{n}^2}{(\bar{n} + 1)^4} \sin^2(Jt) \cos^2(Jt) \right] \\ - \frac{1}{(\bar{n} + 1)^2} \sin^4(Jt) \log \left[\frac{1}{(\bar{n} + 1)^2} \sin^4(Jt) \right] \quad (4.33)$$

which is shown in figures(4.3(a,b,c)).

Case-3(b.) For four photon system as an input (i.e., $N = 4$): For the four photon system the entropy of entanglement see in the figures(4.3(d,e,f)). We see that the system sustains entanglement for the lower values of $\bar{n}_a$ & $\bar{n}_b$, but as the system gets more thermalized, it decoheres and the entanglement entropy tends to zero.

For thermal states, its seen that decoherence is introduced by a thermal bath and eventually the entanglement vanishes. Therefore in the practical use of cavity coupled entangled states this effect has to be factored in. This understanding of thermal effects can help us making the system more efficient.

4.3 Two coupled waveguides with damping ($\gamma \neq 0$)

For coupled waveguides losses, due to “system-reservoir interaction” are quantified by ‘ γ ’, which is “the rate of loss due to the material of the waveguide”. To calculate the decoherence we solve the master equation

$$|\dot{\rho}\rangle = -i\hat{H}'|\rho\rangle \quad (4.34)$$

with,

$$\begin{aligned}\hat{H}' = & \omega(a^\dagger a + b^\dagger b - \tilde{a}^\dagger \tilde{a} - \tilde{b}^\dagger \tilde{b}) + J(a^\dagger b + b^\dagger a - \tilde{a}^\dagger \tilde{b} - \tilde{b}^\dagger \tilde{a}) \\ & - i\gamma(a^\dagger a - a\tilde{a} + a^\dagger \tilde{a}^\dagger + \tilde{a}^\dagger \tilde{a} + b^\dagger b - b\tilde{b} + b^\dagger \tilde{b}^\dagger + \tilde{b}^\dagger \tilde{b})\end{aligned}\quad (4.35)$$

the following transformations,

$$a = \frac{A+B}{\sqrt{2}}, \quad a^\dagger = \frac{A^\dagger+B^\dagger}{\sqrt{2}}, \quad \tilde{a} = \frac{\tilde{A}+\tilde{B}}{\sqrt{2}}, \quad \tilde{a}^\dagger = \frac{\tilde{A}^\dagger+\tilde{B}^\dagger}{\sqrt{2}} \quad (4.36)$$

$$b = \frac{-A+B}{\sqrt{2}}, \quad b^\dagger = \frac{-A^\dagger+B^\dagger}{\sqrt{2}}, \quad \tilde{b} = \frac{-\tilde{A}+\tilde{B}}{\sqrt{2}}, \quad \tilde{b}^\dagger = \frac{-\tilde{A}^\dagger+\tilde{B}^\dagger}{\sqrt{2}}, \quad (4.37)$$

gives

$$\begin{aligned}\hat{H}'' = & \omega(A^\dagger A + B^\dagger B - \tilde{A}^\dagger \tilde{A} - \tilde{B}^\dagger \tilde{B}) + J(-A^\dagger A + B^\dagger B + \tilde{A}^\dagger \tilde{A} - \tilde{B}^\dagger \tilde{B}) \\ & - i\gamma(A^\dagger A + B^\dagger B + \tilde{A}^\dagger \tilde{A} + \tilde{B}^\dagger \tilde{B} - A\tilde{A} - B\tilde{B} + A^\dagger \tilde{A}^\dagger + B^\dagger \tilde{B}^\dagger)\end{aligned}\quad (4.38)$$

This Hamiltonian is diagonalized by using the squeezing (Bogolubov) transformation which mixes the tilde and tilde fields.

$$D = \mu_1 A + \nu_1^* \tilde{A}^\dagger, \quad D^\dagger = \mu_1^* A^\dagger + \nu_1 \tilde{A}, \quad \tilde{D} = \mu_1 \tilde{A} + \nu_1^* A^\dagger \quad \& \quad \tilde{D}^\dagger = \mu_1^* \tilde{A}^\dagger + \nu_1 A \quad (4.39)$$

$$E = \mu_2 B + \nu_2^* \tilde{B}^\dagger, \quad E^\dagger = \mu_2^* B^\dagger + \nu_2 \tilde{B}, \quad \tilde{E} = \mu_2 \tilde{B} + \nu_2^* B^\dagger \quad \& \quad \tilde{E}^\dagger = \mu_2^* \tilde{B}^\dagger + \nu_2 B, \quad (4.40)$$

where,

$$\mu_1 = \cosh r_1 = \sqrt{\frac{|\omega - J - i\gamma|^2}{|-i\gamma - \sqrt{\gamma^2 + (\omega - J)^2}|^2}}, \quad \nu_1 = \sinh r_1 = \sqrt{\frac{-|i\gamma|^2}{|-i\gamma + \sqrt{\gamma^2 + (\omega - J)^2}|^2}} \quad (4.41)$$

$$\mu_2 = \cosh r_2 = \sqrt{\frac{|\omega + J - i\gamma|^2}{|-i\gamma - \sqrt{\gamma^2 + (\omega + J)^2}|^2}}, \quad \nu_2 = \sinh r_2 = \sqrt{\frac{-|i\gamma|^2}{|-i\gamma + \sqrt{\gamma^2 + (\omega + J)^2}|^2}} \quad (4.42)$$

and r_1 and r_2 are the squeezing parameters and $|\mu_1|^2 - |\nu_1|^2 = 1$ and $|\mu_2|^2 - |\nu_2|^2 = 1$.

The final Hamiltonian is

$$\begin{aligned}\hat{H}_f = & S^{-1}(r_1)H_AS(r_1) + S^{-1}(r_2)H_BS(r_2) \\ & = \Omega_1^2(D^\dagger D) + \Omega_2^2(\tilde{D}^\dagger \tilde{D}) + \Omega_3^2((E^\dagger E) + \Omega_4^2(\tilde{E}^\dagger \tilde{E})).\end{aligned}\quad (4.43)$$

where,

$$S(r_1) = \exp[r_1 K_{A+} - r_1^* K_{A-}] = \text{Exp}[r_1 A^\dagger \tilde{A}^\dagger - r_1^* A \tilde{A}]; \quad (4.44)$$

$$S(r_2) = \exp[r_2 K_{B+} - r_2^* K_{B-}] = \text{Exp}[r_2 B^\dagger \tilde{B}^\dagger - r_2^* B \tilde{B}]; \quad (4.45)$$

and $\Omega_1 = -\frac{\sqrt{\gamma^2 + (\omega - J)^2}}{2} - \frac{i\gamma}{2}$, $\Omega_2 = \frac{\sqrt{\gamma^2 + (\omega - J)^2}}{2} - \frac{i\gamma}{2}$, $\Omega_3 = -\frac{\sqrt{\gamma^2 + (\omega + J)^2}}{2} - \frac{i\gamma}{2}$, $\Omega_4 = \frac{\sqrt{\gamma^2 + (\omega + J)^2}}{2} - \frac{i\gamma}{2}$; The generators of the $SU(1,1)$ algebra in terms of the modes A and B are given by

$$K_{A-} = A\tilde{A}, \quad K_{A+} = A^\dagger \tilde{A}^\dagger, \quad K_{A3} = \frac{A^\dagger A + \tilde{A}^\dagger \tilde{A} + 1}{2} \quad (4.46)$$

$$K_{B-} = B\tilde{B}, \quad K_{B+} = B^\dagger \tilde{B}^\dagger, \quad K_{B3} = \frac{B^\dagger B + \tilde{B}^\dagger \tilde{B} + 1}{2}, \quad (4.47)$$

and satisfy the commutation relations

$$[K_{A-}, K_{A+}] = 2K_{A3}, \quad [K_{A3}, K_{A\pm}] = \pm K_{A\pm}; \quad [K_{B-}, K_{B+}] = 2K_{B3}, \quad [K_{B3}, K_{B\pm}] = \pm K_{B\pm}. \quad (4.48)$$

The “Casimir operators” are “ $K_{Ao} = (A^\dagger A - \tilde{A}^\dagger \tilde{A})$ ”, “ $K_{Bo} = (B^\dagger B - \tilde{B}^\dagger \tilde{B})$ ”. Then the solution of eq.(4.34) becomes

$$|\rho(t)\rangle = K(t) e^{[\eta_{A3} K_{A3} + \eta_{A-} K_{A-} + \eta_{A+} K_{A+} + \eta_{B3} K_{B3} + \eta_{B-} K_{B-} + \eta_{B+} K_{B+}]} |\rho(0)\rangle \quad (4.49)$$

where,

$$K(t) = e^{-i\omega t(K_{A0} + K_{B0}) + 2iJt(N_A + \tilde{N}_B - 1)}, \quad \eta_{A-} = \gamma t = \eta_{B-}, \\ \eta_{A3} = -2(\gamma + iJ)t = \eta_{B3}, \quad \eta_{A+} = -\gamma t = \eta_{B+}. \quad (4.50)$$

By using the $SU(1,1)$ “disentanglement formula”[49], one can write eq.(4.49) as,

$$|\rho(t)\rangle = \{K(t) \exp[\Gamma_{A+} K_{A+}] \exp[\ln(\Gamma_{A3}) K_{A3}] \exp[\Gamma_{A-} K_{A-}] \\ \otimes e^{[\Gamma_{B+} K_{B+}] \exp[\ln(\Gamma_{B3}) K_{B3}] \exp[\Gamma_{B-} K_{B-}]} \} |\rho(0)\rangle \quad (4.51)$$

here,

$$\Gamma_{i\pm} = \frac{2\eta_{i\pm} \sinh \phi_i}{2\phi_i \cosh \phi_i - \eta_{i3} \sinh \phi_i} \quad \& \quad \Gamma_{i3} = \left(\frac{2\phi_i}{2\phi_i \cosh \phi_i - \eta_{i3} \sinh \phi_i} \right)^2 \quad (4.52)$$

with

$$\phi_i^2 = \frac{\eta_{i3}^2}{4} - \eta_{i+} \eta_{i-}, \quad (4.53)$$

subscript i labels A, B.

Now we consider an initial state $|\rho(0)\rangle = \sum_{m,n}^N \rho_{m,n}(0) |m, \tilde{m}, n, \tilde{n}\rangle$, in TFD notation. This gives us an exact solution of the density matrix :

$$\begin{aligned} \rho_{m,n}(t) = & C(t) \sum_{q'=0}^{\min(m',n')} \sum_{p'=0}^{\infty} \left[\binom{m'+p'-q'}{p'} \binom{n'+p'-q'}{p'} \binom{m'}{q'} \binom{n'}{q'} \right]^{\frac{1}{2}} \\ & \times \sum_{q=0}^{\min(m,n)} \sum_{p=0}^{\infty} \left[\binom{m+p-q}{p} \binom{n+p-q}{p} \binom{m}{q} \binom{n}{q} \right]^{\frac{1}{2}} \\ & \times [\Gamma_{A+}]^{p'} [\Gamma_{A3}]^{\frac{(m'+n'-2q'+1)}{2}} [\Gamma_{A-}]^{q'} [\Gamma_{B+}]^p [\Gamma_{B3}]^{\frac{(m+n-2q+1)}{2}} [\Gamma_{B-}]^q \\ & \times \rho_{m+p-q, m'+p'-q', n+p-q, n'+p'-q'}(0) \end{aligned} \quad (4.54)$$

where $C(t)$ is an overall phase factor due to $K(t) = e^{-i\omega t(K_{A0}+K_{B0})+2iJt(N_A+\tilde{N}_B-1)}$.

4.3.1 Entanglement properties of the system with damping

Taking a trace of $\rho(t)$ over the tilde space and the “partial transpose of ρ ”, we calculate the eigenvalues of the resulting matrix to enable as to calculate the entropy of entanglement.

$$\lambda_{m_a, m_a} = \frac{N!}{(N-m_a)!m_a!} \sinh^{2m_a}(\theta) \cosh^{2N-2m_a}(\theta); (for, m_a = m_b) \text{ and } \theta = (\sqrt{2}\gamma + iJ)t. \quad (4.55)$$

$$\lambda_{m_a, m_b} = \pm \frac{N!}{\sqrt{(N-m_a)!(N-m_b)!m_a!m_b!}} \sinh^{m_a+m_b}(\theta) \cosh^{2N-m_a-m_b}(\theta); (for, m_a \neq m_b) \quad (4.56)$$

The entropy of entanglement of the the system is

$$\begin{aligned} S = - \sum_{i=0}^N \lambda_i \log \lambda_i = & - \sum_{m_a=0}^N \left[\frac{N!}{(N-m_a)!m_a!} \sinh^{2m_a}(\theta) \cosh^{2N-2m_a}(\theta) \right] \\ & \times \{ \log \left[\frac{N!}{(N-m_a)!m_a!} \right] + 2m_a \log[\sinh \theta] + (2N-2m_a) \log[\cosh \theta] \} \end{aligned} \quad (4.57)$$

Thus, **the entropy of entanglement of the system** for two photon input state is

$$S = -4 \cosh^4(\theta) \log[\cosh(\theta)] - 2 \sinh^2(\theta) \cosh^2(\theta) \log[2 \cosh^2(\theta) \sinh^2(\theta)] - 4 \sinh^4(\theta) \log[\sinh(\theta)] \quad (4.58)$$

from thin curves of figure(4.4) we see.

For four photon states the entropy of entanglement is shown in thick curves of figure(4.4). In

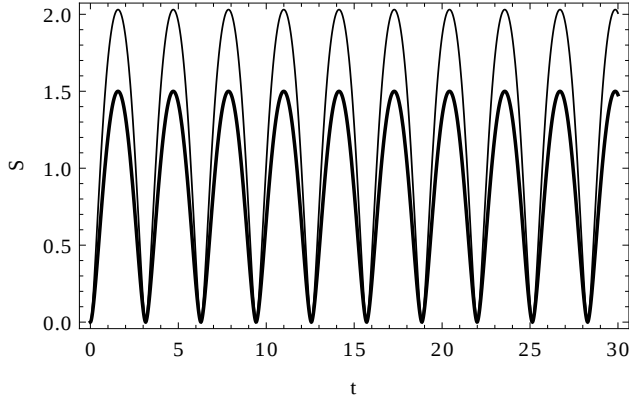
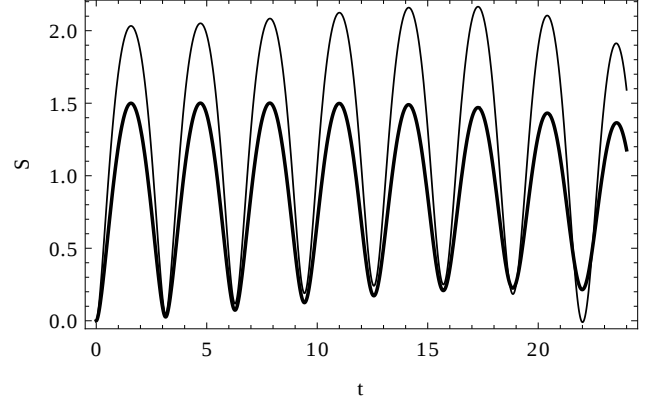
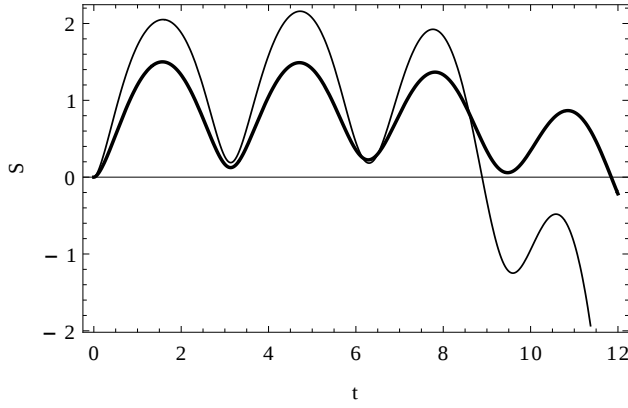
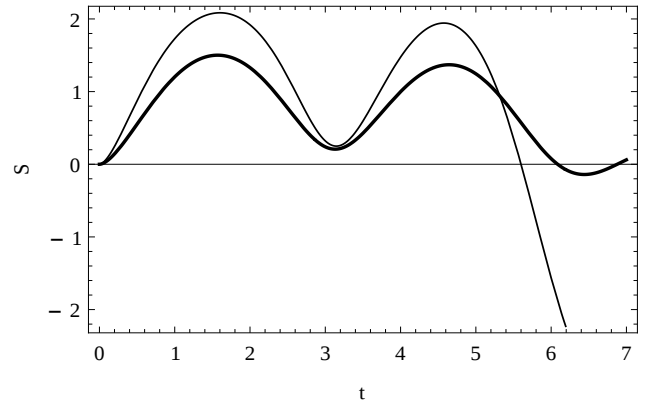
(a) for $\gamma = 0$ (without damping)(b) for $\gamma = 0.01$ (c) for $\gamma = 0.03$ (d) for $\gamma = 0.05$

Figure 4.4: The time evolution of entropy(S) for 2-photons(thick curve) and 4-photons(thin curve) with $J = 0.5$.

the limit γ goes to zero, the undamped case in figure(6.1(d)) is obtain. Increasing the damping effect by adjusting the value of $\gamma = 0.01, 0.03 \& 0.05$, there is more damping for four photon system than the two photon system. This means that increase of input photons, increases the decoherence parameter. For the increase of $\gamma = 0.03, 0.05$, we can see the negative entropy, it is more for four photons than two photons.

Logarithmic negativity entanglement of the damping system:

Since the state is Gaussian, we can use the “**covariance matrix method**” to calculate the log-negativity(entanglement) of the system by using “Simon’s criterion”[41] as mentioned in section-3.2.2. The density matrix can be written as $\rho'(t) = S^\dagger(r)R^\dagger(\phi)\rho(t)R(\phi)S(r)$, where $S(r)$ is the squeezing matrix and $R(\phi)$ is the rotation matrix mixing real and tilde fields. In our case, $\theta = 45^\circ$ (see in eqs(4.36, 4.37)) and ‘r’ is squeezing parameter(see in eqs(5.21, 5.22)), and $\rho(0)$ is the initial state of a two mode system. One can clearly see that this is a product of two mode squeezed states of the four mode Hilbert space. Now we take the initial state $\rho(0)$ to be the two mode vacuum state, $|\rho(0)\rangle = |0, 0, \tilde{0}, \tilde{0}\rangle$. To calculate the entanglement of the time evolved state $\rho'(t)$ we go over to phase space description by following transformations,

$$\begin{aligned} A &= \frac{1}{\sqrt{2}}(x + ip_x), & A^\dagger &= \frac{1}{\sqrt{2}}(x - ip_x), & \tilde{A} &= \frac{1}{\sqrt{2}}(\tilde{x} + i\tilde{p}_x), & \tilde{A}^\dagger &= \frac{1}{\sqrt{2}}(\tilde{x} - i\tilde{p}_x) \\ B &= \frac{1}{\sqrt{2}}(y + ip_y), & B^\dagger &= \frac{1}{\sqrt{2}}(y - ip_y), & \tilde{B} &= \frac{1}{\sqrt{2}}(\tilde{y} + i\tilde{p}_y), & \tilde{B}^\dagger &= \frac{1}{\sqrt{2}}(\tilde{y} - i\tilde{p}_y) \end{aligned} \quad (4.59)$$

Then, the covariance matrix is:

$$V(r_1, r_2) = \begin{pmatrix} p & 0 & 0 & 0 & 0 & 0 & s & 0 \\ 0 & q & 0 & 0 & 0 & 0 & 0 & t \\ 0 & 0 & p^* & 0 & s & 0 & 0 & 0 \\ 0 & 0 & 0 & q^* & 0 & t & 0 & 0 \\ 0 & 0 & s & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & q & 0 & 0 \\ s & 0 & 0 & 0 & 0 & 0 & p^* & 0 \\ 0 & t & 0 & 0 & 0 & 0 & 0 & q^* \end{pmatrix} \quad (4.60)$$

where, $p = e^{2iJt}e^{-2\gamma t}\cosh 2r_1$, $q = e^{2iJt}e^{-2\gamma t}\cosh 2r_2$, $s = e^{-2\gamma t}\sinh 2r_1$, $t = e^{-2\gamma t}\sinh 2r_2$.

