# STUDY OF VARIOUS TRAJECTORIES IN TOPOLOGICAL DYNAMICS WITH AN EMPHASIS ON PERIODIC ONES

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by


**SHARAN GOPAL**

SCHOOL OF MATHEMATICS AND STATISTICS

UNIVERSITY OF HYDERABAD

HYDERABAD - 500 046



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# CERTIFICATE

School of Mathematics and Statistics

University of Hyderabad

Hyderabad- 500 046

Date: March 2013

This is to certify that I, Sharan Gopal, have carried out the research embodied in the present thesis entitled **STUDY OF VARIOUS TRAJECTORIES IN TOPOLOGICAL DYNAMICS WITH AN EMPHASIS ON PERIODIC ONES** for the full period prescribed under Ph.D. ordinance of the University.

I declare that, to the best of my knowledge, no part of this thesis was earlier submitted for the award of research degree of any university.

Sharan Gopal

Candidate (Reg. No. 08MMPP01)

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|---|---|
| Prof. S. Kumaresan | Prof. V. Kannan |
| Dean of the School | Supervisor |

*To my beloved mother*

***Dathu Bai***,

*who means everything to me.*

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- Sharan Gopal

# Contents

# List of Symbols

| | |
|---|---|
| $\mathbb{R}$ | the set of all real numbers |
| $\mathbb{Q}$ | the set of all rational numbers |
| $\mathbb{N}$ | the set of all positive integers |
| $\mathbb{N}_0$ or $\mathbb{W}$ | the set of all non-negative integers |
| $\mathbb{W}_\infty$ | $\mathbb{W} \cup \{\infty\}$ |
| $O^{(n)}$ | $\omega^n.m_n + \omega^{n-1}.m_{n-1} + ... + \omega.m_1 + m_0$, $m_i \in \mathbb{N}_0$ |
| $\mathcal{P}(X)$ | the set of all subsets of $X$ |
| $D(X)$ | the set of limit points of $X$ |
| $X_n$ | $D^n(X) \setminus D^{n+1}(X)$ |
| $\mathcal{O}(x)$ | the orbit of $x$ |
| $P(f)$ | the set of all periodic points of $f$ |
| $Per(f)$ | $\{n \in \mathbb{N} : f$ has a point of period $n\}$ |
| $G_S$ (in $X$) | $\{f : X \to X : f$ is a homeomorphism on $X$ such that $f(S) = S\}$ |

# Abstract

This thesis is in the area of topological dynamics mainly focusing on periodic and some other trajectories. It is divided in to five chapters.

The first chapter starts with an introduction to topological dynamics giving a lot of terminology that one comes across in this area. Some interesting results are also mentioned without giving the proof. Section 1.2 gives a brief account of various types of trajectories. The next section is the explanation of the problem of characterisation of sets of periods and sets of periodic points of a family of dynamical systems. This problem is taken up for two families in Chapter 2 and Chapter 3. Section 1.4 deals with the convergent trajectories, a problem which is dealt with, in Chapter 4.

Chapter 2 gives a characterisation of sets of periods and sets of periodic points of linear operators on $l^2$. The first section gives an introduction to the problem and discusses some Hilbert space theory. Sections 2.2 and 2.3 discuss the sets of periods and sets of periodic points in detail respectively. Then the combined question of characterising the pair of sets that can arise simultaneously as the sets of periods and periodic points is taken up in the section 2.4.

Chapter 3 deals with the same problem of characterising the sets of periodic points for the family of homeomorphisms on compact metric spaces of finite derived length. The problem is introduced in the first section which also gives an account of ordinal numbers. Section 3.2 introduces some definitions and notations used in the chapter. Then follows the main section characterising the sets of periods. As a digression in the above problem, a notion called eventual homeomorphism on metric spaces is defined and the number of subsets of $\omega^n$, up to eventual homeomorphism is counted. Some other equivalence classes on $\omega^2$ are also studied.

A new kind of trajectories, namely the convergent trajectories, which can be the

next simplest trajectories after the eventually periodic ones, is considered in Chapter 4. The main results in this chapter actually list some natural dynamical systems which are void of such simple trajectories. There are two sections in this chapter, the first dealing with the definitions and the next gives the main results of the chapter.

The last chapter of the thesis is about the trajectories in a chaotic system. It shows the possibility of existence of numerous kinds of trajectories in a chaotic system, thus showing the kind of variety a chaotic system exhibits in terms of trajectories. The first section gives an introduction to chaos, followed by a section on main results and the chapter ends with concluding remarks in the last section.

Thus, this thesis studies the dynamics of two particular families of dynamical systems, linear operators on $l^2$ and the homeomorphisms on compact metric spaces with finite derived length, with main focus on periodicity. Then comes the second kind of trajectories, namely the convergent trajectories. Finally, the possibility of various kinds of trajectories in chaotic systems is shown.

**Publications related to this thesis :**

1. K. Ali Akbar, V. Kannan, Sharan Gopal and P. Chiranjeevi, The set of periods of periodic points of a linear operator, Linear Algebra and its Applications 431 (2009) 241 - 246.

2. P.Chiranjeevi, V.Kannan and Sharan Gopal, Periodic points and periods for operators on Hilbert space, Discrete and Continuous Dynamical Systems, 33 (2013) 4233-4237.

3. V.Kannan and Sharan Gopal, 63 Kinds of subsets, Bulletin of Kerala Mathematics Association, Vol.6, No.2, (2010, December) 121 – 130.

# Chapter 1

# Introduction

The modern theory of dynamical systems has its origins in the questions related to the stability of the solar system which were considered by Poincare at the end of the 19th century. Dynamical systems is essentially the study of eventual behavior of evolving systems. Precisely, it is a topological space $X$ together with a family $\mathcal{F} = \{f^t : X \to X : t \in \mathbb{R}\}$ of continuous self maps on $X$ such that $f^t \circ f^s = f^{t+s}$ and $f^0$ is identity on $X$. This is called a continuous dynamical system or a flow . The system $(X, f^t)$ evolves under the maps $f^t$ and $f^t(X)$ is the state of the system at time $t$. Instead of the family $\mathcal{F}$, if we consider $\{f^n : n \in \mathbb{N}_0\}$, where $f$ is a continuous self map on $X$, $f^0$ is identity and $f^n = f \circ f \circ f \circ ... \circ f$ ($n$ times), then $X$ together with this new family is called a discrete dynamical system. This system is simply written as $(X, f)$. Many of the results for flows can be deduced from the results on discrete dynamical systems. In this thesis, only discrete systems are discussed.

Given a point $x$ in a dynamical system $(X, f)$, the sequence of positions of the point $x$ under $f$ as time increases, is called its trajectory. Thus we can say that the study of a dynamical system is in other words study of the trajectories of its points. The present study is of various types of trajectories in different systems.

## 1.1 Definitions

This section gives an account of the terminology that will be used throughout the thesis. In this chapter, $X$ represents a second countable topological space and $f$ a continuous self map on $X$. As mentioned earlier, only the discrete dynamical systems will be considered. So, $(X, f)$ stands for a dynamical system, which will be very often referred briefly to, as a system.

Given a point $x \in X$, the sequence $(x,\ f(x),\ f^2(x), ...)$ is called the trajectory of $x$ and its range i.e., the set $\{f^n(x) : n \in \mathbb{N}_0\}$ is called the orbit of $x$. $x$ is said to be periodic if there is $n \in \mathbb{N}$ such that $f^n(x) = x$ i.e., the point $x$ returns to itself after some finitely many iterations. The least such $n$ is called the period of $x$. A point which "reaches" a periodic point after finitely many iterations is called an eventually periodic point i.e., $y \in X$ is called eventually periodic if $f^n(y)$ is periodic for some $n \in \mathbb{N}_0$. A periodic point of period 1 is called a fixed point i.e., $f(x) = x$. On the same lines, if $f^n(y)$ is a fixed point for some $n \in \mathbb{N}_0$, then $y$ is called an eventually fixed point.

It so happens with the dynamics of some points that they do not return to their position exactly but keep coming arbitrarily close to themselves infinitely many times. A more liberal case would be that every neighborhood of the point meets itself atleast once in its motion. The following definitions make these concepts more precise.

**Definition 1.1.1.** A point $y \in X$ is said to be an $\omega$-limit point of a point $x \in X$ if there is a sequence of natural numbers $(n_k) \to \infty$ (as $k \to \infty$) such that $(f^{n_k}(x)) \to y$. The $\omega-$limit set of $x$ is the set of all $\omega-$limit points of $x$, denoted by $\omega(x)$.

**Definition 1.1.2.** A point $x$ is called recurrent if $x \in \omega(x)$. Equivalently, $(f^{n_k}(x)) \to x$ for some sequence of natural numbers $(n_k) \to \infty$.

**Definition 1.1.3.** A point $x$ is non-wandering if for any neighborhood $U$ of $x$ there exists $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \phi$.

Now the terminology is regarding the interplay between the points. Consider the trajectories of two points $x$ and $y$ in a metric space. A simple case would be that they maintain the same distance between them through out the itineraries i.e., $d(f^n(x), f^n(y)) = d(x, y)$ for every $n \in \mathbb{N}$. This happens, for instance, when $f$ is an isometry. But the proximal points are those which come arbitrarily close to each other at infinitely many times and those which are not proximal are called distal.

**Definition 1.1.4.** (See [7]) Let $X$ be a compact Hausdorff space and $f : X \to X$ be a homeomorphism. $x,\ y \in X$ are said to be proximal if the closure $\overline{\{(f^n(x), f^n(y)) : n \in \mathbb{Z}\}}$ of the full orbit of $(x, y)$ under $f \times f$ intersects the diagonal $\Delta = \{(z, z) \in X \times X : z \in X\}$. Points which are not proximal are called distal points.
$(X, f)$ is called a proximal system if any two points are proximal and it is called distal if any two distinct points are distal.

If $(X, d)$ is a compact metric space, then $x,\ y \in X$ are proximal if there is a sequence $n_k \in \mathbb{Z}$ such that $(d(f^{n_k}(x), f^{n_k}(y))) \to 0$ as $k \to \infty$. Equivalently, $x, y \in X$ are distal if there is $\epsilon > 0$ such that $d(f^n(x), f^n(y)) > \epsilon$ for all $n \in \mathbb{Z}$.

A rather peculiar thing can happen between two trajectories : they come arbitrarily close to each other at infinitely many times (proximality) and yet maintain a minimum positive distance at infinitely many times. This peculiar behavior leads to the concept of scrambledness.

The above classifications classify various kinds of points based on their dynamics. Similar classifications can be done for the subsets of the system. Here, we discuss invariant and scrambled sets. A scrambled set is a set in which every pair of distinct points has the “peculiar” behavior mentioned above.

**Definition 1.1.5.** A subset $Y \subset X$ is said to be scrambled if for any two distinct points $x$ and $y$ in $Y$, $\liminf d(f^n(x), f^n(y)) = 0$ and $\limsup d(f^n(x), f^n(y)) > 0$.

The notion of chaos by Li and Yorke [16] uses the idea of scrambled set.

A subset $A \subset X$ is said to be forward $f - invariant$ if $f(A) \subset A$ and backward $f - invariant$ if $f^{-1}(A) \subset A$. If $A$ is both forward $f - invariant$ and backward $f - invariant$, then $A$ is said to be $f - invariant$. In case $f$ is a homeomorphism, $A$ is $f-$invariant $\Leftrightarrow$ $f(A) = A$.

If a subset $A \subset X$ is forward $f - invariant$, then the dynamics can be restricted to $A$ and if $A$ is topologically a nice set, then $(A, f)$ can be considered as a dynamical system in its own respect. Thus we have the following definition.

**Definition 1.1.6.** A closed, non-empty, forward $f-$invariant subset $Y \subset X$ is called a subsystem of $(X, f)$. A subsystem is a minimal system if it contains no proper subsystem.

Note that a subsystem $Y$ is minimal if and only if the orbit of every point in $Y$ is dense in $Y$.

The dynamics on the entire system as a whole is also of several kinds. The system $(X, f)$ when $f$ is the identity map on $X$ is a simple and trivial system. Here, each point is a fixed point and it itself forms a subsystem. A little more interesting case is each point being a periodic point, in which case the system is union of finite subsystems. In general, several systems can be divided in to proper subsystems. But the dynamics on some systems may be so intruding that from any "part" of the system to any other "part", there is a point moving. A minimal system is a best instance of such dynamics. Here, it is impossible to divide the system in to two parts to which the dynamics can be restricted. There is a weaker notion called topological transitivity.

**Definition 1.1.7.** A system is said to be transitive if it has a dense orbit.

Contrast this with the fact that in a minimal system, every orbit is dense. If $X$ is a locally compact Hausdorff space and if for any two non-empty open sets $U$ and $V$ of $X$, there is $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \varphi$ then $f$ is topologically transitive.

There are some stronger notions of transitivity. A system $(X, f)$ is said to be **totally transitive** if $(X, f^n)$ is transitive for every $n \in \mathbb{N}$. Another stronger version requires that any two non-empty open sets meet at every point of time after a certain stage. To put it rigorously, for any two non-empty open sets $U$ and $V$, there is $n \in \mathbb{N}$ such that $f^k(U) \cap V \neq \varphi$ for every $k \geq n$. Such a system $(X, f)$ is called **topologically mixing**. $(X, f)$ is called **weak mixing** if $(X \times X, f \times f)$ is transitive, where $(f \times f)(x, y) = (f(x), f(y))$.

Coming to the equivalence of systems, we need a map between two dynamical systems which preserves first the topological structures on the underlying spaces and then the dynamics. This notion called topological conjugacy is defined as follows.

**Definition 1.1.8.** Let $(X, f)$ and $(Y, g)$ be two dynamical systems. If there is a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$, then $X$ and $Y$ are said to be topologically conjugate (or briefly conjugate). $h$ is called a topological conjugacy or briefly a conjugacy.

A weaker notion of conjugacy is semi-conjugacy, wherein the map $h$ need not be necessarily injective but has to be surjective. In this case, $Y$ is called a factor of $X$ and $X$ an extension of $Y$.

## 1.2 Some examples of dynamical systems

**Example 1.2.1.** (**Circle rotations**)(See [7]) Consider the unit circle $S^1 = [0,1]/\sim$, where $\sim$ indicates that 0 and 1 are identified. The natural distance on $[0,1]$ induces a metric on $S^1$; specifically, $d(x,y) = min(|x-y|, 1-|x-y|)$. For $\alpha \in \mathbb{R}$, let $R_\alpha$ be the rotation of $S^1$ by angle $2\pi\alpha$, i.e., $R_\alpha(x) = x + \alpha(mod\ 1)$.
Then $(S^1, R_\alpha)$ is a dynamical system for every $\alpha \in \mathbb{R}$. If $\alpha = p/q$ is rational, then $R_\alpha^q$ is identity on $S^1$, so every orbit is periodic. Such rotations are called rational rotations. On the other hand, if $\alpha$ is irrational, then there are no periodic points and every orbit is dense in $S^1$ and these are called irrational rotations. An irrational circle rotation is a minimal dynamical system.

**Example 1.2.2.** (**Expanding endomorphisms**)(See [7]) For $m \in \mathbb{Z}$, $|m| > 1$, define the $times - m$ map $E_m : S^1 \to S^1$ by $E_m(x) = mx(mod\ 1)$. $(S^1, E_m)$ is a dynamical system.
This map is a non-invertible group endomorphism of $S^1$ and expands the distances between nearby points by a factor of $m$ i.e., if $d(x,y) < \frac{1}{2m}$, then $d(E_m(x), E_m(y)) = md(x,y)$. So, it is called an expanding endomorphism of circle. An expanding endomorphism of the circle has dense orbits and is thus transitive, but is not minimal because it has periodic orbits also.

**Example 1.2.3.** (**Toral automorphisms**)(See [7]) The torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ can be viewed as the unit square $[0,1]\times[0,1]$ with opposite sides identified: $(x,0) \sim (x,1)$ and $(0,y) \sim (1,y)$, for every $x,\ y \in [0,1]$. Now, let $A$ be a $2\times 2$ matrix, say $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with integer entries such that $|det(A)| = 1$ (this ensures that $A$ is invertible and $A^{-1}$ is also an integer matrix). Define $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ as $T_A(x,y) = \begin{pmatrix} ax + by(mod\ 1) \\ cx + dy(mod\ 1) \end{pmatrix}$.

Thus $(\mathbb{T}^2, T_A)$ is a dynamical system.

If the eigen values of $A$ do not lie on the unit circle, then $T_A$ is called a hyperbolic toral automorphism. Hyperbolic toral automorphisms form an important class of dynamical systems. We can consider these automorphisms on higher dimensional tori also.

**Example 1.2.4.** (**Solenoid**)(See [7]) Consider the solid torus $\mathcal{T} = S^1 \times D^2$, where $S^1 = [0, 1](mod\ 1)$ and $D^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Fix $k \in (0, \frac{1}{2})$, and define $F : \mathcal{T} \to \mathcal{T}$ by $F(\phi, x, y) = (2\phi,\ kx + \frac{1}{2}cos\ 2\pi\phi,\ ky + \frac{1}{2}sin\ 2\pi\phi)$. The map $F$ stretches by a factor of 2 in the $S^1$-direction, contracts by a factor of $k$ in the $D^2$-direction, and wraps the image twice inside $\mathcal{T}$. Also, $F^{n+1}(\mathcal{T}) \subset int(F^n(\mathcal{T}))$ for any $n \in \mathbb{N}$. The set $\mathcal{S} = \bigcap_{n=0}^{\infty} F^n(\mathcal{T})$ is called a solenoid. It is a closed $F$-invariant subset of $\mathcal{T}$ on which $F$ is bijective and thus $(\mathcal{S}, F)$ is a dynamical system.

Let $\Phi$ denote the set of sequences $(\phi_i)_{i=0}^{\infty}$, where $\phi_i \in S^1$ and $\phi_i = 2\phi_{i+1}(mod\ 1)$ for all $i$. The product topology on $(S^1)^{\mathbb{N}_0}$ induces the subspace topology on $\Phi$. The map $\alpha : \Phi \to \Phi, (\phi_0, \phi_1, . . . .) \mapsto (2\phi_0, \phi_0, \phi_1, . . . .)$ is a group automorphism and a homeomorphism. For $s \in \mathcal{S}$, the first (angular) coordinates of the preimages $F^{-n}(s) = (\phi_n, x_n, y_n)$ form a sequence $(\phi_0, \phi_1, . . . .) \in \Phi$. This defines a map $h : S \to \Phi$ as $h(s) = (\phi_0, \phi_1, . . . .)$. $h$ is a topological conjugacy from $(S, F)$ to $(\Phi, \alpha)$. This makes the study of Solenoid simpler.

**Example 1.2.5.** (**Shift map**)(See [7]) For an integer $m \geq 1$, let $\mathcal{A}_m = \{0, 1, ..., m-1\}$. We refer to $\mathcal{A}_m$ as an alphabet and its elements as symbols. Let $\Sigma_m = \mathcal{A}_m^{\mathbb{Z}}$ be the set of infinite two-sided sequences of symbols in $\mathcal{A}_m$, and $\Sigma_m^+ = \mathcal{A}_m^{\mathbb{N}}$ be the set of infinite one-sided sequences. The discrete topology on $\mathcal{A}_m$ gives a product topology on $\Sigma_m$ and $\Sigma_m^+$, according to which they are compact. We can define a metric on these spaces which induces the same topology. The metric is $d(x, y) = \frac{1}{2^n}$, where $n = min\ \{|i| : x_i \neq y_i\}$. Given a one-sided or two-sided sequence $x = (x_n)$, let $\sigma(x)$ be the sequence given by

$(\sigma(x))_n = x_{n+1}$. This defines a continuous self-map of both $\Sigma_m$ and $\Sigma_m^+$ called the shift map. The dynamical system $(\Sigma_m, \sigma)$ is called the full two-sided shift and $(\Sigma_m^+, \sigma)$ is called the full one-sided shift. The map $\phi : \Sigma_m^+ \to S^1$ defined as $\phi((x_n)) = \Sigma_{n=1}^{\infty} \frac{x_n}{m^n}$ is a semi-conjugacy from $(\Sigma_m^+, \sigma)$ to $(S^1, E_m)$.

The next two examples are maps on a compact interval in $\mathbb{R}$. There are many interesting dynamical systems on compact intervals of $\mathbb{R}$. We hereafter fix $[0, 1]$ for the interval, also denoting it by $I$ and call these systems as *interval maps*. We also sometimes use $C(I)$ for the space of continuous self maps on $I$.

**Example 1.2.6.** (**Tent map**)(See [15])

Define $T : I \to I$ as $T(x) = \begin{cases} 2x & \text{for } x \in [0, \frac{1}{2}] \\ 2 - 2x & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$. This interval map is called the *tent map*. An interesting feature of this map is that it possesses periodic points of all periods i.e., $\forall n \in \mathbb{N}$, $T$ has a periodic point of period $n$.

