LOCAL CONSTANTS FOR GALOIS REPRESENTATIONS - SOME EXPLICIT RESULTS

A thesis submitted during 2015 to the University of Hyderabad in partial fulfillment of the award of a **Doctor of Philosophy degree** in **Mathematics**

by

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August 2015

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Dedication

To my parents

Chhakina Bibi, Mainuddin Biswas

Acknowledgments

I would like to thank my advisor Prof. Rajat Tandon for teaching me so many things and for his patience, unending encouragements and helps. I extend my gratitude to him for allowing me to visit Berlin during my Ph.D.

I would also like to thank Prof. E.-W. Zink, Humboldt University, Berlin, for suggesting the problems in the thesis and his constant guidance, valuable suggestions and comments. Without his helps, this thesis work would not have been possible. Many ways I am indebted to him. I also express my sincere gratitude to Prof. Elmar Grosse-Klönne, Humboldt University, for providing me a very good mathematical environment during my stay in Berlin.

In the last year of my Ph.D program, I had very useful discussion with Prof. Dipendra Prasad, TIFR, India. He is always very kind to me. I express my deep gratitude to him for his constant support. I also express my gratitude to Prof. Robert Boltje, Department of Mathematics, University of California, Santa Cruz, USA for sending me hard copy of his Ph.D thesis and paper which were very much useful for my works.

I would also like to thank Prof. T. Amaranath (former dean of School of Mathematics and Statistics) for his all kind of help and encouragement. Many thanks to my doctoral committee members- Prof. V. Kannan and Prof. V. Suresh (currently Professor at Emory University, USA) for their advices, discussions and valuable comments.

I express my very deep gratitude to Prof. Kumaresan and all the members of School of Mathematics and Statistics of University of Hyderabad.

I am grateful to CSIR, Delhi and IMU-Berlin Einstein Foundation, Berlin for their financial support during my Ph.D.

I would also like to thank all my colleagues, specially to Deborjoti for his help and very good friendship which made my life enjoyable in Hyderabad. I also thank to Niraj, Srikant, Sampath, Rakhee, Ambika, Chiru, Sharan and Bharbhava for their love and beautiful company and making my stay very easy in Hyderabad.

I am grateful to all my teachers during my school days. At school, specially I thank my teachers Mr. Nurul Islam (General Secretary, Al-Ameen Mission, Howrah), Dr. Musarraf Hossain (Chemistry), Dr. Mrinal Kanti Dwari (Physics), Mr. Sk Hasibul Alam (Maths) and Mr. Amir Hossain Mandal (Maths) for their love, encouragement and very good teaching. At my undergraduate and post graduate level (Jadavpur University, Calcutta, India), I would like to thank Dr. Sujit Kumar Sardar, Dr. Kallol Paul, Dr. Shamik Ghosh and Dr. Indranath Sengupta for their beautiful teaching.

I would also like to thank all my friends, Sudip, Prabir, Somnath, Angshuman, Atanu, Tapas, Subhodip, Tilak, Prabin, Khan, Nasarul, Hira, Biswa, Ekta, for their very good company throughout my life.

Finally, I want to thank my family, specially my elder brother, Manajat Ali Biswas, for their constant unconditional support, encouragement and freedom to achieve any goal to which I desire.

Abstract

We can associate local constant to every continuous finite dimensional complex representation of the absolute Galois group G_F of a non-archimedean local field $F/\mathbb{Q}_p$ by Deligne and Langlands. For linear characters of $F^\times$, Tate [32] gives an explicit formula for the abelian local constants, but in general, there is no such explicit formula of local constant for any arbitrary representation of a local Galois group. To give explicit formula of local constant of a representation, we need to compute λ-functions explicitly. In this thesis we compute $\lambda_{K/F}$ explicitly, where K/F is a finite degree Galois extension of a non-archimedean local field F, except when K/F is a wildly ramified quadratic extension with $F \neq \mathbb{Q}_2$.

Then by using this λ-function computation, in general, we give an invariant formula of local constant of finite dimensional Heisenberg representations of the absolute Galois group G_F of a non-archimedean local field F. But for explicit invariant formula of local constant for a Heisenberg representation, we should have information about the dimension of a Heisenberg representation and the arithmetic description of the determinant of a Heisenberg representation. In this thesis, we give explicit arithmetic description of the determinant of Heisenberg representation.

We also construct all Heisenberg representations of dimensions prime to p, and study their various properties. By using λ-function computation and arithmetic description of the determinant of Heisenberg representations, we give an invariant formula of local constant for a Heisenberg representation of dimension prime to p.

Keywords: Local Fields, Extendible functions, Local constants, Lambda functions, Classical Gauss sums, Heisenberg representations, Transfer map, Determinant, Artin conductors, Swan conductors.

Notation

By $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ we denote the natural numbers, integers, rational numbers, real numbers and complex numbers respectively.

For two integers m, n, we denote $gcd(m, n)$ and lcm(m, n) as the greatest common divisor and least common multiple of m, n respectively.

For two sets A and B, $A \subset B$ means A is contained in B. The number of elements in a finite set A will be denoted by $|A|$.

For any homomorphism (group or ring) φ, we denote $\mathrm{Ker}(\varphi) =$ kernel of φ and $\mathrm{Im}(\varphi) =$ image of the homomorphism φ.

Representation of a group means complex representation, otherwise it will be stated. For a group G we denote by $\widehat{G}$ the group of linear characters of G, by $\mathrm{Irr}(G)$ the set of irreducible representations of G, by PI(G) the set of all projective irreducible representation of G, by $Z(G)$ the center of G and by $[G, G]$ the commutator subgroup of G.

For a group G, $H \leqslant G$, $H < G$, $H \lhd G$ denotes that H is a subgroup, a proper subgroup, a normal subgroup of G and for index of H in G, we write $[G : H]$.

For a representation ρ of G, if $H \leqslant G$ we denote by $\rho|_H$ the restriction of ρ to H. For G-set we denote by M^G the set of G-invariant elements of M. For representation ρ of a group, we define $\dim \rho =$ dimension of ρ.

Throughout this thesis, F will denote a non-archimedean local field, that is a finite extension of $\mathbb{Q}_p$ for some prime p. We write O_F for the ring of integers in F, P_F for the prime ideal in O_F and ν_F for the valuation of F. The residue field O_F/P_F is denoted by k_{q_F}, and its number of elements is denoted by q_F. We write U_F for $O_F^\times$, i.e., the group of units of F and $\overline{F}$ for an algebraic closure of F.

The degree of finite fields extension K/F is denoted by $[K : F]$; if the extension is Galois, we write $\mathrm{Gal}(K/F)$ for the Galois group of K/F. We denote $\det(\rho)$ for the determinant of the representation ρ. We also denote $\Delta_{K/F}$ for the determinant of the representation $\mathrm{Ind_{K/F}}(1)$. $N_{K/F}$ denotes the norm map from $K^\times$ to $F^\times$ and $\mathrm{Tr_{K/F}}$ for the trace map from K to F.

We denote μ_{p^∞} as the group of roots of unity of p-power order, and $T_{G/H}$ as the transfer map from $G \to H/[H, H]$.

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Chapter 1

Introduction

Let F be a non-archimedean local field (i.e., finite extension of the p-adic field $\mathbb{Q}_p$, for some prime p). Let $\overline{F}$ be an algebraic closure of F, and $G_F := \mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F})$ be the absolute Galois group of F. Let

$$\rho : G_F \to \mathrm{Aut}_{\mathbb{C}}(V)$$

be a finite dimensional continuous complex representation of the Galois group G_F. For this ρ, we can associate a constant $W(\rho)$ with absolute value 1 by Langlands (cf. [41]) and Deligne (cf. [38]). This constant is called the **local constant** (also known as local epsilon factor) of the representation ρ. Langlands also proves that these local constants are weakly extendible functions (cf. [32], p. 105, Theorem 1).

The existence of this local constant is proved by Tate for one-dimensional representation in [34] and the general proof of the existence of the local constants is proved by Langlands (see [41]). In 1972 Deligne also gave a proof using global methods in [38]. But in Deligne's terminology this local constant $W(\rho)$ is $\epsilon_D(\rho, \psi_F, \mathrm{dx}, 1/2)$, where dx is the Haar measure on F^+ (locally compact abelian group) which is self-dual with respect to the canonical additive character ψ_F of F. Tate in his article [33] denotes this Langlands convention of local constants as $\epsilon_L(\rho, \psi)$. According to Tate (cf. [33], p. 17), the Langlands factor $\epsilon_L(\rho, \psi)$ is $\epsilon_L(\rho, \psi) = \epsilon_D(\rho\omega_{\frac{1}{2}}, \psi, \mathrm{dx}_\psi)$, where ω denotes the normalized absolute value of F, i.e., $\omega_{\frac{1}{2}}(x) = |x|_F^{\frac{1}{2}} = q_F^{-\frac{1}{2}\nu_F(x)}$ (in Section 2.3 we discuss them in more details) which we may consider as a character of $F^\times$, and where dx_ψ is the self-dual Haar measure corresponding to the additive character ψ and q_F is the cardinality of the residue field of F. According to Tate (cf. [32], p. 105) the relation among three conventions of the local constants is:

$$W(\rho) = \epsilon_L(\rho, \psi_F) = \epsilon_D(\rho\omega_{\frac{1}{2}}, \psi_F, \mathrm{dx}_{\psi_\mathrm{F}}). \tag{1.0.1}$$

In Chapter 2 we recall almost all necessary ingredients for this thesis, and they will be used in our next chapters. First we mention the basic properties of local fields and their extensions. We study the Lemma 2.1.2 of J-P. Serre and it will be used in Chapter 3. This result has great role for the computation of λ-functions. The central keywords of this thesis are the local constants and Heisenberg representations. Therefore in Chapter 2 we spend little more time on them. All properties of local constants are well-studied by various people: Langlands [41], Tate [32], Deligne [38], Bushnell-Henniart [6]. We show the connection between all conventions of local constants. We also know that there is a connection between the classical Gauss sums and the local constants of representations of a local Galois group. When the conductor of a multiplicative character of a local field is one, then by proper choice of additive character we see that the computation of local constants comes down to the computation of the classical Gauss sums. Therefore for our lambda-functions computation we need the Theorem 2.4.1 (regarding classical Gauss sum of quadratic character). In Section 2.6 we discuss the group theoretical structure of Heisenberg representations, and for this we follow the articles [11], [12], [14].

In Chapter 3 we compute the lambda-functions for finite Galois extensions explicitly, except wildly ramified quadratic extensions. Let K/F be a finite Galois extension of a non-archimedean local field $F/\mathbb{Q}_p$ and $G = \mathrm{Gal(K/F)}$. The lambda function for the extension K/F is $\lambda_{K/F}(\psi_F) := W(\mathrm{Ind}_{K/F}1, \psi_F)$, where W denotes the local constant and ψ_F is the canonical additive character of F. We also can define λ-function via Deligne's constant $c(\rho) := \frac{W(\rho)}{W(\det(\rho))}$, where ρ is a representation of G and

$$\lambda_1^G = W(\rho) = c(\rho)\cdot W(\det(\rho)) = c(\rho)\cdot W(\Delta_1^G),$$

where $\rho = \mathrm{Ind_1^G}(1)$ and $\Delta_1^G := \det(\mathrm{Ind_1^G}(1))$.

Firstly, in Section 3.2 we compute the λ-function for odd degree Galois extension by using some functoriality properties of λ-functions and the following lemmas (cf. Lemma 3.2.1, Lemma 3.2.2)

Lemma 1.0.1. *Let L/F be a finite Galois extension with $H = \mathrm{Gal(L/K)}$ and $G = \mathrm{Gal(L/F)}$. If $H \leqslant G$ is normal subgroup and if $[G : H]$ is odd, then $\Delta_{K/F} \equiv 1$ and $\lambda_{K/F}^2 = 1$.*

Lemma 1.0.2. *1. If $H \leqslant G$ is normal subgroup of odd index $[G : H]$, then $\lambda_H^G = 1$.*

2. If there exists a normal subgroup N of G such that $N \leqslant H \leqslant G$ and $[G : N]$ odd, then $\lambda_H^G = 1$.

we obtain the following result (cf. Theorem 3.2.3)

Theorem 1.0.3. *Let F be a non-archimedean local field and E/F be an odd degree Galois extension. If $L \supset K \supset F$ be any finite extension inside E, then $\lambda_{L/K} = 1$.*

And in Section 3.3 we compute λ_1^G, where G is a local Galois group for a finite Galois extension. By using Bruno Kahn's results (cf. [4], Theorem 1) and Theorem 3.1.2 (due to Deligne) we obtain the following result (cf. Theorem 3.3.10).

Theorem 1.0.4. *Let G be a finite local Galois group of a non-archimedean local field F. Let S be a Sylow 2-subgroup of G. Denote $c_1^G = c(\mathrm{Ind_1^G}(1))$.*

1. If $S = \{1\}$, then we have $\lambda_1^G = 1$.

2. *If the Sylow 2-subgroup $S \subset G$ is nontrivial cyclic (**exceptional case**), then*

$$\lambda_1^G = \begin{cases} W(\alpha) & \text{if } |S| = 2^n \geqslant 8 \\ c_1^G \cdot W(\alpha) & \text{if } |S| \leqslant 4, \end{cases}$$

where α is a uniquely determined quadratic character of G.

3. *If S is metacyclic but not cyclic (**invariant case**), then*

$$\lambda_1^G = \begin{cases} \lambda_1^V & \text{if } G \text{ contains Klein's } 4 \text{ group } V \\ 1 & \text{if } G \text{ does not contain Klein's } 4 \text{ group } V. \end{cases}$$

4. *If S is nontrivial and not metacyclic, then $\lambda_1^G = 1$.*

From the above theorem we observe that $\lambda_1^G = 1$, except the **exceptional case** and the **invariant case** with G contains Klein's 4-group. Moreover, here α is the uniquely determined quadratic character of G, then $W(\alpha) = \lambda_{F_2/F}$, where F_2/F is the quadratic extension corresponding to α of $G = \mathrm{Gal(K/F)}$. In fact, in the invariant case we need to compute λ_1^V, where V is Klein's 4-group. If $p \neq 2$ then V corresponds to a tame extension and there we have the explicit computation of λ_1^V in Lemma 3.3.9.

Lemma 1.0.5. *Let $F/\mathbb{Q}_p$ be a local field with $p \neq 2$. Let K/F be the uniquely determined extension with $V = \mathrm{Gal(K/F)}$, Klein's 4-group. Then*
$\lambda_1^V = \lambda_{K/F} = -1$ if $-1 \in F^\times$ is a square, i.e., $q_F \equiv 1 \pmod 4$, and
$\lambda_1^V = \lambda_{K/F} = 1$ if $-1 \in F^\times$ is not a square, i.e., if $q_F \equiv 3 \pmod 4$,
where q_F is the cardinality of the residue field of F.

In [39] Deligne computes the Deligne's constant $c(\rho)$, where ρ is a finite dimensional orthogonal representation of the absolute Galois group G_F of a non-archimedean local field F via second Stiefel-Whitney class of ρ. The computations of these Deligne's constants are important to give explicit formulas of lambda functions. But the second Stiefel-Whitney classes $s_2(\rho)$ are not the same as Deligne's constants, therefore full information of $s_2(\rho)$ will not give complete information about Deligne's constants. Therefore to complete the explicit computations of λ_1^G we need to use the definition

$$\lambda_1^G = \lambda_{K/F}(\psi) = W(\mathrm{Ind_{K/F}}1, \psi),$$

where ψ is a nontrivial additive character of F. When we take the canonical additive character ψ_F, we simply write $\lambda_{K/F} = W(\mathrm{Ind_{K/F}}1, \psi_\mathrm{F})$, instead of $\lambda_{K/F}(\psi_F)$.

When $p \neq 2$, in Theorem 3.3.10, we notice that to complete the whole computation we need to compute $\lambda_{K/F}$, where K/F is a quadratic tame extension. In general, in Section 3.4 we study the explicit computation for even degree local Galois extension.

Moreover, by using the properties of λ-function and Lemma 3.4.3 we give general formula of $\lambda_{K/F}$, where K/F is an even degree **unramified extension** and the result is (cf. Theorem 3.4.5):

$$\lambda_{K/F}(\psi_F) = (-1)^{n(\psi_F)}.$$

When K/F is an even degree Galois extension with odd ramification index we have the following result (cf. Theorem 3.4.7).

Theorem 1.0.6. *Let K be an even degree Galois extension of F with odd ramification index. Let ψ be a nontrivial additive character of F. Then*

$$\lambda_{K/F}(\psi) = (-1)^{n(\psi)}.$$

When K/F is a tamely ramified quadratic extension, by using classical Gauss sum we have an explicit formula (cf. Theorem 3.4.10) for $\lambda_{K/F}$.

Theorem 1.0.7. *Let K be a tamely ramified quadratic extension of $F/\mathbb{Q}_p$ with $q_F = p^s$. Let ψ_F be the canonical additive character of F. Let $c \in F^\times$ with $-1 = \nu_F(c) + d_{F/\mathbb{Q}_p}$, and $c' = \frac{c}{\mathrm{Tr_{F/F_0}(pc)}}$, where $F_0/\mathbb{Q}_p$ is the maximal unramified extension in $F/\mathbb{Q}_p$. Let ψ_{-1} be an additive character with conductor -1, of the form $\psi_{-1} = c' \cdot \psi_F$. Then*

$$\lambda_{K/F}(\psi_F) = \Delta_{K/F}(c') \cdot \lambda_{K/F}(\psi_{-1}),$$

where

$$\lambda_{K/F}(\psi_{-1}) = \begin{cases} (-1)^{s-1} & \text{if } p \equiv 1 \pmod 4 \\ (-1)^{s-1} i^s & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

If we take $c = \pi_F^{-1-d_{F/\mathbb{Q}_p}}$, where π_F is a norm for K/F, then

$$\Delta_{K/F}(c') = \begin{cases} 1 & \text{if } \overline{\mathrm{Tr_{F/F_0}(pc)}} \in k_{F_0}^\times = k_F^\times \text{ is a square,} \\ -1 & \text{if } \overline{\mathrm{Tr_{F/F_0}(pc)}} \in k_{F_0}^\times = k_F^\times \text{ is not a square.} \end{cases} \tag{1.0.2}$$

Here "overline" stands for modulo P_{F_0}.

By the properties of λ-function and the above theorem, we can give complete computation of $\lambda_{K/F}$, where K/F is a tamely ramified even degree Galois extension. But the computation of $\lambda_{K/F}$, where K/F is a wildly ramified quadratic extension, seems subtle. In Subsection 3.4.2 we give few computation in the wild case, and they are:

Lemma 1.0.8. *Let K be the finite abelian extension of $\mathbb{Q}_2$ for which $N_{K/\mathbb{Q}_2}(K^\times) = \mathbb{Q}_2^{\times 2}$. Then $\lambda_{K/\mathbb{Q}_2} = 1$.*

More generally, we show (cf. Theorem 3.4.13) that when $F/\mathbb{Q}_2$, and K is the abelian extension of F for which $N_{K/F}(K^\times) = {F^\times}^2$, then $\lambda_{K/F} = 1$. In Example 3.4.14 we compute $\lambda_{F/\mathbb{Q}_2}$, where F is a quadratic extension of $\mathbb{Q}_2$.

In Chapter 4 we give an invariant formula (group theoretical) of the determinant of a Heisenberg representation. From Gallagher's theorem 2.8.2 we know that to compute the determinant of an induced representation we need to compute the transfer map. Since our Heisenberg representations of degree $\geqslant 2$ are induced representation, we first need to give formula for transfer maps which we give in Lemmas 4.1.1 and 4.1.3. By using them in Proposition 4.2.2 we give an invariant formula of the determinant of Heisenberg representation.

Let G be a finite group and ρ be a Heisenberg representation of G. Let Z be the scalar group for ρ and H be a maximal isotropic subgroup of G for ρ. Let χ_H be a linear character of H which is an extension of the central character χ_Z of ρ. Then we know that $\rho = \mathrm{Ind}_H^G\chi_H$. The main aim of Chapter 4 is to give an invariant formula for:

$$\det(\rho) = \det(\mathrm{Ind}_H^G\chi_H).$$

In other words, we will show that this formula is independent of the choice of the maximal isotropic subgroup H because the maximal isotropic subgroups are **not** unique. From Gallagher's result (cf. Theorem 2.8.2) we know that

$$\det(\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}\chi_{\mathrm{H}})(\mathrm{g}) = \Delta_{\mathrm{H}}^{\mathrm{G}}(\mathrm{g}) \cdot \chi_{\mathrm{H}}(\mathrm{T}_{\mathrm{G/H}}(\mathrm{g})) \text{ for all } g \in G.$$

Therefore for explicit computation of determinant of Heisenberg representation ρ, we first need to compute the transfer map $T_{G/H}$. In Lemma 4.1.1 we compute $T_{G/H}$, when H is an abelian normal subgroup of a finite group G (of odd index in G) with $[G,[G,G]] = \{1\}$, and G/H is an abelian quotient group.

Lemma 1.0.9. *Assume that G is a finite group and H a normal subgroup of G such that*

1. *H is abelian,*

2. *G/H is abelian of odd order d,*

3. *$[G,[G,G]] = \{1\}$.*

Then we have $T_{G/H}(g) = g^d$ for all $g \in G$.
As a consequence one has $[G,G]^d = \{1\}$, in other words, G^d is contained in the center of G.

More generally, combining this above Lemma and the elementary divisor theorem, we obtain the following result (cf. Lemma 4.1.3).

Lemma 1.0.10. *Assume that G is a finite group and H a normal subgroup of G such that*

1. *H is abelian*

2. *G/H is abelian of order d, such that (according to the elementary divisor theorem):*

$$G/H \cong \mathbb{Z}/m_1 \times \cdots \times \mathbb{Z}/m_s$$

where $m_1|\cdots|m_s$ and $\prod_i m_i = d$. Moreover, we fix elements $t_1, t_2, \cdots, t_s \in G$ such that $t_iH \in G/H$ generates the cyclic factor $\cong \mathbb{Z}/m_i$, hence $t_i^{m_i} \in H$.

3. *$[G,[G,G]] = \{1\}$. In particular, $[G,G]$ is in the center $Z(G)$ of G.*

Then each $g \in G$ has a unique decomposition

(i)

$$g = t_1^{a_1} \cdots t_s^{a_s} \cdot h, \quad T_{G/H}(g) = \prod_i^s T_{G/H}(t_i)^{a_i} \cdot T_{G/H}(h),$$

where $0 \leqslant a_i \leqslant m_i - 1$, $h \in H$ *and*

(ii)

$$T_{G/H}(t_i) = t_i^d \cdot [t_i^{m_i}, \alpha_i], \qquad T_{G/H}(h) = h^d \cdot [h, \alpha],$$

where $\alpha_i \in G/H$ *is the product over all elements from* $C_i \subset G/H$, *the subgroup which is complementary to the cyclic subgroup* $< t_i >$ *mod* H, *and where* $\alpha \in G/H$ *is product over all elements from* G/H.
Here we mean $[t_i^{m_i}, \alpha_i] := [t_i^{m_i}, \widehat{\alpha}_i]$, $[h, \alpha] := [h, \widehat{\alpha}]$ *for any representatives* $\widehat{\alpha}_i, \widehat{\alpha} \in G$. *The commutators are independent of the choice of the representatives and are always elements of order* $\leqslant 2$ *because* $\widehat{\alpha}_i^2, \widehat{\alpha}^2 \in H$, *and* H *is abelian.*
As a consequence of (i) *and* (ii) *we can always obtain*

(iii)

$$T_{G/H}(g) = g^d \cdot \varphi_{G/H}(g),$$

where $\varphi_{G/H}(g) \in Z(G)$ *is an element of order* $\leqslant 2$.

As a consequence of the second equality in (ii) *combined with* $[G, G] \subseteq H \cap \mathrm{Ker}(T_{G/H})$, *one has* $[G, G]^d = \{1\}$, *in other words,* G^d *is contained in the center* $Z(G)$ *of* G.

Now let $\rho = (Z, \chi_\rho)$ be a Heisenberg representation of G. Then from the definition of Heisenberg representation we have

$$[[G, G], G] \subseteq \mathrm{Ker}(\rho).$$

Now let $\overline{G} := G/\mathrm{Ker}(\rho)$. Then we obtain

$$[\overline{G}, \overline{G}] = [G/\mathrm{Ker}(\rho), \mathrm{G}/\mathrm{Ker}(\rho)] = [\mathrm{G}, \mathrm{G}] \cdot \mathrm{Ker}(\rho)/\mathrm{Ker}(\rho) = [\mathrm{G}, \mathrm{G}]/[\mathrm{G}, \mathrm{G}] \cap \mathrm{Ker}(\rho).$$

Since $[[G, G], G] \subseteq \mathrm{Ker}(\rho)$, then $[x, g] \in \mathrm{Ker}(\rho)$ for all $x \in [G, G]$ and $g \in G$. Hence we obtain

$$[[\overline{G}, \overline{G}], \overline{G}] = [[G, G]/[G, G] \cap \mathrm{Ker}(\rho), \mathrm{G}/\mathrm{Ker}(\rho)] \subseteq \mathrm{Ker}(\rho),$$

This shows that $\overline{G}$ is a two-step nilpotent group. Hence for computing determinant of a Heisenberg representation of a finite group **modulo** $\mathrm{Ker}(\rho)$ we can use the Lemmas 4.1.1 and 4.1.3 and we obtain the following result (cf. Proposition 4.2.2).

Proposition 1.0.11. *Let* $\rho = (Z, \chi_\rho)$ *be a Heisenberg representation of* G, *of dimension* d, *and put* $X_\rho(g_1, g_2) := \chi_\rho \circ [g_1, g_2]$. *Then we obtain*

$$(\det(\rho))(g) = \varepsilon(g) \cdot \chi_\rho(g^d), \tag{1.0.3}$$

where ε *is a function on* G *with the following properties:*

1. *ε has values in $\{\pm 1\}$.*

2. *$\varepsilon(gx) = \varepsilon(g)$ for all $x \in G^2 \cdot Z$, hence ε is a function on the factor group $G/G^2 \cdot Z$, and in particular, $\varepsilon \equiv 1$ if $[G : Z] = d^2$ is odd.*

3. *If d is even, then the function ε need not be a homomorphism but:*

$$\tfrac{\varepsilon(g_1)\varepsilon(g_2)}{\varepsilon(g_1 g_2)} = X_\rho(g_1, g_2)^{\frac{d(d-1)}{2}} = X_\rho(g_1, g_2)^{\frac{d}{2}}.$$

   Furthermore,

   (a) ***When*** $\mathrm{rk}_2(G/Z) \geqslant 4$***:*** *$\varepsilon$ is a homomorphism, and exactly $\varepsilon \equiv 1$.*

   (b) ***When*** $\mathrm{rk}_2(G/Z) = 2$***:*** *ε is not a homomorphism and ε is a function on G/G^2Z such that*

$$(\det \rho)(g) = \varepsilon(g) \cdot \chi_\rho(g^d) = \begin{cases} \chi_\rho(g^d) & \text{for } g \in G^2Z \\ -\chi_\rho(g^d) & \text{for } g \notin G^2Z. \end{cases}$$

In Chapter 5, our main aim is to compute local constants for Heisenberg representations of a local Galois group. We know that Heisenberg representations are **monomial** (that is, induced from linear character of a finite-index subgroup), and local constants are weakly extendible functions. Therefore to compute local constants for Heisenberg representations we need to compute lambda-functions. In Chapter 3 we compute lambda-functions explicitly for finite Galois extensions (except wildly ramified quadratic extension), and in Chapter 5 we use them.

In Section 5.1, we study the arithmetic descriptions of Heisenberg representations. We define U-isotropic Heisenberg representations and give the following lemma (cf. 5.1.9).

Lemma 1.0.12. *Fix a uniformizer π_F and write $U := U_F$. Then we obtain an isomorphism*

$$\widehat{U} \cong F\widehat{F^\times/U} \wedge U, \quad \eta \mapsto X_\eta, \quad \eta_X \leftarrow X$$

between characters of U and U-isotropic alternating characters as follows:

$$X_\eta(\pi_F^a \varepsilon_1, \pi_F^b \varepsilon_2) := \eta(\varepsilon_1)^b \cdot \eta(\varepsilon_2)^{-a}, \quad \eta_X(\varepsilon) := X(\varepsilon, \pi_F), \tag{1.0.4}$$

where $a, b \in \mathbb{Z}$, $\varepsilon, \varepsilon_1, \varepsilon_2 \in U$, and $\eta : U \to \mathbb{C}^\times$. Then

$$\mathrm{Rad(X_\eta)} =< \pi_{\mathrm{F}}^{\#\eta} > \times \mathrm{Ker}(\eta) =< (\pi_{\mathrm{F}}\varepsilon)^{\#\eta} > \times \mathrm{Ker}(\eta),$$

does not depend on the choice of π_F, where $\#\eta$ is the order of the character η, hence

$$F^\times/\mathrm{Rad(X_\eta)} \cong < \pi_{\mathrm{F}} > / < \pi_{\mathrm{F}}^{\#\eta} > \times \mathrm{U/Ker}(\eta) \cong \mathbb{Z}_{\#\eta} \times \mathbb{Z}_{\#\eta}.$$

Therefore all Heisenberg representations of type $\rho = \rho(X_\eta, \chi)$ have dimension $\dim(\rho) = \#\eta$.

From the above lemma we can construct all Heisenberg representations of dimensions prime to p (cf. Corollary 5.1.12). And we also have the following explicit lemma (cf. Lemma 5.1.14).

Lemma 1.0.13 (Explicit Lemma). *Let $\rho = \rho(X_\eta, \chi_K)$ be a U-isotropic Heisenberg representation of the absolute Galois group G_F of a local field $F/\mathbb{Q}_p$. Let $K = K_\eta$ and let E/F be the maximal unramified subextension in K/F. Then:*

1. *The norm map induces an isomorphism:*

$$N_{K/E} : K_F^\times / I_F K^\times \xrightarrow{\sim} I_F E^\times / I_F \mathcal{N}_{K/E}.$$

2. *Let $c_{K/F} : F^\times/\mathrm{Rad}(X_\eta) \wedge F^\times/\mathrm{Rad}(X_\eta) \cong K_F^\times/I_F K^\times$ be the isomorphism which is induced by the commutator in the relative Weil-group $W_{K/F}$. Then for units $\varepsilon \in U_F$ we explicitly have:*

$$c_{K/F}(\varepsilon \wedge \pi_F) = N_{K/E}^{-1}(N_{E/F}^{-1}(\varepsilon)^{1-\varphi_{E/F}}),$$

where $\varphi_{E/F}$ is the Frobenius automorphism for E/F and where N^{-1} means to take a preimage of the norm map.

3. *The restriction $\chi_K|_{K_F^\times}$ is characterized by:*

$$\chi_K \circ c_{K/F}(\varepsilon \wedge \pi_F) = X_\eta(\varepsilon, \pi_F) = \eta(\varepsilon),$$

for all $\varepsilon \in U_F$, where $c_{K/F}(\varepsilon \wedge \pi_F)$ is explicitly given via (2).

After these, in Subsection 5.1.2, we study the Artin conductors, Swan conductors of Heisenberg representations, and these results are important for giving explicit invariant formulas of local constants for Heisenberg representations of dimensions prime to p.

In the following theorem (cf. Theorem 5.2.4) we give an invariant formula of $W(\rho)$ for the Heisenberg representation ρ.

Theorem 1.0.14. *Let $\rho = \rho(X, \chi_K)$ be a Heisenberg representation of the absolute Galois group G_F of a local field $F/\mathbb{Q}_p$ of dimension d. Let ψ_F be the canonical additive character of F and $\psi_K := \psi_F \circ \mathrm{Tr}_{K/F}$. Denote μ_{p^∞} as the group of roots of unity of p-power order and μ_n as the group of n-th roots of unity, where $n > 1$ is an integer.*

1. *When the dimension d is odd, we have*

$$W(\rho) \equiv W(\chi_\rho)' \mod \mu_d,$$

where $W(\chi_\rho)'$ is any d-th root of $W(\chi_K, \psi_K)$.

2. *When the dimension d is even, we have*

$$W(\rho) \equiv W(\chi_\rho)' \mod \mu_{d'},$$

where $d' = \mathrm{lcm}(4, \mathrm{d})$.

For giving more explicit invariant formula of local constants for the Heisenberg representations, we need the following result (cf. Proposition 5.1.7).

Proposition 1.0.15. *Let* $\rho = \rho(Z, \chi_\rho) = \rho(G_K, \chi_K)$ *be a Heisenberg representation of the absolute Galois group* G_F. *Let* E *be a base field of a maximal isotropic for* ρ. *Then* $F^\times \subseteq \mathcal{N}_{K/E}$, *and*

$$\det(\rho)(x) = \Delta_{E/F}(x) \cdot \chi_K \circ N_{K/E}^{-1}(x) \quad \textit{for all } x \in F^\times, \tag{1.0.5}$$

where, for all $x \in F^\times$,

$$\Delta_{E/F}(x) = \begin{cases} 1 & \textit{when } \mathrm{rk}_2(\mathrm{Gal(E/F)}) \neq 1 \\ \omega_{E'/F}(x) & \textit{when } \mathrm{rk}_2(\mathrm{Gal(E/F)}) = 1, \end{cases} \tag{1.0.6}$$

where E'/F *is a uniquely determined quadratic subextension in* E/F, *and* $\omega_{E'/F}$ *is the character of* $F^\times$ *which corresponds to* E'/F *by class field theory.*

By using the above proposition we have the following invariant formula (cf. Theorem 5.2.7) of local constant for a minimal conductor U-isotopic Heisenberg representation of dimension prime to p.

Theorem 1.0.16. *Let* $\rho = \rho(X, \chi_K)$ *be a minimal conductor Heisenberg representation of the absolute Galois group* G_F *of a non-archimedean local field* $F/\mathbb{Q}_p$ *of dimension* m *with* $gcd(m, p) = 1$. *Let* ψ *be a nontrivial additive character of* F. *Then*

$$W(\rho, \psi) = R(\psi, c) \cdot L(\psi, c), \tag{1.0.7}$$

where

$$R(\psi, c) := \lambda_{E/F}(\psi)\Delta_{E/F}(c),$$

is a fourth root of unity that depends on $c \in F^\times$ *with* $\nu_F(c) = 1 + n(\psi)$ *but not on the totally ramified cyclic subextension* E/F *in* K/F, *and*

$$L(\psi, c) := \det(\rho)(c) q_F^{-\frac{1}{2}} \sum_{x \in k_F^\times} (\chi_K \circ N_{E_1/F}^{-1})^{-1}(x) \cdot (c^{-1}\psi)(mx),$$

where E_1/F *is the unramified extension of* F *of degree* m.