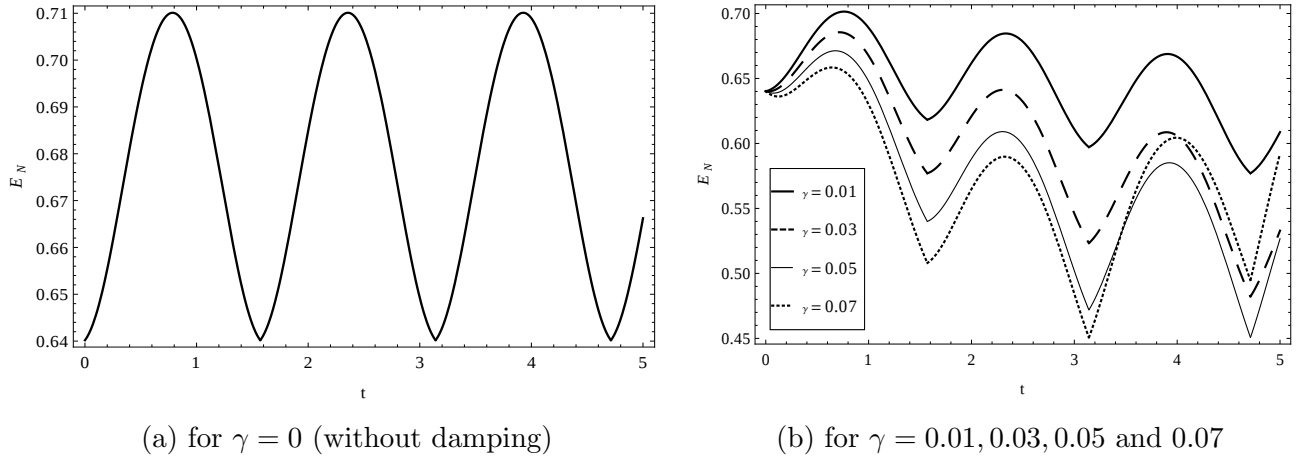


Figure 4.5: Shows the time evolution of entanglement $[E_N(t)]$ for different values of γ with $r = 0.25, J = 0.5$.

Since the tilden fields are fictitious, we trace over them to get the covariance matrix for the physical modes,

$$V(r_1, r_2) = \begin{pmatrix} p+q & 0 & -(s+t) & 0 \\ 0 & p^*+q^* & 0 & s+t \\ -(s+t) & 0 & p+q & 0 \\ 0 & s+t & 0 & p^*+q^* \end{pmatrix} \quad (4.61)$$

The canonical form of covariance matrix is given by,

$$V = \begin{pmatrix} \alpha & \gamma \\ \gamma^\dagger & \beta \end{pmatrix} \quad (4.62)$$

where,

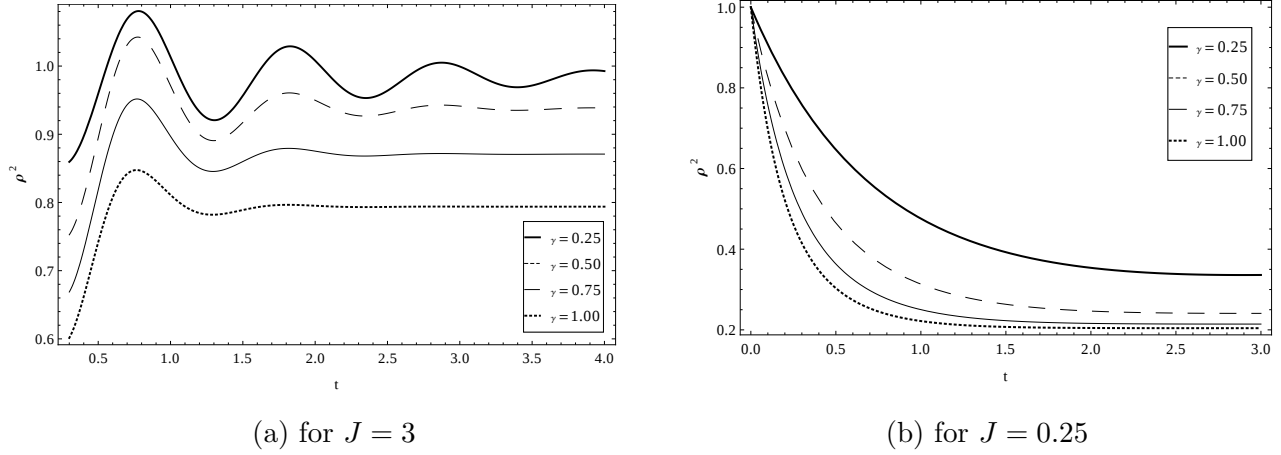
$$\alpha = \begin{pmatrix} p+q & 0 \\ 0 & p^*+q^* \end{pmatrix} = \beta, \quad \text{and} \quad \gamma = \begin{pmatrix} -(s+t) & 0 \\ 0 & (s+t) \end{pmatrix} \quad (4.63)$$

Then the separability condition[41] for any two mode state is

$$Det\alpha Det\beta + \left(\frac{1}{4} - |Det\gamma|\right)^2 - tr(\alpha J \gamma J \beta J \gamma^T J) \geq \frac{1}{4}(Det\alpha + Det\beta) \quad (4.64)$$

The symplectic eigenvalues are defined as,

$$\nu_{\pm} = \sqrt{\frac{1}{2} \left\{ \tilde{\Delta} \pm \sqrt{\tilde{\Delta}^2 - \frac{4}{\mu^2}} \right\}} \quad (4.65)$$

Figure 4.6: Shows the time evolution of Decoherence for different values of γ .

where, $\tilde{\Delta} = \text{Det}\alpha + \text{Det}\beta - 2\text{Det}\gamma = 2(p+q)(p^*+q^*) + 2(s+t)^2$ and $\mu = [\text{Det}V]^{-\frac{1}{2}} = [(p+q)^2(p^*+q^*)^2 + (p+q)^2(s+t)^2 - (p^*+q^*)^2(s+t)^2 - (s+t)^4]^{-\frac{1}{2}}$. The entanglement of the system is

$$E_N = \max\{0, -\log\nu_-\}. \quad (4.66)$$

For $r_1 = r_2 = r$, the entanglement for two mode vacuum states **without damping** (*i.e.*, $\gamma = 0$) is shown in figure(4.6(a)) and **with damping** (*i.e.*, $\gamma \neq 0$) is shown in figure(4.6(b)). We see that the entanglement decreases as damping increases, but, for low damping, the system seems to sustain entanglement to a large extent, so that it is quite robust for applications.

In order to quantify the decoherence effects, we compute ρ^2 as

$$\begin{aligned} \text{Tr}[\rho^2(t)] &= \text{Tr}\left[\sum_{m,n} \langle m, n | \rho^2(t) | m, n \rangle\right] \\ &= \text{Exp}\left[-\frac{4\gamma t \sinh(\sqrt{2}\gamma + iJ)t}{(\sqrt{2}\gamma + iJ)t \cosh(\sqrt{2}\gamma + iJ)t + (\gamma + iJt) \sinh(\sqrt{2}\gamma + iJ)t}\right] \end{aligned} \quad (4.67)$$

The behaviour of decoherence is plotted in figure(5.3). We have considered two cases: figure(5.3(a)), shows the variation of decoherence with time for strong coupling for various values of γ and figure(5.3(b)), shows the evolution of decoherence with weak coupling. For strong coupling, the system decoheres in an oscillatory manner and saturates to a non-zero value, while for weak coupling, one that for even short times, as the value of damping coefficient increases the system decoheres, to a very low value, and very fast.

4.3.2 Entanglement for two mode thermal state with damping

Taking the initial state $\rho(0)$ to be the two mode thermal vacuum state, the covariance matrix is given by,

$$V(r_1, r_2) = \begin{pmatrix} c+d & 0 & e+f & 0 \\ 0 & c^*+d^* & 0 & -(e+f) \\ e+f & 0 & c+d & 0 \\ 0 & -(e+f) & 0 & c^*+d^* \end{pmatrix} \quad (4.68)$$

where,

$$\begin{aligned} c &= e^{2iJt} e^{-2\gamma t} (n_1 \cosh^2 r_1 + n_2 \sinh^2 r_1), \\ d &= e^{2iJt} e^{-2\gamma t} (n_1 \sinh^2 r_2 + n_2 \cosh^2 r_2), \\ e &= \frac{n_1+n_2}{2} e^{-2\gamma t} \sinh 2r_1 \text{ \& } f = \frac{n_1+n_2}{2} e^{-2\gamma t} \sinh 2r_2. \end{aligned}$$

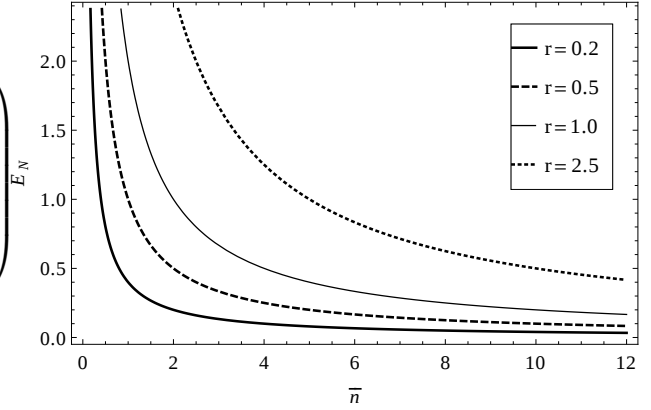


Figure 4.7: Shows entanglement(E_N) vs. thermal distribution fn. ($\bar{n}$) for different values of r

Applying “Simon’s criterion” eq.(5.33) we see that the system is entangled iff

$$(n_1 + n_2)^4 e^{-4\gamma t} [\cosh^2 2r - \sinh^2 2r]^2 + \frac{1}{16} \geq \frac{(n_1 + n_2)^2}{2} e^{-2\gamma t} [\cosh^2 2r + \sinh^2 2r] \quad (4.69)$$

For $r_1 = r_2 = r$, and $n_1 = n_2 = n$, this condition is satisfied for the values of r given in the figure(4.7), in which we plot the logarithmic negativity as a function of $\bar{n}$, for different values of r . We see that the system not only gets less entangled for high values of γ (quantified by r), but also for large n (external heat bath). So that in the presence of a heat bath the effect of damping increases and both have to be considered when generating entanglement in the lab by using coupled cavities.

4.4 Conclusion

In this chapter, we have used TFD to solve the master equation associated with the symmetries, $SU(2)$ & $SU(1,1)$ for coupled lossy waveguides with and without damping. Special attention has been given to the time evolution of the NOON states as inputs and because their entanglement increases with photon number, and survives with time, they are extremely suitable for quantum

information. Our work extends the ref.[48] given, as it gives the exact solution for the master equation, and, in addition, shows how the entanglement behaves for input thermal states. Our results have also shown that the entanglement of the system can withstand a certain amount of damping, suggesting that it can be used for applications such as quantum computation, even if the waveguides are lossy. Furthermore we have shown the effect of an external heat bath on the system, by applying our methods to thermal input states.

Chapter 5

Application of TFD to the entanglement properties of a Coupled two site, two species Bose-Hubbard model

As we have mentioned in earlier chapters, open many-body quantum systems have attracted resumed interest in the subject of quantum information science. In this context, the Bose Hubbard model plays a crucial role because of its link with ultracold atomic gases experiments on optical lattices. It provides the prime ingredient that allows ultracold atomic setups to mimic well-known many-body problems. In particular, it makes those systems needed for modelling quantum simulators and quantum computers. The two site Bose Hubbard model(BHM) that we study in this chapter may be considered the condensed matter analogue of the quantum optical two cavity system we discussed in chapter 4.

In this chapter, we will use the BHM to model bosons trapped on an optical lattice. The Hamiltonian is given by

$$H = \int d^3x \psi^\dagger(x) \left[-\frac{\hbar^2}{2m} \nabla^2 + V_0(x) + V_T(x) \right] \psi(x) + \frac{1}{2} \frac{4\pi a_s \hbar^2}{m} \int d^3x \psi^\dagger(x) \psi^\dagger(x) \psi(x) \psi(x) \quad (5.1)$$

with $\psi(x)$ is a bosonic field for atoms in a given internal atomic state, $V_0(x)$ is the optical lattice potential, and $V_T(x)$ is an additional (slowly varying) external trapping potential, a_s is the s-wave

scattering length and m is the mass of the atoms[55].

When we quantize this model, the atoms are described by harmonic oscillators in individual wells, and the BHM can study the dynamics of these atoms on this optical lattice.

5.1 The Bose-Hubbard model

The Hamiltonian in eq.(5.1) reduces to the Bose Hubbard Hamiltonian(BHH) when we expand the boson field in the Wannier basis(the equivalent of orthonormalized functions on a lattice) and we get,

$$H = -J \sum_{\langle i,j \rangle} b_i^\dagger b_j + \frac{U}{2} \sum_i n_i(n_i - 1) + \sum_i \epsilon_i n_i, \quad (5.2)$$

where, the bosonic annihilation and creation operators at lattice site i are b_i and $b_i^\dagger$ and $n_i = b_i^\dagger b_i$ is the corresponding number of bosonic atoms. The on-site repulsion is parameterised by $U = 4\pi a_s \hbar^2 \int d^3x |w(x)|^4 / m$, and the hopping matrix at adjacent sites i and j is, $J = \int d^3x w^*(x - x_i) [-\frac{\hbar^2}{2m} \nabla^2 + V_0(x)] w(x - x_j)$ and $\epsilon_i = \int d^3x V_T(x) |w(x - x_i)|^2 \approx V_T(x_i)$ is energy offset of each lattice site.

This is the one species BHM, which has been extensively studied in the ref.[55]. We can study the dynamics of the system by considering the ratio U/J , which allows us to have control over the full set of parameters of the system. For high J and low U , this corresponds to the superfluid state and high U and low J , it corresponds to Mott-insulator state. In this chapter, we generalize to the case of boson-boson mixture loaded onto optical lattices, which results in a two species BHM with on-site boson-boson interaction.

The Hamiltonian is given by,

$$H = \omega \sum_k (n_{ak} + n_{bk}) - J \sum_{\langle k,l \rangle} (a_k^\dagger b_l + b_k^\dagger a_l) + \frac{U_{aa}}{2} \sum_k a_k^\dagger^2 a_k^2 + \frac{U_{bb}}{2} \sum_k b_k^\dagger^2 b_k^2 + \frac{U_{ab}}{2} \sum_{\langle k,l \rangle} n_{ak} n_{bl} \quad (5.3)$$

here, $n_{ai} = a_i^\dagger a_i$ and $n_{bi} = b_i^\dagger b_i$ are number operators of the two species of bosons a_i and b_i respectively; U_{aa}, U_{bb} measures the interaction strengths of on-site interacting atoms; U_{ab} measures the interaction strength of nearest neighbour

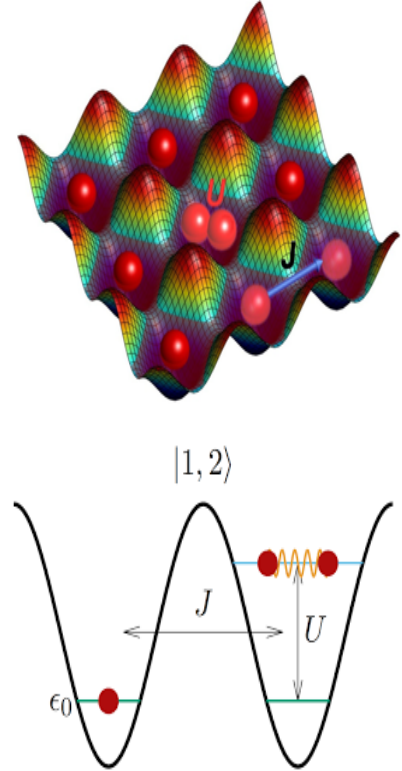


Figure 5.1: A schematic diagram of bosons trapped on an optical lattice.

atoms and J describes the induced hopping between adjacent cells k and l .

To show the difference of entanglement in the two phases, we do not take the whole system; we will restrict ourselves to Bose-Hubbard dimers. Both microscopic tunnelings with a tunable interaction J as well as scattering interaction U of the between the bosons can be studied. The particular realization of the dimer we use is two species a , and b trapped in a two well optical lattice. The Hamiltonian becomes

$$\hat{H} = \omega(a^\dagger a + b^\dagger b) + J(a^\dagger b + b^\dagger a) - \frac{U_{aa}}{2}(a^{\dagger 2} a^2) - \frac{U_{bb}}{2}(b^{\dagger 2} b^2) - \frac{U_{ab}}{2}(a^\dagger b^\dagger ab). \quad (5.4)$$

Chaitanya et. al. [56] have studied this model for entanglement and decoherence in the case $J = U_{aa} = U_{bb} = 0$ and $U_{ab} = U$, i.e., for the localized states (BEC(Bose-Einstein condensed) states) only. This case does not account for the competition between the parameters J and U . the whole idea is to exploit the two phases of this model. Therefore we are motivated to study the more general case $J \neq 0$ and $U \neq 0$ in the presence of non-linear damping and its one of the limiting case $J \neq 0$ and $U = 0$ (Mott-insulator states), so that we can see how entanglement varies across a phase transition. This is an essential extension as it will enable us to compare the entanglement in both the superfluid-like state and the Mott-insulator type state and gives a holistic picture across the phase diagram.

5.2 The master equation for the two species Bose-Hubbard model

The master equation corresponding to two site-two species BHM is given by

$$\begin{aligned} \frac{\partial \rho}{\partial t} = & -i\omega(a^\dagger a \rho - \rho a^\dagger a + b^\dagger b \rho - \rho b^\dagger b) - iJ(a^\dagger b \rho - \rho a^\dagger b + b^\dagger a \rho - \rho b^\dagger a) + \frac{iU_{aa}}{2}(a^{\dagger 2} a^2 \rho - \rho a^{\dagger 2} a^2) \\ & + \frac{iU_{bb}}{2}(b^{\dagger 2} b^2 \rho - \rho b^{\dagger 2} b^2) + \frac{iU_{ab}}{2}(a^\dagger b^\dagger ab \rho - \rho a^\dagger b^\dagger ab) + \frac{\kappa}{2}(2ab\rho a^\dagger b^\dagger - a^\dagger b^\dagger ab \rho - \rho a^\dagger b^\dagger ab) \end{aligned} \quad (5.5)$$

To use the thermofield dynamics(TFD) formalism, the Hilbert space is doubled to $\mathcal{H} \otimes \tilde{\mathcal{H}}$ and the Hamiltonian is $\hat{H} = H + \tilde{H}$; where $\tilde{H}$ is H with operators $\tilde{a}, \tilde{b}$ replaces a, b . The time evolution of the density operator equation is

$$\frac{\partial}{\partial t} |\rho\rangle = -i\hat{H} |\rho\rangle \quad (5.6)$$

with, the BHH

$$\begin{aligned}\hat{H} = & \omega(a^\dagger a - \tilde{a}\tilde{a}^\dagger + b^\dagger b - \tilde{b}\tilde{b}^\dagger) + J(a^\dagger b - \tilde{a}\tilde{b}^\dagger + b^\dagger a - \tilde{b}\tilde{a}^\dagger) - \frac{U_{aa}}{2}(a^{\dagger 2}a^2 - \tilde{a}^2\tilde{a}^{\dagger 2}) \\ & - \frac{U_{bb}}{2}(b^{\dagger 2}b^2 - \tilde{b}^2\tilde{b}^{\dagger 2}) - \frac{U_{ab}}{2}(a^\dagger b^\dagger ab - \tilde{a}\tilde{b}\tilde{a}^\dagger\tilde{b}^\dagger) + \frac{i\kappa}{2}(2ab\tilde{a}\tilde{b} - a^\dagger b^\dagger ab - \tilde{a}\tilde{b}\tilde{a}^\dagger\tilde{b}^\dagger)\end{aligned}\quad (5.7)$$