**Example 1.2.7.** (**Logistic map**)(See [15]) This example actually gives a family of maps called *logistic maps*, defined by $h_r(x) = rx(1 - x)$. However every value of $r$ doesn't give a dynamical system on $I$. $(I, h_r)$ is a system if $r \in [0, 4]$.

We now denote by $\Lambda_n$, the set $\{x \in I : h_r^n(x) \in I\}$. Thus the set $\Lambda = \bigcap_{n=0}^{\infty} \Lambda_n$ along with $h_r$ is a dynamical system for any $r$. There is a detailed description of the set $\Lambda$ in [15].

## 1.3 Trajectories

As the name suggests, the thesis deals with the various kinds of trajectories that arise in topological dynamics. Trajectories with finite orbits are of a special kind. It can be easily seen that any finite orbit is of the form $\{x, f(x), f^2(x), ..., f^n(x)\}$ for some

$n \in \mathbb{N}_0$. Then $f^{n+1}(x) = f^m(x)$ for some $0 \leq m \leq n$. If $m = 0$, then $x$ is a periodic point or otherwise an eventually periodic point. Thus a trajectory with finite orbit is either a periodic trajectory or an eventually periodic trajectory. Periodicity is one of the widely studied dynamical behavior in literature. Especially, the characterization of sets of periods and the sets of periodic points for various families of dynamical systems has been an interesting problem. There are many references listed at the end, which deal with such problems. (See [1], [2], [4], [11], [18], [21], [29], [30], [38] and [39]). In this thesis, this problem is taken up for two different classes. Chapter 2 and Chapter 3 deal with these.

The trajectories that are neither periodic nor eventually periodic have infinite orbits. Suppose $\{f^n(x) : n \in \mathbb{N}_0\}$ is an infinite orbit with a limit point say, $y$. Then there is an increasing sequence $(n_k)$ of positive integers such that $(f^{n_k}(x)) \to y$, which shows that $y$ is an $\omega$-limit point of $x$. A special case is where the trajectory itself converges i.e., the sequence $(f^n(x))$ is convergent. Chapter 4 deals with the convergent trajectories.

Another special kind of trajectory is a trajectory whose corresponding orbit is dense in the space. As seen earlier, presence of a dense orbit in a system shows that it is dynamically indecomposable i.e., a system is transitive if and only if it has a dense orbit and minimal if and only if every orbit is dense. In Chapter 5, we show the variety that chaotic interval maps exhibit, in terms of trajectories.

## 1.4 Periodic Trajectories

Periodicity is one of the important properties of a dynamical system. Periodic trajectories can be considered as the simplest kind of trajectories. With finite orbits, they look very simple. Yet, the knowledge of the periodic trajectories in a system may throw light on many things. For instance, if $X$ is a zero-dimensional metric space such that every continuous self map on $X$ has a periodic point, then $X$ is compact and countable (See [18]).

Motivated by this, the following problem has been well-studied in the literature. Let us first denote by $Per(f)$, the set of periods of periodic points in the system $(X, f)$ i.e., $Per(f) = \{n \in \mathbb{N}$ : there is a periodic point in $(X, f)$ with period $n\}$. It is noted that this subset of $\mathbb{N}$ is not arbitrary. For example, for any continuous self map $f$ on $[0, 1]$, $Per(f) \neq \{1, 2, 3\}$ (See [29], [30]). Now, given a class of dynamical systems on a topological space i.e., a class of continuous self maps on a topological space $X$, we consider the collection $\mathcal{PER}(X) = \{Per(f) : (X, f)$ is a dynamical system in the given class$\}$. As noted earlier, $\mathcal{PER}(X) \neq \mathcal{P}(\mathbb{N})$(the set of all subsets of $\mathbb{N}$) in general. So, we ask the following question : Which subsets of $\mathbb{N}$ can arise as $Per(f)$ for some $(X, f)$ in the given class? In other words, we are looking for a characterization of the sets of periods for the given class of dynamical systems i.e., the characterization of $\mathcal{PER}(X)$. This characterization has been considered for various classes and many of them have been characterized in the literature.

The table below gives some such results. In the table, $\mathcal{P}(\mathbb{N})$ denotes the set of all subsets of $\mathbb{N}$ and $\mathcal{U} = \{A \subset \mathbb{N} : 1 \in A\}$. The collection $\mathcal{S}$ is the collection of non-empty initial segments in the Sharkowskii order $\succ$ on $\mathbb{N}$ which is defined as : $3 \succ 5 \succ 7 \succ ... \succ 2 \times 3 \succ 2 \times 5 \succ 2 \times 7 \succ ... \succ 2^n \times 3 \succ 2^n \times 5 \succ 2^n \times 7 \succ ... \succ 2^n \succ 2^{n-1} \succ 2^2 \succ 2 \succ 1$. An initial segment is a subset $S \subset \mathbb{N}$ such that $m \in S$ and $m \succ n \Rightarrow\ n \in S$.

| **Class of dynamical systems** | $\mathcal{PER}(X)$ |
|---|---|
| continuous maps on $\mathbb{R}$ | $\mathcal{S} \cup \{\varphi\}$ |
| continuous maps on $[0,1]$ | $\mathcal{S}$ |
| Complex polynomials | $\{\mathbb{N},\ \mathbb{N}\setminus\{2\},\ \{1\},\ \varphi\}$ $\cup\{\{1,n\} : n \in \mathbb{N}\setminus\{1\}\}$ |
| transitive interval maps | $\{A \subset \mathbb{N} : 1,2 \in A \text{ and } n \in A\setminus\{1\} \Rightarrow n+2 \in A\}$ |
| continuous maps on $\mathbb{R}^n$ | $\mathcal{P}(\mathbb{N})$ |
| continuous maps on closed unit disc | $\mathcal{U}$ |
| toral automorphisms | $\{\{1\},\ \{1,2\},\ \{1,3\},\ \{1,2,4\},\ \{1,2,3,6\},$ $2\mathbb{N}\cup\{1\},\ \mathbb{N}\setminus\{2\},\ \mathbb{N}\}$ |

This characterization is done for bounded linear operators on $l^2$ in [2] and this result appeared in the thesis [1] also. We quote those results here and use them for further study in Chapter 2. Besides the periods, the study of sets of periodic points has also been an interesting one in the literature (See [4], [11], [39]). These sets are also characterized for some classes of dynamical systems. The following theorem from [39] gives the characterization of sets of periodic points of toral automorphisms.

**Theorem 1.4.1.** *For any toral automorphism $T$, the set $P(T)$ of periodic points of $T$ is one of the following:*

1. $\mathbb{Q}_1 \times \mathbb{Q}_1$, *where* $\mathbb{Q}_1 = \mathbb{Q} \cap [0,1)$
2. $S_r$ *for some* $r \in \mathbb{Q} \cup \{\infty\}$

   *where* $S_r = \{(x,y) \in \mathcal{T}^2 : rx + y \textit{ is rational}\}$ *if* $r \in \mathbb{Q}$ *and* $S_\infty = \mathbb{Q}_1 \times [0,1)$.
3. $\mathcal{T}^2$.

So, the present task is : given a family $\mathcal{F}$ of systems on a space $X$, characterize the family $\{M \subset X : \exists f \in \mathcal{F} \text{ with } P(f) = M\}$, where $P(f)$ is the set of periodic points of $f$. In this thesis, this characterization is done for the bounded linear operators on $l^2$ and the homeomorphisms on compact metric spaces of finite derived length.

The following question which is a combination of the above two characterizations is also considered. What pairs $(A, M)$, $A \subset \mathbb{N}$ and $M \subset X$, can arise as the set of periods and set of periodic points for a system on $X$ simultaneously in the given class of dynamical systems? The answer may not be an arbitrary combination of a possible set of periods and a possible set of periodic points i.e., there can be two different systems say $(X, f)$ and $(X, g)$ in the given class with $Per(f) = A$ and $P(g) = M$ but no system $(X, h)$ in the class with $Per(h) = A$ and $P(h) = M$. This makes the problem more interesting. Section 2.4 answers this question for the class of bounded linear operators on $l^2$.

## 1.5 Other trajectories

The convergent trajectories can be considered as the simplest kind of trajectories among those with infinite orbits. It is proved in Chapter 4 that many interesting kinds of dynamical systems do not possess non-trivial convergent trajectories. These include some well known systems like circle rotation, expanding endomorphism $E_m$ of circle i.e., $E_m(x) = mx \ (mod\ 1)$, where $m \in \mathbb{Z}$ with $|m| > 1$, shift map, the map $F$ on the torus $\mathcal{T}^2$ given by $F(x, y) = (x + \alpha \ (mod\ 1), x + y \ (mod\ 1))$ and tent map. Then the chaotic systems (Devaney chaotic, see [12]) are considered. Chaotic systems exhibit a lot of variety even in terms of trajectories. We prove that, given any (allowed) sequence whose range has finite derived length, there is a chaotic interval map with the given sequence as a trajectory.

# Chapter 2

# Periodic trajectories of bounded linear operators on Hilbert space

We consider three problems on the periodic trajectories of the bounded linear operators on seperable infinite dimensional Hilbert space. In fact, only $l^2$ is considered, as any seperable infinite dimensional Hilbert space is linearly isometric to $l^2$. The first problem is the characterization of sets of periods, the second is the characterization of sets of periodic points and the third is a combination of these two characterizations. The characterization of the sets of periods for this family appeared in [2] and thesis [1]. Besides, these sets are characterized for the linear maps on $\mathbb{R}^n$ and $\mathbb{C}^n$ in this paper. An account of the main results of this paper is given in Section 2.2.

Section 2.3 deals with the problem of characterization of sets of periodic points of the same family of systems. It is very easy to see that the set of periodic points of any system is an $F_\sigma$ set, as the set of fixed points of any system is a closed set in $X$ and the set of periodic points in $(X, f)$ is same as $\bigcup_{n=1}^{\infty} Fix(f^n)$, where $Fix(f^n)$ is the set of fixed points of $f^n$. However the converse is not true i.e., given an $F_\sigma$ set in $X$, there need not be any system on $X$ in the considered family with the given $F_\sigma$ set as the set of periodic points. Here the necessary and sufficient conditions for a subset of $l^2$ to arise as the set of periodic points for some bounded linear operator on $l^2$ is stated and proved.

A step ahead, we ask the following question which is a combination of the above two characterizations. What pairs $(A, M)$, $A \subset \mathbb{N}$ and $M \subset l^2$, can arise as the set of periods and set of periodic points for a bounded linear operator on $l^2$? The answer is not an arbitrary combination of a possible set of periods and a possible set of periodic points i.e., on $l^2$, there can be two different bounded linear operators say $f$ and $g$ with $Per(f) = A$ and $P(g) = M$ but no bounded linear operator $h$ with $Per(h) = A$ and $P(h) = M$. Section 2.4 deals with this problem.

The main results of this chapter are published in the journal : *Discrete and Continuous Dynamical Systems* (See [9]). Before going to the main results, the next section gives a very brief introduction to the theory of Hilbert spaces and bounded linear operators.

## 2.1 Hilbert spaces

If $< .,. >$ is an inner product on a vector space $V$, then the map $V \to \mathbb{R}$ given by $v \mapsto |< v, v >|^{\frac{1}{2}}$ is a norm on $V$. Thus an inner product induces a norm and thereby a metric on a vector space. The completeness property of metric spaces introduces two more terms in this context. A normed space which is complete in the induced norm is called a Banach space and a complete inner product space is called a Hilbert space .

Two vectors $v,\ w$ in an inner product space $(V, < .,. >)$ are said to be orthogonal if $< v, w >= 0$. A set $M \subset V$ is said to be an orthogonal set if any two distinct vectors in $M$ are orthogonal. Further, an orthogonal set in which every vector has norm equal to 1 is called an orthonormal set.

Since the notions of addition and metric are available in an inner product space, infinite series and its convergence are defined in the usual way. Given an orthonormal set $M \subset V$, the set of linear combinations $L(M) = \{\sum_{i \in I} v_i : I \subset \mathbb{N}$ and $v_i \in M$ for every $i \in I\}$ can be considered. If $L(M)$ is dense in $V$, then $M$ is called an orthonormal basis of $V$. This is in general not same as the Hamel basis of the vector space $V$.

A well known example of Hilbert space is the sequence space $l^2$ . By definition $l^2 = \{(x_n) : x_n \in \mathbb{C}$ for every $n \in \mathbb{N}$ and $\sum |x_n|^2 < \infty\}$. The inner product is defined as $< (x_n), (y_n) >= \sum x_n \overline{y_n}$. In fact, there is a family of sequence spaces $l^p$ for every $1 \leq p \leq \infty$ but $l^2$ is the only one among them which has an inner product inducing the norm. Moreover, $l^2$ is the prototype of a Hilbert space . This will be made mathematically precise at the end of this section. We also consider the real Hilbert space $l^2 = \{(x_n) : x_n \in \mathbb{R}$ for every $n \in \mathbb{N}$ and $\sum |x_n|^2 < \infty\}$.

The maps that preserve the linear structure on vector spaces are linear operators. In case of normed spaces, with metric in hand, one can ask for the continuity of these maps. Every linear operator on a finite dimensional normed space is continuous. On any normed space a continuous linear operator $T$ satisfies the following property : $\exists\ c > 0$ such that $||Tx|| \leq c||x||$. (The notations $||.||$ and $< . , . >$ are used generally to denote the norm and inner product on any normed space and inner product space respectively, when there is no room for confusion.) Owing to such property, these operators are also called bounded linear operators (though they need not be bounded in the usual sense).

Now, we shall make precise, the statement that $l^2$ is the prototype of a Hilbert space. If there is a bijective linear operator $T$ from an inner product space $H_1$ to an inner prduct space $H_2$ such that $< Tx, Ty >=< x, y >$, then we say that $H_1$ is linearly isometric to $H_2$. It is known that any seperable infinite dimensional Hilbert space is linearly isometric to $l^2$. In this thesis, the periodic behaviour is studied for bounded linear operators on seperable infinite dimensional Hilbert spaces only, in particular on $l^2$.

## 2.2 Sets of periods

Before proceeding to the main results, we introduce some notations that will be used throughout the chapter.

1. For a subset $A \subset \mathbb{N}$, we denote by $|A|$ the cardinality of $A$.

2. A subset $A$ of $\mathbb{N}$ is said to be closed under lcm if the lcm (least common multiple) of any finitely many elements of $A$ is in $A$. In such case, if $B$ is the smallest subset of $A$ such that every element of $A$ is the lcm of finitely many elements of $B$, then $B$ is called the generator of $A$, denoted by $gen(A)$.

3. For any $A \subset \mathbb{N}$, $\widetilde{A}$ denotes the smallest subset of $\mathbb{N}$ containing $A$ and closed under lcm.

4. For $m,\ n \in \mathbb{N}$, the l.c.m of $m$ and $n$ is denoted by $m \vee n$ and if $A,\ B \subset \mathbb{N}$, we define $A \bigvee B = \{m \vee n : m \in A \text{ and } n \in B\}$.

5. For each $n \in \mathbb{N}$, let $\mathfrak{F}_n$ denote $\{\{1\} \cup \widetilde{A} :\ A \subset \mathbb{N} \text{ and } |A| \leq \frac{n}{2}\} \cup \{\{1\} \cup \widetilde{A} :\ A \subset \mathbb{N} \setminus \{1\},\ 2 \in A \text{ and } |A| = \frac{n+1}{2}\}$.

   A simple observation shows that $\mathfrak{F}_{2m} = \{\{1\} \cup \widetilde{A} : A \subset \mathbb{N} \text{ and } |A| \leq m\}$ and $\mathfrak{F}_{2m+1} = \mathfrak{F}_{2m} \bigcup \{\{1\} \cup \widetilde{B} : B \subset \mathbb{N} \setminus \{1\},\ 2 \in B \text{ and } |B| = m+1\}$ for all $m \in \mathbb{N}$.

The following two simple lemmas followed by two well-known theorems are very useful in the proofs of later theorems. $V$ stands for a vector space in these lemmas.

**Lemma 2.2.1.** The set $P$ of periodic points of a linear operator $T : V \to V$ forms a forward $T-$invariant linear subspace of $V$ and hence $Per(T) = Per(T|_P)$.

**Lemma 2.2.2.** If $T : V \to V$ is a linear operator and if $W_1$ and $W_2$ are forward $T-$invariant linear subspaces of $V$ such that $V = W_1 \bigoplus W_2$, then $Per(T) = Per(T|_{W_1}) \bigvee Per(T|_{W_2})$.

**Theorem 2.2.3.** ***Primary Decomposition Theorem***

*Let $T$ be a linear operator on a finite dimensional vector space $V$ over a field $F$. Let $m_T = p_1^{r_1} \dots p_k^{r_k}$ be the factorization of the minimal polynomial $m_T$ of $T$, where $p_1, p_2, ...p_k$ are distinct irreducible monic polynomials and $r_i \in \mathbb{N}$, $i = 1, 2, .., k$. Let $W_i = \ker p_i^{r_i}(T)$,$i = 1, 2, .., k$. Then*

1. *$V = W_1 \oplus \cdots \oplus W_k$;*
2. *each $W_i$ is forward $T-$invariant;*
3. *if $T_i$ is the operator induced on $W_i$ by $T$, then $m_{T_i} = p_i^{r_i}$.*

**Theorem 2.2.4.** *If $A$ is an $n \times n$ matrix with complex entries and $A^m = I$, where $I$ is the $n \times n$ identity matrix for some $m \in \mathbb{N}$, then $A$ is diagonalizable over $\mathbb{C}$.*

The next two theorems give a characterization of the families of sets of periods of linear transformations on $\mathbb{C}^n$ and $\mathbb{R}^n$ and then follows the characterization of the sets of periods of bounded linear operators on $l^2$.

The theorems are stated and only an idea of the proof is indicated. The complete proofs can be found in [2] and also in the thesis of Ali Akbar [1].

**Theorem 2.2.5.** *For every linear operator $T : \mathbb{C}^n \to \mathbb{C}^n$, $Per(T) \in \mathfrak{F}_{2n}$. Conversely for every $A \in \mathfrak{F}_{2n}$, there is a linear operator $T : C^n \to \mathbb{C}^n$ such that $Per(T) = A$.*

Here, given a linear operator $T$ on $\mathbb{C}^n$, the associated matrix $M$ is considered. If $M$ is diagonalizable, then the proof follows very easily. Otherwise the space $P = P(T)$, the set of periodic points is considered and it can be seen that $(T|_P)^k$ is identity for some $k$. Then by Theorem 2.2.4, it reduces to the previous case and the result follows from Lemma 2.2.1.

For the converse, if $A = \{a_1,\ a_2,\ ...,\ a_m\}$, with $m \leq n$ such that $\{1\} \cup \widetilde{A} \in \mathfrak{F}_{2n}$, then let $D = diag(d_1,\ d_2,\ ...,\ d_m,\ 1,\ ...,\ 1)_{n\times n}$, where $d_i = e^{\frac{2\pi i}{a_i}}$ for $1 \leq i \leq m$. It is proved that $Per(T_D) = \{1\} \cup \widetilde{A}$ where $T_D$ denotes the linear operator on $\mathbb{C}^n$ associated with $D$.

**Theorem 2.2.6.** *If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator, then $Per(T) \in \mathfrak{F}_n$. Conversely for every $A \in \mathfrak{F}_n$, there is a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $Per(T) = A$.*

The proof of the first part of the theorem is based on induction on $n$ and uses the Primary Decomposition theorem (See [14]) and theory of Jordan canonical form. For the converse, given $A \in \mathfrak{F}_n$, a linear operator $T$ is constructed on $\mathbb{R}^n$, considering the cases of $n$ being odd and $n$ being even separately. If $n$ is even and $A = \{1\} \bigcup \widetilde{B}$ with $B = \{b_1, b_2, ...b_k\} \subset \mathbb{N} \setminus \{1\}$, $k \leq \frac{n}{2}$, consider the map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ given by $T(x_1, x_2, ...x_n) = (y_1, y_2, ...y_n)$, where $(y_{2l-1}, y_{2l}) = \rho_{\frac{2\pi}{b_l}}(x_{2l-1}, x_{2l})$ $1 \leq l \leq k$ and $y_i = x_i$ for any $i > 2k$. Here, $\rho_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the rotation map by an angle $\theta$. It follows that $Per(T) = A$. If $n$ is odd, then we can write $\mathfrak{F}_n = \mathfrak{F}_{n-1} \bigcup \{\{1\} \cup \widetilde{B} : B \subset \mathbb{N} \setminus \{1\},$ $2 \in B$ and $|B| = \frac{n+1}{2}\}$. If $A \in \mathfrak{F}_{n-1}$, then consider the map $T = T_A \times I$ on $\mathbb{R}^n$, where $I$ is the identity on $\mathbb{R}$ and $T_A$ is the linear map constructed on $\mathbb{R}^{n-1}$ as in the previous case with $Per(T_A) = A$. Thus $Per(T) = A$. Otherwise, $A = \{1\} \cup \widetilde{B}$ for some $B = \{2, b_1, b_2, ...b_{\frac{n-1}{2}}\} \subset \mathbb{N} \setminus \{1\}$. Here we can consider the map $T(x_1, x_2, ...x_n) = (y_1, y_2, ...y_n)$, where $(y_{2l-1}, y_{2l}) = \rho_{\frac{2\pi}{b_l}}(x_{2l-1}, x_{2l})$, $1 \leq l \leq \frac{n-1}{2}$ and $y_n = -x_n$.