And when $\rho = \rho(X, \chi_K)$ is not minimal conductor, we have the following theorem (cf. Theorem 5.2.9).

Theorem 1.0.17. *Let* $\rho = \rho(X_\rho, \chi_K)$ *be a Heisenberg representation of the absolute Galois group* G_F *of a local field* $F/\mathbb{Q}_p$ *of dimension* m *prime to* p. *Let* ψ *be a nontrivial additive character of* F. *Suppose that the conductor of* ρ *is not minimal,* $\rho = \rho_0 \otimes \widetilde{\chi_F}$ *and* $a(\rho) = m \cdot a(\chi_F)$, *where* $\widetilde{\chi_F} : W_F \to \mathbb{C}^\times$ *corresponds to* $\chi_F : F^\times \to \mathbb{C}^\times$, *and* $h = a(\chi_F) \geqslant 2$.
Case-1: *If* m *is odd, then*

1. when $1+m(h-1)=2d$ is even, we have

$$W(\rho,\psi)=\det(\rho)(c)\psi(mc^{-1}),$$

2. when $1+m(h-1)=2d+1$ is odd, we have

$$W(\rho,\psi)=\det(\rho)(c)\cdot H(\psi,c),$$

where

$$H(\psi,c)=q_F^{-\frac{1}{2}}\sum_{y\in U_F^{h'}/U_F^{h'+1}}(\chi_K|_{E_1}\circ N_{E_1/F}^{-1})^{-1}(y)(c^{-1}\psi)(my),$$

and $h'=[\frac{h}{2}]$, where $[x]$ denotes the largest integer $\leqslant x$.

Case-2: *If m is even, then*

1. when h is odd, we have

$$W(\rho,\psi)=R(\psi,c)\cdot\det(\rho)(c)\cdot H(\psi,c),$$

where $H(\psi,c)$ is the same as in Case-1(2).

2. when h is even, we have

$$W(\rho,\psi)=R(\psi,c)\cdot\det(\rho)(c)\cdot q_F^{\frac{1}{2}}\cdot\psi(c^{-1}m),$$

where $R(\psi,c)=\lambda_{E/F}(\psi)\cdot\Delta_{E/F}(c)$.
Here E_1/F is the maximal unramified subextension in K/F, and E/F is a totally ramified cyclic subextension in K/F and $c\in F^\times$ with $\nu_F(c)=h+n(\psi)$, and

$$\chi_F(1+x)=\psi(x/c),\qquad \text{for all } x\in P_F^{h-h'}/P_F^h.$$

Furthermore, without using λ-function computation, by Deligne-Henniart's result (cf. Lemma 5.2.10), we also give an invariant formula (cf. Theorem 5.2.11) of local constant for a Heisenberg representation of the absolute Galois group G_F of a local field $F/\mathbb{Q}_p$ of dimension prime to p, for sufficiently large conductor characters.

Theorem 1.0.18. *Let $\rho=\rho_0\otimes\widetilde{\chi_F}$ be a Heisenberg representation of G_F of dimension d with $gcd(d,p)=1$, where $\rho_0=\rho_0(X_\eta,\chi_0)$ is a minimal conductor Heisenberg representation. If $a(\chi_F)\geqslant m_\rho\geqslant 2$, a sufficiently large number which depends on ρ, then we have*

$$W(\rho,\psi)=W(\rho_0\otimes\widetilde{\chi_F})=W(\chi_F,\psi)^d\cdot\det(\rho_0)(c),\tag{1.0.8}$$

where ψ is a nontrivial additive character of F, and $c:=c(\chi_F,\psi)\in F^\times$, satisfies

$$\chi_F(1+x)=\psi(c^{-1}x)\text{ for all } x\in P_F^{[\frac{a(\chi_F)}{2}]+1}.$$

In Section 5.3 we use the Tate's **root-of-unity criterion** (cf. [32], p. 112, Corollary 4) in our local constant computation. Let ρ be a Heisenberg representation of the absolute Galois group G_F. In the following proposition (cf. Proposition 5.3.2) we show that if $W(\rho)$ is not a root of unity, then $\dim(\rho)|(\mathrm{q_F}-1)$, and $a_F(\rho)$ is not minimal.

Proposition 1.0.19. *Let $F/\mathbb{Q}_p$ be a local field and let $q_F = p^s$ be the order of its finite residue field. If $\rho = (Z_\rho, \chi_\rho) = \rho(X_\rho, \chi_K)$ is a Heisenberg representation of the absolute Galois group G_F such that $W(\rho)$ is not a root of unity, then $dim(\rho)|(q_F - 1)$ and $a_F(\rho)$ is not minimal.*

Chapter 2

Preliminaries

In this chapter, we will recall some background of local constants, classical Gauss sums, Heisenberg representations that will be relevant to the next chapters. We also state some known facts which we will use in our next chapters. In this chapter for local constants we follow [41], [32], [33] and for extendible functions [19]. For classical Gauss sums and Heisenberg representations we refer [40], [3] and [11], [12] respectively.

2.1 Local Fields and their finite extensions

Let F be a non-archimedean local field, i.e., a finite extension of the field $\mathbb{Q}_p$ (field of p-adic numbers), where p is a prime. Let K/F be a finite extension of the field F. Let $e_{K/F}$ be the ramification index for the extension K/F and $f_{K/F}$ be the residue degree of the extension K/F. The extension K/F is called **unramified** if $e_{K/F} = 1$; equivalently $f_{K/F} = [K : F]$. The extension K/F is **totally ramified** if $e_{K/F} = [K : F]$; equivalently $f_{K/F} = 1$. Let q_F be the cardinality of the residue field k_F of F. If $\gcd(p, [K : F]) = 1$, then the extension K/F is called **tamely ramified**, otherwise **wildly ramified**. The extension K/F is **totally tamely ramified** if it is both totally ramified and tamely ramified.

For a tower of **local** fields $K/L/F$, we have (cf. [21], p. 39, Lemma 2.1)

$$e_{K/F}(\nu_K) = e_{K/L}(\nu_K) \cdot e_{L/F}(\nu_L), \tag{2.1.1}$$

where ν_K is a valuation on K and ν_L is the induced valuation on L, i.e., $\nu_L = \nu_K|_L$. For the tower of fields $K/L/F$ we simply write $e_{K/F} = e_{K/L} \cdot e_{L/F}$. Let O_F be the ring of integers in the local field F and $P_F = \pi_F O_F$ is the unique prime ideal in O_F and π_F is a uniformizer, i.e., an element in P_F whose valuation is one, i.e., $\nu_F(\pi_F) = 1$. Let $U_F = O_F - P_F$ be the group of units in O_F. Let $P_F^i = \{x \in F : \nu_F(x) \geqslant i\}$ and for $i \geqslant 0$ define $U_F^i = 1 + P_F^i$ (with proviso $U_F^0 = U_F = O_F^\times$). We also consider that $a(\chi)$ is the conductor of nontrivial character $\chi : F^\times \to \mathbb{C}^\times$, i.e., $a(\chi)$ is the smallest integer $m \geqslant 0$ such that χ is trivial on U_F^m. We say χ is unramified if the conductor of χ is zero and otherwise ramified. Throughout the thesis when K/F is unramified we choose uniformizers $\pi_K = \pi_F$. And when K/F is totally ramified (both tame and wild) we choose uniformizers $\pi_F = N_{K/F}(\pi_K)$, where $N_{K/F}$ is the norm map from $K^\times$ to $F^\times$.

Definition 2.1.1 (**Different and Discriminant**)**.** Let K/F be a finite separable extension of non-archimedean local field F. We define the **inverse different (or codifferent)** $\mathcal{D}_{K/F}^{-1}$ of K over F to be $\pi_K^{-d_{K/F}} O_K$, where $d_{K/F}$ is the largest integer (this is the exponent of the different $\mathcal{D}_{K/F}$) such that

$$\mathrm{Tr}_{K/F}(\pi_K^{-d_{K/F}} O_K) \subseteq O_F,$$

where $\mathrm{Tr_{K/F}}$ is the trace map from K to F. Then the **different** is defined by:

$$\mathcal{D}_{K/F} = \pi_K^{d_{K/F}} O_K$$

and the **discriminant** $D_{K/F}$ is

$$D_{K/F} = N_{K/F}(\pi_K^{d_{K/F}}) O_F.$$

Thus it is easy to see that $D_{K/F}$ is an **ideal of** O_F.

We know that if K/F is unramified, then $D_{K/F}$ is **a unit in** O_F. If K/F is tamely ramified, then

$$\nu_K(\mathcal{D}_{K/F}) = d_{K/F} = e_{K/F} - 1. \tag{2.1.2}$$

(see [27], Chapter III, for details about different and discriminant of the extension K/F.) We need to mention a very important result of J-P. Serre for our purposes.

Lemma 2.1.2 ([27], p. 50, Proposition 7)**.** *Let K/F be a finite separable extension of the field F. Let I_F (resp. I_K) be a fractional ideal of F (resp. K) relative to O_F (resp. O_K). Then the following two properties are equivalent:*

1. $\mathrm{Tr}_{K/F}(I_K) \subset I_F$.
2. $I_K \subset I_F.\mathcal{D}_{K/F}^{-1}$.

Definition 2.1.3 (**Canonical additive character**)**.** We define the non trivial additive character of F, $\psi_F : F \to \mathbb{C}^\times$ as the composition of the following four maps:

$$F \xrightarrow{\mathrm{Tr}_{F/\mathbb{Q}_p}} \mathbb{Q}_p \xrightarrow{\alpha} \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\beta} \mathbb{Q}/\mathbb{Z} \xrightarrow{\gamma} \mathbb{C}^\times,$$

where

1. $\mathrm{Tr}_{F/\mathbb{Q}_p}$ is the trace from F to $\mathbb{Q}_p$,
2. α is the canonical surjection map,
3. β is the canonical injection which maps $\mathbb{Q}_p/\mathbb{Z}_p$ onto the p-component of the divisible group $\mathbb{Q}/\mathbb{Z}$ and
4. γ is the exponential map $x \mapsto e^{2\pi i x}$, where $i = \sqrt{-1}$.

For every $x \in \mathbb{Q}_p$, there is a rational r, uniquely determined modulo 1, such that $x - r \in \mathbb{Z}_p$. Then $\psi_{\mathbb{Q}_p}(x) = \psi_{\mathbb{Q}_p}(r) = e^{2\pi i r}$. The nontrivial additive character $\psi_F = \psi_{\mathbb{Q}_p} \circ \mathrm{Tr_{F/\mathbb{Q}_p}}$ of F is called the **canonical additive character** (cf. [32], p. 92).

The **conductor** of any nontrivial additive character ψ of the field F is an integer $n(\psi)$ if ψ is trivial on $P_F^{-n(\psi)}$, but nontrivial on $P_F^{-n(\psi)-1}$. So, from Lemma 2.1.2 we can show (cf. Lemma 3.4.3) that

$$n(\psi_F) = n(\psi_{\mathbb{Q}_p} \circ \mathrm{Tr_{F/\mathbb{Q}_p}}) = \nu_{\mathrm{F}}(\mathcal{D}_{\mathrm{F/\mathbb{Q}_p}}),$$

because $d_{\mathbb{Q}_p/\mathbb{Q}_p} = 0$, and hence $n(\psi_{\mathbb{Q}_p}) = 0$.

2.2 Extendible functions

Let G be any finite group. We denote $R(G)$ the set of all pairs (H, ρ), where H is a subgroup of G and ρ is a virtual representation of H . The group G acts on $R(G)$ by means of

$$(H, \rho)^g = (H^g, \rho^g),\ g \in G,$$
$$\rho^g(x) = \rho(gxg^{-1}),\ x \in H^g := g^{-1}Hg$$

Furthermore we denote by $\widehat{H}$ the set of all one dimensional representations of H and by $R_1(G)$ the subset of $R(G)$ of pairs (H, χ) with $\chi \in \widehat{H}$. Here character χ of H we mean always a **linear** character, i.e., $\chi : H \to \mathbb{C}^\times$.

Now define a function $\mathcal{F} : R_1(G) \to \mathcal{A}$, where $\mathcal{A}$ is a multiplicative abelian group with

$$\mathcal{F}(H, 1_H) = 1 \tag{2.2.1}$$

and

$$\mathcal{F}(H^g, \chi^g) = \mathcal{F}(H, \chi) \tag{2.2.2}$$

for all (H, χ), where 1_H denotes the trivial representation of H.
Here a function $\mathcal{F}$ on $R_1(G)$ means a function which satisfies the equation (2.2.1) and (2.2.2).

A function $\mathcal{F}$ is said to be extendible if $\mathcal{F}$ can be extended to an $\mathcal{A}$-valued function on $R(G)$ satisfying:

$$\mathcal{F}(H, \rho_1 + \rho_2) = \mathcal{F}(H, \rho_1)\mathcal{F}(H, \rho_2) \tag{2.2.3}$$

for all $(H, \rho_i) \in R(G), i = 1, 2$, and if $(H, \rho) \in R(G)$ with $\dim \rho = 0$, and Δ is a subgroup of G containing H, then

$$\mathcal{F}(\Delta, \mathrm{Ind}_H^\Delta \rho) = \mathcal{F}(H, \rho), \tag{2.2.4}$$

where $\mathrm{Ind}_H^\Delta \rho$ is the virtual representation of Δ induced from ρ. In general, let ρ be a representation of H with $\dim \rho \neq 0$. We can define a zero dimensional representation of H by ρ and which is: $\rho_0 := \rho - \dim \rho \cdot 1_H$. So $\dim \rho_0$ is zero, then now we use the equation (2.2.4) for ρ_0 and we have,

$$\mathcal{F}(\Delta, \mathrm{Ind}_H^\Delta \rho_0) = \mathcal{F}(H, \rho_0). \tag{2.2.5}$$

Now replace ρ_0 by $\rho - \dim\rho \cdot 1_H$ in the above equation (2.2.5) and we have

$$\mathcal{F}(\Delta, \mathrm{Ind}_H^\Delta(\rho - \dim\rho \cdot 1_H)) = \mathcal{F}(H, \rho - \dim\rho \cdot 1_H)$$
$$\implies \frac{\mathcal{F}(\Delta, \mathrm{Ind}_H^\Delta \rho)}{\mathcal{F}(\Delta, \mathrm{Ind}_H^\Delta 1_H)^{\dim\rho}} = \frac{\mathcal{F}(H, \rho)}{\mathcal{F}(H, 1_H)^{\dim\rho}}.$$

Therefore,

$$\mathcal{F}(\Delta, \mathrm{Ind}_H^\Delta \rho) = \left\{\frac{\mathcal{F}(\Delta, \mathrm{Ind}_H^\Delta 1_H)}{\mathcal{F}(H, 1_H)}\right\}^{\dim\rho} \cdot \mathcal{F}(H, \rho)$$
$$= \lambda_H^\Delta(\mathcal{F})^{\dim\rho} \mathcal{F}(H, \rho), \tag{2.2.6}$$

where

$$\lambda_H^\Delta(\mathcal{F}) := \frac{\mathcal{F}(\Delta, \mathrm{Ind}_H^\Delta 1_H)}{\mathcal{F}(H, 1_H)}. \tag{2.2.7}$$

But by the definition of $\mathcal{F}$, we have $\mathcal{F}(H, 1_H) = 1$, so we can write

$$\lambda_H^\Delta(\mathcal{F}) = \mathcal{F}(\Delta, \mathrm{Ind}_H^\Delta 1_H). \tag{2.2.8}$$

This $\lambda_H^\Delta(\mathcal{F})$ is called **Langlands λ-function** (or simply λ-function) which is independent of ρ. A extendible function $\mathcal{F}$ is called **strongly** extendible if it satisfies equation (2.2.3) and fulfills equation (2.2.4) for all $(H, \rho) \in R(G)$, and if the equation (2.2.4) is fulfilled only when $\dim \rho = 0$, then $\mathcal{F}$ is called **weakly** extendible function. The extendible functions are **unique**, if they exist (cf. [32], p. 103).

Example 2.2.1. Langlands proves the local constants are weakly extendible functions (cf. [32], p. 105, Theorem 1). The Artin root numbers (also known as global constants) are strongly extendible functions (for more examples and details about extendible function, see [32] and [19]).

The next lemma is due to Langlands [41]. This is very important for this thesis. Group theoretically it is not hard to see its proof. But its number theoretical proof is very very difficult and long which can be found in [41].

Lemma 2.2.2. *Let H be a subgroup of a group G and $\mathcal{F}$ an extendible function on $R_1(G)$. Then we have the following properties of λ-factor.*

1. *If $g \in G$, then $\lambda_{g^{-1}Hg}^G(\mathcal{F}) = \lambda_H^G(\mathcal{F})$, where $H \subseteq G$.*
2. *If H' is a subgroup of H then $\lambda_{H'}^G(\mathcal{F}) = \lambda_{H'}^H(\mathcal{F})\lambda_H^G(\mathcal{F})^{[H:H']}$, where $[H : H']$ is index of H' in H.*
3. *If H' is a normal subgroup of G contained in H, then $\lambda_H^G(\mathcal{F}) = \lambda_{H/H'}^{G/H'}(\mathcal{F})$.*

2.3 Local Constants

Let F be a non-archimedean local field and χ be a character of $F^\times$. The $L(\chi)$-functions are defined as follows:

$$L(\chi) = \begin{cases} (1-\chi(\pi_F))^{-1} & \text{if } \chi \text{ is unramified,} \\ 1 & \text{if } \chi \text{ is ramified.} \end{cases}$$

We denote by dx a Haar measure on F, by $d^\times x$ a Haar measure on $F^\times$ and the relation between these two Haar measure is:

$$d^\times x = \tfrac{dx}{|x|},$$

for arbitrary Haar measure dx on F. For given additive character ψ of F and Haar measure dx on F, we have a **Fourier transform** as:

$$\hat{f}(y) = \int f(x)\psi(xy)dx. \tag{2.3.1}$$

where $f \in L^1(F^+)$ (that is, $|f|$ is integrable) and the Haar measure is normalized such that $\hat{\hat{f}}(y) = f(-y)$, i.e., dx is self-dual with respect to ψ. By Tate (cf., [33], p. 13), for any character χ of $F^\times$, there exists a number $W(\chi,\psi,dx) \in \mathbb{C}^\times$ such that it satisfies the following local functional equation:

$$\frac{\int \hat{f}(x)w_1\chi^{-1}(x)d^\times x}{L(w_1\chi^{-1})} = W(\chi,\psi,dx)\frac{\int f(x)\chi(x)d^\times x}{L(\chi)}. \tag{2.3.2}$$

for any such function f for which the both sides make sense. This number $W(\chi,\psi,dx)$ is called the **local epsilon factor or local constant** of χ.

For a nontrivial multiplicative character χ of $F^\times$ and nontrivial additive character ψ of F, we have (cf. [41], p. 4)

$$W(\chi,\psi,c) = \chi(c)\frac{\int_{U_F}\chi^{-1}(x)\psi(x/c)dx}{|\int_{U_F}\chi^{-1}(x)\psi(x/c)dx|} \tag{2.3.3}$$

where the Haar measure dx is normalized such that the measure of O_F is 1 and where $c \in F^\times$ with valuation $n(\psi)+a(\chi)$. The **modified** formula of epsilon factor (cf. [32], p. 94, for proof see [46]) is:

$$W(\chi,\psi,c) = \chi(c)q^{-a(\chi)/2}\sum_{x\in\frac{U_F}{U_F^{a(\chi)}}}\chi^{-1}(x)\psi(x/c). \tag{2.3.4}$$

where $c = \pi_F^{a(\chi)+n(\psi)}$. Now if $u \in U_F$ is unit and replace $c = cu$, then we have

$$W(\chi,\psi,cu) = \chi(c)q^{-\frac{a(\chi)}{2}}\sum_{x\in\frac{U_F}{U_F^{a(\chi)}}}\chi^{-1}(x/u)\psi(x/cu) = W(\chi,\psi,c). \tag{2.3.5}$$

Therefore $W(\chi,\psi,c)$ **depends** only on the exponent $\nu_F(c) = a(\chi)+n(\psi)$. Therefore we can simply write $W(\chi,\psi,c) = W(\chi,\psi)$, because c is determined by $\nu_F(c) = a(\chi)+n(\psi)$ up to a unit u which has **no influence on** $W(\chi, \psi, c)$. If χ is unramified, i.e., $a(\chi) = 0$, therefore $\nu_F(c) = n(\psi)$. Then from the formula of $W(\chi, \psi, c)$, we can write

$$W(\chi, \psi, c) = \chi(c), \tag{2.3.6}$$

and therefore $W(1, \psi, c) = 1$ if $\chi = 1$ is the trivial character.

2.3.1 Some properties of $W(\chi, \psi)$

1. Let $b \in F^\times$ be the uniquely determined element such that $\psi' = b\psi$. Then

$$W(\chi, \psi', c') = \chi(b) \cdot W(\chi, \psi, c). \tag{2.3.7}$$

   Proof. Here $\psi'(x) = (b\psi)(x) := \psi(bx)$ for all $x \in F$. It is an additive character of F. The existence and uniqueness of b is clear. From the definition of conductor of an additive character we have

$$n(\psi') = n(b\psi) = n(\psi) + \nu_F(b).$$

   Here $c' \in F^\times$ is of valuation

$$\nu_F(c') = a(\chi) + n(\psi') = a(\chi) + \nu_F(b) + n(\psi) = \nu_F(b) + \nu_F(c) = \nu_F(bc).$$

   Therefore $c' = bcu$ where $u \in U_F$ is some unit. Now

$$\begin{aligned} W(\chi, \psi', c') &= W(\chi, b\psi, bcu) \\ &= W(\chi, b\psi, bc) \\ &= \chi(bc) q_F^{-\frac{a(\chi)}{2}} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x)((bc)^{-1}(b\psi))(x) \\ &= \chi(b) \cdot \chi(c) q_F^{-\frac{a(\chi)}{2}} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x)\psi(xc^{-1}) \\ &= \chi(b) \cdot W(\chi, \psi, c). \end{aligned}$$

   □

2. Let $F/\mathbb{Q}_p$ be a local field inside $\overline{\mathbb{Q}_p}$. Let χ and ψ be a character of $F^\times$ and F^+ respectively, and $c \in F^\times$ with valuation $\nu_F(c) = a(\chi) + n(\psi)$. If $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ is an automorphism, then:

$$W_F(\chi, \psi, c) = W_{\sigma^{-1}(F)}(\chi^\sigma, \psi^\sigma, \sigma^{-1}(c)),$$

   where $\chi^\sigma(y) := \chi(\sigma(y))$, $\psi^\sigma(y) := \psi(\sigma(y))$, for all $y \in \sigma^{-1}(F)$.

Proof. Let $L := \sigma^{-1}(F)$. Since σ is an automorphism of $\overline{\mathbb{Q}_p}$, then we have $O_F/P_F \cong O_L/P_L$, hence $q_F = q_L$. We also can see that $a(\chi^\sigma) = a(\chi)$ and $n(\psi^\sigma) = n(\psi)$. Then from the formula of local constant we have

$$\begin{aligned} W_{\sigma^{-1}(F)}(\chi^\sigma, \psi^\sigma, \sigma^{-1}(c)) &= W_L(\chi^\sigma, \psi^\sigma, \sigma^{-1}(c)) \\ &= \chi^\sigma(\sigma^{-1}(c)) q_L^{-\frac{a(\chi^\sigma)}{2}} \sum_{y \in \frac{U_L}{U_L^{a(\chi^\sigma)}}} (\chi^\sigma)^{-1}(y) \cdot ((\sigma^{-1}(c))^{-1}\psi^\sigma(y) \\ &= \chi(c) q_F^{-\frac{a(\chi)}{2}} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x)\psi(\frac{x}{c}) \\ &= W_F(\chi, \psi, c). \end{aligned}$$

Here we put $y = \sigma^{-1}(x)$ and use $(\sigma^{-1}(c))^{-1}\psi^\sigma = (c^{-1}\psi)^\sigma$.

□

Remark 2.3.1. We can simply write as before $W_F(\chi, \psi) = W_{\sigma^{-1}(F)}(\chi^\sigma, \psi^\sigma)$. Tate in his paper [32] on local constants defines the local root number as:

$$W_F(\chi) := W_F(\chi, \psi_F) = W_F(\chi, \psi_F, d),$$

where ψ_F is the canonical character of $F^\times$ and $d \in F^\times$ with $\nu_F(d) = a(\chi) + n(\psi_F)$. Therefore after fixing canonical additive character $\psi = \psi_F$, we can rewrite

$$W_F(\chi) = \chi(d(\psi_F)), \text{ if } \chi \text{ is unramified,}$$
$$W_F(\chi) = W_{\sigma^{-1}(F)}(\chi^\sigma).$$

The last equality follows because the canonical character $\psi_{\sigma^{-1}(F)}$ is related to the canonical character ψ_F as: $\psi_{\sigma^{-1}(F)} = \psi_F^\sigma$.

So we see that

$$(F, \chi) \to W_F(\chi) \in \mathbb{C}^\times$$

is a function with the properties (2.2.1), (2.2.2) of extendible functions.

3. If $\chi \in \widehat{F^\times}$ and $\psi \in \widehat{F}$, then

$$W(\chi, \psi) \cdot W(\chi^{-1}, \psi) = \chi(-1).$$

Furthermore if the character $\chi : F^\times \to \mathbb{C}^\times$ is unitary (in particular, if χ is of finite order), then

$$|W(\chi, \psi)|^2 = 1.$$

Proof. We prove this properties by using equation (2.3.4). We know that the additive characters are always unitary, hence

$$\psi(-x) = \psi(x)^{-1} = \overline{\psi}(x).$$

On the other hand we write $\psi(-x) = ((-1)\psi)(x)$, where $-1 \in F^\times$. Therefore $\overline{\psi} = (-1)\psi$. We also have $a(\chi) = a(\chi^{-1})$. Therefore by using equation (2.3.4) we have

$$\begin{aligned} W(\chi,\psi)\cdot W(\chi^{-1},\psi) &= \chi(-1)\cdot q_F^{-a(\chi)} \sum_{x,y\in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x)\chi(y)\psi(\frac{x-y}{c}) \\ &= \chi(-1)\cdot q_F^{-a(\chi)} \sum_{x,y\in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x)\psi(\frac{xy-y}{c}), \quad \text{replacing } x \text{ by } xy \\ &= \chi(-1)\cdot q_F^{-a(\chi)} \sum_{x\in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x)\varphi(x), \end{aligned}$$

where

$$\varphi(x) = \sum_{y\in \frac{U_F}{U_F^{a(\chi)}}} \psi(y\frac{x-1}{c}). \tag{2.3.8}$$

Since $\frac{U_F}{U_F^{a(\chi)}} = (\frac{O_F}{P_F^{a(\chi)}})^\times = \frac{O_F}{P_F^{a(\chi)}} \setminus \frac{P_F}{P_F^{a(\chi)}}$, therefore $\varphi(x)$ can be written as the difference

$$\begin{aligned} \varphi(x) &= \sum_{y\in \frac{U_F}{U_F^{a(\chi)}}} \psi(y\frac{x-1}{c}) \\ &= \sum_{y\in \frac{O_F}{P_F^{a(\chi)}}} \psi(y\frac{x-1}{c}) - \sum_{y\in \frac{P_F}{P_F^{a(\chi)}}} \psi(y\frac{x-1}{c}) \\ &= \sum_{y\in \frac{O_F}{P_F^{a(\chi)}}} \psi(y\frac{x-1}{c}) - \sum_{y\in \frac{O_F}{P_F^{a(\chi)-1}}} \psi(y\frac{(x-1)\pi_F}{c}) \\ &= A - B, \end{aligned}$$

where $A = \sum_{y\in \frac{O_F}{P_F^{a(\chi)}}} \psi(y\frac{x-1}{c})$ and $B = \sum_{y\in \frac{O_F}{P_F^{a(\chi)-1}}} \psi(y\frac{(x-1)\pi_F}{c})$. It is easy to see that (cf. [26], p. 28, Lemma 2.1)

$$\sum_{y\in \frac{O_F}{P_F^{a(\chi)}}} \psi(y\alpha) = \begin{cases} q_F^{a(\chi)} & \text{when } \alpha \in P_F^{-n(\psi)} \\ 0 & \text{otherwise} \end{cases}$$

Therefore $A = q_F^{a(\chi)}$ when $x \in U_F^{a(\chi)}$ and $A = 0$ otherwise. Similarly $B = q_F^{a(\chi)-1}$ when

$x \in U_F^{a(\chi)-1}$ and $B = 0$ otherwise. Therefore we have

$$W(\chi,\psi)\cdot W(\chi^{-1},\psi) = \chi(-1)\cdot q_F^{-a(\chi)}\cdot\{q_F^{a(\chi)} - q_F^{a(\chi)-1}\sum_{x\in\frac{U_F^{a(\chi)-1}}{U_F^{a(\chi)}}}\chi^{-1}(x)\}$$

$$= \chi(-1) - \chi(-1)\cdot q_F^{-1}\sum_{x\in\frac{U_F^{a(\chi)-1}}{U_F^{a(\chi)}}}\chi^{-1}(x).$$

Since the conductor of χ is $a(\chi)$, then it can be proved that $\sum_{x\in\frac{U_F^{a(\chi)-1}}{U_F^{a(\chi)}}}\chi^{-1}(x) = 0$. Thus we obtain

$$W(\chi,\psi)\cdot W(\chi^{-1},\psi) = \chi(-1). \tag{2.3.9}$$

The right side of equation (2.3.9) is a sign, hence we may rewrite (2.3.9) as

$$W(\chi,\psi)\cdot\chi(-1)W(\chi^{-1},\psi) = 1.$$

But we also know from our earlier property that

$$\chi(-1)W(\chi^{-1},\psi) = W(\chi^{-1},(-1)\psi) = W(\chi^{-1},\overline{\psi}).$$

So the identity (2.3.9) rewrites as

$$W(\chi,\psi)\cdot W(\chi^{-1},\overline{\psi}) = 1.$$

Now we assume that χ is unitary, hence

$$W(\chi^{-1},\overline{\psi}) = W(\overline{\chi},\overline{\psi}) = \overline{W(\chi,\psi)}$$

where the last equality is obvious. Now we see that for unitary χ the identity (2.3.9) rewrites as

$$|W(\chi,\psi)|^2 = 1.$$

□

Remark 2.3.2. From the functional equation (2.3.2), we can directly see the first part of the above property of local constant. Denote

$$\zeta(f,\chi) = \int f(x)\chi(x)d^\times x. \tag{2.3.10}$$

Now replacing f by $\hat{\hat{f}}$ in equation (5.2.15), and we get

$$\zeta(\hat{\hat{f}},\chi) = \int \hat{\hat{f}}(x)\chi(x)d^\times x = \chi(-1)\cdot\zeta(f,\chi), \tag{2.3.11}$$

because dx is self-dual with respect to ψ, hence $\hat{\hat{f}}(x) = f(-x)$ for all $x \in F^+$.

Again the functional equation (2.3.2) can be written as follows:

$$\zeta(\hat{f}, w_1\chi^{-1}) = W(\chi, \psi, dx) \cdot \frac{L(w_1\chi^{-1})}{L(\chi)} \cdot \zeta(f, \chi). \tag{2.3.12}$$

Now we replace f by $\hat{f}$, and χ by $w_1\chi^{-1}$ in equation (2.3.12), and we obtain

$$\zeta(\hat{\hat{f}}, \chi) = W(w_1\chi^{-1}, \psi, dx) \cdot \frac{L(\chi)}{L(w_1\chi^{-1})} \cdot \zeta(\hat{f}, w_1\chi^{-1}). \tag{2.3.13}$$

Then by using equations (5.2.16), (2.3.12), from the above equation (2.3.13) we obtain:

$$W(\chi, \psi, dx) \cdot W(w_1\chi^{-1}, \psi, dx) = \chi(-1). \tag{2.3.14}$$

Moreover, the convention $W(\chi, \psi)$ is actually as follows (cf. [33], p. 17, equation (3.6.4)):

$$W(\chi w_{s-\frac{1}{2}}, \psi) = W(\chi w_s, \psi, dx).$$

By using this relation from equation (2.3.14) we can conclude

$$W(\chi, \psi) \cdot W(\chi^{-1}, \psi) = \chi(-1).$$

4. **Twisting formula of abelian local constants:**

   (a) If χ_1 and χ_2 are two unramified characters of $F^\times$ and ψ is a nontrivial additive character of F, then from equation (2.3.6) we have

   $$W(\chi_1\chi_2, \psi) = W(\chi_1, \psi)W(\chi_2, \psi). \tag{2.3.15}$$

   (b) Let χ_1 be ramified and χ_2 unramified then (cf. [33], (3.2.6.3))

   $$W(\chi_1\chi_2, \psi) = \chi_2(\pi_F)^{a(\chi_1)+n(\psi)} \cdot W(\chi_1, \psi). \tag{2.3.16}$$

   Proof. By the given condition $a(\chi_1) > a(\chi_2) = 0$. Therefore $a(\chi_1\chi_2) = a(\chi_1)$. Then we have

$$\begin{aligned} W(\chi_1\chi_2, \psi) &= \chi_1\chi_2(c)q_F^{-a(\chi_1)/2} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} (\chi_1\chi_2)^{-1}(x)\psi(x/c) \\ &= \chi_1(c)\chi_2(c)q_F^{-a(\chi_1)/2} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi_1^{-1}(x)\chi_2^{-1}(x)\psi(x/c) \\ &= \chi_2(c)\chi_1(c)q_F^{-a(\chi_1)/2} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi_1^{-1}(x)\psi(x/c), \quad \text{since } \chi_2 \text{ unramified} \\ &= \chi_2(c)W(\chi_1, \psi) \\ &= \chi_2(\pi_F)^{a(\chi_1)+n(\psi)} \cdot W(\chi_1, \psi). \end{aligned}$$

□

(c) We also have twisting formula of epsilon factor by Deligne (cf. [38], Lemma 4.16) under some special condition and which is as follows (for proof, see Corollary 6.1.2(3)):
Let α and β be two multiplicative characters of a local field F such that $a(\alpha) \geqslant 2 \cdot a(\beta)$. Let ψ be an additive character of F. Let $y_{\alpha,\psi}$ be an element of $F^\times$ such that

$$\alpha(1+x) = \psi(y_{\alpha,\psi}x)$$

for all $x \in F$ with valuation $\nu_F(x) \geqslant \frac{a(\alpha)}{2}$ (if $a(\alpha) = 0$, $y_{\alpha,\psi} = \pi_F^{-n(\psi)}$). Then

$$W(\alpha\beta,\psi) = \beta^{-1}(y_{\alpha,\psi}) \cdot W(\alpha,\psi). \tag{2.3.17}$$

2.3.2 Connection of different conventions for local constants

Mainly there are two conventions for local constants. They are due to Langlands ([41]) and Deligne ([38]). Recently Bushnell and Henniart ([6]) also give a convention of local constants. In this subsection we shall show the connection among all three conventions for local constants[1] We denote ϵ_{BH} as local constant of Bushnell-Henniart (introduced in Bushnell-Henniart, [6], Chapter 6).