To retain all the necessary features yet simplify calculations we apply the self-consistent Hartree-Fock approximation[56, 31] as follows,

$$\begin{aligned}2ab\tilde{a}\tilde{b} &= ab \langle \tilde{a}\tilde{b} \rangle + \langle ab \rangle \tilde{a}\tilde{b}; \text{ with } \langle \hat{\alpha}, \hat{\beta} \rangle = \Delta(t) \text{ and we get,} \\ \hat{H} &= \omega(a^\dagger a + b^\dagger b - \tilde{a}\tilde{a}^\dagger - \tilde{b}\tilde{b}^\dagger) - \frac{\Delta(t)}{4}[U_{aa}(aa + a^\dagger a^\dagger - \tilde{a}\tilde{a} - \tilde{a}^\dagger\tilde{a}^\dagger) + U_{bb}(bb + b^\dagger b^\dagger - \tilde{b}\tilde{b} - \tilde{b}^\dagger\tilde{b}^\dagger)] \\ &+ J(a^\dagger b + b^\dagger a - \tilde{a}\tilde{b}^\dagger - \tilde{b}\tilde{a}^\dagger) - \frac{\Delta(t)}{4}[U_{ab}(ab + a^\dagger b^\dagger - \tilde{a}\tilde{b} - \tilde{a}^\dagger\tilde{b}^\dagger) - i\kappa(ab - a^\dagger b^\dagger + \tilde{a}\tilde{b} - \tilde{a}^\dagger\tilde{b}^\dagger)]\end{aligned}\quad (5.8)$$

Now we consider, $U_{aa} = U_{bb} = 0$, and $U_{ab} = U$, then, the BHH becomes

$$\begin{aligned}\hat{H} = & \omega(a^\dagger a + b^\dagger b - \tilde{a}\tilde{a}^\dagger - \tilde{b}\tilde{b}^\dagger) + J(a^\dagger b + b^\dagger a - \tilde{a}\tilde{b}^\dagger - \tilde{b}\tilde{a}^\dagger) \\ & - \frac{\Delta(t)}{4}[U(ab + a^\dagger b^\dagger - \tilde{a}\tilde{b} - \tilde{a}^\dagger\tilde{b}^\dagger) - i\kappa(ab - a^\dagger b^\dagger + \tilde{a}\tilde{b} - \tilde{a}^\dagger\tilde{b}^\dagger)]\end{aligned}\quad (5.9)$$

5.3 Calculation of Entanglement of the system for two-mode state: for non-zero values of J , U & κ

Now we consider a system with damping coefficient κ , the interaction strength of nearest neighbour atoms U and hopping parameter J , then we can consider the Hamiltonian in Eq.(5.9) for this system and now the following transformations made on the Hamiltonian,

$$a = \frac{A+B}{\sqrt{2}}, \quad a^\dagger = \frac{A^\dagger+B^\dagger}{\sqrt{2}}, \quad b = \frac{-A+B}{\sqrt{2}}, \quad \text{and} \quad b^\dagger = \frac{-A^\dagger+B^\dagger}{\sqrt{2}} \quad (5.10)$$

similarly, we consider the tilde transformations also. Then the BHH will be,

$$\begin{aligned}\hat{H} = & \omega(A^\dagger A + B^\dagger B - \tilde{A}\tilde{A}^\dagger - \tilde{B}\tilde{B}^\dagger) + J(-A^\dagger A + B^\dagger B + \tilde{A}\tilde{A}^\dagger - \tilde{B}\tilde{B}^\dagger) \\ & - \frac{\Delta(t)}{8}[U(-AA + BB - A^\dagger A^\dagger + B^\dagger B^\dagger + \tilde{A}\tilde{A} - \tilde{B}\tilde{B} + \tilde{A}^\dagger\tilde{A}^\dagger - \tilde{B}^\dagger\tilde{B}^\dagger) \\ & - i\kappa(-AA + BB + A^\dagger A^\dagger - B^\dagger B^\dagger - \tilde{A}\tilde{A} + \tilde{B}\tilde{B} + \tilde{A}^\dagger\tilde{A}^\dagger - \tilde{B}^\dagger\tilde{B}^\dagger)]\end{aligned}\quad (5.11)$$

This Hamiltonian is decoupling into tilden and non-tilden parts, $\hat{H} = H_1 + H_2$.

where,

$$H_1 = (\omega - J)A^\dagger A + (\omega + J)B^\dagger B + \frac{(U - i\kappa)\Delta(t)}{8}(AA - BB) + \frac{(U + i\kappa)\Delta(t)}{8}(A^\dagger A^\dagger - B^\dagger B^\dagger), \quad (5.12)$$

$$H_2 = (J - \omega)\tilde{A}\tilde{A}^\dagger - (\omega + J)\tilde{B}\tilde{B}^\dagger - \frac{(U + i\kappa)\Delta(t)}{8}(\tilde{A}\tilde{A} - \tilde{B}\tilde{B}) - \frac{(U - i\kappa)\Delta(t)}{8}(\tilde{A}^\dagger\tilde{A}^\dagger - \tilde{B}^\dagger\tilde{B}^\dagger) \quad (5.13)$$

Then the solution of Eq.(5.6) is given by

$$|\rho(t)\rangle = \{exp[-iH_1 t] \otimes exp[iH_2 t]\}|\rho(0)\rangle, \quad (5.14)$$

here, H_1 and H_2 are independent. The Hamiltonian has SU(1,1) symmetry spectrum generating algebras in terms of the modes A and B are given by,

$$L_- = \frac{AA}{2}, \quad L_+ = \frac{A^\dagger A^\dagger}{2}, \quad \& \quad L_3 = \frac{1}{2}(A^\dagger A + \frac{1}{2}), \quad (5.15)$$

$$M_- = \frac{BB}{2}, \quad M_+ = \frac{B^\dagger B^\dagger}{2}, \quad \& \quad M_3 = \frac{1}{2}(B^\dagger B + \frac{1}{2}), \quad (5.16)$$

which satisfy the commutation relations,

$$[L_-, L_+] = 2L_3, \quad [L_3, L_\pm] = \pm L_\pm; \quad \& \quad [M_-, M_+] = 2M_3, \quad [M_3, M_\pm] = \pm M_\pm; \quad (5.17)$$

and also $N_A = A^\dagger A, N_B = B^\dagger B, N = N_A + N_B$.

The Hamiltonian from Eq.(5.12) in terms of the SU(1,1) generators is,

$$H_1 = \frac{(\omega - J)}{2} + \frac{(U - i\kappa)\Delta(t)}{4}L_- + 2(\omega - J)L_3 + \frac{(U + i\kappa)\Delta(t)}{4}L_+ \\ \frac{(\omega + J)}{2} - \frac{(U - i\kappa)\Delta(t)}{4}M_- + 2(\omega + J)M_3 - \frac{(U + i\kappa)\Delta(t)}{4}M_+ \quad (5.18)$$

The Hamiltonian is diagonalized with the help of the underline SU(1,1) symmetry by a series of transformations given by

$$D = \mu_1 A + \nu_1^* A^\dagger, \quad D^\dagger = \mu_1^* A^\dagger + \nu_1 A, \quad E = \mu_2 B + \nu_2^* B^\dagger, \quad E^\dagger = \mu_2^* B^\dagger + \nu_2 B. \quad (5.19)$$

with $\mu_1 = e^{i\phi_{\mu_1}}|\mu_1|$, $\nu_1 = e^{i\phi_{\nu_1}}|\nu_1|$, $\mu_2 = e^{i\phi_{\mu_2}}|\mu_2|$ and $\nu_2 = e^{i\phi_{\nu_2}}|\nu_2|$; ($\phi_{\mu_1}, \phi_{\nu_1}, \phi_{\mu_2}, \phi_{\nu_2} \in R$), and we should consider the condition that, $t = 0$.

Similarly by diagonalizing H_2 , we get the final diagonalized Hamiltonian H_{1f} ;

$$H_{1f} = S^{-1}(r_1)H'_{1A}S(r_1) + S^{-1}(r_2)H'_{1B}S(r_2) = \Omega_1^2(D^\dagger + \frac{1}{2}) + \Omega_2^2(E^\dagger + \frac{1}{2}), \quad (5.20)$$

here,

$$S(r_1) = \exp(r_1 L_+ - r_1 L_-) \text{ \& } S(r_2) = \exp(r_2 M_+ - r_2 M_-)$$

and $\Omega_1^2 = 64(\omega - J)^2 - (U^2 + \kappa^2)\Delta^2(0)$ and r_1 is related to μ_1, ν_1 ; \&

$\Omega_2^2 = 64(\omega + J)^2 - (U^2 + \kappa^2)\Delta^2(0)$ and r_2 is related to μ_2, ν_2 , in Eq.(5.19) via the following Bogolybov coefficients :

$$\mu_1 = \cosh r_1 = \frac{(\omega - J)}{\sqrt{(\omega - J)^2 - \frac{(U^2 + \kappa^2)\Delta^2(0)}{64}}}, \quad \nu_1 = \sinh r_1 = \frac{\sqrt{U^2 + \kappa^2}\Delta(0)}{8\sqrt{(\omega - J)^2 - \frac{(U^2 + \kappa^2)\Delta^2(0)}{64}}} \quad (5.21)$$

$$\mu_2 = \cosh r_2 = \frac{(\omega + J)}{\sqrt{(\omega + J)^2 - \frac{(U^2 + \kappa^2)\Delta^2(0)}{64}}}, \quad \nu_2 = \sinh r_2 = \frac{\sqrt{U^2 + \kappa^2}\Delta(0)}{8\sqrt{(\omega + J)^2 - \frac{(U^2 + \kappa^2)\Delta^2(0)}{64}}} \quad (5.22)$$

here, r_1 and r_2 are the squeezing parameters, $|\mu_1|^2 - |\nu_1|^2 = 1$ and $|\mu_2|^2 - |\nu_2|^2 = 1$.

Hence the solution of the master Eq.(5.6) from Eq.(5.14) will becomes

$$\begin{aligned} |\rho(t) > &= \exp[-i\hat{H}t]|\rho(0) > \\ &= C(t)\{ \exp(\zeta_{a-}L_- + \zeta_{a3}L_3 + \zeta_{a+}L_+) \otimes \exp(\zeta_{a'-}\tilde{L}_- + \zeta_{a'3}\tilde{L}_3 + \zeta_{a'+}\tilde{L}_+) \\ &\quad \times \exp(\zeta_{b-}M_- + \zeta_{b3}M_3 + \zeta_{b+}M_+) \otimes \exp(\zeta_{b'-}\tilde{M}_- + \zeta_{b'3}\tilde{M}_3 + \zeta_{b'+}\tilde{M}_+) \} |\rho(0) > \end{aligned} \quad (5.23)$$

here, $C(t) = \exp[\{i(\omega - J)(N_A - \tilde{N}_A) + i(\omega + J)(N_B - \tilde{N}_B)\}t]$;

$$\zeta_{a-} = -\frac{(iU+\kappa)\Delta(t)t}{8}, \quad \zeta_{a3} = -i(\omega - J)t, \quad \zeta_{a+} = -\frac{(iU-\kappa)\Delta(t)t}{8};$$

$$\zeta_{a'-} = \frac{(iU-\kappa)\Delta(t)t}{8}, \quad \zeta_{a'3} = i(\omega - J)t, \quad \zeta_{a'+} = \frac{(iU+\kappa)\Delta(t)t}{8};$$

$$\zeta_{b-} = \frac{(iU+\kappa)\Delta(t)t}{8}, \quad \zeta_{b3} = -i(\omega + J)t, \quad \zeta_{b+} = \frac{(iU-\kappa)\Delta(t)t}{8};$$

$$\zeta_{b'-} = -\frac{(iU-\kappa)\Delta(t)t}{8}, \quad \zeta_{b'3} = i(\omega + J)t, \quad \zeta_{b'+} = -\frac{(iU+\kappa)\Delta(t)t}{8};$$

$$\begin{aligned} |\rho(t) > &= C(t)\{ \exp(\Gamma_{a+}L_+) \exp(\ln(\Gamma_{a3})L_3) \exp(\Gamma_{a-}L_-) \otimes \exp(\Gamma_{a'+}\tilde{L}_+) \exp(\ln(\Gamma_{a'3})\tilde{L}_3) \exp(\Gamma_{a'-}\tilde{L}_-) \\ &\quad \times \exp(\Gamma_{b+}M_+) \exp(\ln(\Gamma_{b3})M_3) \exp(\Gamma_{b-}M_-) \otimes \exp(\Gamma_{b'+}\tilde{M}_+) \exp(\ln(\Gamma_{b'3})\tilde{M}_3) \exp(\Gamma_{b'-}\tilde{M}_-) \} |\rho(0) > \end{aligned} \quad (5.24)$$

here,

$$\Gamma_{i\pm} = \frac{2\zeta_{i\pm} \sinh \phi_i}{2\phi_i \cosh \phi_i - \zeta_{i3} \sinh \phi_i} \quad \& \quad \Gamma_{i3} = \left(\frac{2\phi_i}{2\phi_i \cosh \phi_i - \zeta_{i3} \sinh \phi_i} \right)^2 \quad (5.25)$$

with

$$\phi_i^2 = \frac{\zeta_{i3}^2}{4} - \zeta_{i+}\zeta_{i-}, \quad (5.26)$$

here, 'i' stands for a, b, a' and b' and one can calculate, $\phi_i = \frac{(\Delta(0)\sqrt{\kappa^2+U^2}+4i(\omega-J))t}{8}$ (for $i = a$) and ϕ_i^* is it's conjugate.

By considering an initial state $|\rho(0)\rangle = \sum_{m,n}^N \rho_{m,n}(0) |m, \tilde{m}, n, \tilde{n}\rangle$, we get the density matrix :

$$\begin{aligned} \rho_{m,n}(t) = & K(t) \sum_{q'=0}^{\min(m',n')} \sum_{p'=0}^{\infty} \left[\binom{m'+p'-q'}{p'} \binom{n'+p'-q'}{p'} \binom{m'}{q'} \binom{n'}{q'} \right]^{\frac{1}{2}} \\ & \times \sum_{q=0}^{\min(m,n)} \sum_{p=0}^{\infty} \left[\binom{m+p-q}{p} \binom{n+p-q}{p} \binom{m}{q} \binom{n}{q} \right]^{\frac{1}{2}} \\ & \times [\Gamma_{A+}]^{p'} [\Gamma_{A3}]^{\frac{(m'+n'-2q'+1)}{2}} [\Gamma_{A-}]^{q'} [\Gamma_{B+}]^p [\Gamma_{B3}]^{\frac{(m+n-2q+1)}{2}} [\Gamma_{B-}]^q \\ & \times \rho_{m+p-q, m'+p'-q', n+p-q, n'+p'-q'}(0) \end{aligned} \quad (5.27)$$

here, $K(t)$ is an overall phase factor due to $C(t) = \exp[\{i(\omega - J)(N_A - \tilde{N}_A) + i(\omega + J)(N_B - \tilde{N}_B)\}t]$.

5.3.1 Entropy of entanglement of the system

The entropy of entanglement of system with damping is

$$\begin{aligned} S = & - \sum_{i=0}^N \lambda_i \log \lambda_i = - \sum_{n=0}^N \left[\frac{N!}{(N-n)!n!} \sin^{2n}(\phi) \cosh^{2N-2n}(\phi) \right] \\ & \times \{ \log \left[\frac{N!}{(N-n)!n!} \right] + (2n) \log[\sinh(\phi)] + (2N-2n) \log[\cosh(\phi)] \} \end{aligned} \quad (5.28)$$

where, $\phi = \frac{(\Delta(0)\sqrt{\kappa^2+U^2}+4i(\omega-J))t}{8}$ and ϕ^* is it's conjugate.

Case-1 For two boson system as an input(i.e., $N = 2$): For the input two boson state the entanglement entropy is

$$\begin{aligned} S = & -4 \cosh^4(\phi) \log[\cosh(\phi)] - 2 \sinh^2(\phi) \cosh^2(\phi) \\ & x \log[\cosh(\phi) \sinh(\phi)] - 4 \sinh^4(\phi) \log[\sinh(\phi)] \end{aligned} \quad (5.29)$$

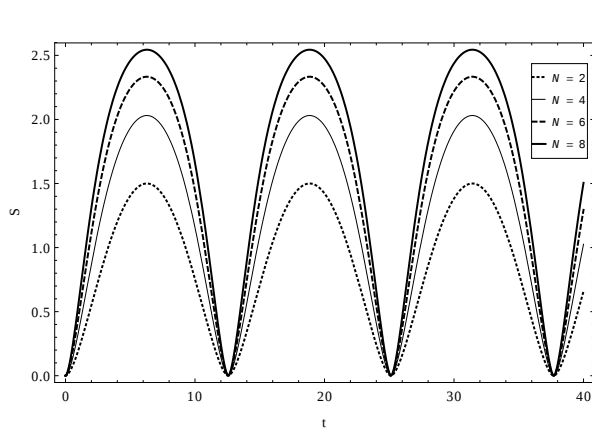
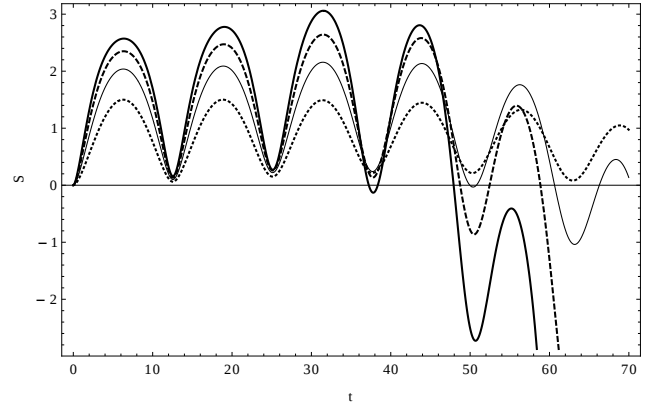
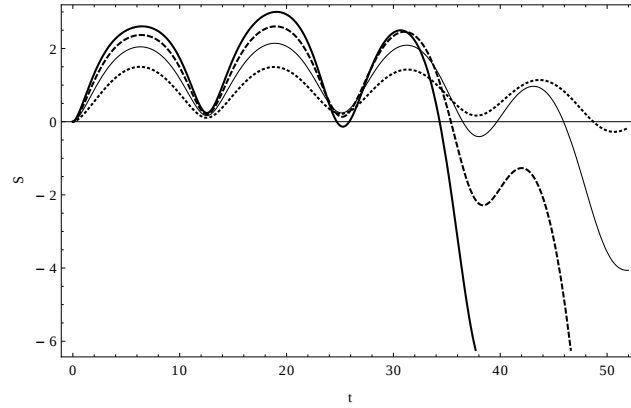
(a) with damping parameter, $\kappa = 0$ (b) with $\kappa = 0.2$ (c) with $\kappa = 0.3$

Figure 5.2: Shows the time evolution of entropy of entanglement $[S(t)]$ for various input boson systems: $N = 2$ (dotted line), $N = 4$ (thin line), $N = 6$ (dashed line), $N = 8$ (thick line) with, $J = 0.5$.

which is shown as dotted line in Fig.5.2, for $J = 0.5$. Here we have considered both cases, in Fig.5.2a, shows the system with no damping, and in Fig.5.2b & 5.2c, shows the system with damping. Increasing the value of κ , causes more damping in the system.

In the similar way, in further cases, the entropy of entanglement for 4-, 6-, 8- boson input states is shown in Fig.5.2 for $J = 0.5$, as thin line (for 4-boson), dashed line (for 6-boson), and thick line (for 8-boson) respectively. Here also the same, we have considered both cases, in Fig.5.2a.

