

UNIVERSITY OF HYDERABAD

DOCTORAL THESIS

ZERO WEIGHT SPACE FOR
TORI INSIDE A DIVISION
ALGEBRA

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Abstract

School of Mathematics and Statistics

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ZERO WEIGHT SPACE FOR TORI INSIDE A DIVISION ALGEBRA

by Sampath LONKA

Let F be a non-Archimedean local field of residue characteristic p . Consider a division algebra D over F of index ℓ , where ℓ is an odd prime and $\ell \neq p$. Let π be an irreducible representation of D^* with trivial central character. Let K/F be any field extension of degree ℓ in D . We compute the complex dimension of K^* -invariant vectors of π as both π and K vary.

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- Sampath Lonka

To my parents
Komuraiah and Ellamma

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1 Introduction

In this chapter we give the motivation for our work and explain the main theorem of this thesis.

1.1 The basic problem

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ with Cartan subalgebra denoted by $\mathfrak{h}$. We denote the space of linear functionals on $\mathfrak{h}$ by $\mathfrak{h}^*$. It is known that finite dimensional irreducible complex representations $L(\lambda)$ of $\mathfrak{g}$ are parametrized by the set of dominant integral weights in the dual $\mathfrak{h}^*$. Here the notion of dominance depends on fixing first a set of simple roots for the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$ in $\mathfrak{h}^*$. For more details on Lie algebras and root systems, we refer [AF],[FH] and [JH].

Let $L(\lambda)$ be a finite dimensional irreducible complex representation of $\mathfrak{g}$ and μ a linear functional on $\mathfrak{h}$. Let $L(\lambda)_\mu$ be the subspace of $L(\lambda)$ which is defined as follows

$$L(\lambda)_\mu = \{v \in L(\lambda) \mid \text{for every } \xi \in \mathfrak{h}, \xi.v = \mu(\xi)v\}.$$

We say that $L(\lambda)_\mu$ is the weight space attached to the functional μ and μ is called a weight of the representation $L(\lambda)$ if $L(\lambda)_\mu \neq \{0\}$. Also we say that the elements of $L(\lambda)_\mu$ are weight vectors. The *dimension* $\dim L(\lambda)_\mu$ is called the multiplicity of the weight μ in $L(\lambda)$. The *zero weight space* is the weight space attached to the weight 0.

Let G be a Lie group and $\mathfrak{g}$ be the Lie algebra of G . Let T be a maximal torus of G which is associated to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. A representation ρ

of G of finite dimension induces a representation of $\mathfrak{g}$. We note that any irreducible representation of torus T is of dimension one. The restriction of an n dimensional representation of G to the torus T will be a sum of n one dimensional representations of the torus T . Then we make the following definition of weight and weight space for Lie groups (cf.[AF]).

Definition 1.1.1 (Weights). *An irreducible representation of T is called a weight. Let (π, W) be a representation of G and χ a weight then the weight space attached to χ is the subspace of W defined by*

$$W_\chi = \{w \in W \mid \pi(t)(w) = \chi(t)w \text{ for every } t \in T\}.$$

We note that if (π, W) is a representation of G then the weight of π is defined as a weight of the representation of $\mathfrak{g}$ associated to π . Hence the zero weight space for T means the weight space corresponding the weight 0 of T , i.e, the trivial character of T . The study of the zero weight space of T in G is of considerable interest and there is a good deal of relevant literature (cf. [Gut], [Ko], [DAG],[H],[Tu], [MR], [Kac] and [KP]) over many decades. We do not as yet have a definitive answer for all Lie groups, even those of classical type. In recent years, the dimension of this space has been studied systematically in [KP] for a connected, adjoint, simple algebraic group G defined over the complex numbers $\mathbb{C}$.

1.2 Motivation

In [P1], Dipendra Prasad essentially shows that if D is a quaternion tame division algebra over a non-Archimedean local field F and π is an irreducible admissible representation of D^* of dimension greater than 1 then there exists a quadratic extension L of F embedded in D such that the restriction of π to L^* contains the trivial representation of L^* and also a field extension K of F of degree two such that the trivial representation of K^* does not occur in $\pi|_{K^*}$.

Wee Teck Gan (cf. [P1]) asks if such results are also true for higher dimensional division algebras over F . The work of Savin [Sa] shows that if this is true for cubic division algebras then it implies the non vanishing of some theta lifts from D^* to G_2 . Indeed Prasad's own work implies the non vanishing of some theta lifts. It is therefore of interest to know when an irreducible admissible representation of a general division algebra over F has K^* -fixed vectors where K is a field inside D . We call the multiplicative group of a maximal subfield in D^* *tori* in D^* . In this thesis, we look at the case of division algebras of dimension ℓ^2 over F where ℓ is an odd prime unequal to the residue characteristic of F .

1.3 Basic setup

1.3.1 Structure of Local fields

Let F be a non-Archimedean local field with residue field $\mathbb{F}$, of characteristic p , containing q elements. Let $\mathcal{O}_F$ be the maximal compact subgroup of F and P_F the unique maximal ideal of $\mathcal{O}_F$, generated by a prime element ϖ_F .

Given a non-Archimedean local field F , its multiplicative group F^* can be written as a product of subgroups, namely the subgroup generated by ϖ_F , a prime element of F , the 1-unit group and the group of root of unity μ_F of order coprime to p .

$$F^* = \langle \varpi_F \rangle \times \mu_F \times U_F^1$$

We can identify μ_F and $\mathbb{F}^*$ in canonical way. For more details on the structure of local fields we refer to [KZ], [L]. Let K/F be a finite field extension of degree n such that $n = fe$, where f is the residue degree and e is the ramification index of K/F . In this thesis, the field extensions K/F are always tame extensions, which means $p \nmid e$. By [KZ], we can then choose prime elements ϖ_F and ϖ_K

such that

$$\varpi_K^e = \zeta_{K/F} \varpi_F \text{ for some } \zeta_{K/F} \in \mu_F.$$

Let K/F be a tame field extension. We say that K/F is *unramified* if $e = 1$, and *totally ramified* if $f = 1$. In the case of an unramified extension K/F , we have $K = F(\zeta_K)$ for any generator ζ_K of μ_K . Further if K/F is unramified, we may take $\varpi_K = \varpi_F$. We will do so throughout the thesis.

1.3.2 Structure of Division algebras

Assume that D be a central division algebra over a non-Archimedean field F of dimension n^2 for some $n \in \mathbb{N}$. Here n is called the index of D . The ring of integers of D is denoted by $\mathcal{O}_D$, the unique maximal ideal is denoted by P_D and residue field is denoted by $\mathbb{D} = \mathcal{O}_D/P_D$, which is a finite field extension of degree n over $\mathbb{F}$. The set of coset representatives of $\mathbb{D}$ is denoted by $\tilde{\mathbb{D}}$ and we write $\tilde{\mathbb{F}} = F \cap \tilde{\mathbb{D}}$.

We write $\text{Tr}_{D/F}$ for reduced trace and $N_{D/F}$ for reduced norm of D over F . Similarly we can denote the trace and norm for the residue fields by $\text{Tr}_{\mathbb{D}/\mathbb{F}}$ and $N_{\mathbb{D}/\mathbb{F}}$ respectively. Also for field extension K/F , we write $\text{Tr}_{K/F}$ and $N_{K/F}$ for the trace and norm maps from K to F . Let ψ_F be a non-trivial additive character of F . For any field extension K/F , we write $\psi_K = \psi_F \circ \text{Tr}_{K/F}$. Assume that the additive character ψ_F is trivial on P_F but non trivial on $\mathcal{O}_F$. Then the additive character ψ_K is trivial on P_K but non trivial on $\mathcal{O}_K$ for any tamely ramified extension K/F (see pg.868 [M]). Let K/F be a field extension of degree n then K can be embedded in D and by the Skolem-Noether theorem [Mu], the embedding is unique upto conjugacy.

We denote $\mathbb{T} = \{\gamma \in \mathbb{D} | \text{Tr}_{\mathbb{D}/\mathbb{F}}(\gamma) = 0\} = \text{kernel of } \text{Tr}_{\mathbb{D}/\mathbb{F}}$. Then $\mathbb{D} = \mathbb{F} \oplus \mathbb{T}$. For a division algebra D of prime index ℓ over F , we write

$$N_\ell(0, 1) = \{x \in \mathbb{D} | \text{Tr}_{\mathbb{D}/\mathbb{F}}(x) = 0 \text{ and } N_{\mathbb{D}/\mathbb{F}}(x) = 1\}.$$

Let the extension K/F be an unramified extension of degree n inside D and σ be a generator of the Galois group $\text{Gal}(K/F)$. The following holds (cf. [W], [SP]).

Theorem 1.3.1 (Structure of Division algebras). *Let D be a division algebra over F of index n . Then D contains an unramified extension K of degree n over F . Moreover there exists a prime element ϖ in D such that*

1. As a K vector space $D = \bigoplus_{j=0}^{n-1} \varpi^j K$
2. For any $x \in K$ we have that $\varpi^{-1} x \varpi = x^\sigma$
3. $\text{Tr}_{D/F}(x \varpi^j) = 0$ for any $x \in K$ and $1 \leq j \leq n-1$
4. $\varpi^n = \varpi_F$ (say) is a prime element in F .

There is a natural filtration U_D^n of D^* for $n \geq 0$ with $U_D^0 = \mathcal{O}_D^*$, and $U_D^n = 1 + P_D^n$ for $n > 0$. Similarly, for field extension K/F , there is a natural filtration U_K^n of K^* . Also we write $K(m) = K^* \cap U_D^m$ for $m \in \mathbb{N}$ for any extension K/F . Note that U_K^m not necessarily equal to $K(m)$. Let θ be a quasi-character of K^* , then the *conductor* of θ is defined by $f(\theta) = \min\{n \mid \ker \theta \supset U_K^n\}$. We denote the greatest integer function of a real number x by $[x]$.