On page 142 of [6], the authors define a rational function $\epsilon_{BH}(\chi,\psi,s) \in \mathbb{C}(q_F^{-s})$. From Theorem 23.5 on p. 144 of [6] for ramified character $\chi \in \widehat{F^\times}$ and conductor[2] $n(\psi) = -1$ we have

$$\epsilon_{BH}(\chi,s,\psi) = q_F^{n(\frac{1}{2}-s)} \sum_{x \in \frac{U_F}{U_F^{n+1}}} \chi(\alpha x)^{-1}\psi(\alpha x)/q_F^{\frac{n+1}{2}}, \tag{2.3.18}$$

where $n = a(\chi)-1$, and $\alpha \in F^\times$ with $\nu_F(\alpha) = -n$.

Also from the Proposition 23.5 of [6] on p. 143 for unramified character $\chi \in \widehat{F^\times}$ and $n(\psi) = -1$ we have

$$\epsilon_{BH}(\chi,s,\psi) = q_F^{s-\frac{1}{2}}\chi(\pi_F)^{-1}. \tag{2.3.19}$$

1. **Connection between ϵ_{BH} and $W(\chi,\psi)$.**

$$W(\chi,\psi) = \epsilon_{BH}(\chi,\tfrac{1}{2},\psi).$$

Proof. From [6], p. 143, Lemma 1 we see:

$$\epsilon_{BH}(\chi,\tfrac{1}{2},b\psi) = \chi(b)\epsilon_{BH}(\chi,\tfrac{1}{2},\psi)$$

[1] The convention $W(\chi,\psi)$ is actually due to Langlands [41], and it is:

$$\epsilon_L(\chi,\psi,\tfrac{1}{2}) = W(\chi,\psi).$$

See equation (3.6.4) on p. 17 of [33] for $V = \chi$.

[2] The definition of level of an additive character $\psi \in \widehat{F}$ in [6] on p. 11 is the negative sign with our conductor $n(\psi)$, i.e., level of $\psi = -n(\psi)$.

for any $b \in F^\times$. But we have seen already that $W(\chi, b\psi) = \chi(b)W(\chi,\psi)$ has the same transformation rule. If we fix one nontrivial ψ then all other nontrivial ψ' are uniquely given as $\psi' = b\psi$ for some $b \in F^\times$. Because of the parallel transformation rules it is now enough to verify our assertion for a single ψ. Now we take $\psi \in \widehat{F^+}$ with $n(\psi) = -1$, hence $\nu_F(c) = a(\chi) - 1$. Then we obtain

$$W(\chi,\psi) = W(\chi,\psi,c) = \chi(c)q_F^{-\frac{a(\chi)}{2}} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x)\psi(c^{-1}x).$$

We compare this to the equation (2.3.18). There the notation is $n = a(\chi) - 1$ and the assumption is $n \geqslant 0$. This means we have $\nu_F(c) = n$, hence we may take $\alpha = c^{-1}$ and then comparing our formula with equation (2.3.18), we see that

$$W(\chi,\psi) = \epsilon_{BH}(\chi, \tfrac{1}{2}, \psi)$$

in the case when $n(\psi) = -1$.
We are still left to prove our assertion if χ is unramified, i.e., $a(\chi) = 0$. Again we can reduce to the case where $n(\psi) = -1$. Then our assertion follows from equation 2.3.19.

□

Remark 2.3.3. From Corollary 23.4.2 of [6], on p. 142, for $s \in \mathbb{C}$, we can write

$$\epsilon_{BH}(\chi, s, \psi) = q_F^{(\frac{1}{2}-s)n(\chi,\psi)} \cdot \epsilon_{BH}(\chi, \frac{1}{2}, \psi), \tag{2.3.20}$$

for some $n(\chi,\psi) \in \mathbb{Z}$. In fact here $n(\chi,\psi) = a(\chi) + n(\psi)$. From above connection, we just see $W(\chi,\psi) = \epsilon_{BH}(\chi, \frac{1}{2}, \psi)$. Thus for arbitrary $s \in \mathbb{C}$, we obtain

$$\epsilon_{BH}(\chi, s, \psi) = q_F^{(\frac{1}{2}-s)(a(\chi)+n(\psi))} \cdot W(\chi,\psi). \tag{2.3.21}$$

This equation (2.3.21) is very important for us. We shall use this to connect with Deligne's convention.

In [33] there is defined a number $\epsilon_D(\chi, \psi, dx)$ depending on χ, ψ and a Haar measure dx on F. This notion is due to Deligne [38]. We write ϵ_D for Deligne's convention in order to distinguish it from the $\epsilon_{BH}(\chi, \frac{1}{2}, \psi)$ introduced in Bushnell-Henniart [6].

In the next Lemma we give the connection between Bushnell-Henniart and Deligne conventions for local constants.

2. **The connection between ϵ_D and ϵ_{BH}:**

Lemma 2.3.4. *We have the relation*

$$\epsilon_{BH}(\chi, s, \psi) = \epsilon_D(\chi \cdot \omega_s, \psi, dx_\psi),$$

where $\omega_s(x) = |x|_F^s = q^{-s\nu_F(x)}$ is unramified character of $F^\times$ corresponding to complex number s, and where dx_ψ is the self-dual Haar measure corresponding to the additive character ψ.

Proof. From equation equation (3.6.4) of [33], we know that

$$\epsilon_L(\chi, s, \psi) := \epsilon_L(\chi\omega_{s-\frac{1}{2}}, \psi) = \epsilon_D(\chi\omega_s, \psi, dx_\psi). \tag{2.3.22}$$

We prove this connection by using the relations (2.3.21) and (2.3.22). From equation (2.3.22) we can write our $W(\chi, \psi) = \epsilon_D(\chi\omega_{\frac{1}{2}}, \psi, dx_\psi)$. Therefore when $s = \frac{1}{2}$, we have the relation:

$$\epsilon_{BH}(\chi, \frac{1}{2}, \psi) = \epsilon_D(\chi\omega_{\frac{1}{2}}, \psi, dx_\psi), \tag{2.3.23}$$

since $W(\chi, \psi) = \epsilon_{BH}(\chi, \frac{1}{2}, \psi)$.

We know that $\omega_s(x) = q_F^{-s\nu_F(x)}$ is an unramified character of $F^\times$. So when χ is also unramified, we can write

$$W(\chi\omega_{s-\frac{1}{2}}, \psi) = \omega_{s-\frac{1}{2}}(c) \cdot \chi(c) = q_F^{(\frac{1}{2}-s)n(\psi)}\epsilon_{BH}(\chi, \frac{1}{2}, \psi) = \epsilon_{BH}(\chi, s, \psi). \tag{2.3.24}$$

And when χ is ramified character, i.e., conductor $a(\chi) > 0$, from Tate's theorem for unramified twist (see property 2.3.1(4b)) , we can write

$$\begin{aligned} W(\chi\omega_{s-\frac{1}{2}}, \psi) &= \omega_{s-\frac{1}{2}}(\pi_F^{a(\chi)+n(\psi)}) \cdot W(\chi, \psi) \\ &= q_F^{-(s-\frac{1}{2})(a(\chi)+n(\psi))} \cdot W(\chi, \psi) \\ &= q_F^{(\frac{1}{2}-s)(a(\chi)+n(\psi))} \cdot \epsilon_{BH}(\chi, \frac{1}{2}, \psi) \\ &= \epsilon_{BH}(\chi, s, \psi). \end{aligned}$$

Furthermore from equation (2.3.22), we have

$$W(\chi\omega_{s-\frac{1}{2}}, \psi) = \epsilon_D(\chi\omega_s, \psi, dx_\psi). \tag{2.3.25}$$

Therefore finally we can write

$$\epsilon_{BH}(\chi, s, \psi) = \epsilon_D(\chi\omega_s, \psi, dx_\psi). \tag{2.3.26}$$

□

Corollary 2.3.5. *For our W we have :*

$$\begin{aligned} W(\chi, \psi) = \epsilon_{BH}(\chi, \tfrac{1}{2}, \psi) = \epsilon_D(\chi\omega_{\frac{1}{2}}, \psi, dx_\psi) \\ W(\chi\omega_{s-\frac{1}{2}}, \psi) = \epsilon_{BH}(\chi, s, \psi) = \epsilon_D(\chi\omega_s, \psi, dx_\psi). \end{aligned}$$

Proof. From the equations (3.6.1) and (3.6.4) of [33] for χ and above two connections the assertions follow. □

2.3.3 Local constants for virtual representations

1. To extend the concept of local constant, we need to go from 1-dimensional to other virtual representations ρ of the Weil groups W_F of non-archimedean local field F. According to Tate [32], the root number $W(\chi) := W(\chi, \psi_F)$ extensions to $W(\rho)$, where ψ_F is the canonical additive character of F. More generally, $W(\chi, \psi)$ extends to $W(\rho, \psi)$, and if E/F is a finite separable extension then one has to take $\psi_E = \psi_F \circ \mathrm{Tr}_{E/F}$ for the extension field E.

   According to Bushnell-Henniart [6], Theorem on p. 189, the functions $\epsilon_{BH}(\chi, s, \psi)$ extend to $\epsilon_{BH}(\rho, s, \psi_E)$, where $\psi_E = \psi \circ \mathrm{Tr_{E/F}}$ [3]. According to Tate [33], Theorem (3.4.1) the functions $\epsilon_D(\chi, \psi, dx)$ extends to $\epsilon_D(\rho, \psi, dx)$. In order to get **weak inductivity** we have again to use $\psi_E = \psi \circ \mathrm{Tr}_{E/F}$ if we consider extensions. Then according to Tate [33] (3.6.4) the Corollary 2.3.5 turns into

   Corollary 2.3.6. *For the virtual representations of the Weil groups we have*

   $$W(\rho\omega_{E,s-\frac{1}{2}}, \psi_E) = \epsilon_{BH}(\rho, s, \psi_E) = \epsilon_D(\rho\omega_{E,s}, \psi_E, dx_{\psi_E}).$$
   $$W(\rho, \psi_E) = \epsilon_{BH}(\rho, \tfrac{1}{2}, \psi_E) = \epsilon_D(\rho\omega_{E,\frac{1}{2}}, \psi_E, dx_{\psi_E}).$$

   Note that on the level of field extension E/F we have to use $\omega_{E,s}$ which is defined as

   $$\omega_{E,s}(x) = |x|_E^s = q_E^{-s\nu_E(x)}.$$

   We also know that $q_E = q_F^{f_{E/F}}$ and $\nu_E = \frac{1}{f_{E/F}} \cdot \nu_F(N_{E/F})$ (cf. [21], p. 41, Theorem 2.5), therefore we can easily see that

   $$\omega_{E,s} = \omega_{F,s} \circ N_{E/F}.$$

   Since the norm map $N_{E/F} : E^\times \to F^\times$ corresponds via class field theory to the injection map $G_E \hookrightarrow G_F$, Tate [33] beginning from (1.4.6), simply writes $\omega_s = ||^s$ and consider ω_s as an unramified character of the Galois group (or of the Weil group) instead as a character on the field. Then Corollary 2.3.6 turns into

   $$W(\rho\omega_{s-\frac{1}{2}}, \psi_E) = \epsilon_{BH}(\rho, s, \psi_E) = \epsilon_D(\rho\omega_s, \psi_E, dx_{\psi_E}), \tag{2.3.27}$$

   for all field extensions, where ω_s is to be considered as 1-dimensional representation of the Weil group $W_E \subset G_E$ if we are on the E-level. The left side equation (2.3.27) is the ϵ-factor of Langlands (see [33], (3.6.4)).

2. The functional equation (2.3.9) extends to

   $$W(\rho, \psi) \cdot W(\rho^V, \psi) = \det_\rho(-1), \tag{2.3.28}$$

[3] Note that they fix a base field F and a nontrivial $\psi = \psi_F$ (which not to be the canonical character used in Tate [32]) but then if E/F is an extension they always use $\psi_E = \psi \circ \mathrm{Tr}_{E/F}$.

where ρ is any virtual representation of the Weil group W_F, ρ^V is the contragredient and ψ is any nontrivial additive character of F. This is formula (3) on p. 190 of [6] for $s = \frac{1}{2}$.

3. Moreover, the transformation law [33] (3.4.5) can (on the F-level) be written as: **unramified character twist**

$$\epsilon_D(\rho\omega_s, \psi, dx) = \epsilon_D(\rho, \psi, dx) \cdot \omega_{F,s}(c_{\rho,\psi}) \tag{2.3.29}$$

for any $c = c_{\rho,\psi}$ such that $\nu_F(c) = a(\rho) + n(\psi)\dim(\rho)$. It implies that also for the root number on the F-level we have

$$W(\rho\omega_s, \psi) = W(\rho, \psi) \cdot \omega_{F,s}(c_{\rho,\psi}). \tag{2.3.30}$$

2.4 Classical Gauss sums

Let k_q be a finite field. Let p be the characteristic of k_q; then the prime field contained in k_q is k_p. The structure of the **canonical** additive character ψ_q of k_q is the same as the structure of the canonical character ψ_F, namely **it comes by trace** from the canonical character of the base field, i.e.,

$$\psi_q = \psi_p \circ \mathrm{Tr}_{\mathrm{k_q/k_p}},$$

where

$$\psi_p(x) := e^{\frac{2\pi i x}{p}} \quad \text{for all } x \in k_p.$$

Gauss sums: Let χ be a multiplicative and ψ an additive character of k_q. Then the Gauss sum $G(\chi, \psi)$ is define by

$$G(\chi, \psi) = \sum_{x \in k_q^\times} \chi(x)\psi(x). \tag{2.4.1}$$

In general, computation of this Gauss is very difficult, but for certain characters, the associated Gauss sums can be evaluated explicitly. In the following theorem for quadratic characters of k_q we can give explicit formula of Gauss sums.

Theorem 2.4.1 ([40], p. 199, Theorem 5.15)**.** *Let k_q be a finite field with $q = p^s$, where p is an odd prime and $s \in \mathbb{N}$. Let χ be the quadratic character of k_q and let ψ be the canonical additive character of k_q. Then*

$$G(\chi, \psi) = \begin{cases} (-1)^{s-1} q^{\frac{1}{2}} & \text{if } p \equiv 1 \pmod 4, \\ (-1)^{s-1} i^s q^{\frac{1}{2}} & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{2.4.2}$$

Let ψ be an additive and χ a multiplicative character of k_q. Let E be a finite extension field of k_q. Then ψ and χ can be **lifted** to E by the setting

$$\psi'(x) = \psi(\mathrm{Tr}_{E/k_q}(x)) \text{ for all } x \in E \text{ and } \chi'(x) = \chi(N_{E/k_q}(x)) \text{ for all } x \in E^\times.$$

From the additivity of the trace and multiplicativity of the norm it follows that χ' is an additive character and χ' is a multiplicative character of E. The following theorem gives the relation between the Gauss sum $G(\chi, \psi)$ in k_q and the Gauss sum $G(\chi', \psi')$ in E.

Theorem 2.4.2 (Davenport-Hasse, [40], p. 197, Theorem 5.14)**.** *Let χ be a multiplicative and ψ an additive character of k_q, not both of them trivial. Suppose χ and ψ are lifted to character χ' and ψ', respectively, of the finite extension field E of k_q with $[E : k_q] = s$. Then*

$$G(\chi', \psi') = (-1)^{s-1} \cdot G(\chi, \psi)^s. \tag{2.4.3}$$

2.5 Witt ring and square class group of a local field

Let F be a field. Let M be any commutative cancellation monoid under addition. We define a relation $\sim$ on $M \times M$ by

$$(x, y) \sim (x', y') \iff x + y' = x' + y \in M.$$

The cancellation law in M implies that $\sim$ is an equivalence relation on $M \times M$. We define the **Grothendieck group** of M to be $\mathrm{Groth(M)} = (\mathrm{M} \times \mathrm{M})/\sim$ (the set of equivalence classes) with addition induced by

$$(x, y) + (x', y') = (x + x', y + y').$$

We can prove that Groth(M) is the additive group generated by M.

Now let $M(F)$ be the set of all isometry classes of (nonsingular) quadratic forms of F, and replace M by $M(F)$ in the definition of Grothendieck group. Then we denote $\widehat{W}(F) =$ Groth(M(F)) which is called the **Witt-Grothendieck** ring of quadratic forms over the field F. Every element of $\widehat{W}(F)$ has the formal expression $q_1 - q_2$, where q_1, q_2 are nonsingular quadratic forms, or rather, isometry classes of such forms.

Now, consider the dimension map $\dim : \mathrm{M(F)} \to \mathbb{Z}$, which is a semiring homomorphism on $M(F)$. This extends uniquely (via the universal property) to a ring homomorphism $\dim : \widehat{\mathrm{W}}(\mathrm{F}) \to \mathbb{Z}$, by

$$\dim(\mathrm{q}_1 - \mathrm{q}_2) = \dim(\mathrm{q}_1) - \dim(\mathrm{q}_2).$$

The kernel of this ring homomorphism, denoted by $\widehat{I}F$, is called the **the fundamental ideal of** $\widehat{W}(F)$. We have $\widehat{W}(F)/\widehat{I}F = \mathbb{Z}$.

Let $\mathbb{Z} \cdot \mathbb{H}$ be the set which consists of all hyperabolic spaces and their additive inverses, and they form an ideal of $\widehat{W}(F)$. The vector ring

$$W(F) = \widehat{W}(F)/\mathbb{Z} \cdot \mathbb{H}$$

is called the **Witt ring** of F. The image of the ideal $\widehat{I}F$ under the natural projection $\widehat{W}(F) \to W(F)$ is denoted by IF; this is called the **fundamental ideal** of $W(F)$. It can be shown that $W(F)/IF \cong \mathbb{Z}/2\mathbb{Z}$ (cf. [47], p. 30, Corollary 1.6).

The group $F^\times/F^{\times 2}$ is called the **square class group** of F, and $Q(F) = \mathbb{Z}_2 \times F^\times/F^{\times 2}$ is called the **extended** square class group of F. We also have $W(F)/I^2F \cong Q(F)$ (cf. [47], p. 31, Proposition 2.1), where I^2F denotes the square of IF. By **Pfister's result** (cf. [47], p. 32, Corollary 2.3), we have the square class group $F^\times/F^{\times 2} \cong IF/I^2F$.

Now we come to our local field case. Let $F/\mathbb{Q}_p$ be a local field. When $F/\mathbb{Q}_p$ is a local field with $p \neq 2$, from Theorem 2.2(1) of [47] we have $F^\times/F^{\times 2} \cong V$, where V is Klein's 4-group. More generally, we need the following results for computing λ-function in the wild case.

Theorem 2.5.1 ([47], p. 162, Theorem 2.22). *Let $F/\mathbb{Q}_p$ be a local field with q_F as the cardinality of the residue field of F. Let $s = \nu_F(2)$. Then $|F^\times/F^{\times 2}| = 4 \cdot q_F^s$.*
In particular, if $p \neq 2$, i.e., $s = \nu_F(2) = 0$, we have $|F^\times/F^{\times 2}| = 4$.

When $p = 2$ and $F/\mathbb{Q}_2$ is a local field of degree n over $\mathbb{Q}_2$, we have $s = \nu_F(2) = e_{F/\mathbb{Q}_2}$ because 2 is a uniformizer in $\mathbb{Q}_2$. We also know that $q_F = q_{\mathbb{Q}_2}^{f_{F/\mathbb{Q}_2}} = 2^{f_{F/\mathbb{Q}_2}}$. We also know that $e_{F/\mathbb{Q}_2} \cdot f_{F/\mathbb{Q}_2} = n$. Then from the above Theorem 2.5.1 we obtain

$$|F^\times/F^{\times 2}| = 4 \cdot q_F^{e_{F/\mathbb{Q}_2}} = 4 \cdot (2^{f_{F/\mathbb{Q}_2}})^{e_{F/\mathbb{Q}_2}} = 4 \cdot 2^n = 2^{2+n}. \tag{2.5.1}$$

Theorem 2.5.2 ([47], p. 165, Theorem 2.29). *Let $F/\mathbb{Q}_2$ be a dyadic local field, with $|F^\times/F^{\times 2}| = 2^m$ ($m \geqslant 3$). Then:*

1. *Case 1: When $-1 \in F^{\times 2}$, we have $W(F) \cong \mathbb{Z}_2^{m+2}$ (here $\mathbb{Z}_n^k$ denotes the direct product of k copies of $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$).*
2. *Case 2: When $-1 \notin F^{\times 2}$, but -1 is a sum of two squares, we have $W(F) \cong \mathbb{Z}_4^2 \times \mathbb{Z}_2^{m-2}$.*
3. *Case 3: When -1 is not a sum of two squares, we have $W(F) = \mathbb{Z}_8 \times \mathbb{Z}_2^{m-1}$.*

2.6 Heisenberg representations

Let ρ be an irreducible representation of a (pro-)finite group G. Then ρ is called a **Heisenberg representation** if it represents commutators by scalar matrices. Therefore higher commutators are represented by 1. We can see that the linear characters of G are Heisenberg representations as the degenerate special case. To classify Heisenberg representations we need to mention two invariants of an irreducible representation $\rho \in \mathrm{Irr(G)}$:

1. Let Z_ρ be the **scalar** group of ρ, i.e., $Z_\rho \subseteq G$ and $\rho(z) =$ scalar matrix for every $z \in Z_\rho$. If $V/\mathbb{C}$ is a representation space of ρ we get Z_ρ as the kernel of the composite map

$$G \xrightarrow{\rho} GL_{\mathbb{C}}(V) \xrightarrow{\pi} PGL_{\mathbb{C}}(V) = GL_{\mathbb{C}}(V)/\mathbb{C}^\times E, \tag{2.6.1}$$

   where E is the unit matrix and denote $\overline{\rho} := \pi \circ \rho$. Therefore Z_ρ is a normal subgroup of G.

2. Let χ_ρ be the character of Z_ρ which is given as $\rho(g) = \chi_\rho(g)\cdot E$ for all $g \in Z_\rho$. Apparently χ_ρ is a G-invariant character of Z_ρ which we call the central character of ρ.

Let A be a profinite abelian group. Then we know that (cf. [14], p. 124, Theorem 1 and Theorem 2) the set of isomorphism classes PI(A) of projective irreducible representations (for projective representation, see [7], §51) of A is in bijective correspondence with the set of continuous alternating characters Alt(A). If $\rho \in \mathrm{PI(A)}$ corresponds to $X \in \mathrm{Alt(A)}$ then

$$\mathrm{Ker}(\rho) = \mathrm{Rad(X)} \qquad \text{and} \quad [A : \mathrm{Rad(X)}] = \dim(\rho)^2,$$

where $\mathrm{Rad(X)} := \{\mathrm{a} \in \mathrm{A} |\, \mathrm{X(a,b)} = 1, \text{ for all}\, \mathrm{b} \in \mathrm{A}\}$, the **radical of** X.

Let $A := G/[G,G]$, so A is abelian. We also know from the composite map (2.6.1) $\overline{\rho}$ is a projective irreducible representation of G and Z_ρ is the kernel of $\overline{\rho}$. Therefore **modulo commutator group** $[G,G]$, we can consider that $\overline{\rho}$ is in PI(A) which corresponds an alternating character X of A with kernel of $\overline{\rho}$ is $Z_\rho/[G,G] = \mathrm{Rad(X)}$. We also know that

$$[A : \mathrm{Rad(X)}] = [\mathrm{G/[G,G]} : \mathrm{Z_\rho/[G,G]}] = [\mathrm{G} : \mathrm{Z_\rho}].$$

Then we observe that

$$\dim(\overline{\rho}) = \dim(\rho) = \sqrt{[\mathrm{G} : \mathrm{Z_\rho}]}.$$

Let H be a subgroup of A, then we define the orthogonal complement of H in A with respect to X

$$H^\perp := \{a \in A : \quad X(a,H) \equiv 1\}.$$

An **isotropic** subgroup $H \subset A$ is a subgroup such that $H \subseteq H^\perp$ (cf. [10], p. 270, Lemma 1(v)). And when isotropic subgroup H is maximal, we call H is a **maximal isotropic** for X. Thus when H is maximal isotropic we have $H = H^\perp$.

We also can show that the Heisenberg representations ρ are fully characterized by the corresponding pair (Z_ρ, χ_ρ).

Proposition 2.6.1 ([12], Proposition 4.2). *The map $\rho \mapsto (Z_\rho, \chi_\rho)$ is a bijection between equivalence classes of Heisenberg representations of G and the pairs (Z_ρ, χ_ρ) such that*

(a) $Z_\rho \subseteq G$ is a coabelian normal subgroup,

(b) χ_ρ is a G-invariant character of Z_ρ,

*(c) $X(\hat{g_1}, \hat{g_2}) := \chi_\rho(g_1 g_2 g_1^{-1} g_2^{-1})$ is a nondegenerate **alternating character** on G/Z_ρ where $\hat{g_1}, \hat{g_2} \in G/Z_\rho$ and their corresponding lifts $g_1, g_2 \in G$.*

For pairs (Z_ρ, χ_ρ) with the properties $(a)-(c)$, the corresponding Heisenberg representation ρ is determined by the identity:

$$\sqrt{[G : Z_\rho]} \cdot \rho = \mathrm{Ind}_{Z_\rho}^{G} \chi_\rho. \tag{2.6.2}$$

Moreover, the character tr_ρ of the Heisenberg representation ρ is

$$\mathrm{tr}_\rho(g) = \begin{cases} 0 & \text{if } g \in G - Z_\rho \\ \dim(\rho)\cdot\chi_\rho(g) & \text{if } g \in Z_\rho \end{cases}$$

Let $C^1G = G$, $C^{i+1}G = [C^iG, G]$ denote the descending central series of G. Now assume that every projective representation of A lifts to an ordinary representation of G. Then by I. Schur's results (cf. [7], p. 361, Theorem 53.7) we have (cf. [14], p. 124, Theorem 2):

1. Let $A \wedge_{\mathbb{Z}} A$ denote the alternating square of the $\mathbb{Z}$-module A. The commutator map

$$A \wedge_{\mathbb{Z}} A \cong C^2G/C^3G, \quad a \wedge b \mapsto [\hat{a}, \hat{b}] \tag{2.6.3}$$

   is an isomorphism.

2. The map $\rho \to X_\rho \in \mathrm{Alt(A)}$ from Heisenberg representations to alternating characters on A is surjective.

Proof. (**of the equation (2.6.2)**)

Let H be a maximal **isotropic** subgroup of G for the Heisenberg representation ρ and choose a character $\chi_H : H \to \mathbb{C}^\times$ such that $\chi_H|_{Z_\rho} = \chi_\rho$. Then we have (cf. [12], p. 193, Proposition 5.3):

$$\rho = \mathrm{Ind}_H^G \chi_H. \tag{2.6.4}$$

This induced representation from χ_H does not depend on the choice of H and the extension of χ_ρ to H. We also know from the transitivity of induction:

$$\mathrm{Ind}_{Z_\rho}^G \chi_\rho = \mathrm{Ind}_H^G \mathrm{Ind}_{Z_\rho}^H \chi_\rho. \tag{2.6.5}$$

Furthermore, we can also write

$$\begin{aligned} \mathrm{Ind}_{Z_\rho}^H \chi_\rho &= \chi_H \otimes Ind_{Z_\rho}^H 1_{Z_\rho}, \quad \text{where } 1_{Z_\rho} \text{ is the trivial representation of } Z_\rho, \\ &= \sum_{\text{all } \chi'_H \text{ which are extension of } \chi_\rho} \chi'_H. \end{aligned} \tag{2.6.6}$$

Here the total number of χ'_H is exactly equal to $[H : Z_\rho]$. Putting this above result in the equation (2.6.5), we have:

$$\begin{aligned} \mathrm{Ind}_{Z_\rho}^G \chi_\rho &= \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\mathrm{Ind}_{\mathrm{Z}_\rho}^{\mathrm{H}} \chi_\rho) \\ &= \mathrm{Ind}_H^G\Big(\sum_{\text{all } \chi'_H \text{ which are extension of } \chi_\rho} \chi'_H\Big), \\ &= \{\text{no. of } \chi_H \text{ which are extended from } \chi_\rho\} \times \mathrm{Ind}_H^G \chi'_H, \\ &= [H : Z_\rho] \cdot \mathrm{Ind}_H^G \chi'_H, \\ &= [H : Z_\rho] \cdot \rho \quad \text{since } \mathrm{Ind}_H^G \chi'_H = \rho. \end{aligned} \tag{2.6.7}$$

We also know that

$$[G:H]=[H:Z_\rho]=\sqrt{[G:Z_\rho]}=\dim\rho. \tag{2.6.8}$$

From the equations (2.6.7) and (2.6.8) we have our desired result which is:

$$\begin{aligned}\mathrm{Ind}_{Z_\rho}^{G}\chi_\rho &= \sqrt{[G:Z_\rho]}\cdot\rho \\ &= \dim\rho\cdot\rho\end{aligned} \tag{2.6.9}$$

Therefore, the equation (2.6.2) is proved. □

Remark 2.6.2. Let χ_ρ be a character of Z_ρ. All extensions $\chi_H\supset\chi_\rho$ are conjugate with respect to G/H. This can be easily seen, since we know $\chi_H\supset\chi_\rho$ and $\chi_H^g(h)=\chi_H(ghg^{-1})$. If we take $z\in Z_\rho$, then we obtain

$$\begin{aligned}\chi_H^g(z)&=\chi_H(gzg^{-1})=\chi_\rho(gzg^{-1})=\chi_\rho(gzg^{-1}z^{-1}z)\\ &=\chi_\rho([g,z]z)=X(g,z)\cdot\chi_\rho(z)=\chi_\rho(z),\end{aligned}$$

since Z_ρ is a normal subgroup of G and the radical of X (i.e., $X(g,z)=\chi_\rho([g,z])=1$ for all $z\in Z_\rho$ and $g\in G$). Therefore, χ_H^g are extensions of χ_ρ for all $g\in G/H$. It can also be seen that the conjugates χ_H^g are all different, because $\chi_H^{g_1}=\chi_H^{g_2}$ is the same as $\chi_H^{g_1g_2^{-1}}=\chi_H$. So it is enough to see that $\chi_H^{g-1}\not\equiv 1$ if $g\neq 1\in G/H$. But

$$\chi_H^{g-1}(h)=\chi_\rho(ghg^{-1}h^{-1})=X(g,h),$$

and therefore $\chi_H^{g-1}\equiv 1$ on H implies $g\in H^\perp=H$, where "$\perp$" denotes the orthogonal complement with respect to X. Then for a given one extension χ_H of χ_ρ all other extensions are of the form χ_H^g for $g\in G/H$.

Remark 2.6.3. Let F be a non-archimedean local field and $G_F:=\mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F})$ be the absolute Galois group. The Heisenberg representations of G_F have arithmetic structure due to E.-W. Zink [13], [14]. For Chapter 4, we just need its group theoretical structure, that is why here we discuss this group theoretical definition. But in Chapter 5 we also need to see its arithmetic structure and we will study them in Chapter 5.

2.7 Some useful results from finite Group Theory

Let G be a finite abelian group and put $\alpha=\prod_{g\in G}g$. By the following theorem we can compute α. It is very much essential for our computation. In the Heisenberg setting for computing transfer map we have to deal with abelian group G/H and $\prod_{t\in G/H}t$, where H is a normal subgroup of G.

Theorem 2.7.1 ([37], Theorem 6 (Miller)). *Let G be a finite abelian group and $\alpha=\prod_{g\in G}g$.*

1. *If G has no element of order 2, then $\alpha=e$.*

2. *If G has a unique element t of order 2, then $\alpha = t$.*

3. *If G has at least two elements of order 2, then $\alpha = e$.*

Let G be a two-step nilpotent group[4]. For this two-step nilpotent group, we have the following lemma.

Lemma 2.7.2 ([25], p. 77, Lemma 9)**.** *Let G be a two-step nilpotent group and let $x, y \in G$. Then*

1. $[x^n, y] = [x, y]^n$, *and*

2. $x^n y^n = (xy)^n [x, y]^{\frac{n(n-1)}{2}}$,

for any $n \in \mathbb{N}$.

We also need the elementary divisor theorem for this article which we take from [8]. Let G be a finite abelian group. So G is finitely generated.

Theorem 2.7.3 ([8], p. 160, Theorem 3 (Invariant form))**.** *Let G be a finite abelian group. Then*

$$G \cong \mathbb{Z}/n_1 \times \mathbb{Z}/n_2 \times \cdots \times \mathbb{Z}/n_s. \tag{2.7.1}$$

for some integers $n_1, n_2, \cdots, n_s$ satisfying the following conditions:

(a) $n_j \geqslant 2$ *for all* $j \in \{1, 2, \cdots, s\}$, *and*

(b) $n_{i+1} | n_i$ *for all* $1 \leqslant i \leqslant s - 1$.

And the expression in (2.7.1) is unique: if $G \cong \mathbb{Z}/m_1 \times \mathbb{Z}/m_2 \times \cdots \times \mathbb{Z}/m_r$, where $m_1, m_2, \cdots, m_r$ satisfies conditions (a) and (b), i.e., $m_j \geqslant 2$ for all j and $m_{i+1} | m_i$ for all $1 \leqslant i \leqslant r - 1$, then $s = r$ and $m_i = n_i$ for all i.

