Effective Dimensions and Dimensional Flow in κ-Spacetime

Thesis submitted for the degree of
Doctor of Philosophy in Physics

by

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Declaration

I hereby declare that, this thesis titled **Effective Dimensions and Dimensional Flow in κ-Spacetime** submitted by me, under the guidance and supervision of Prof. E. Harikumar, is a bona fide research work and is free from plagiarism. I also declare that it has not been submitted previously, in part or in full to this University or any other University or Institution, for the award of any degree or diploma. I hereby agree that my thesis can be deposited in Shodhganga/INFLIBNET.

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(1) Anjana V. and E. Harikumar, Spectral dimension of kappa-deformed spacetime, Phys. Rev. D **91** 065026 (2015), arXiv:1501.00254 [hep-th].

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Further, the student has passed the following courses towards fulfillment of coursework requirement for Ph.D:

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Abstract

The concept of dimension is a fundamental property of space-time. The study of structure of space-time at quantum gravity level suggest that our usual notion of dimension, which is the concept of topological dimension, do not remain valid at the Planck scale. In fact, dimensional flow is an intriguing feature of most of the quantum gravity models. One of the natural way to capture the structure of space-time at Planck scale is by using the idea of non-commutativity.

In this thesis, we study the dimensional flow for a particular example of non-commutative space-time, known as κ-deformed space-time. The investigation of dimension flow is carried out in κ-space-time by considering two distinct definitions of effective dimension, namely the spectral dimension and Hausdorff dimension.

Contents

CHAPTER 1

Introduction

A problem of foremost interest in physics is to understand quantum theory of gravity. Newton's law of gravitation was successful in explaining planitary motion and many other known features of gravitational force. This theory, which works well in large length scales has been found to be correct even upto length of 0.4mm [1]. At large length scale, deviation from Newton's law of gravity, such as perihelion of Mercury were known for a long time and those issues were explained by general theory of relativity. But the nature of this weakest force among four fundamental force is yet be understood at microscopic scale, unlike other three fundamental force of nature.

General relativity describes the structure of space-time at macroscopic level. It presents a continum space-time on which all dynamics of matter occur. In contrast, quantum mechanics describes the behaviour of physical system at very small length scales. These two theories are conceptually and structurally very different. A theory which reconciles general relativity with quantum mechanics is known as quantum theory of gravity and attempts to construct such a consistent quantum theory of gravity, is an active area of research in theoretical physics. There have been many attempts to study different aspects of quantum gravity such as string theory [2], loop quantum gravity [3, 4], non-commutative geometry [5, 6], causal dynamical triangulations [7], Horova-Lifschitz gravity [8], asymptotically safte gravity [9, 10], etc.

There are many interesting reasons for our quest for a quantum theory of gravity. First, grand unified theory (GUT) requires a theory of quantum gravity. As is well known, we have four fundamental forces of nature. Of this, all four but gravity have been combined with quantum mechanics to form a quantum description of these elementary forces. But if we need to combine all the four forces under same platform, we need to understand how gravity behaves at microscopic scales. Another reason for the need to formulate a quantum gravity theory is the prediction of singularities in general relativity [11]. Einstein's theory is a classical theory and hence, we require to incorporate principles of quantum mechanics into the description of gravity to have a better grasp of black hole solutions. In a similar manner, the singularity present at the big bang appears to be an artifact of classical description. In fact, when time scale becomes smaller than $10^{-42}s$, our universe was incredibly small and extremely dense. Such a situation will have quantum effects in the presence of intense gravitational force. Thus, we have to conclude that for a proper physical description of nature, we have to look for a quantum version of gravity.

Einstein's equation tells us the connection between space-time geometry and gravity. Thus in order to understand gravity at short distance (or very high energy scales), we need to understand the structure of space-time at short distance. Now, to probe the space-time at such short length scales, we require the probe to have a comparable wavelength. To be precise, for looking at length scales where quantum gravity effects are dominant, we should have probes of wavelength of the order of Planck scale. Then, from Heisenberg's uncertainty principle we deduce that the probe's momentum and thus the probe's energy shoot up rapidy and ultimately event horizons would be formed. This imples the presence of minimum length beyond which we can not probe the structure of the space-time [12]. It is an established fact that any attempts to have quantum theory of gravity implies the existence of minimal length, which is usually expected to be of the order of Planck length. This has been predicted in various approach to quantum gravity such as string theory [13, 14], loop quantum gravity [15], non-commutative space-time [16], etc.

1.1 Non-commutative space-time

Non-commutative geometry provides a possible way to capture the space-time structure at microscopic level [12, 5]. It is an interesting formulation of geometry which naturally provides place for a fundamental length scale. The concept of non-commutativity was introduced by H. S. Snyder in 1940s to deal with ultraviolet divergence in quantum field theory [17] based on a proposal by Heisenberg. In [17], he proposed a model of discrete space-time in which space-time coordinates are invariant under Lorentz transformation just as in the Minkowski space-time. There are different models of non-commutative space-times discussed in literature. One of them is Moyal space-time [12] whose coordinates satisfy

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \tag{1.1}$$

where $\theta_{\mu\nu}$ are real and constant anti-symmetric tensor, having a dimension of L^2.

Another space-time that has been studied with interest is the κ-deformed space-time. The coordinates of kappa-deformed space-time obey the commutation relation given by

$$[\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu). \tag{1.2}$$

In this thesis, we discuss certain aspects of the κ-space-time. The significance of this space-time comes from the fact that it appears naturally in the low energy limit of loop quantum gravity [18] as well as in the context of doubly special relativity (DSR) theories [19]. Doubly special relativity is a modification of special theory of relativity which incorporates two dimensionfull parameters, which are speed of light, c and another fundamental constant having length dimension. According to special theory of relativity, such a minimum length can not be a fundamental quantity. The introduction of constant having length dimension, apart from the speed of light, necessitates the modification of the relativity principle. DSR is such a relativity principle that introduces a fundamental minimum length scale without singling out any preferred frame.

1.2 Kappa Euclidean space and its realizations

Until now, we have discussed various motivations for studying non-commutative space-time, in particular κ-space-time. In this section, we will summarise various details about the formalism of κ-space-time. This will provide the necessary background for the discussions in the remaining chapters and also set our notations. For working with a non-commutative space-time one may use non-commutative variables and the corresponding mathematical structures such as differential calculus on non-commutative space-time. Alternatively, it is possible to establish a map from non-commutative variables to commutative variables and using this, commutative models equivalent to non-commutative models can be constructed and studied. In doing this, one works with ordinary calculus. In the works reported in this thesis, will be following the latter approach. We adopt the approach developed in [20] and use the realizations of coordinate of κ-space-time construced there. Below we summarise this approach where we closely follow the discussions of [20].

The n-dimensional kappa-deformed space is an example of non-commutative space-time, where the coordinates obey Lie algebra type commutation relations, i.e.,

$$\begin{aligned} [\hat{x}_\mu, \hat{x}_\nu] &= iC_{\mu\nu\lambda}\hat{x}_\lambda \\ &= i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu), \end{aligned} \tag{1.3}$$

where a_μ's are constant, real parameters, which describe the deformation of Euclidean space. The structure constant is related to a_μ through

$$C_{\mu\nu\lambda} = a_\mu \delta_{\nu\lambda} - a_\nu \delta_{\mu\lambda}. \tag{1.4}$$

We choose $a_1 = a_2 = ...a_{n-1} = 0$ and $a_n = a$. Thus the algebra of the non-commutative coordinates becomes

$$[\hat{x}_i, \hat{x}_j] = 0, \qquad [\hat{x}_n, \hat{x}_i] = ia\hat{x}_i, \qquad i, j = 1, 2, ..., n-1. \tag{1.5}$$

Note that the Euclidean time coordinate $\hat{x}_n$ do not commute with corresponding space-coordinate $\hat{x}_i$, where as space coordinates commute among themselves.

It is possible to find a realization of the non-commutative coordinates $\hat{x}_\mu$ in terms of ordinary commutative coordinates x_μ and their derivatives ∂_μ [20]. One such realization which satisfy eqn.(1.5) is

$$\begin{aligned} \hat{x}_i &= x_i \varphi(A), \\ \hat{x}_n &= x_n \psi(A) + iax_i \partial_i \gamma(A), \end{aligned} \tag{1.6}$$

where $A = ia\partial_n$. The commutation relations between the non-commutative coordinates, given in eqn.(1.5) imply the relation

$$\frac{\varphi'(A)}{\varphi(A)}\psi(A) = \gamma(A) - 1 \tag{1.7}$$

where $\varphi' = \frac{d\varphi}{dA}$. The undeformed space appears as a smooth limit when deformation parameter a goes to zero. This gives condition on φ, ψ and γ as

$$\varphi(0) = 1, \qquad \psi(0) = 1$$

and

$$\gamma(0) = \varphi'(0) + 1. \tag{1.8}$$

In the study of physics on κ-space-time, two distinct approaches have been widely used. One is to work with deformed κ-Poincare algebra [21, 22, 23] and other is to work with undeformed κ-Poincare algebra [20]. In the first case, the defining relations of the algebra are modified. But in the second approach, one modifies the form of genetators of the algebra by keeping the defining relations exactly same as the usual Poincare algebra. To distinguish this symmetry algebra from the deformed Poincare algebra, it is called as undeformed κ-Poincare algebra [20, 24, 25].

In this thesis we will be following the second approach. In this approach, for the kappa-deformed Euclidean space, the symmetry algebra is the undeformed $SO_a(n)$ algebra. The modified generators $M_{\mu\nu}$ of this algebra satisfy the relation

$$[M_{\mu\nu}, M_{\lambda\rho}] \;=\; \delta_{\nu\lambda}M_{\mu\rho} - \delta_{\mu\lambda}M_{\nu\rho} - \delta_{\nu\rho}M_{\mu\lambda} + \delta_{\mu\rho}M_{\nu\lambda}. \tag{1.9}$$

We demanded that the extended Lie algebra is linear in the generators, i.e.,

$$[M_{\mu\nu}, \hat{x}_\lambda] = \hat{x}_\mu\delta_{\nu\lambda} - \hat{x}_\nu\delta_{\mu\lambda} - ia_\mu M_{\nu\lambda} + ia_\nu M_{\mu\lambda}. \tag{1.10}$$

The consistency with the Jacobi identities

$$[M_{\alpha\beta}, [\hat{x_\mu}, \hat{x_\nu}]] + [\hat{x_\mu}, [\hat{x_\nu}, M_{\alpha\beta}]] + [\hat{x_\nu}, [M_{\alpha\beta}, \hat{x_\mu}]] = 0,$$

$$[M_{\alpha\beta}, [M_{\gamma\delta}, \hat{x_\mu}] + [M_{\gamma\delta}, [\hat{x_\mu}, M_{\alpha\beta}]] + [\hat{x_\mu}, [M_{\alpha\beta}, M_{\gamma\delta}]] = 0, \tag{1.11}$$

along with the the requirement that generators $M_{\mu\nu}$ to be linear in x_μ and ∂_μ, set the explicit form of $M_{\mu\nu}$ to be

$$\begin{aligned} M_{ij} &= x_i\partial_j - x_j\partial_i, \\ M_{in} &= x_i\partial_n\varphi\frac{e^{2A}-1}{2A} - x_n\partial_i\frac{1}{\varphi} + iax_i\nabla^2\frac{1}{2\varphi} - iax_k\partial_k\partial_i(\frac{\varphi'}{\varphi^2} + \frac{1}{\varphi}). \end{aligned} \tag{1.12}$$

It is easy to see that the usual partial derivative do not transform like a vector under this $M_{\mu\nu}$. This necessiates the introduction of a new definition of derivative, which have the required property and is known in the literature as Dirac derivatives [20]. In the component form, they are written as

$$D_i = \partial_i\frac{e^{-A}}{\varphi(A)}, \qquad D_n = \partial_n\frac{sinhA}{A} + ia\nabla^2\frac{e^{-A}}{2\varphi^2(A)}, \tag{1.13}$$

where $\nabla^2 = \Sigma_{i=1}^{n-1}\partial_i^2$.

The quadratic Casimir of this algebra, $D_\mu D_\mu$ is given by

$$D_\mu D_\mu = D_i D_i + D_n D_n = \Box(1 - \frac{a^2}{4}\Box), \tag{1.14}$$

where

$$\Box = \nabla^2 \frac{e^{-A}}{\varphi^2(A)} - \partial_n^2 \frac{2[1 - coshA]}{A^2}. \tag{1.15}$$

Different realizations of the undeformed κ-Poincare algebra are obtained by different, allowed choices of $\varphi(A)$ [20, 24, 26, 27]. Different choices for $\varphi(A)$ are equivalent to different choices for the $*$-products on deformed space-time [26, 27, 28]. $e^{-\frac{A}{2}}$, e^{-A}, 1, $\frac{A}{e^A-1}$ are some possible choices for $\varphi(A)$ [20].

We can extend these studies to the case of kappa-deformed Minkowski space-time in a straight forward manner.

Many interesting aspects of kappa-deformed space-time have been studied in recent times [25, 29, 24, 30, 31, 32, 33, 34, 35, 36, 37]. Using the Dirac derivatives, the kappa-deformed Klein-Gordon equation was constructed and studied in [24, 25]. Constuction and analysis of κ-deformed Dirac equation is discussed in [29]. One of the consequences of non-commutativity is that particles obey twisted statistics [38, 25]. In non-commutative space-time, the symmetry is implemented by a twisted Hopf algebra [38] and this leads to twisted statistics in non-commutative space-time.

Another universal feature (apart from the existence of a length scale) of almost all quantum gravity models is dimensional flow, i.e., the effective dimension of the space-time changes with the probe scale [39, 40, 41]. One method to measure the dimensionality of a space is using a diffusion process, which is termed as spectral dimension [42]. The notion of spectral dimension is used to study the space-time dimension of the quantum gravity models. Another measure of effective dimension is Hausdorff dimension, which indicate the dimension of in quantum paths [43]. A detailed summary of the dimensional flow and Hausdorff dimension are given in chapter 2.

1.3 Organization of thesis

This thesis is organized as follows. In chapter 2, We give a brief summary of the notion of effective dimensions such as spectral dimension and Hausdorff dimension. As an illustration of the concept of spectral dimension, we present the calculation of spectral dimension experienced by a point particle in Euclidean space. Then, we review the possible generalizations of diffusion process in different quantum gravity models. More specifically, we present alternative choices of Laplacian for kappa-deformed space-time and analyze the corresponding diffusion equations. After this we introduce Hausdorff dimension as yet another example of effective dimension. Calculation of Hausdorff dimension for a quantum path is summarized. This chapter introduce the conventions and notations used in the remaining chapters.

Spectral dimension for an n-dimensional kappa-deformed Euclidean space is investigated in chapter 3 [44, 45]. In calculating the spectral dimension we have chosen a specific realization of κ-space-time. That is, we have used $\varphi(A) = e^{-\frac{A}{2}}$ in these calculations. We then construct the deformed diffusion equation for two different choices of Laplacians which are expressed in terms of variables of commutative space-time. The obtained diffusion equations are solved perturbatively using delta function initial condition which is valid for a point particle probe. This solution is then used to calculate the spectral dimensions of κ-deformed Euclidean space. It is shown that the spectral dimension decreases with the probe scale (σ). For the limit $\sigma \to \infty$, it is observed that the spectral dimension is same as the topological dimension where as the spectral dimension goes to $-\infty$ for vanishing σ. We have also analyzed the dimensional flow for the case where the probe has a finite extension, unlike a point particle. By demanding the positivity of spectral dimension, we have obtained an upper cut-off for deformation parameter a.

In chapter 4, we analyse the spectral dimension of a Wick-rotated κ-Minkowski space [46]. We begin this chapter by reviewing the relationship between Schrödinger equation and diffusion equation. These two are interconnected by Wick-rotation ($t \to -it$). Next, we derive the deformed diffusion equations using kappa-deformed Schrödinger equations obtained from the non-relativistic limits of different choices of Klein-Gordon equations in the κ-deformed space-time. The spectral dimensions are calculated using these deformed diffusion equations and the dependence on diffusion parameter of spectral dimension is studied. In the limit of commutative space-time, we recover the well known equality of spectral dimension and topological dimension. We showed that the higher derivative terms in the deformed diffusion equations make the spectral dimension unbounded (from below) at high energies. We have also analyzed the effect of finite size of the probe on the spectral dimension.

In chapter 5, we study the dimensional flow of (the Wick-rotated) κ-Minkowski space [47], with two different realizations. In the first case, we used a realization $\varphi(A) = e^{-A}$, which is related to bi-cross product basis of κ-space-time. Here, the diffusion equation in the kappa-deformed space-time is obtained by using the corresponding Beltrami-Laplace operator. The perturbatively obtained heat kernel is then used to calculate the spectral dimension. Our results indicate that the spectral dimension varies with the scale of the diffusion and also depend on an integer l. The analysis also shows that, the spectral dimension diverges to $+\infty$ for $l \geq 0$ and to $-\infty$ for $l < 0$ at high energies. We have also used the (α, β, γ) realization to understand about spectral dimension. In this realization, we first constructed Klein-Gordon operator and then it was used to obtain the corresponding diffusion equation. The solution of this equation was then used to analyze the spectral dimension.

Hausdorff dimension of the path of a quantum particle in non-commutative space-time is discussed in chapter 6 [48]. By using the definition of Hausdorff length, we calculate the Hausdorff dimension for non-relativistic and relativistic quantum particle in kappa-space-time. In both cases the Hausdorff dimension depend on the deformation parameter a and the resolution Δx of the measuring apparatus. For the relativistic case we showed that non-commutative correction to Dirac equation brings in the spinorial nature of the wave function into play, unlike in the commutative space-time. We have also derived the generalized uncertainty relations, by imposing self-similarity condition on quantum path.

In chapter 7, we summarize our results and discuss possible questions to be taken up in the future.

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CHAPTER 2

Effective dimension of quantum geometry

2.1 Introduction

Many different paradigms have been developed and employed to unravel the structure of space-time at microscopic scale, in order to obtain a consistent quantum mechanical description of gravity. Change in the space-time dimension with probe scale [1] is a characterising behaviour that is common to many approaches towards quantum gravity such as causal dynamical triangulation [1], asymptotically safe gravity [2], loop gravity [3], deformed(or doubly) special relativity [4], non-commutative space-times [5, 6, 7], Horava-Lifschits gravity [8], and relative-locality [9, 10, 11, 12]. All these approaches show that the effective dimensions felt by a fictitious test particle reduces at high energies (i.e., when the length scale probed is very small). This variation of dimensions with the probe scale at high energies suggest the possibility of a reduction of space-time dimension at extremely short distances. This reduction of the effective dimensions known as dimensional flow is of important consequences as gravity is known to be renormalizable in two dimensions.

The usual notion of the dimension of space is the scale by which the volume of an object changes with change in its size. Thus one defines the topological dimension n as

$$n = \lim_{size \to 0} \frac{\log V}{\log size}. \tag{2.1}$$

Although our world looks (3+1) dimensional macroscopically, the description of quantum gravity phenomena may alter the number of dimensions at microscopic scales. Due to quantum effects, the usual notions of space-time points get blurred and space-time becomes fuzzy [13]. A particle undertaking a random walk in such a fuzzy space-time may not be able to access all the dimensions. Thus, the effective dimensions felt by this test particle can be different from the usual topological dimensions of the space-time. In order to study the nature of space-time at extremely short distances one needs to use new definitions for effective dimension. Two such definition which are commonly used are spectral and Hausdorff dimensions. We now give a brief summary of these two notions of dimensions, which are studied in the context of specific non-commutative space-time in this thesis.

2.2 Spectral Dimension

The concept of spectral dimension is based on the idea of diffusion processes. In order to obtain a grasp of this notion of spectral dimension we will first briefly overview diffusion processes. The idea of diffusion process was initially introduced to study the transport of fluids through membranes [14]. In [14], the diffusion equation for the concentration of diffusing species was derived by combining the relation between diffusion flux and gradient of concentration with the continuity equation. This same equation also describes the time evolution of probability distribution of a position of particle ($p(x,t)$) and is given by

$$\frac{\partial}{\partial t}p(x,t) = D\frac{\partial^2}{\partial x^2}p(x,t), \tag{2.2}$$

where D is the diffusion coefficient. Transport phenomenon in which the random motion of atoms and molecules are discussed in [15] by A. Einstein where a connection between the properties of the system with the diffusion coefficient were established.

A more general treatment of random motion of particle in presence of viscous force is given by the Langevin equation [16]. This equation is written as

$$m\frac{d^2x}{dt^2} = -\gamma\frac{dx}{dt} + \eta(t), \tag{2.3}$$

where m is the particle's mass, γ is the friction coefficient and $\eta(t)$ is the random force. A different model for Brownian motion is give by Fokker-Planck equation. This treatment takes into account both the drag force and the random force in the Brownian motion.

In our study we consider a diffusion process in the absence of a drag force and thus we restrict our focus to the diffusion process described by eqn.(2.2).

Now we present essential details of the definition of spectral dimension, which is used to analyse the dimensional flow in different approaches to quantum gravity. The basic notion of the spectral dimension is introduced using a diffusion process. For this, a test particle is allowed to diffuse in the space under study. By analysing the motion of this particle from one point to another, one calculates the return probability of the particle to come back to its starting point. The spectral dimension is then defined as the scale by which return probability changes with diffusion time. In other words, spectral dimension corresponds to the dimension that a random walker experiences in a diffusion process [1, 5, 6]. The motion of a particle undergoing diffusion on an n-dimensional Euclidean space is described by the equation

$$\frac{\partial}{\partial\sigma}U(x,y;\sigma) = \mathcal{L}U(x,y;\sigma). \tag{2.4}$$

Here σ is fictitious diffusion time with dimension of length squared (L^2), $\mathcal{L}$ is the Laplace operator defined in the space whose metric is $g_{\mu\nu}$. U(x,y;σ) is the probability density of the test particle to diffuse from the initial position x to another point y, in diffusion time σ. One solves the above equation with initial condition that the particle is localized at a fixed point, i.e.,

$$U(x,y;0) = \frac{\delta^n(x-y)}{\sqrt{\det g_{\mu\nu}}}. \tag{2.5}$$

The spectral dimension of this space is related to the trace of the diffusion probability, known as return probability, which measures the probability to find the particle returning back to the starting point after a diffusion time σ. Thus, using the solution for the diffusion equation, the return probability is defined as

$$P_g(\sigma) = \frac{\int d^nx\sqrt{detg_{\mu\nu}}U(x,x;\sigma)}{\int d^nx\sqrt{detg_{\mu\nu}}}, \tag{2.6}$$

where $g_{\mu\nu}$ is the metric of the corresponding space. Spectral dimension D_s is calculated by taking the logarithmic derivative of return probability $P_g(\sigma)$, i.e.,

$$D_s = -2\frac{\partial \ln P_g(\sigma)}{\partial \ln \sigma}, \tag{2.7}$$

which shows the way return probability scales with diffusion time.

To familiarize the notations, concepts and calculational scheme of spectral dimension, we first discuss the diffusion in a flat space. The diffusion equation associated with an n-dimensional flat space is [1]

$$\frac{\partial}{\partial\sigma}U(x,y;\sigma) = \nabla^2 U(x,y;\sigma), \tag{2.8}$$

where $\nabla^2 = \Sigma_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$ is the usual, well defined Laplacian in n-dimensional Euclidean space. Note the metric here is $g_{\mu\nu} = \delta_{\mu\nu}$. We assume that the particle is initially localized at a point in this n-dimensional space. i.e., $U(x,y;0) = \delta^n(x-y)$. With this boundary condition, the solution for the diffusion equation is obtained as

$$U(x,y;\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}}e^{-\frac{|x-y|^2}{4\sigma}}. \tag{2.9}$$

Return probability, which measures the probability to find the particle returning back to the starting point after a diffusion time σ, associated with this heat kernel is attained by setting $x = y$. Thus from eqn.(2.9), one gets

$$P_g(\sigma) = (4\pi\sigma)^{-\frac{n}{2}}. \tag{2.10}$$

By taking the logarithmic derivative of this return probability we obtain the spectral dimension to be

$$D_s = -2\frac{\partial \ln P(\sigma)}{\partial \ln \sigma} = n. \tag{2.11}$$

Thus the spectral dimension of n-dimensional Euclidean space is found to be 'n'. Hence, for the Euclidean space, the spectral dimension is exactly same as the topological dimension of the space.

There have been attempts to study possible generalizations of the diffusion process described by the eqn.(2.4), in order to capture possible quantum gravity effects. Such generalizations include changes in the Beltrami-Laplace operator $\mathcal{L}$, modification in the initial conditions as well as the modification of diffusion operator $\frac{\partial}{\partial\sigma}$ [17, 18, 19].

Studies of the diffusion process in different quantum gravity models show that the effective dimension is different from the usual topological dimension. It is well known that the spectral dimension of quantum gravity models depend on the diffusion time σ and in [1] it was shown to decrease smoothly to a lower value from four, at small values of σ. In space-time models motivated by different approaches to quantum gravity, it was shown, by explicit calculation that the space-time dimension reduces from the low energy value of four to two at high energies (i.e., when probed at extremely short length scales) [1, 2, 3, 8]. In the case of non-commutative space-times, for certain choices of the Laplacian, the spectral dimension reduces to a lower value (but not 2) and for other choices, one gets higher values for spectral dimension, showing super diffusion[4, 5, 6, 7].

A plausible way to capture the nature of quantum space-time is by incorporating the idea of minimal length, which inturn leads to the loss of resolution at high energy scale. The blurriness of space-time at Planck scale is studied by considering a diffusion process in which the initial configuration spread over a small region of space, as opposed to a point like configuration [6]. Here the initial configuration contains the information about the minimal length associated with the space-time manifold. It was shown that for length scales grater than the minimal length, the universe appears to have a smooth space-time and when the length scale approaches the minimal length, the fractal nature of space-time begins to emerge [6]. In the limit where the diffusion time (σ) is of the same order as the (square of) minimal length (l_{min}), the spectral dimension becomes *two*. For trans-Planck regime (where $\sigma < l_{min}$), it was argued that the space-time dissolves completely, leading the spectral dimension to be zero.

In [4], change in the dimension of space in a model which is compatible with deformed special relativity principle (DSR) is analyzed. A Laplacian in momentum space was constructed and using this, scale dependence of spectral dimension was studied numerically. It was shown that the spectral dimension increases to six in the high energy scale showing super diffusion and take the expected value of four in the low energy scale.

Diffusion on a non-commutative space-time with quantum group symmetry was studied in [5]. It was shown that dynamical dimensional reduction appears at short length scale in κ-Minkowski space-time. In [7], the diffusion process on Euclidean kappa-Minkowski space is studied by constructing a momentum group manifold. This construction allows different possible choices for Laplacian. It is found that the behaviour of spectral dimension at different length scales depends on the choice of Laplacian.

Principal of relative locality postulates phase space as more fundamental and consider space-time as emergent phenomenon. This approach is another plaussible way to study quantum theory of gravity. Using principal of relative locality, the relevant metric in the momentum space was obtained in [7]. This metric is then used to construct the Laplace operator in the momentum space. It was shown that the notion of relative locality leads to higher powers of momenta in the dispersion relation. Further, it was shown that the spectral dimension goes to infinity at high energies. Thus, it is seen that the notion of diffusion completely breaks-down at high energies due to the absence of an absolute locality.

We note that the dimensional flow in the non-commutative space-time, and in particular for the case of κ-deformed space-time, changes its characteristic behaviour with change in the form of Laplacians. Also, all these studies[5, 6, 7, 4], where the variation of spectral dimension of κ-space-time were analysed, used the Laplacians in the momentum space. Further, the invariant measure in the momentum space had an important role in deciding the behaviour of spectral dimension as a function of the probe scale[5, 7]. In [1, 2, 3], solutions to diffusion equation were obtained in the coordinate space and thus it is of interest to analyze the change in the spectral dimension of κ-space-time also using the probability density of the test particle undergoing diffusion in coordinate space. This issue is investigated and results are reported in this thesis.

In this thesis, we derive the spectral dimension of the κ-space-time, by setting up various possible kappa-deformed diffusion equations and analyzing them. All our results are valid upto first non-vanishing order in the deformation parameter a.

2.3 Hausdorff Dimension

In the context of quantum gravity models the concepts of Hausdorff dimension has been analysed to study effective dimension [20]. Hausdorff dimension indicates the amount of uncertainty in the path for a quantum particle [20, 21, 22]. The concept of Hausdorff dimension was introduced by Felix Hausdorff in 1918. It was introduced in the context of measure theory to generalize the concept of length in metric spaces by taking into account spaces having fractal dimensions (i.e., metric spaces made out of cantor sets)[23]. It generalizes the notion of topological dimension of space. More specifically, using the definition of Hausdorff dimension one is able to deal with spaces having fractional dimension. We will now briefly discuss the concepts of the Hausdorff dimension.

In order to understand the notion of Hausdorff dimension, consider the measurement of length of an arbitrary curve using an apparatus with resolution Δx. It is reasonable to assume that this measured length will depend on the resolution of the apparatus. This means that, the change in resolution will in turn modify the measured length (See section 3.1). However, it is possible to have a meaning of length which is independent of resolution and this length will be more useful. Such a definition of length was proposed by Hausdorff and is given by

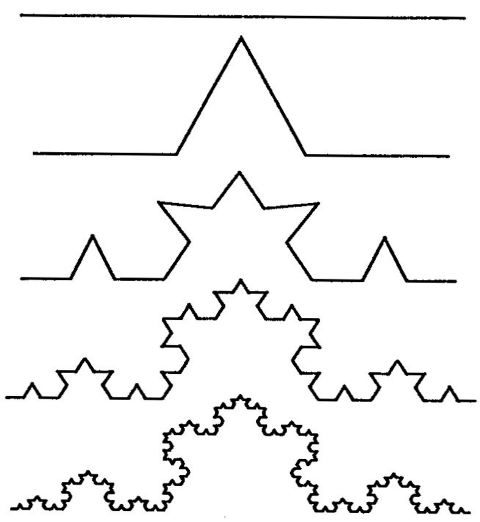
$$L_H = l(\Delta x)^{D_H - 1}, \tag{2.12}$$

where l is the length measured when the resolution of the measuring apparatus is Δx. The parameter D_H is chosen so that the Hausdorff length L_H will be independent of Δx and this D_H is known as Hausdorff dimension.

2.3.1 Koch curve as an illustration of Hausdorff dimension

In this section, we use Koch curve (see fig.[2.1]) as an example to explain the concept of Hausdorff dimension. Koch curve is an example of everywhere continuous but nowhere differentiable curve [24]. The construction of Koch curve is as follows. Consider a straight line with a finite length.

Figure 2.1: Construction of Koch curve



Divide it into three segments of equal length and replace the middle segment by the two sides of an equilateral triangle of the same length as the segment being removed. This procedure is repeated infinitely many times for each segment such that in each step the length of the curve gets multiplied by a factor $\frac{4}{3}$. That is for a curve with unit length, after the first step the length will be $\frac{4}{3}$. In the second step this become $(\frac{4}{3})^2$ and so on. This results, after infinite iteration process, in a curve of infinite length.

Let us suppose that we find the length of the curve to be l using an apparatus of resolution Δx. Now if we refine the apparatus resolution so that the new resolution is $\Delta x' = \frac{\Delta x}{3}$, the apparatus will be able to detect wiggles present at smaller scales. Hence the improved resolution will led to the measurement of a more accurate length, $l' = \frac{4}{3}l$. Thus the Hausdorff length corresponding to this new measured length is given by (see eqn.(2.12))

$$L'_H = l'(\Delta x')^{D_H - 1} = \frac{4}{3}\, l \, \left(\frac{\Delta x}{3}\right)^{D_H - 1}. \tag{2.13}$$

Since Hausdorff length is independent of resolution by very definition, we demand $L_H \;=\; L'_H$, i.e.,

$$\frac{4}{3}\, l \, \left(\frac{\Delta x}{3}\right)^{D_H - 1} = l(\Delta x)^{D_H - 1}. \tag{2.14}$$

This sets the condition

$$(D_H - 1) = \frac{\ln 4 - \ln 3}{\ln 3}, \tag{2.15}$$

which gives

$$D_H = \frac{\ln 4}{\ln 3} = 1.2618. \tag{2.16}$$

We thus conclude that Koch curve has fractional dimension as opposed to an integer dimension. Such curves are generally known as fractals.

2.3.2 Dimension of a Quantum path

We know from Heisenberg's uncertainty principle, the uncertainty in position, Δx and uncertainty in momentum, Δp in a simultaneous measurement of position and momentum are related by

$$\Delta x \; \Delta p \; \geq \; \hbar. \tag{2.17}$$

This relation means that, whenever we try to measure the position (or momentum) of the particle with greater accuracy, we find the error in the measurement of momentum (or position) becoming high. If we repeat this process of measurement of position at regular intervals of time say Δt, we obtain a path which is erratic as the momentum of the particle increases. As we reduce the time interval smaller and smaller, we obtain a path which is more and more irregular. Interestingly such a path can be modeled as a fractal curve [20].