The representations of D^* that we will be considering will have $F^* U_D^t$ (for some non negative t) in the kernel and hence will essentially be representations of the finite group $D^*/F^* U_D^t$. Let G be a locally compact, totally disconnected group and H be a closed subgroup of G . If ρ be a representation of H then the induced representation of ρ to G is denoted by $\text{Ind}_H^G \rho$ and the set of irreducible admissible representations of the group G is denoted by $\text{Irr}(G)$. We denote $\widehat{F}$ for the set of all quasi-characters of multiplicative group of F . We denote the trivial representation of G by $\mathbb{1}_G$. If π be a representation of G , the restriction of π to H is denoted by $\pi|_H$ and the character of the representation π is denoted by Θ_π .

1.4 Main result of the thesis

We begin this section with the following definition.

Definition 1.4.1 (Tori). *We say that the multiplicative group of a maximal subfield in a division algebra D is a tori in D^* . Let K/F be a maximal extension in D then we say that K^* is an unramified tori in D^* if K/F is unramified and a totally ramified tori in D^* if K/F is totally ramified.*

In this thesis, we study the dimension of zero weight space of tori inside the division algebra over F of index ℓ , where ℓ is an odd prime and $\ell \neq p$. In this case, ℓ being a prime, any extension K/F of degree ℓ is either a totally ramified or an unramified extension and hence we have either totally ramified tori or unramified tori. From the structure theory of local fields we know that we have (upto conjugacy) either one unramified tori and one totally ramified tori or one unramified tori and ℓ totally ramified tori in D^* depending on whether F has a primitive ℓ th root of unity or not. We study the dimension of the zero weight space for tori K^* by computing the number $d(\pi, K) = \text{complex dimension of } \text{Hom}_{K^*}(\pi, \mathbb{1}_{K^*})$, where π is an irreducible representaion of D^* .

With the above notation we state the main theorem of this thesis, which is the joint work of author and Rajat Tandon [SR].

Theorem 1.4.2. *Let D be a division algebra of odd prime index ℓ over the non-Archimedean local field F with residue characteristic p , $\ell \neq p$ and π be an irreducible representation of D^* attached to the admissible pair (E, θ) of conductor $m + 1$ (see 2.1.1) with central character $\omega_\pi = \mathbb{1}_{F^*}$. Let K/F be any extension of degree ℓ in D . Then*

$$d(\pi, K) = \begin{cases} 0 & \text{if } E = K \text{ and } K/F \text{ is unramified;} \\ q^{\frac{1}{2}(\ell-1)(\ell-2)m} & \text{if } E/F \text{ is unramified and} \\ & K/F \text{ totally ramified;} \\ q^{(\ell-1)(\frac{m-1}{2}-[\frac{m-1}{\ell}])} & \text{if } E/F \text{ is totally ramified and} \\ & K/F \text{ unramified;} \\ \frac{N_\ell(0, 1)}{\ell} q^{\frac{1}{2}(m-1)(\ell-3)+[\frac{m-1}{\ell}]} & \text{if both } E/F, K/F \text{ are totally ramified.} \end{cases}$$

1.5 Outline of the article

The content of this thesis is structured as follows: In chapter 2, we recall the construction of representations of D^* . The set of irreducible representations of $D^*(\text{Irr}(D^*))$ is parameterized by the set of admissible pairs (E, θ) over F . The construction is well-known and is due to Howe, Moy (see [Ho], [M]) and P. Broussess (see [Br]). In chapter 3, we recall the character formulas for the unramified case [Tak1] and the totally ramified case [CMS]. In chapter 4, we discuss the existence of trace 0 and norm 1 elements in finite field extensions and, in particular, the Katz bound [Kat] for the number of such elements. Finally in chapter 5, we use the character formulas given in [Tak1] and [CMS] to obtain the dimension of zero weight spaces.

2 Representation theory of D^*

In this chapter we recall Howe's construction of admissible representations of D^* in the tame case, which means that p and ℓ are coprime. These irreducible representations are known as tame supercuspidal representations of D^* . For more details of the construction, we refer to ([M],[HR],[LC],[CH] and [CMS]). For tame supercuspidal representations of arbitrary p -adic groups, we refer [Yu] and [Kim]. Also we refer to [BK] for the construction of admissible representations of arbitrary p -adic groups via types.

Let E be an extension of F embedded in D . In the tame case, the representations of D^* are parametrized by admissible pairs (E, θ) where θ is an admissible quasi character of E^* . The one dimensional representations of D^* factor through the norm. Hence 1-dimensional representations of D^* are of the form $\chi \circ N_{D/F}$ for some $\chi \in \widehat{F}$. Howe's construction gives us the representations of D^* of dimension greater than 1.

2.1 Admissible pairs

For a finite extension E/F , of nonarchimedean local fields, we denote the ramification degree by $e = e_{E/F}$ and the residue degree by $f = f_{E/F}$. Assume that ψ_E is an additive character of E of conductor P_E i.e, it is trivial on P_E but non trivial on $\mathcal{O}_E$.

Definition 2.1.1. (see [M], 2.2.3) *Let E over F be a tamely ramified extension. A quasi character θ of E^* is said to be admissible over F if*

1. θ is not of the form $\chi \circ N_{E/L}$ for fields $F \subsetneq L \subsetneq E$ and
2. if $\theta|_{U_E^1}$ factors through $N_{E/L}$ where L is a proper subfield of E containing F then E/L is unramified.

In this thesis by the conductor of the admissible pair (E, θ) we mean the conductor of the quasi character θ of E^* .

Note that if E/F is tamely ramified of *prime* degree then θ is admissible if and only if it is not of the form $\chi \circ N_{E/F}$ for some $\chi \in \widehat{F}$ which is equivalent to $\theta|_{U_E^1}$ not factoring through norm. In particular $\theta|_{U_E^1}$ should not be trivial. Furthermore, if E/F is tamely ramified and the residue fields of F and E have q and q^f elements respectively then E has a primitive (q^f-1) st root ζ of unity and a prime element ϖ_E such that $\varpi_E^e = \zeta^i \varpi_F \in F$ for some e and i . If θ is a quasi character of E^* with conductor $m+1$, i.e., θ is trivial on U_E^{m+1} but non trivial on U_E^m , then if $m > 0$, there exists γ_θ in the group generated by ζ and ϖ_E such that $\theta(x) = \psi_E(\gamma_\theta(x-1))$ for $x \in U_E^m$. Moreover when the conductor of the additive character ψ_E is one, as will be the case in the rest of this thesis, then the valuation of γ_θ is $-m$. We write $\gamma_\theta = \gamma'_\theta \varpi_E^{-m}$, where γ'_θ is a power of ζ . Furthermore we know, by the remark at the beginning of this paragraph that if the degree of E over F is a prime and θ is admissible then m can be zero only when E is unramified over F .

The following definition due to Moy [M]:

Definition 2.1.2. : “A quasi-character θ of E^* is said to be generic over F of conductor $m+1$,

1. if $m = 0$ then E is unramified over F and θ does not factorise through the norm of a proper sub extension of E/F and
2. if $m > 0$ then $E = F(\gamma_\theta)$.”

Let $C_E = \langle \zeta, \varpi_E \rangle$, where ζ be a primitive $(q^f - 1)$ root of unity in E and ϖ_E is a prime element of E such that $\varpi_E^e \in F$ so that $\gamma_\theta \in C_E$ and it can be written in the form $\gamma'_\theta \varpi_E^{-m}$, where $\gamma'_\theta \in \langle \zeta \rangle$. Observe that when E is a tame extension of prime degree and θ is admissible then θ is generic if and only if

1. when E/F is totally ramified then $\ell \nmid m$
2. when E is unramified over F then either $m = 0$ or $\gamma'_\theta \notin F$ i.e, γ'_θ is not a $(q - 1)$ st root of unity.

We remark that θ admissible with $m = 0$ is possible if and only if $\theta(\zeta)$ is not a $(q - 1)$ st root of unity.

For if θ is trivial on U_E^1 then $\theta = \alpha \circ N_{E/F}$ where α is a quasicharacter of F^* which is trivial on U_F^1 , then $\alpha(\varpi_F) = \text{an } \ell\text{th root of } \theta(\varpi_F)$ and $\alpha(\zeta^{\frac{q^\ell - 1}{q - 1}}) = \theta(\zeta)$. The last equality is only possible if $\theta(\zeta)$ is a $(q - 1)$ st root of unity.

Proofs of the following results exist in the literature but in our simple case when $[E : F] = \ell$, a prime not equal to p much simpler proofs can be given which are instructive in themselves.

Proposition 2.1.3. *If θ is generic then θ is admissible. (This is true in general)*

Proof. Case 1: E/F is totally ramified : Then $\ell \nmid m$. We claim that $\theta|_{U_E^m}$ does not factor through norm and therefore θ does not factor through norm. Observe that [JPS]

$$N(U_E^m) = \begin{cases} U_F^{m/\ell} & \text{if } \ell \mid m \\ U_F^{[m/\ell]+1} & \text{if } \ell \nmid m \end{cases}$$

Hence if $\ell \nmid m$ then $N(U_E^m) = N(U_E^{m+1})$. If $\theta = \alpha \circ N_{K/F}$ then

$$\begin{aligned} \theta(U_E^m) &= \alpha(N(U_E^m)) \\ &= \alpha(N(U_E^{m+1})) \\ &= \theta(U_E^{m+1}) \\ &= \{1\} \end{aligned}$$

which contradicts the fact that the conductoral exponent of θ is $m + 1$.