This theorem is known as the **elementary divisor theorem** of a finite abelian group. Moreover, since G is direct product of $\mathbb{Z}/n_i$, $1 \leqslant i \leqslant s$, then we can write

$$|G| = n_1 n_2 \cdots n_s.$$

We also need a structure theorem for finite abelian groups which come provided with an alternating character:

Lemma 2.7.4 ([10], p. 270, Lemma 1(VI))**.** *Let G be a finite abelian group and assume the existence of an alternating bi-character $X : G \times G \to \mathbb{C}^\times$ ($X(g, g) = 1$ for all $g \in G$, hence $1 = X(g_1 g_2, g_1 g_2) = X(g_1, g_2) \cdot X(g_2, g_1)$) which is nondegenerate. Then there will exist elements $t_1, t_1', \cdots, t_s, t_s' \in G$ such that*

[4]Its derived subgroup, i.e., commutator subgroup $[G, G]$ is contained in its center. In other worlds, $[G, [G, G]] = \{1\}$, i.e., any triple commutator gives identity. If ρ is a Heisenberg representation of a finite group G, then $G/\mathrm{Ker}(\rho)$ is a two-step nilpotent group (cf. p. 6).

1. $G = < t_1 > \times < t'_1 > \times \cdots \times < t_s > \times < t'_s >$
   $\cong \mathbb{Z}/m_1 \times \mathbb{Z}/m_1 \times \cdots \times \mathbb{Z}/m_s \times \mathbb{Z}/m_s$ *and* $m_1 | \cdots | m_s$.

2. *For all* $i = 1, 2, \cdots, s$ *we have* $X(t_i, t'_i) = \zeta_{m_i}$ *a primitive* m_i*-th root of unity.*

3. *If we say* $g_1 \perp g_2$ *if* $X(g_1, g_2) = 1$, *then* $(< t_i > \times < t'_i >)^{\perp} = \prod_{j \neq i}(< t_j > \times < t'_j >)$.

2.8 Transfer map

Let H be a subgroup of a finite group G. Let $\{t_1, t_2, \cdots, t_n\}$ be a left transversal for H in G. If $g \in G$ then for all $i = 1, 2, \cdots, n$ we obtain,

$$gt_i \in t_{g(i)}H, \tag{2.8.1}$$

where the map $i \mapsto g(i)$ is a well-defined permutation of the set $\{1, 2, \cdots, n\}$. Assume that $f : H \to A$ is a homomorphism from H to an abelian group A. Then **transfer** of f, written T_f, is a mapping

$$T_f : G \to A \quad \text{given by}$$
$$T_f(g) = \prod_{i=1}^{n} f(t_{g(i)}^{-1} g t_i) \quad \text{for all } g \in G.$$

Since A is abelian, the order of the factors in the product is irrelevant. Since f is a homomorphism from H to A, from above we can see that T_f is a homomorphism G with abelian image, and therefore always: $[G, G] \subseteq \mathrm{Ker}(T_f)$. Now we take f the canonical homomorphism, i.e.,

$$f : H \to H/[H, H], \text{ where } [H, H] \text{ is the commutator subgroup of } H.$$

And we denote $T_f = T_{G/H}$. Thus by definition of transfer map $T_{G/H} : G \to H/[H, H]$, given by

$$T_{G/H}(g) = \prod_{i=1}^{n} f(t_{g(i)}^{-1} g t_i) = \prod_{i=1}^{n} t_{g(i)}^{-1} g t_i [H, H], \tag{2.8.2}$$

for all $g \in G$.

Moreover, if H is any subgroup of finite index in G, then (cf. [9], Chapter 13, p. 183)

$$T_{G/gHg^{-1}}(g') = gT_{G/H}(g')g^{-1}, \tag{2.8.3}$$

for all $g, g' \in G$. Now let H be an abelian normal subgroup of G. Let $H^{G/H}$ be the set consisting the elements which are invariant under conjugation. So it is clear that these elements are central elements and $H^{G/H} \subseteq Z(G)$, the center of G. When H is abelian normal subgroup of G, from equation (2.8.3) we can conclude that (cf. [9], Chapter 13, p. 183) that

$$\mathrm{Im}(T_{G/H}) \subseteq H^{G/H} \subseteq Z(G). \tag{2.8.4}$$

We also need to mention the generalized **Furtwängler**'s theorem for this thesis.

Theorem 2.8.1 ([22], p. 320, Theorem 10.25)**.** *Let G be a finite group, and let $T_{G/K} : G \to K/[K,K]$ be the transfer homomorphism, where $[G,G] \subseteq K \subseteq G$. Then $T_{G/K}(g)^{[K:[G,G]]} = 1$ for all elements $g \in G$.*

Now if $[K : [G,G]] = 1$, i.e., $K = [G,G]$, we have $T_{G/[G,G]}(g) = 1$ for all $g \in G$, i.e., the transfer homomorphism of a finite group to its commutator is **trivial**. This is due to Furtwängler. This is also known as **Principal Ideal Theorem** (cf. [9], p. 194).

To compute the determinant of an induced representation of a finite group, we need the following theorem.

Theorem 2.8.2 (Gallagher, [17], Theorem 30.1.6)**.** *Let G be a finite group and H a subgroup of G. Let ρ be a representation of H and denote $\Delta_H^G = \det(\mathrm{Ind}_H^G 1_H)$. Then*

$$\det(\mathrm{Ind}_H^G\rho)(g) = (\Delta_H^G)^{\dim(\rho)}(g) \cdot (\det(\rho) \circ T_{G/H})(g), \quad \textit{for all } g \in G. \tag{2.8.5}$$

Let T be a left transversal for H in G. Here $\mathrm{Ind}_H^G\rho$ is a block monomial representation (cf. [17], p. 956) with block positions indexed by pairs $(t,s) \in T \times T$. For $g \in G$, the (t,s)-block of $\mathrm{Ind}_H^G\rho$ is zero unless $gt \in sH$, i.e., $s^{-1}gt \in H$ and in which case the block equal to $\rho(s^{-1}gt)$. Then we can write for $g \in G$

$$T_{G/H}(g) = \prod_{t\in T} s^{-1}gt[H,H]. \tag{2.8.6}$$

Thus we can write for all $g \in G$

$$\begin{aligned}\det(\mathrm{Ind}_H^G\rho)(g) &= (\Delta_H^G)^{\dim(\rho)}(g) \cdot (\det(\rho) \circ T_{G/H}(g) \\ &= (\Delta_H^G)^{\dim(\rho)}(g) \cdot (\det(\rho)(\prod_{t\in T} s^{-1}gt[H,H]) \\ &= (\Delta_H^G)^{\dim(\rho)}(g) \cdot \prod_{t\in T} \det(\rho)(s^{-1}gt[H,H]),\end{aligned} \tag{2.8.7}$$

where in each factor on the right, $s = s(t)$ is uniquely determined by $gt \in sH$.

Chapter 3

Computation of λ-functions

In this chapter we give explicit computation of $\lambda_{K/F}$, where K/F is a finite local Galois extension. Two different ways we can define λ-functions: One is directly from local constants, and another one is via Deligne's constants. We will use both of them according to our convenience. In Section 3.2 we first compute λ-function for odd degree Galois extension. And in Section 3.4 we compute λ-functions for even degree tamely ramified Galois extensions. The whole computation is based on the article [43].

3.1 Deligne's Constants

Let K/F be a finite Galois extension of a local field F of characteristic zero. Let $G = \mathrm{Gal}(K/F)$, and let $\rho : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ be a representation. Then for this representation, Deligne (cf. [32], p. 119) defines:

$$c(\rho) := \frac{W(\rho,\psi)}{W(\det(\rho),\psi)}, \tag{3.1.1}$$

where ψ is some additive character of F. If we change the additive character ψ to $\psi' = b\psi$, where $b \in F^\times$, then from [6], p. 190, part (2) of the Proposition, we see:

$$W(\rho, b\psi) = \epsilon(\rho, \frac{1}{2}, b\psi) = \det_\rho(b)\cdot\epsilon(\rho, \frac{1}{2}, \psi) = \det_\rho(b)W(\rho,\psi). \tag{3.1.2}$$

Also, from the property of abelian local constants we have $W(\det(\rho), b\psi) = \det_\rho(b)\cdot W(\det_\rho, \psi)$, hence

$$\frac{W(\rho,\ b\psi)}{W(\det_\rho,\ b\psi)} = \frac{W(\rho,\ \psi)}{W(\det_\rho,\ \psi)} = c(\rho).$$

This shows that the Deligne's constant $c(\rho)$ does not depend on the choice of the additive character ψ. We also have the following properties of Deligne's constants:

Proposition 3.1.1 ([32], p. 119, Proposition 2)**.** *(i) If* $\dim(\rho) = 1$*, then* $c(\rho) = 1$.

(ii)

$$c(\rho_1+\rho_2) = c(\rho_1)c(\rho_2)W(\det(\rho_1))W(\det(\rho_2))\cdot W(\det(\rho_1)\cdot\det(\rho_2))^{-1}. \tag{3.1.3}$$

(iii) $c(\rho+\overline{\rho}) = \det(\rho)(-1)$.

(iv) $c(\overline{\rho}) = \overline{c(\rho)}$, *and* $|c(\rho)| = 1$.

(v) Suppose $\rho = \overline{\rho}$. *Then* $c(\rho) = \pm 1$.

Now G be a finite group. Let ρ be an orthogonal representation of G, i.e., $\rho : G \to O(n)$. We denote the i-th **Stiefel-Whitney class** of ρ by

$$s_i(\rho) \in H^i(G, \mathbb{Z}/2\mathbb{Z}).$$

In low dimensions i, the Stiefel-Whitney class is given algebraically as follows (cf. [24], [39]): Under the canonical isomorphism:

$$H^1(G, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom(G}, \{\pm 1\}),$$

the image of $s_1(\rho)$ is $\det_\rho$. If $s_1(\rho)$ is trivial, i.e., $\det_\rho \equiv 1$, then $s_2(\rho)$ is the element of $H^2(G, \{\pm1\}) = H^2(G, \mathbb{Z}/2\mathbb{Z})$ which is inverse image under $\rho : G \to SO(n)$ of the class of the extension:

$$1 \to \{\pm 1\} \to \mathrm{Spin(n)} \to \mathrm{SO(n)} \to 1,$$

where SO denotes the special orthogonal group and Spin the spinor group.

Now take $G = \mathrm{Gal(K/F)}$, for some finite Galois extension of local fields. In the following theorem due to Deligne for an orthogonal representation $\rho : G \to O(n)$, we know a procedure how to obtain out of $s_2(\rho)$ the constant $c(\rho)$.

Theorem 3.1.2 (Deligne, [32], p. 129, Theorem 3). *Let ρ be an **orthogonal representation** of the finite group G and let $s_2(\rho) \in H^2(G, \mathbb{Z}/2\mathbb{Z})$ be the second Stiefel-Whitney class of ρ. The Galois group $G = \mathrm{Gal}(K/F)$ is a quotient group of the full Galois group $G_F = \mathrm{Gal}(\overline{F}/F)$ which induces an inflation map*

$$\mathrm{Inf} : H^2(G, \mathbb{Z}/2\mathbb{Z}) \to H^2(G_F, \mathbb{Z}/2\mathbb{Z}) \cong \{\pm 1\}. \tag{3.1.4}$$

Then

$$c(\rho) = cl(s_2(\rho)) \in \{\pm 1\} \tag{3.1.5}$$

is the image of the second Stiefel-Whitney class $s_2(\rho)$ under the inflation map (3.1.4). In particular, we have $c(\rho) = 1$ if $s_2(\rho) = 0 \in H^2(G, \mathbb{Z}/2\mathbb{Z})$.

3.2 When K/F is an odd degree Galois extension

We know that our local constant satisfies the following functional equation (cf. equation (2.3.28))

$$W(\rho)W(\widetilde{\rho}) = \det_\rho(-1), \tag{3.2.1}$$

where ρ is a representation of G and $\widetilde{\rho}$ is the contragredient representation of ρ. In this equation (3.2.1), we plug the orthogonal representation $\rho = \mathrm{Ind}_H^G 1_H$ in the place of ρ and we have

$$W(\mathrm{Ind}_H^G 1_H) W(\widetilde{\mathrm{Ind}_H^G 1_H}) = \det_{\mathrm{Ind}_H^G 1_H}(-1) \tag{3.2.2}$$

Now we have from the definition of λ-factor,

$$\begin{aligned}
\lambda_H^G(W)\lambda_H^G(W) &= \det_{\mathrm{Ind}_H^G 1_H}(-1), \quad \text{since } \widetilde{\mathrm{Ind}_H^G 1_H} = \mathrm{Ind}_H^G 1_H \\
(\lambda_H^G(W))^2 &= \det_{\mathrm{Ind}_H^G 1_H}(-1) \\
(\lambda_H^G(W))^4 &= (\det_{\mathrm{Ind}_H^G 1_H}(-1))^2 \\
(\lambda_H^G(W))^4 &= 1, \quad \text{since } \det_{\mathrm{Ind}_H^G 1_H}(-1) \text{ is a sign}
\end{aligned}$$

Therefore, our λ-factor $\lambda_H^G(W)$ is always a **fourth root of unity**.

We also know that

$$\det(\mathrm{Ind}_H^G \chi_H)(s) = \varepsilon_{G/H}(s) \cdot (\chi_H \circ T_{G/H}(s)),$$

where $T_{G/H}$ is the transfer map from $G/[G,G]$ to $H/[H,H]$ and $\varepsilon_{G/H}(s)$ is the sign of $s \in G$ understood as permutation of finite set $G/H : \{gH \mapsto sgH\}$. If we take $\chi_H = 1_H$ the trivial character of H then in particular see that

$$\Delta_H^G(s) := \det(\mathrm{Ind}_H^G 1_H)(s)$$

is a character of G of order 2. If $H \subset G$ is a normal subgroup then it is a character of the factor group G/H and therefore it is **trivial** if G/H is of odd order.

More generally, from Gallagher's Theorem 2.8.2, we have:
If ρ is a (virtual) representation of H, then

$$\det(\mathrm{Ind}_H^G \rho)(s) = \Delta_H^G(s)^{\dim(\rho)} \cdot (\det\rho \circ T_{G/H}(s)), \tag{3.2.3}$$

for $s \in G$.

We assume now that the Galois groups $H \subset G$ have the fields $K \supset F$ as their base fields. Then by class field theory we may interpret $\det(\rho)$ of equation (3.2.3) as a character of $K^\times$ and $\det(\mathrm{Ind}_H^G \rho)$ as a character of $F^\times$, and then the equation (3.2.3) turns into an equality of two characters of $F^\times$:

$$\det(\mathrm{Ind}_H^G \rho) = \Delta_{K/F}^{\dim(\rho)} \cdot \det\rho|_{F^\times}, \quad \text{where} \quad \Delta_{K/F} : F^\times \to \{\pm 1\} \tag{3.2.4}$$

is the discriminant character[1] with respect to the extension K/F. If we consider $Z \subset H \subset G$ corresponding to the base fields $E \supset K \supset F$ then we have

[1] From **example (III) on p. 104 of [32]**, if $H = \mathrm{Gal}(L/K) < G = \mathrm{Gal}(L/F)$ corresponds to an extension $F \subset K \subset L$ of local fields then

$$\det \circ \mathrm{Ind}_H^G 1_H = \Delta_{K/F}$$

can be interpreted by class-field theory as a character of $F^\times$. It is then character of $F^\times$ corresponding to the quadratic extension $F(\sqrt{d_{K/F}})/F$, which is obtained by adjoining the square root of the discriminant $d_{K/F}$ of K/F.

$$\Delta_{E/F} = \det(\mathrm{Ind}_H^G(\mathrm{Ind}_Z^H 1_Z),$$

and with $\rho = \mathrm{Ind}_Z^H 1_Z$ we conclude from (3.2.4) that

$$\Delta_{E/F} = \Delta_{E/K}|_{F^\times} \cdot \Delta_{K/F}^{[E:K]}. \tag{3.2.5}$$

Moreover, in terms of Deligne's constant, we can write:

$$\lambda_H^G := W(\mathrm{Ind}_H^G 1_H) = c(\mathrm{Ind}_H^G 1_H) \cdot W(\det \circ \mathrm{Ind}_H^G 1_H). \tag{3.2.6}$$

Replacing Galois groups by the corresponding local fields we may write the lambda function of finite extension K/F as

$$\lambda_{K/F} = c(\mathrm{Ind}_{K/F} 1) \cdot W(\Delta_{K/F}), \tag{3.2.7}$$

where $c(\mathrm{Ind}_{K/F}1)$ is Deligne's sign, and $\Delta_{K/F}$ is a quadratic character of $F^\times$ related to the discriminant.

Lemma 3.2.1. *Let L/F be a finite Galois extension of a non-archimedean local field F and $G = \mathrm{Gal(L/F)}$, $H = \mathrm{Gal(L/K)}$. If $H \leqslant G$ is normal subgroup and if $[G : H]$ is odd, then $\Delta_{K/F} \equiv 1$ and $\lambda_{K/F}^2 = 1$.*

Proof. If H is a normal subgroup, then $\mathrm{Ind}_H^G 1_H = \mathrm{Ind}_{\{1\}}^{G/H} 1$ is the regular representation of G/H, hence $\det \circ \mathrm{Ind}_H^G 1_H = \Delta_{K/F}$ is the quadratic character of the group G/H. By the given condition order of G/H is odd, then $\Delta_{K/F} \equiv 1$, hence $\lambda_{K/F}^2 = \Delta_{K/F}(-1)$. Thus $\lambda_{K/F}^2 = 1$. □

Note: Since $\Delta_{K/F} \equiv 1$, then $W(\Delta_{K/F}) = 1$. We also know that $c(\mathrm{Ind_{K/F}}(1)) \in \{\pm 1\}$. Then from equation (3.2.7) we can easily see that $\lambda_{K/F}^2 = 1$.

In the next lemma we state some important results for our next Theorem 3.2.3. These are the consequences of Deligne's formula for the local constant of orthogonal representations.

Lemma 3.2.2. *1. If $H \leqslant G$ is normal subgroup of odd index $[G : H]$, then $\lambda_H^G = 1$.*

2. If there exists a normal subgroup N of G such that $N \leqslant H \leqslant G$ and $[G : N]$ odd, then $\lambda_H^G = 1$.

Proof. 1. To prove (1) we use the equation (3.2.6)

$$\lambda_H^G = W(\mathrm{Ind}_H^G 1_H) = c(\mathrm{Ind}_H^G 1_H) \cdot W(\det \circ \mathrm{Ind}_H^G 1_H). \tag{3.2.8}$$

Since $\rho = \mathrm{Ind}_H^G 1_H$ is orthogonal we may compute $c(\rho)$ by using the second Stiefel-Whitney class $s_2(\rho)$ [2]. From Proposition 3.1.1(v) we know that $c(\rho) = W(\rho)/W(det_\rho)$ is a sign. If $cl(s_2(\rho))$ is the image of $s_2(\rho)$ under inflation map (which is injective), then according to Deligne's theorem 3.1.2, we have:

[2] This Stiefel-Whitney class $s_2(\rho)$ is easy accessible only if $\det_\rho \equiv 1$, and this is in general wrong for $\rho = \mathrm{Ind}_H^G 1_H$. But it is true for $\rho = \mathrm{Ind}_H^G 1_H$ if $H \leqslant G$ is normal subgroup and $[G : H]$ is odd (by using Lemma 3.2.1).

$$c(\rho) = cl(s_2(\rho))$$

if ρ is orthogonal. Moreover, we have

$$s_2(\mathrm{Ind}_H^G 1_H) \in H^2(G/H, \mathbb{Z}/2\mathbb{Z}) = \{1\},$$

which implies that in equation (3.2.8) both factors are $= 1$, hence $\lambda_H^G = 1$.

2. From $N \leqslant H \leqslant G$ we obtain

$$\lambda_N^G = \lambda_N^H \cdot (\lambda_H^G)^{[H:N]} \tag{3.2.9}$$

From (1) we obtain $\lambda_N^G = \lambda_N^H = 1$ because N is normal and the index $[G : N]$ is odd, hence $(\lambda_H^G)^{[H:N]} = 1$. Finally this implies $\lambda_H^G = 1$ because λ_H^G is 4th root of unity and $[H : N]$ is odd.

□

Note: In the other words we can state this above Lemma 3.2.2 as follows:
Let $H' = \cap_{x\in\Delta} xHx^{-1} \subset \Delta$ be the largest subgroup of H which is normal in $\Delta \subseteq G$. Then $\lambda_H^\Delta(W) = 1$ if the index $[\Delta : H']$ is odd, in particular if H itself is normal subgroup of Δ of odd index.

Now we are in a position to state the main theorem for odd degree Galois extension of a non-archimedean local field.

Theorem 3.2.3. *Let F be a non-archimedean local field and E/F be an odd degree Galois extension. If $L \supset K \supset F$ be any finite extension inside E, then $\lambda_{L/K} = 1$.*

Proof. By the given condition $|\mathrm{Gal}(E/F)|$ is odd. Then the degree of extension $[E : F]$ of E over F is odd. Let L be any arbitrary intermediate field of E/F which contains K/F. Therefore, here we have the tower of fields $E \supset L \supset K \supset F$. Here the degree of extensions are all odd since $[E : F]$ is odd. By assumption E/F is Galois, then also the extension E/L and E/K are Galois and $H = \mathrm{Gal}(E/L)$ is subgroup of $G = \mathrm{Gal}(E/K)$.

By the definition we have $\lambda_{L/K} = \lambda_H^G$. If H is a normal subgroup of G then $\lambda_H^G = 1$ because $|G/H|$ is odd. But H need not be normal subgroup of G therefore L/K need not be Galois extension. Let N be the **largest** normal subgroup of G contained in H and N can be written as:

$$N = \cap_{g\in G} gHg^{-1}$$

Therefore, the fixed field E^N is the **smallest normal** extension of K containing L. Now we have from properties of λ-function(cf. 2.2(2)),

$$\lambda_N^G = \lambda_N^H \cdot (\lambda_H^G)^{[H:Z]}. \tag{3.2.10}$$

This implies $(\lambda_H^G)^{[H:N]} = 1$ since $[H : N]$ and $[G : N]$ are odd and N is normal subgroup of G contained in H. Therefore $\lambda_H^G = 1$ because λ_H^G is 4th root of unity and $[H : N]$ is odd.

Moreover, if $N = \{1\}$, it is then clear that N is a normal subgroup of G which sits in H and $[G : N] = |G|$ is odd. Therefore Lemma 3.2.2(2) implies that $\lambda_H^G = 1$.

Then we may say $\lambda_{L/K} = 1$ all possible cases if $[E^N : K]$ is odd. When the big extension E/F is odd then all intermediate extensions will be odd. Therefore, the theorem is proved for all possible cases.

□

Remark 3.2.4. **(1).** If the Galois extension E/F is infinite then we say it is **odd** if $[K : F]$ is odd for all sub-extensions of finite degree. This means the pro-finite group $\mathrm{Gal}(E/F)$ can be realized as the projective limit of finite groups which are all odd order. If E/F is Galois extension of odd order in this more general sense, then again we will have $\lambda_{L/K} = 1$ in all cases where λ-function is defined.

(2). When the order of a finite local Galois group is odd, all weak extensions are strong extensions, because from the above Theorem 3.2.3 we have $\lambda_1^G = 1$, where G is the odd order local Galois group. Let H be a any arbitrary subgroup of G, then from the properties of λ-functions we have

$$\lambda_1^G = \lambda_1^H \cdot (\lambda_H^G)^{|H|}.$$

This implies $\lambda_H^G = 1$, because $|H|$ is odd, hence $\lambda_1^H = 1$ and λ-functions are fourth roots of unity.

(3). But this above Theorem 3.2.3 is not true if K/F is not **Galois**. Guy Henniart gives **"An amusing formula"** (cf. [16], p. 124, Proposition 2) for $\lambda_{K/F}$, when K/F is arbitrary odd degree extension, and this formula is:

$$\lambda_{K/F} = W(\Delta_{K/F})^n \cdot \left(\frac{2}{q_F}\right)^{a(\Delta_{K/F})}, \tag{3.2.11}$$

where K/F is an extension in $\overline{F}$ with finite odd degree n, and $\left(\frac{2}{q_F}\right)$ is the Legendre symbol if p is odd and is 1 if $p = 2$. Here a denotes the exponent Artin-conductor.

3.3 Computation of λ_1^G, where G is a finite local Galois group

Let G be a finite local Galois group of a non-archimedean local field F. Now we consider **Langlands'** λ-function:

$$\lambda_H^G := W(\mathrm{Ind}_H^G 1_H) = c_H^G \cdot W(\Delta_H^G), \tag{3.3.1}$$

where

$$c_H^G := c(\mathrm{Ind}_H^G 1_H).$$

From the equation (3.3.1) we observe that to compute λ_H^G we need to compute the Deligne's constant c_H^G and $W(\Delta_H^G)$. By the following theorem due to Bruno Kahn we will get our necessary information for our further requirement.

Theorem 3.3.1 ([4], Série 1-313, Theorem 1). *Let G be a finite group, r_G its regular representation. Let S be any 2-Sylow subgroup of G. Then $s_2(r_G) = 0$ in the following cases:*

1. *S is cyclic group of order $\geqslant 8$;*
2. *S is generalized quaternion group;*
3. *S is not metacyclic group.*

We also need Gallagher's result.

Theorem 3.3.2 (Gallagher, [17], Theorem 30.1.8). *Assume that H is a normal subgroup of G, hence $\Delta_H^G = \Delta_1^{G/H}$, then*

1. *$\Delta_H^G = 1_G$, where 1_G is the trivial representation of G, unless the Sylow 2-subgroups of G/H are cyclic and nontrivial.*
2. *If the Sylow 2-subgroups of G/H are cyclic and nontrivial, then Δ_H^G is the only linear character of G of order 2.*

Definition 3.3.3 (2-rank of a finite abelian group). Let G be a finite abelian group. Then from elementary divisor Theorem 2.7.3 we can write

$$G \cong \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_s} \tag{3.3.2}$$

where $m_1|m_2|\cdots|m_s$ and $\prod_{i=1}^{s} m_i = |G|$. We define

$$\text{the 2-rank of } G := \text{the number of } m_i\text{-s which are even}$$

and we set

$$\mathrm{rk}_2(G) = \text{2-rank of } G.$$

When the order of an abelian group G is odd, from the structure of G we have $\mathrm{rk}_2(G) = 0$, i.e., there is no even m_i-s for G. We also denote

$$G[2] := \{x \in G| \quad 2x = 0\}, \text{ i.e., set of all elements of order at most 2.}$$

If $G = G_1 \times G_2 \times \cdots \times G_r$, $r \in \mathbb{N}$, is an abelian group, then we can show that

$$|G[2]| = \prod_{i=1}^{r} |G_i[2]|. \tag{3.3.3}$$

Remark 3.3.4 (**Remark on Theorem 3.3.2**). If G is a finite group with subgroups $H' \subset H \subset G$ then for $\Delta_H^G = \det(\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}} 1_{\mathrm{H}})$ we know from Gallagher's Theorem 2.8.2

$$\begin{aligned}\Delta_{H'}^G &= \det(\mathrm{Ind}_{\mathrm{H'}}^{\mathrm{G}} 1_{\mathrm{H'}}) = \det(\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\mathrm{Ind}_{\mathrm{H'}}^{\mathrm{H}} 1_{\mathrm{H'}}))\\ &= (\Delta_H^G)^{[H:H']} \cdot \det(\mathrm{Ind}_{\mathrm{H'}}^{\mathrm{H}} 1_{\mathrm{H'}}) \circ \mathrm{T}_{\mathrm{G/H}}\\ &= (\Delta_H^G)^{[H:H']} \cdot (\Delta_{H'}^H \circ T_{G/H}).\end{aligned} \tag{3.3.4}$$

Now we use equation (3.3.4) for $H' = \{1\}$ and $H = [G,G] = G'$. Then we have

$$\Delta_1^G = (\Delta_{G'}^G)^{|G'|} \cdot \Delta_1^{G'} \circ T_{G/G'} = (\Delta_{G'}^G)^{|G'|}, \tag{3.3.5}$$

because by Theorem 2.8.1, $T_{G/G'}$ is the trivial map.

We also know that G' is a normal subgroup of G, then we can write $\mathrm{Ind}_{G'}^{G} 1_{G'} \cong \mathrm{Ind}_1^{G/G'} 1$, hence $\Delta_{G'}^G = \Delta_1^{G/G'}$. So we have

$$\Delta_1^G = (\Delta_{G'}^G)^{|G'|} = (\Delta_1^{G/G'})^{|G'|}. \tag{3.3.6}$$

From the above equation (3.3.6) we observe that Δ_1^G always reduces to the **abelian case** because G/G' is abelian. Moreover, we know that:
If G is abelian then $\mathrm{Ind}_1^G 1 = r_G$ is the sum of all characters of G, hence from Miller's result (cf. [37], Theorem 6) for the abelian group $\widehat{G}$ we obtain:

$$\begin{aligned}\Delta_1^G &= \det(\mathrm{Ind}_1^G 1) = \det(\sum_{\chi\in\widehat{G}} \chi)\\ &= \prod_{\chi\in\widehat{G}} \det(\chi) = \prod_{\chi\in\widehat{G}} \chi\\ &= \begin{cases} \alpha & \textit{if } \mathrm{rk}_2(G) = 1\\ 1 & \textit{if } \mathrm{rk}_2(G) \neq 1,\end{cases}\end{aligned} \tag{3.3.7}$$

where α is the uniquely determined quadratic character of G.

Lemma 3.3.5. *The lambda function for a finite unramified extension of a non-archimedean local field is always a sign.*

Proof. Let K be a finite unramified extension of a non-archimedean local field F. We know that the unramified extensions are Galois, and their corresponding Galois groups are cyclic. Let $G = \mathrm{Gal}(K/F)$, hence G is cyclic.

When the degree of K/F is odd, from Theorem 3.2.3 we have $\lambda_1^G = \lambda_{K/F} = 1$ because K/F is Galois.

When the degree of K/F is even, we have $\mathrm{rk}_2(G) = 1$ because G is cyclic. So from equation (3.3.7) we can write $\Delta_1^G = \alpha$, where α corresponds to the quadratic unramified extension. Then $\Delta_1^G(-1) = \alpha(-1) = 1$, because -1 is a norm, hence from the functional equation (2.3.28) we have

$$(\lambda_1^G)^2 = 1.$$

□

Moreover, since G/G' is abelian, by using equation (3.3.7) for G/G', from equation (3.3.6) we obtain:

Lemma 3.3.6. *Let G be a finite group and S be a Sylow 2-subgroup of G. Then the following are equivalent:*

1. $S < G$ *is nontrivial cyclic;*

2. $\Delta_1^G \neq 1$, *is the unique quadratic character of* G*;*

3. $\mathrm{rk_2(G/G')} = 1$ *and* $|G'|$ *is odd.*

Proof. Take $H = \{1\}$ in Gallagher's Theorem 3.3.2 and we can see that (1) and (2) are equivalent. From equation (3.3.6) we can see (2) implies the condition (3).

Now we are left to show that (3) implies (1). Let S' be a Sylow 2-subgroup of G/G'. Since $\mathrm{rk_2(G/G')} = 1$, hence $\mathrm{rk_2(S')} = 1$, and therefore S' is cyclic. Moreover, $|G'|$ is odd, hence $|S| = |S'|$. Let $f : G \to G/G'$ be the canonical group homomorphism. Since $|G'|$ is odd, and $\mathrm{rk_2(G/G')} = 1$, $f|_S$ is an isomorphism from S to S'. Hence S is a nontrivial cyclic Sylow 2-subgroup of G.

This completes the proof.

□

Theorem 3.3.7 (**Schur-Zassenhaus**)**.** *If* $H \subset G$ *is a normal subgroup such that* $|H|$ *and* $[G : H]$ *are relatively prime, then* H *will have a complement* S *that is a subgroup of* G *such that*

$$G = H \rtimes S$$

is a semidirect product.

Let G be a local Galois group. Then it is known that G has Hall-subgroups (because G is solvable), $H \subset G$ of all types such that $[G : H]$ and $|H|$ are relatively prime. In particular, G will have an odd Hall subgroup $H \subset G$ such that $|H|$ is odd and $[G : H]$ is power of 2. From this we conclude

Proposition 3.3.8. *Let* G *be a finite local Galois group. Let* $H \subset G$ *be an odd order Hall subgroup of* G *(which is unique up to conjugation). Then we have*

$$\lambda_1^G = (\lambda_H^G)^{|H|}. \tag{3.3.8}$$

Hence $\lambda_1^G = \lambda_H^G$ *if* $|H| \equiv 1 \pmod 4$ *and* $\lambda_1^G = (\lambda_H^G)^{-1}$ *if* $|H| \equiv 3 \pmod 4$.
If the local base field F *has residue characteristic* $p \neq 2$*, then the odd order Hall subgroup* $H \subset G$ *is a normal subgroup and therefore* $\lambda_H^G = \lambda_1^{G/H}$*, where* $G/H \cong S$ *is isomorphic to a Sylow* 2*-subgroup of* G*. For* $G = \mathrm{Gal(E/F)}$ *this means that we have a unique normal extension* K/F *in* E *such that* $\mathrm{Gal(K/F)}$ *is isomorphic to a Sylow* 2*-subgroup of* G*, and we will have*

$$\lambda_{E/F} = \lambda_{K/F}^{[E:K]}.$$

Proof. We know that our local Galois group G is solvable, then G has an odd order Hall subgroup $H \subset G$. Then the formula (3.3.8) follows because $\lambda_1^H = 1$ (here $|H|$ is odd and H is a subgroup of the local Galois group G).

Let now $p \neq 2$ and let H be an odd order Hall subgroup of G. The ramification subgroup $G_1 \subset G$ is a normal subgroup of order a power of p, hence $G_1 \subset H$, and $H/G_1 \subset G/G_1$ will be an odd order Hall subgroup of G/G_1. But the group G/G_1 is **supersolvable**. It is also well known that the odd order Hall subgroup of a supersolvable group is normal. Therefore H/G_1 is normal in G/G_1, and this implies that H is normal in G. Now we can use Theorem 3.3.7 and we obtain $G/H \cong S$, where S must be a Sylow 2-subgroup. Therefore when $p \neq 2$ we have

$$\lambda_1^G = \lambda_{E/F} = (\lambda_1^{G/H})^{|H|} = \lambda_{K/F}^{[E:K]},$$

where $G = \mathrm{Gal(E/F)}$, $H = \mathrm{Gal(E/K)}$ and $G/H = \mathrm{Gal(K/F)} \cong \mathrm{S}$.