To study the properties of quantum particle, the notion of quantum path is introduced as follows. One construct the path of a quantum particle by measuring the position of the particle with a resolution Δx in equal intervals of time t_0, $t_0 + \Delta t$, $t_0 + 2\Delta t$, ... ,$t_0 + N\Delta t$. By quantum path we mean the curve obtained by connecting positions at time separated by Δt. The average length of the quantum path, as defined above, travelled by the particle is

$$\langle L \rangle = N \langle \Delta l \rangle, \tag{2.18}$$

where $\langle \Delta l \rangle$ is the average distance travelled by the particle in time Δt and N is the number of measurements of position, made on the particle. The wave function which embeds the fact that the particle is localized with in a region of size Δx is of the form [1]

$$\Psi_{\Delta x}(\mathbf{x}, 0) = \frac{(\Delta x)^{\frac{3}{2}}}{\hbar^3} \int_{\mathbb{R}^3} \frac{d^3 p}{(2\pi)^{\frac{3}{2}}} f\left(\frac{|\mathbf{p}|\Delta x}{\hbar}\right) e^{\frac{i\mathbf{p}.\mathbf{x}}{\hbar}}. \tag{2.19}$$

Note that here we have assumed the average momentum of the particle to be zero. After time Δt the above wave function evolves to

$$\Psi_{\Delta x}(\mathbf{x}, \Delta t) = \frac{(\Delta x)^{\frac{3}{2}}}{\hbar^3} \int_{\mathbb{R}^3} \frac{d^3 p}{(2\pi)^{\frac{3}{2}}} f\left(\frac{|\mathbf{p}|\Delta x}{\hbar}\right) e^{i\mathbf{p}.\mathbf{x}/\hbar - i|\mathbf{p}|^2 \Delta t/2m\hbar}. \tag{2.20}$$

Using this wave function, we find the average distance traveled by the particle in a time Δt as

$$\langle \Delta l \rangle = \int_{\mathbb{R}^3} d^3 x |\mathbf{x}| |\Psi_{\Delta x}(\mathbf{x}, \Delta t)|^2. \tag{2.21}$$

Since qualitative features of the results are independent of the specific form of $f\left(\frac{|\mathbf{p}|\Delta x}{\hbar}\right)$, provided $f\left(\frac{|\mathbf{p}|\Delta x}{\hbar}\right)$ corresponds to a distribution localized to a region of size Δx. Hence we choose a Gaussian distribution and this has the advantage that the integral can be done exactly.

Using a change of variables, $\mathbf{k} = \mathbf{p}\Delta x/\hbar$, $\mathbf{z} = \mathbf{x}/\Delta x$ and $f(|\mathbf{k}|) = (\frac{2}{\pi})^{3/4} e^{-|\mathbf{k}|^2}$, $\langle \Delta l \rangle$ is obtained as

$$\langle \Delta l \rangle \propto \frac{\hbar \Delta t}{m \Delta x} \sqrt{1 + \left(\frac{2m(\Delta x)^2}{\hbar \Delta t}\right)^2}. \tag{2.22}$$

When $\Delta x \ll \sqrt{\hbar \Delta t / 2m}$, the Hausdorff length $\langle L_H \rangle = N \langle \Delta l \rangle (\Delta x)^{D_H - 1}$ become

$$\langle L_H \rangle \propto N \frac{\hbar \Delta t}{m \Delta x} (\Delta x)^{D_H - 1}. \tag{2.23}$$

[1] For example, $f(k) = (\frac{2}{\pi})^{\frac{1}{4}} e^{-k^2}$, the wave function in 1-dimension takes the form $\psi_{\Delta x}(x) = \frac{1}{\sqrt{\Delta x}(2\pi)^{1/4}} e^{-\frac{x^2}{4\Delta x^2}}$. The expectation value of x with this wave function is $\langle x \rangle = \int_{-\infty}^{\infty} dx \psi^* x \psi = \frac{1}{\sqrt{2\pi}} \frac{1}{\Delta x} \int_{-\infty}^{\infty} dx x e^{-\frac{x^2}{2\Delta x^2}} = \Delta x$.

From the above expression one sees the Hausdorff dimension of a quantum particle is two (2).

Now consider the case where the particle has a nonzero average momentum $\mathbf{p}_{av}$. Then the corresponding wave function can be written as

$$\Psi_{\Delta x}(\mathbf{x}, 0) = \frac{(\Delta x)^{\frac{3}{2}}}{\hbar^3} \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^{\frac{3}{2}}} f\left(\frac{|\mathbf{p}|\Delta x}{\hbar}\right) e^{i(\mathbf{p}+\mathbf{p}_{av}).\mathbf{x}/\hbar}. \tag{2.24}$$

The average distance traveled by the particle in time Δt is

$$\langle \Delta l \rangle = \int_{\mathbb{R}^3} d^3x |\mathbf{x}| |\Psi_{\Delta x}(\mathbf{x}, \Delta t)|^2, \tag{2.25}$$

where

$$\Psi_{\Delta x}(\mathbf{x}, \Delta t) = \frac{(\Delta x)^{\frac{3}{2}}}{\hbar^3} \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^{\frac{3}{2}}} f\left(\frac{|\mathbf{p}|\Delta x}{\hbar}\right) e^{i(\mathbf{p}+\mathbf{p}_{av}).\mathbf{x}/\hbar} e^{-i|\mathbf{p}+\mathbf{p}_{av}|^2\Delta t/2m\hbar}. \tag{2.26}$$

Re-writing the above in terms of new variables $\mathbf{k} = \mathbf{p}\Delta x/\hbar$ and $\mathbf{z} = \frac{\mathbf{x}}{\Delta x} - \frac{\Delta t}{m\Delta x}\mathbf{p}_{av}$, one obtains

$$\langle \Delta l \rangle = \Delta x \int_{\mathbb{R}^3} d^3z |\mathbf{z} + \frac{\Delta t}{m\Delta x}\mathbf{p}_{av}| |F(z, \frac{\hbar \Delta t}{2m(\Delta x)^2})|^2, \tag{2.27}$$

where

$$F(z, \frac{\hbar\Delta t}{2m(\Delta x)^2}) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} f(|k|) e^{-i\frac{\Delta t\hbar}{2m\Delta x^2}|k|^2} e^{ik.z}. \tag{2.28}$$

If $\frac{\hbar\Delta t}{2m(\Delta x)^2}$ is a constant (say c), then the function $F(z, c)$ will be independent of Δx. Then

$$\langle \Delta l \rangle = \frac{|\mathbf{p}_{av}|\Delta t}{m} \int_{\mathbb{R}^3} d^3z \left|\frac{\mathbf{p}_{av}}{|\mathbf{p}_{av}|} + \frac{\hbar \mathbf{z}}{2|\mathbf{p}_{av}|(\Delta x)c}\right| \mid F(\mathbf{z}, c) \mid^2 \tag{2.29}$$

Using this in the definition of Hausdorff length, we find

$$\begin{aligned} \langle L_H \rangle &= N \langle \Delta l \rangle (\Delta x)^{D_H - 1} \\ &= \frac{|\mathbf{p}_{av}|T}{m} \int_{\mathbb{R}^3} d^3z \left|\frac{\mathbf{p}_{av}}{|\mathbf{p}_{av}|} + \frac{\hbar \mathbf{z}}{2|\mathbf{p}_{av}|(\Delta x)c}\right| \mid F(\mathbf{z}, c) \mid^2 (\Delta x)^{D_H - 1} \end{aligned} \tag{2.30}$$

If the distance being resolved are much larger than the particle's wavelength, i.e., $\Delta x \gg \hbar / \mid \mathbf{p}_{av} \mid$, then we obtain the Hausdorff dimension to be one (1) and when the distances being resolved are much smaller than the particle's wavelength ($\Delta x \ll \hbar / \mid \mathbf{p}_{av} \mid$), Hausdorff dimension is two (2).

In the above discussion, it has been demonstrated that the path of a particle in quantum mechanics as defined above is a fractal curve with Hausdorff dimension two [21]. This study has been generalised to the case of a particle governed by relativistic quantum mechanics in [22]. For the relativistic particle, the path was defined in [22], in terms of Newton-Wigner operator [25]. It was shown that the Hausdorff dimension in the ultra-relativistic limit is one, where as in the non-relativistic limit, this dimension is two. It is to be noted that the length of the path as well as the Hausdorff dimension of the path is independent of the spinorial structure of the relativistic particle [22].

It is conceivable that the structure of space-time do undergo drastic modification at high energies. Thus it is of interest to investigate the Hausdorff dimension in the context of quantum gravity models. Motion of quantum particle in the presence of a minimal length is discussed in [20]. It is seen that the presence of minimal length modifies the space-time geometry. Interestingly, using the definition of Hausdorff dimension one could account for uncertainty due to quantum mechanics as well as modification of geometry [20].

In this thesis, we extended this idea of Hausdorff dimension to κ-deformed space-time and study the modification in the Hausdorff dimension of the path of a particle moving in the kappa-space-time. We show that the Hausdorff dimension depends on the deformation parameter and the resolution for both non-relativistic and relativistic quantum particle. Another interesting observation is that the path travelled by a relativistic quantum particle depends on the spinorial character of the wave function. This new feature is due to the non-commutative nature of space-time.

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CHAPTER 3

Spectral Dimension of kappa-deformed space-time

3.1 Introduction

The concept of non-commutative space-times emerges naturally in the study of quantum gravity. In such space-times, usual notion of space-time points looses their meaning and space-time becomes fuzzy [1]. The non-commutative space-time introduces a minimal length scale in to the discussion, which is a common feature of all approaches to quantum gravity. Thus, the effective dimension of the non-commutative space-time can be different from the usual topological dimension of space-time. Spectral dimension has been used to study this change in the effective dimension at high energies [2, 3, 4]. In this chapter, we discuss the geometry of kappa-deformed Euclidean space by analyzing the corresponding spectral dimension.

Variation of spectral dimension with the probe scale (σ - which is the diffusion time) in a non-commutative space-time was analysed in [2]. In this paper, the entire effect of non-commutativity was introduced through a modification of the initial condition satisfied by the solution to the diffusion equation. Since non-commutativity will lead to smearing of point objects, in this paper the initial condition was taken as a Gaussian (instead of a Dirac delta function) with width controlled by the minimal length (l) introduced by the non-commutativity of the space-time. i.e. the form of initial condition used in [2] is

$$U(x, y; 0) = \frac{1}{(4\pi l^2)^{\frac{n}{2}}} e^{-\frac{(x-y)^2}{4l^2}}. \tag{3.1}$$

Using this initial condition, one obtain the spectral dimension [2] as

$$D_s = \frac{\sigma}{\sigma + l^2} n. \tag{3.2}$$

Here, the spectral dimension was shown to be a function of diffusion time (σ) and it also depends on the minimal length parameter (l). It is found that at very high energies, space-time dimension reduces to zero. For $n = 4$, spectral dimension becomes two, when $\sigma = l^2$.

In [3], using numerical methods, fractal nature of the κ-Minkowski space-time was studied. Here, the spectral dimension is calculated from the return probability obtained using a specific form of κ-deformed Laplacian, in the momentum space. Also in this study, a modified integration measure, in the momentum space, which is invariant under the κ-deformed Lorentz algebra was used. After numerically carrying out the integrations, it was shown that the spectral dimension reduced to three (3) in the limit of diffusion time going to zero. Further, using a different basis for the κ-deformed Lorentz algebra, same behaviour of spectral dimension was obtained, but with a different choice for the Laplacian.

In [4], change of spectral dimension of κ-Minkowski space-time was studied. Starting from the Euclidean momentum space associated with the κ-Minkowski space-time, possible Laplacians in the momentum space were constructed. These Laplacians were constructed by demanding them to be the Casimir of the κ-Poincare algebra in the momentum space. These Laplacians were constructed as the Casimirs using bi-covariant differential calculus and bi-crossproduct basis, respectively. A third form for the Lpalacian was derived as the geodesic distance in the κ-momentum space. These three possible forms of the Laplacians were then used to study the spectral dimension associated with diffusion process in κ-Minkowski space-time. For the Laplacian written in bi-covariant differential calculus, it was shown that the spectral dimension flow from four at low energies to three in the high energies. For the second situation where the Laplacian was written in bi-crossproduct basis, spectral dimension was shown to increase from 4 to 6 with energy. In the third case, Laplacian was constructed using the notion of relative locality, and it was shown that the spectral dimension goes to infinity (∞) as energy increases.

The studies on spectral dimension gives more inherent characterization of the properties of space-time at quantum gravity level. Most of the above works on spectral dimension in non-commutative space-time were done in momentum space by using Casimir of the algebra as the generalised Laplacian [3, 4]. In this chapter we study the effect of non-commutativity on spectral dimension. Here we have chosen κ-deformed space-time as a platform to study spectral dimension by explicitly constructing and solving non-commutative diffusion equation.

In this chapter, we derive the spectral dimension of the κ-space-time, by setting up the κ-deformed diffusion equation in terms of commutative co-ordinates. This is achieved by first writing down the κ-deformed Laplacian in the Euclidean version of the n-dimensional κ-space-time, in terms of the derivatives with respect to commutative coordinates and deformation parameter a. We then obtain the solution to this heat equation, perturbatively. Using this solution, we calculate the return probability, valid upto second order in the deformation parameter. We then calculate the spectral dimension, as a function of diffusion length and the deformation parameter a. We then analyze the variation of the spectral dimension with probe scale. Since the framework we use here allow us to set up the diffusion equation valid for the κ-deformed space-time, entirely in the commutative space-time, we can use the well established methods of commutative space-time to solve this deformed heat equation. Also, we can use the same initial condition of demanding the test particle to be localized at a fixed point in space at the initial time, as in the commutative space-time. We have also investigated the effect of finite extension of the probe on the spectral dimension, by using a modified initial condition.

This chapter is organised as follows. In the next section, a brief summary of the κ-deformed Euclidean space and its realization is given [5]. In section 3, we set up the heat equation for a fictitious particle in this κ-deformed space-time and obtain its solution. This solution is derived as a perturbative series in the deformation parameter and we obtain the solution valid upto first non-vanishing terms in the deformation parameter. Using this solution, we calculate the return probability and spectral dimension which is valid upto second order in the deformation parameter. We also discuss various limits of the spectral dimension. Then using a different initial condition encoding the finite extension of the probe, we calculate the spectral dimension. We find that the generic feature of the spectral dimension is not altered by the extended nature of the probe. In subsection 3.2, we start with a different, possible Laplacian in the κ-deformed space-time and evaluate corresponding spectral dimension and discuss its limits. Here again, we investigate the effect of a probe with a finite extension on the spectral dimension. Our concluding remarks are given in section 4.

3.2 Kappa-deformed Euclidean space

The n-dimensional kappa-deformed space-time is an example of non-commutative space-time, where the coordinate commutation relations are of the Lie algebra type.

The algebra of the non-commutative coordinates is of the form

$$[\hat{x}_i, \hat{x}_j] = 0, \qquad [\hat{x}_n, \hat{x}_i] = ia\hat{x}_i, \qquad i, j = 1, 2, ..., n-1. \tag{3.3}$$

Thus we note that the Euclidean time coordinate $\hat{x}_n$ do not commute with corresponding space-coordinate $\hat{x}_i$, where as space coordinates commute among themselves.

We can study the non-commutative space-time by re-expressing the non-commutative coordinates $\hat{x}_\mu$ in terms of the commutative coordinates x_μ and their derivatives ∂_μ [5, 6]. One such realization which satisfy eqn.(3.3) is

$$\hat{x}_i = x_i \varphi(A), \tag{3.4}$$
$$\hat{x}_n = x_n \psi(A) + iax_i \partial_i \gamma(A), \tag{3.5}$$

where $A = ia\partial_n$. By demanding that the deformed Poincare transformations must be linear in the coordinates and their derivatives, one obtain the generators of the underlying symmetry algebra. These modified generators satisfy the same relations as the generators of the usual Poincare algebra and hence their algebra is known as undeformed κ-Poincare algebra. These deformed generators are given in terms of the commutative coordinates x_μ, their derivatives ∂_μ and also depend on the deformation parameter a. In the limit of vanishing a, one recover the Poincare algebra. The details of this algebra is discussed in Chapter 1.

The derivatives which transform as vector under κ-Poincare algebra, called Dirac derivatives, are given explicitly as

$$D_i = \partial_i \frac{e^{-A}}{\varphi(A)}, \qquad D_n = \partial_n \frac{sinhA}{A} + ia\nabla^2 \frac{e^{-A}}{2\varphi^2(A)}, \tag{3.6}$$

where $\nabla^2 = \Sigma_{i=1}^{n-1} \partial_i^2$. For different choices of $\varphi(A)$ we can construct different realizations of the undeformed κ-Poincare algebra. $e^{-\frac{A}{2}}$, e^{-A} are some possible choices for $\varphi(A)$.

The quadratic Casimir of this undeformed κ-Poincare algebra $D_\mu D_\mu$ [5, 6] is given by

$$D_\mu D_\mu = D_i D_i + D_n D_n = \Box(1 - \frac{a^2}{4}\Box), \tag{3.7}$$

where

$$\Box = \nabla^2 \frac{e^{-A}}{\varphi^2(A)} - \partial_n^2 \frac{2[1 - coshA]}{A^2}. \tag{3.8}$$

Note that $\Box$ operator is quadratic in space derivatives, thus $D_\mu D_\mu$ has quartic powers of space derivatives. By expanding e^{-A}, one easily see that it has higher powers of time derivatives.

Note that $\Box$ operator will also be an invariant quantity for κ-Poincare algebra. In the limit of vanishing a, $D_\mu D_\mu$ and $\Box$ reduces to usual Laplace-Beltrami operator.

3.3 Spectral dimension of kappa-deformed Euclidean space

In this section, we calculate and analyze the spectral dimension of the κ-deformed space-time, using the kappa-deformed diffusion equation. We solve this deformed diffusion equation, perturbatively and obtain the solution valid upto second order in the deformation parameter. Using this deformed probability density, we calculate the deformed return probability and then spectral dimension of κ-space-time.

Let us start with a diffusion process on a n-dimensional κ-deformed Euclidean space. The motion of the diffused particle is governed by the diffusion equation

$$\frac{\partial}{\partial\sigma} U(x, y; \sigma) = \mathcal{L} U(x, y; \sigma), \tag{3.9}$$

where $\mathcal{L}$ is the generalized Laplacian with respect to the kappa-deformed Euclidean space.

Generalizing the notion of Laplacian being the Casimir of the Poincare algebra to the κ-deformed space-time, we use $D_\mu D_\mu$ as the κ-deformed Laplacian. In the commutative limit ($a \to 0$), we recover the standard Laplacian in the commutative space-time. But if we relax this condition that the Laplacian must be the Casimir and require only that the κ-deformed operator should reduce to the usual Laplacian in the commutative space-time, we can use the $\Box$-operator defined in eqn.(3.8) also as the κ-deformed Laplacian.

Note that both the operators $D_\mu D_\mu$ and $\Box$ are written completely in terms of commutative variables and thus the Laplacian we use are expressed in the commutative space-time. Thus the κ-deformed diffusion equations we study are written in commutative space-time. This simplification is due to the fact that we are using the realization given in eqns.(3.4) and (3.5) and also use undeformed κ-Poincare algebra. This should be contrasted with the previous studies [2, 3, 4].

3.3.1 Spectral dimension from $D_\mu D_\mu$ operator

In this section, we construct the diffusion equation using $D_\mu D_\mu$ (Casimir of the undeformed κ-Poincare algebra) as the Laplace operator. Thus the kappa-deformed diffusion equation for an n-dimensional κ-deformed Euclidean space will be

$$\begin{aligned}\frac{\partial}{\partial\sigma}U(x,y;\sigma) &= D_\mu D_\mu U(x,y;\sigma)\\ &= \Box(1-\frac{a^2}{4}\Box)U(x,y;\sigma), \qquad (3.10)\end{aligned}$$

where the explicit form of $\Box$ is given in eqn.(3.8). We obtain different realizations for κ-Poincare algebra with the choice of $\varphi(A)$. Here we choose $\varphi(A) = e^{-\frac{A}{2}}$.

Now rewrite the Laplace operator $D_\mu D_\mu$ using this realization ($\varphi(A) = e^{-\frac{A}{2}}$) and by taking $A = ia\partial_n$, we find

$$\begin{aligned}D_\mu D_\mu &= \Box(1-\frac{a^2}{4}\Box) \qquad (3.11)\\ &= \left(\nabla^2+\frac{2}{a^2}[1-\cosh(ia\partial_n)]\right)\left(1-\frac{a^2}{4}\left(\nabla^2+\frac{2}{a^2}[1-\cosh(ia\partial_n)]\right)\right)\\ &= \nabla^2-\frac{a^2}{4}\nabla^4+(\frac{2}{a^2}-\nabla^2)[1-\cosh(ia\partial_n)]-\frac{1}{a^2}[1-\cosh(ia\partial_n)]^2. \qquad (3.12)\end{aligned}$$

We expand eqn.(3.12) in powers of deformation parameter a as

$$\begin{aligned}D_\mu D_\mu &= \nabla^2-\frac{a^2}{4}\nabla^4-(\frac{2}{a^2}-\nabla^2)\left[-\frac{a^2\partial_n^2}{2}+\frac{a^4\partial_n^4}{24}-...\right]\\ &- \frac{1}{a^2}\left[\frac{a^4\partial_n^4}{4}-\frac{a^6\partial_n^6}{24}+...\right]. \qquad (3.13)\end{aligned}$$

We restrict our attention to first non-vanishing corrections due to non-commutativity. Thus we obtain the Laplace operator for κ-Euclidean space, valid upto the second order in the deformation parameter a as

$$D_\mu D_\mu = \nabla^2 + \partial_n^2 - \frac{a^2}{3}\partial_n^4 - \frac{a^2}{2}\nabla^2\partial_n^2 - \frac{a^2}{4}\nabla^4. \tag{3.14}$$

Using this in the κ-deformed diffusion equation eqn.(3.10), we find

$$\frac{\partial}{\partial\sigma}U(x,y;\sigma) = \left(\nabla^2 + \partial_n^2 - \frac{a^2}{3}\partial_n^4 - \frac{a^2}{2}\nabla^2\partial_n^2 - \frac{a^2}{4}\nabla^4\right)U(x,y;\sigma) \tag{3.15}$$

Note that all the a dependent terms are of higher derivatives; two of them having quartic derivatives while another involves product of quadratic derivatives in space and time.

For convenience we define the Laplacian in the n-dimensional commutative Euclidean space-time as

$$\nabla^2 U + \partial_n^2 U = \nabla_n^2 U. \tag{3.16}$$

Using this, eqn.(3.15) is re-written as

$$\frac{\partial U}{\partial\sigma} = \nabla_n^2 U - \frac{a^2}{3}\partial_n^4 U - \frac{a^2}{2}\nabla^2\partial_n^2 U - \frac{a^2}{4}\nabla^4 U. \tag{3.17}$$

We use perturbative approach to solve this equation to obtain the heat kernel U(x,y;σ). Thus we start with the probability density, valid upto second order in the deformation parameter a as

$$U = U_0 + aU_1 + a^2U_2. \tag{3.18}$$

Note that we have the following relations between dimensions of terms in the above perturbative series,

$$[U_1] = \frac{1}{L}[U_0], \quad [U_2] = \frac{1}{L^2}[U_0]. \tag{3.19}$$

Using eqn.(3.18) in eqn.(3.17) gives

$$\begin{aligned}\frac{\partial}{\partial\sigma}(U_0 + aU_1 + a^2U_2) &= \nabla_n^2(U_0 + aU_1 + a^2U_2) - \frac{a^2}{3}\partial_n^4(U_0 + aU_1 + a^2U_2) \\ &- \frac{a^2}{2}\nabla^2\partial_n^2(U_0 + aU_1 + a^2U_2) \\ &- \frac{a^2}{4}\nabla^4(U_0 + aU_1 + a^2U_2).\end{aligned} \tag{3.20}$$

Equating the zeroth order terms in a on both sides of above equation, we find that U_0 satisfy the usual heat equation,

$$\frac{\partial}{\partial\sigma}U_0(x,y;\sigma) = \nabla_n^2 U_0(x,y;\sigma). \tag{3.21}$$

The solution to above equation is

$$U_0(x,y;\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}}. \tag{3.22}$$

Next, we equate the first order terms in a in eqn.(3.20) to obtain

$$\frac{\partial}{\partial\sigma} U_1(x,y;\sigma) = \nabla_n^2 U_1(x,y;\sigma) \tag{3.23}$$

showing that U_1 also satisfy the same heat equation as U_0. Thus we find U_1 to be of the same form as U_0, i.e.,

$$U_1(x,y;\sigma) = \frac{\alpha}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}} \tag{3.24}$$

where α has the dimensions of L^{-1}. Thus, for solving both U_0 and U_1, we have used the usual initial condition, i.e., $U_0(x,y;0) = \delta^n(x-y) = U_1(x,y;0)$. This is possible because we could write the deformed diffusion equation in the commutative space-time.

Now we solve for the next term (order a^2) in eqn.(3.18). For this we equate the second order terms in a in eqn.(3.20) and obtain

$$\begin{aligned}\frac{\partial}{\partial\sigma} U_2(x,y;\sigma) &= \nabla_n^2 U_2(x,y;\sigma) - \frac{1}{3}\partial_n^4 U_0(x,y;\sigma) \\ &- \frac{1}{2}\nabla^2\partial_n^2 U_0(x,y;\sigma) - \frac{1}{4}\nabla^4 U_0(x,y;\sigma)\end{aligned} \tag{3.25}$$

Note that the last three terms on the RHS of above equation show the change in the diffusion equation due to the κ-deformation. Using the solution for U_0 obtained in eqn.(3.22) in the above, and after straight forward simplification, we get the equation satisfied by U_2 as

$$\begin{aligned}\frac{\partial}{\partial\sigma} U_2(x,y;\sigma) &= \nabla_n^2 U_2(x,y;\sigma) + \Big[-\frac{(n+1)^2}{16\sigma^2} + \frac{n+2}{16\sigma^3}\Sigma_{i=1}^{n}(x_i-y_i)^2 \\ &+ \frac{(x_n-y_n)^2}{16\sigma^3} - \frac{(x_n-y_n)^4}{48\sigma^4} - \frac{(x_n-y_n)^2}{32\sigma^4}\Sigma_{i=1}^{n-1}(x_i-y_i)^2 \\ &- \frac{1}{64\sigma^4}(\Sigma_{i=1}^{n-1}(x_i-y_i)^2)^2\Big] \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}}.\end{aligned} \tag{3.26}$$

Using Duhamel's principle [7], we solve the above equation which is of the generic form

$$\frac{\partial}{\partial\sigma} U_2(X,\sigma) = \nabla_n^2 U_2(X,\sigma) + f(X,\sigma) \tag{3.27}$$

where $X = x - y$. With the initial condition

$$U_2(X, 0) = g(X), \tag{3.28}$$

the solution to eqn.(3.27) is given by

$$U_2(X, \sigma) = \int_{R^n} \Phi(X - X', \sigma) g(X') dX' + \int_0^\sigma \int_{R^n} \Phi(X - X', \sigma - s) f(X', s) dX' ds \tag{3.29}$$

where $\Phi(X, \sigma)$ will satisfy the equation

$$\left(\frac{\partial}{\partial\sigma} - \nabla_n^2\right) \Phi(X, \sigma) = 0. \tag{3.30}$$

So the function $\Phi(X, \sigma)$ will be of the form

$$\Phi(X, \sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|X|^2}{4\sigma}}. \tag{3.31}$$

In our case we have the initial condition satisfied by the solution as

$$U_2(X, 0) = g(X) = \delta^n(X). \tag{3.32}$$

Note that we are using the same boundary condition as in the usual diffusion equation. Since, the κ-deformed Laplacian and hence the diffusion equation are written fully in the commutative space-time and all the effects of the non-commutativity is included in the modified Laplacian and thus we are justified in using this initial condition. Using eqn(3.31) and eqn.(3.32), the first term on RHS of eqn.(3.29), U_{21} is calculated as

$$\begin{aligned} U_{21}(X, \sigma) &= \int_{-\infty}^{\infty} \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|X-X'|^2}{4\sigma}} \delta^n(X) dX' \\ &= \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|X|^2}{4\sigma}} \\ U_{21}(x, y; \sigma) &= \frac{\beta}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}}, \end{aligned} \tag{3.33}$$

where β has the dimensions of L^{-2}. The second term on RHS of eqn.(3.29),

U_{22} is evaluated as

$$
\begin{aligned}
U_{22}(x,y;\sigma) &= \int_0^\sigma \int_{R^n} \Phi(X-X',\sigma-s) f(X',s) dX' ds \\
&= \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}} \Bigg\{ \Bigg(-\frac{(x_n-y_n)^4}{48\sigma^4} - \frac{(x_n-y_n)^2}{32\sigma^4} \Sigma_{i=1}^{n-1}(x_i-y_i)^2 \\
&- \frac{1}{64\sigma^4}\left(\Sigma_{i=1}^{n-1}(x_i-y_i)^2\right)^2 \Bigg)(\sigma-\epsilon) + \left(\frac{(n+1)^2}{16}\right)\left(\frac{1}{\sigma}-\frac{1}{\epsilon}\right) \\
&+ \left(\frac{(x_n-y_n)^2}{16\sigma^2} + \frac{n+2}{16\sigma^2}\Sigma_{i=1}^{n}(x_i-y_i)^2\right)\ln(\sigma/\epsilon) \\
&- \Bigg(\frac{(x_n-y_n)^2}{4\sigma^3} + \frac{1}{16\sigma^3}\Sigma_{i=1}^{n-1}(x_i-y_i)^2 + \frac{(n-1)}{16\sigma^3}(x_n-y_n)^2 \\
&+ \frac{(n+1)}{16\sigma^3}\Sigma_{i=1}^{n-1}(x_i-y_i)^2\Bigg)\mathbb{A} - \frac{1}{2\pi\sigma^3}\Big((x_n-y_n)\Sigma_{i=1}^{n-1}(x_i-y_i) \\
&+ (x_1-y_1)\Sigma_{i=2}^{n-1}(x_i-y_i) + ... + (x_{n-2}-y_{n-2})(x_{n-1}-y_{n-1})\Big)\mathbb{A} \\
&- \Bigg(\frac{(x_n-y_n)^3}{6\sigma^3\sqrt{\sigma\pi}} + \frac{(x_n-y_n)}{8\sigma^3\sqrt{\sigma\pi}}\Sigma_{i=1}^{n-1}(x_i-y_i)^2 \\
&+ \frac{(x_n-y_n)^2}{8\sigma^3\sqrt{\sigma\pi}}\Sigma_{i=1}^{n-1}(x_i-y_i)\Bigg)\mathbb{B} \\
&- \left(\frac{1}{8\sigma^3\sqrt{\sigma\pi}}\Sigma_{i=1}^{n-1}(x_i-y_i)\Sigma_{j=1}^{n-1}(x_j-y_j)^2\right)\mathbb{B} \\
&+ \left(\frac{2(x_n-y_n)}{3\sigma^2\sqrt{\sigma\pi}} - \frac{(n+1)}{4\sigma^2\sqrt{\sigma\pi}}\Sigma_{i=1}^{n-1}(x_i-y_i) - \frac{(n-1)}{4\sigma^2\sqrt{\sigma\pi}}(x_n-y_n)\right) \\
&\times \left((2\sigma+\epsilon)\sqrt{\frac{\sigma}{\epsilon}-1} - 3\sigma\tan^{-1}\sqrt{\frac{\sigma}{\epsilon}-1}\right) \\
&- \frac{(n+1)^2}{16\sigma^2}\left(-2\sigma\ln(\sigma/\epsilon) + \frac{\sigma^2}{\epsilon} - \epsilon\right) \\
&+ \frac{(n+1)^2}{8\sigma}\left[-1 + \frac{\sigma}{\epsilon} - \ln(\sigma/\epsilon)\right] \\
&+ \left(\frac{n+2}{2\sigma\sqrt{\sigma\pi}}\Sigma_{i=1}^{n}(x_i-y_i)\right)\left(tan^{-1}(\sqrt{\sigma/\epsilon-1}) - \sqrt{\sigma/\epsilon-1}\right) \\
&+ \left(\frac{(x_n-y_n)}{2\sigma\sqrt{\sigma\pi}}\right)\left(tan^{-1}(\sqrt{\sigma/\epsilon-1}) - \sqrt{\sigma/\epsilon-1}\right)\Bigg\} \qquad (3.34)
\end{aligned}
$$

where $\mathbb{A} = \sigma \ln(\sigma/\epsilon) - \sigma + \epsilon$ and $\mathbb{B} = (\sigma \tan^{-1}\sqrt{\frac{\sigma}{\epsilon} - 1} - \epsilon\sqrt{\frac{\sigma}{\epsilon} - 1})$.

Note that U_{22} treats Euclidean time and space coordinate on a different footing. This is apparent from the fact that we have terms depending on $(x_n - y_n)$ and $(x_i - y_i)$, separately. The ϵ appearing in the above is a lower cut-off introduced in evaluating the integral in eqn.(3.29) and we will set the limit $\epsilon \to 0$ after calculating the spectral dimension.

Using eqn.(3.33) and eqn.(3.34), we get the solution to the second order correction as $U_2(x, y; \sigma) = U_{21}(x, y; \sigma) + U_{22}(x, y; \sigma)$. Using eqns. (3.22), (3.24), (3.33) and eqn.(3.34) in eqn.(3.18), we find the heat kernel, valid upto second order in a.

Now we want to calculate the return probability which measure the probability to find the particle returning back to the starting point after a diffusion time σ. In terms of the heat kernel, the return probability is defined as

$$P_g(\sigma) = \frac{\int d^n x \sqrt{det g_{\mu\nu}} U(x, x; \sigma)}{\int d^n x \sqrt{det g_{\mu\nu}}} \tag{3.35}$$

where $g_{\mu\nu}$ is the metric of the underlying space. Here it is $g_{\mu\nu} = \delta_{\mu\nu}$. Using this definition, we obtain the return probability as

$$\begin{aligned} P_g(\sigma) &= \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} + a\frac{\alpha}{(4\pi\sigma)^{\frac{n}{2}}} + a^2\frac{\beta}{(4\pi\sigma)^{\frac{n}{2}}} + \frac{a^2}{(4\pi\sigma)^{\frac{n}{2}}}\left[\frac{(n+1)^2}{16}\left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right)\right. \\ &- \frac{(n+1)^2}{16\sigma^2}\left(-2\sigma\ln\left(\frac{\sigma}{\epsilon}\right) + \frac{\sigma^2}{\epsilon} - \epsilon\right) \\ &+ \left.\frac{(n+1)^2}{8\sigma}\left(-\ln\left(\frac{\sigma}{\epsilon}\right) + \frac{\sigma}{\epsilon} - 1\right)\right] \\ &= \frac{1}{(4\pi\sigma)^{\frac{n}{2}}}\left[1 + a\alpha + a^2\beta + a^2\left(-\frac{(n+1)^2}{16\sigma} + \frac{(n+1)^2}{16\sigma^2}\epsilon\right)\right]. \end{aligned} \tag{3.36}$$

Note that the return probability, calculated above, do have first order as well as second order corrections due to the non-commutativity.