Now the theorem which gives a characterization of family of sets of periods of bounded linear operators on $l^2$ is given.

**Theorem 2.2.7.** *A subset $A \subset \mathbb{N}$ is the set of periods of periodic points of a bounded linear operator on $l^2$ if and only if $A$ is closed under l.c.m and $1 \in A$.*

Here again, the rotation map on $\mathbb{R}^2$ is used in proving the theorem.

## 2.3 Sets of periodic points

We now consider the problem of characterizing the sets of periodic points of bounded linear operators on $l^2$. As already observed, these sets are algebraically, subspaces of $l^2$ and from the topological point of view, they are $F_\sigma$ sets. The observation that it is an $F_\sigma$ set is easy to make and little more examination shows that it is infact an increasing union of countably many closed subspaces (increasing union of countably many sets means the union of members of a sequence of sets, say $(A_n)$ such that $A_n \subset A_{n+1}$ for every $n \in \mathbb{N}$). It is proved here that it is also a sufficient condition for a subspace to arise as the set of periodic points for a bounded linear operator on $l^2$. Proposition 2.3.1 gives a kind of non-triviality to this characterization. It shows that not every subspace of $l^2$ is an increasing union of countably many closed subspaces. This proposition is first stated and proved, then followed by the main theorem and its proof.

**Proposition 2.3.1.** *There is a subspace of a seperable infinite dimensional Hilbert space $H$ which is not a union of countably many closed subspaces of $H$.*

*Proof.* It is known that every Hamel basis of $H$ is of cardinality $\mathfrak{c}$ (cardinality of $\mathbb{R}$). Then the cardinality of the collection of subsets of a Hamel basis of $H$ is $2^{\mathfrak{c}}$ and thus the cardinality of the collection of subspaces of $H$ is $2^{\mathfrak{c}}$. Since $H$ is seperable, it has a countable basis and thus the cardinality of the collection of open sets as well as the collection of closed sets is $\mathfrak{c}$. Therefore the cardinality of the collection of subspaces which are countable unions of closed subspaces is $\mathfrak{c}$. Since $\mathfrak{c} < 2^{\mathfrak{c}}$, there is a subspace of $H$ which is not a union of countably many closed subspaces of $H$. □

**Theorem 2.3.2.** *Let $M$ be a subspace of $l^2$ (complex Hilbert space). $M$ is the set of periodic points of a bounded linear operator on $l^2$ if and only if $M$ is an increasing union of countably many closed subspaces.*

*Proof.* Let $M = \bigcup_{i \in J} M_i$ for some $J \subset \mathbb{N}$, where $M_i$ is a closed subspace of $l^2$ for each $i \in J$ such that $M_i \subset M_{i+1}$, if $i + 1 \in J$. Take an orthonormal basis $B_1$ for $M_1$, extend this to an orthonormal basis $B_2$ for $M_2$. Inductively, extend the orthonormal basis of $M_i$ to an orthonormal basis of $M_{i+1}$. The union of these basis, say $B$ is an orthonormal set in $M$. Extend $B$ to an orthonormal basis $B'$ of $l^2$. Then every element of $l^2$ is of the form $\Sigma_{x_i \in B'} a_i x_i$ for some $a_i \in \mathbb{C}$. Now define a map $T : l^2 \longrightarrow l^2$ as $T(\Sigma_{x_i \in B'} a_i x_i) = \Sigma_{x_i \in B} \alpha_i a_i x_i$, where $\alpha_i$ is a primitive $n^{th}$ root of unity and $n$ is the least positive integer such that $x_i \in M_n$. Then $T$ is a bounded linear operator on $l^2$.

Observe that each element in the basis of $M_n$ is periodic with period atmost $n$. So, every element of $M_n$ is periodic with period atmost $n!$. Since this is true for every $n$, $M \subset P(T)$. It follows from the definition of $T$ that the range of $T$ is a subspace of $\overline{M}$. So, there is no periodic point outside $\overline{M}$. Let $x \in \overline{M} \setminus M$. Then $x = \Sigma_{x_i \in B} a_i x_i$, where $a_i \in \mathbb{C}$ such that $a_i \neq 0$ for infinitely many $i$'s. Again if $\exists k \in \mathbb{N}$ such that $x_i \in M_k$ $\forall i$, then $x \in M_k$ because $M_k$ is closed, which is a contradiction to the fact that $x \notin M$. Therefore for each $j \in \mathbb{N}$, $\exists$ $k > j$ such that $x_i \in M_k \setminus M_{k-1}$ for some $i$ and $a_i \neq 0$. So $x$ is not periodic. Hence $M = P(T)$.

Let $M = P(T)$ for some bounded linear operator $T : l^2 \longrightarrow l^2$. Since the set of fixed points of a bounded linear operator is a closed subspace, $M_k = Fix(T^{k!})$ is closed for each $k \in \mathbb{N}$. The result follows from the observations that $M = \bigcup_{k \in \mathbb{N}} M_k$ and $M_k \subset M_{k+1}$ $\forall k \in \mathbb{N}$. □

The same result is true for real Hilbert spaces also.

**Theorem 2.3.3.** *A subspace $M$ of $l^2$ (real Hilbert space) is the set of periodic points of a bounded linear operator on $l^2$ if and only if $M$ is an increasing union of countably many closed subspaces.*

The proof of this theorem is similar to the above proof. The necessary condition on $M$ to be an increasing union of countably many closed subspaces can be proved exactly as above. For the other part, given a subspace $M$ satisfying the hypothesis, if $M$ is finite dimensional, then the result follows from [2]. Otherwise, we again choose orthonormal bases $B_n$ of $M_n$, an orthonormal set $B$ in $M$ and an orthonormal basis $B'$ of $l^2$ as done above, but with a slight difference so that a map $T$ can be defined on $B$ such that on each $B_n$, it is the product of a cyclic permutaion of length $n$ and cycles of length one. $T$ is extended to $B'$ and then to $l^2$ such that the range is again a subspace of $\overline{M}$. It can be proved that $P(T) = M$. These ideas are again reflected in the proof of Theorem 2.4.4.

## 2.4 Sets of periods and periodic points

This section deals with the following question: Given two sets $A \subset \mathbb{N}$ and $M \subset l^2$, when does there exist a bounded linear operator $T$ on $l^2$ such that $Per(T) = A$ and $P(T) = M$? As already mentioned, a mere combination of the results of above sections does not give this characterization.

We now prove a lemma, which will be used in proving Theorem 2.4.2. After proving Theorem 2.4.2, the main theorems of the section are stated and proved.

**Lemma 2.4.1.** If $M$ is a strictly increasing union of infinitely many closed subspaces, then $M$ is not closed.

*Proof.* Let $M = \bigcup_{k\in\mathbb{N}} M_k$, where $M_k \subsetneq M_{k+1}$ and $M_k$ is closed $\forall k \in \mathbb{N}$. Choose an orthonormal basis $B_i$ of $M_i$ for each $i \in \mathbb{N}$ as done in the proof of Theorem 2.3.2. The union of these bases, say $B$ is an orthonormal set in $M$. Choose $v_1 \in B_1$ and $v_k \in B_k \setminus B_{k-1} \forall k \geq 2$. The infinite series $v = \Sigma_{k=1}^{\infty} \frac{1}{2^k} v_k$ being convergent is in $\overline{M}$. If $v \in M$, then $v \in M_l$ for some $l \in \mathbb{N}$. Then $v$ is a linear combination of elements of $B_l$, which is a contradiction to the unique representation of $v$ in terms of the elements of $B$. Therefore $v \notin M$. Hence $M$ is not closed. □

**Theorem 2.4.2.** *Let $A = Per(T)$ and $M = P(T)$ for some bounded linear operator $T : l^2 \longrightarrow l^2$ (complex Hilbert space).*

1. *$A$ is finite if and only if $M$ is closed.*

2. *$|gen(A)| \leq dim(M)$.*

*Proof.*

1. If $A$ is finite, say $A = \{n_1, n_2, ..., n_r\}$, then $M = \bigcup_{i=1}^{r} Fix(T^{n_i})$. Since $Fix(T^{n_i})$ is closed for each $i$, $M$ is closed.

   To prove that $A$ is finite if $M$ is closed, we prove its contrapositive. Let $A = \{n_i : i \in \mathbb{N}\}$ such that $n_i < n_{i+1}$ $\forall i \in \mathbb{N}$. Let $M_k = Fix(T^{n_1 n_2 \ldots n_k})$. Then $M = \bigcup_{k\in\mathbb{N}} M_k$ and $M_k \subsetneq M_{k+1}$ $\forall k \in \mathbb{N}$. So $M$ is a strictly increasing union of infinitely many closed subspaces. Hence from the above Lemma, $M$ is not closed.

2. The result follows immediately if $M$ is infinite dimensional. If $dim(M) = n \in \mathbb{N}$, then it is proved in [2] that $A \in \mathfrak{F}_{2n}$. Therefore, it follows by the definition of $\mathfrak{F}_{2n}$, that $|gen(A)| \leq dim(M)$.

□

**Theorem 2.4.3.** ***Characterisation for complex Hilbert spaces***

*The following are equivalent for a pair $(A, M)$, where $A \subset \mathbb{N}$ and $M$ is a subspace of the complex Hilbert space $l^2$.*

1. *$1 \in A$, $A$ is closed under l.c.m, $M$ is an increasing union of countably many closed subspaces of $l^2$ such that $|gen(A)| \leq dim(M)$ and $A$ is finite if and only if $M$ is closed.*

2. *There is a bounded linear operator $T$ on $l^2$ such that $Per(T) = A$ and $P(T) = M$.*

*Proof.* Let $gen(A) = \{n_i : i \in J\}$ (it can be assumed that $n_i < n_{i+1}$ if $i+1 \in J$) and $M = \bigcup_{i \in K} M_i$ for some $J, K \subset \mathbb{N}$. Suppose $J$ is finite, say $|J| = r$. Let $\{x_1, x_2, ..., x_r\} \subset S$, where $S$ is an orthonormal basis of $M$. Then by replacing $M_i$, if necessary by the linear span of $\{x_1, x_2, ...x_i\}$ in $l^2$, we can assume that $|J| \leq |K|$. If $J$ is infinite, then $K$ is also infinite, otherwise $M$ will be closed. So, we can assume that $J \subset K$. Choose an orthonormal basis $B_i$ of $M_i$ for each $i \in K$ as done in proofs of earlier theorems, such that $B_i \subset B_{i+1}$. Then $B = \bigcup_{i \in K} B_i$ is an orthonormal set in $M$. Extend $B$ to an orthonormal basis $B'$ of $l^2$.

Define a map $T' : B' \longrightarrow l^2$ as

$$T'(x) = \begin{cases} 0 & \text{if } x \in B' \setminus B \\ x & \text{if } x \in B_i \text{ and } i \notin J \\ \alpha x & \text{if } x \in B \text{ and } \alpha \text{ is the } n_i^{th} \text{ root of unity where} \\ & \quad i \text{ is the least positive integer such that } i \in J \text{ and } x \in B_i \end{cases}.$$

Let $T$ be the bounded linear operator on $l^2$ obtained by extending $T'$.

It can be easily seen that every element of $B_k$ is periodic with period atmost $n_k$. Since $B_k$ is an orthonormal basis of $M_k$, every element $x \in M_k$ can be uniquely written as $x = \Sigma_{x_i \in B_k} a_i x_i$ for some $a_i \in \mathbb{C}$. Now, for any $l \in \mathbb{N}$, $\Sigma_{x_i \in B_k} a_i T_i^l(x_i)$ is the unique representation of $T^l(x)$ in terms of the elements of $B_k$ (where $\alpha_i$ is a root of unity). Since $x_i$ is periodic with period 1 or $n_j$ for some $j \in J$ and $j \leq k$, $x$ is periodic with period equal to the l.c.m of periods of these $n_j$'s. So, every element of $M_k$ is periodic with period equal to the l.c.m of some $n_i$'s with $i \leq k$. Thus $M \subset P(T)$. As done in

the proof of Theorem 2.3.2, it can be shown that $P(T) \subset M$. Hence $P(T) = M$.
Since every periodic element is in $M_k$ for some $k$, it follows from the above discussion that $Per(T) \subset A$. Since $Per(T)$ is closed under l.c.m and $gen(A) \subset Per(T)$, $A \subset Per(T)$. Hence $Per(T) = A$.
The converse part of the proof follows from Theorem 2.2.7, Theorem 2.3.2 and Theorem 2.4.2. □

The following is an analogous theorem for the real Hilbert spaces.

**Theorem 2.4.4.** ***Characterisation for real Hilbert spaces***

*The following are equivalent for a pair $(A, M)$, where $A \subset \mathbb{N}$ and $M$ is a subspace of the real Hilbert space $l^2$.*

1. *$1 \in A$, $A$ is closed under lcm, $M$ is an increasing union of countably many closed subspaces of $l^2$ such that $|gen(A)| \leq \frac{dim(M)+1}{2}$ and $A$ is finite if and only if $M$ is closed.*

2. *There is a bounded linear operator $T$ on $l^2$ such that $Per(T) = A$ and $P(T) = M$.*

*Proof.* Let $gen(A) = \{n_i : i \in J\}$ (it can be assumed that $n_i < n_{i+1}$ if $i + 1 \in J$) and $M = \bigcup_{i \in K} M_i$ for some $J, K \subset \mathbb{N}$. If $M$ is finite dimensional then the result follows from [2]. If $M$ is infinite dimensional, by changing the $M_i$'s, we can assume that $J \subset K$, $dim(M_1) \geq n_1$ and $dim(M_{i+1}) - dim(M_i) \geq n_{i+1}$ if $i + 1 \in J$. Choose orthonormal bases $B_i$ of $M_i$ for each $i \in K$, $B$ of $M$ and $B'$ of $l^2$ as done in the proof of Theorem 2.3.2. Define a permutation $T'$ on $B$ such that for each $i \in K$, $T'$ produces a cycle of length $n_i$ and fixes all other elements in $B_i$. Extend $T'$ to $B'$ by defining $T'(x) = 0$ $\forall x \in B' \setminus B$. If $T$ is the bounded linear operator on $l^2$ obtained by extending $T'$, then by arguing as in the proof of the Theorem 2.4.3, it can be proved that $Per(T) = A$ and $P(T) = M$.

Note that Theorem 2.2.7, a part of the Theorem 2.3.2 (that $M$ is an increasing union of countably many closed subspaces) and first part of the Theorem 2.4.2 (that $A$ is finite if and only if $M$ is closed) are true for real Hilbert spaces also. So, it remains to prove that $|gen(A)| \leq \frac{dim(M)+1}{2}$. If $M$ is infinite dimensional, then it is obvious; otherwise the result follows again from [2]. $\square$

## 2.5 Conclusion

Thus the following question is answered completely for the family of bounded linear operators on $l^2$: Given a family $\mathcal{F}$ of self maps on a space $X$, which sets $A \subset \mathbb{N}$ and $M \subset X$ can arise as the set of periods and set of periodic points of some map $f$ in the family $\mathcal{F}$ i.e., characterize the pairs $(A, M)$, $A \subset \mathbb{N}$, $M \subset X$ such that $\exists f \in \mathcal{F}$ with $Per(f) = A$ and $P(f) = M$.

In fact, the sets of periodic points of bounded linear operators on $l^2$ are also characterized separately and then the above question is answered, while the characterization of sets of periods for these maps is already known (See [1], [2]).

# Chapter 3

# Homeomorphisms on compact metric spaces with finite derived length

This chapter is concerned with the characterization of sets of periodic points of homeomorphisms on compact metric spaces of finite derived length. It can be proved that a compact metric space with finite derived length is countable and a countable compact metric space is homeomorphic to a countable ordinal (See [17], [24]). So the sets of periodic points of self homeomorphisms on a countable ordinal with finite derived length are characterized here. It can be seen that any ordinal of finite derived length can be written as $\omega^n.m_n + \omega^{n-1}.m_{n-1} + ... + \omega.m_1 + m_0$, $m_i \in \mathbb{N}_0$. In the rest of this Chapter, we use the notation $O^{(n)} = \omega^n.m_n + \omega^{n-1}.m_{n-1} + ... + \omega.m_1 + m_0$ . To study the main problem of the chapter, a partition is induced on the ordinal $O^{(n)}$. The partition is so natural that its definition involves only elementary operations: intersection, complementation and formation of derived set. At the same time, its significance is that it is used to understand three more problems: one on group actions, one in topology and the other in algebras of sets. The next paragraph elaborates on this point.

This partition is done with respect to a given subset of $O^{(n)}$ i.e., every subset $S \subset O^{(n)}$ gives rise to a partition $\mathcal{P}_S$ of $O^{(n)}$. This is finer than the partition of $O^{(n)}$ in to the different levels of limit points. $\mathcal{P}_S$ helps to prove four different results as follows:

- A subset $S \subset O^{(n)}$ arises as a set of periodic points of a homeomorphism on $O^{(n)}$ if and only if every finite partition class in $\mathcal{P}_S$ is contained in $S$.

- Using this partition $\mathcal{P}_S$, it is proved that under the action on a metric space $X$, of the group $G_S = \{h : X \to X : h \text{ is a homeomorphism such that } h(S) = S\}$, the countable spaces with finite derived length are the only seperable metric spaces which consist of finitely many $G_S$-orbits for every $S \subset X$.

- $\mathcal{P}_S$ is the smallest partition of $\omega^n$ such that
  (i)$S$ is a union of partition classes and
  (ii)if $A \subset \omega^n$ is a union of partition classes, then $D(A)$ is also a union of partition classes.

- Starting from a subset $S \subset \omega^n$, go on forming many other sets using the operation of derived set, complement and union i.e, $S$, $\omega^n \setminus S$, $\overline{S}$, $\overline{\omega^n \setminus S}$ and so on. In other words, if $n$ sets are already formed, the $(n+1)^{th}$ set can be the derived set of any of the $n$ sets, or the complement of any of the $n$ sets, or the union of any two of these $n$ sets. Among the distinct subsets that can be formed in this way, the minimal ones are precisely the partition classes in $\mathcal{P}_S$.

The first two results form the main part of this chapter and the last two follow from these. The last result is similar to, and is a natural sequel to, the closure-complementation problem described in [22] and further investigated in [13].

Before going to the main results of the chapter, Section 3.1 gives some preliminaries. In this section, some definitions and notations are introduced that will be used throughout. There is a subsection on ordinal numbers. The partition $\mathcal{P}_S$ which is mentioned above is also defined here. It is also proved that the partition classes are invariant under those homeomorphisms which leave $S$ invariant - a result that will be extensively used thereafter.

Section 3.2 gives a characterization of the sets of periodic points of homeomorphisms on an ordinal of finite derived length. We first prove the result for the ordinal of type $\omega^n$ (Theorem 3.2.3) and then for a general ordinal $O^{(n)}$ (Theorem 3.2.4). Lemma 3.2.1 does the major task in proving Theorem 3.2.3.

In Section 3.3, a group action and an equivalence relation are defined on $O^{(n)}$ based on a given subset $S \subset O^{(n)}$. The group orbits and equivalence classes are described in terms of the partition classes of $\mathcal{P}_S$. It is proved that the separable metric spaces with a property related to this group action are characterized by those spaces which have finite derived length.

Shifting the focus on to the ordinal numbers, some natural equivalence relations are defined between the subsets of $\omega^n$ and the equivalence classes are studied. Besides the homeomorphism, a weaker notion of homeomorphism called eventual homeomorphism and a stronger notion called "homeomorphism in $X$" are also considered. Section 3.4 describes these results. These results are published in Bulletin of Kerala Mathematics Association. (See [20])

## 3.1 Preliminaries

### 3.1.1 Ordinal numbers

A set with a binary relation on its elements that is antisymmetric, transitive and total (for any distinct $a$ and $b$ in the set, $a \sim b$ or $b \sim a$) is called an ordered set. Here, we use the symbol $<$ to denote the order relation. If every non-empty subset of an ordered set has a least element, then it is called a well-ordered set.

Define a relation $\sim$ as follows : if $A$ and $B$ are two ordered sets, then we say that $A \sim B$ if there is an order preserving bijection from $A$ to $B$. This is an equivalence relation and each equivalence class (infact a representative of this class) is called an ordinal number.