□

Let $F/\mathbb{Q}_p$ be a local field with $p \neq 2$. Let K/F be the extension such that $\mathrm{Gal(K/F)} = \mathrm{V}$ Klein's 4-group. In the following lemma we give explicit formula for $\lambda_1^V = \lambda_{K/F}$.

Lemma 3.3.9. *Let $F/\mathbb{Q}_p$ be a local field with $p \neq 2$. Let K/F be the uniquely determined extension with $V = \mathrm{Gal(K/F)}$, Klein's 4-group. Then*
$\lambda_1^V = \lambda_{K/F} = -1$ *if $-1 \in F^\times$ is a square, i.e., $q_F \equiv 1 \pmod 4$, and*
$\lambda_1^V = \lambda_{K/F} = 1$ *if $-1 \in F^\times$ is not a square , i.e., if $q_F \equiv 3 \pmod 4$,*
where q_F is the cardinality of the residue field of F.

Proof. If $p \neq 2$ then from Theorem 2.5.1 the square class group $F^\times/F^{\times 2}$ is Klein's 4-group, and K/F is the unique abelian extension such that $N_{K/F}(K^\times) = F^{\times 2}$, hence

$$\mathrm{Gal(K/F)} \cong \mathrm{F}^\times/\mathrm{F}^{\times 2} = \mathrm{V}.$$

Since V is abelian, we can write $\widehat{V} \cong V$. This implies that there are exactly three nontrivial characters of V and they are quadratic. By class field theory we can consider them as quadratic characters of $F^\times$. This each quadratic character determines a quadratic extension of F. Thus there are three quadratic subextensions L_i/F in K/F, where $i = 1, 2, 3$. We denote L_1/F the unramified extension whereas L_2/F and L_3/F are tamely ramified. Then we can write

$$\lambda_{K/F} = \lambda_{K/L_i} \cdot \lambda_{L_i/F}^2 \tag{3.3.9}$$

for all $i \in \{1, 2, 3\}$. The group V has four characters χ_i, $i = 0, \cdots, 3$, where $\chi_0 \equiv 1$ and $\chi_i (i = 1, 2, 3)$ are three characters of V such that $\mathrm{Gal}(K/L_i)$ is the kernel of χ_i, in other words, χ_i is the quadratic character of $F^\times/N_{L_i/F}(L_i^\times)$.

Let $r_V = \mathrm{Ind}_{\{1\}}^V 1$, then

$$\Delta_1^V = \det(r_V) = \textstyle\prod_{i=0}^3 \chi_i \equiv 1,$$

because $\chi_3 = \chi_1 \cdot \chi_2$. Therefore $W(\Delta_1^V) = 1$ and

$$\lambda_{K/F} = c(r_V)$$

is Deligne's constant. More precisely we have

$$\lambda_{K/F} = W(\chi_1) \cdot W(\chi_2) \cdot W(\chi_1\chi_2). \tag{3.3.10}$$

But here χ_1 is unramified and therefore $W(\chi_1) = \chi_1(c_1)$ (see equation (2.3.6)) and by using unramified character twisting formula, $W(\chi_1\chi_2) = \chi_1(c_2) \cdot W(\chi_2)$, where $c_2 = \pi_F c_1$ because $a(\chi_2) = 1 + a(\chi_1) = 1$. Therefore the equation (3.3.10) implies:

$$\lambda_{K/F} = \chi_1(c_1)^2 \cdot \chi_1(\pi_F) \cdot W(\chi_2)^2 = -\chi_2(-1), \tag{3.3.11}$$

since $\chi_1(\pi_F) = -1$. Similarly putting $\chi_2 = \chi_1^{-1}\chi_3 = \chi_1\chi_3$ and $\chi_1\chi_2 = \chi_3$ in the equation (3.3.10) we have

$$\lambda_{K/F} = -\chi_3(-1). \tag{3.3.12}$$

Therefore we have $\lambda_{K/F} = -\chi_i(-1)$ for $i = 2, 3$.

Moreover, we know that

$$\chi_i(-1) = \begin{cases} 1 & \text{if } -1 \in F^\times \text{ is a square, i.e., } q_F \equiv 1 \pmod 4 \\ -1 & \text{if } -1 \in F^\times \text{ is not a square , i.e., } q_F \equiv 3 \pmod 4 \end{cases}$$

Thus finally we conclude that

$$\lambda_{K/F} = -\chi_i(-1) = \begin{cases} -1 & \text{if } -1 \in F^\times \text{ is a square, i.e., } q_F \equiv 1 \pmod 4 \\ 1 & \text{if } -1 \in F^\times \text{ is not a square, i.e., } q_F \equiv 3 \pmod 4 \end{cases}$$

□

In the following theorem we give a general formula of λ_1^G, where G is a finite local Galois group.

Theorem 3.3.10. *Let G be a finite local Galois group of a non-archimedean local field F. Let S be a Sylow 2-subgroup of G.*

1. *If $S = \{1\}$, then we have $\lambda_1^G = 1$.*
2. *If the Sylow 2-subgroup $S \subset G$ is nontrivial cyclic (**exceptional case**), then*

$$\lambda_1^G = \begin{cases} W(\alpha) & \text{if } |S| = 2^n \geqslant 8 \\ c_1^G \cdot W(\alpha) & \text{if } |S| \leqslant 4, \end{cases} \tag{3.3.13}$$

where α is a uniquely determined quadratic character of G.

3. *If S is metacyclic but not cyclic (**invariant case**), then*

$$\lambda_1^G = \begin{cases} \lambda_1^V & \text{if } G \text{ contains Klein's 4 group } V \\ 1 & \text{if } G \text{ does not contain Klein's 4 group } V. \end{cases} \tag{3.3.14}$$

4. If S is nontrivial and not metacyclic, then $\lambda_1^G = 1$.

Proof. **(1).** When $S = \{1\}$, i.e., $|G|$ is odd, we know from Theorem 3.2.3 that $\lambda_1^G = 1$.
(2). When $S =< g >$ is a nontrivial cyclic subgroup of G, Δ_1^G is nontrivial (because $\Delta_1^G(g) = (-1)^{|G|-\frac{|G|}{|S|}} = -1$) and by Lemma 3.3.6, $\Delta_1^G = \alpha$, where α is a uniquely determined quadratic character of G. Then we obtain

$$\lambda_1^G = c_1^G \cdot W(\Delta_1^G) = c_1^G \cdot W(\alpha).$$

If S is cyclic of order $2^n \geqslant 8$, then by Theorem 3.3.1(case 1) and Theorem 3.1.2 we have $c_1^G = 1$, hence $\lambda_1^G = W(\alpha)$.
(3). The Sylow 2-subgroup $S \subset G$ is metacyclic but not cyclic (invariant case):
If G contains Klein's 4-group V, then $V \subset S$ because all Sylow 2-subgroups are conjugate to each other. Then we have $V < S < G$. So from the properties of λ-function we have

$$\lambda_1^G = \lambda_1^V \cdot (\lambda_V^G)^4 = \lambda_1^V.$$

Now assume that G does not contain Klein's 4-group. Then by assumption S is metacyclic, not cyclic and does not contain Klein's 4-group. We are going to see that this implies: S is generalized quaternion, and therefore by Theorem 3.3.1, $s_2(\mathrm{Ind}_1^{\mathrm{G}}(1)) = 0$, hence $c_1^G = 1$.

We use the following criterion for generalized quaternion groups: A finite p-group in which there is a unique subgroup of order p is either cyclic or generalized quaternion (cf. [36], p. 189, Theorem 12.5.2).
So it is enough to show: If S does not contain Klein's 4-group then S has precisely one subgroup of order 2. Indeed, we consider the center $Z(S)$ which is a nontrivial abelian 2-group. If it would be non-cyclic then $Z(S)$, hence S would contain Klein's 4-group. So $Z(S)$ must be cyclic, hence we have precisely one subgroup Z_2 of order 2 which sits in the center of S. Now assume that S has any other subgroup $U \subset S$ which is of order 2. Then Z_2 and U would generate a Klein-4-group in S which by our assumption cannot exist. Therefore $Z_2 \subset S$ is the only subgroup of order 2 in S. But S is not cyclic, so it is generalized quaternion.
Thus we can write $\lambda_1^G = c_1^G \cdot W(\Delta_1^G) = W(\Delta_1^G)$. Now to complete the proof we need to show that $W(\Delta_1^G) = 1$. This follows from Lemma 3.3.6.
(4). The Sylow 2-subgroup S is nontrivial and not metacyclic.
We know that every cyclic group is also a metacyclic group. Therefore when S is nontrivial and not metacyclic, we are **not** in the position: $\mathrm{rk}_2(\mathrm{G/G'}) = 1$ and $|G'|$ is odd. This gives $\Delta_1^G = 1$, hence $W(\Delta_1^G) = 1$. Furthermore by using the Theorem 3.3.1 and Theorem 3.1.2 we obtain the second Stiefel-Whitney class $s_2(\mathrm{Ind}_1^{\mathrm{G}}(1)) = 0$, hence $\lambda_1^G = c_1^G \cdot W(\Delta_1^G) = 1$.

This completes the proof.

□

In the above Theorem 3.3.10 we observe that if we are in the **Case 3**, then by using Lemma 3.3.9 we can give complete formula of λ_1^G for $p \neq 2$. Moreover, by using Proposition 3.3.8 in **case 2**, we can come down the computation of $\lambda_{K/F}$, where K/F is quadratic.

Corollary 3.3.11. *Let $G = \mathrm{Gal(E/F)}$ be a finite local Galois group of a non-archimedean local field $F/\mathbb{Q}_p$ with $p \neq 2$. Let $S \cong G/H$ be a nontrivial Sylow 2-subgroup of G, where H is a uniquely determined Hall subgroup of odd order. Suppose that we have a tower $E/K/F$ of fields such that $S \cong \mathrm{Gal(K/F)}$, $H = \mathrm{Gal(E/K)}$ and $G = \mathrm{Gal(E/F)}$.*

1. *If $S \subset G$ is cyclic, then*

   (a)
$$\lambda_1^G = \lambda_{K/F}^{\pm 1} = \begin{cases} \lambda_{K/F} = W(\alpha) & \text{if } [E:K] \equiv 1 \pmod 4 \\ \lambda_{K/F}^{-1} = W(\alpha)^{-1} & \text{if } [E:K] \equiv -1 \pmod 4, \end{cases}$$
   (here $\alpha = \Delta_{K/F}$ corresponds to the unique quadratic subextension in K/F) if $[K:F] = 2$, hence $\alpha = \Delta_{K/F}$.

   (b)
$$\lambda_1^G = \beta(-1)W(\alpha)^{\pm 1} = \beta(-1) \times \begin{cases} W(\alpha) & \text{if } [E:K] \equiv 1 \pmod 4 \\ W(\alpha)^{-1} & \text{if } [E:K] \equiv -1 \pmod 4 \end{cases}$$
   if K/F is cyclic of order 4 with generating character β such that $\beta^2 = \alpha = \Delta_{K/F}$.

   (c)
$$\lambda_1^G = \lambda_{K/F}^{\pm 1} = \begin{cases} \lambda_{K/F} = W(\alpha) & \text{if } [E:K] \equiv 1 \pmod 4 \\ \lambda_{K/F}^{-1} = W(\alpha)^{-1} & \text{if } [E:K] \equiv -1 \pmod 4 \end{cases}$$
   if K/F is cyclic of order $2^n \geqslant 8$.

   And if the 4th roots of unity are in the F, we have the same formulas as above but with 1 instead of ± 1. Moreover, when $p \neq 2$, a precise formula for $W(\alpha)$ will be obtained in Theorem 3.4.10.

2. *If S is metacyclic but not cyclic and the 4th roots of unity are in F, then*

   (a) $\lambda_1^G = -1$ if $V \subset G$,

   (b) $\lambda_1^G = 1$ if $V \not\subset G$.

3. *The* **Case 4** *of Theorem 3.3.10 will not occur in this case.*

Proof. **(1).** In the case when $p \neq 2$ we know from Proposition 3.3.8 that the odd Hall-subgroup $H < G$ is actually a normal subgroup with quotient $G/H \cong S$. So if $G = \mathrm{Gal(E/F)}$ and K/F is the maximal 2-extension inside E then $\mathrm{Gal(K/F) = G/H \cong S}$. And we obtain:

$$\lambda_1^G = (\lambda_1^{G/H})^{|H|} = \begin{cases} \lambda_{K/F} & \text{if } [E:K] = |H| \equiv 1 \pmod 4 \\ \lambda_{K/F}^{-1} & \text{if } [E:K] = |H| \equiv -1 \pmod 4. \end{cases} \tag{3.3.15}$$

So it is enough to compute $\lambda_{K/F}$ for $\mathrm{Gal(K/F) \cong S}$, i.e., we can reduce the computation to the case where $G = S$.

We know that $\lambda_{K/F} = W(\mathrm{Ind_{K/F}}(1)) = \prod_{\chi} \mathrm{W}(\chi)$, where χ runs over all characters of the cyclic group Gal(K/F). If $[K:F] = 2$ then $\mathrm{Ind_{K/F}}(1) = 1 + \alpha$, where α is a quadratic character of F associated to K by class field theory, hence $\alpha = \Delta_{K/F}$. Thus $\lambda_{K/F} = W(\alpha)$.

If $[K:F] = 4$ then $\mathrm{Ind_{K/F}}(1) = 1 + \beta + \beta^2 + \beta^3$, where $\beta^2 = \alpha = \Delta_{K/F}$ and $\beta^3 = \beta^{-1}$, hence by the functional equation of local constant we have:

$$W(\beta)W(\beta^{-1}) = \beta(-1).$$

We then obtain:

$$\lambda_{K/F} = W(\mathrm{Ind_{K/F}}(1)) = \mathrm{W}(\beta)\mathrm{W}(\beta^2)\mathrm{W}(\beta^3) = \beta(-1) \times \mathrm{W}(\alpha).$$

If S is cyclic of order $2^n \geqslant 8$, then by Theorem 3.1.2, $c_1^S = 1$. Again from equation (3.3.7) we have $W(\Delta_1^S) = W(\alpha)$ because $\mathrm{rk_2(S)} = 1$, where α is the uniquely determined quadratic character of F. Thus we obtain

$$\lambda_{K/F} = c_1^S \cdot W(\Delta_1^S) = W(\alpha).$$

Finally by using the equation (3.3.15) we obtain our desired results.

Now we denote $i = \sqrt{-1}$ and consider it in the algebraic closure of F. If $i \notin F$ then $p \neq 2$ implies that $F(i)/F$ is unramified extension of degree 2. Then we reach the case $i \in F$ which we have assumed.

Then first of all we know that λ_H^G is always is a **sign** because

$$(\lambda_H^G)^2 = \Delta_H^G(-1) = \Delta_H^G(i^2) = 1.$$

Then the formula (3.3.15) turns into

$$\lambda_1^G = (\lambda_1^{G/H})^{|H|} = \lambda_1^{G/H},$$

where $G/H = \mathrm{Gal(K/F)} \cong \mathrm{S}$. Therefore in **Case 2** of Theorem 3.3.10 we have now same formulas as above but with 1 instead of ± 1.

(2). Moreover, when $p \neq 2$ we know that always $\lambda_1^V = -1$ if $i \in F$ (cf. Lemma 3.3.9). Again if $V \subseteq S$, hence $V \subseteq G$, and we have

$$\lambda_1^G = \lambda_1^V \cdot (\lambda_V^G)^4 = \lambda_1^V.$$

Therefore, when S is metacyclic but not cyclic we can simply say:
$\lambda_1^G = \lambda_1^V = -1$, if $V \subset G$,
$\lambda_1^G = 1$, if $V \not\subset G$.

(3). If the base field F is p-adic for $p \neq 2$ then as a Galois group S corresponds to a tamely ramified extension (because the degree 2^n is prime to p), and therefore S must be metacyclic. Therefore the **Case 4** of Theorem 3.3.10 can never occur if $p \neq 2$.

□

Remark 3.3.12. If S is cyclic of order $2^n \geqslant 8$, then we have two formulas: $\lambda_1^G = W(\alpha)$ as obtained in Theorem 3.3.10(2), and $\lambda_1^G = W(\alpha)^{\pm 1}$ in Corollary 3.3.11. So we observe that for $|S| = 2^n \geqslant 8$ and $|H| \equiv -1 \pmod 4$ the value of $W(\alpha)$ must be a sign for $p \neq 2$.

In **Case 3** of Theorem 3.3.10 we notice that $\Delta_1^G \equiv 1$, hence $\lambda_1^G = c_1^G$. We know also that this Deligne's constant c_1^G takes values ± 1 (cf. Proposition 3.1.1(v)). Moreover, we also notice that the Deligne's constant of a representation is independent of the choice of the additive character. Therefore in Case 3 of Theorem 3.3.10, $\lambda_1^G = c_1^G \in \{\pm 1\}$ will **not** depend on the choice of the additive character. Since in Case 3 the computation of λ_1^G does not depend on the choice of the additive characters, hence we call this case as **invariant case**.

Furthermore, Bruno Kahn in his second paper (cf. [4]) deals with $s_2(r_G)$, where r_G is a regular representation of G in the invariant case. For metacyclic S of order $\geqslant 4$, we have the presentation

$$S \cong G(n,m,r,l) = < a, b : a^{2^n} = 1, b^{2^m} = a^{2^r}, bab^{-1} = a^l >$$
$$\text{with } n, m \geqslant 1,\ 0 \leqslant r \leqslant n,\ l \text{ an integer} \equiv 1 \pmod{2^{n-r}},\ l^{2^m} \equiv l \pmod{2^n}.$$

When S is **metacyclic but not cyclic** with $n \geqslant 2$, then $s_2(r_G) = 0$ if and only if $m = 1$ and $l \equiv -1 \pmod 4$ (cf. [4], on p. 575 of the second paper). In this case our $\lambda_1^G = c_1^G = 1$.

Corollary 3.3.13. *Let G be a finite abelian local Galois group of $F/\mathbb{Q}_p$, where $p \neq 2$. Let S be a Sylow 2-subgroup of G.*

1. *If* $\mathrm{rk}_2(\mathrm{S}) = 0$, *we have* $\lambda_1^G = 1$.
2. *If* $\mathrm{rk}_2(\mathrm{S}) = 1$, *then*
$$\lambda_1^G = \begin{cases} W(\alpha) & \text{if } |S| = 2^n \geqslant 8 \\ c_1^G \cdot W(\alpha) & \text{if } |S| \leqslant 4, \end{cases} \tag{3.3.16}$$
where α is a uniquely determined quadratic character of G.
3. *If* $\mathrm{rk}_2(\mathrm{S}) = 2$, *we have*
$$\lambda_1^G = \begin{cases} -1 & \text{if } -1 \in F^\times \text{ is a square element} \\ 1 & \text{if } -1 \in F^\times \text{ is not a square element.} \end{cases} \tag{3.3.17}$$

Proof. This proof is straightforward from Theorem 3.3.10 and Corollary 3.3.11. Here S is abelian and normal because G is abelian. When $\mathrm{rk}_2(\mathrm{S}) = 0$, G is of odd order, hence $\lambda_1^G = 1$. When $\mathrm{rk}_2(\mathrm{S}) = 1$, S is a cyclic group because $S \cong \mathbb{Z}_{2^n}$ for some $n \geqslant 1$. Then we are in the Case 2 of Theorem 3.3.10. From the Case 4 of Corollary 3.3.11, we can say that the case $\mathrm{rk}_2(\mathrm{S}) \geqslant 3$ will not occur here because $p \neq 2$ and G tame Galois group[3].

[3]Here tame Galois group means a Galois group of a tamely ramified extension.

So we are left to check the case $\mathrm{rk_2(S)} = 2$. In this case S is metacyclic and contains Klein's 4-group, i.e., $V \subseteq S \subseteq G$. Then we have from the properties of λ-functions and Lemma 3.3.9 we obtain

$$\lambda_1^G = \lambda_1^V \cdot (\lambda_V^G)^4 = \lambda_1^V = \begin{cases} -1 & \text{if } -1 \in F^\times \text{ is a square element} \\ 1 & \text{if } -1 \in F^\times \text{ is not a square element.} \end{cases} \tag{3.3.18}$$

□

Remark 3.3.14. In Corollary 3.3.13 we observe that except the case $\mathrm{rk_2(S)} = 1$, the computation of λ_1^G is explicit. Now let $G = \mathrm{Gal(L/F)}$, where L/F is a finite abelian Galois extension, and K/F be a subextension in L/F for which $\mathrm{Gal(L/K)} = \mathrm{S}$. Since S is normal, K/F is Galois extension of odd degree. Then when $\mathrm{rk_2(S)} = 1$ we have

$$\lambda_1^G = \lambda_1^S \cdot (\lambda_S^G)^{|S|}. \tag{3.3.19}$$

Again S is normal subgroup of G, hence $G/S \cong \mathrm{Gal(K/F)}$. Hence

$$\lambda_S^G = \lambda_1^{G/S} = \lambda_1^{\mathrm{Gal(K/F)}} = \lambda_{K/F} = 1.$$

Moreover, $\lambda_1^S = \lambda_{L/K}$. Then when $\mathrm{rk_2(S)} = 1$ we have

$$\lambda_1^G = \lambda_1^S = \lambda_{L/K},$$

where $[L : K] = 2^n$ $(n \geqslant 1)$.

From Theorem 3.3.10 and Corollaries 3.3.11, 3.3.13 for the **case** $p \neq 2$ we realize that the explicit computation of $W(\alpha)$ gives a complete computation of λ_1^G in the case $p \neq 2$. In the following section we give explicit computation of $\lambda_{K/F}$, when K/F is an even degree Galois extension.

3.4 Explicit computation of $\lambda_{K/F}$, where K/F is an even degree Galois extension

Let K/F be a quadratic extension of the field $F/\mathbb{Q}_p$. Let $G = \mathrm{Gal}(K/F)$ be the Galois group of the extension K/F. Let t be the **ramification break or jump** (cf. [27]) of the Galois group G (or of the extension K/F). Then it can be proved that the conductor of $\omega_{K/F}$ (a quadratic character of $F^\times$ associated to K by class field theory) is $t+1$ (cf. [42], Lemma 3.1). When K/F is unramified we have $t = -1$, therefore the conductor of a quadratic character $\omega_{K/F}$ of $F^\times$ is zero, i.e., $\omega_{K/F}$ is unramified. And when K/F is tamely ramified we have $t = 0$, then $a(\omega_{K/F}) = 1$. In the wildly ramified case (which occurs if $p = 2$) it can be proved that $a(\omega_{K/F}) = t + 1$ is, **up to the exceptional case** $t = 2 \cdot e_{F/\mathbb{Q}_2}$, always an **even number** which can be seen by the following filtration of $F^\times$.

Let K/F be a quadratic wild ramified extension, hence $p=2$. In F, we have the sequence $F^\times \supset U_F \supset U_F^1 \supset U_F^2 \supset \cdots$ of higher principal unit subgroups. Since $\omega_{K/F}$ is a quadratic character, it will be trivial on $F^{\times^2}$ therefore we need to consider the induced sequence

$$F^\times \supset U_F F^{\times^2} \supset U_F^1 F^{\times^2} \supset U_F^2 F^{\times^2} \supset \cdots \supset F^{\times^2}. \tag{S2}$$

In general, for any prime p, for $F/\mathbb{Q}_p$ we have the following series

$$F^\times \supset U_F F^{\times^p} \supset U_F^1 F^{\times^p} \supset U_F^2 F^{\times^p} \supset \cdots \supset F^{\times^p}. \tag{Sp}$$

Now we use Corollary 5.8 of [21] on p. 16, for the following: Let $e = \nu_F(p) = e_{F/\mathbb{Q}_p}$ be the absolute ramification exponent of F. Then

(i) If $i > \frac{pe}{p-1}$ then $U_F^i \subset F^{\times^p}$, hence $U_F^i F^{\times^p} = F^{\times^p}$.

(ii) If $i < \frac{pe}{p-1}$ and i is prime to p, then in

$$1 \to U_F^i \cap F^{\times^p}/U_F^{i+1} \cap F^{\times^p} \xrightarrow{(1)} U_F^i/U_F^{i+1} \xrightarrow{(2)} U_F^i \cdot F^{\times^p}/U_F^{i+1} \cdot F^{\times^p} \to 1, \tag{3.4.1}$$

the arrow (1) is trivial and (2) is an isomorphism.

(iii) If $i < \frac{pe}{p-1}$ and p divides i, then arrow (1) is an isomorphism and (2) is trivial

Therefore the jumps in **(Sp)** occur at $U_F^i F^{\times^p}$, where i is prime to p and $i < \frac{pe}{p-1}$.
(If $\frac{pe}{p-1}$ is an integer, then $i = \frac{pe}{p-1}$ is an **exceptional case.**)

We now take $p=2$, hence $\frac{pe}{p-1} = 2e$. Then the sequence **(Sp)** turns into **(S2)** for all odd numbers $t < 2e$ or for $t = 2e$ (the exceptional case). This means that in the wild ramified case the conductor $a(\omega_{K/F}) = t+1$ will always be an **even number** (except when $t = 2e$).

From the following lemma we can see that λ-function can change by a sign if we change the additive character.

Lemma 3.4.1. *The λ-function can change by sign if we change the additive character.*

Proof. Let K/F be a finite separable extension of the field F and ψ be a nontrivial additive character of F. We know that the local constant $W(\rho, \psi)$ is well defined for all pairs consisting of a virtual representation ρ of the Weil group W_F and a nontrivial additive character ψ of F. If we change the additive character ψ to $b\psi$, where $b \in F^\times$ is a unique element for which $b\psi(x) := \psi(bx)$ for all $x \in F$, then from equation (3.1.2), we have

$$W(\rho, b\psi) = \det_\rho(b) \cdot W(\rho, \psi). \tag{3.4.2}$$

In the definition of λ-function $\rho = \mathrm{Ind}_{K/F} 1$, therefore by using equation (3.4.2), we have

$$\lambda_{K/F}(b\psi) = W(\mathrm{Ind}_{K/F}1, b\psi) = \Delta_{K/F}(b) W(\mathrm{Ind}_{K/F}1, \psi) = \Delta_{K/F}(b)\lambda_{K/F}(\psi), \tag{3.4.3}$$

where $\Delta_{K/F} = \det(\mathrm{Ind}_{K/F}(1))$ is a quadratic character (a sign function), i.e., $\Delta_{K/F}(b) \in \{\pm 1\}$. $\square$

In the following lemma we compute an explicit formula for $\lambda_{K/F}(\psi_F)$, where K/F is a quadratic unramified extension of F. In general for **any quadratic extension** K/F, we can write $\mathrm{Ind}_{K/F}1 = 1_F \oplus \omega_{K/F}$, where $\omega_{K/F}$ is a quadratic character of $F^\times$ associated to K by class field theory and 1_F is the trivial character of $F^\times$. Now by the definition of λ-function we have:

$$\lambda_{K/F} = W(\mathrm{Ind}_{K/F}1) = W(\omega_{K/F}). \tag{3.4.4}$$

So, $\lambda_{K/F}$ is the local constant of the quadratic character $\omega_{K/F}$ corresponding to K/F.

Lemma 3.4.2. *Let K be the quadratic unramified extension of $F/\mathbb{Q}_p$ and let ψ_F be the canonical additive character of F with conductor $n(\psi_F)$. Then*

$$\lambda_{K/F}(\psi_F) = (-1)^{n(\psi_F)}. \tag{3.4.5}$$

Proof. When K/F is quadratic unramified extension, it is easy to see that in equation (3.4.4) $\omega_{K/F}$ is an unramified character because here the ramification break t is -1. Then from equation (2.3.6) have:

$$W(\omega_{K/F}) = \omega_{K/F}(c).$$

Here $\nu_F(c) = n(\psi_F)$. Therefore from equation (3.4.4) we obtain:

$$\lambda_{K/F} = \omega_{K/F}(\pi_F)^{n(\psi_F)}. \tag{3.4.6}$$

We also know that $\pi_F \notin N_{K/F}(K^\times)$, and hence $\omega_{K/F}(\pi_F) = -1$. Therefore from equation (3.4.6), we have

$$\lambda_{K/F} = (-1)^{n(\psi_F)}. \tag{3.4.7}$$

$\square$

For giving the generalized formula for $\lambda_{K/F}$, where K/F is an even degree unramified extension, we need the following lemma, and here we give an alternative proof by using Lemma 2.1.2.

Lemma 3.4.3 ([2], p. 142, Corollary 3)**.** *Let K/F be a finite extension and let $\mathcal{D}_{K/F}$ be the different of this extension K/F. Let ψ be an additive character of F. Then*

$$n(\psi \circ \mathrm{Tr}_{K/F}) = e_{K/F} \cdot n(\psi) + \nu_K(\mathcal{D}_{K/F}). \tag{3.4.8}$$

Proof. Let the conductor of the character $\psi \circ \mathrm{Tr}_{K/F}$ be m. This means from the definition of conductor of additive character we have

$$\psi \circ \mathrm{Tr}_{K/F}|_{P_K^{-m}} = 1 \text{ but } \psi \circ \mathrm{Tr}_{K/F}|_{P_K^{-m-1}} \neq 1,$$
$$\text{i.e., } \psi(\mathrm{Tr}_{K/F}(P_K^{-m})) = 1 \text{ but } \psi(\mathrm{Tr}_{K/F}(P_K^{-m-1})) \neq 1.$$

This implies

$$\mathrm{Tr}_{K/F}(P_K^{-m}) \subseteq P_F^{-n(\psi)},$$

since $n(\psi)$ is the conductor of ψ. Then by Lemma 2.1.2 we have

$$P_K^{-m} \cdot \mathcal{D}_{K/F} \subseteq P_F^{-n(\psi)} \Leftrightarrow P_K^{-m+d_{K/F}} O_K \subseteq P_F^{-n(\psi)}, \tag{3.4.9}$$

since $\mathcal{D}_{K/F} = \pi_K^{d_{K/F}} O_K$. From the definition of ramification index we know that

$$\pi_K^{e_{K/F}} O_K = \pi_F O_K.$$

Therefore from the equation (3.4.9) we obtain:

$$P_K^{-m+d_{K/F}} O_K \subseteq P_K^{-n(\psi)e_{K/F}} O_K.$$

This implies

$$m \leqslant n(\psi) \cdot e_{K/F} + d_{K/F} = n(\psi) \cdot e_{K/F} + \nu_K(\mathcal{D}_{K/F}), \tag{3.4.10}$$

since $d_{K/F} = \nu_K(\mathcal{D}_{K/F})$.

Now we have to prove that the equality $m = n(\psi) \cdot e_{K/F} + \nu_K(\mathcal{D}_{K/F})$.
Let $m = n(\psi) \cdot e_{K/F} + \nu_K(\mathcal{D}_{K/F}) - r$, where $r \geqslant 0$. Then we have

$$\mathrm{Tr}_{K/F}(P_K^{-n(\psi)\cdot e_{K/F} - d_{K/F} + d_{K/F} + r}) \subseteq P_K^{-n(\psi)\cdot e_{K/F}}.$$

This implies $r \leqslant 0$. Therefore r must be $r = 0$, because by assumption $r \geqslant 0$. This proves that

$$m = n(\psi \circ \mathrm{Tr}_{K/F}) = e_{K/F} \cdot n(\psi) + \nu_K(\mathcal{D}_{K/F}).$$

□

Remark 3.4.4. If K/F is unramified, then $e_{K/F} = 1$ and $\mathcal{D}_{K/F} = O_K$, hence $\nu_K(\mathcal{D}_{K/F}) = \nu_K(O_K) = 0$, therefore from the above Lemma 3.4.3 we have $n(\psi \circ \mathrm{Tr}_{K/F}) = n(\psi)$. Moreover, if $\psi_F = \psi_{\mathbb{Q}_p} \circ \mathrm{Tr}_{F/\mathbb{Q}_p}$, is the canonical additive character of F, then $n(\psi_{\mathbb{Q}_p}) = 0$ and therefore

$$n(\psi_F) = \nu_F(\mathcal{D}_{F/\mathbb{Q}_p})$$

is the exponent of the absolute different.

Theorem 3.4.5. *Let K/F be a finite unramified extension with even degree and let ψ_F be the canonical additive character of F with conductor $n(\psi_F)$. Then*

$$\lambda_{K/F} = (-1)^{n(\psi_F)}. \tag{3.4.11}$$

Proof. When K/F is a quadratic unramified extension, by Lemma 3.4.2, we have $\lambda_{K/F} = (-1)^{n(\psi_F)}$. We also know that if K/F is unramified of even degree then we have precisely one subextension K'/F in K/F such that $[K : K'] = 2$. Then

$$\lambda_{K/F} = \lambda_{K/K'} \cdot (\lambda_{K'/F})^2 = \lambda_{K/K'} = (-1)^{n(\psi_{K'})} = (-1)^{n(\psi_F)},$$

because in the unramified case λ-function is always a sign (cf. Lemma 3.3.5), and from Lemma 3.4.3, $n(\psi_{K'}) = n(\psi_F)$.

This completes the proof.

□

In the following corollary, we show that the above Theorem 3.4.5 is true for any nontrivial arbitrary additive character.