The spectral dimension D_s is extracted by taking the logarithmic derivative of return probability $P_g(\sigma)$. i.e.,

$$\begin{aligned} D_s &= -2\frac{\partial \ln P_g(\sigma)}{\partial \ln \sigma} \qquad (3.37) \\ &= \frac{-2\sigma}{P_g(\sigma)}\frac{\partial P_g(\sigma)}{\partial \sigma} \\ &= \frac{n + na\alpha + na^2\beta - (n+2)(n+1)^2\frac{a^2}{16\sigma} + (n+4)(n+1)^2\frac{a^2}{16\sigma^2}\epsilon}{1 + a\alpha + a^2\beta - a^2\frac{(n+1)^2}{16\sigma}(1 - \frac{\epsilon}{\sigma})}. \end{aligned} \tag{3.38}$$

Keeping upto first non-vanishing terms in a, we obtain the spectral dimension as $D_s = n-(n+1)^2\frac{a^2}{8\sigma}+(n+1)^2\frac{a^2}{4\sigma^2}\epsilon$. After setting the cut-off parameter ϵ to zero, we obtain spectral dimension of the κ-deformed space-time as

$$D_s = n - (n+1)^2\frac{a^2}{8\sigma}. \quad (3.39)$$

Note that the first non-vanishing correction due to non-commutativity is second order in a and this correction also depends on the initial topological dimension n (apart from the diffusion parameter σ). Now we will restrict our attention to the case $n = 4$, so the spectral dimension will be $D_s = 4 - \frac{25}{8}\frac{a^2}{\sigma}$.

Figure 3.1: Spectral dimension as a function of σ with $a = 1$ and $n = 4$ for $D_\mu D_\mu$ as Laplacian.

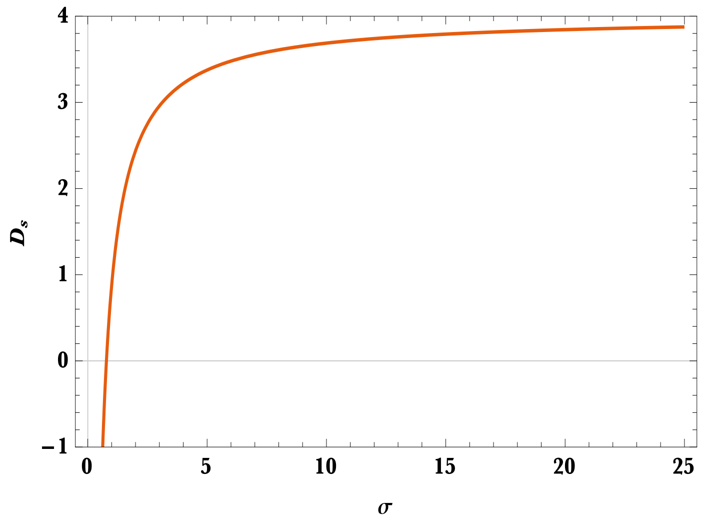


Fig.[3.1] indicate that the spectral dimension is changing from $-\infty$ to 4 (n=4) continuously with diffusion parameter σ. We see that in the limit of large diffusion parameter, the spectral dimension is same as the topological dimension ($\lim_{\sigma\to\infty} D_s = 4$), i.e., the space-time behaves like a smooth differential manifold at large distances. For small σ, where $\sigma > a^2$, the effect of kappa-deformation become significant. This has the consequence that the spectral dimension D_s become less than 4. Also it can be easily verified from $D_s = 4 - \frac{25}{8}\frac{a^2}{\sigma}$ that the effective dimension becomes 2 when $\sigma \simeq 1.56a^2$. Since quantum gravity is renormalizable for $n = 2$, this particular value of σ is a critical value for understanding the quantum nature of gravity. We obtain the spectral dimension to be zero for $\sigma =$

$0.78a^2$. In the limit of vanishing σ, we find that the spectral dimension goes to $-\infty$.

In general, for $n = 4$, the spectral dimension becomes negative for $\sigma < 25a^2/32$. Since the deformation parameter a is related to the minimum length scale associated with the κ-space-time, negative spectral dimension for $\sigma < 25a^2/32$ seems to indicate that the diffusion equation and/or spectral dimension looses its meaning below this scale of $25a^2/32$.

Demanding the spectral dimension to be positive definite ($D_s > 0$) we obtain an upper cut-off for deformation parameter a in the form $a^2 < 32\sigma/25$. Finally, if we let the deformation parameter to go to zero we get back the usual topological dimension.

Extended probe instead of point probe : We have started with the diffusion equation in the κ-deformed space-time, written in commutative space-time and effects of non-commutativity was incorporated through higher derivative terms appearing in the κ-deformed Laplacian(see eqn.(3.15)). Here, since the diffusion equation is written in the commutative space-time itself, we have used point particle as the probe. But the notion of points are meaningless in the non-commutaive space-time. So we might improve the previous result, by considering a probe with extended nature instead of point probe. The extended nature of probe can be incorporated using an initial condition which takes into account of this finite length of the probe particle. One such allowed function for initial condition is Gaussian distribution [2]. i.e., the initial condition for the diffusion equation (eqn.(3.15)) will be modified as

$$U(x, y; 0) = \frac{1}{(4\pi a^2)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4a^2}}. \tag{3.40}$$

Note that the width of the above Gaussian depends on the deformation parameter a which has the dimension of length. With this initial condition, we first solve for U_0 satisfying eqn.(3.21) and we get

$$U_0(x, y; \sigma) = \frac{1}{(4\pi(\sigma + a^2))^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4(\sigma+a^2)}}. \tag{3.41}$$

Since we are interested in the solution valid upto second order in a, keeping terms only up to second order terms in a we find

$$U_0(x, y; \sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}} \left[1 - \frac{na^2}{2\sigma} + \frac{a^2 \mid x - y \mid^2}{4\sigma^2}\right]. \tag{3.42}$$

Next, we solve eqn.(3.23) using the initial condition given in eqn.(3.40) and obtain

$$U_1(x, y; \sigma) = \frac{\alpha}{(4\pi(\sigma + a^2))^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4(\sigma+a^2)}} \tag{3.43}$$

Here we keep terms up to first order in a since the solution U contains aU_1. Thus U_1, valid up to first order in a is given by

$$U_1(x,y;\sigma) = \frac{\alpha}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}}. \tag{3.44}$$

Next we solve for U_2 satisfying eqn.(3.25). In order to solve this we use U_0 obtained in eqn.(3.42). We also use the initial condition give in eqn (3.40), which incorporate the extended nature of the probe. But in this case, we need to consider only zeroth order terms in a (since U_2 comes with a coefficient a^2 in U) and thus the solution for U_2, valid up to zeroth order in a, is same as the one satisfying eqn.(3.25), obtained earlier.

Thus, with the probe having an extension, we find the solution for heat kernel, valid upto second order in a as

$$U(x,y;\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}} \left[1 - \frac{na^2}{2\sigma} + \frac{a^2 \mid x-y \mid^2}{4\sigma^2} + a\alpha + a^2\beta\right] + a^2 U_{22}. \tag{3.45}$$

Using this solution we obtain the return probability as

$$P(\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} \left[1 + a\alpha + a^2\beta - a^2\frac{(n+1)^2}{16\sigma} - \frac{na^2}{2\sigma}\right]. \tag{3.46}$$

Using this, we evaluate the spectral dimension to be

$$D_s = \frac{n + na\alpha + na^2\beta - (n+2)(n+1)^2\frac{a^2}{16\sigma} - n(n+2)\frac{a^2}{2\sigma}}{1 + a\alpha + a^2\beta - a^2\frac{(n+1)^2}{16\sigma} - \frac{na^2}{2\sigma}}. \tag{3.47}$$

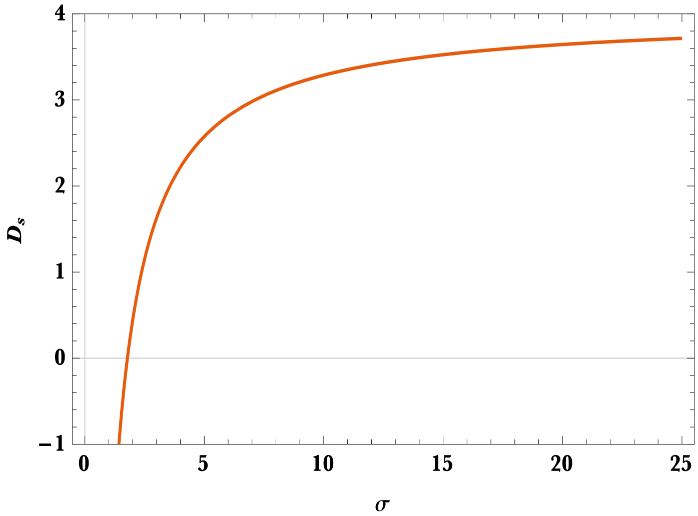
Keeping upto first non-vanishing terms in a, we find

$$D_s = n - (n+1)^2\frac{a^2}{8\sigma} - \frac{na^2}{\sigma} \tag{3.48}$$

By comparing the above expression for D_s with the eqn.(3.39) we see that the extended probe leads to an additional term $-\frac{na^2}{\sigma}$. This additional term depends on the topological dimension n. We note that, as for the particle like probe used earlier, here too, the spectral dimension is same as the topological dimension in the limit $\sigma \to \infty$. Similarly, in the limit $\sigma \to 0$ we see that $D_s \to -\infty$, exactly same as in the case of particle like probe.

For $n = 4$, we find the spectral dimension obtained in eqn.(3.48) vanishes when $\sigma = 1.781a^2$ and it become negative for $\sigma < \frac{57}{32}a^2$. Demanding that the spectral dimension should be positive definite imply a cut-off on

Figure 3.2: Spectral dimension as a function of σ with $a = 1$ and $n = 4$ for $D_\mu D_\mu$ as Laplacian with extended probe.



the deformation parameter given by $a^2 < \frac{32\sigma}{57}$. Thus we see that the effect of extended nature of the probe is to modify the cut-off values of σ and a^2, but do not change the general feature of the dimensional flow and also do not affect the values of spectral dimension in the limit $\sigma \to 0$ as well as in the limit $\sigma \to \infty$. Variation of D_s with σ is plotted in fig.[3.2].

3.3.2 Spectral dimension from $\Box$ operator

In this section, we use $\Box$ operator as the Laplacian in the κ-deformed space-time, which reduces to the usual Laplacian in the commutative space-time. Thus we start with the diffusion equation

$$\frac{\partial}{\partial\sigma}U(x,y;\sigma) = \Box U(x,y;\sigma), \tag{3.49}$$

where $\Box = \nabla^2 \frac{e^{-A}}{\varphi^2} - \partial_n^2 \frac{2(1-coshA)}{A^2}$, and $A = ia\partial_n$. Also, we choose $\varphi = e^{-\frac{A}{2}}$. Now we expand the $\Box$ operator in terms of deformation parameter a.

$$\begin{aligned} \Box &= \nabla^2 + \frac{2}{a^2}[1 - \cosh(ia\partial_n)] \\ &= \nabla^2 + \frac{2}{a^2}\left[\frac{a^2\partial_n^2}{2!} - \frac{a^4\partial_n^4}{4!} + \frac{a^6\partial_n^6}{6!} - ...\right] \\ &= \nabla^2 + \partial_n^2 - \frac{a^2}{12}\partial_n^4 + \frac{a^4}{360}\partial_n^6 - ... \end{aligned} \tag{3.50}$$

We restrict our attention to first non-vanishing correction due to non-commutativity. Thus the above diffusion equation becomes

$$\frac{\partial}{\partial\sigma}U(x,y;\sigma) = \left(\nabla_n^2 - \frac{a^2}{12}\partial_n^4\right)U(x,y;\sigma), \tag{3.51}$$

where $\nabla_n^2 = \nabla^2 + \partial_n^2$. Here, we note that the a dependent term involves quartic time derivative, but there are no terms involving product of derivatives with respect to different coordinates, unlike the κ-diffusion equation obtained in eqn.(3.15). As earlier, we solve the above equation perturbatively using eqn.(3.18) for U, i.e.,

$$\frac{\partial}{\partial\sigma}(U_0 + aU_1 + a^2U_2) = \left(\nabla_n^2 - \frac{a^2}{12}\partial_n^4\right)(U_0 + aU_1 + a^2U_2) \tag{3.52}$$

Collecting the zeroth order terms in a shows that U_0 satisfy the usual heat equation, i.e.,

$$\frac{\partial}{\partial\sigma}U_0(x,y;\sigma) = \nabla_n^2 U_0(x,y;\sigma) \tag{3.53}$$

with solution

$$U_0(x,y;\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}}e^{-\frac{|x-y|^2}{4\sigma}} \tag{3.54}$$

and the first order terms in a leads to the equation

$$\frac{\partial}{\partial\sigma}U_1(x,y;\sigma) = \nabla_n^2 U_1(x,y;\sigma) \tag{3.55}$$

showing that U_1 also satisfy the usual heat equation and therefore the solution is

$$U_1(x,y;\sigma) = \frac{\alpha}{(4\pi\sigma)^{\frac{n}{2}}}e^{-\frac{|x-y|^2}{4\sigma}} \tag{3.56}$$

where α has the dimensions of L^{-1}. Note that in obtaining the solutions for U_0 and U_1, we have used the initial condition of the test particle being localized at a given point in the space, as in the commutative space-time.

Next, by equating the second order terms in a, we find that the equation satisfied by U_2 as

$$\frac{\partial}{\partial\sigma}U_2(x,y;\sigma) = \nabla_n^2 U_2(x,y;\sigma) - \frac{1}{12}\partial_n^4 U_0(x,y;\sigma). \tag{3.57}$$

Note that the above equation has a quartic term in the Euclidean time derivative. Now substituting eqn.(3.54) in the above, we obtain

$$\frac{\partial U_2}{\partial\sigma} = \nabla_n^2 U_2 + \left[-\frac{1}{16\sigma^2} + \frac{(x_n - y_n)^2}{16\sigma^3} - \frac{(x_n - y_n)^4}{192\sigma^4}\right]\frac{1}{(4\pi\sigma)^{\frac{n}{2}}}e^{-\frac{|x-y|^2}{4\sigma}} \tag{3.58}$$

We solve this equation using Duhamel's principle, which is discussed in the above section. Then we obtain the solution U_2 as

$$\begin{aligned}
U_2(x,y;\sigma) &= \frac{e^{-\frac{|x-y|^2}{4\sigma}}}{(4\pi\sigma)^{\frac{n}{2}}}\Big[\beta - \frac{(x_n-y_n)^4}{192\sigma^4}(\sigma-\epsilon) + \frac{(x_n-y_n)^2}{16\sigma^2}\ln(\sigma/\epsilon) \\
&+ \frac{(\epsilon-\sigma)}{16\sigma\epsilon} - \frac{(x_n-y_n)^2}{16\sigma^3}\left(\sigma\ln(\sigma/\epsilon)-\sigma+\epsilon\right) \\
&- \frac{(x_n-y_n)^3}{24\sqrt{\sigma^7\pi}}\left(\sigma tan^{-1}\sqrt{\sigma/\epsilon-1}-\epsilon\sqrt{\sigma/\epsilon-1}\right) \\
&- \frac{(x_n-y_n)}{6\sqrt{\sigma^5\pi}}\left[(2\sigma+\epsilon)\sqrt{\sigma/\epsilon-1}-3\sigma tan^{-1}\sqrt{\sigma/\epsilon-1}\right] \\
&- \frac{1}{16\sigma^2}\left(-2\sigma\ln(\sigma/\epsilon)+\frac{\sigma^2-\epsilon^2}{\epsilon}\right) \\
&+ \frac{(x_n-y_n)}{2\sqrt{\sigma^3\pi}}\left(tan^{-1}\sqrt{\sigma/\epsilon-1}-\sqrt{\sigma/\epsilon-1}\right) \\
&+ \frac{1}{8\sigma}\left[-1-\ln(\sigma/\epsilon)+\sigma/\epsilon\right]\Big]. \qquad (3.59)
\end{aligned}$$

As in eqn.(3.34), we see that the Euclidean time and space coordinate are treated differently. Here also, the ϵ is the lower cut-off introduced in evaluating the integral and we will take the limit $\epsilon \to 0$ after calculating the spectral dimension.

Using this in eqn.(3.35), we obtain the return probability

$$P_g(\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}}\left[1 + a\alpha + a^2\beta + a^2\left(-\frac{1}{16\sigma}+\frac{\epsilon}{16\sigma^2}\right)\right] \qquad (3.60)$$

and from eqn.(3.37), we find the spectral dimension as

$$D_s = \frac{n + na\alpha + na^2\beta - (n+2)\frac{a^2}{16\sigma} + (n+4)\frac{a^2\epsilon}{16\sigma^2}}{1 + a\alpha + a^2\beta + a^2(\frac{-1}{16\sigma}+\frac{\epsilon}{16\sigma^2})}. \qquad (3.61)$$

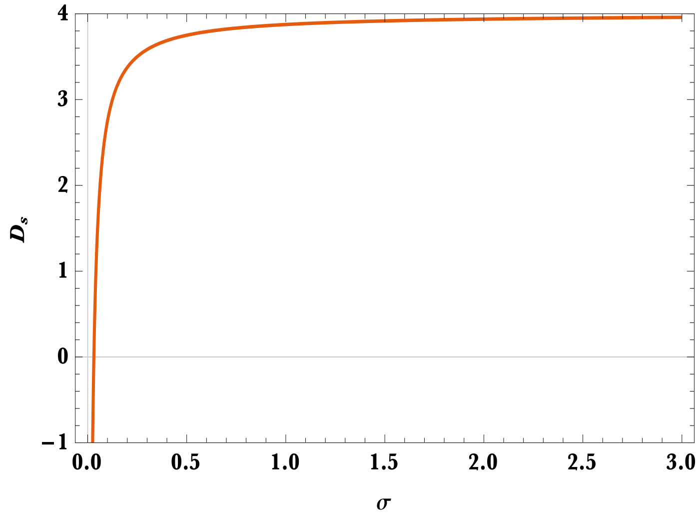
Using binomial expansion, we re-write this expression as

$$D_s = n - \frac{a^2}{8\sigma} - \frac{a^2\epsilon}{4\sigma^2} + O(a^3). \qquad (3.62)$$

We note that all the first order terms in a cancel with each other and thus the first non-vanishing correction due to the non-commutativity is in the second order in the deformation parameter a. After setting the cut-off parameter to zero, we find the spectral dimension to be

$$D_s = n - \frac{a^2}{8\sigma}. \qquad (3.63)$$

Figure 3.3: Spectral dimension as a function of σ with $a = 1$ and $n = 4$ for $\Box$ as Laplacian.



We note that the non-commutative correction is independent of the topological dimension n. Thus the change in the effective dimension is same irrespective of the topological dimension of the space-time we start with. This should be contrasted with the spectral dimension we found in eqn.(3.39).

It is easy to see (Fig.[3.3]) that in the limit $\sigma \to \infty$, we find that spectral dimension and topological dimension to be equal. For $\sigma = a^2/8n$, the spectral dimension vanishes and for smaller values ($\sigma < a^2/8n$), the spectral dimension becomes negative. As in the case of eqn.(3.39), here to, the the spectral dimension seems to loose its meaning below this scale. Here again, we can set the bound on the deformation parameter as $a^2 < 8n\sigma$ to guarantee the positivity of the spectral dimension.

Extended probe instead of point probe : We can also consider a probe with finite extension in place of probe localized at a single point. With the initial condition which imply the extended nature of the probe

$$U(x, y; 0) = \frac{1}{(4\pi a^2)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4a^2}}, \tag{3.64}$$

we solve eqn.(3.51), then we obtain the solution to U_0 satisfying eqn.(3.53) as

$$U_0(x, y; \sigma) = \frac{1}{(4\pi(\sigma + a^2))^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4(\sigma+a^2)}}. \tag{3.65}$$

This can be expanded in terms of deformation parameter a as

$$U_0(x,y;\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}} \left[1 - \frac{na^2}{2\sigma} + \frac{a^2 \mid x-y \mid^2}{4\sigma^2}\right]. \tag{3.66}$$

Similarly, solving eqn.(3.55) with the modified initial condition given above, we obtain the solution for U_1, valid up to first order in a as

$$U_1(x,y;\sigma) = \frac{\alpha}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}}. \tag{3.67}$$

Since we only need solution of U_2 valid only up to zeroth order in a, we find that the solution is same as the one satisfying eqn.(3.57). Thus we have the solution for heat kernel valid upto second order in a as

$$U(x,y;\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}} \left[1 - \frac{na^2}{2\sigma} + \frac{a^2 \mid x-y \mid^2}{4\sigma^2} + a\alpha + a^2\beta\right] + a^2 U_{22}. \tag{3.68}$$

Using this solution we obtain the return probability

$$P_g(\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} \left[1 + a\alpha + a^2\beta - \frac{a^2}{16\sigma} - \frac{na^2}{2\sigma}\right], \tag{3.69}$$

and the spectral dimension

$$D_s = \frac{n + na\alpha + na^2\beta - (n+2)\frac{a^2}{16\sigma} - n(n+2)\frac{a^2}{2\sigma}}{1 + a\alpha + a^2\beta - \frac{a^2}{16\sigma} - \frac{na^2}{2\sigma}}. \tag{3.70}$$

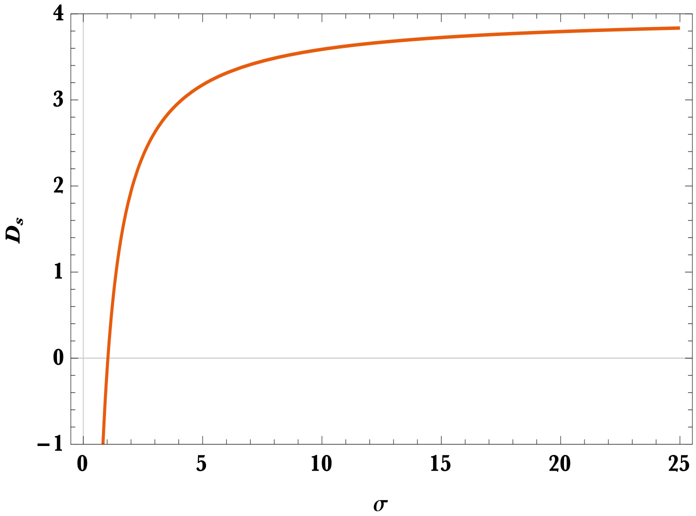
Keeping terms valid up to first non-vanishing terms in a, we find

$$D_s = n - \frac{a^2}{8\sigma} - \frac{na^2}{\sigma} \tag{3.71}$$

By comparing with the eqn.(3.63) we see an additional term $-\frac{na^2}{\sigma}$ in the above, which depends on the topological dimension. We also note that the limiting values of the spectral dimension is same as in that for the spectral dimension obtained for point probe in eqn.(3.63).

For n = 4, we note that the spectral dimension D_s equal to zero when $\sigma = 1.031a^2$ and it become negative for $\sigma < \frac{33}{32}a^2$. As earlier, the requirement of positivity of the spectral dimension will lead to the condition $a^2 < \frac{32}{33}\sigma$. We see that the effect of extended nature of the probe only modify the cut-off values of σ and a^2, while the general feature of the dimensional flow and the values of spectral dimension in the limit $\sigma \to 0$ as well as in the limit $\sigma \to \infty$ are unaffected, in the case of Laplacian being $\Box$ also. The dimensional flow in this case is plotted in fig.[3.4].

Figure 3.4: Spectral dimension as a function of σ with $a = 1$ and $n = 4$ for $\Box$ as Laplacian with extended probe.



3.4 Conclusion

Table 3.1: Summary of the Results

Laplace operator	Nature of the Probe	D_s	$\lim_{\sigma\to\infty} D_s$	$\lim_{\sigma\to 0} D_s$	Bound on a (for $n = 4$)
$D_\mu D_\mu$	Point probe	$D_s = n - (n+1)^2 \frac{a^2}{8\sigma}$	n	$-\infty$	$a^2 < \frac{32}{25}\sigma$
	Extended probe	$D_s = n - (n+1)^2 \frac{a^2}{8\sigma} - \frac{na^2}{\sigma}$	n	$-\infty$	$a^2 < \frac{32}{57}\sigma$
$\Box$	Point probe	$D_s = n - \frac{a^2}{8\sigma}$	n	$-\infty$	$a^2 < 32\sigma$
	Extended probe	$D_s = n - \frac{a^2}{8\sigma} - \frac{na^2}{\sigma}$	n	$-\infty$	$a^2 < \frac{32}{33}\sigma$

In this chapter we have investigated the spectral dimension of κ-space-time using the kappa-deformed diffusion equation. The deformed diffusion equation is constructed for two different choices of Laplacians in n-dimensional, κ-deformed Euclidean space. We use an approach where the deformed Laplacians are expressed in the commutative space-time itself. A distinct feature of this diffusion equation is the presence of higher derivative terms associated with the non-commutative nature of space-time. The diffusion equation so obtained is solved perturbatively in order to find the heat kernel. This is then used to calculate the spectral dimension. Inspecting the expression for spectral dimension in both cases, we notice that the spectral dimension depends on the combination, $\frac{a^2}{\sigma}$ of deformation parameter a and probe scale σ. Also, we see that for large diffusion parameter

σ the spectral dimension is same as the topological dimension n and on the other hand, the spectral dimension goes to $-\infty$ as σ vanishes. We obtained an upper cut-off for deformation parameter a, by demanding the positivity of spectral dimension.

For our analysis, we have chosen the topological dimension of space-time to be 4. When the Laplacian is chosen to be $D_\mu D_\mu$, we find that spectral dimension becomes zero for $\sigma = 0.78a^2$. Also, we see that spectral dimension is positive when $\sigma > \frac{25}{32}a^2$. For the choice, $\mathcal{L} = \Box$, the spectral dimension is positive when $\sigma > \frac{a^2}{32}$ and below which it becomes negative. Since, spectral dimension should be always positive for having valid physical interpretation, we can find an upper cut-off for deformation parameter a. It is found that the positivity condition leads to the cut-off $a^2 < \frac{32}{25}\sigma$ for $D_\mu D_\mu$ operator and $a^2 < 32\sigma$ for $\Box$. If we replace a point probe by an extended probe we get an additional term, $-\frac{4a^2}{\sigma}$ for both choices of Laplacian. However, the qualitative behaviour of the spectral dimension remains the same. These results are given in table [3.1] above.

Vanishing of spectral dimension was argued to be sign of space-time loosing its meaning at trans-Planckian scales[2]. We note that the deformation parameter a, having length dimension, is related to the minimal length associated with the κ-space-time and expected to be of the order of Planck length. For both the cases we studied here, we find that the spectral dimension is positive much below the scale set by the deformation parameter a and become negative when the probe scale is less than $0.78a^2$ and $0.031a^2$, respectively. As discussed in the introduction, some of the earlier studies on the dimensional flow in κ-space-time, using an approach different from the one used here, showed the growth of spectral dimension to $+\infty$, as the probe scale vanishes[4], indicating super diffusion at small probe scales. In the present case, we have the opposite situation where the spectral dimension goes to $-\infty$.

The probability density becoming negative for diffusion equations involving higher derivatives have been noted earlier and a possible re-interpretation of the spectral dimension in such scenarios was discussed in [8]. It was noted that at low energies, the spectral dimension matchs the topological dimension only for certain values of the deformation parameter and it was argued that there should be a threshold value of deformation parameter below which the return probability is not a physically acceptable solution. We can take a similar approach in the cases studied here. We have seen that for certain values of the diffusion parameter we get the spectral dimension to be negative. By inverting this inequality, we can get condition on the deformation parameter ($a^2 < 32\sigma/25$ and $a^2 < 32\sigma$, respectively for the cases we considered here with $n = 4$) for which the spectral dimension is positive. This may also be an indication of the existence of multi-scale structure in the κ-space-time[9].

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CHAPTER 4

Dimensional flow in the kappa-deformed space-time

4.1 Introduction

In chapter 3, we constructed diffusion equation in κ-space-time by using the Casimir of the undeformed κ-Poincare algebra as the Beltrami-Laplace operator appearing in the diffusion equation. In this case, we have not considered possible changes of $\frac{\partial}{\partial \sigma}$ operator appearing in diffusion equation. In this chapter we address this issue of modification of diffusion equation wherein the derivative operator with respect to diffusion time σ also gets kappa-deformed correction. We present such a construction of diffusion equation in kappa-space-time in this chapter. The relation between Schrödinger equation and diffusion equation will play a crucial role in this construction.

It is well known that Schrödinger equation and diffusion equation are related by a Wick rotation [1]. The time-dependent Schrödinger equation for a free particle is

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2\mu}\nabla^2\psi(x,t), \tag{4.1}$$

where μ is the particle's mass, ∇^2 is the Laplacian, and ψ is the wave function. By replacing t in eqn.(4.1) by $-it$ one obtain

$$\frac{\partial \psi}{\partial t} = \frac{\hbar}{2\mu}\nabla^2\psi. \tag{4.2}$$

Dimension of $\frac{\hbar}{2\mu}$ is $length^2/time$ which is same as the dimension of diffusion coefficient D. By re-defining $\frac{\hbar}{2\mu} = D$, one re-express the above equation as the standard diffusion equation,

$$\frac{\partial \psi}{\partial t} = D\nabla^2\psi. \tag{4.3}$$

That is, one can obtain the diffusion equation (4.3) from the Schrödinger equation (4.1) by mapping time(t) to imaginary time ($-it$).

In this chapter, we construct possible modifications to the heat equation due to κ-deformation and study its implication on the scale-dependence of the space-time dimension. We use the connection between the Schrödinger equation to diffusion equation in constructing deformed diffusion equation valid in κ-space-time. Thus, we start from well studied κ-deformed Klein-Gordon equations written using Beltrami-Laplace operator and derive its non-relativistic limit. From thus obtained κ-deformed Schrödinger equation, we construct the deformed heat equation by a Wick rotation (by implementing the map $t \to -it$). Here, we use the κ-deformed Klein-Gordon equation written in commutative space-time [2] as our starting point. Thus note that the Wick rotation is applied to the theory written in the commutative space-time and all the effects of non-commutativity are included through the deformation parameter 'a' dependent terms. This allows us to side-step the non-trivial issues associated with the Wick rotation in non-commutative space-time. The problem associated with the Wick rotation in non-commutative space-time has been analyzed, particularly for the case of moyal space-time in [3, 4, 5]. It was shown in [3] that the naive Wick rotation will lead to the theory being non-unitary and a consistant way to map non-commutative theory from Euclidean to Minkowski signature was obtained [3, 4, 5, 6]. Note that the deformation parameter 'a' is unaffected by the Wick rotation. This allows us to investigate two related issues namely (i) spectral dimension of κ-deformed space and its scaling with energy (ii) dimensional flow of the full κ-deformed space-time. The first problem is studied by evaluating the spectral dimension of κ-space using the κ-deformed heat equation derived from the κ-deformed Schrödinger equation. Here we use the (d-1) dimensional Laplacian constructed as the space derivative part of the non-relativistic limit of κ-deformed Klein-Gordon equation. For investigating the second problem, we take the κ-deformed heat equation obtained by the Wick rotation of the κ-deformed Schrödinger equation and replace the Laplacian with the Euclideanised Beltrami-Laplace operator defined in the d-dimensional κ-deformed space-time. We have carried out this study by different choices of κ-deformed Klein-Gordon equations.

In the first case, thus, we study the spectral dimension of the spatial part of the κ-deformed space-time. Since the spatial coordinates of the κ-space-time commute among themselves and it is the time coordinate which does not commute with the space-coordinates, this approach is appropriate to study how the space dimension of the κ-space-time changes as the probe scale is changed due to the non-commutativity between time and space coordinates. We see that the effect of non-commutativity is to introduce higher spatial derivative terms as well as terms involving both spatial and temporal derivatives in the deformed diffusion equation. The role of these terms on the spectral dimension is brought out here.

In the second case, we take the κ-deformed heat equation as the starting point of the analysis and replace the Laplacian $\mathcal{L}$ by the Euclideanised κ-deformed Beltrami-Laplace operator. Thus here, the non-commutativity shows itself in two ways, by introducing the higher derivative terms in the deformed heat equation and also through the additional terms appearing in the deformed Baltrami-Laplace operator. Here also we do the analysis for different choices of Baltrami-Laplace operator.

Organization of this chapter is as follows. In the second section, we setup the diffusion equation using the deformed Klein-Gordon equation. We start with the Klein-Gordon equation in κ-Minkowski space-time written in terms of commuting coordinates and all the effects of non-commutativity are contained in the 'a'(deformation parameter) dependent terms. By taking the non-relativistic limit of this theory written in terms of the commutative variables and applying Wick rotation, we derive the diffusion equation in the κ-deformed Euclidean space, valid upto first non-vanishing terms in the deformation parameter a. We then solve this diffusion equation perturbatively and use this solution to calculate the spectral dimension. We have also analyzed the change in the spectral dimension due to extended nature of the probe. In the next subsection, we start with a different choice of generalized κ-deformed Klein-Gordon equation and arrive at the κ-deformed diffusion equation. The spectral dimension is calculated using its solution and dimensional flow is analyzed. In section 3, we replace the Laplacian in the modified diffusion equation with the two different choices of Beltrami-Laplace operator and use this diffusion equation to calculate the spectral dimension. The analysis of results and summary are presented in the last section.