We use two interpretations of ordinal numbers : as sets and as numbers. Any finite well-ordered set of cardinality $n$ is order isomorphic to the set $\{0, 1, 2, ..., n-1\}$ with the usual ordering. So, for any $n \in \mathbb{N}$, the ordered set $\{0, 1, 2, ..., n-1\}$ is an ordinal which will be denoted again by $n$. The ordinal 0 (zero) represents empty set. Thus any non-negative integer can be regarded as an ordinal and such ordinals are called as finite ordinals. The ordinals can again be ordered : an ordinal $\alpha$ precedes $\beta(\neq \alpha)$ if there is an injective map from $\alpha$ into $\beta$. Thus the finite ordinals in the increasing order are 0,1,2,3,... . The least infinite ordinal is denoted by $\omega$ (which is equivalent to the set of all non-negative integers).

Let $\alpha$ and $\beta$ be two ordinals. Define $\alpha + \beta$ to be the set containing all the elements of $\alpha$ and $\beta$, where the elements of $\alpha$ are distinguished from those of $\beta$. Define an order on $\alpha+\beta$ as $a < b$ for any $a \in \alpha$, $b \in \beta$ and preserving the orders of $\alpha$ and $\beta$ among their elements. Thus $\alpha+\beta$ is an ordinal. The cartesian product $\alpha \times \beta$ with the lexicographic order is a well-ordered set and the corresponding ordinal is called $\alpha.\beta$. Thus we have the ordinals $0, 1, 2, 3, ...\omega, \omega+1, ..., \omega+\omega(=\omega.2), ..., \omega.n, ..., \omega.\omega(=\omega^2), ..., \omega^n, ....$ There is a pictorial description of the ordinal numbers in [10].

**Definition 3.1.1.** If there exists a smallest ordinal number greater than an ordinal $\alpha$, then it is called the successor of $\alpha$. An ordinal number that is a successor is called a successor ordinal. A limit ordinal is an ordinal number which is neither zero nor a successor ordinal.

If $(X, <)$ is an ordered set, the order topology on $X$ is the topology generated by the base of "open rays" : $(a, \infty) = \{x \in X : a < x\}$ and $(-\infty, b) = \{x \in X : x < b\}$ for all $a, b \in X$ together with the open intervals $(a, b) = \{x \in X : a < x < b\}$.

Here, we consider the order topology on ordinal spaces. The set of limit points of an ordinal $\alpha$ is precisely the set of limit ordinals less than $\alpha$. Zero and the successor ordinals less than $\alpha$ are isolated points in $\alpha$.

### 3.1.2 The partition $\mathcal{P}_S$

For a topological space $X$, $X_k$ denotes the set of $k^{th}$ level limit points of $X$ in $X$ $\forall k \in \mathbb{N}$ (i.e., $D^k(X) \setminus D^{k+1}(X)$, where $D(Z)$ denotes the set of limit points of $Z \subset X$ in $X$ and $D^n(Z) = D(D^{n-1}(Z))$ ) and $X_0$ denotes the set of isolated points of $X$.

We first define a sequence $\mathcal{J}^{(l)}$ , $l \in \mathbb{N}_0$ inductively as follows : $\mathcal{J}^{(0)} = \{a, b\}$, $\mathcal{J}^{(l+1)} = (\mathcal{P}(\mathcal{J}^{(l)}) \setminus \{\varphi\}) \times \mathcal{J}^{(0)}$ ($\mathcal{P}(X)$ stands for the set of all subsets of $X$).

Let $S \subset O^{(n)}$, $n \in \mathbb{N}$. For every $V \in \bigcup_{l=0}^{n} \mathcal{J}^{(l)}$, we associate a subset $S_V$ of $O^{(n)}$ as : Define $S_a = (O^{(n)} \setminus S) \cap (O^{(n)})_0$ and $S_b = S \cap (O^{(n)})_0$. If $V \in \mathcal{J}^{(k+1)}$ for some $0 \leq k \leq n-2$ and say $V = (W, i)$ ($i \in \{a, b\}$) for some $W \subset \mathcal{J}^{(k)}$, then define $S_V = [(\bigcap_{A \in W} \overline{S_A}) \setminus (\bigcup_{A \in \mathcal{J}^{(k)} \setminus W} \overline{S_A})] \cap S^i \cap (O^{(n)})_{k+1}$, where $S^i = \begin{cases} O^{(n)} \setminus S, & i = a \\ S, & i = b \end{cases}$.

Here, $O^{(n)}$ is partitioned in to $(O^{(n)})_k$'s and each $(O^{(n)})_k$ is partitioned into $S \cap (O^{(n)})_k$ and $(O^{(n)} \setminus S) \cap (O^{(n)})_k$. This partition is further refined in such a way that the common limit points of some and only those partition classes in $(O^{(n)})_{k-1}$ constitute the partition classes of $(O^{(n)})_k$; these are denoted by $S_V$, $V \in \bigcup_{l=0}^{n-1} \mathcal{J}^{(l)}$.

### 3.1.3 Invariance of partition classes

**Proposition 3.1.2.** *If $X$ is a topological space with finite derived length $n$ and $h$ is a homeomorphism on $X$, then $X_k$ is $h-$invariant for every $0 \leq k \leq n$.*

*Proof.* Since, being an isolated point is a topological property, $X_0$ is $h-$invariant for any homeomorphism $h$ on $X$. Note that $X_1 = (X \setminus X_0)_0$. Since $X_0$ is $h-$invariant, $h$ is a self homeomorphism on $X \setminus X_0$. Thus by the above argument, it follows that $X_1$ is $h-$invariant. Now let $1 \leq k < n$. Suppose that $X_l$ is $h-$invariant for every $0 \leq l \leq k$. Again $X_{k+1} = (X \setminus \bigcup_{i=0}^{k} X_i)_0$ and $h$ is a self homeomorphism on $X \setminus \bigcup_{i=0}^{k} X_i$. Thus, $X_{k+1}$ is $h-$invariant. Hence $X_k$ is $h-$invariant for every $0 \leq k \leq n$. $\square$

**Theorem 3.1.3.** *If $S \subset O^{(n)}$ is $h-$invariant for some homeomorphism $h$ on $O^{(n)}$, then $S_V$ is $h-$invariant $\forall V \in \bigcup_{k=0}^{n-1} \mathcal{J}^{(k)}$.*

*Proof.* From the above proposition, it is clear that $S_a$ and $S_b$ are $h-$invariant. We now prove that if $S_W$ is $h-$invariant $\forall W \in \mathcal{J}^{(k)}$ for some $0 \leq k \leq n-2$, then $S_V$ is $h-$invariant $\forall V \in \mathcal{J}^{(k+1)}$.

Consider $S_V$, where $V \in \mathcal{J}^{(k+1)}$. Then $V = (W, i)$ for some $W \subset \mathcal{J}^{(k)}$ and $i \in \{a, b\}$. By definition, $S_V = (\bigcap_{A \in W} \overline{S_A} \setminus \bigcup_{A \in \mathcal{J}^{(k)} \setminus W} \overline{S_A}) \cap S^i \cap (O^{(n)})_{k+1}$. Since $h$ is a homeomorphism, observe that for any $A \in \mathcal{J}^{(k)}$, $h(\overline{S_A}) = \overline{h(S_A)} = \overline{S_A}$. Then $h(\bigcap_{A \in W} \overline{S_A}) = \bigcap_{A \in W} \overline{S_A}$ and $h(\bigcap_{A \in \mathcal{J}^{(k)} \setminus W} \overline{S_A}) = \bigcap_{A \in \mathcal{J}^{(k)} \setminus W} \overline{S_A}$. Hence $S_V$ is $h-$invariant. $\square$

## 3.2 Sets of periodic points

**Lemma 3.2.1.** Let $n \in \mathbb{N}$ and $S \subset \omega^{n+1}$. If $1 \leq k \leq n$ and $(\omega^{n+1} \setminus S) \cap S_A$ is either empty or infinite for every $A \in \mathcal{J}^{(k-1)} \cup \mathcal{J}^{(k)}$ then any bijection $f$ on $(\omega^{n+1})_k$ such that

1. $f$ has no orbit of even length

2. $S_V$ is $f-$invariant $\forall V \in \mathcal{J}^{(k)}$ and

3. $P(f) = S \cap (\omega^{n+1})_k$

can be extended to a homeomorphism $h$ on $(\omega^{n+1})_{k-1} \cup (\omega^{n+1})_k$ such that $P(h) = S \cap ((\omega^{n+1})_{k-1} \cup (\omega^{n+1})_k)$.

*Proof.* $f$ being a bijection on $(\omega^{n+1})_k$ with no orbits of even length, defines three types of orbits in $(\omega^{n+1})_k$ :

1. an infinite orbit $(\{x_l : l \in \mathbb{Z} \text{ and } f(x_i) = x_{i+1}\})$,

2. a periodic orbit with odd length greater than 1 $(\{x_l : -m \leq l \leq m$ for some fixed $m \in \mathbb{N}$; $f(x_i) = x_{i+1}\ \forall i < m$ and $f(x_m) = x_{-m}\})$ and

3. a singleton orbit $(\{x : f(x) = x\})$ i.e., a fixed point.

Since $P(f) = S \cap (\omega^{n+1})_k$, $(\omega^{n+1})_k \setminus S$ is a union of infinite orbits and $S$ is a union of finite orbits.

We now define a homeomorphism $h$ on $(\omega^{n+1})_{k-1} \cup (\omega^{n+1})_k$ with $P(h) = S \cap ((\omega^{n+1})_{k-1} \cup (\omega^{n+1})_k)$ in such a way that $h = f$ on $(\omega^{n+1})_k$ and moreover the orbit of each point in $(\omega^{n+1})_{k-1}$ falls in to one of the three types. To cover the general case, we assume that each point in $(\omega^{n+1})_k$ is in both $\overline{S}$ as well as $\overline{(\omega^{n+1})_{k-1} \setminus S}$; and the same method works for other cases also.

**Type I :** $\{x_i : i \in \mathbb{Z} \text{ and } f(x_i) = x_{i+1}\}$

Here, $x_i \in S_V$ $\forall i \in \mathbb{Z}$ for some $V \in \mathcal{J}^{(k)}$. Let $B_i$ and $B_i'$ be two deleted clopen neighborhoods of $x_i$ (i.e., $x_i \notin B_i \cup B_i'$ and $B_i \cup \{x_i\}$, $B_i' \cup \{x_i\}$ are clopen neighborhoods of $x_i$) such that $B_i \subset S$ and $B_i' \subset (\omega^{n+1} \setminus S)$. Choose $B_i$'s and $B_i'$'s in such a way that $B_i \cap S_W$ and $B_i' \cap S_W$ are either empty or infinite $\forall W \in \mathcal{J}^{(k-1)}$. Since all the $x_i$'s are in the same $S_V$, we can assume that for any $W \in \mathcal{J}^{(k-1)}$, $|B_i \cap S_W| = |B_j \cap S_W|$ and $|B_i' \cap S_W| = |B_j' \cap S_W|$, $\forall i, j \in \mathbb{Z}$. Say $B_i \cap (\omega^{n+1})_{k-1} = \{x_{ij} : j \in \mathbb{N}\}$ and $B_i' \cap (\omega^{n+1})_{k-1} = \{x_{ij}' : j \in \mathbb{N}\}$.

Define a bijection $h_I$ on $(\bigcup_{i \in \mathbb{Z}} (B_i \cup B_i')) \cap (\omega^{n+1})_{k-1}$ as

$$h_I(x_{ij}) = \begin{cases} x_{(i+1)j} & \text{if } -j \leq i < j \\ x_{(-j)j} & \text{if } i = j \\ x_{ij} & \text{if } |i| > j \end{cases} \quad \text{and } h_I(x_{ij}') = x_{(i+1)j}'.$$

Extend $h_I$ to $(\bigcup_{i \in \mathbb{Z}} B_i) \bigcup (\bigcup_{i \in \mathbb{Z}} B_i') \cup \{x_i : i \in \mathbb{Z}\}$ by defining $h_I(x_i) = f(x_i)$.

**Type II :** $\{x_i : -m \leq i \leq m$ for some fixed $m \in \mathbb{N}$ and $f(x_i) = x_{i+1}$ $\forall i < m$ and $f(x_m) = x_{-m}\}$

Let $B_i$ and $B_i'$ be two deleted clopen neighborhoods of $x_i$ such that $B_i \subset S$ and $B_i' \subset (\omega^{n+1} \setminus S)$. Choose $B_i$'s and $B_i'$'s in such a way that $B_i \cap S_W$ and $B_i' \cap S_W$ are either empty or infinite $\forall W \in \mathcal{J}^{(k-1)}$. Here again, $x_i \in S_V$ $\forall -m \leq i \leq m$ for some $V \in \mathcal{J}^{(k)}$. Also, $|B_i \cap S_W| = |B_j \cap S_W|$ and $|B_i' \cap S_W| = |B_j' \cap S_W|$, $\forall -m \leq i, j \leq m$ and for any $W \in \mathcal{J}^{(k-1)}$. Say $B_i \cap (\omega^{n+1})_{k-1} = \{x_{ij} : j \in \mathbb{N}\}$ and $B_i' \cap (\omega^{n+1})_{k-1} = \{x_{ij}' : j \in \mathbb{N}\}$.

Define a bijection $h_{II}$ on $[\bigcup_{-m \leq i \leq m} (B_i \cup B_i')] \cap (\omega^{n+1})_{k-1}$ as $h_{II}(x_{ij}) = x_{\psi(i)j}$ and $h_{II}(x_{ij}') = x_{\psi(i)\phi(j)}'$, where $\psi$ is a bijection on $\{-m, -m+1, .., 0, 1, ..., m\}$ defined as $\psi(i) = \begin{cases} i+1 & \forall i < m \\ -m & \text{if } i = m \end{cases}$ and $\phi : \mathbb{N} \to \mathbb{N}$ is defined as $\phi(2i-1) = 2i+1$ $\forall i \in \mathbb{N}$, $\phi(2i) = 2i-2$ $\forall i \geq 2$ and $\phi(2) = 1$.

Extend $h_{II}$ to $(\bigcup_{-m \leq i \leq m} B_i) \bigcup (\bigcup_{-m \leq i \leq m} B_i') \cup \{x_i : -m \leq i \leq m\}$ by defining $h_{II}(x_i) =$

$f(x_i)$.

**Type III :** $\{x : f(x) = x\}$ (i.e., $x$ is a fixed point).

Let $B$ and $B'$ be two deleted clopen neighborhoods of $x$ such that $B \subset S$ and $B' \subset (\omega^{n+1} \setminus S)$. Say $B = \{x_i : i \in \mathbb{N}\}$ and $B' = \{x'_i : i \in \mathbb{N}\}$. Define $h_{III}$ on $(B \cup B') \cap (\omega^{n+1})_{k-1}$ as $h_{III}(x_i) = x_i$ and $h_{III}(x'_i) = \phi(x'_i)$, where $\phi$ is defined as above. Extend $h_{III}$ to $B \cup B' \cup \{x\}$ by defining $h_{III}(x) = x$.

The neighborhoods considered above will form a partition of a clopen set, say $X \subset (\omega^{n+1})_{k-1}$ and we can assume that for every $W \in \mathcal{J}^{(k-1)}$, $S_W \subset X$ or $S_W \subset (\omega^{n+1})_{k-1} \setminus X$. Thus $(\omega^{n+1})_{k-1} \setminus X$ is a union of $S_U$'s for some $U \in \mathcal{J}^{(k-1)}$ such that $D(S_U) = \varphi$. Thus any bijection on $(\omega^{n+1})_{k-1} \setminus X$ will be a homeomorphism on it. So, we define a bijection on $(\omega^{n+1})_{k-1} \setminus X$ so that $S_U$ is a single infinite orbit (type I) if $S_U \subset (\omega^n \setminus S)$ or otherwise each point of $S_U$ is fixed. The homeomorphisms $h_I$, $h_{II}$ and $h_{III}$ can be pasted to get a homeomorphism on $X$ and by pasting this homeomorphism and the bijection on $(\omega^{n+1})_{k-1} \setminus X$, we get a homeomorphism $h$ on $(\omega^{n+1})_k \cup (\omega^{n+1})_{k-1}$ such that $P(h) = S \cap ((\omega^{n+1})_{k-1} \cup (\omega^{n+1})_k)$. $\square$

**Remark 3.2.2.** Similar to the bijection $f$, the extended homeomorphism $h$ satisfies the following :

1. $h$ has no orbit of even length
2. $S_V$ is $h-$invariant $\forall V \in \mathcal{J}^{(k)} \cup \mathcal{J}^{(k-1)}$ and
3. $P(h) = S \cap ((\omega^{n+1})_{k-1} \cup (\omega^{n+1})_k)$.

**Theorem 3.2.3.** *Let $S \subset \omega^n$ for some $n \in \mathbb{N}$. The following are equivalent:*

1. *$S = P(h)$ for some self-homeomorphism $h$ on $\omega^n$.*
2. *$S_V \cap (\omega^n \setminus S)$ is either empty or infinite for every $V \in \bigcup_{k=0}^{n-1} \mathcal{J}^{(k)}$.*

*Proof.* If $S = P(h)$, then $S$ is $h-$invariant and thus by Theorem 3.1.3, $S_V$ is $h-$invariant for every $V$. Also, either $S_V \subset S$ or $S_V \subset (\omega^n \setminus S)$. Since a finite non-empty invariant set certainly contains a periodic point, it follows that $S_V \cap (\omega^n \setminus S)$ is either empty or infinite for every $V$.

For the converse, let $S \subset \omega^n$ such that $S_V \cap (\omega^n \setminus S)$ is either empty or infinite for every $V \in \bigcup_{k=0}^{n-1} \mathcal{J}^{(k)}$.

Suppose $V_1,\ V_2, ..., V_m \in \mathcal{J}^{(n-1)}$ such that $(\omega^n)_{n-1} \setminus S = \bigcup_{j=1}^{m} S_{V_j}$ and $S_{V_j} \neq \varphi\ \forall j \in \{1, 2, ..., m\}$. Say $S_{V_j} = \{x_{jl} : l \in \mathbb{Z}\}$.

Define $f : (\omega^n)_{n-1} \to (\omega^n)_{n-1}$ as $f(x) = \begin{cases} x & \text{if } x \in S \\ x_{j(l+1)} & \text{if } x = x_{jl} \in (\omega^n)_{n-1} \setminus S \end{cases}$. If $m$ does not exist i.e., if there is no $V \in \mathcal{J}^{(n-1)}$ such that $S_V \subset (\omega^n \setminus S)$, then define $f(x) = x\ \forall x \in (\omega^n)_{n-1}$. Then $f$ is a bijection on $(\omega^n)_{n-1}$.

If $n = 1$, then $f$ is a homeomorphism on $\omega$ such that $P(f) = S$. Otherwise, using the above Lemma, this $f$ can be extended to a homeomorphism $h_{n-2}$ on $(\omega^n)_{n-1} \cup (\omega^n)_{n-2}$ such that $P(h_{n-2}) = S \cap ((\omega^n)_{n-1} \cup (\omega^n)_{n-2})$.

$h_{n-2}$ defines a bijection on $(\omega^n)_{n-2}$ which can be further extended to a homeomorphism $h_{n-3}$ on $(\omega^n)_{n-2} \cup (\omega^n)_{n-3}$ such that $P(h) = S \cap ((\omega^n)_{n-2} \cup (\omega^n)_{n-3})$. By pasting $h_{n-2}$ and $h_{n-3}$, we get a homeomorphism on $(\omega^n)_{n-1} \cup (\omega^n)_{n-2} \cup (\omega^n)_{n-3}$ with $S \cap ((\omega^n)_{n-1} \cup (\omega^n)_{n-2} \cup (\omega^n)_{n-3})$ as the set of periodic points. Continuing this way, we get a homeomorphism $h$ on $\omega^n$ with $P(h) = S$. □

Now, we have the main theorem:

**Theorem 3.2.4.** *The following are equivalent for a subset $S \subset O^{(n)}$:*

1. *$S = P(h)$ for some self-homeomorphism $h$ on $O^{(n)}$.*

2. *$S_V \cap (O^{(n)} \setminus S)$ is either empty or infinite for every $V \in \bigcup_{k=0}^{n} \mathcal{J}^{(k)}$.*

*Proof.* The first part of the proof is same as that in the above proof. For the converse, recall that $O^{(n)} = \omega^n.m_n + \omega^{n-1}.m_{n-1} + ... + \omega.m_1 + m_0$ where $m_i \in \mathbb{N}_0$. Here, the highest level of limit points is $n$ and since $(O^{(n)})_n$ is a finite set, $(O^{(n)})_n \subset S$. The rest of the proof follows by taking $f$ to be the identity map on $(O^{(n)})_n$ in Lemma 3.2.1 and also using the ideas of Theorem 3.2.3. □

## 3.3 An equivalence class and a group action

**Definition 3.3.1.** Let $Z$ be a topological space and $S \subset Z$. Let $x,\ y \in Z$. $x$ is said to be topologically same as $y$ in $Z$ with respect to $S$ if there exists a homeomorphism $h$ on $Z$ such that $h(S) = S$ and $h(x) = y$.