5.3.2 Logarithmic negativity of the system:

We can calculate the log-negativity of the system by using **the covariance matrix method** and "Simon's criterion" [41] as mentioned in section-3.2.2. The density matrix can be written as $\rho'(t) = S^\dagger(r)R^\dagger(\phi)\rho(t)R(\phi)S(r)$, where the squeezing matrix is $S(r)$ and $R(\phi)$ is the rotation matrix mixing real and tilde fields. The initial

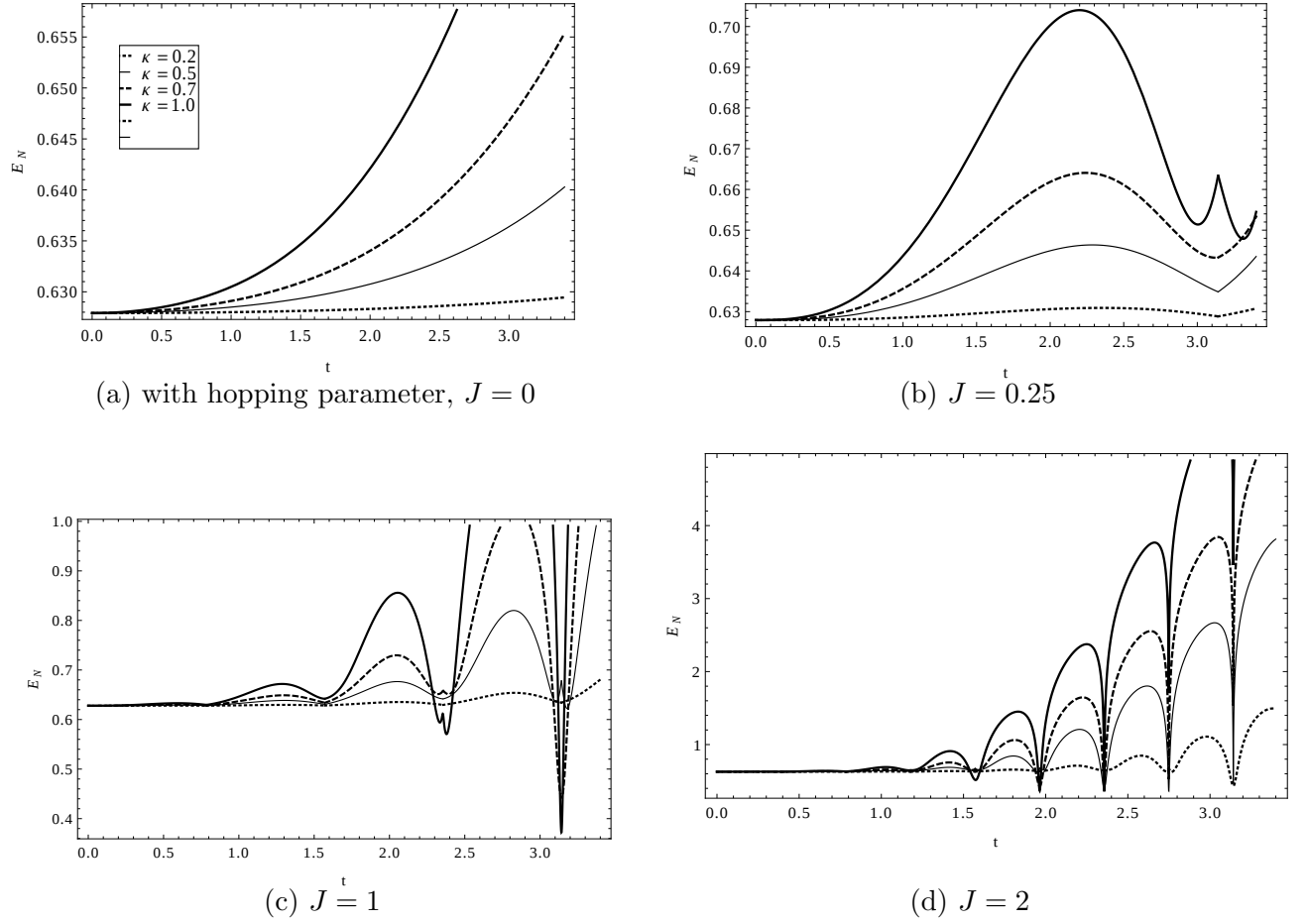


Figure 5.3: Show the time evolution of entanglement(E_N) for different values of $\kappa(= 0.2, 0.5, 0.7$ and 1) correspondingly for $\Delta(0) = 0.25, \omega = 0.25$ and $U = 0.2$.

state of two mode system is the two mode vacuum state($|\rho(0)\rangle = |0, 0, \tilde{0}, \tilde{0}\rangle$).

Using the covariance matrix method to calculate the log-negativity of the time evolved state $\rho'(t)$, the reduced covariance matrix is,

$$V(r_1, r_2) = \begin{pmatrix} p+q & 0 & -(s+t) & 0 \\ 0 & p^*+q^* & 0 & s+t \\ -(s+t) & 0 & p+q & 0 \\ 0 & s+t & 0 & p^*+q^* \end{pmatrix} \quad (5.30)$$

where, $p = e^{2iJt} \cosh 2r_1$, $q = e^{2iJt} \cosh 2r_2$, $s = \sinh 2r_1$, $t = \sinh 2r_2$.

The covariance matrix in canonical form is given by,

$$V = \begin{pmatrix} \alpha & \gamma \\ \gamma^\dagger & \beta \end{pmatrix} \quad (5.31)$$

where,

$$\alpha = \begin{pmatrix} p+q & 0 \\ 0 & p^*+q^* \end{pmatrix} = \beta, \quad \text{and} \quad \gamma = \begin{pmatrix} -(s+t) & 0 \\ 0 & (s+t) \end{pmatrix} \quad (5.32)$$

Then the “separability condition” [41] for any two mode state is

$$\text{Det}\alpha\text{Det}\beta + \left(\frac{1}{4} - |\text{Det}\gamma|^2\right) - \text{tr}(\alpha J \gamma J \beta J \gamma^T J) \geq \frac{1}{4}(\text{Det}\alpha + \text{Det}\beta) \quad (5.33)$$

The symplectic eigenvalues are defined as,

$$\nu_{\pm} = \sqrt{\frac{1}{2} \left\{ \tilde{\Delta} \pm \sqrt{\tilde{\Delta}^2 - \frac{4}{\mu^2}} \right\}} \quad (5.34)$$

where, $\tilde{\Delta} = \text{Det}\alpha + \text{Det}\beta - 2\text{Det}\gamma = 2(p+q)(p^*+q^*) + 2(s+t)^2$

and $\mu = [\text{Det}V]^{-\frac{1}{2}} = [(p+q)^2(p^*+q^*)^2 + (p+q)^2(s+t)^2 - (p^*+q^*)^2(s+t)^2 - (s+t)^4]^{-\frac{1}{2}}$

The **log-negativity entanglement** of the system is

$$E_N = \max\{0, -\log\nu_{-}\}. \quad (5.35)$$

For $r_1 = r_2 = r = -\frac{\Delta(0)}{2} \left(1 + \frac{(\omega-J)^2 t^2}{4}\right) \sqrt{(U^2 + \kappa^2)} t$, which is shown in the Fig. ??(a,b,c,d).

5.3.3 Decoherence of the system

To quantify the **decoherence** effects for damping system, we compute ρ^2 and then,

$$\text{Tr}[\rho^2(t)] = \text{Tr}\left[\sum_{m,n} \langle m, n | \rho^2(t) | m, n \rangle\right] = \text{Exp} \left[\frac{2\zeta_{i\pm} \sinh \phi_i}{2\phi_i \cosh \phi_i - \zeta_{i3} \sinh \phi_i} \right] \quad (5.36)$$

where, $\zeta_{i\pm} = \frac{-(iU \pm \kappa)\Delta(t)t}{8}$, $\zeta_{i3} = -i(\omega - J)t$, and $\phi_i = \frac{(\Delta(0)\sqrt{\kappa^2 + U^2} + 4i(\omega - J)t)}{8}$ and ϕ_i^* is it's conjugate.

The decoherence of the system is shown in Figs.5.4. For weak coupling Fig.5.4a, shows the time evolution of decoherence for various values of κ and Fig.5.4b, shows the same for strong coupling.

For weak coupling, see that for even short times, as the value of damping coefficient increases the system decoheres, to a too low value, very fast, while for strong coupling, the system decoheres in an oscillatory manner and saturates to a non-zero value.

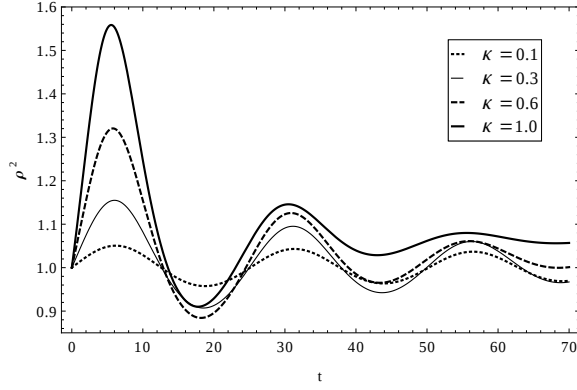
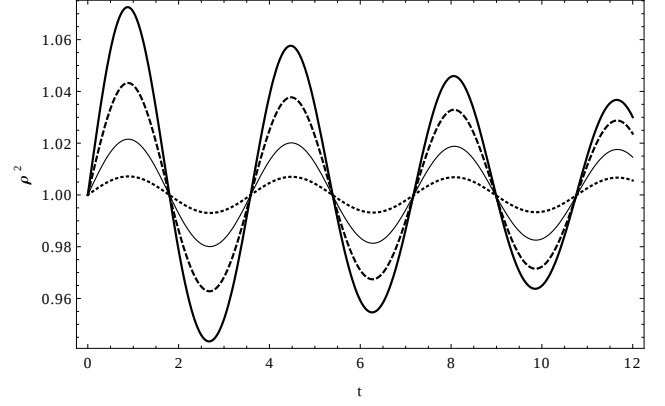

 (a) with hopping parameter, $J = 0.5$

 (b) with $J = 2.0$

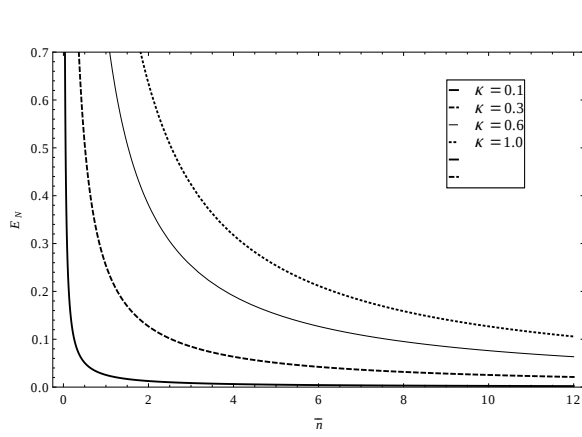
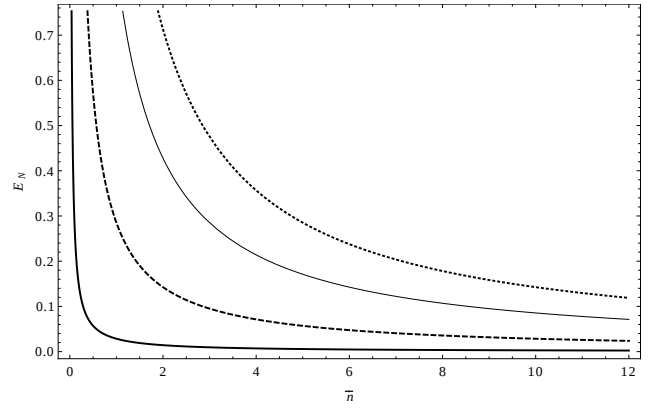
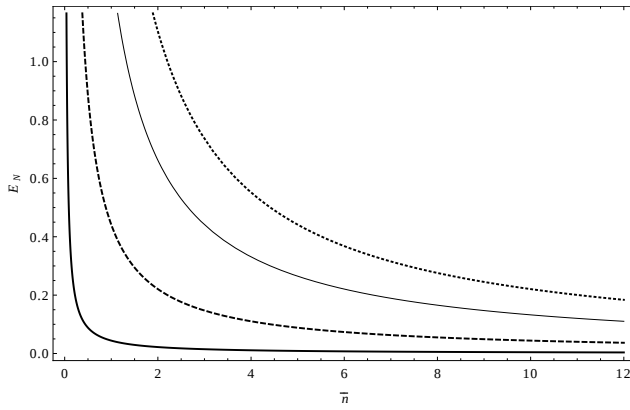
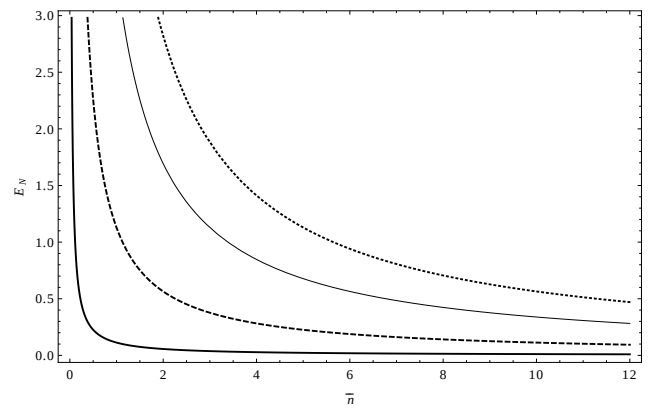
 Figure 5.4: Shows the time evolution of Decoherence for different values of $\kappa (= 0.1, 0.3, 0.6$ and $1.0)$ with $\Delta(0) = 0.25$, $\omega = 0.25$ and $U = 0$.

 (a) with hopping parameter, $J = 0$

 (b) $J = 1$

 (c) $J = 2$

 (d) $J = 4$

 Figure 5.5: Plot of Entanglement(E_N) vs. thermal distribution function($\bar{n}$) for different values of $\kappa (= 0.1, 1, 3$ and $5)$ correspondingly for $\Delta(0) = 0.25$, $\omega = 0.25$.

5.3.4 Entanglement for two-mode thermal state

Taking the initial state $\rho(0)$ to be the two mode thermal vacuum state, the covariance matrix is given by,

$$V(r_1, r_2) = \begin{pmatrix} c+d & 0 & e+f & 0 \\ 0 & c^*+d^* & 0 & -(e+f) \\ e+f & 0 & c+d & 0 \\ 0 & -(e+f) & 0 & c^*+d^* \end{pmatrix} \quad (5.37)$$

where,

$$c = e^{2iJt}(n_1 \cosh^2 r_1 + n_2 \sinh^2 r_1), \quad d = e^{2iJt}(n_1 \sinh^2 r_2 + n_2 \cosh^2 r_2), \quad e = \frac{n_1+n_2}{2} \sinh 2r_1 \quad \text{and} \quad f = \frac{n_1+n_2}{2} \sinh 2r_2.$$

Applying "Simon's criterion" Eq.(5.33) we see that the system is entangled iff

$$(n_1 + n_2)^4 [\cosh^2 2r - \sinh^2 2r]^2 + \frac{1}{16} \geq \frac{(n_1 + n_2)^2}{2} [\cosh^2 2r + \sinh^2 2r] \quad (5.38)$$

For $r_1 = r_2 = r$, and $n_1 = n_2 = n$, the logarithmic negativity entanglement is

$$E_N(r) = -\frac{1}{2} [\log_2(e^{-4r}/n)]. \quad (5.39)$$

which is shown in the Fig. 5.5(a,b,c,d).

We observed that as the system not only gets less entangled for high values of κ (quantified by r), but also for large n (the external heat bath) and entanglement increase for higher values of J .

5.4 Calculation of entanglement of the system for two mode state: for $J \neq 0$

This is one of the limiting case of the most general case in above section, here the Hamiltonian $\hat{H}$ from the Eq.(5.9) with hopping parameter $J \neq 0$, the interaction strength of nearest neighbour atoms $U = 0$ and damping coefficient $\kappa = 0$ is given by

$$\hat{H} = \omega(a^\dagger a + b^\dagger b - \tilde{a}\tilde{a}^\dagger - \tilde{b}\tilde{b}^\dagger) + J(a^\dagger b + b^\dagger a - \tilde{a}\tilde{b}^\dagger - \tilde{b}\tilde{a}^\dagger) \quad (5.40)$$

and can be decoupled into tilden and non-tilden parts: $\hat{H} = H + \tilde{H}$.

where,

$$H = \omega(a^\dagger a + b^\dagger b) + J(a^\dagger b + b^\dagger a) \quad (5.41)$$

$$\tilde{H} = -\omega(\tilde{a}\tilde{a}^\dagger + \tilde{b}\tilde{b}^\dagger) - J(\tilde{a}\tilde{b}^\dagger + \tilde{b}\tilde{a}^\dagger) \quad (5.42)$$

Then the solution of Eq.(5.6) is given by

$$|\rho(t)\rangle = \exp[-iHt] \otimes \exp[i\tilde{H}t] |\rho(0)\rangle \quad (5.43)$$

where, $|\rho(0)\rangle$ is an initial state in $\mathcal{H} \otimes \tilde{\mathcal{H}}$.

To solve the master equation and calculate the entanglement properties of the system, we use the underlying symmetries of the Hamiltonians(Eqs.(5.41), (5.42)), by defining the following operators.

$$L_+ = a^\dagger b, \quad L_- = b^\dagger a \quad \& \quad L_3 = \frac{1}{2}(a^\dagger a - b^\dagger b) \quad (5.44)$$

which satisfy the $SU(2)$ algebra,

$$[L_3, L_\pm] = \pm L_\pm \quad \& \quad [L_+, L_-] = 2L_3 \quad (5.45)$$

with number operator, $\mathcal{N} = a^\dagger a + b^\dagger b$.

The Hamiltonian from Eq.(5.41) in terms of the $SU(2)$ generators is,

$$H = \omega\mathcal{N} + J(L_+ + L_-) \quad (5.46)$$

Hence, the underlying symmetry of the Schrodinger like Eq.(5.6) is $SU(2) \otimes SU(2)$ and $|\rho(t)\rangle$ is given by

$$|\rho(t)\rangle = e^{\alpha(t)L_+ - \alpha^*(t)L_-} |\rho(0)\rangle; \quad \text{here, } \alpha(t) = iJt. \quad (5.47)$$

The $SU(2)$ disentanglement formula gives,

$$|\rho(t)\rangle = e^{\xi L_+} e^{\log(1+|\xi|^2)L_3} e^{-\xi^* L_-} |\rho(0)\rangle \quad (5.48)$$

here, $\xi = \xi(\alpha(t)) = \frac{\alpha(t) \tan(|\alpha(t)|)}{|\alpha(t)|}$

The density matrix in the number state basis (for $n_a \neq n_b$) is given by

$$|\rho(t)\rangle = \sum_{n_a, n_b=0}^N C_{n_a, n_a} C_{n_b, n_b}^* |n_a, N - n_a\rangle \langle n_b, N - n_b|$$

where,

$$C_{n_a, n_a} = \frac{\xi^{n_a}}{(1 + |\xi|^2)^{N/2}} \binom{N}{n_a}^{\frac{1}{2}}; \quad C_{n_b, n_b} = \frac{\xi^{n_b}}{(1 + |\xi|^2)^{N/2}} \binom{N}{n_b}^{\frac{1}{2}}; \quad \& \quad N = n_a + n_b. \quad (5.49)$$

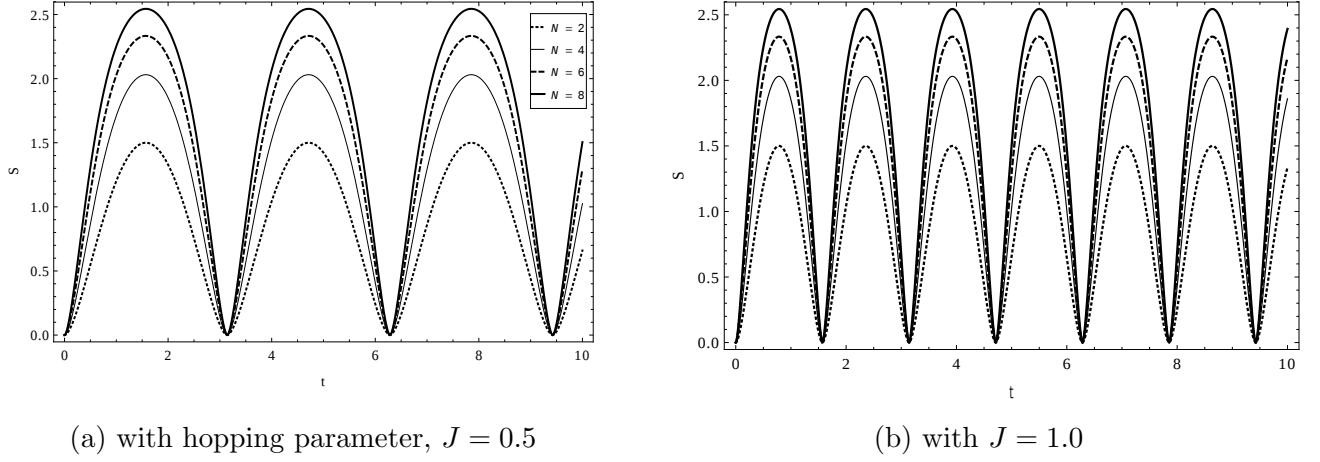


Figure 5.6: Shows the time evolution of entropy entanglement $S(t)$ for various input boson number states ($N = 2$ (dotted line), 4(thin line), 6(dashed line), 8(thick line)).

Using the results of the previous chapters, we can get the eigenvalues of the “partial transpose of ρ ” and calculate the entanglement entropy in the next section.