4.2 κ-deformed diffusion equation and spectral dimension

In this section, we derive the κ-deformed diffusion equations starting from two possible choices of κ-deformed Klein-Gordon equations. The diffusion equation is related to the Schrödinger equation under the mapping $t \to -it$ and we use this map to derive the deformed diffusion equation. By replacing t with $-it$ in the κ-deformed Schrödinger equations, derived by taking the non-relativistic limit of κ-deformed Klein-Gordon equation, we obtain the κ-deformed diffusion equations. Using perturbative method, we obtain their solutions valid upto second order in the deformation parameter. From these solutions, we calculate the return probability which is a measure of finding a particle back at the starting point after a finite time gap. Using this, we calculate the spectral dimension.

4.2.1 Diffusion equation from the κ-deformed Klein-Gordon equation $(D_\mu D^\mu - m^2)\phi = 0$

Here we derive the deformed diffusion equation from the non-relativistic limit of κ-deformed Klein-Gordon equation $(D_\mu D^\mu - m^2)\phi = 0$. Consider an n-dimensional κ-deformed Minkowski space with signature (-++...+). The Generalized Klein-Gordon equation on κ-deformed space-time[2] is

$$\Box(1 + \frac{a^2}{4}\Box)\phi = \frac{m^2c^2}{\hbar^2}\phi, \tag{4.4}$$

where

$$\Box = \nabla_{n-1}^2 \frac{e^{-A}}{\varphi^2} + \partial_0^2 \frac{2(1-\cosh A)}{A^2}. \tag{4.5}$$

Here $\nabla_{n-1}^2 = \Sigma_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2}$, $A = -ia\partial_0$, and we choose $\varphi = e^{-\frac{A}{2}}$. We expand LHS of eqn.(4.4) in terms of the deformation parameter a as

$$\begin{aligned}
\Box(1 + \frac{a^2}{4}\Box) &= \left[\nabla_{n-1}^2 - \frac{2}{a^2}(1-\cosh A)\right]\left[1 + \frac{a^2}{4}\nabla_{n-1}^2 - \frac{1}{2}(1-\cosh A)\right] \\
&= \nabla_{n-1}^2 - \nabla_{n-1}^2(1-\cosh A) - \frac{2}{a^2}(1-\cosh A) \\
&+ \frac{a^2}{4}\nabla_{n-1}^4 + \frac{1}{a^2}(1-\cosh A)^2 \\
&= \nabla_{n-1}^2 + \nabla_{n-1}^2\left(\frac{A^2}{2!} + \frac{A^4}{4!} + ...\right) + \frac{2}{a^2}\left(\frac{A^2}{2!} + \frac{A^4}{4!} + ...\right) \\
&+ \frac{a^2}{4}\nabla_{n-1}^4 + \frac{1}{a^2}\left(\frac{A^2}{2!} + \frac{A^4}{4!} + ...\right)^2.
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
\Box(1+\frac{a^2}{4}\Box) &= \nabla_{n-1}^2 + \left(\nabla_{n-1}^2 + \frac{2}{a^2}\right)\left(-\frac{a^2\partial_0^2}{2!} + \frac{a^4\partial_0^4}{4!} + ...\right) \\
&+ \frac{a^2}{4}\nabla_{n-1}^4 + \frac{1}{a^2}\left(-\frac{a^2\partial_0^2}{2!} + \frac{a^4\partial_0^4}{4!} + ...\right)^2 \\
&= \nabla_{n-1}^2 - \partial_0^2 + \frac{a^2}{4}\nabla_{n-1}^4 - \frac{a^2}{2}\nabla_{n-1}^2\partial_0^2 + \frac{a^2}{12}\partial_0^4 + \frac{a^2}{4}\partial_0^4 \\
&+ \frac{a^4}{24}\nabla_{n-1}^2\partial_0^4 - \frac{a^4}{360}\partial_0^6 - \frac{a^4}{24}\partial_0^6 + O(a^6) + ... \qquad (4.7)
\end{aligned}$$

We restrict our attention to first non-vanishing corrections due to non-commutativity. Thus we obtain the equation valid upto second-order in a as

$$\left(\nabla_{n-1}^2 - \partial_0^2 + \frac{a^2}{4}\nabla_{n-1}^4 - \frac{a^2}{2}\nabla_{n-1}^2\partial_0^2 + \frac{a^2}{3}\partial_0^4\right)\phi = \frac{m^2c^2}{\hbar^2}\phi. \qquad (4.8)$$

We next construct κ-deformed Schrödinger equation by taking the non-relativistic limit of the above κ-deformed Klein-Gordon equation. Note that the eqn.(4.8) is written, completely in the commutative space-time. This allows us to use the well known calculation scheme to obtain the non-relativistic limit [7]. Thus we start with the ansatz wave function ϕ where one separates out the rest-mass dependence and further we use the fact that in the non-relativistic limit, kinetic energy is very small compared to rest mass energy. So we start with the ansatz

$$\phi(x,t) = \varphi(x,t)e^{-i\frac{mc^2}{\hbar}t} \qquad (4.9)$$

in eqn.(4.8). Here x is a point in the (n-1) space. Effectively, this ansatz allows us to extract a term containing the rest mass m. In the non-relativistic limit, $KE << mc^2$ and hence we have

$$\mid i\hbar\frac{\partial\varphi}{\partial t} \mid \ll mc^2\varphi. \qquad (4.10)$$

Substituting eqn.(4.9) in eqn.(4.8) and after using the fact that KE is much smaller than the rest mass energy (stated in eqn.(4.10)), we get the κ-deformed Schrödinger equation as

$$\begin{aligned}
\nabla_{n-1}^2\varphi + i\frac{2m}{\hbar}\frac{\partial\varphi}{\partial t} + \frac{a^2}{4}\nabla_{n-1}^4\varphi + ia^2\frac{m}{\hbar}\frac{\partial}{\partial t}\nabla_{n-1}^2\varphi + \frac{a^2}{2}\frac{m^2c^2}{\hbar^2}\nabla_{n-1}^2\varphi \\
+ia^2\frac{4m^3c^2}{3\hbar^3}\frac{\partial\varphi}{\partial t} + a^2\frac{m^4c^4}{3\hbar^4}\varphi = 0.
\end{aligned} \qquad (4.11)$$

By changing t to $-it$ in the above, we get the deformed diffusion equation as

$$\nabla^2_{n-1}\varphi - \frac{2m}{\hbar}\frac{\partial\varphi}{\partial t} + \frac{a^2}{4}\nabla^4_{n-1}\varphi - a^2\frac{m}{\hbar}\frac{\partial}{\partial t}\nabla^2_{n-1}\varphi + \frac{a^2}{2}\frac{m^2c^2}{\hbar^2}\nabla^2_{n-1}\varphi - a^2\frac{4m^3c^2}{3\hbar^3}\frac{\partial\varphi}{\partial t} + a^2\frac{m^4c^4}{3\hbar^4}\varphi = 0. \tag{4.12}$$

Redefining $kt = \sigma$ with $k = \frac{\hbar}{2m}$ and after some rearrangements, we obtain κ-deformed diffusion equation as

$$\frac{\partial\varphi}{\partial\sigma} = \nabla^2_{n-1}\varphi + \frac{a^2c^2}{8k^2}\nabla^2_{n-1}\varphi + \frac{a^2}{4}\nabla^4_{n-1}\varphi - \frac{a^2}{2}\frac{\partial}{\partial\sigma}\nabla^2_{n-1}\varphi - \frac{a^2c^2}{6k^2}\frac{\partial\varphi}{\partial\sigma} + \frac{a^2c^4}{48k^4}\varphi, \tag{4.13}$$

where $\nabla^2_{n-1} = \Sigma_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}$. In the above equation φ is a function of x and σ. It is clear from the eqn.(4.13) that, in the commutative limit ($a \to 0$), we obtain the usual diffusion equation. Note that in deriving κ-deformed diffusion equation from κ-deformed Schrödinger equation, we only replace t with -it and absorb $\frac{\hbar}{2m}$ factor into the diffusion scale σ. The κ-deformation parameter 'a' does not get any modification under this mapping.

Reordering the terms in eqn.(4.13) we get

$$\left(1 + \frac{a^2c^2}{6k^2}\right)\frac{\partial\varphi}{\partial\sigma} = \nabla^2_{n-1}\varphi + \frac{a^2c^2}{8k^2}\nabla^2_{n-1}\varphi + \frac{a^2}{4}\nabla^4_{n-1}\varphi - \frac{a^2}{2}\frac{\partial}{\partial\sigma}\nabla^2_{n-1}\varphi + \frac{a^2c^4}{48k^4}\varphi. \tag{4.14}$$

In this case, in addition to the modification in the Laplacian we also have a modification to the diffusion operator ($\frac{\partial}{\partial\sigma}$). This result is to be contrasted with deformed diffusion equation studied in chapter 3, where the modification only occurred in the Laplacian. Also note that the deformed diffusion equation has higher order spatial derivative (∇^4_{n-1}) and term involving products of temporal and spatial derivatives i.e., $\frac{\partial}{\partial\sigma}\nabla^2_{n-1}$. But there are no higher derivative terms with respect to (scaled) time (σ). These features would turn out to be significant in the calculation of spectral dimension of κ-deformed space-time.

4.2.2 Spectral dimension

The diffusion equation in κ-deformed space-time, constructed in the above section, is used to calculate the spectral dimension. We solve the diffusion equation by using delta function as the initial condition. To find the heat kernel $\varphi(x, y; \sigma)$ of the κ-deformed diffusion equation obtained in eqn.(4.13), we express the solution as a perturbative series in a as

$$\varphi = \varphi_0 + a\varphi_1 + a^2\varphi_2. \tag{4.15}$$

We note that the dimension of the terms satisfy the relations, $[\varphi_1] = \frac{1}{L}[\varphi_0]$ and $[\varphi_2] = \frac{1}{L^2}[\varphi_0]$.

Using eqn.(4.15) in eqn.(4.13) gives

$$\begin{aligned} \frac{\partial}{\partial\sigma}\left(\varphi_0 + a\varphi_1 + a^2\varphi_2\right) &= \Bigg[\nabla^2_{n-1} + \frac{a^2c^2}{8k^2}\nabla^2_{n-1} + \frac{a^2}{4}\nabla^4_{n-1} - \frac{a^2}{2}\frac{\partial}{\partial\sigma}\nabla^2_{n-1} \\ &- \frac{a^2c^2}{6k^2}\frac{\partial}{\partial\sigma} + \frac{a^2c^4}{48k^4}\Bigg]\left(\varphi_0 + a\varphi_1 + a^2\varphi_2\right) \end{aligned} \tag{4.16}$$

By equating the terms of same order in a, we solve the above equation. The zeroth order terms in a leads to

$$\frac{\partial}{\partial\sigma}\varphi_0(x, y; \sigma) = \nabla^2_{n-1}\varphi_0(x, y; \sigma). \tag{4.17}$$

The Laplacian ∇^2_{n-1} is with respect to x coordinates and will act on the x dependence of the heat kernel. The solution to this equation is given by

$$\varphi_0(x, y; \sigma) = \frac{1}{(4\pi\sigma)^{\frac{n-1}{2}}}e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4\sigma}}. \tag{4.18}$$

Next, equating the first order terms in a gives us the equation

$$\frac{\partial}{\partial\sigma}\varphi_1(x, y; \sigma) = \nabla^2_{n-1}\varphi_1(x, y; \sigma). \tag{4.19}$$

Note that here too, ∇^2_{n-1} is the Laplacian with respect to x coordinates (and this notation is used in the remaining part of this chapter) and this will act on the first argument of φ_1, namely x.

The solution $\varphi_1(x, y; \sigma)$ satisfying the above equation also have the same form as $\varphi_0(x, y; \sigma)$ since both satisfy the same heat equation. Thus we get

$$\varphi_1(x, y; \sigma) = \frac{\alpha}{(4\pi\sigma)^{\frac{n-1}{2}}}e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4\sigma}}. \tag{4.20}$$

where the constant α has dimension of L^{-1}. Now by equating the second order terms in a in eqn.(4.13), we find

$$\frac{\partial\varphi_2}{\partial\sigma} = \nabla^2_{n-1}\varphi_2 + \frac{c^2}{8k^2}\nabla^2_{n-1}\varphi_0 + \frac{1}{4}\nabla^4_{n-1}\varphi_0 - \frac{1}{2}\frac{\partial}{\partial\sigma}\nabla^2_{n-1}\varphi_0 - \frac{c^2}{6k^2}\frac{\partial\varphi_0}{\partial\sigma} + \frac{c^4}{48k^4}\varphi_0. \tag{4.21}$$

Substituting the solution for φ_0 from eqn.(4.18) in the above equation and after straight forward manipulations, we get

$$\frac{\partial\varphi_2}{\partial\sigma} = \nabla^2_{n-1}\varphi_2 + [\frac{c^4}{48k^4} + \frac{c^2}{48k^2}\frac{(n-1)}{\sigma} - \frac{(n^2-1)}{16\sigma^2} - \frac{c^2}{96k^2}\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{\sigma^2}$$
$$+\frac{(n+1)}{16\sigma^3}\Sigma_{i=1}^{n-1}(x_i-y_i)^2 - \frac{1}{64\sigma^4}(\Sigma_{i=1}^{n-1}(x_i-y_i)^2)^2]\frac{1}{(4\pi\sigma)^{\frac{n-1}{2}}}e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4\sigma}}. \tag{4.22}$$

The above equation is of the generic form

$$\frac{\partial}{\partial\sigma}\varphi_2(X,\sigma) = \nabla^2_{n-1}\varphi_2(X,\sigma) + f(X,\sigma). \tag{4.23}$$

For a given initial condition, $\varphi_2(X,0) = g(X)$, the solution to above equation can be written as [8]

$$\begin{aligned}\varphi_2(X,\sigma) &= \int_{R^{n-1}}\Phi(X-X',\sigma)g(X')dX' \\ &+ \int_0^\sigma\int_{R^{n-1}}\Phi(X-X',\sigma-s)f(X',s)dX'ds,\end{aligned} \tag{4.24}$$

where

$$\Phi(X,\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n-1}{2}}}e^{-\frac{\left(X_1^2+X_2^2+\ldots+X_{n-1}^2\right)}{4\sigma}}. \tag{4.25}$$

Using the initial condition $\varphi_2(X,0) = \delta^{n-1}(X)$, we obtain the first term φ_{21} of the RHS of eqn.(4.24) as

$$\begin{aligned}\varphi_{21}(X,\sigma) &= \int_{R^{n-1}}\Phi(X-X',\sigma)g(X')dX' \\ &= \int_{R^{n-1}}\frac{1}{(4\pi\sigma)^{\frac{n-1}{2}}}e^{-\frac{\left((X_1-X_1')^2+(X_2-X_2')^2+\ldots+(X_{n-1}-X_{n-1}')^2\right)}{4\sigma}}\delta^{n-1}(X')dX' \\ &= \frac{1}{(4\pi\sigma)^{\frac{n-1}{2}}}e^{-\frac{\Sigma_{i=1}^{n-1}X_i^2}{4\sigma}} \\ \varphi_{21}(x,y;\sigma) &= \frac{\beta}{(4\pi\sigma)^{\frac{n-1}{2}}}e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4\sigma}}.\end{aligned} \tag{4.26}$$

where β has dimension L^{-2}. The second term on RHS of eqn.(4.24), φ_{22} is calculated as

$$
\begin{aligned}
\varphi_{22}(x,y;\sigma) &= \int_0^\sigma \int_{R^{n-1}} \Phi(X-X',\sigma-s)f(X',s)dX'ds \\
&= \frac{e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4\sigma}}}{(4\pi\sigma)^{\frac{n-1}{2}}}\Bigg[\left(\frac{c^4}{48k^4}-\frac{n^2-1}{16\sigma^2}+\frac{c^2}{48k^2}\frac{(n-1)}{\sigma}\right)(\sigma-\epsilon) \\
&- \left(\frac{c^2}{96k^2\sigma^2}+\frac{1}{64\sigma^4}\Sigma_{i=1}^{n-1}(x_i-y_i)^2\right)\Sigma_{i=1}^{n-1}(x_i-y_i)^2(\sigma-\epsilon) \\
&+ \frac{(n+1)}{16\sigma^3}\Sigma_{i=1}^{n-1}(x_i-y_i)^2(\sigma-\epsilon) \\
&- \left(\frac{c^2}{24k^2}\frac{1}{\sigma\sqrt{\sigma\pi}}\right)\Sigma_{i=1}^{n-1}(x_i-y_i)(\sigma\tan^{-1}q-\epsilon q) \\
&- \left(\frac{1}{8\sigma^3\sqrt{\sigma\pi}}\Sigma_{i=1}^{n-1}(x_i-y_i)^2\right)\Sigma_{i=1}^{n-1}(x_i-y_i)(\sigma\tan^{-1}q-\epsilon q) \\
&- \frac{1}{2\sigma^3\pi}\Big[(x_1-y_1)\Sigma_{i=2}^{n-1}(x_i-y_i)+(x_2-y_2)\Sigma_{i=3}^{n-1}(x_i-y_i) \\
&+ \ldots+(x_{n-2}-y_{n-2})(x_{n-1}-y_{n-1})\Big]A \\
&+ \frac{\Sigma_{i=1}^{n-1}(x_i-y_i)}{4\sigma^2\sqrt{\sigma\pi}}\Big((5n+2)\sigma\tan^{-1}q-(4n+2)\sigma q-nq\epsilon\Big)\Bigg]
\end{aligned}
\tag{4.27}
$$

where $q=\sqrt{\frac{\sigma}{\epsilon}-1}$ and $A=\sigma\ln(\sigma/\epsilon)-\sigma+\epsilon$.

Using eqns.(4.18), (4.20), (4.26) and (4.27) in eqn.(4.15), we find the heat kernel valid upto second order in a. Using the definition of return probability we obtain $P_g(\sigma)$ (in the limit $\epsilon\to 0$) as

$$
\begin{aligned}
P_g(\sigma) &= \frac{\int d^n x\sqrt{detg_{\mu\nu}}\varphi(x,x;\sigma)}{\int d^n x\sqrt{detg_{\mu\nu}}} \\
&= \frac{1}{(4\pi\sigma)^{\frac{n-1}{2}}}\left[1+a\alpha+a^2\beta+a^2\frac{c^4}{48k^4}\sigma-a^2\frac{(n^2-1)}{16\sigma}+a^2\frac{c^2}{48k^2}(n-1)\right].
\end{aligned}
\tag{4.28}
$$

The spectral dimension is found by taking the logarithmic derivative of the above return probability. Thus we get

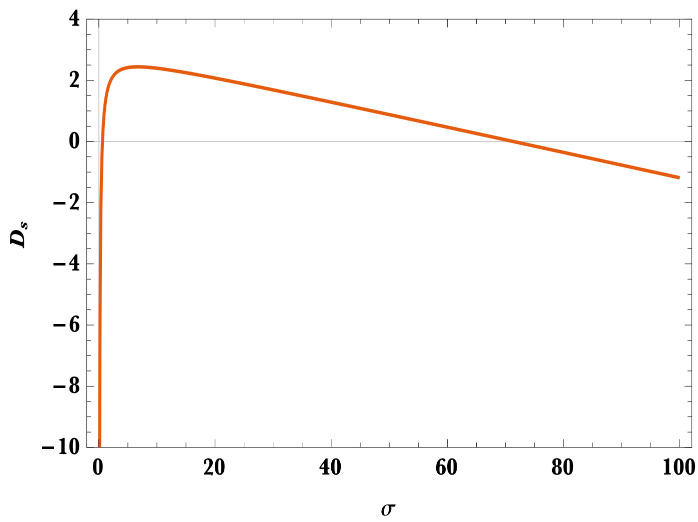
$$\begin{aligned} D_s &= \frac{-2\sigma}{P_g(\sigma)}\frac{\partial P_g(\sigma)}{\partial \sigma} \\ &= \frac{(1+a\alpha+a^2\beta)(n-1) - a^2(3-n)\frac{c^4\sigma}{48k^4} - a^2\frac{(n^2-1)(n+1)}{16\sigma} + \frac{a^2c^2}{48k^2}(n-1)^2}{1+a\alpha+a^2\beta+a^2\frac{c^4}{48k^4}\sigma - a^2\frac{(n^2-1)}{16\sigma} + a^2\frac{c^2}{48k^2}(n-1)} \end{aligned} \tag{4.29}$$

Keeping upto first non-vanishing terms in a, we obtain the spectral dimension as

$$D_s = (n-1) - \frac{a^2}{8}\frac{(n^2-1)}{\sigma} - \frac{a^2c^4}{24k^4}\sigma. \tag{4.30}$$

From the above expression, we see that apart from the usual (n-1) term, we have two additional terms and both of them are of second order in a. They arise due to the non-commutative nature of κ-space-time. One term is dependent on the topological dimension n we started with and the other term is independent of the initial dimension. Note that the diffusion scale σ appear in the n dependent correction as inverse where as in the second correction term, it appears linearly. In the commutative limit, we see that the spectral dimension is same as the topological dimension i.e $D_s = n-1$.

Figure 4.1: Spectral dimension as a function of σ for n=4, a=1, c=k=1.



For n=4 with a=c=k=1, we obtain $D_s = 3 - \frac{\sigma}{24} - \frac{15}{8\sigma}$. From this, we see that in the limit $\sigma \to 0$, the spectral dimension $D_s \to -\infty$. As σ increases the spectral dimension also increases and reaches a maximum value $D_s \sim 2.44$ for $\sigma \sim 6.7$. As we go further, the spectral dimension start decreasing (see fig.[4.1]).

In general, for n=4 and with c=k=1, we get an inequality for σ, $\frac{36}{a^2} - \sqrt{\frac{1296}{a^4} - 45} < \sigma < \frac{36}{a^2} + \sqrt{\frac{1296}{a^4} - 45}$ where the spectral dimension become positive and it takes negative value outside this range. The condition on the deformation parameter, $a^2 < \frac{72\sigma}{45+\sigma^2}$, implies that the spectral dimension is positive.

Extended probe instead of point probe : We now investigate the effect of the extended nature of the probe on the spectral dimension. For this purpose, we consider Gaussian distribution as our initial condition in solving eqn.(4.13), i.e., we take

$$\varphi(x, y; 0) = \frac{1}{(4\pi a^2)^{(\frac{n-1}{2})}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i - y_i)^2}{4a^2}}, \tag{4.31}$$

instead of the delta function condition used to obtain φ_0, φ_1 and φ_2. Using eqn.(4.31) and eqn.(4.15), we solve eqn.(4.13) and obtain the zeroth order solution as

$$\varphi_0(x, y; \sigma) = \frac{1}{(4\pi(\sigma + a^2))^{(\frac{n-1}{2})}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i - y_i)^2}{4(\sigma + a^2)}}. \tag{4.32}$$

Keeping terms upto second order in a, we find

$$\varphi_0(x, y; \sigma) = \frac{1}{(4\pi\sigma)^{(\frac{n-1}{2})}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i - y_i)^2}{4\sigma}} \left(1 + \frac{a^2}{4\sigma^2}\Sigma_{i=1}^{n-1}(x_i - y_i)^2 - (n-1)\frac{a^2}{2\sigma}\right). \tag{4.33}$$

Similarly we obtain φ_1 from eqn.(4.19), valid up to first order in a as

$$\varphi_1(x, y; \sigma) = \frac{\alpha}{(4\pi\sigma)^{(\frac{n-1}{2})}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i - y_i)^2}{4\sigma}}. \tag{4.34}$$

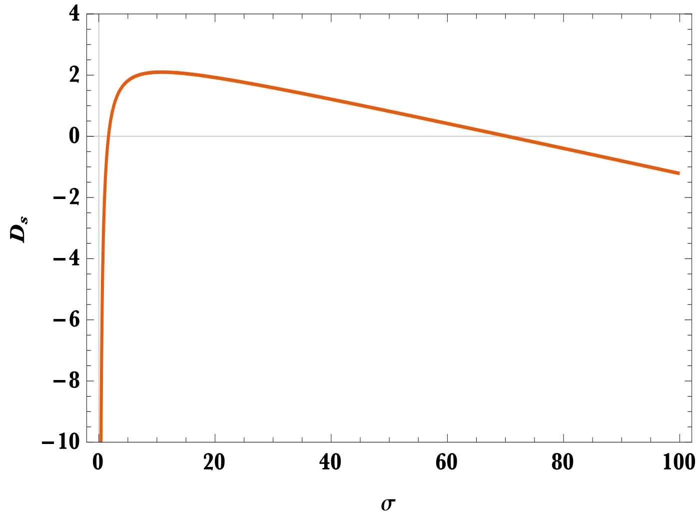
For φ_2 we need to consider only the zeroth order terms in a, since expression for φ contains $a^2\varphi_2$, and thus the solution for φ_2 will be the same as we obtained in eqns.(4.26) and (4.27). Thus we get the return probability as

$$\begin{aligned} P_g(\sigma) &= \frac{1}{(4\pi\sigma)^{\frac{n-1}{2}}}\Big[1 + a\alpha + a^2\beta + \frac{a^2c^4}{48k^4}\sigma - a^2\frac{(n^2-1)}{16\sigma} \\ &+ \frac{a^2c^2}{48k^2}(n-1) - (n-1)\frac{a^2}{2\sigma}\Big]. \end{aligned} \tag{4.35}$$

Using this we calculate the spectral dimension as

$$D_s = (n-1) - \frac{a^2c^4}{24k^4}\sigma - \frac{a^2}{8}\frac{(n^2-1)}{\sigma} - \frac{a^2}{\sigma}(n-1). \tag{4.36}$$

Figure 4.2: Spectral dimension as a function of σ for n=4, a=1, c=k=1 with extended probe.



By comparing with the eqn.(4.30), we note that we have an extra term $-\frac{a^2}{\sigma}(n-1)$, which is due to the extended nature of the probe. Further, we note that the dimensional flow has the same general behaviour as the one obtained with point particle probe in eqn.(4.30) (see fig.[4.2]). Here again, we note that there are terms with σ^{-1} dependence and one term with linear dependence on σ, the diffusion scale. The finite size effect of the test particle introduces a correction which is proportional to the inverse power of σ.

4.2.3 Diffusion equation for $(\Box - m^2)\phi = 0$ and spectral dimension

Eqn.(4.4) and eqn.(4.5) show that both $D_\mu D^\mu$ and $\Box$ operator have the same commutative limit. Thus, the requirement of correct commutative limit allow

$$\Box\phi = \frac{m^2c^2}{\hbar^2}\phi. \tag{4.37}$$

as a possible κ-deformed Klein-Gordon equation. Expand this equation in terms of deformation parameter a to obtain

$$\begin{aligned}\left[\nabla_{n-1}^2 - \frac{2}{a^2}(1-\cosh A)\right]\phi &= \frac{m^2c^2}{\hbar^2}\phi \\ \left[\nabla_{n-1}^2 + \frac{2}{a^2}\left(-\frac{a^2}{2}\partial_0^2 + \frac{a^4}{24}\partial_0^4 - \frac{a^6}{720}\partial_0^6 + ...\right)\right]\phi &= \frac{m^2c^2}{\hbar^2}\phi. \quad (4.38)\end{aligned}$$

Keeping the terms up to first non-vanishing terms in a, we find

$$\left(\nabla_{n-1}^2 - \partial_0^2 + \frac{a^2}{12}\partial_0^4\right)\phi = \frac{m^2c^2}{\hbar^2}\phi. \quad (4.39)$$

Using eqn.(4.9) and eqn.(4.10) in the above, we obtain the non-relativistic limit of eqn.(4.39) as

$$\nabla_{n-1}^2\varphi + i\frac{2m}{\hbar}\frac{\partial\varphi}{\partial t} + i\frac{a^2}{3}\frac{m^3c^2}{\hbar^3}\frac{\partial\varphi}{\partial t} + \frac{a^2}{12}\frac{m^4c^4}{\hbar^4}\varphi = 0. \quad (4.40)$$

After mapping t to $-it$ and redefining $kt = \sigma$ (where $k = \frac{\hbar}{2m}$) we re-express the above equation as

$$\frac{\partial\varphi}{\partial\sigma} = \nabla_{n-1}^2\varphi - \frac{a^2c^2}{24k^2}\frac{\partial\varphi}{\partial\sigma} + a^2\frac{c^4}{192k^4}\varphi. \quad (4.41)$$

Unlike eqn.(4.13), here we do not have higher derivatives terms. We perturbatively solve this deformed diffusion equation using the series expansion of φ given in eqn.(4.15). i.e.,

$$\frac{\partial}{\partial\sigma}[\varphi_0 + a\varphi_1 + a^2\varphi_2] = \left(\nabla_{n-1}^2 - \frac{a^2c^2}{24k^2}\frac{\partial}{\partial\sigma} + a^2\frac{c^4}{192k^4}\right)[\varphi_0 + a\varphi_1 + a^2\varphi_2]. \quad (4.42)$$

Equating the zeroth order terms in eqn.(4.42) gives

$$\frac{\partial}{\partial\sigma}\varphi_0(x,y;\sigma) = \nabla_{n-1}^2\varphi_0(x,y;\sigma), \quad (4.43)$$

whose solution is

$$\varphi_0(x,y;\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n-1}{2}}}e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4\sigma}}. \quad (4.44)$$

Equating the first order terms in a on both sides of eqn.(4.42) gives

$$\frac{\partial}{\partial\sigma}\varphi_1(x,y;\sigma) = \nabla_{n-1}^2\varphi_1(x,y;\sigma). \quad (4.45)$$

The solution to this equation is

$$\varphi_1(x,y;\sigma) = \frac{\alpha}{(4\pi\sigma)^{\frac{n-1}{2}}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4\sigma}}. \tag{4.46}$$

Note α has the dimension of inverse length. Next we collect terms of having a^2 from both sides of eqn.(4.42) to get

$$\frac{\partial\varphi_2}{\partial\sigma} = \nabla_{n-1}^2\varphi_2 - \frac{c^2}{24k^2}\frac{\partial\varphi_0}{\partial\sigma} + \frac{c^4}{192k^4}\varphi_0. \tag{4.47}$$

Substituting for φ_0 from eqn.(4.44) in the above, reduces eqn.(4.47) to

$$\begin{aligned}\frac{\partial\varphi_2}{\partial\sigma} &= \nabla_{n-1}^2\varphi_2 + \Big[\frac{c^2}{48k^2}\frac{(n-1)}{\sigma} - \frac{c^2}{96k^2}\frac{1}{\sigma^2}\Sigma_{i=1}^{n-1}(x_i-y_i)^2 \\ &+ \frac{c^4}{192k^4}\Big]\frac{1}{(4\pi\sigma)^{\frac{n-1}{2}}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4\sigma}}.\end{aligned} \tag{4.48}$$

Using eqn.(4.24), we solve this differential equation. Then the first term of eqn.(4.24) will give φ_{21} as

$$\varphi_{21}(x,y;\sigma) = \frac{\beta}{(4\pi\sigma)^{\frac{n-1}{2}}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4\sigma}} \tag{4.49}$$

where β has dimension L^{-2}. The second term on RHS of eqn.(4.24), φ_{22} is evaluated as

$$\begin{aligned}\varphi_{22}(x,y;\sigma) &= \frac{1}{(4\pi\sigma)^{\frac{n-1}{2}}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4\sigma}} \Bigg[\Big(\frac{c^4}{192k^4} - \frac{c^2}{96k^2\sigma^2}\Sigma_{i=1}^{n-1}(x_i-y_i)^2 \\ &+ \frac{c^2}{48k^2\sigma}(n-1)\Big)(\sigma-\epsilon) \\ &- \frac{c^2}{24k^2}\frac{1}{\sigma\sqrt{\sigma\pi}}\Sigma_{i=1}^{n-1}(x_i-y_i)^2\left(\sigma\tan^{-1}\sqrt{\frac{\sigma}{\epsilon}-1} - \epsilon\sqrt{\frac{\sigma}{\epsilon}-1}\right)\Bigg].\end{aligned} \tag{4.50}$$

Using eqns.(4.44),(4.46),(4.49) and eqn.(4.50) in eqn.(4.15) we find the heat kernel valid upto second order in a. From this we calculate the return probability (in the limit $\epsilon \to 0$) as

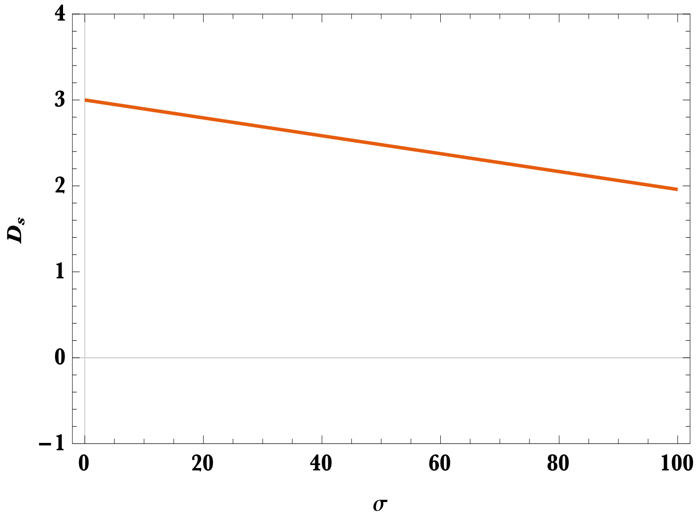
$$P_g(\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n-1}{2}}}\left[1 + a\alpha + a^2\beta + a^2\frac{c^4}{192k^4}\sigma + a^2\frac{c^2}{48k^2}(n-1)\right]. \tag{4.51}$$

Using this we find the Spectral dimension to be

$$D_s = (n-1) - \frac{a^2c^4}{96k^4}\sigma. \tag{4.52}$$

The correction of the spectral dimension is of second order in a and it is independent of the initial dimension. Thus we see that the change in spectral dimension is same for space-times of all dimensions. Here we see that the a dependent correction to the spectral dimension is linear in the diffusion scale σ. Unlike the spectral dimension obtained in eqn.(4.30), there is no term involving σ^{-1} in eqn.(4.52).