Given $S \subset Z$, this definition induces an equivalence relation $R_S$ on $Z$ as : $x\ R_S\ y$ if $x$ is topologically same as $y$ with respect to $S$. It is easy to see that this is an equivalence relation on $Z$ and the following theorem describes the equivalence classes of $O^{(n)}$.

**Theorem 3.3.2.** *The family* $\{S_V : V \in \bigcup_{l=0}^{n} \mathcal{J}^{(l)},\ S_V \neq \varphi\}$ *is the set of equivalence classes of* $O^{(n)}$ *with respect to* $R_S$.

*Proof.* From the proof of Theorem 3.1.3, it follows that $S_V$ is invariant $\forall V \in \bigcup_{l=0}^{n} \mathcal{J}^{(l)}$, under any homeomorphism $h$ on $O^{(n)}$ such that $h(S) = S$. So if $V \neq V'$, then $x \in S_V$ and $y \in S_{V'}$ are not related.

Now, let $x, y \in S_V$ for $V \in \mathcal{J}^{(k)}$ for some $k \in \{0, 1, ..., n\}$. Define a bijection $h$ on $(O^n)_k$ such that $h(z) = \begin{cases} z & \text{if } z \notin \{x, y\} \\ y & \text{if } z = x \\ x & \text{if } z = y \end{cases}$. Using the ideas of the proof of Lemma 3.2.1, this map can be extended to a homeomorphism $h$ on $\bigcup_{i=0}^{k} (O^{(n)})_i$ such that

$h(S \cap (\bigcup_{i=0}^{k}(O^{(n)})_i)) = S \cap (\bigcup_{i=0}^{k}(O^{(n)})_i)$, which can be further extended to $O^{(n)}$ by defining $h(z) = z, \forall z \in O^{(n)} \setminus \bigcup_{i=0}^{k}(O^{(n)})_i$. Hence the family $\{S_V : V \in \bigcup_{k=0}^{n} \mathcal{J}^{(k)},\ S_V \neq \varphi\}$ is the set of equivalence classes. $\square$

If a group $G$ is acting on a set $X$ and $x \in X$, then the set $Gx = \{gx : g \in G\}$ is called the $G-$orbit of $x$. When a finite group acts on a finite set, one natural problem is to count the number of orbits. Burnside's lemma and Polya's theorem are in this direction. Occasionally, there are some infinite groups acting on infinite sets, but having only finitely many orbits. The problem of the present section has such background.

To every topological space $X$, we can associate a natural group, namely the group of self homeomorphisms on it. Here, we consider a subgroup of this group. Given a subset $S$ of a topological space $X$, let $G_S = \{f : X \to X : f$ is a homeomorphism on $X$ such that $f(S) = S\}$ , i.e., the collection of self-homeomorphisms on $X$ under which $S$ is invariant. It can be easily seen that $G_S$ is a group.

The following theorem gives a neat description of the $G_S-$orbits in $O^{(n)}$ in terms of $S_V$'s.

**Theorem 3.3.3.** *The family $\{S_V : V \in \bigcup_{k=0}^{n} \mathcal{J}^{(k)},\ S_V \neq \varphi\}$ is precisely the collection of $G_S-$orbits on $O^{(n)}$ i.e., $\forall V \in \bigcup_{k=0}^{n} \mathcal{J}^{(k)}$, $S_V$ is either empty or a $G_S-$orbit and every $G_S-$orbit is $S_V$ for some $V \in \bigcup_{k=1}^{n} \mathcal{J}^{(k)}$.*

The proof follows from Theorem 3.3.2.

Now we ask the following question : Which are the separable metric spaces $X$, for which there are finitely many $G_S$ orbits for every $S \subset X$ ?

The above theorem shows that compact metric spaces with finite derived length are among these. In fact, the answer is complete. It is a very neat and elegant one, given by the following theorem. In proving this theorem, we use the fact that a countable metric space with finite derived length is homeomorphic to a subspace of $\omega^n$.

**Theorem 3.3.4.** *Among the separable metric spaces, those with finite derived length are the only spaces which have finitely many $G_S$-orbits for every subset $S$.*

*Proof.* If $X$ has finite derived length and is compact, then the result follows from Theorem 3.3.3.

Now, suppose $X$ is not compact and has finite derived length. Since $X$ is separable, it is known that $X$ is homeomorphic to a subspace of $\omega^n$ for some $n$. So, it is enough to consider the number of $G_S-$orbits in $X$, where $S \subset X \subset \omega^n$. Let $H = \{h : \omega^n \to \omega^n : h(X) = X \text{ and } h(S) = S\}$. Then $H$ is isomorphic to a subgroup of $G_S$. Thus the number of $G_S-$orbits in $\omega^n$ cannot exceed the number of $H-$orbits in $X$. It follows from Theorem 3.3.3 and the proof of Theorem 3.3.2, that the non-empty members of the family $\{S \cap X_V : V \in \bigcup_{k=0}^{n} \mathcal{J}^{(k)}\} \cup \{(\omega^n \setminus S) \cap X_V : V \in \bigcup_{k=0}^{n} \mathcal{J}^{(k)}\}$ are precisely the collection of $H-$orbits on $\omega^n$. So, the number of $H-$orbits is atmost twice the number of $X_V$'s, which are finite in number. Thus there are finitely many $H-$orbits in $\omega$ and hence finitely many $G_S-$orbits in $X$.

Let $X$ be a separable metric space with infinite derived length. For any homeomorphism $h$ on $X$ and for any $k \in \mathbb{N}$, $X_k$ is $h-$invariant. Thus, for any $S \subset X$, each $G_S$-orbit is contained in some $X_k$ and every non-empty $X_k$ contains a $G_S$ orbit. So, if $X_k \neq \varphi$ for every $k \in \mathbb{N}_0$, then there are infintely many $G_S$-orbits for any $S \subset X$. Otherwise, choose $n \in \mathbb{N}$ such that $X_n \neq \varphi$ and $X_{n+1} = \varphi$. Then $D^{n+1}(X) = D^{n+2}(X)$, which shows that $D^{n+1}(X)$ is a perfect space. So, it is enough to prove the result for a perfect space.

Now, suppose that $X$ is perfect. It is known that there is an embedding of $\omega^\omega + 1$ into $X$. Since $(\omega^\omega + 1)_n \neq \varphi$ for every $n \in \mathbb{N}_0$, there are infinitely many $G_S$-orbits in $\omega^\omega + 1$ for any $S \subset \omega^\omega + 1$. Let $Y$ be a homeomorphic image of $\omega^\omega + 1$ in $X$. Then there are infinitely many $G_Y$-orbits in $Y$. If $Z \subset Y$ is a $G_Y$-orbit in $Y$ i.e., for any $x, y \in Z$, there is a homeomorphism $h : Y \to Y$ such that $h(x) = y$, then extending $h$ to $X$, we get a homeomorphism $h' : X \to X$ such that $h'(Y) = Y$ and $h'(x) = y$. Thus every $G_Y$-orbit in $Y$ gives rise to a $G_Y$-orbit in $X$. Hence, there are infinitely many $G_Y$-orbits in $X$.

□

## 3.4 Homeomorphism classes

In this section, some natural equivalence relations are defined between the subsets of $\omega^2$ and the equivalence classes are studied. Besides the homeomorphism, a weaker notion of homeomorphism called eventual homeomorphism and a stronger notion called "homeomorphism in $X$" are also considered. For elegance, we consider the space $X = \overline{\{m - \frac{1}{n} : m, n \in \mathbb{N}\}}$ (closure in $\mathbb{R}$) as a substitute for $\omega^2$. Note that $X_0 = \{m - \frac{1}{n} : m, n \in \mathbb{N}\}$ and $X_1 = \mathbb{N}_0$. A natural one-one correspondence between a subset of $(\mathbb{W}_\infty)^6$ and the set of equivalence classes of $\omega^2$ is described under the equivalence relation $\approx$ given by : $S, T \subset X$, $S \approx T$ if $S$ is homeomorphic to $T$ in $X$, thus showing that there are countably infinite number of subsets of $\omega^2$ upto homeomorphism in $\omega^2$. It is proved that there are exactly 63 subsets of $X$ upto eventual homeomorphism in $X$ and we generalize this result for $\omega^n$ for every $n \in \mathbb{N}$.

Continuing this study, we describe a natural one-one correspondence between a subset of $(\mathbb{W}_\infty \times \{0, \infty\}) \cup \mathbb{W}$ (where $\mathbb{W}_\infty$ is the extended whole number system i.e, $\mathbb{W}_\infty = \mathbb{N} \cup \{0, \infty\}$) and the homeomorphism classes of subspaces of $\omega^2$. This is a classification of countable metric spaces with derived length one. This problem of understanding all countable metric spaces has been made by many in the last hundred years. Complete success has been obtained among compact spaces (See [24]) and among self dense spaces (See [32]). Here, a description is given for a narrow class, namely, the countable metric spaces with derived length one.

**Definition 3.4.1.** Two subsets $S$ and $T$ of a topological space $Z$ are said to be homeomorphic in $Z$ if there exists a homeomorphism $h : Z \longrightarrow Z$ such that $h(S) = T$.

Being homeomorphic in $Z$ is stronger than being homeomorphic; when $S$ and $T$ are homeomorphic in $Z$, then $Z \setminus S$ and $Z \setminus T$ are also homeomorphic.

**Definition 3.4.2.** $S$ and $T$ are said to be eventually homeomorphic in $Z$ if there exist bounded subsets of $Z$, say $B_1$ and $B_2$ such that $S \setminus B_1$ and $T \setminus B_2$ are homeomorphic in $Z$.

Note that being eventually homeomorphic in $Z$ is an equivalence relation on the set of all subsets of $Z$.

Given $Y \subset X$, the partition $\mathcal{P}_Y$ consists of atmost 8 partition classes. So, we adopt the following convenient notation (here, $Y^c$ stands for the complement of $Y$ in $X$ i.e., $X \setminus Y$):

1. $C_{1,Y} = Y \cap X_0$
2. $C_{2,Y} = Y^c \cap X_0$
3. $C_{3,Y} = (\overline{Y \cap X_0} \setminus \overline{Y^c \cap X_0}) \cap Y \cap X_1$
4. $C_{4,Y} = \overline{Y \cap X_0} \cap \overline{Y^c \cap X_0} \cap Y \cap X_1$
5. $C_{5,Y} = (\overline{Y^c \cap X_0} \setminus \overline{Y \cap X_0}) \cap Y \cap X_1$
6. $C_{6,Y} = (\overline{Y \cap X_0} \setminus \overline{Y^c \cap X_0}) \cap Y^c \cap X_1$
7. $C_{7,Y} = \overline{Y \cap X_0} \cap \overline{Y^c \cap X_0} \cap Y^c \cap X_1$
8. $C_{8,Y} = (\overline{Y^c \cap X_0} \setminus \overline{Y \cap X_0}) \cap Y^c \cap X_1$

### 3.4.1 Homeomorphism in $X$

**Theorem 3.4.3.** *Let $S, T \subset X$. $S$ and $T$ are homeomorphic in $X$ if and only if $|C_{i,S}|$ $= |C_{i,T}|$ for every $i \in \{1, 2, ..., 8\}$.*

*Proof.* We consider a partition of $X$ as follows.

Given $Y \subset X$, let $X' = (\bigcup_{x \in X_1 \setminus (C_{4,Y} \cup C_{7,Y})} A_{x,Y}) \bigcup (\bigcup_{x \in C_{4,Y} \cup C_{7,Y}} B_{x,Y}) \bigcup (\bigcup_{x \in C_{4,Y} \cup C_{7,Y}} D_{x,Y} \setminus \{x\})$, where $A_{x,Y}$ is a clopen neighborhood of $x$ contained in either $Y$ or $X \setminus Y$ and $B_{x,Y} \cup D_{x,Y}$ is a clopen neighborhood of $x$ such that $B_{x,Y} \setminus \{x\} \subset Y$ and $D_{x,Y} \setminus \{x\} \subset X \setminus Y$. All these neighborhoods are chosen to be mutually disjoint. If $C_{1,Y}$ and $C_{2,Y}$ are infinite, then we can assume that $X = X'$ or otherwise $X = X' \cup C_{1,Y}$ (if $C_{1,Y}$ is finite) and $X = X' \cup C_{2,Y}$ (if $C_{2,Y}$ is finite). The non-empty ones among the $C_{i,Y}$ s are used in giving a partition of $C_{1,Y}$ as: $C_{1,Y} = (\bigcup_{x \in C_{3,Y} \cup C_{6,Y}} A_{x,Y} \setminus \{x\}) \bigcup (\bigcup_{x \in C_{4,Y} \cup C_{7,Y}} B_{x,Y} \setminus \{x\})$.

Suppose $|C_{i,S}| = |C_{i,T}|$ for every $i \in \{1, 2, ..., 8\}$. Let $\varphi$ be a bijection on $X_1$ such that $\varphi(C_{i,S}) = C_{i,T}$ for every $i \in \{3, 4, ..., 8\}$. Define homeomorphisms $h_x : A_{x,S} \longrightarrow A_{x,T}$ for every $x \in X_1 \setminus (C_{4,S} \cup C_{7,S})$ and $h_x : B_{x,S} \cup D_{x,S} \longrightarrow B_{x,T} \cup D_{x,T}$ such that $h_x(B_{x,S}) = B_{x,T}$ and $h_x(D_{x,S}) = D_{x,T}$ for every $x \in C_{4,S} \cup C_{7,S}$ and also that $h_x(x) = \varphi(x)$ for every $x \in X_1$.

Now define $h : X' \longrightarrow X'$ as $h(y) = h_x(y)$ where $y \in A_{x,S}$ or $y \in B_{x,S} \cup D_{x,S}$. Since each $h_x$ is a homeomorphism on a clopen subset, $h$ is a homeomorphism on $X'$. In case $C_{i,S}$ for $i \in \{1, 2\}$ is finite, define a bijection $\psi : C_{i,S} \to C_{i,T}$ and extend $h$ to $X$ by defining $h(x) = \psi(x)\ \forall x \in C_{i,S}$. Since $C_{i,S}$ for $i \in \{1, 2\}$ contains only isolated points, $h$ is still a homeomorphism on $X$. It is clear that $h(C_{i,S}) = C_{i,T}\ \forall i \geq 3$ and also $h(C_{1,S}) = C_{1,T}$. Since $S = \bigcup_{i \in \{1,3,4,5\}} C_{i,S}$, $h(S) = T$.

Conversely, suppose that $S$ and $T$ are homeomorphic in $X$. Then it follows that $S$ is homeomorphic to $T$ as well as $X \setminus S$ is homeomorphic to $X \setminus T$. Now, each $C_{i,S}$ and $C_{i,T}$ is characterized by a topological property in $S$ and $T$ respectively. $h$

being a homeomorphism preserves topological properties. So, $h(C_{i,S}) = C_{i,T}$ and thus $|C_{i,S}| = |C_{i,T}|$ for every $i \in \{1, 2, ..., 8\}$. $\square$

Given a subset $S \subset \omega^2$, let $a_i = |C_{i,S}|$ for $i \in \{1, 2, ..., 8\}$. Since $X_0$ and $X_1$ are infinite sets and $X_0 = C_{1,S} \cup C_{2,S}$, $X_1 = \bigcup_{i=3}^{8} C_{i,S}$, it follows that $a_1 + a_2 = \infty$ and $\Sigma_{i=3}^{8} a_i = \infty$. It is also clear that $a_1 < \infty \Leftrightarrow a_3 = a_4 = a_6 = a_7 = 0$ and $a_2 < \infty \Leftrightarrow a_4 = a_5 = a_7 = a_8 = 0$. Thus for every $S \subset \omega^2$, we get an element in $W = \{(a_1, a_2, ..., a_8) \in (\mathbb{W}_\infty)^8 : a_1 + a_2 = \infty,\ \Sigma_{i=3}^{8} a_i = \infty,\ a_1 < \infty \Leftrightarrow a_3 = a_4 = a_6 = a_7 = 0 \text{ and } a_2 < \infty \Leftrightarrow a_4 = a_5 = a_7 = a_8 = 0\}$.

Conversely, given any element $(a_1, a_2, ..., a_8)$ in $W$, we can construct a subset $S \subset \omega^2$ such that $a_i = |C_{i,S}|$ for $i \in \{1, 2, ..., 8\}$. Moreover it follows from Theorem 3.4.3 that two subsets $S, T \subset \omega^2$ with the corresponding 8-tuples $(a_1, a_2, ..., a_8)$ and $(b_1, b_2, ..., b_8)$ are homeomorphic in $X$ if and only if $a_i = b_i$ for $i \in \{1, 2, ..., 8\}$.

Thus there is a one to one correspondence between the set of equivalence classes of $\omega^2$ (where the equivalence relation $\approx$ is given by : $S, T \subset X$, $S \approx T$ if $S$ is homeomorphic to $T$ in $X$) and the set $W$. Hence the following theorem.

**Theorem 3.4.4.** *There are countably infinite number of subsets of $\omega^2$ upto homeomorphism in $\omega^2$.*

### 3.4.2 Eventual homeomorphism in $\omega^2$

**Lemma 3.4.5.** If $S \subset Z$ and $B$ is a bounded subset of $Z$, then $S$ is eventually homeomorphic to $S \Delta B$ in $Z$.

*Proof.* Let $B_1 = S \cap B$ and $B_2 = B \setminus S$, then $S \Delta B = (S \setminus B_1) \cup B_2$. Since $B_1$ is bounded, $S$ is eventually homeomorphic to $S \setminus B_1$ and this in turn is eventually homeomorphic to $(S \setminus B_1) \cup B_2$ because $B_2$ is bounded. Hence $S$ is eventually homeomorphic to $S\Delta B$ in $Z$. □

**Lemma 3.4.6.** Let $S, T \subset X$. If $C_{i,S}$ and $C_{i,T}$ are both finite or infinite for every $i \in \{3, 4, ..., 8\}$, then $S$ is bounded if and only if $T$ is bounded and $X \setminus S$ is bounded if and only if $X \setminus T$ is bounded.

*Proof.* $S$ is bounded if and only if $C_{1,S}$ is bounded and this is same as saying that $C_{3,S} \cup C_{4,S} \cup C_{6,S} \cup C_{7,S}$ is finite. Then $C_{3,T} \cup C_{4,T} \cup C_{6,T} \cup C_{7,T}$ is finite which in turn is same as saying that $C_{1,T}$ is bounded and thus $T$ is bounded. Replacing $C_{3,S} \cup C_{4,S} \cup C_{6,S} \cup C_{7,S}$ by $C_{4,S} \cup C_{5,S} \cup C_{7,S} \cup C_{8,S}$, it follows that $X \setminus S$ is bounded if and only if $X \setminus T$ is bounded. □

**Theorem 3.4.7.** *Let $S, T \subset X$. $S$ and $T$ are eventually homeomorphic in $X$ if and only if both $|C_{i,S}|$ and $|C_{i,T}|$ are finite or both infinite for every $i \in \{3, 4, ..., 8\}$.*

*Proof.* Suppose $S$ and $T$ are eventually homeomorphic in $X$. Then there exist bounded sets $B_1,\ B_2 \subset X$ such that $|C_{i,S\setminus B_1}| = |C_{i,T\setminus B_2}|\ \forall i \in \{3, 4, ..., 8\}$. It can be verified that $C_{i,S\setminus B_1} \Delta C_{i,S} \subset \overline{B_1} \cap X_1$. Since $\overline{B_1} \cap X_1$ is finite, it follows that both $C_{i,S\setminus B_1}$ and $C_{i,S}$ are finite or both infinite. Similarly both $C_{i,T\setminus B_2}$ and $C_{i,T}$ are finite or both infinite. Since $|C_{i,S\Delta B_1}| = |C_{i,T\Delta B_2}|$ for each $i \in \{3, 4, .., 8\}$, both $|C_{i,S}|$ and $|C_{i,T}|$ are finite or both infinite for every $i \in \{3, 4, ..., 8\}$.

For the converse, fix a $k \in \{3, 4, ..., 8\}$ for which $C_{k,S}$ and $C_{k,T}$ are infinite (such $k$ exists because $X_1$ is infinite). Now for every $i \in \{3, 4, .., 8\}$ such that $C_{i,S}$ is finite, add to $S$ or remove from $S$ a bounded set $B_i$ such that $C_{i,S\Delta B_i} = \varphi$ and $C_{i,S} \subset C_{k,S\Delta B_i}$. Note that in order to make $C_{i,S}$ empty, we are adding to (or removing from) $C_{k,S}$ a finite set which does not alter its cardinality because it is an infinite set. If $C_{1,S}$ is bounded, then $C_{3,S} \cup C_{4,S} \cup C_{6,S} \cup C_{7,S}$ is finite. So after the above steps, we can assume that

$C_{1,S\Delta B} = \varphi$. Similarly, if $C_{2,S}$ is bounded, then we can assume that $C_{2,S\Delta B} = \varphi$. Thus we get a bounded subset $B \subset X$ such that for any $i \in \{1, 2, .., 8\}$, $C_{i,S\Delta B}$ is either infinite or empty and it follows from Lemma 3.4.5 that $S\Delta B$ and $S$ are eventually homeomorphic in $X$.