5.4.1 Entropy of entanglement of the system

Now, the entropy for this system will be (as shown in the eq.(3.13),

$$S = - \sum_{i=0}^N \lambda_i \log \lambda_i = - \sum_{n=0}^N \left[\frac{N!}{(N-n)!n!} \sin^{2n}(Jt) \cos^{2N-2n}(Jt) \right] \times \{ \log \left[\frac{N!}{(N-n)!n!} \right] + (2n) \log[\sin(Jt)] + (2N-2n) \log[\cos(Jt)] \} \quad (5.50)$$

Now we consider different cases of optical input states,

Case-1 For the input two boson system (i.e., $N = 2$), the entropy of entanglement (from eq(5.50)) is shown as dotted line in Fig.5.6a, for $J = 0.5$. Starting with a separable state at $t = 0$, S increases from zero to a maximum value of 1.5 at $Jt = 1.55$. This is a maximally entangled value, after which it eventually becomes zero at $Jt = 3.144$. Thus at this point, the state becomes disentangled. We see a periodic behavior no interference effects. And also, for $J = 1.0$ dotted line in Fig.5.6b, shows quite similar results of $J = 0.5$ case but, repeated twice within the period of a cycle.

In similar way, in further cases, we have calculated entropy of entanglement for 4-boson, 6-boson and 8-boson systems and shown in thin line(4-boson), dashed line(6-boson) and thick line(8-boson) in Fig.(5.6a) for $J = 0.5$ and in Fig.5.6b) for $J = 1.0$ respectively. This is similar to the case-1, with increasing amplitude values as increasing boson number and also see the periodic behavior that, the system got entangled, disentangled periodically and there are no interference effects.

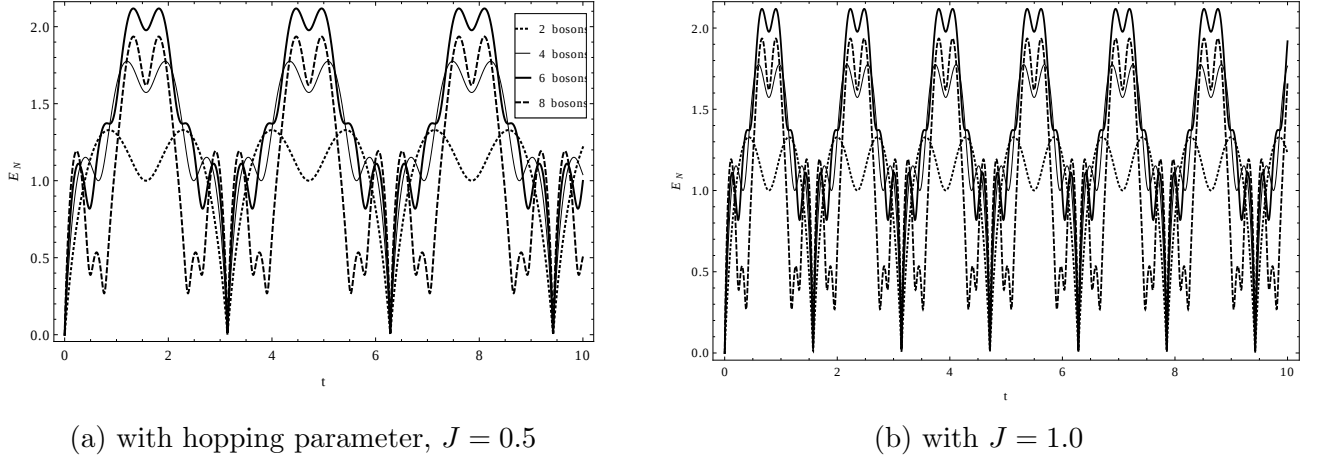


Figure 5.7: Shows the time evolution of log negativity $[E_N(t)]$ for various input boson number states: $N = 2$ (dotted line) for $|1, 1\rangle$ state, $N = 4$ (thin line) for $|2, 2\rangle$ state, $N = 6$ (thick line) for $|3, 3\rangle$ state, $N = 8$ (dashed line) for $|4, 4\rangle$ state.

5.4.2 Logarithmic negativity of the system

We can also measure the logarithmic negativity of the system as defined in eq.(4.22) as follows: Now we consider different cases of optical input states,

Case-1 For the input two boson system (i.e., $N = 2$):, the log-negativity of the input two boson NOON-state $|\psi_{in}\rangle = |1, 1\rangle$, is shown as dotted line in Fig.5.7a, for $J = 0.5$. Starting with a separable state at $t = 0$, (E_N) increases from zero to a maximum value of 1.33 at $Jt = 0.8514$. This is a maximally entangled state. Then E_N decreases and attains a value of 1.001 at $Jt = 1.571$, this is entangled state. Again E_N increases to a maximum of 1.33 at $Jt = 2.281$ and finally it decreases and eventually becomes equal to zero at $Jt = 3.14$. Thus at this point, the state becomes disentangled. We see a periodic behavior and continue the interference effects. And also, for $J = 1.0$, dotted line in Fig.5.7b, shows quite similar results of $J = 0.5$ case, with decreasing to half of the bandwidth.

In similar way, in further cases ($N = 4, 6, 8$), we have calculated the log-negativity entanglement for 4-boson, 6-boson and 8-boson systems and shown in thin line(4-boson), thick line(6-boson) and dashed line(8-boson) in Fig.(5.6a) for $J = 0.5$ and in Fig.5.6b) for $J = 1.0$ respectively. These are quite different to the case-1, in raising dips with time laps as increasing boson number and also see the periodic behavior that, the system got entangled and disentangled periodically and continue the interference effects.

5.5 Conclusion

In this chapter, we extended the work of Chaitanya et al[56] in the presence of non-linear damping with coupling strength. This is of interest because ultracold atoms are a natural resource for quantum information process. Using TFD, we have exactly solved the master equation associated with $SU(1,1)$ symmetries for two site Bose-Hubbard model. The different entanglement properties and decoherence of the system for coupling strength(J) and with damping($\kappa \neq 0$) and without damping($\kappa = 0$) have been shown. Our results have also shown that the entanglement of the system can withstand a certain amount of damping, suggesting that it can use for applications such as in quantum computation. We compare the role of the hopping parameter(J) and the repulsive on-site interaction parameter(U) on these features and shown how we can tune the system for optimum quantum information processing.

Chapter 6

Application of TFD to study thermal sources through pion correlations and multiplicities in high energy collisions

In the previous chapters we have applied the operator formalism of Thermo-field dynamics to many-body quantum systems those are used for quantum communications and shown how TFD helps us to understand, and hence control, their decoherence. The quantum correlations in such systems are easy to quantify. However the full armour of Thermal field theory reveals itself in particle physics, since the theoretical framework of particle physics is quantum field theory[2] and particles are excitations of the quantum field theoretic vacuum. The dynamics of their interaction is given by the underlying field theory. Protons, neutrons, pions, and many other hadrons (strongly interacting particles) discovered in the last 80-years are now understood to be rather complicated systems which must be explained in terms of their fundamental point like constituents: quarks (and anti-quarks) bound together through strong interactions mediated by gluons. The field theory that describe these strong nuclear interactions is called QCD. It has led to the prediction that at high temperature and high densities, quarks and gluons form a state of matter called quark gluon plasma(QGP)[57]. QCD vacuum states produce quark-antiquark pairs and gluons, unlike the photons, the gluons interact with one another. In QCD, coupling increases at large separations and decreases at small separations. This existence of direct coupling of the gluons has dramatic implications causing QCD to have the following unusual properties: Asymptotic freedom[58] and Confinement[59]. Asymptotic freedom is dominant in high-energy scales but, as energy decreases, confinement becomes dominant. The exact nature of the phase transition from confinement to asymptotic freedom is still under investigation, although results from Lattice gauge theory[60] show promising results. Experiments at the Relativistic heavy ion

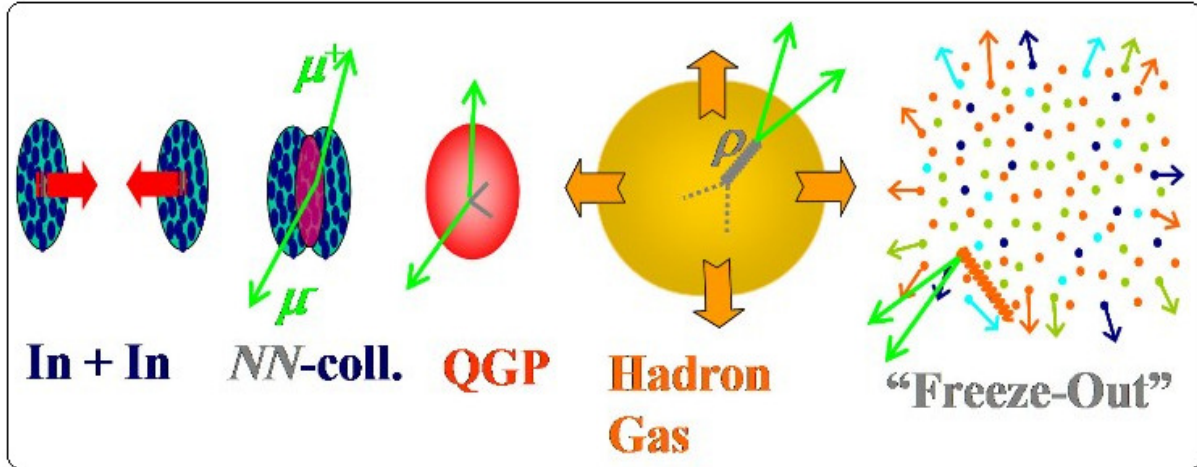


Figure 6.1: Relativistic heavy-ion collisions: quark gluon plasma(QGP)

collider(RHIC) in April 2005, tentatively confirmed the existence of a state where quarks and gluons are deconfined and form the QGP[61]. In such experiments, because of the phenomenon of confinement, the QGP cannot be directly detected. Since this plasma survives for a short time, we cannot use traditional methods to study its properties. Its detection is based on the properties of the particles detected in detectors. Pions, photons, muons and electrons are the main particles that survive to be detected. Thus, both theoretical and experimental tools must be developed as certain whether a region of high temperature (called a "fireball") has been formed in the collision. Among the experimental measurables are the correlations and distributions of pions and photons that emerge from the short lived formation region[62]. The pion multiplicity distributions and two pion correlations are examples of indirect signals which carry information about the initial stages of the collision and can give some information whether a hot thermal state was formed in the initial stages of the collision. **Our aim is to study how such quantities are modified by high temperature effects using various statistical and quantum optical techniques.** High-energy heavy ion collision processes (quark-gluon plasma) is a crucial test for the double-time TFD. With the formalism developed so far, we attempt to give a possible way of looking at the phenomenon.

6.1 Determining multiplicity distributions and correlations for non-thermal states using quantum statistical methods

Measurement of the source size in high energy interactions gives an idea of the corresponding energy density reached in the event and gives information about a possible QGP phase transition. We calculate the different

multiplicity distributions for different sources and the degree of second order correlation function (which is related to the second order moments of the distribution), by using quantum statistical methods. An important measurable is the chaoticity parameter, which characterizes the amount of non-coherent thermal excitations in the source is given by $\lambda = g^{(2)} - 1$ and has been found in various experiments by fitting experimental data, this gives a direct measurement of the thermal nature of the source and we would like to quantify its temperature dependence by using Thermo-field Dynamics. For a coherent source, $g^{(2)}$ is always one and hence $\lambda = 0$. Now we calculate λ for some mixed state distributions with the help of density matrices and discuss the different probability distributions and second order correlations.

All these moments and the multiplicity distribution can be obtained by the differentiation of an analytic generating function, $G(z)$ defined by the formula:

$$G(z) = \sum_{n=0}^{\infty} P_n z^n. \quad (6.1)$$

The probability distribution function is,

$$P_n = \frac{1}{n!} \left. \frac{d^n G(z)}{dz^n} \right|_{z=0}. \quad (6.2)$$

The average multiplicity is,

$$\langle n \rangle = \sum_{n=0}^{\infty} n P_n. \quad (6.3)$$

Then, the second order correlation is given by

$$g^2(0) = \left\{ \frac{\partial^2 G(z)}{\partial z^2} \right\} / \left\{ \left(\frac{\partial G(z)}{\partial z} \right)^2 \right\} \Big|_{z=1} \quad (6.4)$$

We will use these techniques to calculate the second order correlation for the various states that we consider in this chapter.

The two-particle correlation function in momentum space can be written as

$$C(k, k') = \frac{P(k, k')}{P(k)P(k')} \quad (6.5)$$

where $P(k, k')$ is the two particle probability density, subject to Bose Einstein symmetrization and $P(k), P(k')$ are the single-particle probability distribution for the particles with four momentum k and k' respectively. Thus the value of C at concurrent momenta can characterize the distribution source. Now we generalize from two point sources x (corresponding to momentum k) and x' (corresponding to momentum k') to the distribution of pion production points described by $f(x)$. If $G[f(x)] = \int f(x) e^{-(k-k')x} dx$ is the fourier

transform of $f(x)$, then the momentum space correlation function is,

$$C(k, k') = 1 + |G[f(x)]|^2 \quad (6.6)$$

. The momentum correlation function of pion pairs can give the distribution function of the points of origin of the pions. The above discussion is for classical pions. Since pions are by nature quantum mechanical and we would like to use the well established tools of quantum optics, therefore, we require the knowledge of the quantum mechanical correlation function. For a quantum systems characterized by ρ (the density operator) and $a^\dagger$ and a (the particle creation and annihilation operators). If we do the mode expansion in rapidities y and y' , then the two particle correlation function is given by,

$$C(y, y') = \langle a^\dagger(y)a(y) \rangle \langle a^\dagger(y')a(y') \rangle + |\langle a^\dagger(y)a(y') \rangle|^2 + |\langle a(y)a(y') \rangle|^2. \quad (6.7)$$

Therefore, the normalized correlation function viz. degree of second order coherence is,

$$\begin{aligned} g^{(2)} &= \frac{\langle a^\dagger a a^\dagger a \rangle - \langle a^\dagger a \rangle^2}{\langle a^\dagger a \rangle^2} \\ &= \frac{\langle a^{\dagger 2} a^2 \rangle}{\langle a^\dagger a \rangle^2} \end{aligned} \quad (6.8)$$

Using the density matrix formalism, the chaoticity parameter(λ) has a natural interpretation in terms of second order correlation, $g^2(0)$ and can give information about the nature of the pion emission other than chaotic, coherent or partially coherent. In fact situations such as $g^2(0) > 2$ or $g^2(0) < 1$ can be constitute within certain states giving information about inherently quantum mechanical processes such as squeezing and anti-bunching which can be encountered in many pion emission processes.

In this chapter, we shown how incorporation of temperature is particularly simple in the thermofield dynamical approach to study quantum systems and apply it to multiplicity distributions and correlations of hadrons by adopting quantum optical phenomenon and statistical methods approach to high density QCD.

One can also use the density matrix formalism in statistical quantum mechanics to calculate the probability distribution. The density matrix for a statistical quantum ensemble and the probability distribution P_n are related by the formula

$$\rho = \sum P_n |n\rangle \langle n|, \quad (6.9)$$

where, sum is with respect to a complete set of states having a closure property, $\sum_n |n\rangle \langle n| = 1$. The average value of an operator A [63] can be written as,

$$\langle A \rangle = \text{Tr}(\rho A) = \sum_n \langle n | \rho A | n \rangle = \sum_n \sum_m \langle n | A | m \rangle \langle m | n \rangle \quad (6.10)$$

As we have seen in the introduction from eq.(6.7), for the stationary condition of equal rapidities, the degree of second order coherence is given in the density operator formalism by,

$$g^{(2)} = \frac{\text{Tr}(\rho a^{\dagger 2} a^2)}{[\text{Tr}(\rho a^{\dagger} a)]^2} = \frac{\langle a^{\dagger} a a^{\dagger} a \rangle - \langle a^{\dagger} a \rangle^2}{\langle a^{\dagger} a \rangle^2} = \frac{\langle a^{\dagger 2} a^2 \rangle}{\langle a^{\dagger} a \rangle^2} \quad (6.11)$$

which is the same as the second order correlation function given by equation(6.4). Thus, using the quantum statistical formalism, the degree of second order coherence for different distributions in Fourier space can be calculated. This gives us the value of two particle correlation $g^{(2)}(0)$ to compare with the experiment.

6.1.1 The generalized Glauber Lachs Distribution

If the source is completely coherent then the two particle correlation function is one(as in Poissonian), where as if the source is completely chaotic(noisy) $g^{(2)} = 2$ (as in Bose Einstein distribution). But in reality, the source is a mixture of these extremes (i.e., usually the coherent and the chaotic sources are superimposed). Such sources are described through the Glauber-Lachs distribution is considered. The generalized Glauber Lachs distribution is,

$$P_k(n) = \frac{(p\bar{n}/k)^n}{(1 + p\bar{n}/k)^{n+k}} \exp\left[-\frac{\gamma p\bar{n}}{1 + p\bar{n}/k}\right] L_n^{(k-1)}\left(-\frac{\gamma p\bar{n}}{1 + p\bar{n}/k}\right) \quad (6.12)$$

where $\gamma = |\zeta|^2/A$ (“the ratio of the ratio of the average value of the coherent hadrons to that of the chaotic hadrons”), The chaoticity parameter $p = 1/(1 + \gamma)$ and it is defined as ‘p’ is defined by

$$p = \frac{\bar{n} - |\alpha|^2}{\bar{n}} = \frac{\bar{n}_{ch}}{\bar{n}} \quad (6.13)$$

and here, $\bar{n} = \bar{n}_c + \bar{n}_{ch}$ and $\bar{n}_c = |\alpha|^2$ where, n_c , n_{ch} are the mean numbers of coherent and chaotic quanta respectively and $\bar{n}$ is the total mean number and $L_n^{(k-1)}$ stands for the Associated Laguerre polynomials respectively. This distributions shown in fig.(6.2)

The density operator for this distribution is

$$\rho = \frac{(p\bar{n}/k)^n}{(1 + p\bar{n}/k)^{(n+k)}} e^{\frac{-\gamma p\bar{n}}{1 + p\bar{n}/k}} L_n^{k-1}\left(\frac{-\gamma k}{1 + p\bar{n}/k}\right) |n\rangle \langle n| \quad (6.14)$$

The second order correlation is given by

$$g^{(2)} = \frac{\sum_n n(n-1) \frac{(p\bar{n}/k)^n}{(1 + p\bar{n}/k)^{(n+k)}} e^{\frac{-\gamma p\bar{n}}{1 + p\bar{n}/k}} L_n^{k-1}\left(\frac{-\gamma k}{1 + p\bar{n}/k}\right)}{\left(\sum_n n \frac{(p\bar{n}/k)^k}{(1 + p\bar{n}/k)^{n+k}} e^{\frac{-\gamma p\bar{n}}{1 + p\bar{n}/k}} L_n^{k-1}\left(\frac{-\gamma k}{1 + p\bar{n}/k}\right)\right)^2} \quad (6.15)$$

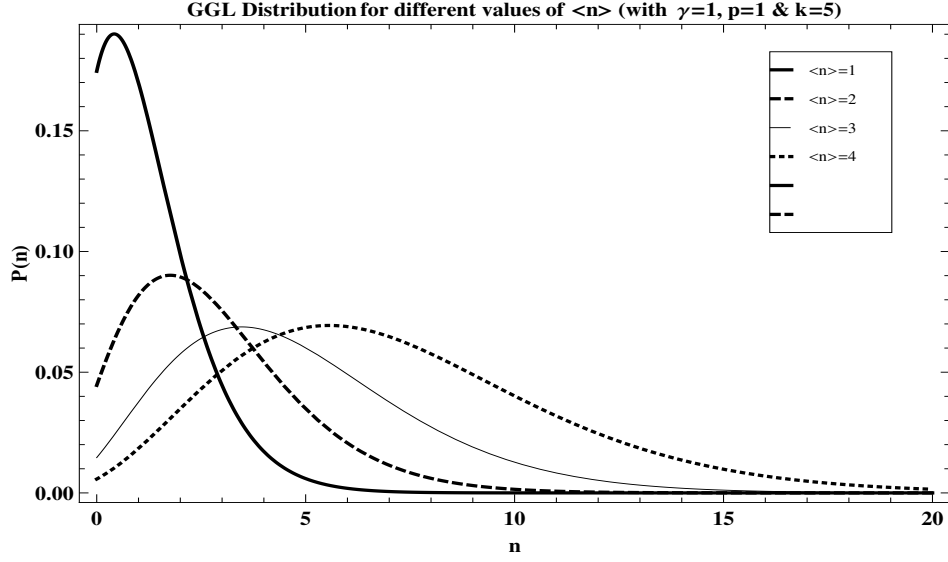


Figure 6.2: Generalized Glauber Lachs Distribution as a function of 'n'

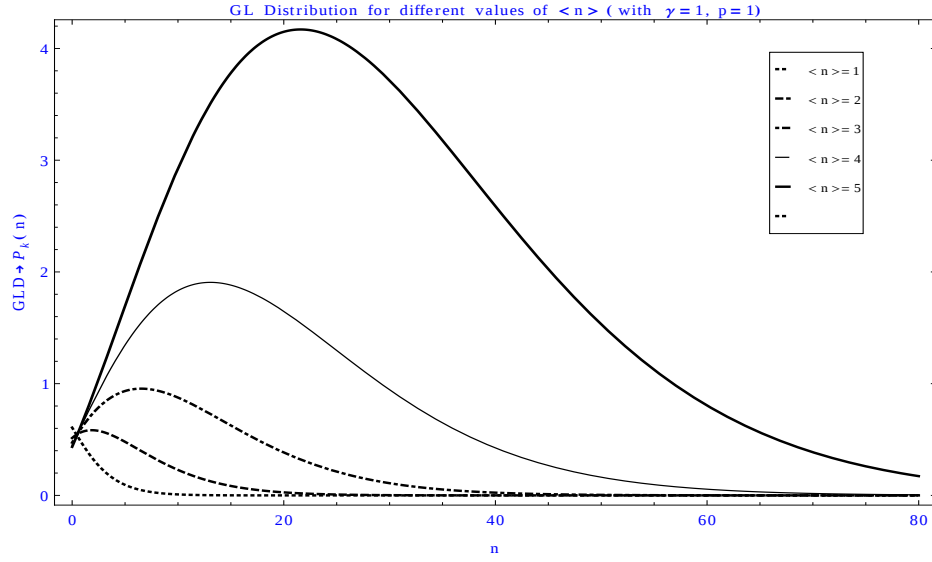


Figure 6.3: Glauber Lach Distribution as a function of 'n'

By using some algebra we get,

$$g^{(2)} = \frac{k(k+1) - \gamma k(k+3) + \gamma^2 k^2}{k^2 - 2\gamma k^2 + \gamma^2 k^2} \quad (6.16)$$

Therefore, the second order correlation and chaoticity for this distribution are:

$1 < g^{(2)} < 2 \rightarrow 0 < \lambda < 1$, so, this Perina-Mc Gill(GGL) distribution describes a partially chaotic source.