Case 2: E is unramified over F : Then either $m = 0$ or $\gamma'_\theta \notin F$. If $m = 0$ then the definition of generic means that θ does not factor through norm and so θ is admissible. If $\gamma'_\theta \notin F^*$ we claim that $\theta|_{U_E^m}$ does not factor through norm and therefore θ does not factor through norm. Observe that, we have

$$\begin{aligned}
\theta(1 + x\varpi_E^m) &= \psi_E(\gamma_\theta x\varpi_E^m) \text{ for every } x \in \mathcal{O}_E \\
&= \psi_E(\gamma'_\theta \varpi_E^{-m} x\varpi_E^m) \\
&= \psi_E(\gamma'_\theta x) \text{ for every } x \in \mathcal{O}_E.
\end{aligned}$$

If σ generates the Galois group of E over F and if $\theta|_{U_E^m}$ factors through norm then $\theta^\sigma = \theta$ on U_E^m . Hence

$$\begin{aligned}
\theta^\sigma(1 + x\varpi_E^m) &= \theta(1 + x\varpi_E^m) \\
\Rightarrow \psi_E(\gamma_\theta'^\sigma x) &= \psi_E(\gamma'_\theta x) \text{ for every } x \in \mathcal{O}_E \\
\Rightarrow \psi_E((\gamma_\theta'^\sigma - \gamma'_\theta)x) &= 1 \text{ for every } x \in \mathcal{O}_E
\end{aligned}$$

But when E is unramified over F and $\gamma'_\theta \notin F$ then $\gamma_\theta'^\sigma - \gamma'_\theta$ must be a unit in E so by the above if θ factors through norm on U_E^m we get that ψ_E is trivial on $\mathcal{O}_E$ which contradicts the fact that the conductor of ψ_E is P_E . $\square$

In what follows we will usually assume that θ is trivial on F^* i.e., θ is a quasi character of E^*/F^* . In this case the converse is also true i.e if θ is admissible then it is generic.

Proposition 2.1.4. *If $\theta|_{F^*} = 1$ and θ is admissible then it is generic. (E is of prime degree over F .)*

Proof. Case 1: If E/F is totally ramified, we need to show that $\ell \nmid m$. But if $\ell \mid m$ then $U_E^m \subset F^*U_E^{m+1}$. Since θ is trivial on $F^*U_E^{m+1}$ it would mean that θ is trivial on U_E^m which contradicts the fact that $m+1$ is the conductor of θ .

Case 2: If E is unramified over F and if $m = 0$ then θ is generic by definition; otherwise we need to show $\gamma'_\theta \notin F$. Without loss of generality we may assume that $\psi_E = \psi_F \circ \text{Tr}_{E/F}$, where ψ_F is an additive character of F^* of conductor P_F . Then for any $x \in \mathcal{O}_F$, we have

$$\begin{aligned}
1 &= \theta(1 + x\varpi_F^m) \\
&= \psi_F(\text{Tr}_{E/F} \gamma'_\theta \varpi_F^{-m} \varpi_F^m x) \\
&= \psi_F(\ell \gamma'_\theta x) \text{ if } \gamma'_\theta \in F.
\end{aligned}$$

But then ψ_F is trivial on $\ell x \mathcal{O}_F = \mathcal{O}_F$ which contradicts the fact that the conductor of ψ_F is P_F . Hence $\gamma'_\theta \notin F$ which implies that θ is generic.

□

So for characters of E^*/F^* genericity and admissability are the same.

The following proposition says more about genericity. We do not assume here that $\theta|_{F^*}$ is trivial.

Proposition 2.1.5. *Let E/F be a tame extension of degree ℓ (prime) and (E, θ) an admissible pair of conductoral exponent $m+1$. Then θ is generic is equivalent to $\theta|_{U_E^m}$ does not factor through norm. Furthermore*

1. *If E/F is totally ramified then $\theta|_{U_E^m}$ does not factor through norm if and only if $m > 0$ and $\ell \nmid m$.*
2. *If E/F is unramified then $\theta|_{U_E^m}$ does not factor through norm if and only if either $m = 0$ and $\theta(\zeta)$ is not a $(q-1)$ st root of 1 or $m > 0$ and θ is non trivial on elements of the form $1 + x\varpi_F^m$, where $x \in \mathcal{O}_E$ and $\text{Tr}_{E/F}(x) = 0$.*

Thus when θ is generic then θ is minimal, i.e., its conductoral exponent cannot be lowered by twisting it by a quasi character of E^* which factors through norm.

Proof. We have already proved that when the extension E/F is as in the proposition that $\theta|_{U_E^m}$ does not factor through norm (Proposition 2.1.3). For the converse suppose that $\theta|_{U_E^m}$ does not factor through norm.

Case (i): E is totally ramified over F : Then $m+1 > 1$ and $\varpi_E^\ell = \varpi_F \in F$.

We claim that $\ell \nmid m$ and therefore θ is generic. Suppose to the contrary that $\ell \mid m$, we have

$$U_E^m = U_F^{\frac{m}{\ell}} U_E^{m+1}.$$

Define $\alpha_1 : U_F^{\frac{m}{\ell}} / U_F^{\frac{m}{\ell}+1} \rightarrow \mathbb{C}^*$

such that

$$\alpha_1(1 + x\varpi_F^{\frac{m}{\ell}}) = \theta(1 + \frac{x}{\ell}\varpi_E^m).$$

Then

$$\alpha_1 \circ N_{E/F}(1 + x\varpi_E^m) = \alpha_1 \circ N_{E/F}\left((1 + x_0\varpi_F^{\frac{m}{\ell}})(1 + y\varpi_E^{m+1})\right)$$

for some $x_0 \in \mathcal{O}_F$, $y \in \mathcal{O}_E$. So

$$\begin{aligned}
\alpha_1 \circ N_{E/F}(1 + x\varpi_E^m) &= \alpha_1((1 + x_0\varpi_F^{\frac{m}{\ell}})^\ell) \\
&= \alpha_1(1 + \ell x_0\varpi_F^{\frac{m}{\ell}}) \\
&= \theta(1 + x_0\varpi_E^m) \\
&= \theta((1 + x_0\varpi_F^{\frac{m}{\ell}})(1 + y\varpi_E^{m+1})) \\
&= \theta(1 + x\varpi_E^m) \quad (\because f(\alpha_1) = \frac{m}{\ell} + 1.)
\end{aligned}$$

Therefore $\theta = \alpha_1 \circ N_{E/F}$ on U_E^m which gives a contradiction.

Case (ii): E/F is unramified: The statement for $m = 0$ has already been proved.

Observe that if the Galois group of E/F is generated by σ then

$$\theta^\sigma \theta^{-1}(1 + x\varpi_F^m) = \psi_E(\gamma'_\theta(\sigma(x) - x)) = \psi_E((\sigma(\gamma'_\theta) - \gamma'_\theta)x)$$

since ψ_E factors through trace. Clearly then if $\theta|_{U_E^m}$ does not factor through norm, i.e., $\theta^\sigma \theta^{-1}$ is not trivial on U_E^m we must have $\gamma'_\theta \notin F$ so that θ is generic.

Finally observe that $\theta|_{U_E^m}$ factors through norm if and only if $\theta^\sigma = \theta$ on U_E^m .

Moreover

$$\theta^\sigma(1 + x\varpi_F^m) = \theta(1 + x\varpi_F^m) \text{ if and only if } \theta(1 + (\sigma x - x)\varpi_F^m) = 1$$

where $x \in \mathcal{O}_E$.

Suppose θ is not trivial on all elements of the form $1 + x\varpi_F^m$, where $x \in \mathcal{O}_E$ and $\text{Tr}_{E/F}(\gamma) = 0$. Observe that, by a slight refinement of the additive Hilbert's 90, the elements of $\mathcal{O}_E$ of trace 0 are precisely of the form $\sigma(y) - y$ for some $y \in \mathcal{O}_E$. These facts clearly show that $\theta|_{U_E^m}$ does not factor through norm precisely when θ is non trivial on elements of the form $1 + x\varpi_F^m$ for $x \in \mathcal{O}_E$ of trace zero. $\square$

If $\theta|_{U_E^m}$ is trivial on elements of the form $1 + x\varpi_F^m$ for $x \in \mathcal{O}_E$ of trace 0 then the following explains how we can get a quasi character α of F^* such that $\theta = \alpha \circ N_{E/F}$ on U_E^m . Define $\alpha : U_F^m/U_F^{m+1} \rightarrow \mathbb{C}^*$ by $\alpha(1 + u\varpi_F^m) = \theta(1 + \frac{u}{\ell}\varpi_F^m)$, $u \in \mathcal{O}_F$. Extend α to a character of F^* . (Recall that if G is a finite abelian group with subgroup H then any character of H can be extended to G . Hence a character of U_E^m/U_E^{m+1} can be extended to U_E/U_E^{m+1} and further extended to E^* by giving $\alpha(\varpi)$ any arbitrary complex value.) We claim that $\theta = \alpha \circ N_{E/F}$ on U_E^m . The proof of the claim is as follows.

Let $u = \frac{1}{\ell}\text{Tr}_{E/F}(x)$ and $y = x - \frac{1}{\ell}\text{Tr}_{E/F}(x)$. When $x \in \mathcal{O}_E$ then $u \in \mathcal{O}_F$ and $y \in \ker \text{Tr}_{E/F} \cap \mathcal{O}_E$.

Then $\alpha \circ N_{E/F}(1 + x\varpi_F^m) = \alpha \circ N_{E/F}(1 + (u + y)\varpi_F^m)$

$$= \alpha(N_{E/F}(1 + u\varpi_F^m))\alpha_1(N_{E/F}(1 + y\varpi_F^m))$$

$$= \alpha((1 + u\varpi_F^m)^\ell)\alpha(1 + \text{Tr}_{E/F}y\varpi_F^m)$$

$$= \alpha((1 + \ell u\varpi_F^m))$$

$$= \theta(1 + u\varpi_F^m)\theta(1 + y\varpi_F^m)$$

$$= \theta(1 + x\varpi_F^m).$$

Since θ is non trivial on U_E^m then α is non trivial on U_F^m .

Therefore the conductor of α is $m + 1$.

Finally we show in general that any admissible quasi character of E^* is the product of a generic character and a character that factors through norm.