Similarly there exists $B' \subset X$ such that $T\Delta B'$ and $T$ are eventually homeomorphic in $X$ and $C_{i,T\Delta B'}$ is either empty or infinite $\forall i \in \{1, 2, ..., 8\}$.

Since both $|C_{i,S}|$ and $|C_{i,T}|$ are finite or both infinite for every $i \in \{3, 4, ..., 8\}$, $|C_{i,S}| = |C_{i,T}|$. It follows from Lemma 3.4.6 that $C_{i,S\Delta B}$ is bounded if and only if $C_{i,T\Delta B'}$ is bounded for $i \in \{1, 2\}$. But $C_{i,S\Delta B}$ and $C_{i,T\Delta B'}$ are either empty or unbounded. Thus $|C_{i,S\Delta B}| = |C_{i,T\Delta B'}|$ $\forall i \in \{1, 2\}$. Hence $|C_{i,S\Delta B}| = |C_{i,T\Delta B'}|$ $\forall i \in \{1, 2, ..., 8\}$. From Theorem 3.4.3, we can see that $S\Delta B$ and $T\Delta B'$ are homeomorphic in $X$. Thus $S$ is eventually homeomorphic to $T$ in $X$. □

**Theorem 3.4.8.** *There are exactly 63 subsets of $X$ upto eventual homeomorphism in $X$.*

*Proof.* Since there are two possibilities for each set $C_{i,S}$, $i \in \{3, 4, ..., 8\}$, one being finite and the other infinite, there are totally 64 possible combinations for these 6 sets. However all sets cannot be finite because $X_1$ is infinite. So there are atmost 63 subsets up to eventual homeomorphism in $X$. It follows from the list of sets given in the Appendix at the end of this chapter, that there are exactly 63 of them. □

**Eventual homeomorphism in $\omega^n$ for $n > 2$**

Given $S \subset \omega^n$, consider the topologically same points in $\omega^n$ with respect to $S$. Let $l_k$ denote the maximum number of equivalence classes possible in $(\omega^n)_k$, when $n > k$ (note that $(\omega^{n_1})_k = (\omega^{n_2})_k$ for any $n_1, n_2 > k$). We have proved that $l_0 = 2$ and $l_1 = 6$.

Let $D_1, D_2, ..., D_{l_k}$ be the possible equivalence classes of $(\omega^n)_k$ and let $k+1 < n$. Consider the sets $(\cap_{i\in I}\overline{D_i})\setminus(\cap_{j\notin I}\overline{D_j})$ for every non-empty subset $I$ of $\{1, 2, ..., l_k\}$ which are $2^{l_k} - 1$ in number. Similar to the case of $\omega^2$, it can be proved that each equivalence class in $(\omega^n)_{k+1}$ is an intersection of a set from the above collection with either $S$ or $\omega^n \setminus S$ (if non-empty). Thus there are atmost $2(2^{l_k} - 1)$ i.e., $2^{l_k+1} - 2$ equivalence classes in $(\omega^n)_{k+1}$ i.e., $l_{k+1} = 2^{l_k+1} - 2$. It also follows from the proof of Theorem 3.4.8 that the number of subsets of $\omega^n$ upto eventual homeomorphism in $\omega^n$ is $2^{l_{n-1}} - 1$. Having proved that $l_0 = 2$ and $l_1 = 6$, $l_k$ can be found inductively for every $k \in \mathbb{N}$ and thus the number of subsets of $\omega^n$ up to eventual homeomorphism in $\omega^n$ can be determined.

### 3.4.3 Homeomorphism classes of $\omega^2$

Given a finite subset of $X$, we will assign a unique whole number, namely its cardinality and conversely, for any given $n \in \mathbb{W}$, we can have a finite subset of $X$ with cardinality $n$. Moreover, it is clear that two sets with different cardinalities cannot be homeomorphic. Now, let $S$ be an infinite subset. $S$ can be written as $S \setminus S_1 = (\cup_{x\in S_1} B_x) \cup B'$, where $B_x = \{x - \frac{1}{q} : q \in \mathbb{N}\} \cap S$, $B' \cap B_x = \varphi$ $\forall x \in S_1$ and $(B')_1 = \varphi$. If $B'$ is finite, then replace $B_x$ for some $x \in S_1$ by $B_x \cup B'$ and if both $S_1$ and $B'$ are infinite say $S_1 = \{x_q : q \in \mathbb{N}\}$ and $B' = \{y_q : q \in \mathbb{N}\}$, then replace $B_{x_q}$ by $B_{x_q} \cup \{y_q\}$ $\forall q \in \mathbb{N}$. Thus we can write $S = (\cup_{x\in S_1} C_x) \cup B$, where $C_x$ is a clopen neighborhood of $x$ such that $C_x \cap C_y = \varphi$ for any $x \neq y$ and $B$ is either an empty or an infinite clopen subset of $S$ such that $B_1 = \varphi$, both $S_1$ and $B$ cannot be empty simultaneously and whenever $|S_1| = \infty$, $B = \varphi$. Thus given an infinite subset $S$ of $X$, we can assign a unique element $(m, n)$ in $W' = \{(p, q) \in \mathbb{W}_\infty \times \{0, \infty\} : p = \infty \Rightarrow q = 0 \text{ and } p^2 + q^2 \neq 0\}$, where $m = |S_1|$ and $n = |B|$. Also given any element $(m, n)$ in $W'$, we can construct a subset $S$ of $\omega^2$ satisfying the above conditions as

$$S = \begin{cases} \{1 - \frac{1}{q} : q \in \mathbb{N}\} & \text{if } m = 0 \text{ and } n = \infty \\ \{1, 2, ..., m\} \cup \{p - \frac{1}{q} : 1 \leq p \leq m,\ q \in \mathbb{N}\} & \text{if } m \in \mathbb{N} \text{ and } n = 0 \\ \{1, 2, ..., m\} \cup \{p - \frac{1}{q} : 1 \leq p \leq m,\ q \in \mathbb{N}\} & \\ \quad \cup \{m + 1 - \frac{1}{q} : q \in \mathbb{N}\} & \text{if } m \in \mathbb{N} \text{ and } n = \infty \\ X & \text{if } m = \infty \text{ and } n = 0 \end{cases}$$

Now, given any two subsets $S, T \subset X$, let $S = (\cup_{x \in S_1} C_x) \cup B$ and $T = (\cup_{y \in T_1} D_y) \cup E$ be the similar representations as above. If $|S_1| = |T_1|$ and $|B| = |E|$, let $\varphi : S_1 \longrightarrow T_1$ and $\psi : B \longrightarrow E$ be any bijections. By defining homeomorphisms $h_x : C_x \to D_{\varphi(x)}$ and then pasting these homeomorphisms and the bijection $\psi$, we get a homeomorphism from $S$ to $T$. Conversely, if $S$ and $T$ are homeomorphic, then $S_1 = T_1$ and $|B| = |E|$. Thus there is a natural one-one correspondence between the set $\mathbb{W} \cup W'$ and the homeomorphism classes of subspaces of $\omega^2$.

**Theorem 3.4.9.** *For any $S \subset X$, there are atmost 8 minimal subsets of $X$ which are invariant under all those homeomorphisms under which $S$ is invariant.*

*Proof.* Let $\mathcal{H}_S = \{h : X \to X : h \text{ is a homeomorphism on } X \text{ such that } h(S) = S\}$. Consider $C_{i,S}$ for $i \in \{1, 2, ..., 8\}$. It is clear that each $C_{i,S}$ is $h-$invariant $\forall h \in \mathcal{H}_S$. Further, it follows from the proof of Theorem 3.4.3 that for any $x, y \in C_{i,S}$, $\exists h \in \mathcal{H}_S$ such that $h(x) = y$. So, no proper subset of $C_{i,S}$ is $h-$invariant for all $h \in \mathcal{H}_S$. Therefore all these sets $C_{i,S}$, if non-empty, are minimal sets which are $h-$invariant $\forall h \in \mathcal{H}_S$. Hence the theorem. $\square$

In this section, we have thus counted the number of subsets of $\omega^n$ upto eventual homeomorphism in $\omega^n$ and have given a natural one-one correspondence between the countable set $\mathbb{W} \cup W'$ and the homeomorphism classes of subspaces of $\omega^2$. We ask for a natural correspondence in the case of $\omega^n$ for $n > 2$.

## 3.5 Conclusion

We conclude this chapter by posing a question. The problem of characterizing the sets of periodic points of homeomorphisms on $O^{(n)}$ leads to a very natural and interesting question of characterizing the same sets for continuous maps on $O^{(n)}$. It is hoped that the ideas in this paper will be useful in discussing the later question.

**Appendix : 63 examples**

We use the following notation in the examples :

For any $M, N \subset \mathbb{N}$, $M + N = \{m + n : m \in M, n \in N\}$,

$pM = \{pm : m \in M\}$ $\forall p \in \mathbb{N}$,

$\frac{1}{M} = \{\frac{1}{m} : m \in M\}$ and

$M - \frac{1}{N} = \{m - \frac{1}{n} : m \in M, n \in N\}$.

With an abuse of notation, $S_n$ in the following list and the table represents the $n^{th}$ set among the 63 sets, but not the $n^{th}$ level of limit points of $S$.

Following is a list of 63 subsets, no two of which are eventually homeomorphic in $X$.

1. $S_1 = \varphi$.
2. $S_2 = X_1$.
3. $S_3 = 2\mathbb{N}$.
4. $S_4 = X_0$.
5. $S_5 = X_0 \cup X_1$.
6. $S_6 = X_0 \cup 2\mathbb{N}$.
7. $S_7 = 2\mathbb{N} - \frac{1}{\mathbb{N}}$.
8. $S_8 = (2\mathbb{N} - \frac{1}{\mathbb{N}}) \cup (4\mathbb{N} + \{-1\})$.
9. $S_9 = (2\mathbb{N} - \frac{1}{\mathbb{N}}) \cup (2\mathbb{N} + \{-1\})$.
10. $S_{10} = (2\mathbb{N} - \frac{1}{\mathbb{N}}) \cup 4\mathbb{N}$.
11. $S_{11} = (2\mathbb{N} - \frac{1}{\mathbb{N}}) \cup 4\mathbb{N} \cup (4\mathbb{N} + \{-1\})$.

12. $S_{12} = (2\mathbb{N} - \frac{1}{\mathbb{N}}) \cup 4\mathbb{N} \cup (2\mathbb{N} + \{-1\})$.

13. $S_{13} = (2\mathbb{N} - \frac{1}{\mathbb{N}}) \cup 2\mathbb{N}$.

14. $S_{14} = (2\mathbb{N} - \frac{1}{\mathbb{N}}) \cup 2\mathbb{N} \cup (4\mathbb{N} + \{-1\})$.

15. $S_{15} = (2\mathbb{N} - \frac{1}{\mathbb{N}}) \cup 2\mathbb{N} \cup (2\mathbb{N} + \{-1\})$.

16. $S_{16} = (2\mathbb{N} - \frac{1}{2\mathbb{N}})$.

17. $S_{17} = (2\mathbb{N} - \frac{1}{2\mathbb{N}}) \cup (4\mathbb{N} + \{-1\})$.

18. $S_{18} = (2\mathbb{N} - \frac{1}{2\mathbb{N}}) \cup (2\mathbb{N} + \{-1\})$.

19. $S_{19} = (2\mathbb{N} - \frac{1}{2\mathbb{N}}) \cup 4\mathbb{N}$.

20. $S_{20} = (2\mathbb{N} - \frac{1}{2\mathbb{N}}) \cup 4\mathbb{N} \cup (4\mathbb{N} + \{-1\})$.

21. $S_{21} = (2\mathbb{N} - \frac{1}{2\mathbb{N}}) \cup 4\mathbb{N} \cup (2\mathbb{N} + \{-1\})$.

22. $S_{22} = (2\mathbb{N} - \frac{1}{2\mathbb{N}}) \cup 2\mathbb{N}$.

23. $S_{23} = (2\mathbb{N} - \frac{1}{2\mathbb{N}}) \cup 2\mathbb{N} \cup (4\mathbb{N} + \{-1\})$.

24. $S_{24} = (2\mathbb{N} - \frac{1}{2\mathbb{N}}) \cup 2\mathbb{N} \cup (2\mathbb{N} + \{-1\})$.

25. $S_{25} = (2\mathbb{N} - \frac{1}{\mathbb{N}}) \cup (2\mathbb{N} + \{-1\} - \frac{1}{2n})$.

26. $S_{26} = (2\mathbb{N} - \frac{1}{\mathbb{N}}) \cup (2\mathbb{N} + \{-1\} - \frac{1}{2n}) \cup (4\mathbb{N} + \{-1\})$.

27. $S_{27} = (2\mathbb{N} - \frac{1}{\mathbb{N}}) \cup (2\mathbb{N} + \{-1\} - \frac{1}{2n}) \cup (2\mathbb{N} + \{-1\})$.

28. $S_{28} = (2\mathbb{N} - \frac{1}{\mathbb{N}}) \cup (2\mathbb{N} + \{-1\} - \frac{1}{2n}) \cup 4\mathbb{N}$.

29. $S_{29} = (2\mathbb{N} - \frac{1}{\mathbb{N}}) \cup (2\mathbb{N} + \{-1\} - \frac{1}{2n}) \cup 4\mathbb{N} \cup (4\mathbb{N} + \{-1\})$.

30. $S_{30} = (2\mathbb{N} - \frac{1}{\mathbb{N}}) \cup (2\mathbb{N} + \{-1\} - \frac{1}{2n}) \cup 4\mathbb{N} \cup (2\mathbb{N} + \{-1\})$.

31. $S_{31} = (2\mathbb{N} - \frac{1}{\mathbb{N}}) \cup (2\mathbb{N} + \{-1\} - \frac{1}{2n}) \cup 2\mathbb{N}$.

32. $S_{32} = (2\mathbb{N} - \frac{1}{\mathbb{N}}) \cup (2\mathbb{N} + \{-1\} - \frac{1}{2n}) \cup 2\mathbb{N} \cup (4\mathbb{N} + \{-1\})$.

33. $S_{33} = (2\mathbb{N} - \frac{1}{\mathbb{N}}) \cup (2\mathbb{N} + \{-1\} - \frac{1}{2n}) \cup 2\mathbb{N} \cup (2\mathbb{N} + \{-1\})$.

34. $S_{34} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}})$.

35. $S_{35} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup 6\mathbb{N}$.

36. $S_{36} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup 3\mathbb{N}$.

37. $S_{37} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup (6\mathbb{N} + \{-2\})$.

38. $S_{38} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup (6\mathbb{N} + \{-2\}) \cup 6\mathbb{N}$.

39. $S_{39} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup (6\mathbb{N} + \{-2\}) \cup 3\mathbb{N}$.

40. $S_{40} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup (3\mathbb{N} + \{-2\})$.

41. $S_{41} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup (3\mathbb{N} + \{-2\}) \cup 6\mathbb{N}$.

42. $S_{42} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup (3\mathbb{N} + \{-2\}) \cup 3\mathbb{N}$.

43. $S_{43} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup (6\mathbb{N} + \{-1\})$.

44. $S_{44} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup (6\mathbb{N} + \{-1\}) \cup 6\mathbb{N}$.

45. $S_{45} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup (6\mathbb{N} + \{-1\}) \cup 3\mathbb{N}$.

46. $S_{46} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup (6\mathbb{N} + \{-1\}) \cup (6\mathbb{N} + \{-2\})$.

47. $S_{47} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup (6\mathbb{N} + \{-1\}) \cup (6\mathbb{N} + \{-2\}) \cup 6\mathbb{N}$.

48. $S_{48} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup (6\mathbb{N} + \{-1\}) \cup (6\mathbb{N} + \{-2\}) \cup 3\mathbb{N}$.

49. $S_{49} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup (6\mathbb{N} + \{-1\}) \cup (3\mathbb{N} + \{-2\})$.

50. $S_{50} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup (6\mathbb{N} + \{-1\}) \cup (3\mathbb{N} + \{-2\}) \cup 6\mathbb{N}$.

51. $S_{51} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup (6\mathbb{N} + \{-1\}) \cup (3\mathbb{N} + \{-2\}) \cup 3\mathbb{N}$.

52. $S_{52} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup (3\mathbb{N} + \{-1\})$.

53. $S_{53} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup (3\mathbb{N} + \{-1\}) \cup 6\mathbb{N}$.

54. $S_{54} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup (3\mathbb{N} + \{-1\}) \cup 3\mathbb{N}$.

55. $S_{55} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup (3\mathbb{N} + \{-1\}) \cup (6\mathbb{N} + \{-2\})$.

56. $S_{56} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup (3\mathbb{N} + \{-1\}) \cup (6\mathbb{N} + \{-2\}) \cup 6\mathbb{N}$.

57. $S_{57} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup (3\mathbb{N} + \{-1\}) \cup (6\mathbb{N} + \{-2\}) \cup 3\mathbb{N}$.

58. $S_{58} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup (3\mathbb{N} + \{-1\}) \cup (3\mathbb{N} + \{-2\})$.

59. $S_{59} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup (3\mathbb{N} + \{-1\}) \cup (3\mathbb{N} + \{-2\}) \cup 6\mathbb{N}$.

60. $S_{60} = (3\mathbb{N} + \{-2\} - \frac{1}{\mathbb{N}}) \cup (3\mathbb{N} + \{-1\} - \frac{1}{2\mathbb{N}}) \cup (3\mathbb{N} + \{-1\}) \cup (3\mathbb{N} + \{-2\}) \cup 3\mathbb{N}$.

61. $S_{61} = \mathbb{N} - \frac{1}{2\mathbb{N}}$.

62. $S_{62} = (\mathbb{N} - \frac{1}{2\mathbb{N}}) \cup 2\mathbb{N}$.

63. $S_{63} = (\mathbb{N} - \frac{1}{2\mathbb{N}}) \cup \mathbb{N}$.

Theorem 3.4.7 and the following table make it clear that no two of the above 63 subsets are eventually homeomorphic (note that for any $k < l$, $S_{k \cdots l}$ in the table means the sets $S_k$,$S_{k+1}$,...,$S_l$). For any $j, k \in \{1, 2, ...63\}$, $j \neq k$ there is a row in the following table containing only $S_j$ but not $S_k$.

| $i$ | **Subsets $S$ for which $C_{i,S}$ is finite** |
|---|---|
| 3 | $S_{1\cdots4}, S_{7\cdots9}, S_{16\cdots27}, S_{34\cdots36}, S_{43\cdots45}, S_{52\cdots54}, S_{61\cdots63}$ |
| 4 | $S_{1\cdots18}, S_{25}, S_{28}, S_{31}, S_{34\cdots42}, S_{61}$ |
| 5 | $S_1, S_{4\cdots7}, S_{10}, S_{13}, S_{16}, S_{19}, S_{22}, S_{25\cdots33}, S_{34}, S_{37}, S_{40}, S_{43}, S_{46}, S_{49}, S_{52}, S_{55}, S_{58}, S_{61\cdots63}$ |
| 6 | $S_{1\cdots3}, S_5, S_{13\cdots15}, S_{16\cdots24}, S_{31\cdots33}, S_{40\cdots42}, S_{49\cdots51}, S_{58\cdots63}$ |
| 7 | $S_{1\cdots15}, S_{22\cdots24}, S_{27}, S_{30}, S_{33}, S_{52\cdots60}, S_{63}$ |
| 8 | $S_2, S_{4\cdots6}, S_9, S_{12}, S_{15}, S_{18}, S_{21}, S_{24\cdots33}, S_{36}, S_{39}, S_{42}, S_{45}, S_{48}, S_{51}, S_{54}, S_{57}, S_{60\cdots63}$ |

# Chapter 4

# Systems without convergent trajectories

Dealing further with trajectories, we consider a simple kind of trajectories namely the convergent ones. It is evident that the trajectories of fixed points and eventually fixed points are eventually constant and thus convergent. So, we will be interested in those convergent trajectories which are not eventually constant. Hereafter, such trajectories will be referred to as non-trivial convergent trajectories. Another convention that will be followed is that the underlying space $X$ is a compact metric space and $d$ always denotes the metric on it.

It is proved in this chapter that there are no non-trivial convergent trajectories in many classes of dynamical systems. For example, there are no non-trivial convergent trajectories for circle rotation (which is a positively equicontinuous non-wandering map), expanding endomorphism $E_m$ of circle i.e., $E_m(x) = mx \ (mod\ 1)$, where $m \in \mathbb{Z}$ with $|m| > 1$ (an expanding map), shift map (a positively expansive map), the map $F$ on the torus $\mathcal{T}^2$ given by $F(x, y) = (x + \alpha \ (mod\ 1), x + y \ (mod\ 1))$ (a distal map) and tent map (a chaotic piecewise monotonic interval map). Despite having a complex behaviour, some chaotic systems may still contain non-trivial convergent trajectories. Two examples of chaotic systems are given, one of which contains a non-trivial convergent trajectory and the other doesn't.