We consider three cases:

Case-1: Glauber Lachs Distribution

For a single source ($k = 1$) GGLD gives the Glauber Lachs distribution which is expressed as,

$$P(n) = \frac{(p\bar{n})^n}{(1 + p\bar{n})^{(n+1)}} e^{(\frac{-|\alpha|^2}{1+p\bar{n}})} L_n\left(\frac{-|\alpha|^2}{1 + p\bar{n}}\right) \quad (6.17)$$

where, L_n is the Laguerre polynomial. It was shown in fig(6.3)

The density operator for this distribution is,

$$\rho = \frac{\bar{n}^n}{(1 + \bar{n})^{(n+1)}} e^{(\frac{-|\alpha|^2}{1+\bar{n}})} L_n\left(\frac{-|\alpha|^2}{\bar{n}(1 + \bar{n})}\right) \quad (6.18)$$

Similarly as in GGLD, we can get **the second order correlation** as

$$g^{(2)} = \frac{2 - \frac{4|\alpha|^2}{\bar{n}} + \frac{|\alpha|^4}{\bar{n}^2}}{1 - \frac{2|\alpha|^2}{\bar{n}} + \frac{|\alpha|^4}{\bar{n}^2}} \quad (6.19)$$

* When $|\alpha| \rightarrow 0$, the value of $g^{(2)} = 2$ it turns out to be Geometric distribution.

* When $|\alpha| \rightarrow \infty$, the value is $g^{(2)} = 1$ it turns out to be Poisson distribution.

Therefore, the degree of second order coherence is $1 < g^{(2)} < 2$, so, this distribution describes a partially coherent source.

Case-2: Poisson distribution(PD)

When $\gamma = \infty$ and $k = 1$ (or) $\gamma = 0$ and $k = \infty$, then $g^{(2)} = 1$. So, GGLD turns out to be Poisson distribution.

Case-3: Negative binomial distribution(NBD)

When ($\gamma = 0$), $1 < g^{(2)} < 2$, so, GGLD turns out to be NBD. The distribution most used in describing particles in collision experiments is the phenomenological NBD. This distribution has been successfully used for fits of main features of experimental data in particle physics. In birth and death processes the generalized basic Markovian equation [64] is, $\frac{dP_n(t)}{dt} = a_{n+1}P_{n+1} + c_{n-1}P_{n-1} - (a_n + c_n)P_n$. here, a_n is the birth-emission coefficient (creation of particle) and c_n is the death (annihilation of particle) coefficient.

Now we take $a_n = \alpha n$ and $c_n = \beta n + \gamma$ (i.e., both the birth and the death processes) then the above equation turns out to be, $\frac{dP_n}{dt} = \alpha(n+1)P_{n+1} - (\beta n + \gamma)P_n + (\beta n - \beta + \gamma)P_{n-1} - \alpha n P_n$. For solving this equation with the condition, $dP_n/dt = 0$ and define $q = \beta/\alpha$; $k = \gamma/\beta$ we get the two solutions $(n+1)P_{n+1} = (n+k)qP_n$,

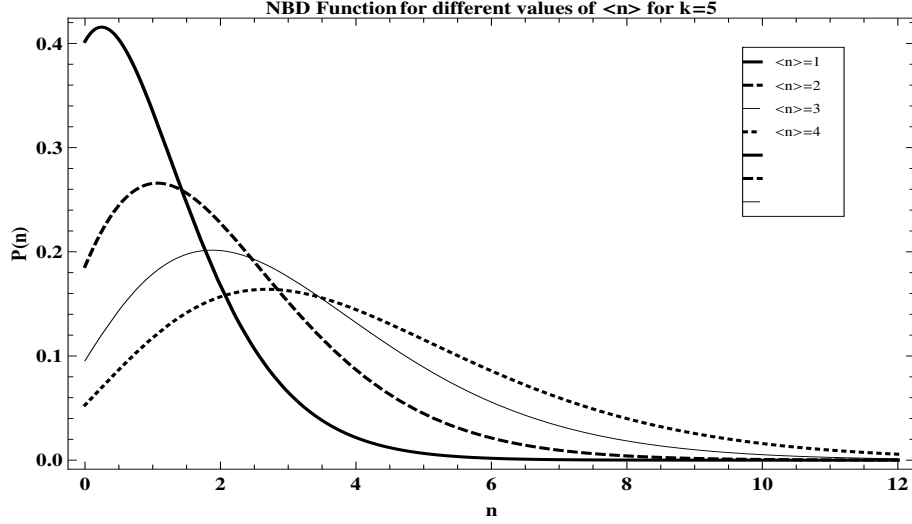


Figure 6.4: Negative Binomial Distribution as a function of 'n'

$(n + k - 1)qP_{n-1} = nP_n$ and the joint solution is,

$$P_k(n) = \frac{\Gamma(n + k)}{\Gamma(n + 1)\Gamma(k)} p^k q^n \quad (6.20)$$

where $p = 1 - q$. This is **the Negative Binomial distribution** or also called as a **generalized Bose-Einstein distribution**. If $\bar{n} = \frac{qk}{1-q} = \frac{\gamma}{\alpha-\beta}$ which gives, $q = \frac{\bar{n}/k}{1+\bar{n}/k}$ and $p = \frac{1}{1+\bar{n}/k}$ and we get

$$P_k(n) = \frac{\Gamma(n + k)}{\Gamma(n + 1)\Gamma(k)} \frac{(\bar{n}/k)^n}{(1 + \bar{n}/k)^{n+k}} \quad (6.21)$$

where $\bar{n}$ and k are the average multiplicity and the intrinsic parameter respectively. With these parameters the NBD has many possible applications in different areas (like, biology, optics, etc.) also.

In the case of gluon emission and absorption in a cascade, this distribution has the following interpretation. Now suppose initially we have 'k' gluons, these are participate in a interaction and gives 'n' gluons, then a similar birth-death equation can be written and gives a NBD and shown in fig(6.4).

The second order correlation is

$$g^{(2)} = (\bar{n}/k)^k \frac{k+1}{k} \left(\frac{1-z}{z}\right)^k = \frac{k+1}{k}. \quad (6.22)$$

* When $k = 1$, $g^{(2)} = 2$, ie it turns out to be Geometric distribution.

* When $k = \infty$, $g^{(2)} = 1$ ie it turns out to be Poisson distribution.

Therefore, the second order correlation and chaoticity for this distribution are:

$1 < g^{(2)} < 2 \Rightarrow 0 < \lambda < 1$, so, this NBD describes a partially chaotic source. However in no way can we

determine the temperature or fluctuations of the thermal chaotic part of the source, an attempt was made to separate the chaotic (thermal) component from the coherent component by using the Glauber Lachs Distribution found in quantum optics.

6.1.2 KNO-scaling limit of GGLD and NBD

As we known the GGLD is defined as in eq.(??), in the KNO-scaling limit (n and $\langle n \rangle$ are large, but the ratio $z = \frac{n}{\langle n \rangle}$ is finite), for the quantity $\langle n \rangle P(n, \langle n \rangle)$ the following gamma distribution is derived from the refered GGLD equation as[65, 66],

$$\psi_k(z, p) = \left(\frac{k}{p}\right)^k \left[\frac{z}{\sqrt{z(k/p)^2(1-p)}}\right]^{k-1} \exp\left[-\frac{k}{p}(1-p+z)\right] I_{k-1}(2\sqrt{z(k/p)^2(1-p)}) \quad (6.23)$$

where, I_{k-1} is the modified Bessel function. If $\gamma = 0$ this becomes the gamma distribution. The KNO-scaling limit of GGLD plotted in the fig.s(6.6,6.7)

And also, we known the NBD is defined as in eq.(6.21), in the KNO-scaling limit, for the quantity $\langle n \rangle P(n, \langle n \rangle)$ the following gamma distribution is derived from the refered NBD equation as[66]

$$\Psi_k(z) = \frac{k^k}{\Gamma(k)} z^{k-1} \exp(-kz) \quad (6.24)$$

The KNO-scaling limit of NBD plotted in the fig.(6.5).

The above inferences show that at certain conditions certain distributions are used and which all are worked at absolute zero temperature only.

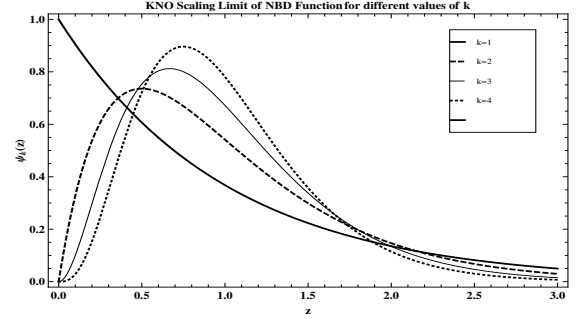


Figure 6.5: KNO-scaling of NBD(z)

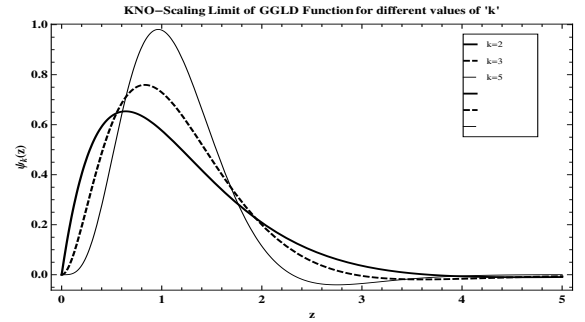


Figure 6.6: KNO-scaling of GGLD(z,k)

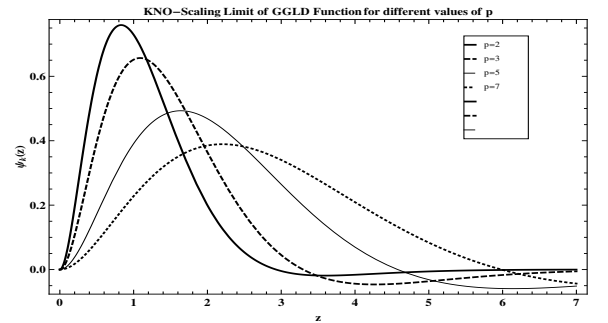


Figure 6.7: KNO-scaling of GGLD(z,p)

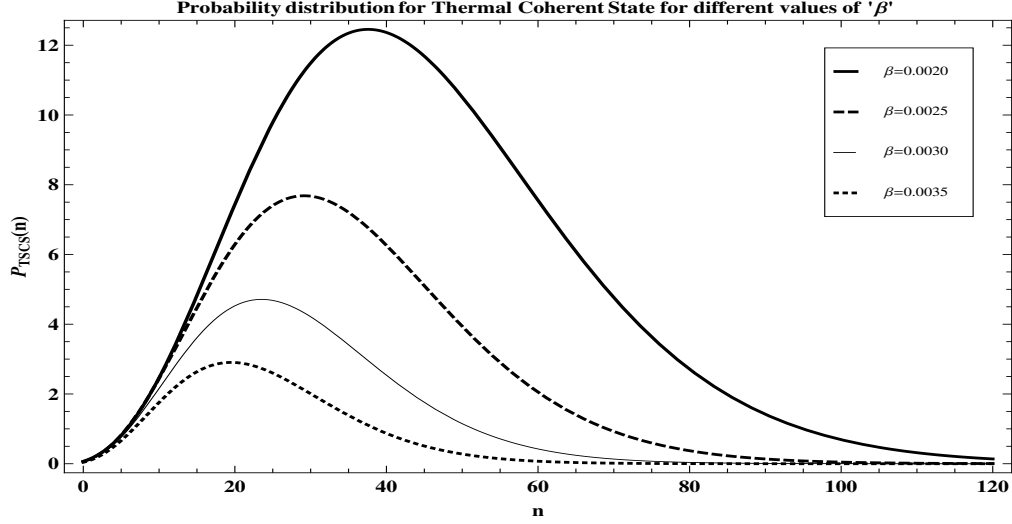


Figure 6.8: Multiplicity distribution for TCS with different values of ' β '

6.2 Calculating multiplicity distributions and correlations of thermal states using TFD

Although the distributions considered above give an idea that the source can be characterized as chaotic(thermal) or coherent by looking at the distribution and correlation there is no direct thermal dependence in the distributions and correlations, so a direct thermal link cannot be established. A thermal link can be provided by using the TFD formalism and various types of thermal states such as thermal coherent states and thermal squeezed coherent states to calculate multiplicity distributions and correlations.

Here, we have given a certain proportion to the above GGLD and NBD to few thermal multiplicities and correlations described by taking chaotic parameter(γ) has on temperature dependance (depends on thermal distribution function, $f(\beta)$).

6.2.1 Thermal Coherent State(TCS)

Here, we will give the multiplicity distribution of a thermal coherent state and show that, not only can temperature dependence be manifest, but that in various limits the distributions reduce to the NBD and Glauber lachs distributions. **The probability distribution for TCS** is derived as (in appendix-B) given by,

$$P_{TCS}(n) = \frac{2}{1+f(\beta)} \left(\frac{f(\beta)}{1+f(\beta)} \right)^n \text{Exp} \left(-\frac{|\alpha|^2}{1+f(\beta)} \right) L_n \left(-\frac{|\alpha|^2}{1+f(\beta)} \right) \quad (6.25)$$

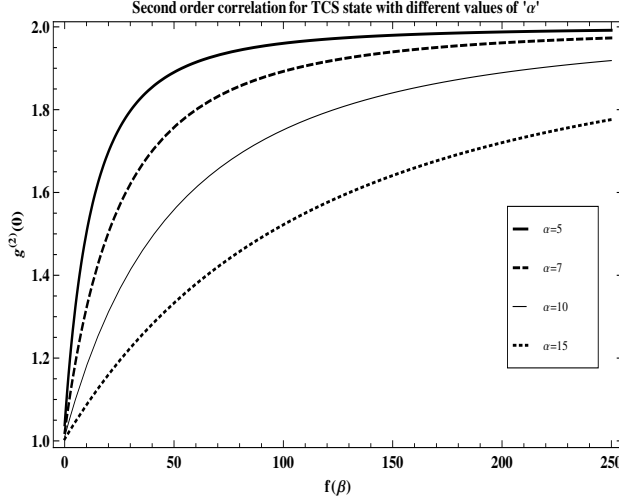


Figure 6.9: Second order correlation for TCS

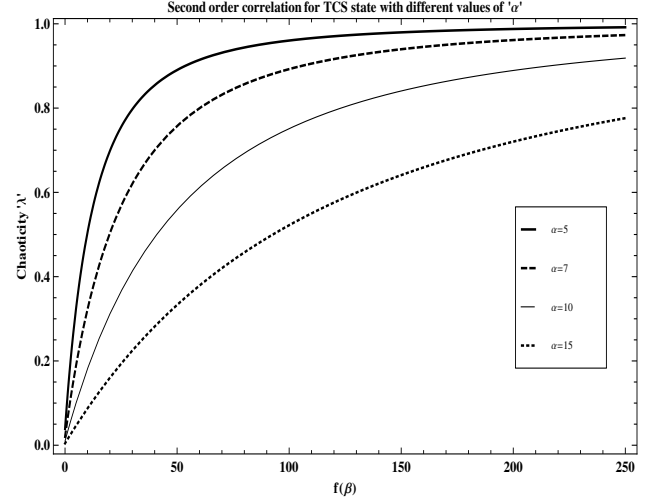


Figure 6.10: Chaoticity of TCS

This distribution shown in the fig.(6.8). Notice that in the limit $\beta \rightarrow \infty$; $T = 0$ we retrieve the zero temperature result. Comparing this expression with the Glauber Lachs distribution, we find that finite temperature effects.

The second order correlation for TCS can be written as (with the help of appendix-B),

$$\begin{aligned}
 g_{TCS}^{(2)}(0) &= \frac{\langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle}{(\langle \hat{a}^{\dagger} \hat{a} \rangle)^2} = \frac{\{|\alpha|^4 + 4(A + \frac{1}{2})|\alpha|^2 + 2(A + \frac{1}{2})^2\}}{\{|\alpha|^2 + (A + \frac{1}{2})\}^2} \\
 &= \frac{\{|\alpha|^4 + 4((1 + 2f(\beta))\sinh^2 r + f(\beta) + \frac{1}{2})|\alpha|^2 + 2((1 + 2f(\beta))\sinh^2 r + f(\beta) + \frac{1}{2})^2\}}{\{|\alpha|^2 + (1 + 2f(\beta))\sinh^2 r + f(\beta) + \frac{1}{2}\}^2}
 \end{aligned} \quad (6.26)$$

This correlation shown in fig.(6.9)

Then **the chaoticity for TCS** is

$$\begin{aligned}
 \lambda_{TCS} &= \frac{\{A + \frac{1}{2}\}^2 + 2(A + \frac{1}{2})|\alpha|^2}{\{(A + \frac{1}{2})|\alpha|^2\}^2} \\
 &= \frac{\{(1 + 2f(\beta))\sinh^2 r + f(\beta) + \frac{1}{2}\}^2 + 2\{(1 + 2f(\beta))\sinh^2 r + f(\beta) + \frac{1}{2}\}|\alpha|^2}{\{((1 + 2f(\beta))\sinh^2 r + f(\beta) + \frac{1}{2})|\alpha|^2\}^2}
 \end{aligned} \quad (6.27)$$

This is shown in fig.(6.10). Here, we observe that as increasing the temperature, chaoticity is increases up to one for any value of α . But, for lower values of ' α ' chaoticity is high. We observe that on incorporation of temperature, $g^2(0)$ even exceeds the chaotic limit showing the over bunching effect. Bose-Einstein correlations can be an important tool to distinguish between pions emitted from cold and hot sources. This may be of use in the studies of the QGP.