Proposition 2.1.6. *Let E/F be a tame extension of prime degree and θ be an admissible character of E^* . Then*

$$\theta = (\chi \circ N_{E/F})\theta',$$

where $\chi \in \widehat{F}$ and θ' is a generic character of E^* .

Proof. We first make the elementary remark that if we twist (multiply) an admissible quasi character of E^* by a character that factors through norm then the product is again admissible. Let the conductor of θ be $m+1$. If θ is generic we let $\theta' = \theta$ and $\chi = 1$. If θ is not generic then $\theta|_{U_E^m}$ must factor through norm. Let $\theta = \alpha \circ N_{E/F}$ on U_E^m . Let $\theta_1 = \theta(\alpha^{-1} \circ N_{E/F})$. Clearly θ_1 is trivial on U_E^m and so its conductor $m_1 + 1$ is less than $m + 1$. If $\theta_1|_{U_E^{m_1}}$ does not factor through norm then it is generic and we let $\theta' = \theta_1$ and $\chi = \alpha_1$. If, on the other hand, $\theta_1 = \beta_2 \circ N_{E/F}$ on $U_E^{m_1}$ we let $\theta_2 = \theta_1(\beta_2^{-1} \circ N_{E/F}) = \theta(\alpha_2^{-1} \circ N_{E/F})$ where $\alpha_2 = \alpha_1\beta_2$. Clearly the conductor $m_2 + 1$ of θ_2 is less than $m_1 + 1$. We now have $\theta = \theta_2(\alpha_2 \circ N_{E/F})$ and $m_2 < m_1$. Continuing in this way we will finally get a character θ_n which is generic and a character α_n of F^* such that $\theta = \theta_n(\alpha_n \circ N_{E/F})$. We let $\theta' = \theta_n$ and $\chi = \alpha_n$. $\square$

2.2 Construction of π_θ

Let D be a tame division algebra over its centre F of dimension ℓ^2 and (E, θ) an admissible pair such that E is embedded in D and has dimension ℓ over F and the conductor of θ is $m+1$. Let us first assume that E/F is unramified. If $m = 0$ then θ may be considered to be a representation of F^*U_D since $F^*U_D/U_D^1 \simeq E^*/U_E^1$. We define π_θ to be $\text{Ind}_{F^*U_D}^{D^*} \theta$. If $m > 0$ we let ψ_F be the character of $(F, +)$ as in §1.3.2. Then there exists an element $\gamma_\theta \in E$ such that $\theta(x) = \psi_F \circ \text{Tr}_{E/F}(\gamma_\theta(x-1))$ for all $x \in U_E^t$ where $t = \lfloor \frac{\ell m + 2}{2} \rfloor$. We define ρ_θ on $E^*U_D^t$ by $\rho_\theta(xy) = \theta(x)\psi_F \circ \text{Tr}_{D/F}(\gamma_\theta(y-1))$ for $x \in E^*$ and $y \in U_D^t$. We

then define $\pi_\theta = \text{Ind}_{E^*U_D^t}^{D^*} \rho_\theta$. In both cases π_θ is an irreducible representation of D^* .

Proposition 2.2.1 (cf.[Tak1]). *Let π_θ be an irreducible representation of D^* as above. Then we have the following*

1. *dimension of $\pi_\theta = \ell q^{\ell(\ell-1)m/2}$.*
2. *$f(\pi) = \ell f(\theta)$.*
3. *Any irreducible representations of D^* whose conductor is divisible by ℓ is of the form π_θ for some unramified pair (E, θ) as above.*

Let the extension E/F be totally ramified of prime degree ℓ and (E, θ) be an admissible pair over F . We next construct π_θ attached to the pair (E, θ) . We follow [CMS]. There is a non trivial additive character ψ_0 of $\mathbb{F}$ such that

$$\theta(1 + \alpha \varpi_E^m) = \psi_0(\text{Tr} \alpha) = \psi_0(\ell \alpha), \text{ for } \alpha \in \tilde{\mathbb{F}}.$$

Define

$$\begin{cases} m' = m'' = \frac{m+1}{2} & \text{if } m \text{ is odd;} \\ m' = \frac{m}{2} + 1, m'' = \frac{m}{2} & \text{if } m \text{ is even;} \end{cases}$$

let $H' = E^*U_D^{m'}/U_D^{m+1}$ and $H'' = E^*U_D^{m''}/U_D^{m+1}$, We set

$$N_0 = \{y \in P_D^{m'} \mid \text{Tr}_{D/F} yz = 0 \text{ for all } z \in E\},$$

and define

$$N = U_D^{m+1}(1 + N_0)/U_D^{m+1}.$$

Then we have the following:

1. $N \triangleleft H'$
2. $N \cap E^*U_D^{m+1}/U_D^{m+1}$ is trivial.
3. $H' = N.E^*U_D^{m+1}/U_D^{m+1}$

4. We extend θ to H' by setting

$$\theta^\#|_N \equiv 1, \theta^\#|_{E^*} = \theta.$$

Observe that θ may be considered to be a character of $E^*U_D^{m+1}/U_D^{m+1}$. If m is odd then we define $\pi_\theta = \text{Ind}_{H'}^{D^*/U_D^{m+1}} \theta^\#$. If on the other hand m is even then modulo the kernel of $\theta^\#$, H'' is a Heisenberg p -group in which H' has index $q^{\ell-1}$. Then there exists a subgroup W of index $q^{\frac{\ell-1}{2}}$ in H'' such that (modulo kernel $\theta^\#$) W is abelian and $\theta^\#$ extends to a character θ' of W . Moreover $\text{Ind}_W^{H''} \theta'$ is irreducible. We let $\pi_\theta = \text{Ind}_W^{D^*} \theta'$. The following proposition is due to Takahashi and Moy.

Proposition 2.2.2 (cf. [Tak2], [M]). *Let π_θ be the irreducible representation of D^* as above. Then we have the following*

1. *dimension of $\pi_\theta = \frac{q^\ell - 1}{q - 1} q^{\frac{1}{2}(\ell-1)(m-1)}$.*
2. *$f(\pi) = f(\theta) + \ell$.*
3. *Any irreducible representations of D^* whose conductor is not divisible by ℓ , of dimension greater than 1, is of the form π_θ for some totally ramified pair (E, θ) as above.*

3 Character formulas

In this chapter we recall the character formulas from [Tak1] and [CMS]. Recall that ℓ is an odd prime.

3.1 Unramified Case

In this section we consider irreducible representations π of D^* attached to some unramified pair. Then the following theorem is due to Takahashi.

Theorem 3.1.1 ([Tak1], Corollary 1.2.2). *Let E/F be an unramified extension of degree over ℓ in D and the pair (E, θ) admissible over F . Assume that π is the irreducible representation attached to the pair (E, θ) . Then*

$$\Theta_\pi(x) = \begin{cases} q^{\frac{1}{2}\ell(\ell-1)j} \sum_{\sigma \in \text{Gal}(E/F)} \theta(\sigma(x)) & \text{if } x \in C_j^* \ (0 \leq j < m) \\ q^{\frac{1}{2}\ell(\ell-1)m} \sum_{\sigma \in \text{Gal}(E/F)} \theta(\sigma(x)) & \text{if } x \in C_m \end{cases}$$

where $C_j = U_F^1 U_E^j / U_F^1 U_E^{\ell m+1}$ and $C_j^* = C_j - C_{j+1}$ for $0 \leq j < m$.

The following theorem gives character formulas when we restrict π to totally ramified tori.

Theorem 3.1.2 ([Tak1], Theorem 1.2.17(2)). *Let E, π be as in the theorem above and K/F be a totally ramified extension of degree ℓ . Then*

$$\Theta_\pi(x) = \begin{cases} 0 & \text{if } x \notin F^* U_K^{\ell m+1} \\ \theta(c) \ell q^{\frac{1}{2}\ell(\ell-1)m} & \text{if } x = c(1+y) \in F^* U_K^{\ell m+1} \end{cases}$$

3.2 Totally ramified case

Note that every non zero element x of the division algebra D has a unique expression of the form

$$x = \sum_{j=j_0}^{\infty} \alpha_j \varpi^j, \quad \alpha_j \in \tilde{\mathbb{D}} \text{ and } j_0 = \nu(x)$$

where ν be the valuation of D .

The following definition due to [CMS].

Definition 3.2.1. *Let x be as above. We say that x is a normal element if the monomials $\alpha_j \varpi^j$ all commute.*

We note that every element of D^* is conjugate to a normal element. Given any normal $x \in D \setminus F$,

write

$$x = \alpha_0 \varpi^{j_0} (1 + \alpha_1 \varpi + \cdots)$$

and define

$$\nu_0(x) = \begin{cases} 0 & \text{if } \alpha_0 \varpi^{j_0} \notin F; \\ j & \text{if } \alpha_0 \varpi^{j_0} \in F \text{ and } j \text{ is the smallest index with } \alpha_j \varpi^j \notin F. \end{cases}$$

Let the extension E/F be totally ramified of degree ℓ and π be the irreducible representation attached to the admissible pair (E, θ) . The following theorem is taken from [CMS]. There is a typographical error in [Theorem 4.2 (c), [CMS]] which is acknowledged in (pg.58, [ACS]). In view of [ACS], we have the following theorem.

Theorem 3.2.2 ([CMS], Theorem 4.2). *Let $x \in D^*$ be normal and $\nu(x) = j$. Then we have the following:*

1. *If $j > m$, $\Theta_\pi(x) = \frac{q^\ell - 1}{q - 1} q^{\frac{1}{2}(\ell-1)(m-1)} \theta^\#(x)$.*
2. *If $j < m$ and x is not conjugate to an element of E , then $\Theta_\pi(x) = 0$.*

3. If $j < m$ and $x \in E$, then

$$\Theta_\pi(x) = q^{\frac{1}{2}(\ell-1)j} \sum_{\sigma \in \text{Gal}(E/F)} G(\sigma(x), \theta)\theta(\sigma(x)),$$

where $\text{Gal}(E/F)$ is the Galois group of E/F .