Corollary 3.4.6. *Let K/F be a finite unramified extension of even degree and let ψ be any nontrivial additive character of F with conductor $n(\psi)$. Then*

$$\lambda_{K/F}(\psi) = (-1)^{n(\psi)}. \tag{3.4.12}$$

Proof. We know that any nontrivial additive character ψ is of the form, for some unique $b \in F^\times$, $\psi(x) := b\psi_F(x)$, for all $x \in F$. By the definition of conductor of additive character of F, we obtain:

$$n(\psi) = n(b\psi_F) = \nu_F(b) + n(\psi_F).$$

Now let $G = \mathrm{Gal(K/F)}$ be the Galois group of the extension K/F. Since K/F is unramified, then G is **cyclic**. Let S be a Sylow 2-subgroup of G. Here S is nontrivial cyclic because the degree of K/F is even and G is cyclic. Then from Lemma 3.3.6 we have $\Delta_1^G = \Delta_{K/F} \not\equiv 1$. Therefore $\Delta_{K/F}(b) = (-1)^{\nu_F(b)}$ is the uniquely determined unramified quadratic character of $F^\times$. Now from equation (3.4.3) we have:

$$\begin{aligned}
\lambda_{K/F}(\psi) &= \lambda_{K/F}(b \cdot \psi_F) \\
&= \Delta_{K/F}(b)\lambda_{K/F}(\psi_F) \\
&= (-1)^{\nu_F(b)} \times (-1)^{n(\psi_F)}, \quad \text{from Theorem 3.4.5} \\
&= (-1)^{\nu_F(b)+n(\psi_F)} \\
&= (-1)^{n(\psi)}.
\end{aligned}$$

Therefore when K/F is an unramified extension of even degree, we have

$$\lambda_{K/F}(\psi) = (-1)^{n(\psi)}, \tag{3.4.13}$$

where ψ is any nontrivial additive character of F.

□

In the following theorem we give an explicit formula of $\lambda_{K/F}$, when K/F is an even degree Galois extension with odd ramification index.

Theorem 3.4.7. *Let K be an even degree Galois extension of a non-archimedean local field F of odd ramification index. Let ψ be a nontrivial additive character of F. Then*

$$\lambda_{K/F}(\psi) = (-1)^{n(\psi)}. \tag{3.4.14}$$

Proof. In general, any extension K/F of local fields has a uniquely determined maximal subextension F'/F in K/F which is unramified. If the degree of K/F is even, then certainly we have $e_{K/F} = [K : F']$ because $e_{K/F} = e_{F'/F} \cdot e_{K/F'} = e_{K/F'}$ and K/F' is a totally ramified extension. By the given condition, here K/F is an even degree Galois extension with odd ramification index $e_{K/F}$, hence K/F' is an odd degree Galois extension. Now from the properties of λ-function and Theorem 3.2.3 we have

$$\lambda_{K/F} = \lambda_{K/F'} \cdot (\lambda_{F'/F})^{e_{K/F}} = (-1)^{e_{K/F}\cdot n(\psi_F)} = (-1)^{n(\psi_F)},$$

because K/F' is an odd degree Galois extension and F'/F is an unramified extension.

□

3.4.1 Computation of $\lambda_{K/F}$, where K/F is a tamely ramified quadratic extension

The existence of a tamely ramified quadratic character (which is not unramified) of a local field F implies $p \neq 2$ for the residue characteristic. But then by Theorem 2.5.1

$$F^\times/{F^\times}^2 \cong V$$

is isomorphic to Klein's 4-group. So we have only 3 nontrivial quadratic characters in that case, corresponding to 3 quadratic extensions K/F. One is unramified and other two are ramified. The unramified case is already settled (cf. Lemma 3.4.2). Now we have to address the quadratic ramified characters. These two ramified quadratic characters determine two different quadratic ramified extensions of F.
In the ramified case we have $a(\chi) = 1$ because it is tame, and we take ψ of conductor -1. Then we have $a(\chi)+n(\psi) = 0$ and therefore in the definition of $W(\chi,\psi)$ (cf. equation (2.3.4)) we can take $c = 1$. So we obtain:

$$W(\chi,\psi) = q_F^{-\frac{1}{2}} \sum_{x\in U_F/U_F^1} \chi^{-1}(x)\psi(x) = q_F^{-\frac{1}{2}} \sum_{x\in k_F^\times} \overline{\chi}^{-1}(x)\overline{\psi}(x), \tag{3.4.15}$$

where $\overline{\chi}$ is the quadratic character of the residue field $k_F^\times$, and $\overline{\psi}$ is an additive character of k_F. When $n(\psi) = -1$, we observe that both **the ramified characters χ give the same $\overline{\chi}$, hence the same $W(\chi,\psi)$**, because one is different from other by a quadratic unramified character twist. To compute an explicit formula for $\lambda_{K/F}(\psi_{-1})$, where K/F is a tamely ramified quadratic extension and ψ_{-1} is an additive character of F with conductor -1, we need to use classical quadratic Gauss sums.

Now let F be a non-archimedean local field. Let ψ_{-1} be an additive character of F of conductor -1, i.e., $\psi_{-1}: F/P_F \to \mathbb{C}^\times$. Now restrict ψ_{-1} to O_F, it will be one of the characters $a\cdot\psi_{q_F}$, for some $a \in k_{q_F}^\times$ and usually it will not be ψ_{q_F} itself. Therefore choosing ψ_{-1} is very important and we have to choose ψ_{-1} such a way that its restriction to O_F is exactly ψ_{q_F}. Then we will be able to use the quadratic classical Gauss sum in our λ-function computation. We also know that there exists an element $c \in F^\times$ such that

$$\psi_{-1} = c\cdot\psi_F \tag{3.4.16}$$

induces the canonical character ψ_{q_F} on the residue field k_F.

Now question is: **Finding proper $c \in F^\times$ for which** $\psi_{-1}|_{O_F} = c\cdot\psi_F|_{O_F} = \psi_{q_F}$, i.e., the canonical character of the residue field k_F.

From the definition of conductor of the additive character ψ_{-1} of F, we obtain from the construction (3.4.16)

$$-1 = \nu_F(c) + n(\psi_F) = \nu_F(c) + d_{F/\mathbb{Q}_p}. \tag{3.4.17}$$

In the next two lemmas we choose the proper c for our requirement.

Lemma 3.4.8. *Let $F/\mathbb{Q}_p$ be a local field and let ψ_{-1} be an additive character of F of conductor -1. Let ψ_F be the canonical character of F. Let $c \in F^\times$ be any element such that $-1 = \nu_F(c) + d_{F/\mathbb{Q}_p}$, and*

$$\mathrm{Tr_{F/F_0}(c)} = \frac{1}{\mathrm{p}}, \tag{3.4.18}$$

where $F_0/\mathbb{Q}_p$ is the maximal unramified subextension in $F/\mathbb{Q}_p$. Then the restriction of $\psi_{-1} = c\cdot\psi_F$ to O_F is the canonical character ψ_{q_F} of the residue field k_F of F.

Proof. Since $F_0/\mathbb{Q}_p$ is the maximal unramified subextension in $F/\mathbb{Q}_p$, we have $\pi_{F_0} = p$, and the residue fields of F and F_0 are isomorphic, i.e., $k_{F_0} \cong k_F$, because F/F_0 is totally ramified extension. Then every element of O_F/P_F can be considered as an element of O_{F_0}/P_{F_0}. Moreover, since $F_0/\mathbb{Q}_p$ is the maximal unramified extension, then from Proposition 2 of [2] on p. 140, for $x \in O_{F_0}$ we have

$$\rho_p(\mathrm{Tr_{F_0/\mathbb{Q}_p}(x)}) = \mathrm{Tr_{k_{F_0}/k_{\mathbb{Q}_p}}}(\rho_0(\mathrm{x})),$$

where ρ_0, ρ_p are the canonical homomorphisms of O_{F_0} onto k_{F_0}, and of $O_{\mathbb{Q}_p}$ onto $k_{\mathbb{Q}_p}$, respectively. Then for $x \in k_{F_0}$ we can write

$$\mathrm{Tr_{F_0/\mathbb{Q}_p}(x)} = \mathrm{Tr_{k_{F_0}/k_{\mathbb{Q}_p}}(x)}. \tag{3.4.19}$$

Furthermore, since F/F_0 is totally ramified, we have $k_F = k_{F_0}$, then the trace map for the tower of the residue fields $k_F/k_{F_0}/k_{\mathbb{Q}_p}$ is:

$$\mathrm{Tr_{k_F/k_{\mathbb{Q}_p}}(x)} = \mathrm{Tr_{k_{F_0}/k_{\mathbb{Q}_p}}} \circ \mathrm{Tr_{k_F/k_{F_0}}(x)} = \mathrm{Tr_{k_{F_0}/k_{\mathbb{Q}_p}}(x)}, \tag{3.4.20}$$

for all $x \in k_F$. Then from the equations (3.4.19) and (3.4.20) we obtain

$$\mathrm{Tr_{F_0/\mathbb{Q}_p}(x)} = \mathrm{Tr_{k_F/k_{\mathbb{Q}_p}}(x)} \tag{3.4.21}$$

for all $x \in k_F$.

Since the conductor of ψ_{-1} is -1, for $x \in O_F/P_F (= O_{F_0}/P_{F_0}$ because F/F_0 is totally ramified) we have

$$\begin{aligned}
\psi_{-1}(x) &= c\cdot\psi_F(x)\\
&= \psi_F(cx)\\
&= \psi_{\mathbb{Q}_p}(\mathrm{Tr_{F/\mathbb{Q}_p}(cx)})\\
&= \psi_{\mathbb{Q}_p}(\mathrm{Tr_{F_0/\mathbb{Q}_p}\circ Tr_{F/F_0}(cx)})\\
&= \psi_{\mathbb{Q}_p}(\mathrm{Tr_{F_0/\mathbb{Q}_p}(x\cdot Tr_{F/F_0}(c))})\\
&= \psi_{\mathbb{Q}_p}(\mathrm{Tr_{F_0/\mathbb{Q}_p}(\frac{1}{p}x)}), \quad \text{since } x\in O_F/P_F = O_{F_0}/P_{F_0} \text{ and } \mathrm{Tr_{F/F_0}(c)} = \mathrm{\frac{1}{p}}\\
&= \psi_{\mathbb{Q}_p}(\frac{1}{p}\mathrm{Tr_{F_0/\mathbb{Q}_p}(x)}), \quad \text{because } \frac{1}{p}\in\mathbb{Q}_p\\
&= e^{\frac{2\pi i \mathrm{Tr_{F_0/\mathbb{Q}_p}(x)}}{p}}, \quad \text{because } \psi_{\mathbb{Q}_p}(x) = e^{2\pi i x}\\
&= e^{\frac{2\pi i \mathrm{Tr_{k_F/k_{\mathbb{Q}_p}}(x)}}{p}}, \quad \text{using equation (3.4.21)}\\
&= \psi_{q_F}(x).
\end{aligned}$$

This competes the lemma.

□

The next step is to produce good elements c more explicitly. By using Lemma 3.4.8, in the next lemma we see more general choices of c.

Lemma 3.4.9. *Let $F/\mathbb{Q}_p$ be a tamely ramified local field and let ψ_{-1} be an additive character of F of conductor -1. Let ψ_F be the canonical character of F. Let $F_0/\mathbb{Q}_p$ be the maximal unramified subextension in $F/\mathbb{Q}_p$. Let $c\in F^\times$ be any element such that $-1 = \nu_F(c) + d_{F/\mathbb{Q}_p}$, then*

$$c' = \frac{c}{\mathrm{Tr_{F/F_0}(pc)}},$$

fulfills conditions (3.4.17), (3.4.18), and hence $\psi_{-1}|_{O_F} = c'\cdot\psi_F|_{O_F} = \psi_{q_F}$.

Proof. By the given condition we have $\nu_F(c) = -1 - d_{F/\mathbb{Q}_p} = -1-(e_{F/\mathbb{Q}_p}-1) = -e_{F/\mathbb{Q}_p}$. Then we can write $c = \pi_F^{-e_{F/\mathbb{Q}_p}}u(c) = p^{-1}u(c)$ for some $u(c)\in U_F$ because $F/\mathbb{Q}_p$ is tamely ramified, hence $p = \pi_F^{e_{F/\mathbb{Q}_p}}$. Then we can write

$$\mathrm{Tr_{F/F_0}(pc) = p\cdot Tr_{F/F_0}(c) = p\cdot p^{-1}u_0(c) = u_0(c)\in U_{F_0}\subset U_F},$$

where $u_0(c) = \mathrm{Tr_{F/F_0}(u(c))}$, hence $\nu_F(\mathrm{Tr_{F/F_0}(pc)}) = 0$. Then the valuation of c' is:

$$\begin{aligned}
\nu_F(c') &= \nu_F(\tfrac{c}{\mathrm{Tr_{F/F_0}(pc)}}) = \nu_F(c) - \nu_F(\mathrm{Tr_{F/F_0}(pc)})\\
&= \nu_F(c) - 0 = \nu_F(c) = -1 - d_{F/\mathbb{Q}_p}.
\end{aligned}$$

Since $\mathrm{Tr_{F/F_0}(pc) = u_0(c)\in U_{F_0}}$, we have

$$\mathrm{Tr_{F/F_0}(c') = Tr_{F/F_0}(\tfrac{c}{Tr_{F/F_0}(pc)}) = \tfrac{1}{Tr_{F/F_0}(pc)}\cdot Tr_{F/F_0}(c) = \tfrac{1}{p\cdot Tr_{F/F_0}(c)}\cdot Tr_{F/F_0}(c) = \tfrac{1}{p}}.$$

Thus we observe that here $c' \in F^\times$ satisfies equations (3.4.17) and (3.4.18). Therefore from Lemma 3.4.8 we can see that $\psi_{-1}|_{O_F} = c' \cdot \psi_F|_{O_F}$ is the canonical additive character of k_F. □

By Lemmas 3.4.8 and 3.4.9 we get many good (in the sense that $\psi_{-1}|_{O_F} = c\cdot\psi_F|_{O_F} = \psi_{q_F}$) elements c which we will use in our next theorem to calculate $\lambda_{K/F}$, where K/F is a tamely ramified quadratic extension.

Theorem 3.4.10. *Let K be a tamely ramified quadratic extension of $F/\mathbb{Q}_p$ with $q_F = p^s$. Let ψ_F be the canonical additive character of F. Let $c \in F^\times$ with $-1 = \nu_F(c) + d_{F/\mathbb{Q}_p}$, and $c' = \frac{c}{\mathrm{Tr_{F/F_0}(pc)}}$, where $F_0/\mathbb{Q}_p$ is the maximal unramified extension in $F/\mathbb{Q}_p$. Let ψ_{-1} be an additive character of F with conductor -1, of the form $\psi_{-1} = c' \cdot \psi_F$. Then*

$$\lambda_{K/F}(\psi_F) = \Delta_{K/F}(c') \cdot \lambda_{K/F}(\psi_{-1}),$$

where

$$\lambda_{K/F}(\psi_{-1}) = \begin{cases} (-1)^{s-1} & \text{if } p \equiv 1 \pmod 4 \\ (-1)^{s-1} i^s & \text{if } p \equiv 3 \pmod 4). \end{cases}$$

If we take $c = \pi_F^{-1-d_{F/\mathbb{Q}_p}}$, where π_F is a norm for K/F, then

$$\Delta_{K/F}(c') = \begin{cases} 1 & \text{if } \overline{\mathrm{Tr_{F/F_0}(pc)}} \in k_{F_0}^\times = k_F^\times \text{ is a square,} \\ -1 & \text{if } \overline{\mathrm{Tr_{F/F_0}(pc)}} \in k_{F_0}^\times = k_F^\times \text{ is not a square.} \end{cases} \tag{3.4.22}$$

Here "overline" stands for modulo P_{F_0}.

Proof. From equation (3.4.3) we have

$$\lambda_{K/F}(\psi_{-1}) = \lambda_{K/F}(c'\psi_F) = \Delta_{K/F}(c') \cdot \lambda_{K/F}(\psi_F).$$

Since $\Delta_{K/F}$ is quadratic, we can write $\Delta_{K/F} = \Delta_{K/F}^{-1}$. So we obtain

$$\lambda_{K/F}(\psi_F) = \Delta_{K/F}(c') \cdot \lambda_{K/F}(\psi_{-1}).$$

Now we have to compute $\lambda_{K/F}(\psi_{-1})$, and which we do in the following:
Since $[K : F] = 2$, we have $\mathrm{Ind_{K/F}}(1) = 1_\mathrm{F} \oplus \omega_\mathrm{K/F}$. The conductor of $\omega_{K/F}$ is 1 because K/F is a tamely ramified quadratic extension, and hence $t = 0$, so $a(\omega_{K/F}) = t + 1 = 1$. Therefore we can consider $\omega_{K/F}$ as a character of $F^\times/U_F^1$. So the restriction of $\omega_{K/F}$ to U_F, $\mathrm{res}(\omega_\mathrm{K/F}) := \omega_\mathrm{K/F}|_\mathrm{U_F}$, we may consider as the uniquely determined character of $k_F^\times$ of order 2. Since c' satisfies equations (3.4.17), (3.4.18), then from Lemma 3.4.9 we have $\psi_{-1}|_{O_F} = c' \cdot \psi_F|_{O_F} = \psi_{q_F}$, and this is the canonical character of k_F. Then from equation (3.4.15) we can write

$$\begin{aligned}\lambda_{K/F}(\psi_{-1}) &= q_F^{-\frac{1}{2}} \sum_{x\in k_F^\times} \mathrm{res}(\omega_\mathrm{K/F})(\mathrm{x})\psi_\mathrm{q_F}(\mathrm{x}) \\ &= q_F^{-\frac{1}{2}} \cdot G(\mathrm{res}(\omega_\mathrm{K/F}), \psi_\mathrm{q_F}).\end{aligned}$$

Moreover, by Theorem 2.4.1 we have

$$G(\mathrm{res}(\omega_{\mathrm{K/F}}), \psi_{\mathrm{q_F}}) = \begin{cases} (-1)^{s-1} q_F^{\frac{1}{2}} & \text{if } p \equiv 1 \pmod 4 \\ (-1)^{s-1} i^s q_F^{\frac{1}{2}} & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{3.4.23}$$

By using the classical quadratic Gauss sum we obtain

$$\lambda_{K/F}(\psi_{-1}) = \begin{cases} (-1)^{s-1} & \text{if } p \equiv 1 \pmod 4 \\ (-1)^{s-1} i^s & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{3.4.24}$$

We also can write $\Delta_{K/F} = \det(\mathrm{Ind_{K/F}}(1)) = \det(1_{\mathrm{F}} \oplus \omega_{\mathrm{K/F}}) = \omega_{\mathrm{K/F}}$. So we have

$$\Delta_{K/F}(\pi_F) = \omega_{K/F}(\pi_F) = 1,$$

because $\pi_F \in N_{K/F}(K^\times)$.

Under the assumption of the Theorem 3.4.10 we have $\pi_F \in N_{K/F}(K^\times)$, $\Delta_{K/F} = \omega_{K/F}$ and $c' = \frac{c}{\mathrm{Tr_{F/F_0}(pc)}}$, where $c \in F^\times$ with $\nu_F(c) = -1 - d_{F/\mathbb{Q}_p}$. Then we can write

$$\begin{aligned}
\Delta_{K/F}(c') &= \omega_{K/F}(c') \\
&= \omega_{K/F}\left(\frac{c}{\mathrm{Tr_{F/F_0}(pc)}}\right) \\
&= \omega_{K/F}\left(\frac{\pi_F^{-e_{F/\mathbb{Q}_p}} u(c)}{u_0(c)}\right), \quad \text{where } c = \pi_F^{-e_{F/\mathbb{Q}_p}} u(c),\ \mathrm{Tr_{F/F_0}(pc) = u_0(c) \in U_{F_0}} \\
&= \omega_{K/F}(\pi_F^{-e_{F/\mathbb{Q}_p}})\omega_{K/F}(v), \quad \text{where } v = \frac{u(c)}{u_0(c)} \in U_F \\
&= \omega_{K/F}(x) \\
&= \begin{cases} 1 & \text{when } x \text{ is a square element in } k_F^\times \\ -1 & \text{when } x \text{ is not a square element in } k_F^\times, \end{cases}
\end{aligned}$$

where $v = xy$, with $x = x(\omega_{K/F}, c) \in U_F/U_F^1$, and $y \in U_F^1$.

In particular, if we choose c such a way that $u(c) = 1$, i.e., $c = \pi_F^{-1-d_{F/\mathbb{Q}_p}}$, then we have $\Delta_{K/F}(c') = \Delta_{K/F}(\mathrm{Tr_{F/F_0}(pc)})$. Since $\mathrm{Tr_{F/F_0}(pc) \in O_{F_0}}$ is a unit and $\Delta_{K/F} = \omega_{K/F}$ induces the quadratic character of $k_F^\times = k_{F_0}^\times$, then for this particular choice of c we obtain

$$\Delta_{K/F}(c') = \begin{cases} 1 & \text{if } \overline{\mathrm{Tr_{F/F_0}(pc)}} \text{ is a square in } k_{F_0}^\times \\ -1 & \text{if } \overline{\mathrm{Tr_{F/F_0}(pc)}} \text{ is not a square in } k_{F_0}^\times. \end{cases}$$

□

Note: When we are in the Case 1 of Corollary 3.3.11, by using this above Theorem 3.4.10 we can give explicit formula for $\lambda_{K/F}$, because here the quadratic extensions are tamely ramified (since $p \neq 2$).

Remark 3.4.11. When $p \neq 2$, for any non-archimedean local field $F/\mathbb{Q}_p$ we know that the square class group $F^\times/F^{\times 2} \cong V$ Klein's 4-group. If K is the abelian extension of F with $N_{K/F}(K^\times) = F^{\times 2}$, then from Lemma 3.3.9 we can see that $\lambda_{K/F}$ depends on the choice of the base field F. Now we put $\lambda_i = \lambda_{L_i/F}$, where L_i/F are quadratic extension of F, $i = 1, 2, 3$. Let ψ be an additive character of F. From a direct computation we can write

$$\lambda_{K/F}(\psi) = W(\mathrm{Ind}_{K/F}1, \psi) = W(\chi_1, \psi)W(\chi_2, \psi)W(\chi_3, \psi) = \lambda_1(\psi)\lambda_2(\psi)\lambda_3(\psi). \quad (3.4.25)$$

On the other hand from Lemma 3.3.9 we see:

$$\lambda_{K/F}(\psi) = -\lambda_2(\psi)^2 = -\lambda_3(\psi)^2. \quad (3.4.26)$$

Comparing these two expressions we obtain:

$$\lambda_1(\psi)\lambda_3(\psi) = -\lambda_2(\psi), \text{ and } \lambda_1(\psi)\lambda_2(\psi) = -\lambda_3(\psi).$$

Moreover, since L_1/F is unramified, therefore from Lemma 3.4.2 we have $\lambda_1(\psi) = (-1)^{n(\psi)}$. So we observe:

(a) The three conditions: $n(\psi)$ odd, $\lambda_1(\psi) = -1$, and $\lambda_2(\psi) = \lambda_3(\psi)$ are equivalent.

(b) In the same way, the conditions: $n(\psi)$ even, $\lambda_1(\psi) = 1$ and $\lambda_2(\psi) = -\lambda_3(\psi)$ are equivalent.

Let μ_4 denote the group generated by a fourth root of unity. Then we have more equivalences:

(c) $q_F \equiv 1 \pmod 4$, $\mu_4 \subset F^\times$ and $\lambda_2(\psi)^2 = \lambda_3(\psi)^2 = 1$ are three equivalent conditions.

(d) $q_F \equiv 3 \pmod 4$, $\mu_4 \not\subset F^\times$ and $\lambda_2(\psi)^2 = \lambda_3(\psi)^2 = -1$ are also three equivalent conditions.

This gives us four disjoint cases: We can have (a) and (c) or (a) and (d) or (b) and (c) or (b) and (d).

We can put all these four disjoint cases into the following table:

q_F	$n(\psi)$	$\lambda_{K/F}$	λ_1	λ_2	λ_3
$q_F \equiv 1 \pmod 4$	*odd*	-1	-1	± 1	± 1
$q_F \equiv 1 \pmod 4$	*even*	-1	1	1	-1
$q_F \equiv 3 \pmod 4$	*odd*	1	-1	$\pm i$	$\pm i$
$q_F \equiv 3 \pmod 4$	*even*	1	1	$+i$	$-i$

For the two cases where λ_2, λ_3 have different sign, and we take (up to permutation) λ_2 with positive sign.

3.4.2 Computation of $\lambda_{K/F}$, where K/F is a wildly ramified quadratic extension

In the case $p = 2$, the square class group of F, i.e., $F^\times/F^{\times 2}$ can be large, so we can have many quadratic characters but they are wildly ramified, not tame. Let $F = \mathbb{Q}_2$, then we have $\mathbb{Q}_2^\times/\mathbb{Q}_2^{\times 2} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. If $K/\mathbb{Q}_2$ is the abelian extension for which $N_{K/\mathbb{Q}_2}(K^\times) = \mathbb{Q}_2^{\times 2}$, then we have the following lemma.

Lemma 3.4.12. *Let K be the finite abelian extension of $\mathbb{Q}_2$ for which $N_{K/\mathbb{Q}_2}(K^\times) = \mathbb{Q}_2^{\times 2}$. Then $\lambda_{K/\mathbb{Q}_2} = 1$.*

Proof. Let $G = \mathrm{Gal(K/\mathbb{Q}_2)}$. We know that $\mathbb{Q}_2/\mathbb{Q}_2^{\times 2} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore from class field theory $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. So the 2-rank of G is 3, i.e., $\mathrm{rk}_2(\mathrm{G}) = 3$, and hence from the equation 3.3.7 we have $\Delta_1^G = 1$. Moreover, it is easy to see that G is not metacyclic, because $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not cyclic. So from Theorems 3.3.1, and 3.1.2 we have $c_1^G = 1$. Then finally we obtain

$$\lambda_{K/\mathbb{Q}_2} = \lambda_1^G = c_1^G \cdot W(\Delta_1^G) = 1.$$

□

Moreover, from Theorem 2.5.2, if $F/\mathbb{Q}_2$, we have $|F^\times/F^{\times 2}| = 2^m$, $(m \geqslant 3)$, therefore more generally we obtain the following result.

Theorem 3.4.13. *Let F be an extension of $\mathbb{Q}_2$. Let K be the abelian extension for which $N_{K/F}(K^\times) = F^{\times 2}$. Then $\lambda_{K/F} = 1$.*

Proof. This proof is same as the above Lemma 3.4.12. Let $G = \mathrm{Gal(K/F)}$. From Theorem 2.5.2 we have $\mathrm{rk}_2(\mathrm{G}) \neq 1$ and G is not metacyclic. Therefore by using Theorems 3.3.1, and 3.1.2 we can conclude that $\lambda_{K/F} = \lambda_1^G = 1$. □

Example 3.4.14 (Computation of $\lambda_{L/\mathbb{Q}_2}$, where $L/\mathbb{Q}_2$ is a quadratic extension). Let $F = \mathbb{Q}_2$. For the principal unit filtration we write $U^i := U^i_{\mathbb{Q}_2}$. Then we have

$$\mathbb{Q}_2^\times \supset U^0 = U^1 \supset U^2 \supset U^3,$$

and $U^3 \subset \mathbb{Q}_2^{\times 2}$, therefore we can write

$$\mathbb{Q}_2^\times \supset U^0\mathbb{Q}_2^{\times 2} = U^1\mathbb{Q}_2^{\times 2} \supset U^2\mathbb{Q}_2^{\times 2} \supset U^3\mathbb{Q}_2^{\times 2} = \mathbb{Q}_2^{\times 2}.$$

Now take modulo $\mathbb{Q}_2^{\times 2}$ we have

$$\mathbb{Q}_2^\times/\mathbb{Q}_2^{\times 2} > U^1\mathbb{Q}_2^{\times 2}/\mathbb{Q}_2^{\times 2} > U^2\mathbb{Q}_2^{\times 2}/\mathbb{Q}_2^{\times 2} > \{1\},$$

and the index is always 2. So we have

$$2 - 1 = 1 \text{ character } \chi_1 \text{ with } a(\chi_1) = 0,\ \chi_1 \neq \chi_0, \text{ the trivial character,}$$
$$2^2 - 2 = 2 \text{ characters } \chi_2,\ \chi_3 \text{ with } a(\chi_i) = 2,\ i = 2, 3,$$
$$2^3 - 2^2 = 4 \text{ characters } \chi_4, \cdots, \chi_7 \text{ with } a(\chi_i) = 3,\ i = 4, \cdots, 7.$$

The last case is the **exceptional case** (cf. p. 50) because $p=2$, $e=1$ gives $i=\frac{pe}{p-1}=2$. Here we will have odd conductor. We can simplify as follows:

$$\chi_1,\ \chi_2,\ \chi_3=\chi_1\chi_2,\ \chi_4,\ \chi_5=\chi_1\chi_4,\ \chi_6=\chi_2\chi_4,\ \text{and}\ \chi_7=\chi_1\chi_2\chi_4.$$

We denote $G=\mathrm{Gal(K/F)}\cong\mathbb{Q}_2^\times/\mathbb{Q}_2^{\times 2}$. Since G is abelian, then $G\cong\widehat{G}$, namely $\widehat{G}=\{1=\chi_0,\chi_1,\chi_2,\cdots,\chi_7\}$, where $\chi_i^2=1$, $i=1,2,\cdots,7$. So we can write

$$\mathrm{Ind_{K/F}}(1)=1\oplus\sum_{\mathrm{i=1}}^{7}\chi_{\mathrm{i}}.$$

Again from Lemma 3.4.12 we have $\lambda_{K/F}=1$. Thus we can write

$$\begin{aligned}\lambda_1^G=\lambda_{K/F}=W(\mathrm{Ind_{K/F}}(1))&=\prod_{\mathrm{i=1}}^{7}\mathrm{W}(\chi_{\mathrm{i}})\\ &=\prod_{i=1}^{7}\lambda_i=1,\end{aligned}$$

where $\lambda_i=\lambda_{K_i/F}$ and K_i/F is the corresponding quadratic extension of character χ_i for $i=1,2,\cdots,7$. Moreover, there is an unramified quadratic extension of $\mathbb{Q}_2$, namely $K_1/\mathbb{Q}_2$ which corresponds χ_1. Then $\lambda_1=\lambda_{K_1/\mathbb{Q}_2}=(-1)^{n(\psi_{\mathbb{Q}_2})}=1$, because the conductor $n(\psi_{\mathbb{Q}_2})=0$. We also have

$$\lambda_2=W(\chi_2),\ \lambda_3=W(\chi_3),\cdots,\lambda_7=W(\chi_7).$$

Then we obtain

$$\begin{aligned}\lambda_{K/\mathbb{Q}_2}(\psi_{\mathbb{Q}_2})&=\prod_{i=1}^{7}\lambda_i\\ &=W(\chi_1)W(\chi_2)W(\chi_3)W(\chi_4)W(\chi_5)W(\chi_6)W(\chi_7)\\ &=W(\chi_1)W(\chi_2)W(\chi_1\chi_2)W(\chi_4)W(\chi_1\chi_4)W(\chi_2\chi_4)W(\chi_1\chi_2\chi_4)\\ &=(-1)^{n(\psi_{\mathbb{Q}_2})}\cdot W(\chi_2)\cdot\chi_1(p)^2W(\chi_2)\cdot W(\chi_4)\cdot\chi_1(p)^3W(\chi_4)\cdot W(\chi_2\chi_4)\cdot\chi_1(p)^3W(\chi_2\chi_4)\\ &=W(\chi_2)^2\cdot W(\chi_4)^2\cdot W(\chi_2\chi_4)^2\\ &=1,\end{aligned}$$

since $n(\psi_{\mathbb{Q}_2})=0$ and $\lambda_{K/\mathbb{Q}_2}=1$.

Now we have to give explicit computation of λ_i, where $i=1,\cdots,7$. For this particular example directly we can give explicit computation of λ_i by using the modified formula (2.3.4) of abelian local constant. Before going to our explicit computation we need to recall few facts. Suppose that χ is a multiplicative character of a non-archimedean local field $F/\mathbb{Q}_p$ of conductor n. Then we can write

$$W(\chi,\psi_F)=\chi(\pi_F^{n+n(\psi_F)})\cdot q_F^{-\frac{n}{2}}\cdot\sum_{x\in U_F/U_F^n}\chi^{-1}(x)\psi(\frac{x}{\pi_F^{n+n(\psi_F)}}),\tag{3.4.27}$$

where π_F is a uniformizer of F. By definition we have $\psi_F(x) = e^{2\pi i \mathrm{Tr}_{F/\mathbb{Q}_p}(x)}$, and any element $x \in U_F/U_F^n$ can be written as

$$x = a_0 + a_1\pi_F + a_2\pi_F^2 + \cdots + a_{n-1}\pi_F^{n-1}, \quad \text{where } a_i \in k_F \text{ and } a_0 \neq 0.$$

Then we can consider the following set

$$\{a_0 + a_1\pi_F + a_2\pi_F^2 + \cdots + a_{n-1}\pi_F^{n-1} \mid \quad \text{where } a_i \in k_F \text{ and } a_0 \neq 0\}$$

is a representative of U_F/U_F^n.

When $F = \mathbb{Q}_2$, we have $a(\chi_2) = a(\chi_3) = 2$ and $a(\chi_i) = 3$, $(i = 4, \cdots, 7)$. Therefore we can write

$$U_{\mathbb{Q}_2}/U_{\mathbb{Q}_2}^2 = \{1, 1+2\} = \{1, 3\}, \qquad U_{\mathbb{Q}_2}/U_{\mathbb{Q}_2}^3 = \{1, 1+2, 1+2^2, 1+2+2^2\} = \{1, 3, 5, 7\}.$$

We also know that any square element x in $\mathbb{Q}_2^\times$ is of the form $x = 4^m(1+8n)$, where $m \in \mathbb{Z}$ and n is a 2-adic integer. This tells us $\pm 2, -1$ and ± 5 are not square in $\mathbb{Q}_2^\times$. Again, if we fix $\pi_F = 2$ as a uniformizer, then we can write

$$\mathbb{Q}_2^\times =< 2 > \times U_{\mathbb{Q}_2}^1 =< 2 > \times < \eta > \times U_{\mathbb{Q}_2}^2 =< 2 > \times < -1 > \times U_{\mathbb{Q}_2}^2,$$

where $\eta^2 = 1$. Then we have the following list of seven quadratic extensions of $\mathbb{Q}_2$ (cf. [30], p. 18, Corollary of Theorem 4 and [15], pp. 83-84):

$$\mathbb{Q}_2(\sqrt{5}), \mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{-5}), \mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{10}), \mathbb{Q}_2(\sqrt{-10}).$$

Now our next job is to see the norm groups of the above quadratic extensions of $\mathbb{Q}_2$. For any finite extension K/F, we denote $\mathcal{N}_{K/F} := N_{K/F}(K^\times)$, the norm group of the extension K/F. So we can write:

$$\mathcal{N}_{\mathbb{Q}_2(\sqrt{5})/\mathbb{Q}_2} =< 2^2 > \times U_{\mathbb{Q}_2} =< 2^2 > \times < -1 > \times U_{\mathbb{Q}_2}^2,$$

$$\mathcal{N}_{\mathbb{Q}_2(\sqrt{-1})/\mathbb{Q}_2} =< 2 > \times U_{\mathbb{Q}_2}^2,$$

$$\mathcal{N}_{\mathbb{Q}_2(\sqrt{-5})/\mathbb{Q}_2} =< -2 > \times U_{\mathbb{Q}_2}^2,$$

$$\mathcal{N}_{\mathbb{Q}_2(\sqrt{2})/\mathbb{Q}_2} =< 2 > \times < -1 > \times U_{\mathbb{Q}_2}^3,$$

$$\mathcal{N}_{\mathbb{Q}_2(\sqrt{-2})/\mathbb{Q}_2} =< 2 > \times U_{\mathbb{Q}_2}^3,$$

$$\mathcal{N}_{\mathbb{Q}_2(\sqrt{10})/\mathbb{Q}_2} =< 2 \times 5 > \times < -1 > \times U_{\mathbb{Q}_2}^3,$$

$$\mathcal{N}_{\mathbb{Q}_2(\sqrt{-10})/\mathbb{Q}_2} =< -2 > \times U_{\mathbb{Q}_2}^3.$$

From the above norm groups, we can conclude that:

1. the extension $\mathbb{Q}_2(\sqrt{5})$ is unramified, hence it corresponds the character χ_1.