Figure 4.3: Spectral dimension as a function of σ for n=4, a=1, c=k=1.



In the commutative limit we have $D_s = n-1$, same as the topological dimension. For n=4, k=c=1, it is easy to see from the fig.[4.3] that spectral dimension $D_s = 3$ exactly at $\sigma = 0$, and it start decreasing as σ increases. For $\sigma = \frac{288}{a^2}$ the spectral dimension vanishes and it is negative for higher values of σ.

Extended probe instead of point probe : Now we want to see the change in spectral dimension due to the extended nature of probe. We use Gaussian function as initial condition and solve for the heat kernel. The modified initial condition will be

$$\varphi(x,y;0) = \frac{1}{(4\pi a^2)^{(\frac{n-1}{2})}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4a^2}}. \tag{4.53}$$

Using this initial condition, we solve eqn.(4.41) and obtain the zeroth order term as

$$\varphi_0(x,y;\sigma) = \frac{1}{[4\pi(\sigma+a^2)]^{(\frac{n-1}{2})}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4(\sigma+a^2)}}. \tag{4.54}$$

Since we are interested only upto second order terms in a, we expand this as

$$\varphi_0(x,y;\sigma) = \frac{1}{(4\pi\sigma)^{(\frac{n-1}{2})}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4\sigma}} \left(1 + \frac{a^2}{4\sigma^2}\Sigma_{i=1}^{n-1}(x_i-y_i)^2 - (n-1)\frac{a^2}{2\sigma}\right). \tag{4.55}$$

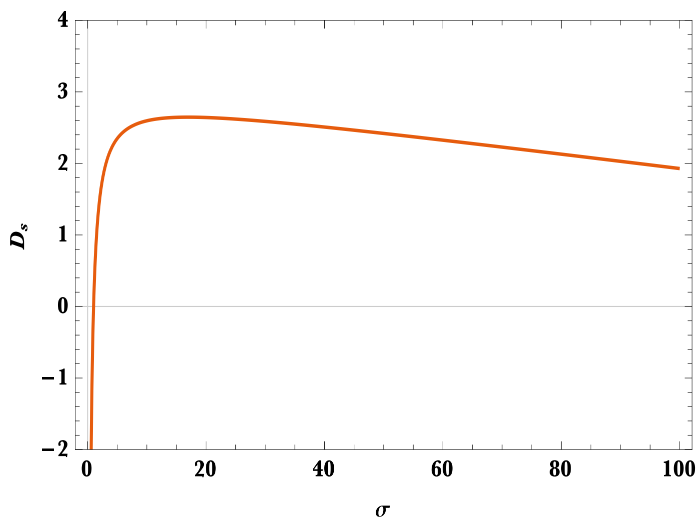
Similarly we obtain φ_1 from eqn.(4.45), valid upto first order in a as

$$\varphi_1(x,y;\sigma) = \frac{\alpha}{(4\pi\sigma)^{(\frac{n-1}{2})}} e^{-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4\sigma}}. \tag{4.56}$$

The equation for φ_2 will be same as eqn.(4.48), since we are interested only in terms of the order a^2. The resultant solution will be same as eqns.(4.49) and (4.50). Using this we obtain the spectral dimension as

$$D_s = (n-1) - \frac{a^2c^4}{96k^4}\sigma - (n-1)\frac{a^2}{\sigma}. \tag{4.57}$$

Figure 4.4: Spectral dimension as a function of σ for n=4, a=1, c=k=1 with extended probe.



By comparing with the eqn.(4.52), here we have an extra term $-\frac{a^2}{\sigma}(n-1)$ due to the extended nature of the probe. Thus we see that the extended nature of the probe introduce a correction to the spectral dimension which depends on the inverse power of the diffusion scale σ. The presence of this additional term modifies the behaviour of spectral dimension (see fig.[4.4]).

4.3 Modified κ-diffusion equation and spectral dimension

In this section, we study alternative diffusion equations than the ones analyzed in the previous section. Here, we generalise the approach where one starts from the diffusion equation and replace the Laplacian with the Beltrami-Laplace operator. Thus, we start with the κ-deformed diffusion equation derived in eqn.(4.13), but use the κ-deformed Beltrami-Laplace operator in place of the Laplacian ∇^2_{n-1}, keeping all other terms of eqn.(4.13) unchanged. Thus in this approach, we include the possible modification of the diffusion equation in κ-space-time coming from two sources. First due to the additional terms in the diffusion equation involving the derivative with respect to the diffusion time σ, and second due to the non-local and higher derivatives terms appearing through the deformed Beltrami-Laplace operator. As earlier, here too we analyze the spectral dimension using two different choices of κ-deformed Beltrami-Laplace operator.

4.3.1 Diffusion equation with Beltrami-Laplace operator and corresponding spectral dimension

In this subsection, we rewrite the diffusion equation eqn.(4.13) using the Casimir (general form of Laplacian) of the kappa-Euclidean space. The Casimir of d-dimensional κ-deformed Euclidean space is given by[9, 10, 11, 12]

$$D_\mu D_\mu = \Box(1 - \tfrac{a^2}{4}\Box) \tag{4.58}$$

$$\Box = \nabla^2_{d-1}\tfrac{e^{-A}}{\varphi^2} - \partial^2_d \tfrac{2(1-coshA)}{A^2} \tag{4.59}$$

where $\nabla^2_{d-1} = \Sigma_{i=1}^{d-1}\frac{\partial^2}{\partial x_i^2}$ and $\partial_d^2 = \frac{\partial^2}{\partial x_d^2}$. Here x_d is the Euclidean time coordinate and x_i, $i = 1, 2, ..., d-1$ are the space coordinates.

Eqn.(4.13), for a generic n-dimensional Euclidean space reads as

$$\frac{\partial\varphi}{\partial\sigma} = \nabla^2_n\varphi + \frac{a^2c^2}{8k^2}\nabla^2_n\varphi + \frac{a^2}{4}\nabla^4_n\varphi - \frac{a^2}{2}\frac{\partial}{\partial\sigma}\nabla^2_n\varphi - \frac{a^2c^2}{6k^2}\frac{\partial\varphi}{\partial\sigma} + \frac{a^2c^4}{48k^4}\varphi. \tag{4.60}$$

Since the above equation is valid for any dimensions, we use $D_\mu D_\mu$ for ∇_n^2, which is the general form of the Beltrami-Laplace operator in the κ-deformed Euclidean space.

We expand eqn.(4.58) upto first non-vanishing terms in a,

$$D_\mu D_\mu = \nabla_{d-1}^2 + \partial_d^2 - \frac{a^2}{3}\partial_d^4 - \frac{a^2}{2}\nabla_{d-1}^2\partial_d^2 - \frac{a^2}{4}\nabla_{d-1}^4, \tag{4.61}$$

and use this in eqn.(4.60) and keep terms upto second order in a

$$\begin{aligned} \frac{\partial\varphi}{\partial\sigma} &= \nabla_{n-1}^2\varphi + \partial_n^2\varphi - \frac{a^2}{3}\partial_n^4\varphi - \frac{a^2}{2}\nabla_{n-1}^2\partial_n^2\varphi - \frac{a^2}{4}\nabla_{n-1}^4\varphi \\ &+ \frac{a^2c^2}{8k^2}\left[\nabla_{n-1}^2\varphi + \partial_n^2\varphi\right] + \frac{a^2}{4}\left[\nabla_{n-1}^4\varphi + \partial_n^4\varphi + 2\nabla_{n-1}^2\partial_n^2\varphi\right] \\ &- \frac{a^2}{2}\frac{\partial}{\partial\sigma}\left[\nabla_{n-1}^2\varphi + \partial_n^2\varphi\right] - \frac{a^2c^2}{6k^2}\frac{\partial\varphi}{\partial\sigma} + \frac{a^2c^4}{48k^4}\varphi. \end{aligned} \tag{4.62}$$

By comparing with eqn.(4.60), we see that there are three extra terms in the above equation and they modify the spectral dimension (obtained in section 2.2). Note that the extra terms are of higher derivatives in space and Euclidean time coordinates. We have a term which is quartic derivative in Euclidean time, terms involving product of derivatives in space and Euclidean time and a term having quartic derivatives in space coordinate. We solve the above diffusion equation perturbatively using eqn.(4.15) for φ. By equating the zeroth order terms in a we obtain

$$\frac{\partial}{\partial\sigma}\varphi_0(x,y;\sigma) = \nabla_{n-1}^2\varphi_0(x,y;\sigma) + \partial_n^2\varphi_0(x,y;\sigma). \tag{4.63}$$

This is the usual diffusion equation in n-dimension whose solution is

$$\varphi_0(x,y;\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}}e^{-\frac{|x-y|^2}{4\sigma}}. \tag{4.64}$$

The first order term in a will give

$$\frac{\partial}{\partial\sigma}\varphi_1(x,y;\sigma) = \nabla_{n-1}^2\varphi_1(x,y;\sigma) + \partial_n^2\varphi_1(x,y;\sigma). \tag{4.65}$$

It is clear that $\varphi_1(x,y;\sigma)$ also satisfy the usual heat equation and thus

$$\varphi_1(x,y;\sigma) = \frac{\alpha}{(4\pi\sigma)^{\frac{n}{2}}}e^{-\frac{|x-y|^2}{4\sigma}}. \tag{4.66}$$

Now equate the second order terms in a in eqn.(4.62) to get

$$\begin{aligned}\frac{\partial\varphi_2}{\partial\sigma} &= \nabla^2_{n-1}\varphi_2+\partial^2_n\varphi_2-\frac{1}{3}\partial^4_n\varphi_0-\frac{1}{2}\nabla^2_{n-1}\partial^2_n\varphi_0-\frac{1}{4}\nabla^4_{n-1}\varphi_0\\ &+ \frac{c^2}{8k^2}\left(\nabla^2_{n-1}\varphi_0+\partial^2_n\varphi_0\right)+\frac{1}{4}\left(\nabla^4_{n-1}\varphi_0+\partial^4_n\varphi_0+2\nabla^2_{n-1}\partial^2_n\varphi_0\right)\\ &- \frac{1}{2}\frac{\partial}{\partial\sigma}\left(\nabla^2_{n-1}\varphi_0+\partial^2_n\varphi_0\right)-\frac{c^2}{6k^2}\frac{\partial\varphi_0}{\partial\sigma}+\frac{c^4}{48k^4}\varphi_0. \end{aligned} \tag{4.67}$$

After substitute for φ_0 in the above equation and solving, we find φ_2 as

$$\begin{aligned}\varphi_2(x,y;\sigma) &= \frac{1}{(4\pi\sigma)^{\frac{n}{2}}}e^{-\frac{\Sigma^n_{i=1}(x_i-y_i)^2}{4\sigma}}\Bigg[\beta+\Bigg(-\frac{(2n^2+4n+1)}{16\sigma}+\frac{c^2n}{48k^2}\\ &+ \frac{c^4\sigma}{48k^4}\Bigg)\left(1-\frac{\epsilon}{\sigma}\right)-\Bigg(\frac{(x_n-y_n)^4}{192\sigma^4}-\frac{(x_n-y_n)^2}{16\sigma^3}+\frac{c^2\Sigma^n_{i=1}(x_i-y_i)^2}{96k^2\sigma^2}\\ &- \frac{(n+2)}{8\sigma^3}\Sigma^n_{i=1}(x_i-y_i)^2+\frac{1}{32\sigma^4}\left(\Sigma^n_{i=1}(x_i-y_i)^2\right)^2\Bigg)(\sigma-\epsilon)\\ &+ \frac{(n+2)}{2\sigma\sqrt{\sigma\pi}}\Sigma^n_{i=1}(x_i-y_i)\left(\tan^{-1}q-q\right)-\frac{1}{2\pi\sigma^3}\Big((x_1-y_1)\Sigma^n_{i=2}(x_i-y_i)\\ &+ (x_2-y_2)\Sigma^n_{i=3}(x_i-y_i)+...+(x_{n-1}-y_{n-1})(x_n-y_n)\Big)A\\ &- \frac{1}{2\pi\sigma^3}\Big((x_n-y_n)\Sigma^n_{i=1}(x_i-y_i)+(x_1-y_1)\Sigma^{n-1}_{i=2}(x_i-y_i)\\ &+ ...+(x_{n-2}-y_{n-2})(x_{n-1}-y_{n-1})\Big)A\\ &- \Bigg(\frac{(x_n-y_n)^3}{24\sigma^3\sqrt{\sigma\pi}}+\frac{1}{4\sigma^3\sqrt{\sigma\pi}}\Sigma^n_{i=1}(x_i-y_i)\Sigma^n_{i=1}(x_i-y_i)^2\\ &+ \frac{c^2}{24k^2}\frac{1}{\sigma\sqrt{\sigma\pi}}\Sigma^n_{i=1}(x_i-y_i)\Bigg)\left(\sigma\tan^{-1}q-\epsilon q\right)\\ &+ \left(\frac{7(x_n-y_n)}{6\sigma^2\sqrt{\sigma\pi}}-\frac{(n+1)}{2\sigma^2\sqrt{\sigma\pi}}\Sigma^n_{i=1}(x_i-y_i)\right)\left((2\sigma+\epsilon)q-3\sigma\tan^{-1}q\right)\Bigg] \end{aligned} \tag{4.68}$$

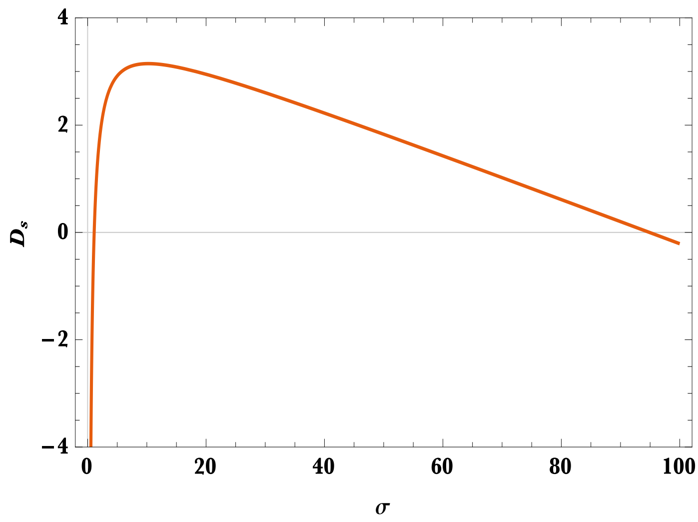
where $q=\sqrt{\frac{\sigma}{\epsilon}-1}$ and $A=\sigma\ln(\sigma/\epsilon)-\sigma+\epsilon$. Using this heat kernel, we obtain the return probability as

$$\begin{aligned}P_g(\sigma) &= \frac{1}{(4\pi\sigma)^{\frac{n}{2}}}\Big[1+a\alpha+a^2\beta-(1+n)^2\frac{a^2}{16\sigma}+a^2\frac{c^4}{48k^4}\sigma\\ &- a^2\frac{n(n+2)}{16\sigma}+\frac{a^2c^2}{48k^2}n\Big]. \end{aligned} \tag{4.69}$$

The logarithmic derivative of the above expression give the spectral dimension as

$$D_s = n - \frac{a^2}{8\sigma}(1 + 4n + 2n^2) - \frac{a^2 c^4}{24k^4}\sigma. \qquad (4.70)$$

Figure 4.5: Spectral dimension as a function of σ for n=4, a=1, c=k=1.



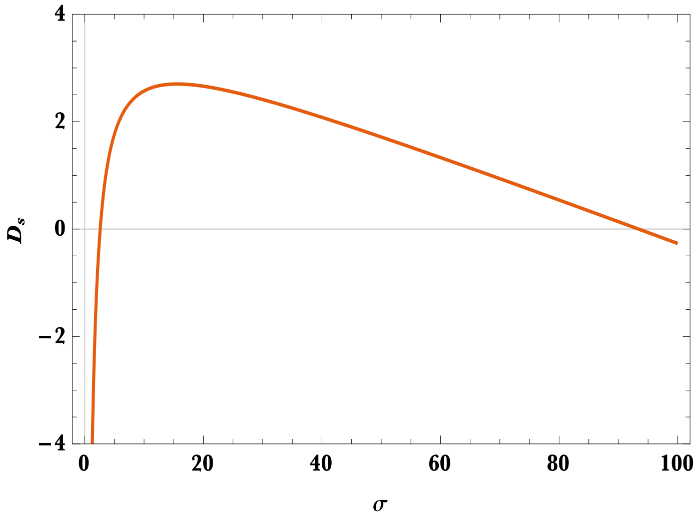
In the commutative limit we find $D_s = n$. Spectral dimension as a function of σ with n=4 and a=k=c=1 is shown in the fig.[4.5]. We see that in the limit $\sigma \to 0$, the spectral dimension $D_s \to -\infty$. As σ increases, D_s reaches a value close to 3 and thereafter decreases with increase in σ. We note that one of the correction depend on the diffusion scale linearly while the other changes as the inverse of σ. This feature is same as the spectral dimension obtained in eqn.(4.30). The requirement of the positivity of spectral dimension gives a bound on the deformation parameter as $a^2 < \frac{96\sigma}{147+\sigma^2}$.

The use of an extended probe would result the spectral dimension given by

$$D_s = n - \frac{a^2}{8\sigma}(1 + 4n + 2n^2) - \frac{a^2 c^4}{24k^4}\sigma - \frac{a^2 n}{\sigma}. \qquad (4.71)$$

By comparing with eqn.(4.70) we find an additional term $-\frac{a^2 n}{\sigma}$ due to the finite width of the probe. This new term is proportional to the initial dimension we start with and inversely proportional to σ. Note that the extended probe does not change the generic behaviour of the dimensional flow (fig.[4.6).

Figure 4.6: Spectral dimension as a function of σ for n=4, a=1, c=k=1 with extended probe.



4.3.2 Spectral dimension with $\Box$ as the Beltrami-Laplace operator

It is easy to see from eqn.(4.58) and eqn.(4.59) that the $\Box$ operator has the same commutative limit as $D_\mu D_\mu$. The eqn.(4.41) in generic n-dimension space-time is of the form

$$\frac{\partial\varphi}{\partial\sigma} = \nabla_n^2\varphi - \frac{a^2c^2}{24k^2}\frac{\partial\varphi}{\partial\sigma} + a^2\frac{c^4}{192k^4}\varphi. \tag{4.72}$$

Now we use $\Box$ as the general form of Beltrami-Laplace operator in the above equation, in place of ∇_n^2. We expand the $\Box$ operator and keep terms upto first non-vanishing terms in a, i.e.,

$$\Box = \nabla_{d-1}^2 + \partial_d^2 - \frac{a^2}{12}\partial_d^4. \tag{4.73}$$

Now substituting eqn.(4.73) in eqn.(4.72) and keeping terms upto second order in a, we get

$$\frac{\partial\varphi}{\partial\sigma} = \nabla_{n-1}^2\varphi + \partial_n^2\varphi - \frac{a^2}{12}\partial_n^4\varphi - \frac{a^2c^2}{24k^2}\frac{\partial\varphi}{\partial\sigma} + a^2\frac{c^4}{192k^4}\varphi. \tag{4.74}$$

We note that eqn.(4.74) has one extra term compared to eqn.(4.72) which is quartic derivative in the Euclidean time. We solve the above diffusion equation perturbatively using eqn.(4.15) for φ, as earlier. By equating the zeroth order terms in a we obtain

$$\frac{\partial}{\partial\sigma}\varphi_0(x,y;\sigma) = \nabla^2_{n-1}\varphi_0(x,y;\sigma) + \partial^2_n\varphi_0(x,y;\sigma) \tag{4.75}$$

and corresponding solution is

$$\varphi_0(x,y;\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}}. \tag{4.76}$$

The first order terms in a give

$$\frac{\partial}{\partial\sigma}\varphi_1(x,y;\sigma) = \nabla^2_{n-1}\varphi_1(x,y;\sigma) + \partial^2_n\varphi_1(x,y;\sigma) \tag{4.77}$$

whose solution is given by

$$\varphi_1(x,y;\sigma) = \frac{\alpha}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}}. \tag{4.78}$$

Second order terms in a will result in

$$\frac{\partial\varphi_2}{\partial\sigma} = \nabla^2_{n-1}\varphi_2 + \partial^2_n\varphi_2 - \frac{1}{12}\partial^4_n\varphi_0 - \frac{c^2}{24k^2}\frac{\partial\varphi_0}{\partial\sigma} + \frac{c^4}{192k^4}\varphi_0. \tag{4.79}$$

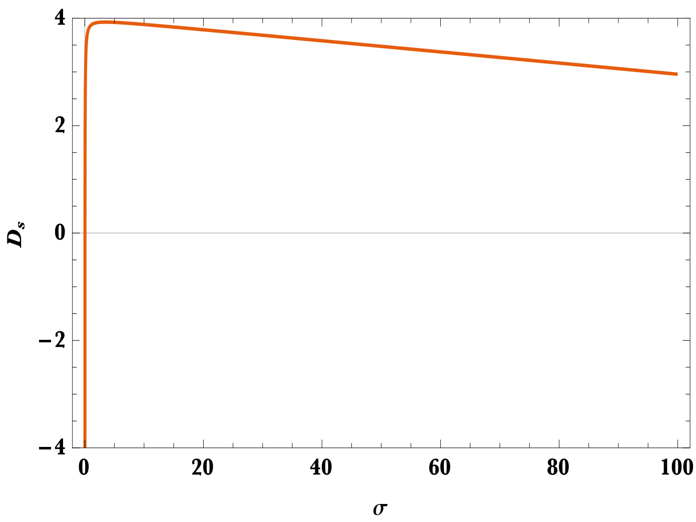
We solve this equation by substituting for φ_0 and using eqn.(4.24).

$$\begin{aligned}
\varphi_2(x,y;\sigma) &= \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{\Sigma^n_{i=1}(x_i-y_i)^2}{4\sigma}} \Bigg[\beta + \left(-\frac{1}{16\sigma} + \frac{c^4\sigma}{192k^4} + \frac{c^2 n}{48k^2}\right)\left(1-\frac{\epsilon}{\sigma}\right) \\
&+ \left(\frac{(x_n-y_n)^2}{16\sigma^3} - \frac{(x_n-y_n)^4}{192\sigma^4} - \frac{c^2}{96k^2\sigma^2}\Sigma^n_{i=1}(x_i-y_i)^2\right)(\sigma-\epsilon) \\
&- \left(\frac{(x_n-y_n)^3}{24\sigma^3\sqrt{\sigma\pi}} + \frac{c^2}{24k^2}\frac{\Sigma^n_{i=1}(x_i-y_i)}{\sigma\sqrt{\sigma\pi}}\right)\left(\sigma\tan^{-1}\sqrt{\frac{\sigma}{\epsilon}-1} - \epsilon\sqrt{\frac{\sigma}{\epsilon}-1}\right) \\
&- \frac{(x_n-y_n)}{6\sigma^2\sqrt{\sigma\pi}}\left((2\sigma+\epsilon)\sqrt{\frac{\sigma}{\epsilon}-1} - 3\sigma\tan^{-1}\sqrt{\frac{\sigma}{\epsilon}-1}\right) \\
&+ \frac{(x_n-y_n)}{2\sigma\sqrt{\sigma\pi}}\left(\tan^{-1}\sqrt{\frac{\sigma}{\epsilon}-1} - \sqrt{\frac{\sigma}{\epsilon}-1}\right)\Bigg].
\end{aligned} \tag{4.80}$$

The solutions of eqns.(4.75, 4.77) and eqn.(4.2) are used to obtain the return probability,

$$P(\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}}\left[1 + a\alpha + a^2\beta - \frac{a^2}{16\sigma} + \frac{a^2c^4}{192k^4}\sigma + \frac{a^2c^2}{48k^2}n\right], \tag{4.81}$$

Figure 4.7: Spectral dimension as a function of σ for n=4, a=1, c=k=1.



and using the definition for spectral dimension, we find D_s as

$$D_s = n - \frac{a^2}{8\sigma} - \frac{a^2 c^4}{96k^4}\sigma. \tag{4.82}$$

Note that the spectral dimension has one term which is linear in σ and another which is proportional to σ^{-1}. For n=4 and k=c=1, it is easy to see from the fig.[4.7] that, the spectral dimension increases with σ initially and then decreases as σ increases. It is clear that the spectral dimension is positive for $\frac{192}{a^2} - \sqrt{\frac{36864}{a^4} - 12} < \sigma < \frac{192}{a^2} + \sqrt{\frac{36864}{a^4} - 12}$.

Spectral dimension with an extended probe is given by

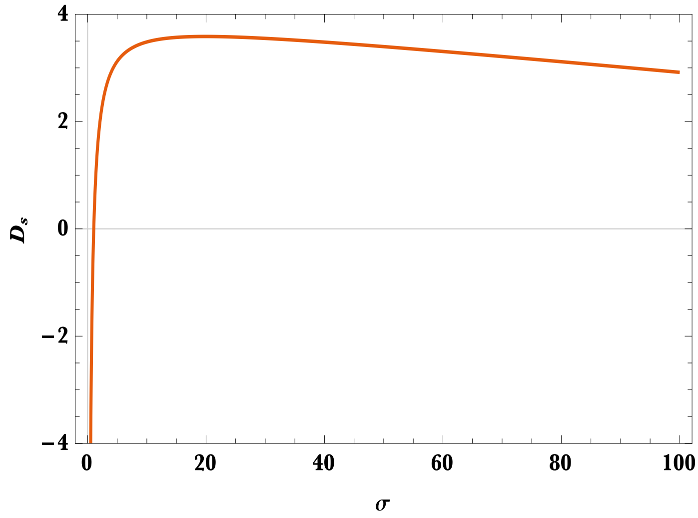
$$D_s = n - \frac{a^2}{8\sigma} - \frac{a^2 c^4}{96k^4}\sigma - \frac{a^2 n}{\sigma} \tag{4.83}$$

which has an extra term, $-\frac{a^2 n}{\sigma}$ compared to eqn.(4.82). Note that this additional term depends on the topological dimension n and it is proportional to σ^{-1}. This will not change the general feature of the dimensional flow (fig.[4.8]).

4.4 Conclusion

In this chapter, we have constructed four different, modified diffusion equations in the κ-space-time and using their solutions, analyzed the dimensional flow in the κ-space-time. In these studies, we have used probes

Figure 4.8: Spectral dimension as a function of σ for n=4, a=1, c=k=1 with extended probe.



which are point-like as well as probes with finite extension. For all these cases, we get the correct commutative limit, where the spectral dimension match with the topological dimension. In all the four cases studied, the spectral dimension changes with the probe scale. We note that for the three cases studied, (see eqns.(4.30, 4.70, 4.82)), in the high energy limit where $\sigma \to 0$, the spectral dimension become infinitely negative $(-\infty)$. Thus for these three cases spectral dimension looses its meaning at high energies. By demanding that the spectral dimension should be positive definite we obtain bounds on the deformation parameter in terms of diffusion time in these three cases. In the case of spectral dimension obtained in the eqn.(4.52), we note that as $\sigma \to 0$ spectral dimension becomes equal to topological dimension. In all the four cases, we see a novel feature of spectral dimension of non-commutative space-time in comparison with the result obtained in chapter 3 as well as in[13, 14, 15, 16]. The new fact emerged here is that the spectral dimension goes to $-\infty$ at low energies (i.e.,$\sigma \to \infty$). We want to emphasis that this feature is absent in the commutative limit and in the commutative limit we get back the equality between the spectral dimension and the topological dimension at low energies. From eqns.(4.30),(4.70) and (4.82), we see that the spectral dimension increases from $-\infty$ as σ rises from zero, reaches a maximum value and then decreases to $-\infty$. The maximum value of spectral dimension in all the three cases is less than the topological dimension.

We note that the major difference in the present analysis from the earlier ones is the use of a modified diffusion equation(s). In our case, we have not just used Beltrami-Laplace operator in the usual diffusion equation, but derived the modified diffusion equation in the κ-deformed space-time. This is done by applying the Wick rotation to the κ-deformed Schrödinger equation, obtained by taking the non-relativistic limit of well studied κ-deformed Klein-Gordon equation. This approach, explicitly introduces finite mass for the particle undertaking diffusion on the deformed space-time. We see from the spectral dimension obtained in eqns.(4.30, 4.52, 4.70) and eqn.(4.82) that in the limit of a massless probe, the spectral dimension and topological dimension coincides at low energies. We note here that the probes used in earlier studies[13, 14, 15, 16] were massless ones. The spectral dimension obtained in eqn.(4.52) shows the interesting property that in the limit of probe mass set to zero, there are no correction to spectral dimension due to the non-commutativity. This feature is unique as the spectral dimension calculated for other three cases, do have a dependent term, even in the limit of vanishing probe mass.

The diffusion equation constructed and analyzed in section 2.3 (see eqn.(4.41)) do not have any higher derivative terms unlike the other three cases studied here (see eqns.(4.13), (4.62) and (4.74)). The deformed diffusion equation given in eqn.(4.41) is obtained from a specific choice of Laplacian (equivalently Klein-Gordon operator in the κ-deformed space-time). The fact that in the massless limit of the probe, the spectral dimension is exactly same as the topological dimension for all probe scales show that the non-commutativity between time and space coordinate do not affect the spectral dimension of space-part of κ-space-time at all.

The eqn.(4.13) and eqn.(4.41) are derived from the Wick rotated non-relativistic limit of two different choices of the κ-deformed Klein-Gordon equation. In the non-relativistic limit, one neglects higher time derivative terms and thus keeps only higher space derivative terms (if any) appearing in the deformed diffusion equation. Thus we do not have any higher time derivatives (equivalently, higher derivatives with respect to σ) in these two equations. Further, for the specific choice of deformed Klein-Gordon equation used in section 2.3, there are no higher order spatial derivatives (upto second order in a). This is why the spectral dimension obtained in eqn.(4.52) has a completely different behaviour at high energies. For both the choice of Beltrami-Laplace operator considered in section 3, higher derivatives with respect to spatial as well as Euclidean time coordinates are present and they do appear in the corresponding diffusion equations (see eqn.(4.62)) and (4.74).

It is interesting to note that the three diffusion equations leading to negative spectral dimension of high energies all have the higher derivative terms. It has been known that such equations result in negative return probabilities [17]. In our formulation, κ-deformed diffusion equations are written down in the commutative space-time. All the effects of nonlocality inherent in the non-commutative space-time are contained in the a-dependent terms of the deformed diffusion equation. As it is clear, these terms are all higher order derivatives and thus non-local (except for the case studied in section 2.3). As discussed above, the higher time derivative terms drops out in the non-relativistic limit and this explain why noncommutativity do not play any role in the limit of vanishing mass of the probe for the spectral dimension obtained in eqn.(4.52). The κ-deformed Laplacian we used do have higher derivative terms. These terms summarise the non-local effects of the non-commutativity of the space-time. In the momentum space representation of Laplacian, this non-locality appear as higher power terms of momentum[13, 14, 16]. Laplacians with higher derivatives were also analyzed in[17, 18, 19, 20].

The negative value of spectral dimension we see in our analysis might be a reflection of the higher derivative terms (and thus related to the inbuilt non-locality of non-commutative space-time). But the higher derivative terms in the Laplacian (equivalently, Beltrami-Laplace operator) is a characteristic feature of κ-deformed space-time. Here we have taken a perturbative approach in the analysis of spectral dimension. A detailed analysis of the issue of higher derivatives require a field theoretic reinterpretation going beyond the usual diffusion equation[17, 18]. The issues related to higher derivative terms and that of the negative return probability have been analyzed in[17] and, field theoretical re-interpretation of spectral dimension as a possible way to avoid the negative return probability was introduced. For the spectral dimension calculated in eqn.(4.30), eqn.(4.70) and eqn.(4.82), imposing the requirement that the spectral dimension should be positive definite at high energies translate in to the conditions $a^2 < \frac{72\sigma}{45+\sigma^2}$, $a^2 < \frac{96\sigma}{147+\sigma^2}$,$a^2 < \frac{384\sigma}{12+\sigma^2}$ respectively. This feature suggest the possibility of multiscale structure of the space-time at high energies and such possibilities have been pointed out in [18].

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CHAPTER 5

Diffusion in κ-deformed space and Spectral Dimension

5.1 Introduction

In this chapter we study spectral dimension in κ-space-time using different choices of realization. The reason for this analysis is to see whether the spectral dimension and dimensional flow depend on the realizations or not. It is known that description of physics depend on the choice of realization. This is evident from construction of κ-deformed Dirac equations [1, 2, 3] where different generalisation of Dirac equation to κ-space-time were obtained leading to different conclusions. Unruh effect in κ-space-time also analysed using different realizations [4, 5]. Different realization can lead to completely contrasting predication and we want to see what is the status of spectral dimension as we use different realizations.

In the present chapter, we choose to work with two different realizations. One choice is the realization used in [6] with $\varphi(A) = e^{-A}$. For different choices of $\varphi(A)$ we can construct different realizations of the undeformed kappa-Poincare algebra [6, 7, 8]. The advantage of our particular choice of φ is its connection with the bi-crossproduct basis [9, 10] and is used in analysing the modification of central potential in the κ-space-time [11].