4. If $j = m$, then $x = fx_0$, $f \in F$ and $x_0 \equiv 1 + \alpha\varpi^m \pmod{P_D^{m+1}}$. Then

$$\Theta_\pi(x) = q^{\frac{1}{2}(\ell-1)(m-1)}\theta(f) \sum_{\delta \in \ker N_{\mathbb{D}/\mathbb{F}}} \psi_0 \circ \text{Tr}_{\mathbb{D}/\mathbb{F}}(\alpha\delta).$$

Where

$$G(x, \theta) = \begin{cases} \left(\frac{q}{\ell}\right) & \text{if } j = 0 \text{ (the Legendre symbol)} \\ 1 & \text{if } j > 0, \text{ and } m \text{ and } j \text{ have opposite parities;} \\ q^{-(\ell-1)/2} \sum_{\gamma \in T} \psi_0 \text{Tr}_{\mathbb{D}/\mathbb{F}}(\alpha_j \gamma^{\sigma^{\frac{m+j}{2}}} (\gamma^{\sigma^j} - \gamma)) & \text{if } j > 0, \text{ and } m \text{ and } j \\ & \text{have the same parity.} \end{cases}$$

The following lemmas are repeatedly used in section 5.1. The first two lemmas can be stated in greater generality. However we state them only when ℓ is a prime and prove them by using the structure theory of local fields [W].

Lemma 3.2.3. *Let K/F be a tame extension of degree ℓ and r a non negative integer. Then*

$$\begin{aligned} K^* \cap F^*U_D^r &= F^*U_K^{\lceil \frac{r-1}{\ell} \rceil + 1} \text{ if } K \text{ is unramified over } F \\ &= F^*U_K^r \text{ if } K \text{ is totally ramified over } F. \end{aligned}$$

Proof. We note that in either case

the right hand side in the above equalities $\subset F^*U_D^r$ for $r \geq 0$.

Let $x \in F^*U_D^r$ then x can be written uniquely as

$$x = \sum_{i \geq 0}^{r-1} \alpha_i \varpi_F^i + \sum_{j \geq r} \beta_j \varpi^j$$

where $\alpha_i \in \tilde{\mathbb{F}}$ and $\beta_j \in \tilde{\mathbb{D}}$.

Case I: If K/F is an unramified extension and $x \in F^*U_D^r \cap K^*$. Then we have the following

$$j \equiv 0 \pmod{\ell} \text{ and } \beta_j \in \tilde{\mathbb{F}}.$$

Then

$$\begin{aligned} K^* \cap F^*U_D^r &\subset F^*U_K^{\lceil \frac{r}{\ell} \rceil} \text{ if } r \equiv 0 \pmod{\ell} \\ &\subset F^*U_K^{\lceil \frac{r}{\ell} \rceil + 1} \text{ if } r \not\equiv 0 \pmod{\ell}. \end{aligned}$$

Hence

$$\begin{aligned} K^* \cap F^*U_D^r &= F^*U_K^{\lceil \frac{r+\ell-1}{\ell} \rceil} \\ &= F^*U_K^{\lceil \frac{r}{\ell} \rceil + 1}. \end{aligned}$$

Case II: If K/F is a totally ramified and $x \in F^*U_D^r \cap K^*$. Then $\beta_j \in \tilde{\mathbb{F}}$ which implies that $x \in F^*U_K^r$. Hence

$$K^* \cap F^*U_D^r = F^*U_K^r.$$

□

Lemma 3.2.4. *Let K/F be as in above lemma and r be a non negative integer.*

Then

$$|K^*/F^*U_K^r| = \begin{cases} \frac{q^\ell - 1}{q - 1} q^{(\ell-1)(r-1)}, & \text{if } K \text{ is unramified over } F; \\ \ell q^{(r-1) - \lceil \frac{r-1}{\ell} \rceil}, & \text{if } K \text{ is totally ramified over } F. \end{cases}$$

Proof. We note that

$$F^* = \varpi_F \times \langle \zeta_F \rangle \times U_F^1$$

where ζ_F be the primitive $(q-1)$ st root of unity in F .

Then we write K^* as

$$K^* = \begin{cases} \varpi_F \times \langle \zeta_K \rangle \times U_K^1, & \text{if } K/F \text{ is unramified;} \\ \varpi_K \times \langle \zeta_F \rangle \times U_K^1, & \text{if } K/F \text{ is totally ramified} \end{cases}$$

where ζ_K be the primitive $(q^\ell - 1)$ st root of unity in K . Also we note that

$$U_F^1 \cap U_K^r = \begin{cases} U_F^r, & \text{if } K/F \text{ is unramified;} \\ U_F^{[\frac{r-1}{\ell}] + 1}, & \text{if } K/F \text{ is totally ramified.} \end{cases}$$

Since $U_F^1 U_K^r / U_K^r \cong U_F^1 / U_F^1 \cap U_K^r$ and for $i > 0$,

$$U_K^i / U_K^{i+1} \cong \begin{cases} \mathbb{D}, & \text{if } K/F \text{ is unramified;} \\ \mathbb{F}, & \text{if } K/F \text{ is totally ramified.} \end{cases}$$

Then

$$\begin{aligned} [U_K^1 : U_F^1 U_K^r] &= [U_K^1 : U_K^r] / [U_F^1 U_K^r : U_K^r] \\ &= [U_K^1 : U_K^r] / [U_F^1 : U_F^1 \cap U_K^r] \\ &= \begin{cases} q^{(\ell-1)(r-1)} & \text{if } K \text{ is unramified over } F; \\ q^{(r-1) - [\frac{r-1}{\ell}]} & \text{if } K \text{ is totally ramified over } F. \end{cases} \end{aligned}$$

If K/F is unramified, the cardinality of the quotient is given by

$$\begin{aligned} [K^* : F^* U_K^r] &= \frac{q^\ell - 1}{q - 1} [U_K^1 : U_F^1 U_K^r] \\ &= \frac{q^\ell - 1}{q - 1} q^{(\ell-1)(r-1)}. \end{aligned}$$

In case of totally ramified K/F , we have

$$\begin{aligned} [K^* : F^* U_K^r] &= \ell [U_K^1 : U_F^1 U_K^r] \\ &= \ell q^{(r-1) - [\frac{r-1}{\ell}]}. \end{aligned}$$

□

Finally, the following lemma is well known.

Lemma 3.2.5. “Let G be a finite group and χ be a non trivial character of G .

Then

$$\sum_{g \in G} \chi(g) = 0.”$$

4 Trace zero and Norm one elements

Let $\mathbb{F}_q$ be a finite field with $q = p^r$ elements. In this chapter we recall the result of Moio ([Mo]) where he obtains bounds for the cardinality of elements of $\mathbb{F}_{q^n}$ with prescribed trace and norm over $\mathbb{F}_q$ for arbitrary n . As a corollary we will see a sharp bound in the case when n is a prime and the trace and norm are 0 and 1 respectively. Also we prove that the number of elements of $\mathbb{F}_{q^\ell}$ with trace zero and norm one is non zero and multiple of ℓ . The existence of trace zero and norm one elements in $\mathbb{F}_{q^3}$ over $\mathbb{F}_q$ is discussed in the following example.

Example 4.0.6. (Existence of trace zero and norm 1 elements)

There exist $x \in \mathbb{F}_{q^3}$ such that $\text{tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q} x = 0$ and $N_{\mathbb{F}_{q^3}/\mathbb{F}_q} x = 1$.

Proof. Consider polynomial $f_a(x) = X^3 + aX - 1$, $a \in \mathbb{F}_q$. If $a \neq b$ and x_1 and x_2 are two roots of f_a and f_b respectively in $\mathbb{F}_q$ then $x_1 \neq x_2$. For suppose $x_1 = x_2$. Then $x_1^3 + ax_1 - 1 = x_2^3 + bx_2 - 1 = 0$, which implies $a = b$. Hence the q different f_a 's can't all have a root in $\mathbb{F}_q$ (since 0 is never a root). Therefore at least one polynomial of the form $X^3 + aX - 1$ must be irreducible. Then in $\mathbb{F}_{q^3} \cong \mathbb{F}_q[X]/\langle X^3 + aX - 1 \rangle$, X has trace 0 and norm 1.

□

4.1 Katz bound

We define for any positive integer t

$$\begin{aligned} N_\ell(0, b) &= |\{x \in \mathbb{F}_{q^t} \mid \text{Tr}_{\mathbb{F}_{q^t}/\mathbb{F}_q}(x) = 0 \text{ and } N_{\mathbb{F}_{q^t}/\mathbb{F}_q}(x) = b\}| \\ &= \text{the number of elements of } \mathbb{F}_{q^\ell} \text{ with trace 0 and norm } b \end{aligned}$$

over $\mathbb{F}_q$.

Then the following theorem is due to Moisio.

Theorem 4.1.1 (cf. [Mo], (Theorem.3.2)). *Let γ_t be a primitive element of $\mathbb{F}_{q^t}$. Assume that $p \nmid \frac{m}{t}$ and $d \mid r$ where $b = g^r$. Then,*

$$N_t(0, b) = d \left(\frac{q^{\ell-1} - 1}{q - 1} + \frac{1}{q} \sum_{x \in \mathbb{F}_{q^t}} \psi_0(\text{Tr} \gamma_t^{i_0} x^s) \right)$$

where $s = \gcd(\ell, q - 1)$, $d = \gcd(\frac{m}{t}, q - 1)$.

In case of $t = \ell$, we have the following corollary.

Corollary 4.1.2. *Assume that $p \nmid \frac{m}{\ell}$. Then,*

$$N_\ell(0, 1) = \frac{q^{\ell-1} - 1}{q - 1} + \frac{1}{q} \sum_{x \in \mathbb{F}_{q^\ell}} \psi_0(\text{Tr} x^s)$$

where $s = \gcd(\ell, q - 1)$.