6.2.2 Thermal Squeezed Coherent State(TSCS)

The characteristic function of TSCS (in eq.2.74) and its density matrix (in eq.2.75) are presented to quantify quantum and thermal fluctuations and connects squeezing and temperature effects. The Characteristic function for TSCS ref.[67] is,

$$C(\beta, s) = \text{Exp}[-(A + \frac{1}{2}[1 - s])|\beta|^2 - \frac{E}{2}(\beta^2 + \beta^{*2})]\{\text{Exp}(\beta\alpha^* - \beta^*\alpha)\} \quad (6.28)$$

where,

$$A = f(\beta) + (1 + 2f(\beta)) \sinh^2 r; E = (1 + 2f(\beta)) \sinh r \cosh r; \text{ and } f(\beta) = \frac{1}{e^{\beta\omega} - 1} \quad (6.29)$$

where s takes the values 1,0 and -1 correspondingly to normally, symmetrically and anti-normally ordered characteristic functions respectively. **The probability distribution function for TSCS** is derived as (in appendix-A) given by

$$P(n) = \frac{1}{[(A + \frac{1}{2})^2 - E^2]^{1/2}} \sum_{m=0}^n (1 - \nu_-)^m (1 - \nu_+)^{n-m} L_m^{(-1/2)}(-X_-) L_{n-m}^{(-1/2)}(-X_+) \times \text{Exp}[-\frac{(A+1)|\alpha|^2 + E(\alpha_1^2 - \alpha_2^2)}{(A+1)^2 - E^2}], \quad (6.30)$$

where,

$$\nu_{\pm} = [A + 1 \pm E]^{-1}; \quad Y_{\pm} = [(A \pm E)(A \pm E + 1)]^{-1} \quad (6.31)$$

$$X_+ = \alpha_2^2[(A + E)(A + E + 1)]^{-1}; \text{ and } X_- = \alpha_1^2[(A - E)(A - E + 1)]^{-1}, \quad (6.32)$$

here, A and E are functions of squeezing(r) and thermal(β) parameters shown in eq.(6.29). This distribution shown in fig.s(6.11, 6.12).

The second order correlation for TSCS can be written as,

$$\begin{aligned} g_{TSCS}^{(2)}(0) &= \frac{\langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle}{(\langle \hat{a}^{\dagger} \hat{a} \rangle)^2} = \frac{|\alpha|^4 + 4(A + \frac{1}{2})|\alpha|^2 - E(\alpha^2 + \alpha^{*2}) + 2(A + \frac{1}{2})^2 + E^2}{\{|\alpha|^2 + (A + \frac{1}{2})\}^2} \\ &= \frac{\{|\alpha|^4 + 4((1 + 2f(\beta))\sinh^2 r + f(\beta) + \frac{1}{2})|\alpha|^2 - \{(1 + 2f(\beta))\sinh r \cosh r\}(\alpha^2 + \alpha^{*2})\}}{\{|\alpha|^2 + (1 + 2f(\beta))\sinh^2 r + f(\beta) + \frac{1}{2}\}^2} \\ &+ \frac{\{2((1 + 2f(\beta))\sinh^2 r + f(\beta) + \frac{1}{2})^2 + \{(1 + 2f(\beta))\sinh r \cosh r\}^2\}}{\{|\alpha|^2 + (1 + 2f(\beta))\sinh^2 r + f(\beta) + \frac{1}{2}\}^2} \end{aligned} \quad (6.33)$$

This correlation shown in fig.(6.13)

Then **the chaoticity for TSCS** is

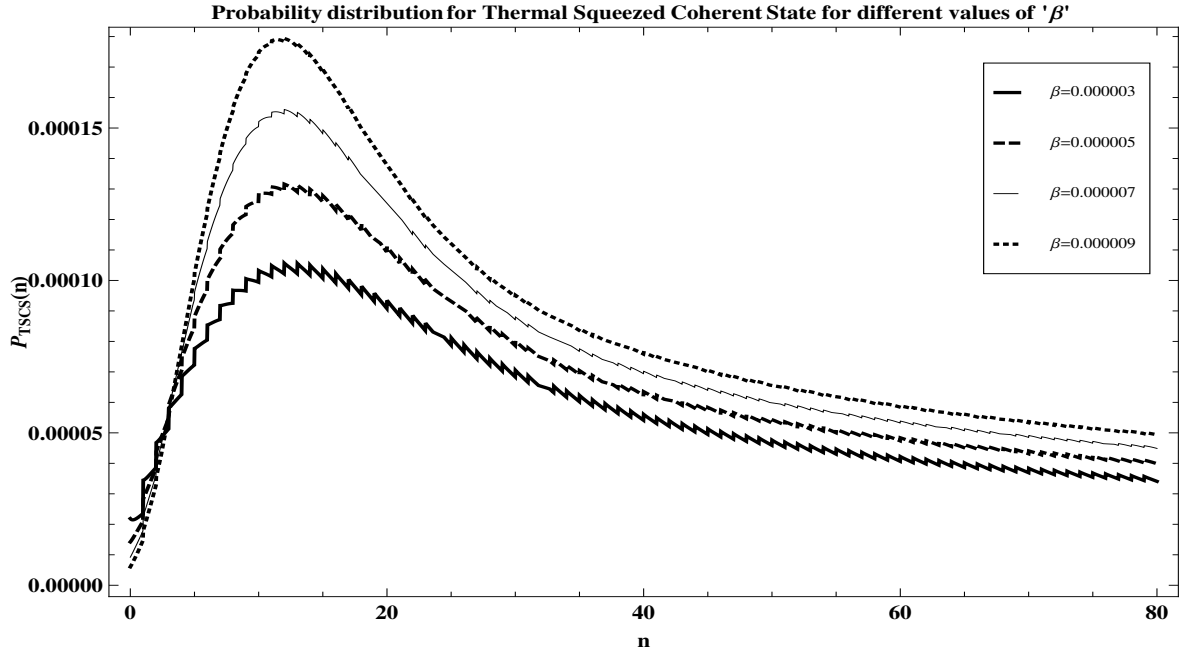


Figure 6.11: Multiplicity distribution for TSCS with different values of ' β ' at ' $r=3.5$ '

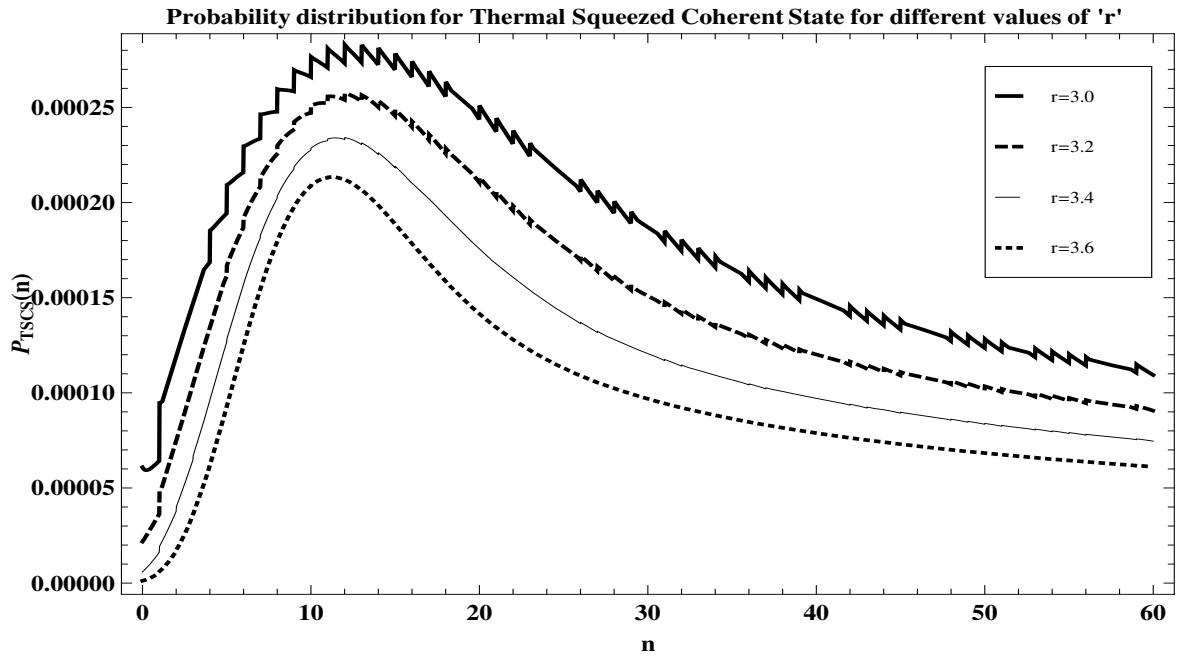


Figure 6.12: Multiplicity distribution for TSCS with different values of ' r ' at ' $\beta = .000008$ '

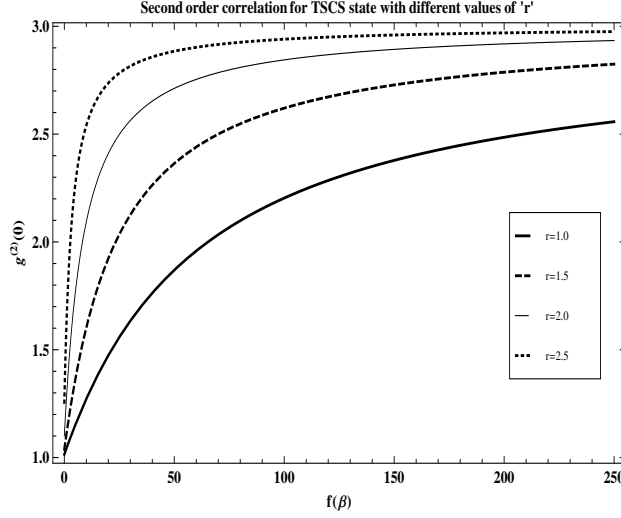


Figure 6.13: Second order correlation for TSCS

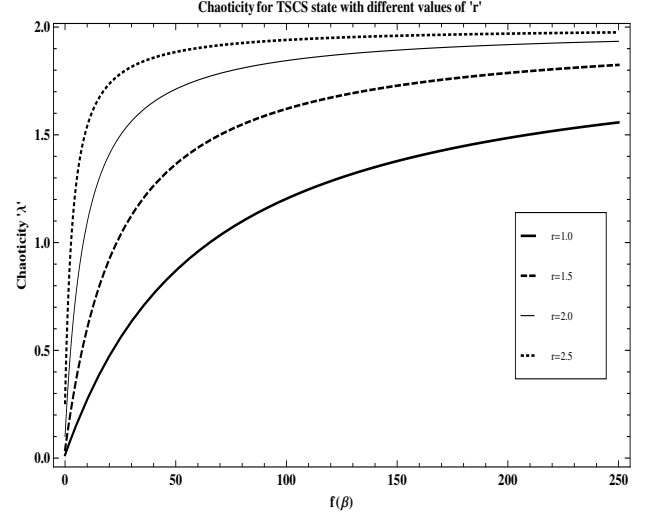


Figure 6.14: Chaoticity of TSCS

$$\begin{aligned}
 \lambda_{TSCS} &= \frac{2(A + \frac{1}{2})|\alpha|^2 - E(\alpha^2 + \alpha^{*2}) + \{A + \frac{1}{2}\}^2 + E^2}{\{(A + \frac{1}{2})|\alpha|^2\}^2} \\
 &= \frac{\{2((1 + 2f(\beta))\sinh^2 r + f(\beta) + \frac{1}{2})|\alpha|^2 - \{(1 + 2f(\beta))\sinh r \cosh r\}(\alpha^2 + \alpha^{*2})\}}{\{|\alpha|^2 + (1 + 2f(\beta))\sinh^2 r + f(\beta) + \frac{1}{2}\}^2} \\
 &+ \frac{\{((1 + 2f(\beta))\sinh^2 r + f(\beta) + \frac{1}{2})^2 + \{(1 + 2f(\beta))\sinh r \cosh r\}^2\}}{\{|\alpha|^2 + (1 + 2f(\beta))\sinh^2 r + f(\beta) + \frac{1}{2}\}^2}
 \end{aligned} \tag{6.34}$$

This is shown in fig.(6.14). Here, we observe that as increasing the temperature, chaoticity is increases up to two for any value of r and also for increasing the values of ' r ' the chaoticity is increases. So, chaoticity is the function of temperature. Even small amount of squeezing the corresponding source is more chaotic. This can be explain how much thermalization occur in fire ball region.

6.3 Conclusion

In this chapter, we have derived and plotted the probability distributions and correlations of Thermal Coherent State(TCS) and Thermal Squeezed Coherent State(TSCS) by using thermal-field theory, then compared with Generalized Glauber-Lachs(GGL) formula and identified the similarities. In future plan, we will compare the both with experimental heavy ion collisions data.

6.4 Appendix: A mathematical approach to obtain the thermal probability distributions and correlations

A: For Thermal squeezed coherent state(TSCS)

We start with constructing the wigner function from the characteristic function of a TSCS. From the Wigner function the probability distribution is in step wise. The Wigner function can be defined as a Fourier transform of the characteristic function. So which can be defined as below,

$$W(\gamma, 0) = \frac{1}{\pi^2} \int d^2\beta \text{Exp}[-(A + \frac{1}{2})|\beta|^2 - \frac{E}{2}(\beta^2 + \beta^{*2}) + \beta(\alpha^* - \gamma^*) - \beta^*(\alpha - \gamma)] \quad (6.35)$$

If we take $\beta = x_1 + ix_2$ in above function it can be written as,

$$\begin{aligned} W(\gamma, 0) = \frac{1}{\pi^2} \int dx_1 dx_2 \text{Exp}[-(A + \frac{1}{2} + E)x_1 x_1 - (A + \frac{1}{2} - E)x_2 x_2 \\ + \{(\alpha^* - \gamma^*) - (\alpha - \gamma)\}x_1 + i\{(\alpha^* - \gamma^*) + (\alpha - \gamma)\}x_2] \end{aligned} \quad (6.36)$$

For solving this double integral, we can use a matrix method described in the appendix 1 to get the Wigner W-function from eq.(??) is,

$$W(\gamma, 0) = \frac{1}{\pi[(A + \frac{1}{2})^2 - E^2]^{1/2}} \text{Exp}[-\frac{(A + \frac{1}{2})|\gamma - \alpha|^2 + \frac{1}{2}E[(\gamma - \alpha)^2 + (\gamma^* - \alpha^*)^2]}{(A + \frac{1}{2})^2 - E^2}] \quad (6.37)$$

From the Wigner function we get the probability distribution function as [67],

$$P(n) = 2(-1)^n \int d^2\gamma W(\gamma, 0) L_n(4|\gamma|^2) \text{Exp}[-(2|\gamma|^2)] \quad (6.38)$$

which can be formally written as,

$$P(n) = 2(-1)^n L_n(-\frac{\partial}{\partial \lambda}) I(\lambda) |_{\lambda=1}; \quad (6.39)$$

Here,

$$I(\lambda) = \int d^2\gamma W(\gamma, 0) \text{Exp}[-(4\lambda|\gamma|^2)]$$

the integral is,

$$I(\lambda) = \frac{1}{\pi[(A + \frac{1}{2})^2 - E^2]^{1/2}} \text{Exp}[-\frac{(A + \frac{1}{2})|\alpha|^2 + \frac{E}{2}(\alpha^2 + \alpha^{*2})}{(A + \frac{1}{2})^2 - E^2}] \frac{\pi}{(\text{Det}[A])^{\frac{1}{2}}} \text{Exp}[\frac{1}{4}B^\dagger . A^{-1} . B]$$

Now substituting $I(\lambda)$ in equation(6.39).

Finally, **The probability distribution function for TSCS** is,

$$P(n) = \frac{1}{[(A + \frac{1}{2})^2 - E^2]^{1/2}} \sum_{m=0}^n (1 - \nu_-)^m (1 - \nu_+)^{n-m} L_m^{(-1/2)}(-X_-) L_{n-m}^{(-1/2)}(-X_+) \times \exp\left[-\frac{(A+1)|\alpha|^2 + E(\alpha_1^2 - \alpha_2^2)}{(A+1)^2 - E^2}\right], \quad (6.40)$$

where,

$$\nu_{\pm} = [A + 1 \pm E]^{-1}; \quad Y_{\pm} = [(A \pm E)(A \pm E + 1)]^{-1} \quad (6.41)$$

$$X_+ = \alpha_2^2[(A + E)(A + E + 1)]^{-1}; \quad \text{and} \quad X_- = \alpha_1^2[(A - E)(A - E + 1)]^{-1}, \quad (6.42)$$

The expectation value of product of the l-ordered creation and annihilation operators can be defined as,

$$\langle \hat{a}^{\dagger l} \hat{a}^l \rangle = \frac{\partial^{2l} C(\beta, s)}{\partial \beta^l \partial (-\beta^*)^l} \Big|_{\beta, \beta^*=0} \quad (6.43)$$

The average number density for TSCS can be derived as,

$$\begin{aligned} \langle n \rangle &= \langle \hat{a}^{\dagger} \hat{a} \rangle = \frac{\partial}{\partial \beta} \frac{\partial C(\beta, s=0)}{\partial (-\beta^*)} \Big|_{\beta, \beta^*=0} \\ &= |\alpha|^2 + A + \frac{1}{2} \\ &= \{|\alpha|^2 + (1 + 2f(\beta)) \sinh^2 r + f(\beta) + \frac{1}{2}\} \end{aligned} \quad (6.44)$$

and

$$\begin{aligned} \langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle &= \frac{\partial^2}{\partial \beta^2} \frac{\partial^2 C(\beta, s=0)}{\partial (-\beta^*)^2} \Big|_{\beta, \beta^*=0} \\ &= \{|\alpha|^4 + 4(A + \frac{1}{2})|\alpha|^2 - E(\alpha^2 + \alpha^{*2}) + 2(A + \frac{1}{2})^2 + E^2\} \end{aligned} \quad (6.45)$$

The variance is one of the moments of a distribution and a measure of how far a set of numbers is spread out, describing how far the numbers lie from the mean (expected value). It forms part of a systematic approach to distinguishing between probability distributions. For this TSCS variance is

$$\begin{aligned} \mu &= \langle n^2 \rangle - \langle n \rangle^2 = \langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle - (\langle \hat{a}^{\dagger} \hat{a} \rangle)^2 \\ &= 2(A + \frac{1}{2})|\alpha|^2 - E(\alpha^2 + \alpha^{*2}) + (A + \frac{1}{2})^2 + E^2 \end{aligned} \quad (6.46)$$

B: For Thermal coherent state(TCS)

For producing the multiplicity distributions for thermal coherent state(TCS), we can take squeezing parameter $r = 0$ in eq.(6.40) and by using associated Laguerre polynomials and its identities,

$$L_n^{(k)}(x) = \sum_{m=0}^n \frac{\Gamma(n+k+1)(-x)^m}{(n-m)!\Gamma(m+k+1)m!}; \sum_{m=0}^n L_m^{k_1}(x)L_{n-m}^{k_2}(y) = L_n^{k_1+k_2+1}(x+y), \quad (6.47)$$

The probability distribution for TCS is

$$P_{TCS}(n) = \frac{2}{1+f(\beta)} \left(\frac{f(\beta)}{1+f(\beta)} \right)^n \text{Exp}\left(-\frac{|\alpha|^2}{1+f(\beta)}\right) L_n\left(-\frac{|\alpha|^2}{1+f(\beta)}\right) \quad (6.48)$$

The average number density for TCS can be derived as,

$$\langle n \rangle = \langle \hat{a}^\dagger \hat{a} \rangle = \frac{\partial}{\partial \beta} \frac{\partial C(\beta, s=0)}{\partial(-\beta^*)} \Big|_{\beta, \beta^*=0} = \{|\alpha|^2 + f(\beta) + \frac{1}{2}\} \quad (6.49)$$

and

$$\langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle = \frac{\partial^2}{\partial \beta^2} \frac{\partial^2 C(\beta, s=0)}{\partial(-\beta^*)^2} \Big|_{\beta, \beta^*=0} = \{|\alpha|^4 + 4(A + \frac{1}{2})|\alpha|^2 + 2(A + \frac{1}{2})^2\} \quad (6.50)$$

Therefore, the variance for TCS is

$$\mu = \langle n^2 \rangle - \langle n \rangle^2 = \langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle - (\langle \hat{a}^\dagger \hat{a} \rangle)^2 = \{(A + \frac{1}{2})^2 + 2(A + \frac{1}{2})|\alpha|^2\} \quad (6.51)$$

Chapter 7

Summary and Conclusions

7.1 Summary

In this thesis, I have used the thermo-field dynamics formalism to study quantum entanglement, decoherence and quantum correlations in a few generic many-body thermal quantum systems in various branches of physics. The aim is to show the simple yet powerful use of the method for solving the master equation, studying thermal decoherence effects and thermal effects on correlations in various systems. The systems discussed elaborately belong to three different branches of physics, illustrating the universality of formalism. The first one is quantum entanglement in coupled waveguides system in quantum optics, the second one, quantum entanglement in coupled two site, two species Bose-Hubbard model in condensed matter physics and the last one is, multiplicity distributions and correlations of hadrons in particle physics.