## 4.1 Definitions

**Definition 4.1.1.** A dynamical system $(X, f)$ is said to be positively equicontinuous if the family $\{f^n : n \in \mathbb{N}\}$ is equicontinuous. In other words, given any $\epsilon > 0$, $\exists \delta > 0$ such that $d(f^n(x), f^n(y)) < \epsilon \ \forall n \in \mathbb{N}$ whenever $x, y \in X$ with $d(x, y) < \delta$.

**Definition 4.1.2.** A point $x \in X$ is said to be non-wandering in the system $(X, f)$ if for every neighborhood $U$ of $x$, $f^n(U) \cap U \neq \varphi$ for some $n \in \mathbb{N}$. $(X, f)$ is said to be non-wandering if every point in $X$ is non-wandering.

**Definition 4.1.3.** $(X, f)$ is said to be expanding if $\exists\ \mu > 1$ and $\epsilon > 0$ such that for any $x,\ y \in X$, $d(x, y) < \epsilon \ \Rightarrow d(f(x), f(y)) \geq \mu d(x, y)$.

**Definition 4.1.4.** $(X, f)$ is said to be positively expansive if $\exists\ r > 0$ such that for any $x,\ y \in X$, $\exists n \in \mathbb{N}_0$ with $d(f^n(x), f^n(y)) \geq r$.

See [7], [25] for the above definitions.

**Remark 4.1.5.** If a trajectory $(f^n(x))$ converges, say to $y$, then $y$ is a fixed point. This is because $(f(f^n(x))) \to f(y)$ and $(f(f^n(x))) = (f^{n+1}(x))$ is a subsequence of $(f^n(x))$.

**Definition 4.1.6.** [12] A system $(X, f)$ is said to be chaotic if

(1)it has a dense set of periodic points

(2)it is transitive

(3)it has sensitive dependence on initial conditions (i.e., $\exists \delta > 0$ such that for any point $x \in X$ and any neighborhood $U$ of $x$, there is a point $y \in U$ and $n \in \mathbb{N}$ with $d(f^n(x), f^n(y)) \geq \delta$).

This definition is due to R.L.Devaney [12]. There are however other definitions of chaos which are not necessarily equivalent to this definition.

## 4.2 Main results

The convergent trajectories can be considered as the simplest kind of trajectories after the periodic and pre-periodic ones. As mentioned earlier, we prove that many of the interesting dynamical systems do not possess these simple trajectories, except for the trivial ones. Though chaotic systems have very complicated dynamics, the absence of non-trivial convergent trajectories is not witnessed in these systems in general (See Example 4.2.1). However, the set of points with convergent trajectories is topologically a small set (Proposition 4.2.3). Further, Theorem 4.2.5 shows that chaotic piecewise monotonic interval maps are void of such trajectories. Here, two examples of chaotic maps are given, the first of which admits a non-trivial convergent trajectory whereas the second doesn't.

**Example 4.2.1.** Let $T$ be a hyperbolic toral auotomorphism on the torus $\mathbb{T}^2$ (See [7]). Then $(\mathbb{T}^2, T)$ is a chaotic system. If $\mathbf{v}$ is an eigen vector corresponding to the eigen value with modulus less than 1, then the trajectory of $\mathbf{v}$ is not eventually constant but converges to $\mathbf{0}$.

**Example 4.2.2.** (Tent map) Define $f : [0,1] \to [0,1]$ as $f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2-2x & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$.
It can be proved that $([0,1], f)$ is chaotic (See [15]). The only convergent trajectories in $([0,1], f)$ are those which are eventually 0 or $\frac{2}{3}$.

**Proposition 4.2.3.** *If $(X, f)$ is a chaotic system, then the set of points with convergent trajectories has empty interior.*

*Proof.* Since $X$ is transitive we can find $x_0 \in X$ such that the orbit of $x_0$ is dense in $X$. Let $S = \{x \in X : x \text{ has a convergent trajectory}\}$.
If $U \subset S$ is a non-empty open set, then $f^n(x_0) \in U$ for some $n \in \mathbb{N}$. Then $f^n(x_0) = x$ for some $x \in S$, which implies that $(f^{n+k}(x_0))_k$ is a convergent sequence. This leads to a contradiction because it is dense in $X$ and $X$ has several other orbits in it ($X$ has a dense set of periodic points). □

A stronger result follows from the above proof. Loosely speaking, we can say that there is only one kind of dominant trajectory, namely the one with dense orbit. The following remark makes this precise.

**Remark 4.2.4.** If $(X, f)$ is a chaotic system, then the set of points whose trajectories are not dense has empty interior.

In the following theorem, an interval map $(I, f)$ is a dynamical system, where the underlying space $I$ is an interval. $f$ is said to be piecewise monotonic if $I$ can be partitioned in to finitely many subintervals, on each of which $f$ is monotonic.

**Theorem 4.2.5.** *In a chaotic piecewise monotonic interval map, there are no non-trivial convergent trajectories.*

*Proof.* Let $f : I \to I$ be an interval map and say $(f^n(x)) \to l$ for some $x$ and $l$ in $I$, where $(f^n(x))$ is not eventually constant.
Assume that $l$ is not an end point of $I$. Since there are finitely many critical points, $\exists \epsilon > 0$ such that $f$ is monotonic on $[l - \epsilon, l]$ and $[l, l + \epsilon]$ and $[l - \epsilon, l + \epsilon] \neq I$. Choose the least positive integer $n_0$ such that $f^n(x) \in (l - \epsilon, l + \epsilon)$ $\forall n \geq n_0$.

Suppose $f^m(x) \in (l-\epsilon, l)$ for some $m \geq n_0$. If $f$ is increasing on $[l-\epsilon, l]$, then for any $y \in (f^m(x), l)$ we have $f^{n_0+n}(x) \leq f^n(y) \leq l$ $\forall n \in \mathbb{N}$ and thus $f^n((f^m(x), l)) \subset [l-\epsilon, l]$ $\forall n \in \mathbb{N}$.

If $f$ is decreasing on $[l-\epsilon, l]$, then for any $y \in (f^m(x), l)$ we have $f^{m+1}(x) \geq f(y) \geq l$ and thus $f(y) \in [l, l+\epsilon]$. On $[l, l+\epsilon]$, if $f$ is increasing, then by a similar argument as above, we get $f^{n+1}(y) \in [l, l+\epsilon]$ for every $n \in \mathbb{N}$. But, if $f$ is decreasing on $[l, l+\epsilon]$ also, then $f^2(y) \in [l-\epsilon, l]$, $f^3(y) \in [l, l+\epsilon]$ and so on. Thus, if $f$ is decreasing on $[l-\epsilon, l]$ as well as $[l, l+\epsilon]$, the sequence $(f^n(y))$ keeps oscillating on either side of $l$ but remains between $l-\epsilon$ and $l+\epsilon$. In any case, $f^n(f^m(x), l) \subset [l-\epsilon, l+\epsilon]$ $\forall n \in \mathbb{N}$ which contradicts the transitivity of $f$.

On the other hand, let $f^n(x) \in [l, l+\epsilon]$ for every $n \geq n_0$. Here, $f$ will be increasing on $[l, l+\epsilon]$, otherwise $f^{n_0+1}(x)$ would lie in $[l-\epsilon, l]$. Again by similar arguments, it can be proved that $f^n(l, f^{n_0}(x)) \subset [l-\epsilon, l+\epsilon]$ $\forall n \in \mathbb{N}$ which contradicts the transitivity of $f$.

If $l$ is an end point of $I$ we consider either $[l, l+\epsilon]$ or $[l-\epsilon, l]$ on which $f$ is monotonic (in fact, $f$ will be increasing on these intervals) and proceed similarly. $\square$

**Theorem 4.2.6.** *If $(X, f)$ is a positively equicontinuous non-wandering dynamical system, then there is no convergent trajectory in it except that of a fixed point.*

*Proof.* Let $(X, f)$ be a positively equicontinuous non-wandering dynamical system. Suppose there is a convergent trajectory say, $(f^n(p)) \to q$ and $p$ is not a fixed point. Since $p \neq q$, we can choose disjoint open sets $U$ and $V$ such that $p \in U$ and $q \in V$. Fix an $\epsilon > 0$ such that the open ball $B_\epsilon(q)$ with centre $q$ and radius $\epsilon$ is contained in $V$. Since $f$ is positively equicontinuous, we can find $\delta > 0$ such that $d(f^n(x), f^n(y)) < \frac{\epsilon}{2}$ $\forall n \in \mathbb{N}$ whenever $d(x, y) < \delta$. Further, choose $\delta < \epsilon$. Since $(f^n(p)) \to q$, there exists $m \in \mathbb{N}$ such that $f^n(p) \in B_{\frac{\delta}{2}}(q)$ $\forall n \geq m$. Now, choose mutually disjoint neighborhoods

$U'_0$, $U'_1$, . . . ,$U'_{m-1}$ of $p$, $f(p)$, ... $f^{m-1}(p)$ respectively such that $U'_0 \subset U$. Let $U_{m-1}$ be a neighborhood of $f^{m-1}(p)$ such that $U_{m-1} \subset U'_{m-1}$ and $f(U_{m-1}) \subset B_{\frac{\delta}{2}}(q)$. Similarly, we can find neighborhoods $U_k$ of $f^k(p)$ for each $0 \leq k < m-1$ such that $U_k \subset U'_k$ and $f(U_k) \subset U_{k+1}$. Observe that $f^k(U_0) \subset U_k \ \forall \ 0 \leq k \leq m-1$.

We now claim that $f^n(U_0) \cap U_0 = \varphi$ for every $n \in \mathbb{N}$. If $1 \leq n \leq m-1$, then $f^n(U_0) \subset U_n$ and thus $f^n(U_0) \cap U_0 = \varphi$. Also since $f^m(U_0) \subset B_{\frac{\delta}{2}}(q) \subset B_\epsilon(q) \subset V$, $f^m(U_0) \cap U_0 = \varphi$. Now, let $n > m$ and let $x \in U_0$. Then, $d(f^n(x), f^n(p)) = d(f^{n-m+m}(x), f^{n-m+m}(p)) < \frac{\epsilon}{2}$ because $f^m(x)$, $f^m(p) \in B_{\frac{\delta}{2}}(q)$. Thus $d(f^n(x), q) \leq d(f^n(x), f^n(p)) + d(f^n(p), q) < \frac{\epsilon}{2} + \frac{\delta}{2} < \epsilon$ i.e., $f^n(x) \in B_\epsilon(q)$ for any $n > m$. This shows that $f^n(x) \notin U_0$ for any $n \in \mathbb{N}$. This proves the claim as $x$ is chosen arbitrarily.

But the proven claim is in contradiction to the fact that $f$ is non-wandering. Thus there is no convergent trajectory other than that of a fixed point. □

Circle rotations are nice examples of equicontinuous non-wandering systems.

**Theorem 4.2.7.** *There are no non-trivial convergent trajectories in an expanding dynamical system.*

*Proof.* If $(X, f)$ is an expanding dynamical system, then there exist $\epsilon > 0$ and $\mu > 1$ such that $d(f(x), f(y)) \geq \mu d(x, y)$ whenever $d(x, y) < \epsilon$. Suppose there is a non-trivial convergent trajectory say, $(f^n(p))$. Then $\exists n_0 \in \mathbb{N}$ such that $d(f^n(p), f^{n+1}(p)) < \epsilon$ for every $n \geq n_0$. In particular, $d(f^{n_0}(p), f^{n_0+1}(p)) < \epsilon$ ; so $d(f^{n_0+1}(p), f^{n_0+2}(p)) \geq \mu d(f^{n_0}(p), f^{n_0+1}(p))$.

Again $d(f^{n_0+2}(p), f^{n_0+3}(p)) \geq \mu d(f^{n_0+1}(p), f^{n_0+2}(p)) \geq \mu^2 d(f^{n_0}(p), f^{n_0+1}(p))$.

Hence it can be proved that $d(f^{n_0+n}(p), f^{n_0+n+1}(p)) \geq \mu^n d(f^{n_0}(p), f^{n_0+1}(p))$ for any $n \in \mathbb{N}$.

This leads to a contradiction because, on one hand $d(f^{n_0+n}(p), f^{n_0+n+1}(p)) \to 0$ and on the other hand $d(f^{n_0+n}(p), f^{n_0+n+1}(p)) > d(f^{n_0}(p), f^{n_0+1}(p)) > 0$ for every $n \in \mathbb{N}$ owing to the facts that $\mu > 1$ and $f^{n_0}(p) \neq f^{n_0+1}(p)$. □

**Theorem 4.2.8.** *There are no non-trivial convergent trajectories in a positively expansive dynamical system.*

*Proof.* If $(X, f)$ is a positively expansive dynamical system, then there exists $r > 0$ such that for any two distinct points $x$ and $y$, $d(f^n(x), f^n(y)) \geq r$ for some $n \in \mathbb{N}_0$. Suppose there is a non-trivial convergent trajectory say, $(f^n(p)) \to q$. Now, $\exists n_0 \in \mathbb{N}$ such that $d(f^n(p), q) < r$ $\forall n \geq n_0$. This leads to a contradiction because, $f^{n_0}(p) \neq q \Rightarrow d(f^{n_0+n}(p), f^n(q)) = d(f^{n_0+n}(p), q) \geq r$ for some $n \in \mathbb{N}_0$. □

Observe that if $x$ and $y$ are distal points in a system $(X, f)$, then $\exists \epsilon > 0$ such that $d(f^n(x), f^n(y)) \geq \epsilon$ for every $n \in \mathbb{N}$ (See [7]). Thus the following result follows from the above proof.

**Proposition 4.2.9.** *There are no non-trivial convergent trajectories in a distal dynamical system.*

**Convergent trajectories in Proximal systems**

In a system $(X, f)$, a point $y$ is called an $\omega-$limit point of a point $x$ if there is a sequence of natural numbers $(n_k) \to \infty$ (as $k \to \infty$) such that $(f^{n_k}(x)) \to y$ (See [7]). Let $(X, f)$ be a proximal system with a fixed point say $x_0$. For any $x \in X$, there is a sequence $(n_k) \to \infty$ such that $d(f^{n_k}(x), f^{n_k}(x_0)) \to 0$ i.e., $d(f^{n_k}(x), x_0) \to 0$ and thus $(f^{n_k}(x)) \to x_0$. So, the fixed point is an $\omega-$limit point of every point in $X$. This also shows that a proximal system has atmost one fixed point. Contractive homeomorphisms on complete metric spaces are nice examples. Thus, if a proximal system has a convergent trajectory, then the limit of the trajectory is an $\omega-$limit point of every point in $X$.

Let $(X, f)$ be a dynamical system. If the orbit $\{f^n(x) : n \in \mathbb{N}_0\}$ of a point $x \in X$ has finitely many limit points, then we can consider mutually disjoint neighbourhoods of the limit points and each of these contains a subsequence of the trajectory of $x$. Using this or considering directly the system $(X, f^n)$ for arbitrary $n$, the following proposition follows from Theorem 4.2.5, Theorem 4.2.7 and Proposition 4.2.9.

**Proposition 4.2.10.** *If $(X, f)$ is one of the following systems*

1. *Chaotic piecewise monotonic interval maps*

2. *Expanding maps*

3. *Distal maps,*

*then every infinite orbit in $(X, f)$ has infinitely many limit points.*

## 4.3 Summary

It is thus proved that the following systems do not have non-trivial convergent trajectories.

- Chaotic piecewise monotonic interval maps
- Expanding maps
- Distal maps
- Positively equicontinuous non-wandering systems
- Positively expansive maps.

Moreover, in the first three kinds of maps above, every infinite orbit has infinitely many limit points.

# Chapter 5

# Trajectories in chaos

This chapter shows the possibility of existence of a numerous kinds of trajectories in a chaotic system. The two kinds of trajectories that are considered in the earlier chapters, namely the periodic and convergent ones can occur in a chaotic system. In fact the periodic ones do occur; the definition of chaoticity given by Devaney, itself demands a dense set of periodic points. The topological chaos implies the abundance of trajectories. By definition, a system is topologically chaotic if it has positive topological entropy and topological entropy is the exponential growth rate of the number of essentially different orbits of length $n$ (See [7]).

Here, in this chapter, we prove that, given any (allowed) sequence in $[0, 1]$ whose range has finite derived length, it can occur as a trajectory of a chaotic map (Devaney chaotic) on $[0, 1]$. Some other kinds of sequences which occur or do not occur as trajectories of chaotic systems are also discussed. So, this can be considered as an attempt to characterize the trajectories of chaotic interval maps. It is interesting to note that there are "many" sequences which can occur as trajectories of interval maps in general but not of chaotic interval maps.

In this chapter, the first section gives an introduction to chaos, followed by a section on main results. The chapter ends with concluding remarks in Section 5.3.

## 5.1 Introduction to chaos

As the dictionary suggests, the term chaos means total disorder and confusion, in other words unpredictability of a situation. For us, a chaotic system is a highly unpredictable dynamical system. The term chaos became popular through the paper : *Period three implies chaos* by Tien-Yien Li and James A Yorke [16]. There were many attempts to make the definition of chaos, mathematically precise. See [12], [16], [31], [23], [28], [34], [35] and [36] for several definitions of chaos. Here, we consider only the Devaney's definition of chaos. The main ingredient of chaos which intuitively shows the unpredictability is the sensitive dependence on initial conditions. The next thing is the topological transitivity, which makes it impossible to divide the system in to smaller subsystems. Amidst the disorders of unpredictability and indivisibility, there is still a kind of uniformity in Devaney's definition of chaos i.e., there are plenty of periodic orbits. In fact, the set of periodic points is dense in the space.

**Definition 5.1.1.** (See [7] and [25]) A map $f$ on a metric space $(X, d)$ is said to have sensitive dependence on initial conditions provided there is an $r > 0$ such that for each point $x \in X$ and for each $\epsilon > 0$ there is a point $y \in X$ with $d(x, y) < \epsilon$ and a $k \geq 0$ such that $d(f^k(x), f^k(y)) \geq r$.

In the expanding endomorphism $(S^1, E_m)$ of the circle, distance between points $x$ and $y$ is expanded by a factor of $m$ if $d(x, y) < \frac{1}{2m}$, so any two nearby points are moved at least $\frac{1}{2m}$ apart by an iterate of $E_m$, so $E_m$ has sensitive dependence on initial conditions.

**Definition 5.1.2.** (See [12]) A dynamical system $(X, f)$ is said to be Devaney chaotic if

1. $f$ has sensitive dependence on initial conditions
2. $(X, f)$ is topologically transitive and
3. the set of periodic points is dense in $X$.

Hereafter, Devaney chaotic systems will be simply called chaotic systems, as we use only the Devaney chaotic systems in this chapter. The tent map and $h_4$ (logistic map) are chaotic on $I$. Irrational circle rotations are transitive but do not exhibit sensitive dependence on initial conditions and thus not chaotic.

However, it is proved in [5] that the first condition in the above definition is redundant. There is however an assumption in proving this, that $X$ is infinite, which is of course very mild. It is also proved (in [3]) that this is the only redundancy in general i.e., (1) and (2) do not imply (3), (1) and (3) do not imply (2). But if $f$ is an interval map, then transitivity alone is sufficient to establish the chaoticity. This is proved in the paper *On intervals, transitivity = chaos* by Vellekoop and Berglund [40]. They also give examples that there are no other redundancies.

## 5.2 Main Results

We quote the following theorem from [40].

**Theorem 5.2.1.** *Let $I$ be a, not necessarily finite interval and $f : I \to I$ a continuous and topologically transitive map. Then (1)the periodic points of $f$ are dense in $I$ and (2) $f$ has sensitive dependence on initial conditions.*

Thanks to this theorem, it is enough to prove the transitivity of an interval map to prove that it is chaotic. In the following proposition, a way of extending a map from a closed subset of $[0,1]$ to the entire interval $[0,1]$ is given. This is used in constructing a chaotic map in Theorem 5.2.5.

**Proposition 5.2.2.** *Let $S \subset [0,1]$ be a closed set and $f : S \to [0,1]$ be uniformly continuous. Extend $f$ to $[0,1]$ as follows:*

1. *if $x,\ y \in S$ such that $(x,y) \cap S = \varphi$, then define $f$ to be a linear map on $[x,y]$.*

2. *define $f$ continuously piecewise linear on $[0,\ inf\ S]$ and $[sup\ S,\ 1]$ such that the modulus of the slope of each piece is greater than 4.*

*Then $f$ is continuous on $[0,1]$.*

**Remark 5.2.3.** In the statement (2) above, the condition on the slope is not necessary here, but is needed in proving the next theorem, where this proposition is used.

*Proof.* Let $p \in [0,1]$. If $p \notin D(S)$, then in a sufficiently small neighborhood of $p$, $f$ is continuous piecewise linear and thus continuous at $p$.