Proof. Follows from the theorem when $b = 1$, $i_0 = 0$ □

Then we have the special case of the Katz bound (cf. [Mo], (1.2)) in case ℓ is a prime.

Lemma 4.1.3 (see [Mo], corollary (3.3)).

$$\left| N_\ell(0, 1) - \frac{q^{\ell-1} - 1}{q - 1} \right| \leq (s - 1) q^{\frac{\ell-2}{2}}$$

where $s = \gcd(\ell, q - 1)$.

Lemma 4.1.4. $N_\ell(0, 1)$ is non zero for all odd primes ℓ .

Proof. By using above lemma we have the following:

Case (i): if $\ell \nmid q-1$, then $N_\ell(0,1) = \frac{q^{\ell-1}-1}{q-1}$.

Case (ii): if $s = \ell$ (i.e., $\ell|(q-1)$) and $\ell \geq 5$, then

$$\begin{aligned} |N_\ell(0,1) - \frac{q^{\ell-1}-1}{q-1}| &\leq (\ell-1)q^{(\ell-2)/2} \\ &< q \cdot q^{(\ell-2)/2} = q^{\ell/2} \leq q^{\ell-2} \\ &< \frac{q^{\ell-1}-1}{q-1}. \end{aligned}$$

Case (iii): if $\ell = 3$, then

$$|N_3(0,1) - (q+1)| \leq 2\sqrt{q}.$$

Therefore, in all cases, we have

$$N_\ell(0,1) \neq 0.$$

□

However, if $\ell = 2$ the following is easy to prove:

Lemma 4.1.5. *If q is odd and $\ell = 2$, then*

$$N_2(0,1) = \begin{cases} 0 & \text{if } q \equiv 1 \pmod{4} \\ 2 & \text{if } q \equiv 3 \pmod{4}. \end{cases}$$

Proof. If $q \equiv 3 \pmod{4}$, then -1 is not a square in $\mathbb{F}_q$ and x^2+1 is irreducible over $\mathbb{F}_q$. Write

$$\mathbb{F}_{q^2} = \mathbb{F}_q + \mathbb{F}_q\alpha$$

where $\alpha^2 = -1$.

Let $x = a + b\alpha \in \mathbb{F}_{q^2}$ ($a, b \in \mathbb{F}_q$) such that $\text{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(x) = 0$ and $N_{\mathbb{F}_{q^2}/\mathbb{F}_q}(x) = 1$. Then $a = 0$ and $-b^2\alpha^2 = 1$, which implies that $b^2 = 1$ i.e., $b = \pm 1$. Therefore $N_2(0, 1) = 2$.

If $q \equiv 1 \pmod{4}$, there exist an $i \in \mathbb{F}_q$ such that $i^2 = -1$. Choose a $\beta \in \mathbb{F}_q$ which is not a square. Then $x^2 - \beta$ is irreducible over $\mathbb{F}_q$. So we write

$$\mathbb{F}_{q^2} = \mathbb{F}_q + \mathbb{F}_q\alpha$$

where $\alpha^2 = \beta$.

For $a + b\alpha$ to have trace 0 we must have $a = 0$. For it to then have norm 1 we must have $-b^2\beta = 1$ which implies that $\beta = (ib^{-1})^2$ which contradicts the fact that β is not a square. Therefore there are no elements of trace zero and norm one. $\square$

Lemma 4.1.6. $N_\ell(0, 1) \equiv 0 \pmod{\ell}$.

Proof. By using the lemma above (4.1.4) and Theorem (3.2) in [Mo], when ℓ is a prime we have the following:

$$N_\ell(0, 1) = \begin{cases} \frac{q^{\ell-1} - 1}{q - 1} & \text{if } q \not\equiv 1 \pmod{\ell}; \\ \frac{q^{\ell-1} - 1}{q - 1} + \frac{1}{q} \sum_{x \in \mathbb{F}_{q^\ell}} \psi_0(\text{Tr}_{\mathbb{F}_{q^\ell}/\mathbb{F}_q}(x^\ell)) & \text{if } q \equiv 1 \pmod{\ell}. \end{cases}$$

Case (i): $q \not\equiv 1 \pmod{\ell}$. By using elementary properties of finite group $\mathbb{Z}_\ell$, we can show that $\frac{q^{\ell-1} - 1}{q - 1} \equiv 0 \pmod{\ell}$.

Case (ii): $q \equiv 1 \pmod{\ell}$. We note that $\frac{q^{\ell-1} - 1}{q - 1} \equiv -1 \pmod{\ell}$. To complete the proof, we prove that $\sum_{x \in \mathbb{F}_{q^\ell}} \psi_0(\text{Tr}_{\mathbb{F}_{q^\ell}/\mathbb{F}_q}(x^\ell)) \equiv 1 \pmod{\ell}$. Let A be the subgroup of $\mathbb{F}_q^*$ consisting of ℓ th roots of unity and let $\mathbb{F}_{q^\ell}^* = \bigcup_i x_i A$ be the left coset decomposition of A in $\mathbb{F}_{q^\ell}^*$. Then

$$\begin{aligned}
\sum_{x \in \mathbb{F}_{q^\ell}} \psi_0(\text{Tr}_{\mathbb{F}_{q^\ell}/\mathbb{F}_q}(x^\ell)) &= 1 + \sum_{x \in \mathbb{F}_{q^\ell}^*} \psi_0(\text{Tr}_{\mathbb{F}_{q^\ell}/\mathbb{F}_q}(x^\ell)) \\
&= 1 + \sum_i \sum_{t \in A} \psi_0(\text{Tr}_{\mathbb{F}_{q^\ell}/\mathbb{F}_q}((x_i t)^\ell)) \\
&= 1 + \sum_i \sum_{t \in A} \psi_0(\text{Tr}_{\mathbb{F}_{q^\ell}/\mathbb{F}_q}(x_i^\ell)) \\
&= 1 + \ell \sum_i \psi_0(\text{Tr}_{\mathbb{F}_{q^\ell}/\mathbb{F}_q}(x_i^\ell)) \\
&\equiv 1 \pmod{\ell}.
\end{aligned}$$

$$[\because \sum_i \psi_0(\text{Tr}_{\mathbb{F}_{q^\ell}/\mathbb{F}_q}(x_i^\ell)) \in \mathbb{Z}]$$

Thus, we have

$$N_\ell(0, 1) \equiv 0 \pmod{\ell}.$$

□

5 Multiplicity Theorem

5.1 Multiplicity Theorem

We are now ready to state and prove our multiplicity theorems. For a subfield K embedded in D , for the trivial representation of K^* to occur in an irreducible representation π of D^* it is necessary that the central character of π should be trivial. If π is parametrized by the admissible pair (E, θ) then the central character is $\theta|_{F^*}$. Hence we assume from now on that $\theta|_{F^*}$ is trivial. By Lemma 2.1.4 above (ℓ being an odd prime) we know that θ is generic. In this section we assume that the conductor of the generic pair (E, θ) is $m + 1$. Recall also that our representations of D^* are representations of a finite quotient of D^* . For a finite group G and for complex valued functions f, g on G we define

$$\langle f, g \rangle_G = \frac{1}{|G|} \sum_{x \in G} f(x) \overline{g(x)}.$$

The following theorem give the dimension of zero weight space of the irreducible representations attached to unramified tori.

Theorem 5.1.1. *Let E/F be an unramified extension of degree ℓ and π be the irreducible representation of D^* attached to the generic pair (E, θ) . Assume that the extension K/F is degree ℓ embedded in D . Then*

$$d(\pi, K) = \begin{cases} 0 & \text{if } K/F \text{ is an unramified extension} \\ q^{\frac{1}{2}(\ell-1)(\ell-2)m} & \text{if } K/F \text{ is a totally ramified extension.} \end{cases}$$

Proof. Note that dimension of π is $\ell q^{\frac{1}{2}\ell(\ell-1)m}$ (see §2.2). The proof divided into two cases.

Case (I): Suppose that K/F is unramified. Without loss of generality, we may assume that $K = E$. We may consider π as a representation of $D^*/F^*U_D^{\ell m+1}$ where $m+1$ is the conductor of θ . Since $E^* \cap F^*U_D^{\ell m+1} = F^*U_E^{m+1}$ (see Lemma 3.2.3). By the definition of $d(\pi, E)$ we have :

$$\begin{aligned} d(\pi, E) &= \langle \Theta_\pi|_{E^*}, \mathbb{1}_{E^*} \rangle_{E^*/F^*U_E^{m+1}} \\ &= \frac{1}{|E^*/F^*U_E^{m+1}|} \sum_{x \in E^*/F^*U_E^{m+1}} \Theta_\pi(x) \\ &= \frac{1}{|E^*/F^*U_E^{m+1}|} \left[\sum_{j=0}^{m-1} q^{\frac{1}{2}\ell(\ell-1)j} \sum_{x \in C_j^*} \sum_{\sigma \in \text{Gal}(E/F)} \theta(\sigma(x)) \right. \\ &\quad \left. + q^{\frac{1}{2}\ell(\ell-1)m} \sum_{x \in C_m} \sum_{\sigma \in \text{Gal}(E/F)} \theta(\sigma(x)) \right] \end{aligned}$$

We note that for $0 \leq j \leq m$, C_j 's are finite groups (see Theorem (3.1.1)). Thus

$$\sum_{x \in C_j} \theta(\sigma(x)) = 0$$

and hence

$$\sum_{x \in C_j^*} \theta(\sigma(x)) = 0.$$

Thus we have,

$$\langle \Theta_\pi|_{E^*}, \mathbb{1}_{E^*} \rangle_{E^*} = 0.$$

Case (II): Suppose that K/F is totally ramified extension. since $\pi|_{K^*}$ is trivial on

$$K^* \cap F^*U_D^{\ell m+1} = F^*U_K^{\ell m+1} \text{ (see Lemma 3.2.3).}$$

Then we have

$$\begin{aligned} d(\pi, K) &= \langle \Theta_\pi|_{K^*}, \mathbb{1}_{K^*} \rangle_{K^*/F^*U_K^{\ell m+1}} \\ &= \frac{1}{|K^*/F^*U_K^{\ell m+1}|} \sum_{x \in K^*/F^*U_K^{\ell m+1}} \Theta_\pi(x) \\ &= \frac{1}{\ell q^{(\ell-1)m}} \Theta_\pi(1) \text{ (see Lemma 3.2.4 & Theorem 3.1.2)} \\ &= \frac{1}{\ell q^{(\ell-1)m}} \dim \pi \end{aligned}$$

$$= q^{\frac{1}{2}(\ell-1)(\ell-2)m}.$$

□

We next look at representations π of D^* (with trivial central character) parametrized by the generic pair (E, θ) where E/F is totally ramified extension of degree ℓ .