Another realization that we work with in this chapter uses a realization of phase space variables associated with κ-space-time to that of phase space variables corresponding to commutative case [12]. This should be contrasted with all other realizations studied in this thesis. We will refer to this realization as (α, β, γ) realization in the rest of this chapter. The (α, β, γ) realization was used in literature, to study various physical problems such as the formulation of electrodynamics in κ-space-time [12], study of kappa-deformed geodesic equation [13], corrections to black hole entropy due to κ-deformation [14], investigation of classical aspects of central potential problems in kappa-space-time [15, 16, 17]. In chapter 3 and chapter 4, we have analyzed the spectral dimension and dimensional flow using the realization $\varphi(A) = e^{-\frac{A}{2}}$.

5.2 Spectral Dimension with e^{-A} realization

In this section, we derive the κ-deformed diffusion equation, using the κ-deformed Beltrami-Laplace operator, which is expressed in terms of ordinary commutative coordinates and their derivatives using the realization $\varphi(A) = e^{-A}$. The heat kernel is obtained using a perturbative method as in the earlier chapters and using this we calculate the spectral dimension of (the Wick-rotated) κ-Minkowski space-time, valid up to first order in a.

Consider a n-dimensional wick-rotated κ-deformed Minkowski space-time. The motion of a diffused particle in this space will be governed by the equation

$$\frac{\partial}{\partial\sigma}U(x, y; \sigma) = D_\mu D_\mu U(x, y; \sigma), \tag{5.1}$$

where $D_\mu D_\mu$ is the Beltrami-Laplace operator in the κ-Euclidean space. We re-express this Laplacian in terms of commutative coordinates and their derivatives, but now in the realization $\varphi(A) = e^{-A}$.

For the kappa-Minkowski space-time, we have the generalized Laplacian

$$D_\mu D^\mu = D_i D_i - D_0 D_0 = \Box(1 + \frac{a^2}{4}\Box), \tag{5.2}$$

where $\Box = \nabla^2_{n-1}\frac{e^{-A}}{\varphi^2(A)} + \partial_0^2\frac{2(1-\cosh A)}{A^2}$. For the realization $\varphi(A) = e^{-A}$, the $\Box$ operator is explicitly given as

$$\Box = \nabla^2_{n-1}e^A + 2\partial_0^2\frac{(1 - \cosh A)}{A^2}. \tag{5.3}$$

Using this in eqn.(5.2) we find

$$D_\mu D^\mu = \left[\nabla^2_{n-1}e^A + 2\partial_0^2\frac{(1 - \cosh A)}{A^2}\right]\left(1 + \frac{a^2}{4}\left[\nabla^2_{n-1}e^A + 2\partial_0^2\frac{(1 - \cosh A)}{A^2}\right]\right). \tag{5.4}$$

With $A = -ia\partial_0$, $D_\mu D^\mu$ can be expressed as

$$\begin{aligned} D_\mu D^\mu &= \left[\nabla^2_{n-1}e^A - \frac{2}{a^2}(1-\cosh A)\right]\left(1 + \frac{a^2}{4}\left[\nabla^2_{n-1}e^A - \frac{2}{a^2}(1-\cosh A)\right]\right) \\ &= \nabla^2_{n-1}e^A - \frac{2}{a^2}(1-\cosh A) - \nabla^2_{n-1}e^A(1-\cosh A) \\ &+ \frac{1}{a^2}(1-\cosh A)^2 + \frac{a^2}{4}\nabla^4_{n-1}e^{2A} \end{aligned} \tag{5.5}$$

We expand this in terms of deformation parameter a as

$$\begin{aligned} D_\mu D^\mu &= \nabla^2_{n-1}\left(1 - ia\partial_0 - \frac{a^2}{2}\partial_0^2 + ...\right) - \frac{2}{a^2}\left(\frac{a^2}{2}\partial_0^2 - \frac{a^4}{24}\partial_0^4 + ...\right) \\ &- \nabla^2_{n-1}\left(1 - ia\partial_0 - \frac{a^2}{2}\partial_0^2 + ...\right)\left(\frac{a^2}{2}\partial_0^2 - \frac{a^4}{24}\partial_0^4 + ...\right) \\ &+ \frac{1}{a^2}\left(\frac{a^2}{2}\partial_0^2 - \frac{a^4}{24}\partial_0^4 + ...\right)^2 + \frac{a^2}{4}\nabla^4_{n-1}\left(1 - 2ia\partial_0 - a^2\partial_0^2 + ...\right) \\ D_\mu D^\mu &= \nabla^2_{n-1} - \partial_0^2 - ia\nabla^2_{n-1}\partial_0 - a^2\nabla^2_{n-1}\partial_0^2 + \frac{a^2}{3}\partial_0^4 + \frac{a^2}{4}\nabla^4_{n-1} + O(a^3) + ... \end{aligned} \tag{5.6}$$

Now by setting $x^0 = -ix^n$, we obtain the Beltrami-Laplace operator in the n-dimensional κ-Euclidean space, i.e.,

$$D_\mu D_\mu = \nabla^2_{n-1} + \partial_n^2 + a\nabla^2_{n-1}\partial_n + a^2\nabla^2_{n-1}\partial_n^2 + \frac{a^2}{3}\partial_n^4 + \frac{a^2}{4}\nabla^4_{n-1} + ... \tag{5.7}$$

Note that the $\Box$-operator also reduces to the usual Laplacian in the commutative limit. Expansion of $\Box$ with $\varphi(A) = e^{-A}$ is given by

$$\begin{aligned} \Box &= \nabla^2_{n-1}e^A - \frac{2}{a^2}(1-\cosh A) \\ &= \nabla^2_{n-1} - \partial_0^2 - ia\nabla^2_{n-1}\partial_0 - \frac{a^2}{2}\nabla^2_{n-1}\partial_0^2 + \frac{a^2}{12}\partial_0^4 + ... \end{aligned} \tag{5.8}$$

This in the Euclidean space is

$$\Box = \nabla^2_{n-1} + \partial_n^2 + a\nabla^2_{n-1}\partial_n + \frac{a^2}{2}\nabla^2_{n-1}\partial_n^2 + \frac{a^2}{12}\partial_n^4 + ... \tag{5.9}$$

Note that, upto first order correction in a, the Casimir and the $\Box$-operator are identical (see eqn.(5.7) and eqn.(5.9)).

By restricting our attention to first non-vanishing corrections due to non-commutativity, we rewrite the diffusion equation in the κ-deformed Euclidean space as

$$\begin{aligned}\frac{\partial}{\partial\sigma}U(x,y;\sigma) &= \nabla^2_{n-1}U(x,y;\sigma)+\partial^2_n U(x,y;\sigma)+a\nabla^2_{n-1}\partial_n U(x,y;\sigma)\\ &= \nabla^2_n U(x,y;\sigma)+a\nabla^2_{n-1}\partial_n U(x,y;\sigma). \qquad (5.10)\end{aligned}$$

Here $\nabla^2_n = \nabla^2_{n-1}+\partial^2_n$ is the Laplacian in n-dimensional space. It is easy to see that if we are considering the first order correction terms only, the $\Box$-operator also leads to the same diffusion equation as in eqn.(5.10). This should be contrasted with the results obtained in the realization $\varphi(A) = e^{-\frac{A}{2}}$, studied in earlier chapters. By comparing the above deformed diffusion equation with the usual diffusion equation in Euclidean space, we note that the above equation has an extra term, involving product of temporal and spacial derivatives i.e., $\nabla^2_{n-1}\partial_n$.

The heat kernel $U(x,y;\sigma)$ is obtained by solving this equation perturbatively, i.e., we start with the ansatz solution as a perturbative series in a (note that the deformation parameter is expected to be of the order of Planck length), given by

$$U = U_0 + aU_1 + ... \qquad (5.11)$$

Using eqn.(5.11) in eqn.(5.10) and equating the terms of same order in a, we solve equation for U perturbatively. Zeroth order terms in a gives usual heat equation,

$$\frac{\partial}{\partial\sigma}U_0(x,y;\sigma) = \nabla^2_n U_0(x,y;\sigma). \qquad (5.12)$$

The solution of the above equation is

$$U_0(x,y;\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}}e^{-\frac{|x-y|^2}{4\sigma}}. \qquad (5.13)$$

By equating the first order terms in a, we obtain

$$\frac{\partial}{\partial\sigma}U_1(x,y;\sigma) = \nabla^2_n U_1(x,y;\sigma)+\nabla^2_{n-1}\partial_n U_0(x,y;\sigma). \qquad (5.14)$$

Substituting the solution for U_0 from eqn.(5.13) in the above, we get

$$\nabla^2_{n-1}U_0(x,y;\sigma) = \left[-\frac{(n-1)}{2\sigma}+\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4\sigma^2}\right]\frac{1}{(4\pi\sigma)^{\frac{n}{2}}}e^{-\frac{|x-y|^2}{4\sigma}} \qquad (5.15)$$

$$\nabla^2_{n-1}\partial_n U_0(x,y;\sigma) = \frac{(x_n-y_n)}{2\sigma}\left[\frac{(n-1)}{2\sigma}-\frac{\Sigma_{i=1}^{n-1}(x_i-y_i)^2}{4\sigma^2}\right]\frac{1}{(4\pi\sigma)^{\frac{n}{2}}}e^{-\frac{|x-y|^2}{4\sigma}}. \qquad (5.16)$$

Substituting eqn.(5.16) in eqn.(5.14), and after a straight forward simplification we end up with the equation for U_1 as

$$\begin{aligned} \frac{\partial}{\partial\sigma}U_1(x,y;\sigma) &= \nabla_n^2 U_1(x,y;\sigma) + \left[\frac{(n-1)}{4\sigma^2}(x_n - y_n)\right]\frac{1}{(4\pi\sigma)^{\frac{n}{2}}}e^{-\frac{|x-y|^2}{4\sigma}} \\ &- \left[\frac{(x_n-y_n)}{8\sigma^3}\Sigma_{i=1}^{n-1}(x_i-y_i)^2\right]\frac{1}{(4\pi\sigma)^{\frac{n}{2}}}e^{-\frac{|x-y|^2}{4\sigma}}. \qquad (5.17) \end{aligned}$$

It is known that for a differential equation which has the general form

$$\frac{\partial}{\partial\sigma}U_1(X,\sigma) = \nabla_n^2 U_1(X,\sigma) + f(X,\sigma), \qquad (5.18)$$

the solution satisfying the initial condition

$$U_1(X,0) = g(X), \qquad (5.19)$$

is given by [18]

$$U_1(X,\sigma) = \int_{R^n}\Phi(X-X',\sigma)g(X')dX' + \int_0^\sigma\int_{R^n}\Phi(X-X',\sigma-s)f(X',s)dX'ds, \qquad (5.20)$$

where

$$\Phi(X,\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}}e^{-\frac{|X|^2}{4\sigma}}. \qquad (5.21)$$

Using this, we solve the equation for U_1 (eqn.(5.17)). The initial condition satisfied by the solution of eqn.(5.17) is

$$U_1(X,0) = g(X) = \delta^n(X), \qquad (5.22)$$

where $X = x - y$. With this initial condition, we calculate the first term on RHS of eqn.(5.20) (defined as U_{11}). Thus we get

$$\begin{aligned} U_{11}(X,\sigma) &= \int_{R^n}\Phi(X-X',\sigma)g(X')dX' \\ &= \int_{R^n}\frac{1}{(4\pi\sigma)^{\frac{n}{2}}}e^{-\frac{|X-X'|^2}{4\sigma}}\delta^n(X')dX' \\ U_{11}(x,y;\sigma) &= \frac{\alpha}{(4\pi\sigma)^{\frac{n}{2}}}e^{-\frac{|x-y|^2}{4\sigma}}, \qquad (5.23) \end{aligned}$$

where α has the dimensions of L^{-1}. The second term on RHS of eqn.(5.20), will diverges for the lower limit of the s integral. We avoid this by inserting a lower cutoff ϵ and after the calculations, take the limit $\epsilon \to 0$. Performing X' integration in the second term on the RHS of eqn.(5.20), we obtain

$$\begin{aligned}
U_{12}(X,\sigma) &= \int_\epsilon^\sigma \int_{R^n} \Phi(X-X',\sigma-s)f(X',s)dX'ds \\
U_{12}(x,y;\sigma) &= \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}} \int_\epsilon^\sigma \Bigg\{ -\frac{1}{8\sigma^3}(x_n-y_n)\Sigma_{i=1}^{n-1}(x_i-y_i)^2 \\
&- \frac{1}{4\sigma^2\sqrt{\sigma\pi}}\left(\Sigma_{i=1}^{n-1}(x_i-y_i)^2 + 2(x_n-y_n)\Sigma_{i=1}^{n-1}(x_i-y_i)\right)\left[\frac{\sigma-s}{s}\right]^{\frac{1}{2}} \\
&- \frac{(n-1)}{2\sigma\sqrt{\sigma\pi}}\left[\frac{\sigma-s}{s}\right]^{\frac{3}{2}} + \frac{(n-1)}{2\sqrt{\sigma\pi}}\frac{1}{s}\left[\frac{\sigma-s}{s}\right]^{\frac{1}{2}} + \frac{n-1}{4\sigma}(x_n-y_n)\frac{1}{s} \\
&- \left(\frac{1}{\sigma^2\pi}\Sigma_{i=1}^{n-1}(x_i-y_i) + \frac{(n-1)}{4\sigma^2}(x_n-y_n)\right)\left[\frac{\sigma-s}{s}\right]\Bigg\}ds \qquad (5.24)
\end{aligned}$$

Integration over s will result in the following expression for U_{12}.

$$\begin{aligned}
U_{12}(x,y;\sigma) &= \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}} \Bigg\{ \frac{(n-1)}{2\sqrt{\sigma\pi}}\left[5\tan^{-1}q - (4+\frac{\epsilon}{\sigma})q\right] \\
&- \frac{1}{4\sigma^2\sqrt{\sigma\pi}}\Sigma_{i=1}^{n-1}(x_i-y_i)^2\left[\sigma\tan^{-1}q - \epsilon q\right] \\
&- \frac{1}{2\sigma^2\sqrt{\sigma\pi}}(x_n-y_n)\Sigma_{i=1}^{n-1}(x_i-y_i)\left[\sigma\tan^{-1}q - \epsilon q\right] \\
&- \frac{1}{\sigma^2\pi}\Sigma_{i=1}^{n-1}(x_i-y_i)\left[\sigma\ln(\sigma/\epsilon) - \sigma + \epsilon\right] \\
&+ \left(-\frac{(x_n-y_n)}{8\sigma^3}\Sigma_{i=1}^{n-1}(x_i-y_i)^2 + \frac{(n-1)}{4\sigma^2}(x_n-y_n)\right)[\sigma-\epsilon]\Bigg\}
\end{aligned} \qquad (5.25)$$

where $q = \sqrt{\frac{\sigma}{\epsilon} - 1}$. Note that the arctan terms in the above equation arise from integrations of the type $\int dx\sqrt{\frac{1}{x} - 1}$. These arctan factors will turn out to be crucial in our calculation of spectral dimension.

Using eqns.(5.13), (5.23) and eqn.(5.25) we find the heat kernel valid upto first order in a. Using this, we obtain the return probability

$$P_g(\sigma) = \frac{\int d^nx\sqrt{detg_{\mu\nu}}U(x,x;\sigma)}{\int d^nx\sqrt{detg_{\mu\nu}}} \qquad (5.26)$$

as

$$P_g(\sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}}\left[1 + a\alpha + a\frac{(n-1)}{2\sqrt{\sigma\pi}}\left(5\tan^{-1}\sqrt{\frac{\sigma}{\epsilon}-1} - (4+\frac{\epsilon}{\sigma})\sqrt{\frac{\sigma}{\epsilon}-1}\right)\right]. \quad (5.27)$$

By taking the logarithmic derivative of $P_g(\sigma)$, we evaluate the spectral dimension

$$D_s = -2\sigma\frac{\partial}{\partial\sigma}\ln P_g(\sigma) \quad (5.28)$$

and obtain

$$D_s = \frac{n + na\alpha + a\frac{(n-1)}{2\sqrt{\sigma\pi}}\left[5(n+1)\tan^{-1}\sqrt{\frac{\sigma}{\epsilon}-1} - [4n + (n+3)\frac{\epsilon}{\sigma}]\sqrt{\frac{\sigma}{\epsilon}-1}\right]}{1 + a\alpha + a\frac{(n-1)}{2\sqrt{\sigma\pi}}\left[5\tan^{-1}\sqrt{\frac{\sigma}{\epsilon}-1} - [4+\frac{\epsilon}{\sigma}]\sqrt{\frac{\sigma}{\epsilon}-1}\right]}. \quad (5.29)$$

Keeping upto first non-vanishing terms in a, we obtain the spectral dimension as

$$D_s = n + a\frac{(n-1)}{2\sqrt{\sigma\pi}}\left[5\tan^{-1}\sqrt{\frac{\sigma}{\epsilon}-1} - 3\frac{\epsilon}{\sigma}\sqrt{\frac{\sigma}{\epsilon}-1}\right]. \quad (5.30)$$

After taking the limit ϵ to zero, we obtain spectral dimension of the κ-deformed space-time as

$$D_s = n + \frac{5}{4}(n-1)(2l+1)\sqrt{\pi}\frac{a}{\sqrt{\sigma}}, \qquad l \in \mathbb{Z}. \quad (5.31)$$

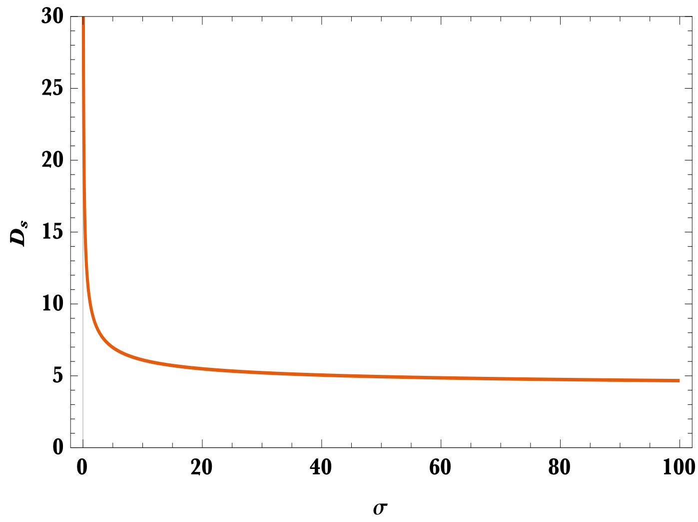
Note that, we have an extra term in the expression for spectral dimension due to non-commutative nature of the space-time. The correction term is proportional to $\frac{a}{\sqrt{\sigma}}$ and is also dependent on the initial topological dimension n. In the commutative limit ($a \to 0$) the spectral dimension is same as the topological dimension 'n'. Here the integer l arises due to the term $\tan^{-1}\sqrt{\frac{\sigma}{\epsilon}-1}$ in eqn.(5.30) (This $\tan^{-1}$ term is due to the mixed derivative term $\nabla^2_{n-1}\partial_n$ in the diffusion equation (5.14)). Limit $\epsilon \to 0$ gives $\tan^{-1}\pm\infty$ which is $(2l+1)\frac{\pi}{2}$. From the expression for spectral dimension (eqn.(5.31)), we see that the behaviour of spectral dimension depends on a as well as on l.

The behaviour of the spectral dimension for n=4, l=0 and a=1 is shown in fig.[5.1].

From this it is easy to see that

$$\lim_{\sigma\to 0} D_s \approx +\infty, \qquad \lim_{\sigma\to\infty} D_s \approx 4.$$

Figure 5.1: spectral dimension as a function of σ with $a = 1$, $n = 4$ and l=0.



Thus we observe that the dimensions of the space-time increases from the usual topological dimension as $\sigma \to 0$ and at high energies $D_s \to \infty$ showing super-diffusion. For all values of $l > 0$, the behaviour of spectral dimension will be same as that of $l = 0$ case.

For negative values of l, the behaviour of spectral dimension is different from what we observed above. The plot for spectral dimension with $l = -1$ is given in fig.[5.2] and we see that the spectral dimension flows to $-\infty$ at high energies. In the limit of large diffusion parameter, the spectral dimension is same as the topological dimension. For $l < 0$, the spectral dimension become negative for $\sqrt{\sigma} < -\frac{15}{16}(2l + 1)\sqrt{\pi}a$ and vanishes for $\sqrt{\sigma} = -\frac{15}{16}(2l + 1)\sqrt{\pi}a$. By demanding the spectral dimension to be positive, we obtain a lower cutoff for the deformation parameter as $-\frac{16}{15}\frac{1}{(2l+1)}\sqrt{\frac{\sigma}{\pi}} < a$.

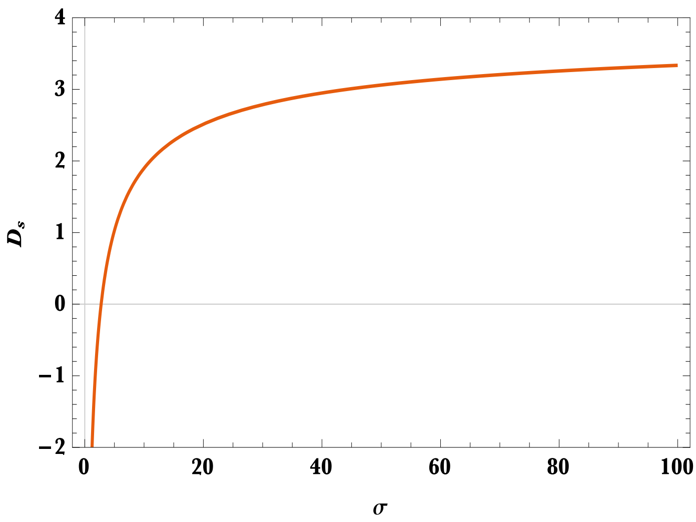
Extended probe : Now we want to analyze the change in spectral dimension due to the extended nature of the probe. The use of an extended probe would result in modification of initial condition as

$$U(x, y; 0) = \frac{1}{(4\pi a^2)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4a^2}}. \tag{5.32}$$

With this we obtain the solution for eqn.(5.12) as

$$U_0(x, y; \sigma) = \frac{1}{(4\pi(\sigma + a^2))^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4(\sigma+a^2)}}. \tag{5.33}$$

Figure 5.2: spectral dimension as a function of σ with $a = 1$, $n = 4$ and l=-1.



Now expand this in powers of a results

$$U_0(x, y; \sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}} \left[1 - \frac{na^2}{2\sigma} + \frac{a^2 \mid x - y \mid^2}{4\sigma^2} + ...\right]. \quad (5.34)$$

Here the modification in $U_0(x, y; \sigma)$ is second order in deformation parameter a. Since the diffusion equation we are considering is first order in a, we are interested in the solution valid up to first order in a. So by neglecting the terms which are higher powers in a gives

$$U_0(x, y; \sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}}, \quad (5.35)$$

which is same as the one obtained with point probe. Thus the spectral dimension will not have any correction up to first order in a due to the finite width in the probe in the present case, unlike the results obtained for the realization $\varphi(A) = e^{-\frac{A}{2}}$ (see chapter 3 and chapter 4).

5.3 Spectral dimension with (α, β, γ) realization

We here present an alternative derivation for spectral dimension using the realization introduced in [12]. The usefulness of this realization comes from the fact that the non-commutative variables are expressed in terms of commutative phase space variables. This enables us to describe phase space dynamics more easily.

Realization of non-commutative variables $\hat{x}^\mu$ and $\hat{p}^\mu$ can be expressed in terms of commutative variables x^μ and p^μ as

$$\hat{x}^\mu = x^\mu + \alpha x^\mu(a.p) + \beta(a.x)p^\mu + \gamma a^\mu(x.p), \tag{5.36}$$

$$\hat{p}^\mu = p^\mu + (\alpha+\beta)(a.p)p^\mu + \gamma a^\mu(p.p), \tag{5.37}$$

where α, β, $\gamma \in \mathbb{R}$ and μ takes values $0, 1, 2, 3$. These $\hat{x}^\mu$ and $\hat{p}^\mu$ satisfy the following relations.

$$[\hat{x}^\mu, \hat{x}^\nu] = i(a^\mu \hat{x}^\nu - a^\nu \hat{x}^\mu), \tag{5.38}$$

$$[\hat{p}^\mu, \hat{p}^\nu] = 0 \tag{5.39}$$

$$[\hat{p}^\mu, \hat{x}^\nu] = i\eta^{\mu\nu}(1 + s(a.p)) + i(s+2)a^\mu p^\nu + i(s+1)a^\nu p^\mu, \tag{5.40}$$

where $s = 2\alpha + \beta$ and a^μ is the deformation parameter. The commutation relation between coordinates (eqn.(5.38)) gives the constraint on α, β and γ

$$\gamma - \alpha = 1, \qquad \alpha, \beta, \gamma \in \mathbb{R} \tag{5.41}$$

In this realization the Klein-Gordon equation in the κ-Minkowski spacetime takes the form [14]

$$\begin{aligned}\partial_\sigma(\eta^{\sigma\nu}\partial_\nu\Phi) = \partial_\alpha\partial_\beta(\mathcal{A}^{\alpha\beta\gamma\delta}\partial_\gamma\partial_\delta\Phi) + \partial_\gamma\partial_\delta(\mathcal{A}^{\alpha\beta\gamma\delta}\partial_\alpha\partial_\beta\Phi) \\ -\frac{1}{2}\Sigma_\alpha\partial_\alpha\partial_\alpha(\mathcal{A}^{\alpha\alpha\gamma\delta}\partial_\gamma\partial_\delta\Phi + \mathcal{A}^{\gamma\delta\alpha\alpha}\partial_\gamma\partial_\delta\Phi)\end{aligned} \tag{5.42}$$

where

$$\mathcal{A}^{\alpha\beta\gamma\delta} = i\eta^{\beta\delta}(\gamma x^\alpha a^\gamma + \beta(a.x)\eta^{\alpha\gamma} + \alpha a^\alpha x^\gamma) \tag{5.43}$$

Expanding this with metric (-,+,...,+) and choose $a^\mu = (a, 0...0)$, we obtain Klein-Gordon equation, up to first order in a, as

$$\begin{aligned} &- \frac{\partial^2\Phi}{\partial x^{0^2}} + \nabla^2_{n-1}\Phi + ia[\alpha+\beta+\gamma]x^0\frac{\partial^4\Phi}{\partial x^{0^4}} + ia\frac{3}{2}[\alpha+\gamma]x^i\frac{\partial}{\partial x^i}\frac{\partial^3\Phi}{\partial x^{0^3}} \\ &+ ia[(n+1)(\alpha+\gamma)+2\beta]\frac{\partial^3\Phi}{\partial x^{0^3}} - ia[2(\alpha+\gamma)+4\beta]x^0\frac{\partial^2}{\partial x^{0^2}}(\nabla^2_{n-1}\Phi) \\ &- ia4[\alpha+\beta+\gamma]\frac{\partial}{\partial x^0}(\nabla^2_{n-1}\Phi) - ia\frac{3}{2}[\alpha+\gamma]x^i\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^0}(\nabla^2_{n-1}\Phi) \\ &+ ia\beta x^0\nabla^4_{n-1}\Phi = 0 \end{aligned} \tag{5.44}$$

where $\nabla^2_{n-1} = \Sigma_{i=1}^{n-1}\frac{\partial^2}{\partial_{x_i}{}^2}$. Now by setting $x^0 \to -ix^n$ in the above expression, we get the Wick rotated Klein-Gordon equation (i.e. Klein-Gordon equation in Euclidean space-time) as

$$\begin{aligned}
&\frac{\partial^2\Phi}{\partial x^{n2}} + \nabla^2_{n-1}\Phi + a[\alpha+\beta+\gamma]x^n\frac{\partial^4\Phi}{\partial x^{n4}} + a\frac{3}{2}[\alpha+\gamma]x^i\frac{\partial}{\partial x^i}\frac{\partial^3\Phi}{\partial x^{n3}} \\
+ &\ a[(n+1)(\alpha+\gamma)+2\beta]\frac{\partial^3\Phi}{\partial x^{n3}} + a[2(\alpha+\gamma)+4\beta]x^n\frac{\partial^2}{\partial x^{n2}}(\nabla^2_{n-1}\Phi) \\
+ &\ a4[\alpha+\beta+\gamma]\frac{\partial}{\partial x^n}(\nabla^2_{n-1}\Phi) + a\frac{3}{2}[\alpha+\gamma]x^i\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^n}(\nabla^2_{n-1}\Phi) \\
+ &\ a\beta x^n\nabla^4_{n-1}\Phi = 0
\end{aligned} \tag{5.45}$$

From this expression, we write the Klein-Gordon operator $\Box_{KG}$ in the form

$$\begin{aligned}
\Box_{KG} &= \nabla^2_{n-1} + \partial^2_n + a[\alpha+\beta+\gamma]x^n\partial^4_n + \frac{3}{2}a[\alpha+\gamma]x^i\partial_i\partial^3_n \\
&+ a[(n+1)(\alpha+\gamma)+2\beta]\partial^3_n + a[2(\alpha+\gamma)+4\beta]x^n\partial^2_n\nabla^2_{n-1} \\
&+ 4a[\alpha+\beta+\gamma]\partial_n\nabla^2_{n-1} + \frac{3}{2}a[\alpha+\gamma]x^i\partial_i\partial_n\nabla^2_{n-1} + a\beta x^n\nabla^4_{n-1}
\end{aligned} \tag{5.46}$$

The above obtained Klein-Gordon operator replaces Laplacian $\mathcal{L}$ in the expression for diffusion equation (see eqn.(4) in chapter 2). This results in the diffusion equation

$$\frac{\partial}{\partial\sigma}U(x,y;\sigma) = \Box_{KG}U(x,y;\sigma) \tag{5.47}$$

$$\begin{aligned}
&= \nabla^2_{n-1}U + \partial^2_nU + a[\alpha+\beta+\gamma]x^n\partial^4_nU + \frac{3}{2}a[\alpha+\gamma]x^i\partial_i(\partial^3_nU) \\
&+ a[(n+1)(\alpha+\gamma)+2\beta]\partial^3_nU + a[2(\alpha+\gamma)+4\beta]x^n\partial^2_n(\nabla^2_{n-1}U) \\
&+ 4a[\alpha+\beta+\gamma]\partial_n(\nabla^2_{n-1}U) + \frac{3}{2}a[\alpha+\gamma]x^i\partial_i\partial_n(\nabla^2_{n-1}U) \\
&+ a\beta x^n\nabla^4_{n-1}U
\end{aligned} \tag{5.48}$$

For convenience, we define $\nabla^2_{n-1} + \partial^2_n = \nabla^2_n$. We now solve this diffusion equation perturbatively by taking

$$U = U_0 + aU_1. \tag{5.49}$$

The resulting equation is solved by equating the expression of same powers of a. The zeroth order terms in a gives

$$\frac{\partial}{\partial\sigma}U_0(x,y;\sigma) = \nabla^2_nU_0(x,y;\sigma). \tag{5.50}$$

This leads to the solution

$$U_0(x, y; \sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}} \tag{5.51}$$

Since, the knowledge of the $U(x, x; \sigma)$ enables us to find the expression for return probability, in the present calculation we first find the expression for $U_0(x, x; \sigma)$ as

$$U_0(x, x; \sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}}. \tag{5.52}$$

In order to find the leading order correction to the return probability, due to κ-deformation, we equate the first order terms in a to obtain

$$\begin{aligned}
\frac{\partial}{\partial\sigma} U_1(x, y; \sigma) &= \nabla_n^2 U_1 + [\alpha + \beta + \gamma] x^n \partial_n^4 U_0 + \frac{3}{2}[\alpha + \gamma] x^i \partial_i (\partial_n^3 U_0) \\
&+ [(n+1)(\alpha + \gamma) + 2\beta] \partial_n^3 U_0 + [2(\alpha + \gamma) + 4\beta] x^n \partial_n^2 (\nabla_{n-1}^2 U_0) \\
&+ 4[\alpha + \beta + \gamma] \partial_n (\nabla_{n-1}^2 U_0) + \frac{3}{2}[\alpha + \gamma] x^i \partial_i \partial_n (\nabla_{n-1}^2 U_0) \\
&+ \beta x^n \nabla_{n-1}^4 U_0.
\end{aligned} \tag{5.53}$$

Plugging the expression for U_0, as given in eqn.(5.51), to the above equation reduces to

$$\begin{aligned}
\frac{\partial}{\partial\sigma} U_1(x, y; \sigma) &= \nabla_n^2 U_1(x, y; \sigma) + \Bigg(\left[9n(\alpha + \gamma) + (n^2 + 8n)\beta\right] \frac{x_n - y_n}{4\sigma^2} \\
&- [(3n+5)(\alpha + \gamma) + 4(n+1)\beta] \frac{(x_n - y_n)^3}{8\sigma^3} \\
&- \left[(\frac{3n}{2} + 12)(\alpha + \gamma) + (2n + 10)\beta \right] \frac{x_n - y_n}{8\sigma^3} \Sigma_{i=1}^{n-1} (x_i - y_i)^2 \\
&+ \left[\frac{7}{2}(\alpha + \gamma) + 4\beta \right] \frac{(x_n - y_n)^3}{16\sigma^4} \Sigma_{i=1}^{n-1} (x_i - y_i)^2 \\
&+ \left[\frac{3}{2}(\alpha + \gamma) + \beta \right] \frac{x_n - y_n}{16\sigma^4} (\Sigma_{i=1}^{n-1} (x_i - y_i)^2)^2 \\
&+ (\alpha + \beta + \gamma) \frac{(x_n - y_n)^5}{16\sigma^4} \Bigg) \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}}
\end{aligned} \tag{5.54}$$

By considering the initial condition $U_1(x, y; 0) = \delta^n(x - y)$, the solution to above equation is given by eqn.(5.20). We obtain the first term in the

RHS of eqn.(5.20), U_{11} as

$$\begin{aligned} U_{11}(x, y; \sigma) &= \int_{R^n} \Phi(X - X', \sigma) g(X') dX' \\ &= \frac{q}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}}, \end{aligned} \tag{5.55}$$

where the constant q has the dimensions of L^{-1}. Note that $U_{11}(x, x; \sigma)$ has the form,

$$U_{11}(x, x; \sigma) = \frac{q}{(4\pi\sigma)^{\frac{n}{2}}}. \tag{5.56}$$

The second term on RHS of eqn.(5.20), U_{12} is evaluated as

$$U12(x, x; \sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} \left[\frac{\mathcal{K}}{\sqrt{\sigma\pi}} \tan^{-1} \sqrt{\frac{\sigma}{\epsilon} - 1} + \frac{1}{\sqrt{\sigma\pi}} \sqrt{\frac{\sigma}{\epsilon} - 1} (\mathcal{L} + \frac{\epsilon}{\sigma} \mathcal{M} + \frac{\epsilon^2}{\sigma^2} \mathcal{N}) \right], \tag{5.57}$$

where,
$\mathcal{K} = (-\frac{9}{16}n^2 + \frac{165}{8}n - \frac{33}{16})(\alpha + \gamma) + (\frac{17}{8}n^2 + 17n - \frac{9}{8})\beta$
$\mathcal{L} = (-\frac{37}{2}n + \frac{3}{2})(\alpha + \gamma) - (2n^2 + 16n - 1)\beta$
$\mathcal{M} = (\frac{15}{16}n^2 - \frac{3}{8}n + \frac{17}{18})(\alpha + \gamma) + (\frac{1}{8}n^2 + n - \frac{1}{8})\beta$
$\mathcal{N} = (-\frac{3}{8}n^2 - \frac{7}{4}n + \frac{1}{8})(\alpha + \gamma) + (-\frac{n^2}{4} - 2n + \frac{1}{4})\beta.$
With the above calculations, we find the return probability as

$$\begin{aligned} P_g(\sigma) &= \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} \Bigg[1 + aq + a \Bigg(\frac{1}{\sqrt{\sigma\pi}} \sqrt{\frac{\sigma}{\epsilon} - 1} \left(\mathcal{L} + \frac{\epsilon}{\sigma} \mathcal{M} + \frac{\epsilon^2}{\sigma^2} \mathcal{N} \right) \\ &+ \frac{\mathcal{K}}{\sqrt{\sigma\pi}} \tan^{-1} \sqrt{\frac{\sigma}{\epsilon} - 1} \Bigg) \Bigg]. \end{aligned} \tag{5.58}$$

The above obtained return probability is then used to derive the expression for the spectral dimension. This results in the expression

$$\begin{aligned} D_s &= \frac{1}{P_g(\sigma)} \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} \Bigg\{ n + naq + a \frac{\mathcal{L}}{\sqrt{\sigma\pi}} \sqrt{\frac{\sigma}{\epsilon} - 1} \left((n+1) - \frac{\sigma}{(\sigma - \epsilon)} \right) \\ &+ a \frac{\mathcal{K}}{\sqrt{\sigma\pi}} \left((n+1) \tan^{-1} \sqrt{\frac{\sigma}{\epsilon} - 1} - \frac{1}{\sqrt{\frac{\sigma}{\epsilon} - 1}} \right) \\ &+ a \frac{\mathcal{M}}{\sqrt{\sigma\pi}} \sqrt{\frac{\sigma}{\epsilon} - 1} \left((n+3) \frac{\epsilon}{\sigma} - \frac{\epsilon}{(\sigma - \epsilon)} \right) \\ &+ a \frac{\mathcal{N}}{\sqrt{\sigma\pi}} \sqrt{\frac{\sigma}{\epsilon} - 1} \left((n+5) \frac{\epsilon^2}{\sigma^2} - \frac{\epsilon^2}{(\sigma^2 - \sigma\epsilon)} \right) \Bigg\}. \end{aligned} \tag{5.59}$$

Keeping first non-vanishing terms in a, we obtain D_s (in the limit $\epsilon \to 0$) as

$$D_s = n + \frac{a\sqrt{\pi}}{2\sqrt{\sigma}}(2p+1)\mathcal{K} \tag{5.60}$$

where p is an arbitrary integer. The presence of this integer is due to the term $\tan^{-1}\sqrt{\frac{\sigma}{\epsilon}-1}$.