Suppose $p \in D(S)$. Let $\epsilon > 0$. Choose $\delta > 0$ such that $x,\ y \in S$, $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{2}$.

In case $p \notin \{inf\ S,\ sup\ S\}$, choose $a,\ b \in S$ such that $p \in (a,b)$ and $b - a < \delta$. Let $x \in (a,b)$. If $x \in S$, then $|f(p) - f(x)| < \epsilon$. Otherwise, choose $c, d \in S$ such that $x \in (c,d)$ and $(c,d) \cap S = \varphi$. Clearly, $(c,d) \subset (a,b)$. Thus $|f(c) - f(d)| < \frac{\epsilon}{2}$. Since $f(x)$ lies between $f(c)$ and $f(d)$, $|f(x) - f(d)| < \frac{\epsilon}{2}$. Also, we have $|f(d) - f(p)| < \frac{\epsilon}{2}$. Thus $|f(x) - f(p)| < \epsilon$ and hence $f$ is continuous at $p$.

If $p = inf\ S$, then $f$ is clearly left continuous at $p$. For the right continuity at $p$, consider a neighborhood $(p,b)$ instead of $(a,b)$ in the above argument. Similarly, to prove the left continuity at $sup\ S$, consider $(a, sup\ S)$ instead of $(a,b)$. □

**Remark 5.2.4.** Given a sequence $(p_k)$ in $[0,1]$, the current problem is to construct a chaotic map with $(p_k)$ as a trajectory. In other words, a sequence $(p_k)$ is given along with the map $f(p_k) = p_{k+1}$ on the set $\{p_k : k \in \mathbb{N}\}$ and $f$ has to be extended to a chaotic map on $[0,1]$. It is well-known that $f$ should be necessarily uniformly continuous on $\{p_k : k \in \mathbb{N}\}$ for $f$ to be atleast continuous on $[0,1]$.
So, hereafter, the sequences $(p_k)$, where the map $p_k \mapsto p_{k+1}$ is uniformly continuous on $\{p_k : k \in \mathbb{N}\}$ are called *allowed sequences*.

We follow the following conventions and notations in this chapter. A sequence $(n_k)$ is said to be increasing if $n_k \leq n_{k+1}$ for every $k \in \mathbb{N}$ and it is decreasing if $n_k \geq n_{k+1}$ for every $k \in \mathbb{N}$. If $(n_k)$ is increasing and converging to $n$, then we write $(n_k) \uparrow n$ and if $(n_k)$ is decreasing and converging to $n$, then it is written as $(n_k) \downarrow n$. The phrases *strictly increasing* (respectively *strictly decreasing*) is used to imply that $n_k < n_{k+1}$ (respectively $n_k > n_{k+1}$) for every $k \in \mathbb{N}$. As defined in Chapter 3, we denote by $D(X)$, the set of limit points of $X$ in $X$ and by $X_k$ the set of $k^{th}$ level limit points. The notation $(a,b)$ is used to denote sometimes an ordered pair and sometimes an open interval. However, there is no ambiguity, as the context makes the matter clear.

### 5.2.1 A method of extending $f$

Let $[a,b] \subset [0,1]$. Given a countable discrete set $X \subset [a,b]$ having a unique limit point and a uniformly continuous function $f : X \to [0,1]$, we give here a particular method of defining a continuous map from $\overline{X} \cup Z$ to $[0,1]$ where $Z$ is a countable subset of $[0,1]$ defined, based on the elements of $X$. Actually, $Z$ is the set of midpoints of terms of sequences defined in an earlier step in the method and $Z$ will be referred briefly to, as *sequence of midpoints*. This method will be used extensively in the proof of next theorem, where it will be referred to, as simply *the method*.

***The Method :***

Let $D(X) = \{p\}$. Note that $f$ is defined at $p$, say $f(p) = q$.

If $p \neq a$ and if $X$ does not contain a strictly increasing sequence, then choose an arbitrary strictly increasing sequence in $(a, p)$ converging to $p$, including the points of $X$ if any. Let $X'$ be the range of this sequence. Extend $f$ continuously to $X \cup X'$. A similar thing can be done if $p \neq b$ and if there is no strictly decreasing sequence. Thus, we can assume that $X$ contains a strictly increasing as well as a strictly decreasing sequence both converging to $p$, unless $p \in \{a, b\}$.

Let $(x_k)$ and $(y_k)$ be the sequences whose terms constitute $X$ such that $(x_k) \uparrow p$, $(y_k) \downarrow p$. If $p = a$, take $x_k = a$ for every $k \in \mathbb{N}$. Otherwise, take $(x_k)$ to be strictly increasing. If $p = b$, take $y_k = b$ for every $k \in \mathbb{N}$. Otherwise, take $(y_k)$ to be strictly decreasing.

Choose two sequences $(r_k)$ and $(s_k)$ such that $(r_k) \uparrow q$ and $(s_k) \downarrow q$.

Let $a_k = \frac{x_k + x_{k+1}}{2}$ and $b_k = \frac{y_k + y_{k+1}}{2}$. Now, $f$ is extended to the set $\overline{X} \cup Z$, where $Z = \{a_k : k \in \mathbb{N}\} \cup \{b_k : k \in \mathbb{N}\}$ in the following way.

(1)If $p = a$, then $a_k = a$ for every $k \in \mathbb{N}$, in which case, $f(a_k)$ is already defined. Otherwise, define $f(a_k) = \begin{cases} r_{\frac{k}{2}} & \text{if } k \text{ is even} \\ s_k & \text{if } k \text{ is odd} \end{cases}$.

(2)If $p = b$, then $b_k = b$ for every $k \in \mathbb{N}$, in which case, $f(b_k)$ is already defined. Otherwise, define $f(b_k) = \begin{cases} s_{\frac{k}{2}} & \text{if } k \text{ is even} \\ r_k & \text{if } k \text{ is odd} \end{cases}$.

**Theorem 5.2.5.** *If $(p_k)$ is an allowed sequence in $[0, 1]$ such that its range has finite derived length, then there exists a chaotic map $f$ on $[0, 1]$ with $(p_k)$ as a trajectory.*

*Proof.* Let $X$ be the range of the given sequence and let $n$ be the derived length of $X$. Define $f : X \to X$ as $f(p_k) = p_{k+1}$ and extend it to $\overline{X}$. Observe that $X$ is a discrete set and also that $X_i$ is forward $f-$invariant for each $0 \leq i \leq n$. We extend this map $f$

to $[0, 1]$ in the following $n + 1$ steps. Before that, say $X_n = \{m_1, m_2, .., m_l\}$ such that $m_i < m_{i+1}$ for every $1 \leq i < l$ and let $n_1 = 0$, $n_{l+1} = 1$ and $n_i = \frac{m_{i-1}+m_i}{2}$ for $2 \leq i \leq l$.

*Step 1 :*

Each of the sets $\overline{X_{n-1}} \cap [n_i, n_{i+1}]$ is a set with unique limit point, namely $m_i$. So, *the method* can be applied to each of these, taking $a = n_i$ and $b = n_{i+1}$. As done in *the method*, choose an increasing and a decreasing sequence in $\overline{X_{n-1}} \cap [n_i, n_{i+1}]$ both converging to $m_i$. Then for each $1 \leq i \leq l$, we define *the sequences of midpoints* as done in *the method*. Let $X^{(1)}$ be the union of the ranges of these *sequences of midpoints*. Now, fix an $i \in \{1, 2, ...l\}$. Let $f(m_i) = m_j$. Now, apply *the method* to the set $\overline{X_{n-1}} \cap [n_i, n_{i+1}]$, by choosing $r_1 = 0$, $s_1 = 1$ and $(r_k)_{k=2}^{\infty}$, $(s_k)_{k=2}^{\infty}$ to be the sequences that constitute $X^{(1)} \cap [n_j, n_{j+1}]$. Doing this for every $i$, $f$ is thus defined on the set $X^{(1)}$. Define $\mathfrak{X}^1 = \{(a, b) \in X^{(1)} \times X^{(1)} : a$ and $b$ are consecutive terms of one of these sequences such that $a \neq b$ $\}$. Note that every point of $X_{n-1}$ lies in some interval $[x, y]$, where $(x, y) \in \mathfrak{X}^1$ and on the other hand, each such interval $[x, y]$ contains exactly one point of $X_{n-1}$.

*Step 2:*

Consider the collection: $\{\overline{X_{n-2}} \cap [x, y] : (x, y) \in \mathfrak{X}^1\}$. *The method* can be applied to each member of the collection. First define the sequences of midpoints for all the sequences as done in *the method* and let $X^{(2)}$ be the union of the ranges of these *sequences of midpoints*. Now, consider a member of the above collection, say $\overline{X_{n-2}} \cap [x, y]$, $(x, y) \in \mathfrak{X}^1$. This contains a unique point of $X_{n-1}$, say $p$. Apply *the method* to this set, choosing the sequences $(r_k)$ and $(s_k)$ such that $r_1 = \frac{x'}{2}$, $s_1 = \frac{y'+1}{2}$ and $(r_k)_{k=2}^{\infty}$, $(s_k)_{k=2}^{\infty}$ are the sequences that constitute $X^{(2)} \cap [x', y']$, where $(x', y') \in \mathfrak{X}^1$ such that $f(p) \in [x', y']$. Apply *the method* in the same way for each member of the collection. Thus $f$ is defined on $X^{(2)}$. Also, define $\mathfrak{X}^2 = \{(a, b) \in X^{(2)} \times X^{(2)} : a$ and $b$ are consecutive terms of one of these sequences, i.e., the sequences defined in Step 2$\}$.

In general, for any $2 \leq k \leq n$, having completed the step $k-1$, we do the following in Step $k$. First define the sequences of midpoints for all the sequences as done in *the method*, considering separately each member of the collection $\{\overline{X_{n-k}} \cap [x,y] : (x,y) \in \mathfrak{X}^{k-1}\}$, where $\mathfrak{X}^{k-1} = \{(a,b) \in X^{(k-1)} \times X^{(k-1)}$ : $a$ and $b$ are consecutive terms of one of the sequences defined in Step $k-1$ $\}$. Apply *the method* to each member of the collection, choosing the sequences $(r_k)$ and $(s_k)$, as done in Step 2. Thus $f$ is defined on the closed set $Y = \overline{X} \bigcup (\cup_{i=1}^{n} X^{(i)})$.

Consider the set $\overline{X_0} \cap [x,y]$, for some $(x,y) \in \mathfrak{X}^{n-1}$. This consists of an increasing and a decreasing sequence. Let $(u_k)$ be the increasing sequence here. By the definition of $X^{(n)}$, it follows that the terms of these sequences are elements of the set $X^{(n)} \cap [x,y]$. If, for some $k \in \mathbb{N}$, the modulus of slope of the line segment joining $(u_k, f(u_k))$ and $(u_{k+1}, f(u_{k+1}))$ is atmost 4, then introduce two more terms $u'_k = \frac{u_k + u_{k+1}}{2}$ and $u'_{k+1} = \frac{u'_k + u_{k+1}}{2}$ and define $f(u'_k) = f(u_{k+1})$ and $f(u'_{k+1}) = f(u_k)$. Now, consider the sequence $(u_1,\ u_2\ , .., u_k, u'_k, u'_{k+1}, u_{k+1}, ..)$. Repeat the process, if necessary, until the above discussed slope has modulus greater than 4, for every pair of consecutive terms. This is done for each of the sequences (both increasing and decreasing) that constitute $X^{(n)}$. Observe that we need to introduce atmost finitely many points to achieve this. So, we can assume for the rest of the proof, without affecting the closedness of $Y$, that if $x$ and $y$ are consecutive terms of any of the sequences that constitute $X^{(n)}$, the slope of the line segment joining $(x, f(x))$ and $(y, f(y))$ is greater than 4 in absolute value.

In the final step, extend the function $f$ to $[0,1]$ as done in the above proposition. Note that $(p_k)$ is a trajectory of $f$.

It is now claimed that $f$ is chaotic. Before proving this, following are four important observations that will be used in the proof later.

1. The graph of $f$ consists of line segments, each of slope greater than 4 in absolute value.

2. $Y$ is the set of critical points of $f$.

3. If $x, y \in X^{(1)}$ such that $x$ and $y$ are the first terms of the increasing and the decreasing sequences respectively in $X^{(1)} \cap [n_i, n_{i+1}]$ for some $i$, then $f(x) = 0$ and $f(y) = 1$. So, if $(a, b)$ is a subinterval containing $x$ and $y$, then $f(a, b) = [0, 1]$.

4. If $u, v \in X_m$, $m < n$, such that $u$ and $v$ are the first terms of an increasing and a decreasing sequence respectively, defined newly in Step $(n - m)$ (i.e., *the sequences of midpoints*), then $f(u) = \frac{x'}{2}$ and $f(v) = \frac{1+y'}{2}$ for some $(x', y') \in \mathfrak{X}^{n-m-1}$. Further, $x'$ and $y'$ are terms of sequences in $X^{(n-m-1)}$ converging to a same point, say $p$. Then $f(p)$ lies between $f(x')$ and $f(y')$. Note that $p, f(p) \in X_{m+1}$. So, if $(a, b)$ is a subinterval containing $u$ and $v$, then $[x', y'] \subset f(a, b)$ and $f^2(a, b) \cap X_{m+1} \neq \varphi$.

Recall that $X_n = \{m_1, m_2, ..., m_l\}$ is a finite $f-$invariant set and $m_i$ is the unique element of $X_n$ in $[n_i, n_{i+1}]$. Observe that each set $X^{(1)} \cap (n_i, n_{i+1})$ consists of an increasing and a decreasing sequence, say $(y_k^{(i)})$ and $(z_k^{(i)})$ respectively, both converging to $m_i$.

Let $(a, b) \subset [0, 1]$. We prove that $f^l(a, b) = [0, 1]$ for some $l$, which establishes the transitivity of the system and thus chaoticity.

Since $X_n$ is a finite $f-$invariant set, it has atleast one periodic point. Suppose that $X_n \cap (a, b) \neq \varphi$. First consider the case, where $m_i \in (a, b)$ for some $i$ and $m_i$ is periodic with period, say $k$. We can assume that $\{m_i, m_{i+1}, ..., m_{i+k-1}\}$ is the orbit of $m_i$. (i.e., $i$ is the least index in the orbit of $m_i$). If $i$ is not the least index in the orbit of $m_i$, then $f^j(m_i)$ for some $j$ has this property and thus we can replace $(a, b)$ by $f^j(a, b)$ and proceed in the similar way.

Choose the least positive integer $m$ such that $y^{(i)}_{2^m},\ z^{(i)}_{2^m} \in (a,b)$. It follows by the definition of $f$ that $[y^{(i')}_{2^m-1},\ z^{(i')}_{2^m-1}] \subset f(a,b)$ where $m_{i'} = f(m_i)$. Iterating further, we get $[y^{(j)}_1, z^{(j)}_1] \subset f^m(a,b)$ for some $j$ and thus $f^{m+1}(a,b) = [0,1]$.

If $(a,b)$ does not contain any periodic point, then the finiteness and $f-$invariance of $X_n$ force that $f^j(a,b)$ contains a periodic point for some $j$ and then the proof follows by replacing $(a,b)$ by $f^j(a,b)$ in the above case.

Suppose that $X_n \cap (a,b) = \varphi$ but $D(X) \cap (a,b) \neq \varphi$. Choose the least positive integer $m$ such that $(a,b) \cap X_{m+1} = \varphi$. Choose $q \in (a,b) \cap X_m$. There exists a unique ordered pair $(x,y) \in \mathfrak{X}^{n-m}$ such that $q \in [x,y]$. We applied *the method* to the set $\overline{X_{m-1}} \cap [x,y]$ in the Step $(n-m+1)$ to get *the sequences of midpoints*, say $(u_k)$ and $(v_k)$ respectively such that $(u_k) \uparrow q$ and $(v_k) \downarrow q$ and the terms of these sequences constitute the set $X^{(n-m+1)} \cap (x,y)$.

Choose the least integer $l$ such that $[u_{2^l}, v_{2^l}] \subset (a,b)$. By a similar argument as done in the previous case, it can be proved that $[u'_1, v'_1] \subset f^l(a,b)$ for some $u'_1, v'_1 \in X^{(n-m+1)}$. (Note that $u'_1$ and $v'_1$ are the first terms of some sequences $(u'_k)$ and $(v'_k)$, whose ranges are contained in the set $X^{(n-m+1)}$). It follows from the definition of $f$ at $u'_1$ and $v'_1$ that $[x', y'] \subset f^{l+1}(a,b)$ for some $(x', y') \in \mathfrak{X}^{n-m}$. Say, $q'$ is the limit of the sequence in $X^{(n-m)}$, of which $x'$ and $y'$ are terms. Then, $q' \in X_{m+1}$ and $f(q')$ lies between $f(x')$ and $f(y')$ and thus $f(q') \in f^{l+2}(a,b)$. Moreover, $f(q') \in X_{m+1}$. Thus, $f^{l+2}(a,b) \cap X_{m+1} \neq \varphi$. Repeating the above argument by choosing a point in $f^{l+2}(a,b) \cap X_{m+1}$, we get $f^{l'}(a,b) \cap X_{m+2} \neq \varphi$ for some $l' \in \mathbb{N}$ and thus finally $f^r(a,b) \cap X_n \neq \varphi$ for some $r \in \mathbb{N}$. Then from the above case, it follows that $f^{r'}(a,b) = [0,1]$ for some $r' \in \mathbb{N}$.

Now, let $D(X) \cap (a,b) = \varphi$. If there are two distinct points $x, y \in X^{(n)} \cap (a,b)$, then by the definition of $f$ on $X^{(n)}$, $f(a,b) \cap X_1 \neq \varphi$ and this falls under the previous case.

So, it is enough to prove that there are atleast two distinct points of $X^{(n)}$ in $f^l(a,b)$ for some $l \in \mathbb{N}$.

If $(a,b)$ contains atmost one point of $X^{(n)}$, then $(a,b)$ can be divided in to atmost four subintervals, on each of which $f$ is linear. Choose a maximal such subinterval, say $(c,d)$. We have $d-c \geq \frac{1}{4}(b-a)$. Since the modulus of the slope of the graph of $f$ on $(c,d)$ is greater than 4, we have $|f(d)-f(c)| > 4(d-c) \geq (b-a)$. Thus the length of the interval $f(a,b)$ is greater than the length of $(a,b)$. By iterating, the length keeps on increasing, till it contains atleast two points of $X^{(n)}$ i.e., the cardinality of the set $X^{(n)} \cap f^l(a,b)$ is atleast 2, for some $l \in \mathbb{N}$. □

In the next proposition, we give a class of allowed sequences which cannot occur as trajectories of chaotic maps i.e., they do occur as trajectories of some interval maps in general, but not as trajectories of chaotic maps. This class consists of those allowed sequences whose ranges are somewhere-dense but not dense in $[0,1]$. A subset $Y$ of a topological space $X$ is said to be somewhere-dense in $X$ if the closure of $Y$ in $X$ has non-empty interior.

**Proposition 5.2.6.** *Let $(x_n)$ be an allowed sequence in $[0,1]$.*

1. *If $\{x_n : n \in \mathbb{N}\}$ is dense in $[0,1]$, then any interval map with $(x_n)$ as a trajectory is chaotic.*

2. *If $\{x_n : n \in \mathbb{N}\}$ is somewhere-dense but not dense in $[0,1]$, then no interval map with $(x_n)$ as a trajectory is chaotic.*

*Proof.* The proof of the first statement is obvious from the fact that $\{x_n : n \in \mathbb{N}\}$ is a dense orbit of the map.

Now, suppose the set $X = \{x_n : n \in \mathbb{N}\}$ is somewhere-dense but not dense in $[0,1]$. Let $f$ be an interval map with $(x_n)$ as a trajectory. It can be easily seen that the closure of any orbit is $f-$invariant; so, $\overline{X}$ is $f-$invariant. As $X$ is somewhere-dense, we can choose a non-empty open set $U \subset \overline{X}$. Then $f^k(U) \subset \overline{X}$ for every $k \in \mathbb{N}$. Observe that $[0,1] \setminus \overline{X}$ is a non-empty open set and $f^k(U) \cap ([0,1] \setminus \overline{X}) = \varphi$ for every $k \in \mathbb{N}$, which shows that $f$ is not transitive and thus not chaotic. $\square$

## 5.3 Conclusion

It is thus proved that any allowed sequence whose range has finite derived length can arise as a trajectory of some chaotic map. It is also observed that if the range of an allowed sequence is dense, then it occurs as a trajectory of only chaotic maps and if it is somewhere-dense but not dense, then any interval map with this sequence as a trajectory is not chaotic. This also shows that there are plenty of sequences that can arise as trajectories of some interval maps in general, but not as trajectories of chaotic interval maps. We thus conclude the chapter with the following natural question : Should every allowed sequence whose range is a scattered space occur as a trajectory of some chaotic map?

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