The following proposition give the dimension of zero weight space of irreducible representation attached to totally ramified tori when it restricted to unramified tori.

Proposition 5.1.2. *Let E/F be a totally ramified extension of degree ℓ and π be a representation of D^* attached to the generic pair (E, θ) . Let K/F be an unramified extension of degree ℓ embedded in D . Then*

$$d(\pi, K) = q^{(\ell-1)(\frac{m-1}{2} - \lfloor \frac{m-1}{\ell} \rfloor)}.$$

Proof. We note that dimension of π is $\frac{q^\ell - 1}{q - 1} q^{\frac{1}{2}(\ell-1)(m-1)}$ and $K^* \cap F^* U_D^{m+1} = F^* U_K^{\lfloor \frac{m}{\ell} \rfloor + 1}$ for $\ell \nmid m$ (see Lemma 3.2.3). Hence we consider $\pi|_{K^*}$ as a representation of the group $K^*/F^* U_K^{\lfloor \frac{m}{\ell} \rfloor + 1}$. From the definition of $d(\pi, K)$, we have

$$\begin{aligned} d(\pi, K) &= \langle \Theta_\pi|_{K^*}, \mathbb{1}_{K^*} \rangle_{K^*/F^* U_K^{\lfloor \frac{m}{\ell} \rfloor + 1}} \\ &= \frac{1}{|K^*/F^* U_K^{\lfloor \frac{m}{\ell} \rfloor + 1}|} \sum_{x \in K^*/F^* U_K^{\lfloor \frac{m}{\ell} \rfloor + 1}} \Theta_\pi(x). \end{aligned}$$

By using Theorem 3.2.2 (case 2) and Lemma 3.2.4, we have

$$\begin{aligned} d(\pi, K) &= \frac{1}{\frac{q^\ell - 1}{q - 1} q^{(\ell-1)\lfloor \frac{m}{\ell} \rfloor}} \Theta_\pi(1) \\ &= \frac{\dim \pi}{\frac{q^\ell - 1}{q - 1} q^{(\ell-1)\lfloor \frac{m}{\ell} \rfloor}} \\ &= q^{(\ell-1)(\frac{m-1}{2} - \lfloor \frac{m-1}{\ell} \rfloor)}. \end{aligned}$$

□

Suppose $\ell \mid (q - 1)$ so that E/F is a Galois extension. Let E_1 be another totally ramified extension of degree ℓ in D such that $E_1 \not\cong E$. Recall that we have a primitive $(q^\ell - 1)$ st root ζ of unity and a prime element ϖ such that $\varpi \zeta \varpi^{-1} = \zeta^q$. Without loss of generality, we may assume that $E = F(\varpi)$ and $E_1 = F(\zeta^i \varpi)$ for some i . The element $\varpi_{E_1} = \zeta^i \varpi$ is a prime element of E_1 . Furthermore $\mathbb{F}$ consists of the intersection with F of the cyclic group $\langle \zeta \rangle$ generated by ζ together with 0.

Then the following two lemmas are useful.

Lemma 5.1.3. *Let E, π be as in the previous proposition and the extension K/F is totally ramified of degree ℓ not isomorphic to E . Then*

$$\sum_{u \in U_F^1 U_K^m / U_F^1 U_K^{m+1}} \Theta_\pi(u) = N_\ell(0, 1) q^{\frac{1}{2}(\ell-1)(m-1)+1}.$$

Proof. Let $\varpi_K = \zeta_1 \varpi$ be a prime element in K for some $(q^\ell - 1)$ st roots of unity ζ_1 and write $u = 1 + \alpha \varpi_K^m = 1 + \alpha \zeta_1^{1-q^m} \varpi^m$, where $\alpha \in \mathbb{F}$. Then

$$\begin{aligned} \sum_{u \in U_F^1 U_K^m / U_F^1 U_K^{m+1}} \Theta_\pi(u) &= \Theta_\pi(1) + \sum_{1 \neq u \in U_F^1 U_K^m / U_F^1 U_K^{m+1}} \Theta_\pi(u) \\ &= \frac{q^\ell - 1}{q - 1} q^{\frac{1}{2}(m-1)(\ell-1)} + q^{\frac{1}{2}(m-1)(\ell-1)} \sum_{\alpha \in \mathbb{F}^*} \sum_{\delta \in \ker N_{\mathbb{D}/\mathbb{F}}} \psi_0(\alpha \text{Tr}_{\mathbb{D}/\mathbb{F}}(\zeta_1^{1-q^m} \delta)) \\ &\quad \text{(case 4 of Theorem 3.2.2)} \\ &= q^{\frac{1}{2}(m-1)(\ell-1)} \left[\frac{q^\ell - 1}{q - 1} + \sum_{\alpha \in \mathbb{F}^*} \sum_{\delta' \in \ker N_{\mathbb{D}/\mathbb{F}}} \psi_0(\alpha \text{Tr}_{\mathbb{D}/\mathbb{F}}(\delta')) \right] \\ &= q^{\frac{1}{2}(m-1)(\ell-1)} \left[\frac{q^\ell - 1}{q - 1} + \sum_{\alpha \in \mathbb{F}^*} \left(\sum_{\substack{N_{\mathbb{D}/\mathbb{F}}(\delta')=1 \\ \text{Tr}_{\mathbb{D}/\mathbb{F}}(\delta')=0}} \psi_0(\alpha \text{Tr}_{\mathbb{D}/\mathbb{F}}(\delta')) + \sum_{\substack{N_{\mathbb{D}/\mathbb{F}}(\delta')=1 \\ \text{Tr}_{\mathbb{D}/\mathbb{F}}(\delta') \neq 0}} \psi_0(\alpha \text{Tr}_{\mathbb{D}/\mathbb{F}}(\delta')) \right) \right] \\ &= q^{\frac{1}{2}(\ell-1)(m-1)} \left\{ \frac{q^\ell - 1}{q - 1} + (q - 1) N_\ell(0, 1) + (-1) \left[\frac{q^\ell - 1}{q - 1} - N_\ell(0, 1) \right] \right\} \\ &= N_\ell(0, 1) q^{\frac{1}{2}(\ell-1)(m-1)+1}. \quad \square \end{aligned}$$

Lemma 5.1.4. *Let E, π be as in the previous proposition. Then*

$$\sum_{x \in U_E^1 / U_F^1 U_E^{m+1}} \Theta_\pi(x) = N_\ell(0, 1) q^{\frac{1}{2}(\ell-1)(m-1)+1}.$$

Proof. For $1 \leq j \leq m$, write

$$C_j = U_F^1 U_E^j / U_F^1 U_E^{m+1}$$

and

$$C_j^* = U_F^1 U_E^j / U_F^1 U_E^{m+1} - U_F^1 U_E^{j+1} / U_F^1 U_E^{m+1}.$$

Then

$$\begin{aligned}
\sum_{x \in U_E^1/U_F^1 U_E^{m+1}} \Theta_\pi(x) &= \sum_{j=1}^{m+1} \left[\sum_{\substack{x \in U_E^1/U_F^1 U_E^{m+1} \\ \nu_0(x)=j}} \Theta_\pi(x) \right] \\
&= \sum_{j=1}^{m-1} q^{\frac{1}{2}(\ell-1)j} \sum_{x \in C_j^*} \sum_{\sigma \in \text{Gal}(E/F)} G(\sigma(x), \theta) \theta(\sigma(x)) \\
&\quad + q^{\frac{1}{2}(\ell-1)(m-1)} \sum_{x \in C_m^*} \sum_{\delta \in \ker N_{\mathbb{D}/\mathbb{F}}} \psi_0(\alpha_m \text{Tr}_{\mathbb{D}/\mathbb{F}}(\delta)) + \Theta_\pi(1).
\end{aligned}$$

Using case 3 of Theorem 3.2.2, we simplify the right hand side terms of above equation.

If m and j have opposite parities then $G(\sigma(x), \theta) = 1$. Then we have

$$\sum_{x \in C_j^*} \sum_{\sigma \in \text{Gal}(K/k)} \theta(\sigma(x)) = 0 \quad \text{for } 1 \leq j < m$$

(since $\sum_{x \in C_j} \theta(\sigma(x)) = 0$ for $\sigma \in \text{Gal}(E/F)$ and hence $\sum_{x \in C_j^*} \theta(\sigma(x)) = 0$).