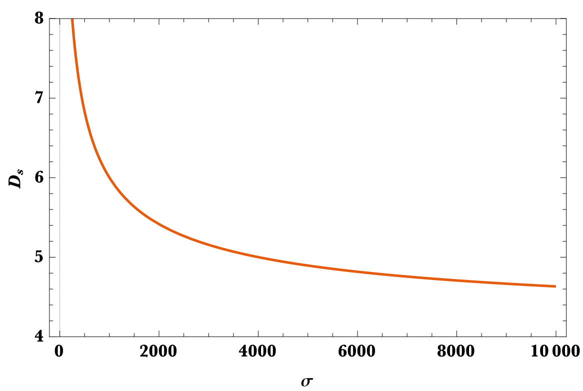
Note that the first non-vanishing correction due to non-commutativity is first order in deformation parameter a. This new term is proportional to $\sigma^{-\frac{1}{2}}$ and it also depends on the initial topological dimension n.

For the choice $n=4$ and recalling that, $\gamma - \alpha = 1$, we find

$$D_s = 4 + \frac{a}{2}\sqrt{\frac{\pi}{\sigma}}(2p+1)\left[\frac{1143}{16}(2\alpha+1) + \frac{807}{8}\beta\right]. \tag{5.61}$$

We have illustrated the behavior of spectral dimension with probe scale for different values of p, α and β with the choices $a=1$ and $n=4$ in fig.[5.3], fig.[5.4], fig.[5.5], fig.[5.6], fig.[5.7], fig.[5.8], fig.[5.9], fig.[5.10] and fig.[5.11].

Figure 5.3: spectral dimension as a function of σ with $a=1$, $n=4$ and $p=\alpha=\beta=0$.



By analyzing the plots, we observe that for α ,$\beta = 0$, the generic behaviour of spectral dimension changes with the integer p. For $p \geq 0$ (with $\alpha=\beta=0$), we note that the spectral dimension increases from 4 as the probe scale decreases and diverges to infinity as $\sigma \to 0$ (see fig.[5.3 and fig.[5.5). For $p < 0$, the dimensional flow shows that $D_s \to -\infty$ at high energies (see fig.[5.4]). Note that we had a similar behaviour for different values of l in the previous section (see fig.[5.1] and [5.2]).

Figure 5.4: spectral dimension as a function of σ with $a = 1$, $n = 4$, $p = -1$ and $\alpha = \beta = 0$.

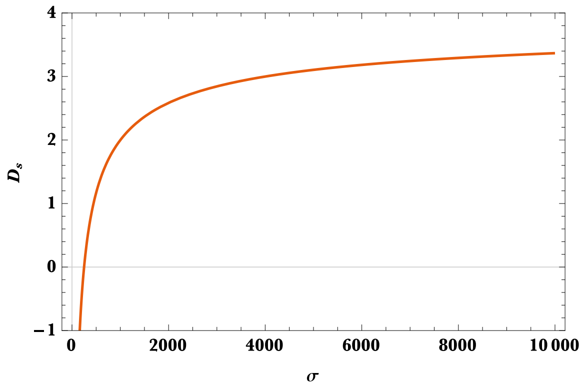
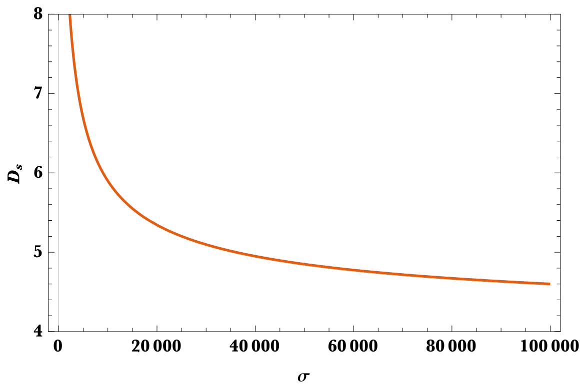


Figure 5.5: spectral dimension as a function of σ with $a = 1$, $n = 4$, $p = 1$ and $\alpha = \beta = 0$.



Inspection of spectral dimension for various values of p, α, and β indicates that, in general, there are two characteristic behaviour for the plots. One class of figures shows super-diffusion where the spectral dimension goes to $+\infty$ (see figures [5.3], [5.5], [5.7], 5.8] and [5.11]) and another class of figures in which spectral dimension goes to $-\infty$ (see figures [5.4], [5.6], [5.9], and [5.10]), at high energies. In all the cases, it is observed that $\lim_{\sigma\to\infty} D_s \approx 4$.

Extended probe : Instead of point particle as probe, one can also use a probe with finite width. The extended nature of the probe can be

Figure 5.6: spectral dimension as a function of σ with $a = 1$, $n = 4$, $p = \alpha = 0$ and $\beta = -1$.

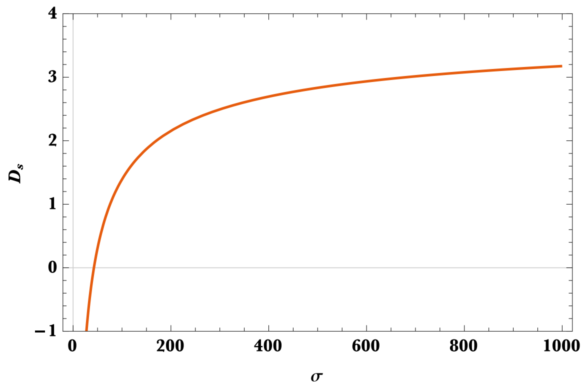
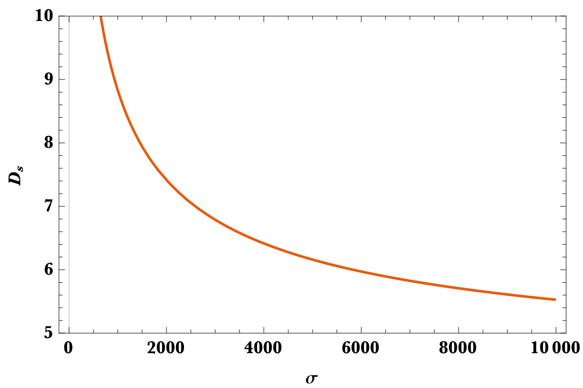


Figure 5.7: spectral dimension as a function of σ with $a = 1$, $n = 4$ and $p = \alpha = \beta = -1$.



incorporated as earlier, using the initial condition

$$U(x, y; 0) = \frac{1}{(4\pi a^2)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4a^2}}. \tag{5.62}$$

With this initial condition we get

$$U_0(x, y; \sigma) = \frac{1}{(4\pi(\sigma + a^2))^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4(\sigma+a^2)}}, \tag{5.63}$$

which is rewritten in terms of a as

$$U_0(x, y; \sigma) = \frac{1}{(4\pi\sigma)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\sigma}} \left[1 - \frac{na^2}{2\sigma} + \frac{a^2 \mid x - y \mid^2}{4\sigma^2} + ...\right]. \tag{5.64}$$

Figure 5.8: spectral dimension as a function of σ with $a = 1$, $n = 4$, $p = 1$, $\alpha = \frac{1}{2}$ and $\beta = -1$.

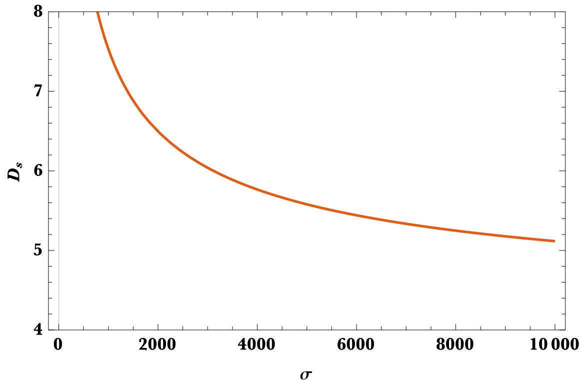
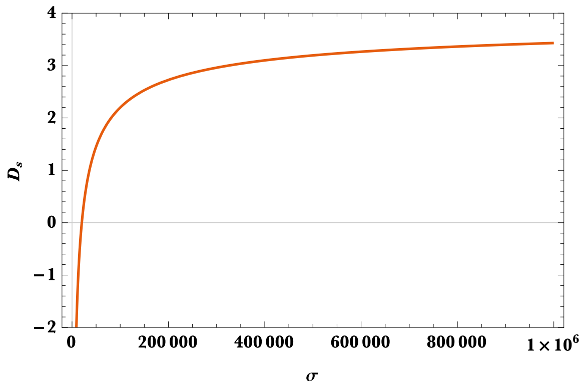


Figure 5.9: spectral dimension as a function of σ with $a = 1$, $n = 4$, $p = 1$, $\alpha = -1$ and $\beta = 0$.



Note that the first non-vanishing correction due to non-commutativity is second order in deformation parameter. The diffusion equation that we are considering contains only first order terms in a. Thus, up to first order in a, extended nature of the probe does not affect the result observed in eqn.(5.61).

5.4 Conclusion

In this chapter, we have constructed the modified diffusion equation for two different realizations of κ-deformed space-time and calculated the corresponding spectral dimension. In the first case we used a specific choice

Figure 5.10: spectral dimension as a function of σ with $a = 1$, $n = 4$, $p = -1$, $\alpha = 0$ and $\beta = 1$.

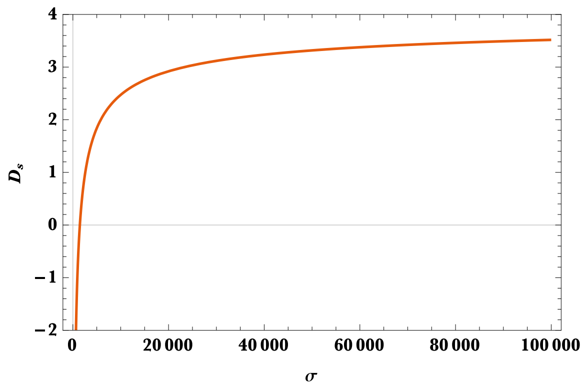
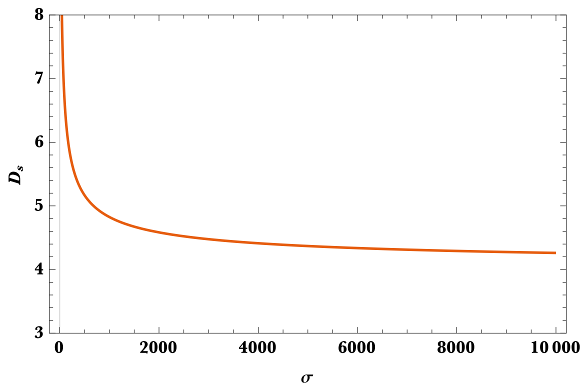


Figure 5.11: spectral dimension as a function of σ with $a = 1$, $n = 4$, $p = -1$, $\alpha = 0$ and $\beta = 1$.



for $\varphi(A)$ which is related to bi-crossproduct basis ($\varphi(A) = e^{-A}$). The deformed diffusion equation is derived using the modified Beltrami-Laplace operator which is written in terms of commutative coordinates and their derivatives. The effects of non-commutativity are included through the a-dependent terms. Here, the first nonvanishing corrections due to κ-deformation are in the first order in deformation parameter. From the analysis, we found that the spectral dimension is changing with the probe scale and it is inversely proportional to $\sqrt{\sigma}$. It also depends on an integer l, which appear due to the third order derivative term appearing in eqn.(5.14). For $l < 0$, the spectral dimension decreases from 4 and reduces to $-\infty$ as σ goes to zero and for $l \geq 0$, the spectral dimension increases to a value higher than the topological dimension in the small probe scale region and in the limit $\sigma \to 0$, D_s goes to infinity for $l \geq 0$. But for the large σ values the spectral dimension approaches to the topological dimension in both the cases. For $l < 0$ case, note that the spectral dimension become negative for $\sqrt{\sigma} < -\frac{15}{16}(2l+1)\sqrt{\pi}a$ which indicate that the spectral dimension loses its meaning below this scale. We have obtained a lower cutoff for the deformation parameter $-\frac{16}{15}\frac{1}{(2l+1)}\sqrt{\frac{\sigma}{\pi}} < a$, by demanding spectral dimension to be positive.

We have also constructed diffusion equation, valid up to first order in a, using (α, β, γ) realization. This diffusion equation is then solved and the solution is used to calculate the spectral dimension. It is found that, generically we have two characteristic types of plots, when we examine the spectral dimension for various values of α, β and p. There are graphs which goes to positive infinity ($+\infty$) at high energies exhibiting a super diffusion and another type of graphs which approaches negative infinity (-∞) for small σ (high energy). It is to be emphasized that, in all these cases, the spectral dimension approaches topological dimension for large probe scale value.

It is interesting to note that, the analysis of the diffusion equation, constructed using the Klein-Gordon operator in (α, β, γ) realization, shows that the generic behaviour of spectral dimension is comparable with the one obtained using the realization $\varphi(A) = e^{-A}$.

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CHAPTER 6

Non-Commutative space-time and Hausdorff dimension

6.1 Introduction

In previous chapters, we have studied spectral dimension of κ-space-time using different choices of Laplace-Beltrami operator and analyzed the change in spectral dimension with the probe scale.

Another interesting definition for effective dimension is Hausdorff dimension and it indicates the amount of uncertainty in the path for a quantum particle [1, 2, 3]. In this chapter, we study the Hausdorff dimension of the path of a particle moving in the κ-deformed space-time. The parameter 'a', which describe the deformation in space, has length dimension, and this sets a minimum length scale below which one cannot probe or localise the particle. This length scale should also restrict the minimum value of the resolution of the measurement of position of the particle. The effects of deformation are incorporated in our analysis through the parameter 'a' appearing in the most general κ-deformed wave function in calculating the position expectation value of the particle moving in κ-space-time. We then generalize this method to study the Hausdorff dimension of path of a relativistic particle, governed by κ-deformed Dirac equation [4]. Applying the self-similarity criterion on the path of non-relativistic and relativistic particle on non-commutative space-time, we then derive generalized uncertainty relation valid up to first order in the deformation parameter.

In [1], dimension of path of particle obeying Schrodinger equation was analyzed. This is done by first defining the path of a non-relativistic quantum mechanical particle as follows; the expectation value of the position operator is measured at equal intervals of time Δt. Now joining the neighbouring points where the particle is localised at times $n\Delta t$ and $(n+1)\Delta t$ (where n is an integer) and repeating this for all the intervals (i.e., for n = 1, 2, 3, ...), one gets a curve made up of $(n-1)$ line segments. This curve is defined as the path followed by the quantum mechanical particle. This definition, naturally leads to dependence of the length of the path to the resolution of the measuring instrument. But one would like to have a definition of length of the path which is independent of the resolution of instrument. The path of quantum mechanical particle as defined above [1] is clearly an everywhere continuous but nowhere differentiable curve which is known as Koch curve, in the study of fractals [5, 6]. Length of such fractal curves which is independent of the resolution of the measuring instrument has been introduced by Hausdorff and using this, dimension of quantum mechanical particle was shown to be 2 in [1]. For a curve whose length is l as measured by an instrument of resolution Δx, Hausdorff introduced a new notion of length, L_H as [1]

$$L_H = l(\Delta x)^{D_H - 1} \tag{6.1}$$

Here, L_H is required to be independent of Δx. This fixes the parameter D_H uniquely. This value of D_H is called the Hausdorff dimension of the fractal curve or path of the quantum particle. This study has been generalized to the case of a particle governed by relativistic quantum mechanics in [3]. For the relativistic particle, the path was defined in [3], in terms of Newton-Wigner operator [7]. It was shown that the Hausdorff dimension in the ultra-relativistic limit is 1, where as in the non-relativistic limit, this dimension is 2, as obtained by [1]. It was further shown in [3] that the length of the path as well as the Hausdorff dimension of the path is independent of the spinorial structure of the relativistic particle. In [3], the Hausdorff dimension for a relativistic particle is shown to be 1. Thus we see that the Hausdorff dimension in non-relativistic regime (which is 2) is reduced to one by relativistic effects, in the commutative space-time.

In [2], the approach of [1] has been adapted for calculating the Hausdorff dimension of a quantum space-time. The quantum nature of space-time was brought in by the introduction of a minimal length, which prevents localisation of particles below this length. The introduction of this minimal length changes the measure of integration in a non-commutative space-time. This change does modify the expectation value of the position operator calculated on such quantum space-time. It was shown in [2] that the Hausdorff dimension of the path is 2 as long as the resolution Δx of the measurement is much above the intrinsic minimum length associated with the quantum space-time. But if the resolution is much below this minimal length, the Hausdorff dimension D_H is $1 - d$, where 'd' is the topological dimension of the space-time. It was shown that the quantum nature of the space-time reduces the Hausdorff dimension from 2, it can be zero or even negative [2].

In this chapter, we analyze the Hausdorff dimension of the path of the quantum particle in κ-deformed space-time. We study this for non-relativistic as well as relativistic quantum particle.

This chapter is organized as follows. In the next section, the path of a non-relativistic particle moving in kappa-deformed space-time is analyzed. Path of the quantum particle is constructed by measuring the position expectation values at different times separated by an interval Δt. We calculate the Hausdorff dimension by measuring the distance travelled by the particle and find that it depends on the deformation parameter. In section 3, we investigate the dimension of relativistic quantum path and find that the spinor nature of the wave function plays a crucial role in calculating the Hausdorff dimension. In section 4, by imposing self-similarity condition on the path of non-relativistic and relativistic quantum particle in non-commutative space-time, we derive the generalized uncertainty relation valid for κ-space-time. Our concluding remarks are given in section 5.

6.2 Dimension of non-relativistic quantum paths

Consider a particle moving in the κ-spacetime. Its path is constructed by measuring the position expectation value of the particle, with resolution Δx, at different times t_0, $t_1 = t_0 + \Delta t$, ..., $t_N = t_0 + N\Delta t$, and joining these points by straight lines. The average distance travelled by the particle in time Δt is $\langle \Delta l \rangle$, and the total distance travelled in time $T = N\Delta t$ is

$$\langle l \rangle = N \langle \Delta l \rangle, \tag{6.2}$$

where N is the number of intervals. It should be noted that, this measurement of length will depend on the spatial resolution Δx of the measuring instrument and also on the deformation of spacetime characterized by the deformation parameter 'a'.

We start with the wave function of a quantum particle in κ-spacetime which is of the form

$$\Psi_{\Delta x,a}(\mathbf{x},0)=\frac{(\Delta x+a)^{\frac{3}{2}}}{\hbar^3}\int_{\mathbb{R}^3}\frac{d^3p}{(2\pi)^{\frac{3}{2}}}f\left(\frac{|\mathbf{p}|(\Delta x+a)}{\hbar}\right)e^{\frac{i}{\hbar}\mathbf{p}.\mathbf{x}}. \tag{6.3}$$

In writing this wave function, we have used the fact that position of the particle is localized within a region of size Δx and in addition there is a modification due to the deformation of space-time which is incorporated through 'a' dependence of the wave function. Note that in the limit $a\to 0$, we get back the wave function in the commutative space [1]. After time Δt the above wave function evolves to

$$\Psi_{\Delta x,a}(\mathbf{x},\Delta t)=\frac{(\Delta x+a)^{\frac{3}{2}}}{\hbar^3}\int_{\mathbb{R}^3}\frac{d^3p}{(2\pi)^{\frac{3}{2}}}f\left(\frac{|\mathbf{p}|(\Delta x+a)}{\hbar}\right)e^{\frac{i}{\hbar}\mathbf{p}.\mathbf{x}}e^{-i\frac{|\mathbf{p}|^2\Delta t}{2\tilde{m}\hbar}}. \tag{6.4}$$

Here $\tilde{m}$ is the modified mass of the particle due to κ-deformation [4, 8, 9] and depends on the realization of κ-space-time coordinate. We next calculate the average distance travelled by the particle Δl in time Δt and using this find the total distance travelled in finite time $T=N\Delta t$. For this, we first restrict our attention to path of particle with zero average momentum (i.e.,in the classical limit, the particle is at rest).

Using the redefinition $\mathbf{k}=\frac{\mathbf{p}(\Delta x+a)}{\hbar}$, the above wave function in eqn.(6.4) is re-written as

$$\Psi_{\Delta x,a}(\mathbf{x},\Delta t)=\frac{1}{(\Delta x+a)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\frac{d^3k}{(2\pi)^{\frac{3}{2}}}f(|\mathbf{k}|)e^{i\mathbf{k}.\frac{\mathbf{x}}{(\Delta x+a)}}e^{-i|\mathbf{k}|^2\frac{\hbar\Delta t}{2\tilde{m}(\Delta x+a)^2}}. \tag{6.5}$$

Now we calculate average distance travelled by the particle in time Δt as

$$\langle\Delta l\rangle=\int_{\mathbb{R}^3}d^3x|\mathbf{x}||\Psi_{\Delta x,a}(\mathbf{x},\Delta t)|^2. \tag{6.6}$$

Substituting (6.5) in (6.6) and letting $\mathbf{y}=\frac{\mathbf{x}}{\Delta x+a}$, one obtains

$$\langle\Delta l\rangle=(\Delta x+a)\int_{\mathbb{R}^3}d^3y|\mathbf{y}|\left|\int\frac{d^3k}{(2\pi)^{\frac{3}{2}}}f(|\mathbf{k}|)e^{i\mathbf{k}.\mathbf{y}}e^{-i|\mathbf{k}|^2\frac{\hbar\Delta t}{2\tilde{m}(\Delta x+a)^2}}\right|^2. \tag{6.7}$$

With $f(|\mathbf{k}|)=Ae^{-(1+\frac{a^2}{\Delta x^2})|\mathbf{k}|^2}$, we get

$$\langle\Delta l\rangle=(\Delta x+a)\int_{\mathbb{R}^3}d^3y|\mathbf{y}|\left|\int\frac{d^3k}{(2\pi)^{\frac{3}{2}}}e^{-[1+\frac{a^2}{\Delta x^2}+i\frac{\hbar\Delta t}{2\tilde{m}(\Delta x+a)^2}]|\mathbf{k}|^2}e^{i\mathbf{k}.\mathbf{y}}\right|^2. \tag{6.8}$$

Performing k integration in the above equation, we find

$$\langle \Delta l \rangle = (\Delta x + a) \int_{\mathbb{R}^3} d^3y |\mathbf{y}| \frac{A^2}{8} \frac{1}{(\alpha^2+\beta^2)^{3/2}} e^{-\frac{\alpha|\mathbf{y}|^2}{2(\alpha^2+\beta^2)}}, \tag{6.9}$$

where A is the normalisation constant of the function $f(|\mathbf{k}|)$, $\alpha = (1+\frac{a^2}{\Delta x^2})$ and $\beta = \frac{\hbar \Delta t}{2\tilde{m}(\Delta x + a)^2}$. This choice of $f(|\mathbf{k}|)$ incooperates the modification of the integration measure [2] to take into account the feature of minimal length associated with quantum space-time. Re-expressing the above equation in polar coordinate, we obtain

$$\begin{aligned} \langle \Delta l \rangle &= (\Delta x + a) \frac{A^2}{8} \frac{1}{(\alpha^2+\beta^2)^{3/2}} \int_0^\infty dr 4\pi r^2 r e^{-\frac{r^2\alpha}{2(\alpha^2+\beta^2)}} \\ &= A^2 \pi (\Delta x + a) \frac{\sqrt{\alpha^2+\beta^2}}{\alpha^2}. \qquad (6.10) \\ &= \frac{\pi A^2}{2} \frac{\Delta t \hbar}{\tilde{m}(\Delta x + a)} \frac{\sqrt{1 + \left(\frac{2\tilde{m}(\Delta x + a)^2}{\hbar \Delta t}\right)^2 \left[1 + \frac{a^2}{\Delta x^2}\right]^2}}{[1 + \frac{a^2}{\Delta x^2}]^2}. \qquad (6.11) \end{aligned}$$

Following the definition $\langle L_H \rangle = N \langle \Delta l \rangle \Delta x^{D_H - 1}$, we find the Hausdorff length for the quantum path as

$$\langle L_H \rangle \propto \Delta x^{D_H - 1} \frac{T\hbar}{\tilde{m}(\Delta x + a)} \frac{\sqrt{1 + \left(\frac{2\tilde{m}(\Delta x + a)^2}{\hbar \Delta t}\right)^2 \left[1 + \frac{a^2}{\Delta x^2}\right]^2}}{[1 + \frac{a^2}{\Delta x^2}]^2}. \tag{6.12}$$

When $(\Delta x + a) \ll \sqrt{\hbar \Delta t / 2\tilde{m}}$ the above equation reduces to

$$\langle L_H \rangle \propto \Delta x^{D_H - 2} \frac{T\hbar}{\tilde{m}(1 + \frac{a}{\Delta x})} \frac{1}{[1 + \frac{a^2}{\Delta x^2}]^2} \tag{6.13}$$

Since Hausdorff length is independent of Δx, we demand

$$\frac{\partial \langle L_H \rangle}{\partial \Delta x} = 0. \tag{6.14}$$

This condition gives

$$\frac{\Delta x^{D_H - 3}}{(1 + \frac{a}{\Delta x})} \left[1 + \frac{a^2}{\Delta x^2}\right]^{-2} \left(D_H - 2 + \frac{\frac{a}{\Delta x}}{(1 + \frac{a}{\Delta x})} + \frac{4\frac{a^2}{\Delta x^2}}{1 + \frac{a^2}{\Delta x^2}} \right) = 0$$

$$D_H - 2 + \frac{\frac{a}{\Delta x}}{(1 + \frac{a}{\Delta x})} + \frac{4\frac{a^2}{\Delta x^2}}{1 + \frac{a^2}{\Delta x^2}} = 0 \tag{6.15}$$

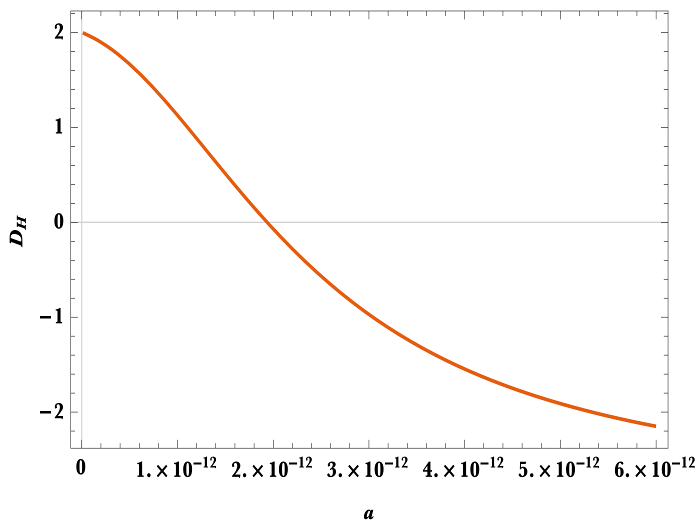
which implies

$$D_H = 2 - \frac{a}{a + \Delta x} - \frac{4a^2}{\Delta x^2 + a^2}, \tag{6.16}$$

which in the commutative limit (a $\to$ 0) reduces to 2 [1]. Note that Hausdorff dimension D_H is independent of the mass of the particle $\tilde{m}$. For $a \ll \Delta x$, we obtain $D_H \simeq 2$, which coincides with the result in the commutative space. But for $\Delta x \ll a$, $D_H = 2 - (1 + \frac{\Delta x}{a})^{-1} - 4(1 + (\frac{\Delta x}{a})^2)^{-1} \simeq -3 + \frac{\Delta x}{a}$. i.e., below the minimal length a, we obtain Hausdorff dimension of the path of the quantum particle as approximately -3. For the specific case in which the resolution coincides with the deformation in space i.e. $a = \Delta x$, we discover that the Hausdorff dimension is $-\frac{1}{2}$.

Taking Δx to be the Compton wavelength of electron, i.e., $\Delta x = 2.426 \times 10^{-12} m$; we plot Hausdorff dimension as a function of a in fig.[6.1]. From this figure, we see that in non-commutative space-time the Hausdorff dimension is always smaller than 2 and decreases with increasing deformation parameter a. The Hausdorff dimension attains negative value after a certain point.

Figure 6.1: Hausdorff dimension as a function of a for a non-relativistic quantum path.



We know that in classical physics the Hausdorff dimension is always one and for a non-relativistic quatum particle it is greater than one [1, 3]. By requiring $D_H \geq 0$, we naturally obtain an upper bound on the deformation parameter which is of the order of $10^{-12} m$ (i.e., $a < 10^{-12} m$). For the case when $a \ll \Delta x$, we find that the Hausdorff dimension is less than 2 and for $a \gg \Delta x$ it is found that D_H is -3. If the deformation is of the order of resolution (i.e., $a \simeq \Delta x$), we obtain D_H as -0.5 (see enq.6.16). Hausdorff dimension attaining zero or negative values, signals lose of meaning of space-time. This feature was also noted in [2] for a quantum space-time.