A brief summary of the thesis is as follows: In the introductory chapters 1, 2 & 3: I have introduced the basic formalisms of thermal quantum field theory and given an explanation of how the TFD formalism is a very powerful tool to study the thermal systems in equilibrium as well as out of equilibrium. A brief description of quantum entanglement and how TFD helps in solving the master equation of the density matrix, to calculate entanglement properties and decoherence is given in the introduction. The second chapter introduces the reader to the TFD formalism in detail: its construction within the doubling of Hilbert space, thermo-algebra for various thermal systems and defined various thermal states. The third chapter discusses various measures of quantum entanglement for a bipartite system and their relationship with each other.

In the fourth chapter, the TFD formalism is used to solve the master equation of coupled waveguide systems. In case of coupled waveguides without damping, the evolution of quantum many body NOON

states is shown for $N = 2, 3, 4, 5, \text{etc.}$ This is important because for a NOON state the decoherence of a damped state is relatively more. The entanglement of the NOON states survives with time upon increasing photon number, making them extremely suitable for quantum information. It is inevitable that the quantum state interacts with the environment which will cause the decoherence of the state, thus in quantum super sensitive measurements, decoherence should be included. We have investigated the performance degradation of the entanglement of various optical states by solving the master equation with the presence of damping using $SU(2)$ and $SU(1,1)$ algebras. Extending upon the work of Rai et. al[48], we have an exact solution for the master equation, and, we can also show how the entanglement behaves for input thermal states. Our results have also shown that the entanglement of the system can withstand a certain amount of damping, suggesting that, even if the waveguides are lossy, it can be used for applications such as quantum computation. Furthermore we have shown the effect of an external heat bath on the system, by applying our methods to thermal input states. Our method shows the usefulness of TFD in quantum entanglement problems, quite orthogonal to the approach given in ref.[?], and allows us to handle damping in entanglement generation properties. We propose to apply this formalism to coupled light-atom systems, to shed further light on the effect of damping on the generation of entanglement.

In fifth chapter, We studied the Bose-Hubbard model by using TFD to extend the work of K.V.S.S. Chaitanya et. al[56] in the presence of non-linear damping with coupling strength. This is of interest because ultracold atoms are a natural resource for quantum information processes. Indeed, we have exactly solved the master equation associated with $SU(1,1)$ symmetries for two-site Bose-Hubbard model using the disentanglement formula. The different entanglement properties and decoherence of the system for coupling strength(J) and with damping($\kappa \neq 0$) and without damping($\kappa = 0$) are illustrated. We compare the role of the hopping parameter(J) and the repulsive on-site interaction parameter(U) on these features and shown how we can tune the system for optimum quantum information processing.

In the sixth chapter, which represents a departure from earlier studies, we show a unique application of TFD to study a truly many particle system of pions produced from a thermal source in a heavy ion collision. Since the thermal source is considered to be the origin of the much sought after quark gluon plasma in the lab, its thermal properties are of interest. Direct measurement of these properties is impossible as it lives only for a fraction of a nano second. Whether a thermal source, hot enough to produce the QGP is formed or not must be deduced from any thermal effects on the features of multiplicity distributions and correlations of pions that are emitted in the epoch of fireball formation. In particular Bose-Einstein correlations(BEC) among pions tell us if a source is chaotic (thermal) or coherent. These properties can be examined using quantum optical techniques such as Hanbury Brown Twiss interferometry. Our work presents thermal effects on BEC phenomenology using TFD. One of the main parameters is the temperature of the particle(emitting) source under the random external forces(fields) influence[68]. We have seen clearly the dependence of the chaoticity parameter on the number of thermal particles. These may be useful as tools to study the properties

of the QGP believed to be produced in the collision of heavy ions at RHIC and LHC (in the future).

7.2 Conclusions

Studying the many-body quantum systems within a thermal environment is an active area of physics. A large amount of experimental progress has been made to realize many-body quantum systems that evolve under their quantum dynamics. Optical cavity systems and ultracold atoms[55] in optical lattices(BEC) have tunability and long coherence times. Therefore one can probe quantum entanglement dynamics in these systems for purposes of quantum computing. These systems can decohere due to interaction with thermal environments and damping effects, therefore solving the master equation of a system in thermal environment using TFD and give us useful information about their sustainability. Most research has been done so far in quantum information theory, quantum cryptography, quantum teleportation, quantum dense coding and quantum computing and so on are done without considering thermal effects, i.e., at absolute zero temperature. The work done in this thesis is a start to more work in this direction. So far only photonic and bosonic fields have been studied. We hope to extend our work to quantum entanglement in atom-photon fields and its evolution and decoherence effects. We also intend to go beyond the two site Bose Hubbard model and look at the entanglement properties in systems which have phase transitions. On the particle physics side, the thermal multiplicity distributions still have to be compared with actual data from heavy ion experiments to show how the temperature of the emitting source can be extracted. This will be a future avenue of our work.

Bibliography

- [1] F. Halzen and A. D. Martin, (1984) “Quarks and Leptons-An Introductory course in Modern Particle Physics”, Wiley, New York.
- [2] J. D. Bjorken and S. D. Drell, (1965) “Relativistic Quantum Mechanics”, McGraw-Hill, New York.
- [3] Gregg Jaeger, (2014) “What in the (quantum) world is macroscopic?” *American Journal of Physics*, **82**, 9, pp.896-905; doi: 10.1119/1.4878358.
- [4] Y. Takahashi and H. Umezawa, (1975) ”Higher order calculation in thermo field theory“, *Collective Phenomena*, **2**, pp.55-80; reprinted in (1996) ’Thermofield Dynamics’ *International Journal of Modern Physics B*, **10**, 13-14, pp.1755-1805.
- [5] H. Umezawa, H. Matsumoto, and M. Tachiki, (1982) ”Thermofield dynamics and condensed states“, North-Holland Publishing Company, Amsterdam-New York-Oxford.
- [6] H. Umezawa, (1993) ”Proceedings of the conference on Thermofield Dynamics“, Banf, Canada.
- [7] Ashok Das, (1997) “Finite temperature field theory“, World Scientific Publishing, University of Rochester, Rochester, New York-14627.
- [8] Faqir C. Khanna, and et al., (2009) Thermal quantum field theory algebraic aspects and applications“, World Scientific publishing Co. Pte. Ltd., 5 Toh Tuck Link, Singapore-596224.
- [9] A.Mann, and M.Revzen, (1989) ”Thermal coherent states“, *Physics Letters A* **134**, 273-275;
- [10] A. Einstein, B. Podolsky, and N. Rosen, (1935) ”Can Quantum mechanical description of physical reality be considered complete?“, *Physical Review* **47**, pp.777.
- [11] Schrödinger, E., 1935, *Naturwissenschaften* 23, 807.
- [12] J. von Neumann, (1932), *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin).
- [13] W.K. Wootters, (1998) ”Entanglement of formation of an arbitrary state of two qubits“, *Physical Review Letters* **80**, pp.2245- ; W.K. Wootters, (2001) ”Entanglement of formation and Concurrence“, *Quantum Information and Computation* **1**, pp.27-44.

- [14] L. Laplae, F. Mancini, and H. Umezawa, (1974) "Derivation and application of the boson method in superconductivity", *Physical Review* **C**, **10**, 4, pp.151-272.
- [15] E.J. Verboven, (1966) "Quantum thermodynamics of an infinite system of harmonic oscillators" *Physics Letters***21**, pp.391-393.
- [16] J.P.Whitehead, H.Matsumoto and H.Umezawa, (1984) "A diagrammatic approach to spin algebras", *Physics Letters* **103**, pp.408-410.
- [17] S.M.Barnett and P.L.Knight, (1985) "Thermofield analysis of squeezing and statistical mixtures in quantum optics", *Journal of Optical Society of America* **B** **2**, pp.467-479.
- [18] S. Chaturvedi, V. Srinivasan, G. S. Agarwal, (1999) "Quantum phase space distributions in thermofield dynamics", *Journal of Physics A: Mathematical and General* **32**, 10, pp.1909-1926. (arXiv:quant-ph/9810011v2)
- [19] S. Chaturvedi, and V. Srinivasan, (1991) "Solution of the Master Equation for an Attenuated or Amplified Nonlinear Oscillator with an Arbitrary Initial Condition" *Journal of Modern Optics* **38**, 4, pp.777-783.
- [20] L.M. Silva, A.E. Santana, and J.D.M. Vianna (1996) *Brazil Journal of Physics*, **27**, pp.619.
- [21] H. Bohr, H.B. Neilson, (1977) "Hadron production from a boiling quark soup: quark model predicting particle ratios in hadronic collisions", *Nuclear Physics***B**, **128**, pp.275-293.
- [22] C.H. Bennett and S.J. Wiesner, (1992) "Communication via one- and two-particle operators on Einstein-Podolsky-Rosen state", *Physical Review Letters* **69**, pp.2881.
- [23] S.Chaturvedi, R.Sandhya, V.Srinivasan and R.Simon, (1990) "Thermal counterparts of non-classical states in quantum optics" *Physical Review* **A** **41**, pp.3969-3974.
- [24] Alber, G., T. Beth, M. Horodecki, P. Horodecki, R. Horodecki, M. Rotteler, H. Weinfurter, R. F. Werner, and A. Zeilinger, 2001a, Quantum Information: An Introduction to Basic Theoretical Concepts and Experiments(Springer).
- [25] Bouwmeester, D., A. K. Ekert, and A. Zeilinger, 2000, The physics of quantum information : quantum cryptography, quantum teleportation, quantum computation (Springer, New York).
- [26] Deutsch, D., (1985) "Quantum theory, the ChurchTuring principle and the universal quantum computer", *Proceedings of the Royal Society* **A** **400**, issue 1818, pp. . <https://doi.org/10.1098/rspa.1985.0070>.
- [27] R.P.Feynman, 1982, "Simulating physics with computers", *International Journal of Theoretical Physics* **21**, pp.467-488.

- [28] C.H.Bennett, et.al, (1993) "Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels", *Physical Review Letters* **70**, 13, pp.1895-1899.
- [29] C.H.Bennett, and G. Brassard, (1984) Proceedings of the IEEE International Conference on Computers, Systems and Signal Processing, IEEE Computer Society, New York, pp.175179.
- [30] P. Shanta, S. Chaturvedi, V. Srinivasan, and A. K. Kapoor, (1996) "Operator methods for master equations in Quantum optics", *International Journal of Modern Physics B*, **10**, pp.1573-1584.
- [31] P. Shanta, S. Chaturvedi, and V. Srinivasan, (1996) "Eigenstates of linear combinations of two-boson creation and annihilation operators: An algebraic approach", *Modern Physics Letters A*, **11**, 29, pp.2381-2396.
- [32] S. Chaturvedi, and V. Srinivasan, (1991) "Class of exactly solvable master equations describing coupled nonlinear oscillators", *Physical Review A*, **43**, 7, pp.4054-4057.
- [33] K. Wodkiewicz, and J. H. Eberly, (1985) "Coherent states, squeezed fluctuations, and the SU(2) and SU(1,1) groups in quantum-optics applications" *Journal of Optical society of America B*, **2**, 3, pp.458-466.
- [34] T. Garavaglia, (1990) "The characteristic functions for the squeezed coherent chaotic photon state with application to the Jaynes-Cummings model" *Physics Letters A*, **145**, 5, pp.215-219.
- [35] Carlos Alexandre Brasil, Felipe Fernandes Fanchini, Reginaldo de Jesus Napolitano, (2013) "A simple derivation of the Lindblad equation", *Revista Brasileira de Ensino de Física*, **35**, 1, pp.1303.
- [36] C.H. Bennett and H.J. Bernstein, S. Popescu and B. Schumacher, (1996) "Concentrating partial entanglement by local operations", *Physical Review A* **53**, pp.2046.
- [37] N.J. Cerf and C. Adami (1997) "Negative entropy and information in quantum mechanics", *Physical Review Letters*, **79**, 26, pp.5194-5197; Cerf N.J., Adami C. (1997) "Negative Entropy in Quantum Information Theory". In: Ferrero M., van der Merwe A. (eds) New Developments on Fundamental Problems in Quantum Physics. Fundamental Theories of Physics (An International Book Series on The Fundamental Theories of Physics: Their Clarification, Development and Application), vol 81. Springer, Dordrecht. https://doi.org/10.1007/978-94-011-5886-2_11
- [38] G. Vidal and R. F. Werner, (2002) "A Computable measure of entanglement", *Physical Review A*, **65**, 2, pp.032314(11). (arXiv:quant-ph/0102117v1)
- [39] HODA ALIJANZADEH BOURA, AURELIAN ISAR, (2015) "Logarithmic negativity of two bosonic modes in the thermal reservoir model", 2015.

- [40] Gerardo Adesso and Fabrizio Illuminati, (2007) "Entanglement in continuous variable systems: Recent advances and current perspectives"
- [41] R. Simon, (2000) Peres-Horodecki "Separability Criterion for Continuous Variable Systems", *Physical Review Letters*, **84**, 12, pp.2726-2729.
- [42] K. Hornberger, "Introduction to Decoherence Theory", Lecture notes, Arnold Sommerfeld Center for Theoretical Physics, Munich, Germany.
- [43] D. N. Christodoulides, F. Lederer, and Y. Silberberg, (2003) "Discretizing light behaviour in linear and nonlinear waveguide lattices", *Nature(London)* **424**, pp.817823.
- [44] S. Longhi, (2009) "Optical analogue of population trapping in the continuum: classical and quantum interference effects", *Physical Review A* **79**, 2, pp.023811(9). (arXiv:quant-ph/1001.1000v1); S. Longhi, (2009) "Quantum Optical analogies using photonic structures", *Laser and Photonic Review*, **3**, pp.243-261.
- [45] A. Politi, M. J. Cryan, J. G. Rarity, S. Yu, and J. L. O'Brien, (2008) "Silica-on-Silicon Waveguide Quantum Circuits", *Science* **320**, pp.646-649.
- [46] Lee A. Rozema, James D. Bateman, Dylan H. Mahler, et. al, (2014) "Scalable Spatial Superresolution Using Entangled Photons", *Physical Review Letters* **112**, pp.223602(5). (arXiv:quant-ph/1312.2012v1)
- [47] W. H. Zurek, (2003) "Decoherence, einselection, and the quantum origin of the classical", *Reviews of Modern Physics* **75**, pp.715-775.
- [48] Amit Rai, Sumanta Das, and G. S. Agarwal, (2010) "Quantum entanglement in coupled lossy waveguides", *Optics Express*, **18**, 6, pp.6241-6254. arXiv:quant-ph/0907.2432; and reprinted in (2013), arXiv:quant-ph/0907.2432v3.
- [49] A. Perelomov, (1986) "Generalized Coherent States and their Applications", Springer-Verlag.
- [50] Kazuyuki Fujii (2002) "Introduction to Coherent States and Quantum Information Theory", Prepared for Conference(C02-03-07.2), Department of Mathematical Sciences, Yokohama City University, Yokohama, JAPAN. (arXiv:quant-ph/0112090v2)
- [51] G. S. Agarwal, and A. Biswas, (2005) "Quantitative measures of entanglement in pair-coherent states", *Journal of Optics B: Quantum Semiclassical Optics* **7**, pp.350-354.
- [52] C. K. Hong, Z. Y. Ou, and L. Mandel, (1987) "Measurement of subpicosecond time intervals between two photons by interference", *Physical Review Letters* **59**, pp.2044-2046.
- [53] I. Afek, O. Ambar, Y. Silberberg, (2010) "High-NOON States by Mixing Quantum and Classical Light", *Science*, **328**, 5980, pp.879-881. doi: 10.1126/science.1188172

- [54] **Naveen Kumar Mogurampally**, K. V. S. Shiv Chaitanya and Bindu A. Bambah, "*Quantum Entanglement in coupled lossy waveguides using $SU(2)$ and $SU(1,1)$ Thermo-algebras*", Journal of Modern Physics, (2015) 6, pp. 1554-1571. doi:10.4236/jmp.2015.611158, arXiv:1507.01539 [quant-ph].
- [55] D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner and P. Zoller, (1998) "Cold bosonic atoms in optical lattices", *Physics Review Letters* **81**, 15, pp.3108; D. Jaksch, P. Zoller, (2005) "The cold atom Hubbard toolbox", *Annals of Physics*, 315, pp.52-79.
- [56] K.V.S. Shiv Chaitanya, Sibasish Gosh, V. Srinivasan, (2014) "Entanglement in two site Bose-Hubbard model", *Journal of Modern Optics*, **61**, issue 17, 1409, arXiv:quant-ph/1302.5238v4.
- [57] Henrik Bohr, and H. B. Nielsen, (1977) "Hadron production from a boiling quark soup: A thermodynamical quark model predicting particle ratios in hadronic collisions" *Nuclear Physics B* **128**, 2, pp.275-293. [https://doi.org/10.1016/0550-3213\(77\)90032-3](https://doi.org/10.1016/0550-3213(77)90032-3)
- [58] J. C. Collins, M. J. Perry, (1975) "Superdense Matter: Neutrons or Asymptotically Free Quarks?", *Physics Review Letters*, **34**, 21, pp.1353-1356. <https://doi.org/10.1103/PhysRevLett.34.1353>
- [59] A. Di Giacomo, (2000) "Confinement in QCD", *Nuclear Physics*, **A 663-664**, pp.199c-205c. [https://doi.org/10.1016/S0375-9474\(99\)00588-6](https://doi.org/10.1016/S0375-9474(99)00588-6)
- [60] John B. Kogut, (1963) "The lattice gauge theory approach to quantum chromodynamics", *Review of Modern Physics*, **55**, 3, pp.775-836. <https://doi.org/10.1103/RevModPhys.55.775>
- [61] Miklos Gyulassy, Larry McLerran, (2005) "New forms of QCD matter discovered at RHIC", *Nuclear Physics A* **750**, pp.30-63. <https://doi.org/10.1016/j.nuclphysa.2004.10.034>
- [62] C. P. Singh, (1993) "Signals of quark-gluon plasma" *Physics Reports (Review Section of Physics Letters)*, **236**,3, pp.147-224, North-Holland.
- [63] R. J. Glauber, (1963) "Coherent and Incoherent States of the Radiation Field", *Physical Review* **131**, 2766.
- [64] N.G. van Kampen, (1981) "Stochastic Processes in Physics and Chemistry", North-Holland, Amsterdam.
- [65] Z. Koba, H. B. Nielsen, P. Olesen, (1972) "Scaling of multiplicity distributions in high-energy hadron collisions", *Nuclear Physics B* **40**, pp.317-334. [https://doi.org/10.1016/0550-3213\(72\)90551-2](https://doi.org/10.1016/0550-3213(72)90551-2).
- [66] Takuya Mizoguchi, Minoru Biyajima, (2010) "Analyses of multiplicity distributions with c and Bose-Einstein correlations at LHC by means of generalized Glauber-Lachs formula", *European Physics Journal C* **70**, 1061-1069. arXiv:1010.1870

- [67] J. Perina, (1984) "Quantum Statics of Linear and Nonlinear Optical Phenomenon", SNTL, Prague.
- [68] G.A. Kozlov, (1998) "Deconfined phase via correlation functions: Some new trends", *Physical Review C* **58**, 1188; G.A. Kozlov, (2001) "The disorder deviation in the deconfined phase", *Journal of Mathematical Physics* **42**, 4749.
- [69] Bindu A. Bambah and **Naveen Kumar Mogurampally**, "*Thermal effects in the hadronic and photonic multiplicity distributions and correlations: A Thermo-Field Dynamic approach*", 6th Asian Triangle Heavy Ion Conference, 15th-19th February 2016, IIC, New Delhi.

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states with thermal noise", Journal of Modern Optics, 1999

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L. P. Guo. "Probing spin exchange interaction in two-species bosons in optical lattices by cavity-enhanced light scattering", The European Physical Journal D, 09/04/2009

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"Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2", Springer Science and Business Media LLC, 2018

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Publication

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T. Arimitsu, J. Pradko, H. Umezawa.
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