If m and j have the same parities, then

$$G(\sigma(x), \theta) = \sum_{\gamma \in T} \psi_0(\alpha_j \text{Tr}_{\mathbb{D}/\mathbb{F}}(\gamma^{\sigma^{\frac{m+j}{2}}} (\gamma^{\sigma^j} - \gamma))).$$

For $1 \leq j < m$,

$$\begin{aligned}
&\sum_{x \in C_j^*} \sum_{\sigma \in \text{Gal}(E/F)} \sum_{\gamma \in T} \psi_0(\alpha_j \text{Tr}_{\mathbb{D}/\mathbb{F}}(\gamma^{\sigma^{i+j}} (\gamma^{\sigma^j} - \gamma))) \theta(\sigma(x)) \\
&= \sum_{\sigma \in \text{Gal}(E/F)} \sum_{\gamma \in T} \sum_{\substack{(\alpha_j, \alpha_{j+1}, \dots, \alpha_m) \in \mathbb{F}^{m-j} \\ \alpha_j \neq 0}} \psi_0(\alpha_j \text{Tr}_{\mathbb{D}/\mathbb{F}}(\gamma^{\sigma^{\frac{m+j}{2}}} (\gamma^{\sigma^j} - \gamma))) \theta(\sigma(1 + \sum_j \alpha_j \varpi^j)) \\
&= \sum_{\substack{\sigma \in \text{Gal}(E/F) \\ \alpha_j \in \mathbb{F} - \{0\}, \gamma \in T}} \left[\psi_0(\alpha_j \text{Tr}_{\mathbb{D}/\mathbb{F}}(\gamma^{\sigma^{\frac{m+j}{2}}} (\gamma^{\sigma^j} - \gamma))) \theta(\sigma(1 + \alpha_j \varpi^j)) \right. \\
&\quad \times \left. \sum_{(\alpha'_{j+1}, \dots, \alpha'_m) \in \mathbb{F}^{m-j}} \theta(\sigma(1 + \sum_{i=j+1}^m \alpha'_i \varpi^i)) \right]
\end{aligned}$$

$= 0$ ($\because$ the later sum is 0).

Thus

$$\sum_{\substack{x \in U_E^1/U_F^1 U_E^{m+1} \\ \nu_0(x)=j}} \Theta_\pi(x) = 0 \quad \text{for } 1 \leq j < m.$$

Therefore, we have

$$\begin{aligned} \sum_{x \in U_E^1 / U_F^1 U_E^{m+1}} \Theta_\pi(x) &= \sum_{x \in U_F^1 U_E^m / U_F^1 U_E^{m+1}} \Theta_\pi(x) \\ &= N_\ell(0, 1) q^{\frac{1}{2}(\ell-1)(m-1)+1} \text{ (see Lemma 5.1.3).} \end{aligned} \quad \square$$

Now the following proposition give the dimension of zero weight space in totally ramified case.

Proposition 5.1.5. *Let E, π be as in the last proposition and K/F be the totally ramified extension of degree ℓ . Then*

$$d(\pi, K) = \frac{N_\ell(0, 1)}{\ell} q^{\frac{1}{2}(m-1)(\ell-3)+[\frac{m-1}{\ell}]}.$$

Proof. Case (I): Assume that $K \not\cong E$. Since $\pi|_{K^*}$ is a representation of $K^*/F^*U_K^{m+1}$ and $F^*U_D^{m+1} \cap K^* = F^*U_K^{m+1}$. Then by the definition of $d(\pi, K)$, we have the following:

$$\begin{aligned} d(\pi, K) &= \langle \Theta_\pi|_{K^*}, \mathbb{1}_{K^*} \rangle_{K^*/F^*U_K^{m+1}} \\ &= \frac{1}{|K^*/F^*U_K^{m+1}|} \sum_{x \in K^*/F^*U_K^{m+1}} \Theta_\pi(x) \\ &= \frac{1}{\ell q^{m-[\frac{m}{\ell}]}} \left(\sum_{u \in U_F^1 U_K^m / U_F^1 U_K^{m+1}} \Theta_\pi(u) \right) \quad (\because \Theta_\pi(x) = 0 \text{ for } \nu_0(x) < m, \\ &\quad \text{Th. 3.2.2.(2)}) \\ &= \frac{1}{\ell q^{m-[\frac{m}{\ell}]}} N_\ell(0, 1) q^{\frac{1}{2}(\ell-1)(m-1)+1} \text{ (see Lemma 5.1.3)} \\ &= \frac{N_\ell(0, 1)}{\ell} q^{\frac{1}{2}(\ell-3)(m-1)+[\frac{m-1}{\ell}]}. \end{aligned}$$

Case (II): Assume that $K \cong E$. Without loss of generality, we may assume that $E = K$.

Again by the definition of $d(\pi, K)$, we have

$$\begin{aligned} d(\pi, K) &= \langle \Theta_\pi|_{K^*}, \mathbb{1}_{K^*} \rangle_{K^*/F^*U_K^{m+1}} \\ &= \frac{1}{|K^*/F^*U_K^{m+1}|} \sum_{x \in K^*/F^*U_K^{m+1}} \Theta_\pi(x) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\ell q^{m-\lceil \frac{m}{\ell} \rceil}} \sum_{i=0}^{\ell-1} \sum_{u \in U_K^1/U_F^1 U_K^{m+1}} \Theta_\pi(\varpi^i u) \\
&= \frac{1}{\ell q^{m-\lceil \frac{m}{\ell} \rceil}} \left(\sum_{u \in U_K^1/U_F^1 U_K^{m+1}} \Theta_\pi(u) + \sum_{i=1}^{\ell-1} \sum_{u \in U_K^1/U_F^1 U_K^{m+1}} \Theta_\pi(\varpi^i u) \right).
\end{aligned}$$

Since $\nu_0(\varpi^i u) = 0$ for $1 \leq i < \ell$, by case 3 of Theorem 3.2.2, we have

$$\Theta_\pi(\varpi^i u) = \left(\frac{q}{\ell}\right) \theta(\varpi^i u).$$

Then

$$\begin{aligned}
d(\pi, K) &= \frac{1}{\ell q^{m-\lceil \frac{m}{\ell} \rceil}} \left(\sum_{u \in U_K^1/U_F^1 U_K^{m+1}} \Theta_\pi(u) + \sum_{i=1}^{\ell-1} \sum_{u \in U_K^1/U_F^1 U_K^{m+1}} \left(\frac{q}{\ell}\right) \theta(\varpi^i u) \right) \\
&= \frac{1}{\ell q^{m-\lceil \frac{m}{\ell} \rceil}} \left(\sum_{u \in U_K^1/U_F^1 U_K^{m+1}} \Theta_\pi(u) + \sum_{i=1}^{\ell-1} \left(\frac{q}{\ell}\right) \theta(\varpi^i) \sum_{u \in U_K^1/U_F^1 U_K^{m+1}} \theta(u) \right).
\end{aligned}$$

Note that θ is a non trivial character of the group $U_K^1/U_F^1 U_K^{m+1}$, and hence

$$\sum_{u \in U_K^1/U_F^1 U_K^{m+1}} \theta(u) = 0.$$

Thus in view of Lemma 5.1.4, we have

$$\begin{aligned}
d(\pi, K) &= \frac{1}{\ell q^{m-\lceil \frac{m}{\ell} \rceil}} \sum_{u \in U_K^1/U_F^1 U_K^{m+1}} \Theta_\pi(u) \\
&= \frac{1}{\ell q^{m-\lceil \frac{m}{\ell} \rceil}} \left(q^{\frac{1}{2}(\ell-1)(m-1)+1} N_\ell(0, 1) \right) \\
&= \frac{N_\ell(0, 1)}{\ell} q^{\frac{1}{2}(m-1)(\ell-3)+\lceil \frac{m}{\ell} \rceil}.
\end{aligned}$$

Remark 5.1.6. If $\ell \nmid q-1$, we have $N_\ell(0, 1) = \frac{q^{\ell-1}-1}{q-1}$ (see Lemma 4.1.6)

then

$$d(\pi, K) = \frac{q^{\ell-1}-1}{\ell(q-1)} q^{\frac{1}{2}(m-1)(\ell-3)+\lceil \frac{m}{\ell} \rceil}.$$

□

Thus in view of Propositions 5.1.2 and 5.1.5, we proved the following theorem.

Theorem 5.1.7. *Let E/F be a totally ramified extension of degree ℓ and (E, θ) be the admissible pair of conductor $m + 1$ with trivial central character. Let π be the irreducible representation of D^* attached to the pair (E, θ) . If K/F be an extension of degree ℓ embedded in D . Then*

$$d(\pi, K) = \begin{cases} q^{(\ell-1)(\frac{m-1}{2} - [\frac{m-1}{\ell}])} & \text{if } K/F \text{ is unramified;} \\ \frac{N_\ell(0, 1)}{\ell} q^{\frac{1}{2}(m-1)(\ell-3) + [\frac{m-1}{\ell}]} & \text{if } K/F \text{ are totally ramified.} \end{cases}$$

Theorem (5.1.1) and theorem (5.1.7) together give the proof of our main theorem.

Let K/F be any field extension of degree ℓ , we denote $K(n) = K^* \cap U_D^n$ for $n \geq 0$. We express our theorem in terms of the dimension of π (see §2.2).

Theorem 5.1.8. *Let D be a division algebra of odd prime index ℓ over the non-archimedean local field F with residue characteristic p , $\ell \neq p$ and π be an irreducible representation of D^* attached to the admissible pair (E, θ) of conductor $m + 1$ with central character $\omega_\pi = \mathbb{1}_{F^*}$. If K/F be any extension of degree ℓ in D . Then*

$$d(\pi, K) = \begin{cases} 0 & \text{if } E = K \text{ and } K/F \text{ is} \\ & \text{unramified;} \\ \frac{\dim \pi}{|K^*/F^*K(\ell m + 1)|} & \text{if } E/F \text{ unramified and} \\ & K/F \text{ totally ramified;} \\ \frac{\dim \pi}{|K^*/F^*K(m + 1)|} & \text{if } E/F \text{ totally ramified and} \\ & K/F \text{ unramified;} \\ \frac{q(q-1)N_\ell(0, 1)}{q^\ell - 1} \cdot \frac{\dim \pi}{|K^*/F^*K(m + 1)|} & \text{if both } E/F, K/F \text{ are} \\ & \text{totally ramified.} \end{cases}$$

Note that in the cases above $F^*K(n)$ is the largest subgroup of K^* on which π is trivial.

In conclusion we remark that in view of Lemma 4.1.5 our formulae agree entirely with those of Prasad [P2] in case D is a tame quaternion division algebra over F .

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