Particle with non-zero average momentum : Next we consider particle with non-zero average momentum (i.e., in the classical limit, the particle is moving) and calculate the Hausdorff dimension of its path. The wave function of a quantum particle with average momentum $\mathbf{p}_{av}$ in the deformed space-time is given by

$$\Psi_{\Delta x,a}(\mathbf{x},0) = \frac{(\Delta x + a)^{\frac{3}{2}}}{\hbar^3} \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^{\frac{3}{2}}} f\left(\frac{|\mathbf{p}|(\Delta x + a)}{\hbar}\right) e^{i\frac{(\mathbf{p}+\mathbf{p}_{av}).\mathbf{x}}{\hbar}}. \quad (6.17)$$

The average distance traveled by the particle in time Δt is

$$\langle \Delta l \rangle = \int_{\mathbb{R}^3} d^3x |\mathbf{x}| |\Psi_{\Delta x,a}(\mathbf{x},\Delta t)|^2. \quad (6.18)$$

where

$$\Psi_{\Delta x,a}(\mathbf{x},\Delta t) = \frac{(\Delta x + a)^{\frac{3}{2}}}{\hbar^3} \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^{\frac{3}{2}}} f\left(\frac{|\mathbf{p}|(\Delta x + a)}{\hbar}\right) e^{i\frac{(\mathbf{p}+\mathbf{p}_{av}).\mathbf{x}}{\hbar}} e^{-i\frac{|\mathbf{p}+\mathbf{p}_{av}|^2\Delta t}{2\tilde{m}\hbar}}. \quad (6.19)$$

Proceeding as in the previous section with $\mathbf{y} = \frac{\mathbf{x}}{a+\Delta x} - \frac{\Delta t}{\tilde{m}(\Delta x+a)}\mathbf{p}_{av}$ and $f(|\mathbf{k}|) = Ae^{-(1+\frac{a^2}{\Delta x^2})|\mathbf{k}|^2}$, we find the length of the quantum path in time Δt as

$$\begin{aligned} \langle \Delta l \rangle &= (a+\Delta x) \int d^3y \left| \mathbf{y} + \frac{\Delta t}{\tilde{m}(a+\Delta x)} \mathbf{p}_{av} \right| \\ &\times \left| \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} Ae^{-|\mathbf{k}|^2} e^{-i|\mathbf{k}|^2[\frac{\hbar\Delta t}{2\tilde{m}(a+\Delta x)^2} - i\frac{a^2}{\Delta x^2}]} e^{i\mathbf{k}.\mathbf{y}} \right|^2 \end{aligned} \quad (6.20)$$

Using the redefinition

$$\frac{\hbar \Delta t}{2\tilde{m}(a+\Delta x)^2} - i\frac{a^2}{\Delta x^2} = b, \quad (6.21)$$

where b is demanded to be a constant, eqn.(6.20) takes the form

$$\begin{aligned} \langle \Delta l \rangle &= \frac{\Delta t |\mathbf{p}_{av}|}{\tilde{m}} \int d^3y \left| \frac{\mathbf{p}_{av}}{|\mathbf{p}_{av}|} + \frac{\hbar \mathbf{y}}{2|\mathbf{p}_{av}|(a+\Delta x)[b + i\frac{a^2}{\Delta x^2}]} \right| \\ &\times \left| \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} Ae^{-|\mathbf{k}|^2} e^{-ib|\mathbf{k}|^2} e^{i\mathbf{k}.\mathbf{y}} \right|^2. \end{aligned} \quad (6.22)$$

From this, using the definition of Hausdorff length ($\langle L_H \rangle$), we find

$$\begin{aligned}\langle L_H \rangle &= N\langle \Delta l \rangle (\Delta x)^{D_H - 1} \\ &= (\Delta x)^{D_H-1} \frac{T|\mathbf{p}_{av}|}{\tilde{m}} \int d^3y \left| \frac{\mathbf{p}_{av}}{|\mathbf{p}_{av}|} + \frac{\hbar \mathbf{y}}{2|\mathbf{p}_{av}|(a+\Delta x)[b + i\frac{a^2}{\Delta x^2}]} \right| \\ &\times \left| \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} A e^{-|\mathbf{k}|^2} e^{-ib|\mathbf{k}|^2} e^{i\mathbf{k}.\mathbf{y}} \right|^2 .\end{aligned} \tag{6.23}$$

For the case, $\Delta x \gg \frac{\hbar}{|\mathbf{p}_{av}|}$ (i.e., the resolution is much greater than the particle's wavelength), the Hausdorff length $\langle L_H \rangle$ is proportional to $(\Delta x)^{D_H-1}$ which sets Hausdorff dimension to be 1, as in the classical case. In this limit, D_H is independent of deformation parameter a.

When $\Delta x \ll \frac{\hbar}{|\mathbf{p}_{av}|}$, we can neglect the term $\frac{\mathbf{p}_{av}}{|\mathbf{p}_{av}|}$ in comparison with the other terms in eqn.(6.23). With $b = b_1 + ib_2$, we obtain

$$\langle L_H \rangle \propto \frac{(\Delta x)^{D_H-2}}{(1+\frac{a}{\Delta x})} \frac{1}{\sqrt{b_1^2 + (b_2 + \frac{a^2}{\Delta x^2})^2}}$$

which gives the Hausdorff dimension

$$D_H = 2 - \frac{a}{a+\Delta x} - \frac{2\frac{a^2}{\Delta x^2}(b_2 + \frac{a^2}{\Delta x^2})}{[b_1^2 + (b_2 + \frac{a^2}{\Delta x^2})^2]}. \tag{6.24}$$

Note that, in the commutative limit we get $D_H = 2$. From the above discussion, for path of a non-relativistic quantum particle, one can see that Hausdorff dimension is not an integer, and also it depends on deformation parameter a as well as on resolution Δx. There are two 'a' dependent correction terms in eqn.(6.24). For $a \ll \Delta x$ we find $D_H = 2$, which coincides with the commutative result. For $a \gg \Delta x$, we obtain $D_H = -1$ and for $a = \Delta x$ the Hausdorff dimension become $D_H = \frac{3}{2} - \frac{2(b_2+1)}{b_1^2+(b_2+1)^2}$. We also note that, D_H for a particle with non-vanishing average momentum depends on the mass of the particle through b (see eqn.(6.21)).

6.3 Dimension of relativistic quantum paths

We will next calculate the effect of non-commutativity in Hausdorff dimension for relativistic Dirac particle in κ-space-time. κ-Dirac equation in terms of Dirac derivatives is studied in [10] and is given as

$$\left(\gamma^0 D_0 + \gamma^i D_i + \frac{mc}{\hbar}\right) \Psi = 0, \tag{6.25}$$

where

$$D_0 = \partial_0 \frac{sinhA}{A} - ia\nabla^2 \frac{e^{-A}}{2\varphi^2(A)} \quad (6.26)$$

$$D_i = \partial_i \frac{e^{-A}}{\varphi(A)} \quad (6.27)$$

$$\gamma^0 = -i\beta \qquad \gamma^i = -i\beta\alpha^i \quad (6.28)$$

With $A = ia\partial_0$, $\varphi(A) = e^{-A}$ and $\eta_{\mu\nu} = diag(-1, +1, +1, +1)$, the Dirac equation can be rewritten as

$$-i\partial_0\Big(1 - \frac{a^2\partial_0^2}{6} + \frac{a^4\partial_0^4}{120} + ...\Big)\Psi - \frac{a}{2}\nabla^2\Big(1 + ia\partial_0 - \frac{a^2\partial_0}{2} - ...\Big)\Psi$$
$$-i\alpha^i\partial_i\Psi + \beta\frac{mc}{\hbar}\Psi = 0 \quad (6.29)$$

We restrict our attention to first non-vanishing correction due to non-commutativity. Thus we obtain the kappa-Dirac equation, valid up to first order in a as

$$i\hbar\frac{\partial\Psi}{\partial t} = -i\hbar\alpha^i\partial_i\Psi + \beta mc^2\Psi - \frac{a}{2}\hbar\nabla^2\Psi. \quad (6.30)$$

This gives the form of κ-deformed Dirac Hamiltonian, valid up to first order in a as

$$H_{DNC} = H_D + \frac{ac}{2\hbar}p^2, \quad (6.31)$$

where $H_D = c\mathbf{p}.\alpha + \beta mc^2$. In the Dirac representation α and β are given as

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (6.32)$$

where σ_i are the Pauli matrices.

Since we want to restrict our calculation to deal with only observables, we will be working in Foldy-Wouthuysen representation(FW-representation) [11]. The change from the Dirac representation to FW-representation is achieved by a unitary transformation implemented by

$$U = \frac{\beta H_D + E_p}{\sqrt{2E_p(mc^2 + E_p)}} \quad (6.33)$$

where $E_p = (p^2c^2 + m^2c^4)^{1/2}$. The Hamiltonian in Foldy-Wouthuysen representation obtained by applying the above unitary transformation is

$$H_{FWNC} = UH_{DNC}U^\dagger \quad (6.34)$$
$$= U\left(H_D + \frac{ac}{2\hbar}p^2\right)U^\dagger$$
$$= E_p\beta + \frac{ac}{2\hbar}p^2, \quad (6.35)$$

where H_{DNC} is given in eqn.(6.31). The modified position operator, $\mathbf{X}$ compatible with the Foldy-Wouthuysen representation, known as Newton-Wigner operator [11] is

$$\mathbf{X} = U\mathbf{x}U^{\dagger}. \tag{6.36}$$

Note that $\mathbf{X}$ and $\mathbf{p}$ satisfy the same commutation relation as $\mathbf{x}$ and $\mathbf{p}$, and thus we find $\Delta X \geq \hbar/\Delta p$. This X satisfies the Ehrenfest equation

$$\frac{d^2X^2}{dt^2} = -\hbar^{-2}[H_{DNC}, [H_{DNC}, X^2]] \tag{6.37}$$

which governs its time evolution.

Using (6.35) and (6.36) in (6.37), we obtain the equation for X^2 as

$$\frac{d^2X^2}{dt^2} = \frac{2c^4p^2}{E_p^2} + \frac{4a}{\hbar}\frac{c^3p^2}{E_p^2}H_D. \tag{6.38}$$

By solving the above equation, we obtain the average value of X^2 at time t as

$$\langle X^2\rangle_t = c^4\langle\frac{p^2}{E_p^2}\rangle_0 t^2 + \frac{2ac^3}{\hbar}\langle\frac{p^2}{E_p^2}H_D\rangle_0 t^2 + \langle\frac{dX^2}{dt}\rangle_0 t + \langle X^2\rangle_0. \tag{6.39}$$

where subscript 0 on terms in RHS stands for values at t=0. The average length $\langle\Delta l\rangle$ travelled by the particle during the time interval $\Delta t \gg \langle\frac{dX^2}{dt}\rangle_0/(c^4\langle\frac{p^2}{E_p^2}\rangle_0)$ will be

$$\langle\Delta l\rangle_{\Delta t} = (\langle X^2\rangle_{\Delta t} - \langle X^2\rangle_0)^{1/2} = c^2\Delta t\langle\frac{p^2}{E_p^2}\rangle_0^{\frac{1}{2}}\left[1 + \frac{2a}{\hbar c}\frac{\langle\frac{p^2}{E_p^2}H_D\rangle_0}{\langle\frac{p^2}{E_p^2}\rangle_0}\right]^{\frac{1}{2}} \tag{6.40}$$

Note that eqn.(6.40) involves expectation value of H_D. Thus above expression for $\langle\Delta l\rangle$ shows that the distance travelled by the relativistic quantum particle depends on the spinorial character of the wave function. This should be contrasted with the commutative space-time result [3], where the spinorial structure of the relativistic wave function does not play any role in the calculation of Hausdorff dimension. This is due to the fact that, in the commutative space $\langle\Delta l\rangle_{\Delta t}$ does not involve expectation value of H_D and thus the result is independent of the spinorial nature of the wave function [3]. But here, we see that the spinorial nature of wave function comes into effect in the non-commutative space-time. Note that in the commutative limit, i.e., $a \to 0$, the $\langle\Delta l\rangle_{\Delta t}$ is independent of H_D and thus all dependence of spinorial wave function drops out. In the momentum representation, the wave function of positive energy, spin up particle, compatible with non-commutative space-time (with minimal length given by a) is of the form

$$\Psi(p)_{NC} = \sqrt{\frac{E_p + mc^2}{2E_p}} \begin{pmatrix} 1 \\ 0 \\ \frac{c}{E_p+mc^2} p_z \\ \frac{c}{E_p+mc^2}(p_x + ip_y) \end{pmatrix} \sqrt{\frac{3}{4\pi}} \left(\frac{1}{\Delta p} + \frac{a}{\hbar}\right)^{3/2}, \quad for \mid p - p_0 \mid < \frac{\hbar \Delta p}{(\hbar + a\Delta p)} \tag{6.41}$$

Using this wave function we obtain the average length $\langle \Delta l \rangle$ travelled by the particle as

$$\begin{aligned}
\langle \Delta l \rangle_{\Delta t} &= c^2 \Delta t \left[\langle \frac{p^2}{E_p^2} \rangle_0 + \frac{2a}{\hbar c} \langle \frac{p^2}{E_p^2} H_D \rangle_0 \right]^{\frac{1}{2}} \\
&= c\Delta t \Bigg\{ 1 - \frac{3}{4}\left(1 + \frac{a\Delta p}{\hbar}\right)^3 x^2 \Bigg(2\eta - 2x \Big[\arctan\left(\frac{\eta y + 1}{xy}\right) \\
&+ \arctan\left(\frac{\eta y - 1}{xy}\right) \Big] + \frac{1}{2}\left[y\left(\eta^2 + x^2\right) - \frac{1}{y}\right] \log\left(\frac{(1+\eta y)^2 + x^2 y^2}{(1-\eta y)^2 + x^2 y^2}\right) \Bigg) \\
&+ \frac{a}{\hbar}\frac{1}{(\Delta p)^3 p_0} \Bigg(\frac{-9}{8} p_0 m^4 c^4 \Big[\log(-p_0 + \sqrt{p_0^2 + m^2c^2}) \\
&+ \log(p_0 + \sqrt{p_0^2 + m^2c^2}) \Big] + \frac{1}{40} \Big[\sqrt{(p_0 - \Delta p)^2 + m^2c^2} \\
&\times \Big(2p_0^4 - 11 p_0^2 m^2 c^2 + 32 m^4 c^4 + 2p_0^3 \Delta p - 13 p_0 m^2 c^2 \Delta p \\
&- 18 p_0^2 \Delta p^2 + 24 m^2 c^2 \Delta p^2 + 22 \Delta p^3 p_0 - 8 \Delta p^4 \Big) \\
&+ \sqrt{(p_0 + \Delta p)^2 + m^2c^2} (-2p_0^4 + 11 p_0^2 m^2 c^2 - 32 m^4 c^4 - 13 p_0 m^2 c^2 \Delta p \\
&+ 2p_0^3 \Delta p + 18 p_0^2 \Delta p^2 - 24 m^2 c^2 \Delta p^2 + 22 \Delta p^3 p_0 + 8 \Delta p^4) \Big] + \frac{9}{8} p_0 m^4 c^4 \\
&\times [\log(-(p_0 - \Delta p) + \sqrt{(p_0 - \Delta p)^2 + m^2c^2}) + \log((p_0 + \Delta p) \\
&+ \sqrt{(p_0 + \Delta p)^2 + m^2c^2})] \Bigg) \Bigg\}^{1/2}
\end{aligned} \tag{6.42}$$

where $x = mc/\Delta p$, $y = \Delta p/p_0$ and $\eta = \frac{\hbar}{a\Delta p + \hbar}$.

From eqn.(6.42), in the commutative limit ($a \to 0$), we obtain

$$\langle \Delta l \rangle_{\Delta t} = c\Delta t \Bigg\{ 1 - \frac{3}{4}x^2 \Bigg(2 - 2x \left[\arctan\left(\frac{y-1}{xy}\right) + \arctan\left(\frac{y+1}{xy}\right)\right] + \frac{1}{2}[y(1+x^2) - y^{-1}] \log\left(\frac{(y+1)^2 + x^2y^2}{(y-1)^2 + x^2y^2}\right)\Bigg)\Bigg\}^{\frac{1}{2}} \quad (6.43)$$

which is same as result obtained in [3].

For $\Delta p \ll p_0$, i.e., in the classical limit, eqn.(6.42) reduces to

$$\langle \Delta l \rangle^{cl}_{\Delta t} = c\Delta t \left[\frac{p_0^2}{(p_0^2 + m^2c^2)} + \frac{2a}{\hbar} \frac{p_0^2}{\sqrt{(p_0^2 + m^2c^2)}} \right]^{\frac{1}{2}}. \quad (6.44)$$

This quantity is independent of ΔX. So the Hausdorff dimension D_H is one as in the commutative case [3].

For the case when $p_0 \ll \Delta p$, we consider two regimes, i.e.,

- when $\Delta p \ll mc$ (non-relativistic limit of quantum regime), eqn.(6.42) reduces to

$$\langle \Delta l \rangle^{QNR}_{\Delta t} = c\Delta t \left[\frac{3}{5}\left(\frac{\Delta p}{mc}\right)^2 \left(\frac{\hbar}{a\Delta p + \hbar}\right)^2 - \frac{3a}{8\hbar}\left(\frac{6m^3c^3}{(\Delta p)^2} + 2mc\right)\right]^{1/2}. \quad (6.45)$$

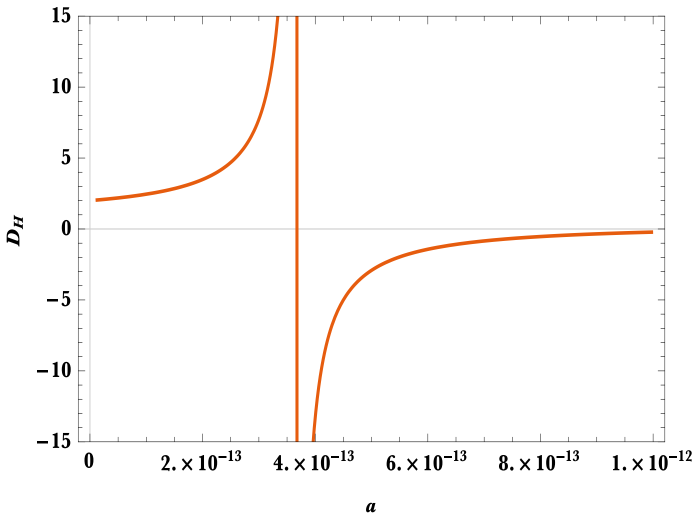
From this expression, we find the Hausdorff dimension to be

$$D_H = 1 + \frac{\frac{3}{5}(\frac{\hbar}{mc})^2 \frac{\Delta X}{(a+\Delta X)^3} + \frac{9}{4}a(\frac{mc}{\hbar})^3(\Delta X)^2}{\left[\frac{3}{5}(\frac{\hbar}{mc})^2 \frac{1}{(a+\Delta X)^2} - \frac{9}{4}a(\frac{mc}{\hbar})^3(\Delta X)^2 - \frac{3}{4}\frac{amc}{\hbar}\right]}. \quad (6.46)$$

We note that Hausdorff dimension depends on the mass of the particle, unlike the non-relativistic case (with $\mathbf{p}_{av} = 0$) or the commutative case. Setting $\Delta X = \lambda_c$ (Compton wavelength of electron $\frac{\hbar}{mc}$), we plot D_H against a in fig.[6.2].

From the figure, we see that the Hausdorff dimension increases rapidly until a approaches value of the order of $10^{-13}m$ (for $\lambda_c = 2.426 \times 10^{-12}m$). At this point, the Hausdorff dimension blows up by approaching positive infinity and abruptly moves to negative infinity such that it becomes lesser negative for higher values of deformation parameter.

Figure 6.2: Hausdorff dimension as a function of a for a particle in non-relativistic quantum regime.



- $\Delta p \gg mc$ (Ultrarelativistic case). In this case, eqn.(6.42) become

$$\langle \Delta l \rangle_{\Delta t}^{QR} = c\Delta t \left[1 + \frac{3}{2} \frac{a\Delta p}{\hbar} \right]^{1/2} \tag{6.47}$$

and the corresponding D_H is

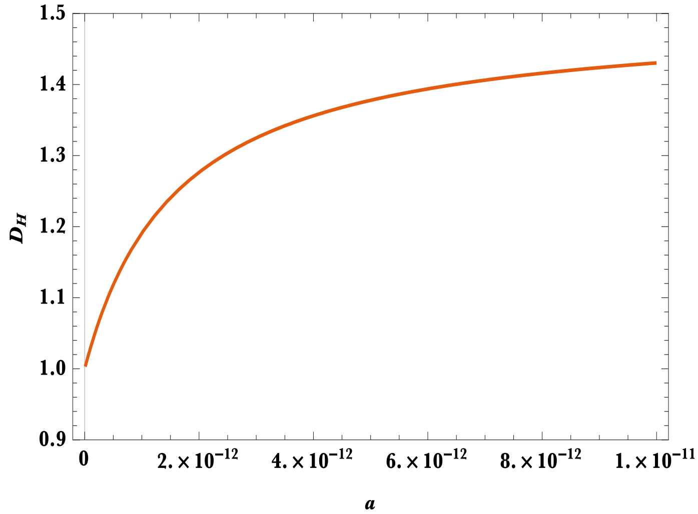
$$D_H = 1 + \frac{1}{2} \frac{3a}{(2\Delta X + 3a)}. \tag{6.48}$$

In the commutative space, the length travelled by the relativistic particle is independent of the resolution ΔX. So the Hausdorff dimension is always one for the case of relativistic particle[3]. In our case, we see that the Hausdorff dimension depends on deformation parameter a as well as resolution ΔX, and it is always greater than one. This modification is due to the non-commutative nature of space-time. Also, note that H_D is independent of the mass of the particle.

In the commutative limit, one can see that the distance travelled by the ultrarelativistic particle is independent of Δp (or ΔX). Hence Hausdorff dimension is always one in this limit.

Hausdorff dimension given in eqn.(6.48) as a function of deformation parameter is plotted in fig.[6.3]. Here, we can see that the Hausdorff dimension increases with increase in deformation parameter and its value is always greater than one.

Figure 6.3: Hausdorff dimension as a function of a for a relativistic quantum path.



In contrast to commutative case [3], we see that the non-commutative nature of space-time results in an increase in the Hausdorff dimension. In the ultra-relativistic limit, Hausdorff dimension increases with a while in the non-relativistic limit D_H increases as a increases and diverges to infinity. Beyond this critical value of a, Hausdorff dimension is negative as shown in fig.[6.2].

6.4 Modified Uncertainty Relation

In [1, 3], it was shown that the self-similarity condition when applied to the path of the quantum particle, naturally leads to Heisenberg's uncertainty relation. Thus it is natural to anticipate that the requirement of self-similarity applied to path of a particle on non-commutative space-time would lead to modified uncertainty relation compatible with non-commutative space-time. The path of a quantum-mechanical particle will be self similar if $\langle \Delta l \rangle \propto \Delta x$.

For the path of a non-relativistic quantum particle (see eqn.(6.10)), along with the condition $(\Delta x + a) \ll \sqrt{\frac{\hbar \Delta t}{2\tilde{m}}}$, we get

$$\langle \Delta l \rangle \quad \propto \quad \frac{\hbar \Delta t}{2\tilde{m}(\Delta x + a)} \frac{1}{[1 + \frac{a^2}{\Delta x^2}]^2} \tag{6.49}$$

By demanding self-similarity condition, we get the uncertainty in time as

$$\Delta t \simeq \frac{\tilde{m}}{\hbar}(\Delta x)^2(1+\frac{a}{\Delta x})(1+\frac{a^2}{\Delta x^2})^2. \tag{6.50}$$

Using this along with $\Delta E \sim \frac{(\Delta p)^2}{\tilde{m}}$ in $\Delta E\ \Delta t \geq \hbar$, we obtain the deformed uncertainty relation as

$$\frac{(\Delta p)^2}{\tilde{m}}\ \frac{\tilde{m}}{\hbar}(\Delta x)^2(1+\frac{a}{\Delta x})(1+\frac{a^2}{\Delta x^2})^2 \geq \hbar$$

$$i.e., \qquad \Delta x\ \Delta p(1+\frac{a^2}{\Delta x^2})\sqrt{1+\frac{a}{\Delta x}} \geq \hbar. \tag{6.51}$$

Keeping terms upto first order in a, we write the modified uncertainty relation in the form

$$\Delta x\ \Delta p(1+\frac{a}{2\Delta x}) \geq \hbar. \tag{6.52}$$

For the specific case where the resolution is comparable with the deformation in space, we have $\frac{3}{2}\Delta x\ \Delta p \geq \hbar$.

Following a similar procedure, using eqn.(6.45), for a spin half particle in the non-relativistic limit, we arrive at a modified uncertainty relation

$$\Delta X\ \Delta p\ \ (1.136)\ \ \left[1+\frac{a}{2\Delta X}+\frac{5}{16}a\Delta X^2\left(\frac{mc}{\hbar}\right)^3\left(1+3\left(\frac{mc}{\hbar}\right)^2\Delta X^2\right)\right] \geq \hbar. \tag{6.53}$$

The corresponding uncertainty relation for a spin half particle in the relativistic regime ($\Delta E = c\Delta p$), derived from eqn.(6.47) is

$$\Delta X\ \Delta p(1-\frac{3}{4}\frac{a}{\Delta X}) \geq \hbar. \tag{6.54}$$

When $a = \Delta X$, the modified uncertainty relation becomes

$$\frac{1}{4}\Delta X\ \Delta p \geq \hbar. \tag{6.55}$$

Note that eqn.(6.52) and eqn.(6.54) show the generalized uncertainty relation, valid upto first order in a is of the form $\Delta X\ \Delta p\ f\left(\frac{a}{\Delta X}\right) \geq \hbar$. The modified uncertainty relation obtained for non-relativistic limit of relativistic particle have other a dependent terms apart from the $\frac{a}{\Delta X}$ term. All these modified relations, in the commutative limit reduce to well known Heisenberg's uncertainty relation. Since the non-relativistic limit of relativistic path retains the information about the spinorial nature of the particle, it is natural that the result derived from this limit given in eqn.(6.45) is different from the result obtained by analyzing the eqn.(6.10) for non-relativistic particle directly.

6.5 Conclusion

In this chapter, we have calculated the Hausdorff dimension for a quantum particle moving in a non-commutative space-time. For a non-relativistic quantum particle, we found that the Hausdorff dimension is decreasing with deformation parameter a for a fixed resolution Δx. We have obtained an upper cutoff for the deformation parameter ($a < 10^{-12}m$) by demanding $D_H \geq 0$. An interesting result we obtained is that the path travelled by a relativistic quantum particle depends on the spinorial character of the wave function, unlike in the commutative space-time. It also depends on the mass of the particle for the relativistic case as well as for the non-relativistic case with $\mathbf{p}_{av} \neq 0$. We also note that the Hausdorff dimension of path of a relativistic particle is larger than one in the κ-deformed space-time which should be contrasted with the value of D_H in the commutative space where it is always one.

We have derived generalized uncertainty relation, by imposing self-similarity condition on the path of the quantum particle. We found that due to kappa-deformation the uncertainty relation is modified and the modification depends on a factor of the form $\frac{a}{\Delta x}$. Generalized uncertainty relations and their implications have been studied in [12, 13, 14]. Generalized uncertainty relation for a specific realisation of κ-deformed space-time has been discussed in [15], where the modified uncertainty relation was found to have a mass dependence. We have shown that the self-similarity condition imposed on path of particle on non-commutative space-time leads to modified uncertainty relation. All the obtained modified uncertainty relations, in the commutative limit reduce to well known Heisenberg's uncertainty relation.

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CHAPTER 7

Conclusion

In this thesis, we have studied effective dimensions of kappa-deformed space-time. The investigations reported here use two notions of dimensions, namely spectral dimension and Hausdorff dimension in analyzing the effect of non-commutativity on the changes in space-time dimension with the probe scale. Spectral dimension has been studied by constructing different possible generalizations of diffusion equation in κ-space-time. These studies are intended to provide better insight into the nature of space-time at Planck scale.

In chapter 1, we motivate the study of kappa-space-time. We have chosen to work directly with realizations of non-commutative coordinates in terms of their commutative counter part and their derivative. We have mainly concentrated on three different realizations in which the first two corresponds to different choices of $\varphi(A)$ and the remaining one is the (α, β, γ)-realization.

In chapter 2, we have summarize two different definitions of effective dimensions, namely spectral dimension and Hausdorff dimension. We started with the concept of spectral dimension and the basic idea of spectral dimension is explained. Since spectral dimension is related to diffusion equation, a modification to diffusion equation in a given space-time leads to the change of behaviour of spectral dimension. In general, there are three types modifications possible for diffusion equation in κ-space-time. More specifically, these are, changes in Laplace operator(∇^2) appearing in the diffusion equation, modification in the initial condition($U(x, y; 0)$) and possibility of modification in diffusion operator(∂_σ).

Chapter 2 also discuss the notion of Hausdorff dimension. Hausdorff length is introduced as a definition of length which is independent of the resolution of measuring apparatus and used widely in the study of fractals. The dimension that arising in this context is known as Hausdorff dimension. Hausdorff dimension has been discussed in the context of quantum path.

In chapter 3, the spectral dimension of kappa-deformed Euclidean space, for a specific choice of realization ($\varphi(A) = e^{-\frac{A}{2}}$) is studied. Here the diffusion equations are constructed using the Casimir of the undeformed κ-Poincare algebra ($D_\mu D_\mu$) and $\Box$ operator which also has the correct commutative limit. We keep terms up to first non-vanishing corrections in the deformation parameter a, which were found to be second order in a. It was shown that the modified diffusion equations have higher order derivative terms. We have solved these diffusion equations perturbatively. The spectral dimensions calculated from these solutions showed a length scale dependence and they take negative values at high energies ($\sigma \to 0$). It was observed that, in the limit $\sigma \to \infty$, the spectral dimension and the topological dimension are same. By introducing a cut-off on the deformation parameter, spectral dimension is guaranteed to be positive definite. We have also analyzed the dimensional flow for the case where the probe particle has an extension, in contrast to point particle. It was found that, the extended nature of the probe will not alter the generic features of the spectral dimension.

In chapter 4, we have used the relation between Schrödinger equation and diffusion equation to study the scale dependence on effective dimension. We have constructed four different modified diffusion equations in kappa-space-time for $\varphi(A) = e^{-\frac{A}{2}}$ and found that, Laplace operator(∇^2) as well as the diffusion operator(∂_σ) get modification due to non-commutative nature of space-time. The analysis of spectral dimension showed that in high energy limit, the spectral dimension become infinitely negative. We have also analyzed the effect of finite width of the probe and found that the generic behaviour of dimensional flow is not changing.

In chapter 5, we have addressed the question, whether different choice of realization of the κ-space-time will affect the dimensional flow. We have constructed modified diffusion equations for two different realizations of kappa-space-time ($\varphi(A) = e^{-A}$ and (α, β, γ) realizations) and calculated the corresponding spectral dimensions. In both the cases, the leading order correction term due to κ-deformation is first order in deformation parameter a. Our results indicated that the spectral dimension varies with the scale of diffusion and also depends on an integer (l for e^{-A} realization and p for (α, β, γ) realizations).

With $\varphi(A) = e^{-A}$ realization, we showed that, for large diffusion times $(\sigma \to \infty)$ the spectral dimension approaches the usual topological dimension where as diverges to positive infinity $(+\infty)$ for $l \geq 0$ and $-\infty$ for $l < 0$ at high energies. Inspection of spectral dimension in the (α, β, γ) realizations showed that the characteristic behaviour of D_s depends on various values of p, α and β.

In chapter 6, we have studied the Hausdorff dimension of the path of a quantum particle in non-commutative space-time. We showed that the Hausdorff dimension depends on the deformation parameter a and the resolution Δx for both non-relativistic and relativistic quantum particle moving on κ-space-time. For the non-relativistic case, it was shown that Hausdorff dimension is always less than two in the non-commutative space-time. For relativistic quantum particle, we found the Hausdorff dimension increases with the non-commutative parameter, in contrast to the commutative space-time. We showed that non-commutative correction to Dirac equation brings in the spinorial nature of the relativistic wave function into play, unlike in the commutative space-time. By imposing self-similarity condition on the path of non-relativistic and relativistic quantum particle in non-commutative space-time, we derived the corresponding generalized uncertainty relation.

In conclusion, our study of effective dimensions of κ-space-time indicates that the structure of space-time significantly changes at high energies. The analysis of the pattern of dimensional flow at these energies show that the spectral dimension become negative, indicating a lose of meaning of space-time itself [1]. This has also lead to the discussion of multiscale structure of spacetime [1]. In order to understand about these new features at the Planck scale physics, we plan to study the following issues in recent future.

A re-interpretation of the spectral dimension was discussed in [2], which avoids the negative probability associated with the higher order, and non-local dispersion relations. Since in our case, the Laplacians do have higher derivatives, it will be interesting to use the approach developed in [2] and analyse the spectral dimension. We plan to take up this issue. Our studies on spectral dimension showed that it becomes negative for high energies. One needs to find possible ways to avoid the negative values of the spectral dimension. This will lead us to study the multiscale structure of the space-time, which has been pointed out earlier in [1]. Another important problem we plan to study is the effect of curvature on the spectral dimension of κ-deformed space-time [3].

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To Whom It May Concern

This is to certify that the thesis titled " Effective Dimensions and Dimensional flow in κ-spacetime" submitted by Ms. Anjana V (12PHPH16), to the University of Hyderabad is based on her research work done under my guidance and has been screened by Turnitin software at Indira Gandhi Memorial Library, University of Hyderabad. The report shows a similarity index of 40%. Of this, about 34% are from published papers where Anjana is the lead author. A through look at the report shows that this 34% similarity as well as the remaining similarity index is due to the standard technical terms and usages used in the published works in this area of research. Use of such technical and scientific terms are unavoidable.

Thanking you

Sincerely yours,

E. Harikumar