2. the extensions $\mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{-5})$ are two wild quadratic extensions which correspond the characters χ_2, χ_3 respectively.

3. and the extensions $\mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{10}), \mathbb{Q}_2(\sqrt{-2})$, and $\mathbb{Q}_2(\sqrt{-10})$ correspond the characters χ_4, χ_5, χ_6 and χ_7 respectively.

Now we have all necessary informations for giving explicit formula of λ-functions, and they are:

1. $\lambda_{\mathbb{Q}_2(\sqrt{5})/\mathbb{Q}_2} = W(\chi_1, \psi_{\mathbb{Q}_2}) = (-1)^{n(\psi_{\mathbb{Q}_2})} = 1.$

2.
$$\begin{aligned}\lambda_{\mathbb{Q}_2(\sqrt{-1})/\mathbb{Q}_2} &= W(\chi_2, \psi_{\mathbb{Q}_2}) = \chi_2(2^2)\cdot\frac{1}{2}\cdot\sum_{x\in U_{\mathbb{Q}_2}/U_{\mathbb{Q}_2}^2}\chi_2(x)\psi_{\mathbb{Q}_2}(\frac{x}{4})\\ &= \frac{1}{2}\cdot\left(\psi_{\mathbb{Q}_2}(\frac{1}{4}) + \chi_2(3)\cdot\psi_{\mathbb{Q}_2}(\frac{3}{4})\right)\\ &= \frac{1}{2}\cdot\left(e^{\frac{\pi i}{2}} - e^{\frac{3\pi i}{2}}\right), \quad \text{since } 3\notin\mathcal{N}_{\mathbb{Q}_2(\sqrt{-1})/\mathbb{Q}_2} \text{ and } \psi_{\mathbb{Q}_2}(x) = e^{2\pi i x}\\ &= \frac{1}{2}\cdot(i+i) = i.\end{aligned}$$

3. $\lambda_{\mathbb{Q}_2(\sqrt{-5})/\mathbb{Q}_2} = W(\chi_3, \psi_{\mathbb{Q}_2}) = W(\chi_1\chi_2, \psi_{\mathbb{Q}_2}) = \chi_1(2^2)\cdot W(\chi_2, \psi_{\mathbb{Q}_2}) = i.$

4.
$$\begin{aligned}\lambda_{\mathbb{Q}_2(\sqrt{2})/\mathbb{Q}_2} &= W(\chi_4, \psi_{\mathbb{Q}_2}) = \chi_4(2^3)\cdot\frac{1}{2\sqrt{2}}\cdot\sum_{U_{\mathbb{Q}_2}/U_{\mathbb{Q}_2}^3}\chi_4(x)\psi_{\mathbb{Q}_2}(\frac{x}{8})\\ &= \frac{1}{2\sqrt{2}}\cdot\left(\psi_{\mathbb{Q}_2}(\frac{1}{8}) + \chi_4(3)\cdot\psi_{\mathbb{Q}_2}(\frac{3}{8}) + \chi_4(5)\cdot\psi_{\mathbb{Q}_2}(\frac{5}{8}) + \chi_4(7)\cdot\psi_{\mathbb{Q}_2}(\frac{7}{8})\right)\\ &= \frac{1}{2\sqrt{2}}\left(e^{\frac{\pi i}{4}} - e^{\frac{3\pi i}{4}} - e^{\frac{5\pi i}{4}} + e^{\frac{7\pi i}{4}}\right)\\ &= \frac{1}{2\sqrt{2}}\cdot(2\sqrt{2} + 0\cdot i)\\ &= 1,\end{aligned}$$

since $3, 5\notin\mathcal{N}_{\mathbb{Q}_2(\sqrt{2})/\mathbb{Q}_2}$ but $7\in\mathcal{N}_{\mathbb{Q}_2(\sqrt{2})/\mathbb{Q}_2}$.

5. $\lambda_{\mathbb{Q}_2(\sqrt{10})/\mathbb{Q}_2} = W(\chi_5, \psi_{\mathbb{Q}_2}) = W(\chi_1\chi_4, \psi_{\mathbb{Q}_2}) = \chi_1(2^3)\cdot W(\chi_4, \psi_{\mathbb{Q}_2}) = (-1)\cdot 1 = -1.$

6. Again $2, 3\in\mathcal{N}_{\mathbb{Q}_2(\sqrt{-2})/\mathbb{Q}_2}$ but $5, 7\notin\mathcal{N}_{\mathbb{Q}_2(\sqrt{-2})/\mathbb{Q}_2}$, so similarly we can write
$$\begin{aligned}\lambda_{\mathbb{Q}_2(\sqrt{-2})/\mathbb{Q}_2} &= W(\chi_6, \psi_{\mathbb{Q}_2})\\ &= \chi_6(2^3)\cdot\frac{1}{2\sqrt{2}}\left(e^{\frac{\pi i}{4}} + e^{\frac{3\pi i}{4}} - e^{\frac{5\pi i}{4}} - e^{\frac{7\pi i}{4}}\right)\\ &= \frac{1}{2\sqrt{2}}\cdot 2\sqrt{2} = i.\end{aligned}$$

7. $\lambda_{\mathbb{Q}_2(\sqrt{-10})/\mathbb{Q}_2} = W(\chi_7, \psi_{\mathbb{Q}_2}) = W(\chi_1\chi_6, \psi_{\mathbb{Q}_2}) = (-1)\cdot W(\chi_6, \psi_{\mathbb{Q}_2}) = -i.$

Remark 3.4.15. Finally we observe that Theorem 3.3.10 and Corollary 3.3.11 are the general results on $\lambda_1^G = \lambda_{E/F}$, where E/F is a Galois extension with Galois group $G = \mathrm{Gal(E/F)}$. And **the general results leave open** the computation of $W(\alpha)$, where α is a quadratic character of G. For such a quadratic character we can have three cases:

1. unramified, this is the Theorem 3.4.5,
2. tamely ramified, this is the Theorem 3.4.10,
3. wildly ramified, its explicit computation is still open.

We also observe from the above example 3.4.14 that giving explicit formula for $\lambda_{K/F}$, where K/F is a wildly ramified quadratic extension, is very subtle. In particular, when $F = \mathbb{Q}_2$, in the above Example (3.4.14) we have the explicit computation of $\lambda_{K/\mathbb{Q}_2}$.

Chapter 4

Determinant of Heisenberg representations

In this chapter we give an invariant formula of determinant of a Heisenberg representation ρ of a finite group G modulo $\mathrm{Ker}(\rho)$. The group G need not be a two-step nilpotent group, but under modulo $\mathrm{Ker}(\rho)$, G is a two-step nilpotent group. In this chapter firstly we compute transfer map for two-step nilpotent group. Then we compute $\det(\rho)$ modulo $\mathrm{Ker}(\rho)$, because G is always a two-step nilpotent group under modulo $\mathrm{Ker}(\rho)$. This chapter is based on the article [45].

4.1 Explicit computation of the transfer map for two-step nilpotent group

Let G be a finite group with $[G,[G,G]] = \{1\}$. Let H be a normal subgroup of G, with abelian quotient group G/H of order d. If d is odd, then in the following lemma we compute $T_{G/H}(g)$ for all $g \in G$.

Lemma 4.1.1. *Assume that G is a finite group and H a normal subgroup such that*

1. *H is abelian,*
2. *G/H is abelian of odd order d,*
3. *$[G,[G,G]] = \{1\}$.*

Then we have $T_{G/H}(g) = g^d$ for all $g \in G$.
As a consequence one has $[G,G]^d = \{1\}$, in other words, G^d is contained in the center of G.

Proof. In general, we know that transfer map is independent of the choice of the left transversal for H in G. So we take T as a transversal[1] for H in G such that T forms a subgroup

[1] Since H is normal, left cosets and right cosets are the same, so we can simply call transversal instead of specifying left or right transversal. Again here H is abelian normal subgroup of G, hence G splits over H (cf. [18], p. 103, Theorem 22 (Artin)).

of G. By the given condition H is normal, we have $T \cong G/H$ and $G = TH$, where $TH = \{th \mid t \in T, h \in H\}$. This shows that every element $g \in G$ can uniquely be written as $g = th$, where $t \in T$ and $h \in H$.

First assume $g = h \in H$. Then we have

$$ht = t \cdot t^{-1}ht \in tH,$$

because H is a normal subgroup of G. Hence $s = t$, where $s = s(t)$ is a function of t which is uniquely determined by $gt \in sH$, for some $g \in G$. Therefore:

$$\begin{aligned} T_{G/H}(h) &= \prod_{t\in T} s^{-1}ht = \prod_{t\in T} t^{-1}ht = \prod_{t\in T} hh^{-1}t^{-1}ht = \prod_{t\in T}(h \cdot [h^{-1}, t^{-1}]) \\ &= h^d \prod_{t\in T}[h^{-1}, t^{-1}] = h^d[h^{-1}, \prod_{t\in T} t^{-1}]. \end{aligned} \tag{4.1.1}$$

We have used the condition (3) in the last two equalities which means that commutators are in the center. Now we use that G/H is of odd order, hence $x = 1$ if $x \in G/H$ is an element such that $x = x^{-1}$, i.e., G/H has no self-inverse element. Therefore from Theorem 2.7.1, we have $\prod_{t\in T} t^{-1} = \prod_{t\in T} t = 1 \in G/H$, hence $T_{G/H}(h) = h^d$. Proceeding with the proof of the Lemma we have now

$$T_{G/H}(th) = T_{G/H}(t) \cdot T_{G/H}(h) = T_{G/H}(t) \cdot h^d. \tag{4.1.2}$$

Moreover, from Lemma 2.7.2 we can write

$$t^d h^d = (th)^d [t^{\frac{d(d-1)}{2}}, h] = (th)^d [e, h] = (th)^d,$$

since $[G, G] \subseteq Z(G)$ and d is odd[2]. So we are left to show that $T_{G/H}(t) = t^d$ for all $t \in T$.

Since G/H is an abelian group of odd order, hence we may write

$$G/H = C \times U,$$

where C is cyclic group of odd order $m|d$, and we assume $t \in T$ such that tH is a generator of C. Then our transversal system can be chosen as

$$T = \{t^i u | i = 0, 1, \cdots, m-1, uH \in U\}.$$

Now if $i \leqslant m-2$ we have $t \cdot t^i u = t^{i+1} \cdot u = s$, hence $s^{-1} \cdot t \cdot t^i u = 1$. But $i = m-1$ we obtain

$$t(t^{m-1}u) = t^m u \in uH, \qquad u^{-1}t(t^{m-1}u) = u^{-1}t^m u,$$

hence

$$\begin{aligned} T_{G/H}(t) &= \prod_{u\in U} u^{-1}t^m u = \prod_{u\in U} t^m[t^{-m}, u^{-1}] = t^d \prod_{u\in U}[t^{-m}, u^{-1}] \\ &= t^d[t^{-m}, \prod_{u\in U} u^{-1}] = t^d[t^{-m}, e] = t^d, \end{aligned} \tag{4.1.3}$$

[2]Here d divides $\frac{d(d-1)}{2}$ and the order of group G/H is d. So for any $t \in G/H$, $t^{\frac{d(d-1)}{2}} = e$, the identity in G/H.

since d is odd, then the order of U is also odd and by Theorem 2.7.1 we have $\prod_{u\in U} u^{-1} = \prod_{u\in U} u = e \in U$.

We also know that any $g \in G$ can uniquely be written as $g = th$, where $t \in T$ and $h \in H$. Then finally we obtain:

$$T_{G/H}(g) = T_{G/H}(th) = T_{G/H}(t) \cdot T_{G/H}(h) = t^d \cdot h^d = (th)^d[t^{\frac{d(d-1)}{2}}, h] = g^d. \tag{4.1.4}$$

Moreover, by our assumption (2), we have G/H is an abelian group, therefore $[G, G] \subseteq H$, in particular, $T_{G/H}(h) = h^d$ for $h \in [G, G] \subseteq H$. On the other hand, from Theorem 2.8.1 $T_{G/[G,G]}$ is trivial. So under the above Lemma's conditions we conclude $[G, G]^d = 1$, in other words, G^d is in the center because due to condition (3) the commutator is bilinear.

□

Remark 4.1.2. From Lemma 4.1.1 we have $T_{G/H}(g) = g^d$ for all $g \in G$. This implies for $g_1, g_2 \in G$

$$T_{G/H}(g_1g_2) = (g_1g_2)^d, \text{ on the other hand,}$$
$$T_{G/H}(g_1g_2) = T_{G/H}(g_1) \cdot T_{G/H}(g_2) = g_1^d \cdot g_2^d,$$

because $T_{G/H}$ is a homomorphism. Hence for all g_1, $g_2 \in G$ we have

$$(g_1g_2)^d = g_1^d g_2^d.$$

This implies G^d is actually a subgroup of G not only a subset.

By combining Lemma 4.1.1 and the elementary divisor theorem, we have the following result.

Lemma 4.1.3. *Assume that G is a finite group and H a normal subgroup such that*

1. *H is abelian*

2. *G/H is abelian of order d, such that (according to the elementary divisor theorem):*

$$G/H \cong \mathbb{Z}/m_1 \times \cdots \times \mathbb{Z}/m_s$$

   where $m_1 | \cdots | m_s$ and $\prod_i m_i = d$. Moreover, we fix elements $t_1, t_2, \cdots, t_s \in G$ such that $t_iH \in G/H$ generates the cyclic factor $\cong \mathbb{Z}/m_i$, hence $t_i^{m_i} \in H$.

3. *$[G, [G, G]] = \{1\}$. In particular, $[G, G]$ is in the center $Z(G)$ of G.*

Then each $g \in G$ has a unique decomposition

(i)

$$g = t_1^{a_1} \cdots t_s^{a_s} \cdot h, \quad T_{G/H}(g) = \prod_i^s T_{G/H}(t_i)^{a_i} \cdot T_{G/H}(h),$$

where $0 \leqslant a_i \leqslant m_i - 1$, $h \in H$, and

(ii)

$$T_{G/H}(t_i) = t_i^d \cdot [t_i^{m_i}, \alpha_i], \qquad T_{G/H}(h) = h^d \cdot [h, \alpha],$$

where $\alpha_i \in G/H$ is the product over all elements from $C_i \subset G/H$, the subgroup which is complementary to the cyclic subgroup $< t_i >$ mod H, and where $\alpha \in G/H$ is product over all elements from G/H.
Here we mean $[t_i^{m_i}, \alpha_i] := [t_i^{m_i}, \widehat{\alpha}_i]$, $[h, \alpha] := [h, \widehat{\alpha}]$ for any representatives $\widehat{\alpha}_i, \widehat{\alpha} \in G$. The commutators are independent of the choice of the representatives and are always elements of order $\leqslant 2$ because $\widehat{\alpha}_i{}^2, \widehat{\alpha}^2 \in H$, and H is abelian. As a consequence of (i) and (ii) we always obtain

(iii)

$$T_{G/H}(g) = g^d \cdot \varphi_{G/H}(g),$$

where $\varphi_{G/H}(g) \in Z(G)$ is an element of order $\leqslant 2$.

As a consequence of the second equality in (ii) combined with $[G,G] \subseteq H \cap \mathrm{Ker}(T_{G/H})$, one has $[G,G]^d = \{1\}$, in other words, G^d is contained in the center $Z(G)$ of G.

Proof. By the given conditions, we have the abelian group G/H of order d with

$$G/H \cong \mathbb{Z}/m_1 \times \cdots \times \mathbb{Z}/m_s$$

where $m_1 | \cdots | m_s$ and $\prod_i m_i = d$. Moreover, we fix elements $t_1, t_2, \cdots, t_s \in G$ such that $t_i H \in G/H$ generates the cyclic factor $\cong \mathbb{Z}/m_i$, hence $t_i^{m_i} \in H$. Therefore for a fixed $i \in \{1, 2, \cdots, s\}$ we can define a subgroup $C_i \subset G/H$ such that C_i is complementary to the cyclic subgroup $< t_i >$ of order m_i mod H, i.e., $G/H =< t_i H > \times C_i$.

Then for a fixed $i \in \{1, 2, \cdots, s\}$, we can choose a transversal system for H in G and which is:

$$T = \left\{ t_i^j c | \quad 0 \leqslant j \leqslant m_i - 1, cH \in C_i \right\}.$$

Therefore from equation (4.1.3) we can write

$$T_{G/H}(t_i) = t_i^d \cdot [t_i^{-m_i}, \prod_{c \in C_i} c] = t_i^d \cdot [t_i^{-m_i}, \alpha_i], \tag{4.1.5}$$

where $\alpha_i = \prod_{c \in C_i} c$.

For $h \in H$, from equation (4.1.1) we have

$$T_{G/H}(h) = h^d \cdot [h^{-1}, \alpha], \tag{4.1.6}$$

where $\alpha = \prod_{t \in T} t$.

We also have $\widehat{\alpha}^2, \widehat{\alpha}_i{}^2 \in H$, and the commutator $[.,.]$ is bilinear by assumption (3), hence $1 = [h, \widehat{\alpha}^2] = [h, \widehat{\alpha}]^2$ and therefore

$$[h, \widehat{\alpha}] = [h, \widehat{\alpha}]^{-1} = [h^{-1}, \widehat{\alpha}].$$

Similarly, we have

$$[t_i^{m_i}, \widehat{\alpha_i}] = [t_i^{m_i}, \widehat{\alpha_i}]^{-1} = [t_i^{-m_i}, \widehat{\alpha_i}].$$

Thus we can rewrite the equations (4.1.5) and (4.1.6) as:

$$T_{G/H}(t_i) = t_i^d \cdot [t_i^{-m_i}, \prod_{c \in C_i} c] = t_i^d \cdot [t_i^{m_i}, \alpha_i], \tag{4.1.7}$$

and

$$T_{G/H}(h) = h^d \cdot [h, \alpha]. \tag{4.1.8}$$

Here $[t_i^{m_i}, \alpha_i] := [t_i^{m_i}, \widehat{\alpha_i}]$ and $[h, \alpha] := [h, \widehat{\alpha}]$ for any representatives $\widehat{\alpha_i}, \widehat{\alpha} \in G$.

We also know that every $g \in G$ can be uniquely written as th, where $t \in G/H$ and $h \in H$. Again, since G/H is abelian, therefore by using elementary divisor decomposition of G/H, we can also uniquely express t as $t = t_1^{a_1} t_2^{a_2} \cdots t_s^{a_s}$, where $0 \leqslant a_i \leqslant m_i - 1$. Thus each g has a unique decomposition

$$g = th = t_1^{a_1} t_2^{a_2} \cdots t_s^{a_s} \cdot h.$$

Then we have

$$\begin{aligned} T_{G/H}(g) &= T_{G/H}(th) = T_{G/H}(t) \cdot T_{G/H}(h) \\ &= T_{G/H}(t_1^{a_1} t_2^{a_2} \cdots t_s^{a_s}) \cdot T_{G/H}(h) \\ &= \prod_{i=1}^{s} T_{G/H}(t_i)^{a_i} \cdot T_{G/H}(h). \end{aligned}$$

By the assumption (2), G/H is an abelian group, hence $[G, G] \subseteq H$. And from equation (4.1.8) we have $T_{G/H}(h) = h^d[h, \alpha]$. This implies for $[G, G] \subseteq \mathrm{Ker}(T_{G/H})$, hence $[G, G] \subseteq H \cap \mathrm{Ker}(T_{G/H})$. On the other hand in general $T_{G/H} : G \to H/[H, H]$ is a homomorphism with values in an abelian group, hence it is trivial on commutators. So under the assumptions we can say $[G, G]^d = \{1\}$, in other words, G^d is in the center $Z(G)$ because due to assumption (3) the commutator is bilinear. Let Z_2 be the set of all elements of $Z(G)$ of order $\leqslant 2$. Since $G^d \subseteq Z(G)$, then by using Lemma 2.7.2(2) with $n = d$, we obtain

$$\prod_{i=1}^{s} t_i^{a_i \cdot d} \equiv (\prod_{i=1}^{s} t_i^{a_i})^d \pmod{Z_2} \equiv t^d \pmod{Z_2} \tag{4.1.9}$$

because combining $G^d \subseteq Z(G)$ and Lemma 2.7.2(2) we can write

$$x^d y^d = (xy)^d \cdot [x, y]^{\frac{d(d-1)}{2}} \equiv (xy)^d \pmod{Z_2} \text{ for all } x, y \in G.$$

Moreover, since $\alpha_i^2 = 1$ and $\alpha^2 = 1$, therefore we have $[t_i^{m_i}, \alpha_i]^{a_i} \in Z_2$ for all $i \in \{1, 2, \cdots, s\}$ and $[h, \alpha] \in Z_2$. Again by using Lemma 2.7.2(2) we can write

$$\prod_{i=1}^{s} t_i^{a_i \cdot d} \cdot [t_i^{m_i}, \alpha_i]^{a_i} \cdot h^d[h, \alpha] \equiv g^d \pmod{Z_2}. \tag{4.1.10}$$

Now by using equations (4.1.7) and (4.1.8), we obtain:

$$\begin{aligned} T_{G/H}(g) &= \prod_{i=1}^{s} T_{G/H}(t_i)^{a_i} \cdot T_{G/H}(h) \\ &= \prod_{i=1}^{s} t_i^{a_i \cdot d} \cdot [t_i^{m_i}, \alpha_i]^{a_i} \cdot h^d [h, \alpha] \\ &\equiv g^d \pmod{Z_2} \quad \text{by equation (4.1.10)} \\ &= g^d \cdot \varphi_{G/H}(g), \end{aligned}$$

where $\varphi_{G/H}$ is a correcting function with values in Z_2.

□

Remark 4.1.4 (**Properties of the correcting function** $\varphi_{G/H}$). **(i)** The correcting function $\varphi_{G/H}$ is a function on $G/G^2[G,G]$ with values in Z_2.

Proof. From Lemma 4.1.3 we have

$$T_{G/H}(g) = g^d \varphi_{G/H}(g), \tag{4.1.11}$$

where $\varphi_{G/H}(g)$ is the correcting function.

We have here $[G,[G,G]] = \{1\}$. This implies $[g,z] = 1$ for all $g \in G$ and $z \in [G,G]$. Since $[G,G] \subseteq \mathrm{Ker}(T_{G/H})$, then for all $x \in [G,G]$ we have

$$T_{G/H}(gx) = T_{G/H}(g)T_{G/H}(x) = T_{G/H}(g) \quad \text{for all } g \in G.$$

Also here we have $[G,G]^d = \{1\}$, then by using Lemma 2.7.2(2) for $x \in [G,G]$ we can write

$$(gx)^d = g^d x^d [g,x]^{-\frac{d(d-1)}{2}} = g^d \quad \text{for all } g \in G.$$

From Lemma 4.1.3 we also have $T_{G/H}((gx)^d) = (gx)^d \varphi_{G/H}(gx)$. By comparing these above equations for $x \in [G,G]$ we obtain

$$\varphi_{G/H}(gx) = \varphi_{G/H}(g) \quad \text{for all } g \in G.$$

Moreover, if $x \in G^2$, from Lemma 4.1.3 we have

$$T_{G/H}(x) = x^d \varphi_{G/H}(x) = x^d \text{ since } \varphi_{G/H}(x) = 1 \in Z_2$$
$$\text{So } T_{G/H}(gx) = T_{G/H}(g) \cdot T_{G/H}(x) = T_{G/H}(g) \cdot x^d \quad \text{for all } g \in G.$$

Again from Lemma 2.7.2(2) we have for $x \in G^2$

$$(gx)^d = g^d x^d [g,x]^{-\frac{d(d-1)}{2}} = g^d x^d \quad \text{for all } g \in G.$$

By comparing we can see that $\varphi_{G/H}(gx) = \varphi_{G/H}(g)$ for all $g \in G$ and $x \in G^2$.

Thus we can conclude that the correcting function $\varphi_{G/H}$ is a function on $G/G^2[G,G]$ with values in Z_2. □

(ii) $G^d \subset Z(G)$ if and only if $\mathrm{Im}(\varphi_{\mathrm{G/H}}) \subset \mathrm{Z(G)}$.

Proof. From Lemma 4.1.3(iii) we have $T_{G/H}(g) = g^d \cdot \varphi_{G/H}(g)$. From relation (2.8.4) we also know that $\mathrm{Im(T_{G/H}) \subseteq H^{G/H} \subseteq Z(G)}$, hence $g^d \cdot \varphi_{G/H}(g) \in Z(G)$. Now if $G^d \subset Z(G)$, then $\varphi_{G/H}(g) \in Z(G)$ for all $g \in G$. Hence $\mathrm{Im(\varphi_{G/H}) \subset Z(G)}$.
Conversely, if $\mathrm{Im(\varphi_{G/H}) \subset Z(G)}$, then from $g^d\varphi_{G/H}(g) \in Z(G)$ we can conclude that $G^d \subset Z(G)$. □

(iii) When d is odd (resp. even), $\varphi_{G/H}$ is a homomorphism (resp. not a homomorphism).

Proof. Since $T_{G/H}$ is a homomorphism we obtain the identity:

$$(g_1g_2)^d\varphi_{G/H}(g_1g_2) = g_1^dg_2^d\varphi_{G/H}(g_1)\varphi_{G/H}(g_2). \tag{4.1.12}$$

This implies

$$\frac{\varphi_{G/H}(g_1g_2)}{\varphi_{G/H}(g_1)\varphi_{G/H}(g_2)} = \frac{g_1^dg_2^d}{(g_1g_2)^d} = \frac{(g_1g_2)^d[g_1,g_2]^{\frac{d(d-1)}{2}}}{(g_1g_2)^d} = [g_1,g_2]^{\frac{d(d-1)}{2}}. \tag{4.1.13}$$

We also have here $[G,G]^d = 1$. When d is odd, then d divides $\frac{d(d-1)}{2}$, hence the right side of equation (4.1.13) is equal to 1. Thus when d is odd, φ is a homomorphism, and exactly $\varphi \equiv 1$. This follows from Lemma 4.1.1.
But when d is even d does not divide $\frac{d(d-1)}{2}$, hence the right side of equation (4.1.13) is not equal to 1. This shows that $\varphi_{G/H}$ is **not** a homomorphism when d is even. □

(iv) If $H' \subset G$ is another normal subgroup such that H' is abelian and G/H' is abelian of order d, then $\varphi_{G/H'}$ is again a function on $G/G^2[G,G]$ with values in Z_2 which satisfies the same identity (4.1.13), hence we will have

$$\varphi_{G/H'} = \varphi_{G/H} \cdot f_{H,H'}$$

for some homomorphism $f_{H,H'} \in \mathrm{Hom}(G/G^2[G,G], Z_2)$.

4.2 Invariant formula of determinant for Heisenberg representations

In general, for the Heisenberg setting G need not be two-step nilpotent group. But $\overline{G} = G/\mathrm{Ker}(\rho)$ is always a two-step nilpotent group, where ρ is a Heisenberg representation of G. The Lemmas 4.1.1 and 4.1.3 hold for two-step nilpotent groups. Therefore to use them in our Heisenberg setting, we have to do our computation under **modulo** $\mathrm{Ker}(\rho)$. **Our determinant computation is under modulo** $\mathrm{Ker}(\rho)$. And we drop **modulo** $\mathrm{Ker}(\rho)$ from our remaining part of this chapter.

Before going to our next proposition we need this following result.

Proposition 4.2.1. *Let G be an abelian group of* $\mathrm{rk}_2(G) = n$. *Then G has $2^n - 1$ nontrivial elements of order* 2.

Proof. We know that $(\mathbb{Z}_n, +)$ is a cyclic group, where $n \in \mathbb{N}$. If n is odd, $\mathbb{Z}_n$ does not have any nontrivial element of order 2. But when n is even, it is clear that $\frac{n}{2} \in \mathbb{Z}_n$ is the only one nontrivial element of order 2. So this tells us when n is even, $\mathbb{Z}_n$ has a unique element of order 2.

Any given abelian group G of $\mathrm{rk}_2(G) = n$ can be written as

$$G \cong \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_s}$$

where $m_1|m_2|\cdots|m_s$ and $m_{s-n+1}, m_{s-n+2}, \cdots, m_s$ are n even, and rest of the m_i-s are odd. Therefore from equation (3.3.3) we conclude that

$$|G[2]| = |\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_s}[2]| = \prod_{i=1}^{n} |\mathbb{Z}_{m_{s-n+i}}[2]| = \underset{n\text{-times}}{2\times\cdots\times 2} = 2^n.$$

Hence we can conclude that when G is abelian with $\mathrm{rk}_2(G) = n$, it has $2^n - 1$ nontrivial elements of order 2.

□

Proposition 4.2.2. *Let $\rho = (Z, \chi_\rho)$ be a Heisenberg representation of G, of dimension d, and put $X_\rho(g_1, g_2) := \chi_\rho \circ [g_1, g_2]$. Then we obtain*

$$(\det(\rho))(g) = \varepsilon(g) \cdot \chi_\rho(g^d), \tag{4.2.1}$$

where ε is a function on G with the following properties:

1. *ε has values in $\{\pm 1\}$.*
2. *$\varepsilon(gx) = \varepsilon(g)$ for all $x \in G^2 \cdot Z$, hence ε is a function on the factor group $G/G^2 \cdot Z$, and in particular, $\varepsilon \equiv 1$ if $[G : Z] = d^2$ is odd.*
3. *If d is even, then the function ε need not be a homomorphism but:*

$$\tfrac{\varepsilon(g_1)\varepsilon(g_2)}{\varepsilon(g_1 g_2)} = X_\rho(g_1, g_2)^{\frac{d(d-1)}{2}} = X_\rho(g_1, g_2)^{\frac{d}{2}}.$$

Furthermore,

(a) ***When** $\mathrm{rk}_2(G/Z) \geqslant 4$**:** ε is a homomorphism, and exactly $\varepsilon \equiv 1$.*

(b) ***When** $\mathrm{rk}_2(G/Z) = 2$**:** ε is not a homomorphism and ε is a function on $G/G^2 Z$ such that*

$$(\det \rho)(g) = \varepsilon(g) \cdot \chi_\rho(g^d) = \begin{cases} \chi_\rho(g^d) & \text{for } g \in G^2 Z \\ -\chi_\rho(g^d) & \text{for } g \notin G^2 Z. \end{cases}$$

Proof. By the given condition, $\rho = (Z, \chi_\rho)$ is a Heisenberg representation of G. Let H be a maximal isotopic subgroup for X_ρ, then we have $\rho = \mathrm{Ind}_H^G(\chi_H)$, where χ_H is a linear character of H which extends χ_ρ. Then modulo $\mathrm{Ker}(\chi_\rho) = \mathrm{Ker}(\rho) \subset Z$ the assumptions of the Lemma 4.1.3 are fulfilled, and therefore:

$$\begin{aligned}(\det\rho)(g) &= \Delta_H^G(g) \cdot \chi_H(T_{G/H}(g)) \\ &= \Delta_H^G(g) \cdot \chi_\rho(T_{G/H}(g)) \quad \text{because the values of } T_{G/H} \text{ are in } Z \\ &= \Delta_H^G(g) \cdot \chi_\rho(g^d)\chi_\rho(\varphi_{G/H}(g)) \quad \text{from Lemma 4.1.3} \\ &= \varepsilon(g) \cdot \chi_\rho(g^d),\end{aligned}$$

where

$$\varepsilon(g) := \Delta_H^G(g) \cdot \chi_\rho(\varphi_{G/H}(g)). \tag{4.2.2}$$

Since Δ_H^G is the quadratic determinant character of G, hence for every $g \in G$, we have $\Delta_H^G(g) \in \{\pm 1\}$. And $\varphi_{G/H}(g) \in Z_2$, then $\chi_\rho(\varphi_{G/H}(g)) \in \{\pm 1\}$. Therefore for every $g \in G$,

$$\varepsilon(g) = \Delta_H^G(g) \cdot \chi_\rho(\varphi_{G/H}(g)) \in \{\pm 1\},$$

which does not depend on H because $\Delta_H^G = \Delta_1^{G/H}$, and χ_ρ does not depend on H.

Here Z is the scalar group of the irreducible representation ρ of dimension d, then by definition of scalar group, elements $z \in Z$ are represented by scalar matrices, i.e.,

$$\rho(z) = \chi_\rho(z) \cdot I_d, \quad \text{where } I_d \text{ is the } d \times d \text{ identity matrix.}$$

This implies

$$(\det\rho)(z) = \chi_\rho(z)^d = \chi_\rho(z^d).$$

We also know that Z is the radical of X_ρ, therefore

$$X_\rho(z, g) = \chi_\rho([z, g]) = 1 \text{ for all } z \in Z \text{ and } g \in G.$$

Moreover, we can consider $\det\rho$ as a linear character of G, therefore

$$(\det\rho)(gz) = (\det\rho)(g) \cdot (\det\rho)(z) = \varepsilon(g)\chi_\rho(g^d)\chi_\rho(z^d). \tag{4.2.3}$$

On the other hand

$$(\det\rho)(gz) = \varepsilon(gz)\chi_\rho((gz)^d) = \varepsilon(gz)\chi_\rho(g^d z^d [g, z]^{-\frac{d(d-1)}{2}}) = \varepsilon(gz)\chi_\rho(g^d)\chi_\rho(z^d). \tag{4.2.4}$$

On comparing equations (4.2.3) and (4.2.4) we get

$$\varepsilon(gz) = \varepsilon(g) \text{ for all } g \in G \text{ and } z \in Z.$$

Moreover, since $\varepsilon(g)$ is a sign, we have

$$(\det\rho)(g^2) = (\det\rho)(g)^2 = \varepsilon(g)^2\chi_\rho(g^d)^2 = \chi_\rho(g^{2d}).$$

Therefore

$$(\det\rho)(gx^2) = (\det\rho)(g) \cdot (\det\rho)(x^2) = \varepsilon(g)\chi_\rho(g^d)\chi_\rho(x^{2d}). \tag{4.2.5}$$

On the other hand

$$(\det\rho)(gx^2) = \varepsilon(gx^2)\chi_\rho((gx^2)^d) = \varepsilon(gx^2)\chi_\rho(g^d)\chi_\rho(x^{2d}). \tag{4.2.6}$$

So we see from equations (4.2.5) and (4.2.6) $\varepsilon(gx^2) = \varepsilon(g)$, hence ε is a function on G/G^2Z.

In particular, when $[G:Z] = d^2$ is odd, i.e., $|G/H| = d$ is odd, we have $\varphi_{G/H}(g) = 1$ as well $\Delta_H^G(g) = 1$ because H is normal subgroup of odd index in G. This shows that $\varepsilon \equiv 1$ when $[G:Z] = d^2$ is odd.

For checking property (iii), we use equation (4.2.1) and $[G,G]^d = \{1\}$. Since $[G,G]^d = \{1\}$, we have for $g_1, g_2 \in G$

$$([g_1,g_2]^{\frac{d(d-1)}{2}})^2 = 1, \text{ i.e., } [g_1,g_2]^{\frac{d(d-1)}{2}} = \frac{1}{[g_1,g_2]^{\frac{d(d-1)}{2}}}. \text{ Also,}$$
$$[g_1,g_2]^{d-1} = [g_1,g_2]^{-1} \text{ and } [g_1,g_2]^{\frac{d}{2}} = [g_1,g_2]^{-\frac{d}{2}}.$$

From equation (4.2.1) we obtain

$$\frac{(\det\rho)(g_1)\cdot(\det\rho)(g_2)}{(\det\rho)(g_1g_2)} = \frac{\varepsilon(g_1)\chi_\rho(g_1^d)\cdot\varepsilon(g_2)\chi_\rho(g_2^d)}{\varepsilon(g_1g_2)\chi_\rho((g_1g_2)^d)}.$$

This implies

$$\begin{aligned}\frac{\varepsilon(g_1)\varepsilon(g_2)}{\varepsilon(g_1g_2)} &= \frac{\chi_\rho((g_1g_2)^d)}{\chi_\rho(g_1^dg_2^d)} = \chi_\rho([g_1,g_2])^{\frac{d(d-1)}{2}} \\ &= X_\rho(g_1,g_2)^{\frac{d(d-1)}{2}} = X_\rho(g_1,g_2)^{\frac{d}{2}}.\end{aligned} \tag{4.2.7}$$

This shows that ε need not be a homomorphism when d is even.

But when $|G/Z| = d^2$ and d is even we can write

$$\begin{aligned}G/Z &\cong (\mathbb{Z}/m_1\times\mathbb{Z}/m_1)\times\cdots\times(\mathbb{Z}/m_s\times\mathbb{Z}/m_s)\\ &\cong (<t_1>\times<t_1'>)\perp\cdots\perp(<t_s>\times<t_s'>),\end{aligned}$$

such that $m_1|\cdots|m_s$ and $\prod_{i=1}^s m_i^2 = d^2$, $X_\rho(t_i,t_i') = \chi_\rho([t_i,t_i']) = \zeta_{m_i}$, a primitive m_i-th root of unity because $[t_i,t_i']^{m_i} = 1$. If m_{s-1}, m_s are both even[3], which means 2-rank of G/Z is $\geqslant 4$ then $\frac{d}{m_s}$ is even and therefore $X_\rho(x,y)^{\frac{d}{2}} \equiv 1$, hence from equation (4.2.7) we see that ε is a homomorphism.

Moreover, from the above we see that

$$H/Z = <t_1>\times\cdots\times<t_s> \cong H'/Z = <t_1'>\times\cdots<t_s'>$$

are two maximal isotropic which are isomorphic. We have $H\cap H' = Z$, hence G is not the direct product of H and H but nevertheless $G = H\cdot H'$. So for any $g\in G$ there must exist a decomposition $g = h\cdot h'$, where $h\in H$ and $h'\in H'$.

Now we assume $\mathrm{rk}_2(G/Z)\neq 2$, hence $\mathrm{rk}_2(H/Z) = \mathrm{rk}_2(H'/Z)\neq 1$. And since $G/H\cong H/Z$ and $G/H'\cong H'/Z$, then $\mathrm{rk}_2(G/H) = \mathrm{rk}_2(G/H')\neq 1$. Then from Proposition 4.2.1 we can

[3]Here $d = m_1\cdots m_{s-1}m_s$, if both m_{s-1}, m_s are even, then $\frac{d}{2} = m_1\cdots(\frac{m_{s-1}}{2})\cdot m_s$. This shows that $m_i|\frac{d}{2}$ for all $i\in\{1,\cdots,s\}$. Therefore, $X_\rho(t_i,t_i')^{\frac{d}{2}} = (\zeta_{m_i})^{\frac{d}{2}} = 1$ for all $i\in\{1,\cdots,s\}$.

say both G/H and G/H' have at least 3 elements of order 2. Then from Theorem 2.7.1 we have $\alpha_{G/H} = 1$ and $\alpha_{G/H'} = 1$. Furthermore from formula (4.1.8) we obtain

$$T_{G/H}(h) = h^d \cdot [h, \alpha_{G/H}] = h^d, \quad \text{and} \quad T_{G/H}(h') = h'^d \cdot [h', \alpha_{G/H'}] = h'^d.$$

So we can write

$$\begin{aligned}
(\det \rho)(g) &= (\det \rho)(h) \cdot (\det \rho)(h'), \quad \text{here } g = h \cdot h' \text{ is a decomposition of } g \text{ with } h \in H,\ h' \in H', \\
&= \chi_\rho(h^d) \cdot \chi_\rho(h'^d), \quad \text{because } \mathrm{rk}_2(G/H') = \mathrm{rk}_2(G/H') \neq 1, \\
&= \chi_\rho(h^d \cdot h'^d) \\
&= \chi_\rho((h \cdot h')^d [h, h']^{\frac{d(d-1)}{2}}) \quad \text{using Lemma 2.7.2(2)} \\
&= \chi_\rho(g^d) \cdot X_\rho(h, h')^{\frac{d(d-1)}{2}} \\
&= \chi_\rho(g^d),
\end{aligned}$$

because all $m_i | \frac{d}{2}$, $i \in \{1, 2, \cdots, s\}$, and then

$$X_\rho(h, h')^{\frac{d(d-1)}{2}} = \chi_\rho([h, h'])^{\frac{d(d-1)}{2}} = \zeta_m^{\frac{d(d-1)}{2}} = 1,$$

where ζ_m is a primitive m-th root of unity and m is some positive integer (which is the order of $[h, h']$) which divides $\frac{d}{2}$. This shows that when $\mathrm{rk}_2(G/Z) \neq 2$ we have $\varepsilon \equiv 1$.

If on the other hand only m_s is even, i.e., G/Z has 2 rank$= 2$, then $\frac{d}{m_s}$ is odd. Therefore $X_\rho(t_s, t'_s)^{\frac{d}{2}} = (\zeta_{m_s})^{\frac{d}{2}} = -1$, since m_s does not divide $\frac{d}{2}$ and $(\zeta_{m_s}^{\frac{d}{2}})^2 = 1$. Therefore ε cannot be a homomorphism when $\mathrm{rk}_2(G/Z) = 2$.

But since ε is a function on G/G^2Z, hence $\varepsilon|_{G^2Z} \equiv 1$. Therefore when $g \in G^2Z$ we have $(\det \rho)(g) = \chi_\rho(g^d)$. So now we are left to show that for $g \notin G^2Z$, $\varepsilon(g) = -1$, i.e., $(\det \rho)(g) = -\chi_\rho(g^d)$. Also, for $\mathrm{rk}_2(G/Z) = 2$, G/G^2Z is Klein's 4-group[4] and ε is a sign function on that group. So up to permutation the possibilities are

1. $++++$
2. $+++-$
3. $++--$
4. $+---$

The cases (1), (3) can be excluded because we know that ε is not a homomorphism. So we have to exclude the case (2) and for this it is enough to see that we must have "$-$" more than once.

[4]Since G/Z is an abelian group, we have $G/Z \cong \widehat{G/Z}$. When rk$_2$(G/Z) = 2, by Proposition 4.2.1, there are exactly three elements of order 2 in G/Z, and this each element (i.e., self-inverse element) corresponds a quadratic character of G/Z. Hence the group G/G^2Z has exactly three quadratic characters. Furthermore, G/G^2Z is a quotient group of the abelian group G/Z, hence G/G^2Z is abelian. Therefore G/G^2Z is isomorphic to the Klein's 4-group.

If we restrict ε to a maximal isotropic subgroup H, then from equation (4.2.7) we can say ε is a homomorphism on H, because $X_\rho|_{H\times H} = 1$. We also have from equation (4.1.8) $T_{G/H}(h) = h^d \cdot [h, \alpha_{G/H}] = h^d \varphi_{G/H}(h)$. This implies $\varphi_{G/H}(h) = [h, \alpha_{G/H}]$ for all $h \in H$. Moreover, since $\Delta_H^G|_H \equiv 1$, then for $h \in H$ we obtain:

$$\varepsilon(h) = \Delta_H^G(h) \cdot \chi_\rho(\varphi_{G/H}(h)) = \chi_\rho([h, \alpha_{G/H}]).$$

If there exists a maximal isotropic subgroup H of $\mathrm{rk}_2(\mathrm{H/Z}) = 1$, then from the Proposition 4.2.1 can say that H/Z has a unique element of order 2. We also know $G/H \cong H/Z$ because G/H and H/Z are both finite abelian groups of same order d, hence $\mathrm{rk}_2(\mathrm{H/Z}) = \mathrm{rk}_2(\mathrm{G/H}) = 1$. Then from the Proposition 4.2.1, G/H has a unique element of order 2, and therefore by Miller's theorem we have $\alpha_{G/H} \neq 1$. Thus for the case $\mathrm{rk}_2(\mathrm{H/Z}) = 1$ we have

$$\varepsilon(h) = \Delta_H^G(h) \cdot \chi_\rho([h, \alpha_{G/H}]) = \chi_\rho([h, \alpha_{G/H}]) = -1 \tag{4.2.8}$$

for all nontrivial $h \in H$.

Moreover, if $\mathrm{rk}_2(\mathrm{G/Z}) = 2$, then from the Lemma 2.7.4 there exists subgroups H, H' with the following properties

1. $\mathrm{rk}_2(\mathrm{H/Z}) = \mathrm{rk}_2(\mathrm{H'/Z}) = 1$
2. $G = H \cdot H'$
3. $Z = H \cap H'$

Then H/G^2Z and H'/G^2Z are two different subgroups of order 2 in Klein's 4-group. Now take the nontrivial elements of these subgroups are h and h' respectively. Then by using equation (4.2.8) we have

$$\varepsilon(h) = \varepsilon(h') = -1,$$

i.e., the nontrivial elements of H/G^2Z and H'/G^2Z give the two "$-$" signs for ε.

Therefore the only possibility is $+ - --$, i.e., ε takes 1 on the trivial coset and -1 on the three other cosets.

This completes the proof. □

Corollary 4.2.3. *1. Let $\rho = (Z, \chi_\rho) \in \mathrm{Irr}(G)$ be a Heisenberg representation of odd dimension d. Then $G^d \subseteq Z$ and*

$$\det(\rho)(g) = \chi_\rho(g^d), \qquad \textit{for all } g \in G.$$

In particular, $\det(\rho) \equiv 1$ if and only if χ_ρ is a character of Z/G^d. This is only possible if $[G, G] \not\subseteq G^d$ and if χ_ρ is a nontrivial character on $G^d[G, G]/G^d \subseteq Z/G^d$.

2. Let ω be a linear character of G, then $\rho \otimes \omega = (Z, \chi_{\rho\otimes\omega})$, where:

$$\chi_{\rho\otimes\omega} = \chi_\rho \cdot \omega_Z, \qquad \det(\rho\otimes\omega) = \det(\rho) \cdot \omega^d,$$

where $\omega_Z = \omega|_Z$. Therefore it is possible to find ω such that $\det(\rho\otimes\omega)\equiv 1$, equivalently $\chi_\rho = \omega_Z^{-1}$ on G^d, if and only if χ_ρ is trivial on $G^d\cap[G,G]$.

Proof. **(1).** We consider H such that $Z\subset H\subset G$ and H is maximal isotropic with respect to

$$X(g_1,g_2) := \chi_\rho\circ[g_1,g_2].$$

By definition, $Z/[G,G]$ is radical of X, hence $\mathrm{Ker}(\rho)=\mathrm{Ker}(\chi_\rho)\subset Z$, and factorizing by $\mathrm{Ker}(\chi_\rho)$ we obtain a group $\overline{G}=G/\mathrm{Ker}(\chi_\rho)$ which satisfies the assumptions of the Lemma 4.1.1. Moreover, $\rho=\mathrm{Ind}_H^G\chi_H$ for any extension χ_H of χ_ρ, hence

$$\det(\rho)=\Delta_H^G\cdot(\chi_H\circ T_{G/H}) = \chi_H\circ T_{G/H},$$

because $H\subset G$ is a normal subgroup of odd index d. Applying the Lemma 4.1.1 we obtain for all $g\in G$:

$$\det(g)=\chi_H\circ T_{G/H}(g)=\chi_H(g^d)=\chi_\rho(g^d),\quad \text{since } g^d\in Z, \tag{4.2.9}$$

here we have used $g^d=T_{G/H}(g)\in\mathrm{Im}(T_{G/H})\subseteq H^{G/H}\subseteq Z$ because our computation is modulo $\mathrm{Ker}(\chi_\rho)$.

If $\det(\rho)\equiv 1$, then from equation (4.2.9), we have $\chi_\rho(g^d)=1$, i.e., $g^d\in\mathrm{Ker}(\chi_\rho)$ for all $g\in G$. This shows $G^d\subseteq\mathrm{Ker}(\chi_\rho)$. Again if $G^d\subseteq\mathrm{Ker}(\chi_\rho)$, then it is easy to see $\det\rho\equiv 1$.

Now if $\chi_\rho: Z/G^d\to\mathbb{C}^\times$ is a character, then we see that $G^d\subseteq\mathrm{Ker}(\chi_\rho)$, i.e., $g^d\in\mathrm{Ker}(\chi_\rho)$. Thus from equation (4.2.9), we conclude $\det(\rho)\equiv 1$.

If $[G,G]\subseteq G^d$ this would imply $[G,G]\subset\mathrm{Ker}(\chi_\rho)$ which means $Z=G$, hence ρ is of dimension 1. Also, if χ_ρ is trivial on $G^d[G,G]/G^d$, i.e., $[G,G]\subseteq\mathrm{Ker}(\rho)$, hence dimension of ρ is 1. Therefore when $\det\rho\equiv 1$ and $[G,G]\not\subseteq G^d$ and χ_ρ is a nontrivial character on $G^d[G,G]/G^d\subseteq Z/G^d$, then we can extend χ_ρ to Z/G^d because χ_ρ is G-invariant and $G^d[G,G]/G^d$ is a normal subgroup of Z/G^d.

(2). Let ω be a linear character of G and $\omega_Z=\omega|_Z$. Then we can write (cf. [28], p. 57, Remark (3))

$$\omega\otimes\mathrm{Ind}_Z^G\chi_\rho=\mathrm{Ind}_Z^G(\chi_\rho\otimes\omega_Z)=\mathrm{Ind}_Z^G\chi_{\rho\otimes\omega}=d\cdot\rho\otimes\omega,$$

where $\chi_{\rho\otimes\omega=\chi_\rho\cdot\omega_Z}$ and $d=\dim(\rho)$. Moreover, it is easy to see $\chi_\rho\otimes\omega_Z$ is a G-invariant. Therefore we can write $\rho\otimes\omega=(Z,\chi_{\rho\otimes\omega})$. Now we are left to compute determinant of $\rho\otimes\omega$, which follows from the properties of determinant function (cf. [17], p. 955, Lemma 30.1.3). Since $\dim(\rho)=d$ and ω is linear, we have

$$\det(\rho\otimes\omega)=\det(\rho)^{\omega(1)}\cdot\det(\omega)^{\rho(1)}=\det(\rho)\cdot\omega^d.$$

Here d is odd and we know $g^d\in Z$, then for every $g\in G$, we have

$$\det(\rho\otimes\omega)(g)=\chi_\rho(g^d)\cdot\omega^d(g)=\chi_\rho\cdot\omega_Z(g^d). \tag{4.2.10}$$

Now if $\det(\rho\otimes\omega)\equiv 1$, then we have $\chi_\rho=\omega_Z^{-1}$ on G^d. This implies, it is possible to find a linear character ω such that $\det(\rho\otimes\omega)\equiv 1$.

Now let $\chi_\rho = \omega_Z^{-1}$ on G^d. Since $G^d \cap [G,G] \subseteq G^d$, $\chi_\rho \cdot \omega_Z(g) = 1$ for $g \in G^d \cap [G,G]$. Then χ_ρ is trivial on $G^d \cap [G,G]$.

Conversely, if χ_ρ is trivial on $G^d \cap [G,G]$, then we are left to show that we can find an ω such that $G^d \subseteq \mathrm{Ker}(\chi_\rho \cdot \omega_Z)$.

Put $Z_1 = G^d \cdot [G,G]$, and $Z_0 = G^d \cap [G,G]$. Then we have $Z \supset Z_1 \supset Z_0$, and $Z_1/Z_0 = G^d/Z_0 \times [G,G]/Z_0$ is a direct product. Now assume that χ_ρ is a character of Z/Z_0. Then the restriction $\chi_{Z_1} = \chi_1 \cdot \chi_2$ comes as the product of two characters of Z_1, where χ_1 is trivial on $[G,G]$ and χ_2 is trivial on G^d. But then we can find ω of $G/[G,G]$ such that $\omega_{Z_1} = \chi_1$, hence $\omega^{-1}\chi_\rho$ restricted to Z_1 is equal χ_2. In particular, $\omega^{-1}\chi_\rho$ is trivial on G^d and therefore $\omega^{-1} \otimes \rho$ has $\det(\omega^{-1} \otimes \rho) \equiv 1$. □

Corollary 4.2.4. *If $\rho = (Z, \chi_\rho)$ is a Heisenberg representation (of dimension > 1) for a nonabelian group of order p^3, $(p \neq 2,)$ then $Z = [G,G]$ is cyclic group of order p, and $G^p = Z$ or $G^p = \{1\}$ depending on the isomorphism type of G. So we have $\det(\rho) \not\equiv 1$ and $\det(\rho) \equiv 1$ depending on the isomorphism type of G.*

Proof. In this particular case for Heisenberg setting we have $|Z| = p$ and G/Z is abelian. This implies $[G,G] \subseteq Z$. Here G is nonabelian and p is prime, therefore $[G,G] = Z$ is a cyclic group of order p. Let $\Psi : G \to G$ be a p-power map, i.e., $g \mapsto g^p$. It can be proved that this map Ψ is a surjective group homomorphism (by using Lemma 2.7.2) and the image is in Z (because from Lemma 4.1.1 and relation (2.8.4) we have $g^p = T_{G/H}(g) \in Z$), hence from the first isomorphism theorem we have

$$G/\mathrm{Ker}(\Psi) \cong \mathrm{Im}(\Psi) = G^p.$$

Thus we can write

$$p^3 = |\mathrm{Ker}(\Psi)| \cdot |G^p|.$$

So we have the possibility

$$|\mathrm{Ker}(\Psi)| = p^3 \text{ or } p^2 \text{ corresponding to } G^p = 1 \text{ or } G^p = Z.$$

Both the cases are possible depending on the isomorphism type of G. So we can conclude that $\det(\rho) \not\equiv 1$ and $\det(\rho) \equiv 1$ depending on the isomorphism type of G.

□

Remark 4.2.5. We know that there are two non-abelian group of order p^3, up to isomorphism (for details see [35]). Now put

$$G_p = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z}/p\mathbb{Z} \right\}$$

We observe that this G_p is a non-abelian group under matrix multiplication with order p^3. We also see that $G_p^p = \{I_3\}$, the identity in G_p. Now if ρ is a Heisenberg representation of group G_p, then we will have $\det(\rho) \equiv 1$.

And when G is extraspecial group of order p^3, where $p \neq 2$ with $G^p = Z$, we will have $\mathrm{Ker}(\Psi) \cong C_p \times C_p$, where C_p is the cyclic group of order p. Therefore $\det(\rho)(g) = \chi_Z(g^p)$. This shows that $\det(\rho) \not\equiv 1$.

From Corollary 4.2.4, we observe that for non-abelian group of order p^3, where p is prime, the determinant of Heisenberg representation of G gives the information about the isomorphism type of G.

Remark 4.2.6. When the dimension d of ρ is odd, the 2-rank of G/Z is 0, and we notice that in this case $\varepsilon \equiv 1$. Therefore we could rephrase our above Proposition 4.2.2 as follows:

1. If the 2-rank of G/Z is different from 2, we have

$$(\det \rho)(g) = \chi_\rho(g^d).$$

2. If the 2-rank of G/Z is equal to 2, then G/G^2Z is Klein's 4-group, and we have a sign function ε on G/G^2Z such that

$$(\det \rho)(g) = \varepsilon(g) \cdot \chi_\rho(g^d).$$

Moreover, the function ε is not a homomorphism and it takes 1 on the trivial coset and -1 on the three other cosets. Thus when $\mathrm{rm}_2(\mathrm{G/Z}) = 2$, we can write

$$(\det \rho)(g) = \begin{cases} \chi_\rho(g^d) & \text{for } g \in G^2Z \\ -\chi_\rho(g^d) & \text{for } g \notin G^2Z. \end{cases}$$

Example 4.2.7. Let G be a dihedral group of order 8 and we write

$$G = \{e, b, a, a^2, a^3, ab, a^2b, a^3b \mid \quad a^4 = b^2 = e, bab^{-1} = a^{-1}\}.$$

It is easy to see that $[G, G] = \{e, a^2\} = Z = Z(G) = G^2$. Thus $G^2Z = \{e, a^2\}$. We also have

$G/G^2Z \cong \{G^2Z, aG^2Z, abG^2Z, bG^2Z\}$, and the subgroups of order 4 are:
$H_1 = \{e, b, a^2, a^2b\}$, $H_2 = \{e, ab, a^2, a^3b\}$ and $H_3 = \{e, a, a^2, a^3\}$.
And the factor groups are: $G/H_1 = \{H_1, aH_1\}$, $G/H_2 = \{H_2, aH_2\}$ and $G/H_3 = \{H_3, bH_3\}$.

Let ρ be a Heisenberg representation of G. The dimension of ρ is 2. In this case the 2-rank of G/Z is 2, then $G/Z = G/G^2Z$ is Klein's 4-group.

For this group we can see that ε is a function on $G/Z \cong \{Z, aZ, bZ, abZ\}$. When $g = a \in G$, we have H_1 for which $g = a \in H_1$, then we have

$$\varepsilon(a) = \Delta_{H_1}^G(a) \cdot \chi_\rho([a, \alpha_{G/H_1}]) = \chi_\rho([a, \alpha_{G/H_1}]) = -1.$$

Similarly, we can see when $g \in bZ$ and $g \in abZ$ we have $\varepsilon(g) = -1$. Thus we can conclude for dihedral group of order 8, that

$$(\det\rho)(g) = \begin{cases} \chi_\rho(g^2) & \text{for } g \in Z \\ -\chi_\rho(g^2) & \text{for } g \notin Z. \end{cases}$$

For our next remark we need the following lemma.

Lemma 4.2.8. *Let $\rho = (Z, \chi_\rho)$ be a Heisenberg representation of G and put $X_\rho(g_1, g_2) := \chi_\rho([g_1, g_2])$. Then for every element $g \in G$, there exists a maximal isotropic subgroup H for X_ρ such that $g \in H$.*

Proof. Let g be a nontrivial element in G. Now we take a cyclic subgroup H_0 generated by g, i.e., $H_0 =< g >$. Then $X_\rho(g, g) = 1$ implies $H_0 \subseteq H_0^\perp$. If H_0 is not maximal isotropic, then the inclusion is proper and H_0 together with some $h \in H_0^\perp \smallsetminus H_0$ generates some larger isotropic subgroup $H_1 \supset H_0$. Again we have $H_1 \subseteq H_1^\perp$, and if H_1 is not maximal then the inclusion is proper, then again we proceed same method and will have another isotopic subgroup and we continue this process step by step come to maximal isotropic subgroup H.

Therefore for every element $g \in G$, we would have a maximal subgroup H such that $g \in H$. □

Remark 4.2.9. From equation (4.2.7) we can say that ε is a homomorphism when it restricts to H because $X_\rho|_{H\times H} = 1$. Also from Lemma 4.2.8, if $g \in G$, then there always exists a maximal isotropic subgroup H such that $g = h \in H$. Since H is normal, then $\Delta_H^G = \Delta_1^{G/H}$ is trivial on H and we see in particular for $h \in H$

$$(\det\rho)(h) = \chi_H(h)^d \chi_H(\varphi_{G/H}(h)) = \chi_\rho(h^d) \cdot \chi_\rho([h, \alpha_{G/H}]), \tag{4.2.11}$$

since $[h, \alpha_{G/H}] \in Z$.

The formula (4.2.11) reformulates as:

$$(\det\rho)(h) = \chi_\rho(h^d) \cdot \chi_\rho([h, \alpha_{G/H}]) = \chi_\rho(h^d) \cdot X_\rho(h, \alpha_{G/H}), \tag{4.2.12}$$

if $g = h$ sits in some maximal isotropic H, and $\alpha_{G/H} \in G/H$ is as above (the product over all elements from G/H). Of course some $g \in G$ can sit in several different maximal isotropic H. So it is a little mysterious that the result (=left side of the formula (4.2.12)) is independent from that H. Moreover, if H is maximal isotropic then

$$G/H \cong \widehat{H/Z}, \qquad g \mapsto \{h \mapsto X(h, g)\}.$$

Now $\alpha_{G/H} \neq 1$ means that G/H has precisely one element of order 2, equivalently H/Z has precisely one character of order 2, equivalently H/H^2Z is of order 2. Therefore equation (4.2.12) can be reformulated as to say that in the critical case:

$$(\det\rho)(h) = \chi_\rho(h^d) \text{ or } = -\chi_\rho(h^d) \text{ depending on } h \in H^2Z \text{ or } h \notin H^2Z.$$

Chapter 5

The local constants for Heisenberg representations

In this chapter we give an invariant formula of local constant for a Heisenberg representation ρ of the absolute Galois group G_F of a non-archimedean local field $F/\mathbb{Q}_p$ (cf. Theorem 5.2.4). But for giving more explicit invariant formula for $W(\rho)$, we need to know the full information about the dimension of a Heisenberg representation. In Theorem 5.1.28, we compute the dimension of a Heisenberg representation ρ of G_F.

In Section 5.1, we define U-isotopic Heisenberg representations and study their properties (e.g., dimensions, Artin conductors, Swan conductors). In Theorem 5.2.7, we give an invariant formula of local constant of a minimal conductor Heisenberg representation ρ of dimension prime to p. And when ρ is not minimal conductor but dimension is prime to p, we have Theorems 5.2.9, 5.2.11.

In Section 5.3, we also discuss Tate's root-of-unity criterion, and by applying this Tate's criterion we give the information when $W(\rho)$ will be a root of unity or not. This chapter is based on the article [44].

5.1 Arithmetic description of Heisenberg representations

In Section 2.6 of Chapter 2, we became familiar with the notion of Heisenberg representations of a (pro-)finite group. These Heisenberg representations have arithmetic structure due to E.-W. Zink (cf. [11], [13], [14]). For this chapter we need to describe the arithmetic structure of Heisenberg representations.

Let $F/\mathbb{Q}_p$ be a local field, and $\overline{F}$ be an algebraic closure of F. Denote $G_F = \mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F})$ the absolute Galois group for $\overline{F}/F$. We know that (cf. [20], p. 197) each representation $\rho : G_F \to GL(n, \mathbb{C})$ corresponds to a projective representation $\overline{\rho} : G_F \to GL(n, \mathbb{C}) \to PGL(n, \mathbb{C})$. On the other hand, each projective representation $\overline{\rho} : G_F \to PGL(n, \mathbb{C})$ can be lifted to a representation $\rho : G_F \to GL(n, \mathbb{C})$. Let $A_F = G_F^{ab}$ be the factor commutator group of G_F.

Define

$$FF^\times := \varprojlim (F^\times/N \wedge F^\times/N)$$

where N runs over all open subgroups of finite index in $F^\times$. Denote by $\mathrm{Alt(F^\times)}$ as the set of all alternating characters $X : F^\times \times F^\times \to \mathbb{C}^\times$ such that $[F^\times : \mathrm{Rad(X)}] < \infty$. Then the local reciprocity map gives an isomorphism between A_F and the profinite completion of $F^\times$, and induces a natural bijection

$$\mathrm{PI(A_F)} \xrightarrow{\sim} \mathrm{Alt(F^\times)}, \tag{5.1.1}$$

where $\mathrm{PI(A_F)}$ is the set of isomorphism classes of projective irreducible representations of A_F. By using class field theory from the commutator map (2.6.3) (cf. p. 125 of [14]) we obtain

$$c : FF^\times \cong [G_F, G_F]/[[G_F, G_F], G_F]. \tag{5.1.2}$$

Let K/F be an abelian extension corresponding to the norm subgroup $N \subset F^\times$ and if $W_{K/F}$ denotes the relative Weil group, the commutator map for $W_{K/F}$ induces an isomorphism (cf. p. 128 of [14]):

$$c : F^\times/N \wedge F^\times/N \to K_F^\times/I_F K^\times, \tag{5.1.3}$$

where

$$K_F^\times := \{x \in K^\times | \quad N_{K/F}(x) = 1\}, \text{ i.e., the norm-1-subgroup of } K^\times,$$
$$I_F K^\times := \{x^{1-\sigma} | \quad x \in K^\times, \sigma \in \mathrm{Gal(K/F)}\} < \mathrm{K_F^\times}, \text{ the augmentation with respect to } K/F.$$

Taking the projective limit over all abelian extensions K/F the isomorphisms (5.1.3) induce:

$$c : FF^\times \cong \varprojlim K_F^\times/I_F K^\times, \tag{5.1.4}$$

where the limit on the right side refers to norm maps. This gives an arithmetic description of Heisenberg representations of the group G_F.

Theorem 5.1.1 (Zink, [11], p. 301, Corollary 1.2)**.** *The set of Heisenberg representations ρ of G_F is in bijective correspondence with the set of all pairs (X_ρ, χ_ρ) such that:*

1. *X_ρ is a character of $FF^\times$,*
2. *χ_ρ is a character of $K^\times/I_F K^\times$, where the abelian extension K/F corresponds to the radical $N \subset F^\times$ of X_ρ, and*
3. *via (5.1.3) the alternating character X_ρ corresponds to the restriction of χ_ρ to $K_F^\times$.*

Given a pair (X, χ), we can construct the Heisenberg representation ρ by induction from $G_K := \mathrm{Gal(\overline{F}/K)}$ to G_F:

$$\sqrt{[F^\times : N]} \cdot \rho = \mathrm{Ind_{K/F}}(\chi), \tag{5.1.5}$$

where N and K are as in (2) of the above Theorem 5.1.1 and where the induction of χ (to be considered as a character of G_K by class field theory) produces a multiple of ρ. From $[F^\times : N] = [K : F]$ we obtain the **dimension formula:**

$$\dim(\rho) = \sqrt{[\mathrm{F}^\times : \mathrm{N}]}, \tag{5.1.6}$$

where N is the radical of X.

Let K/E be an extension of E, and $\chi_K : K^\times \to \mathbb{C}^\times$ be a character of $K^\times$. In the following lemma, we give the conditions of the existence of characters $\chi_E \in \widehat{E^\times}$ such that $\chi_E \circ N_{K/E} = \chi_K$, and the solutions set of this χ_E.

Lemma 5.1.2. *Let K/E be a finite extension of a field E, and $\chi_K : K^\times \to \mathbb{C}^\times$.*

(i) The existence of characters $\chi_E : E^\times \to \mathbb{C}^\times$ such that $\chi_E \circ N_{K/E} = \chi_K$ is equivalent to $K_E^\times \subset \mathrm{Ker}(\chi_\mathrm{K})$.

(ii) In case (i) is fulfilled, we have a well defined character

$$\chi_{K/E} := \chi_K \circ N_{K/E}^{-1} : \mathcal{N}_{K/E} \to \mathbb{C}^\times, \tag{5.1.7}$$

on the subgroup of norms $\mathcal{N}_{K/E} := N_{K/E}(K^\times) \subset E^\times$, and the solutions χ_E such that $\chi_E \circ N_{K/E} = \chi_K$ are precisely the extensions of $\chi_{K/E}$ from $\mathcal{N}_{K/E}$ to a character of $E^\times$.

Proof. **(i)** Suppose that an equation $\chi_K = \chi_E \circ N_{K/E}$ holds. Let $x \in K_E^\times$, hence $N_{K/E}(x) = 1$. Then

$$\chi_K(x) = \chi_E \circ N_{K/E}(x) = \chi_E(1) = 1.$$

So $x \in \mathrm{Ker}(\chi_\mathrm{K})$, and hence $K_E^\times \subset \mathrm{Ker}(\chi_\mathrm{K})$.

Conversely assume that $K_E^\times \subset \mathrm{Ker}(\chi_\mathrm{K})$. Then χ_K is actually a character of $K^\times/K_E^\times$. Again we have $K^\times/K_E^\times \cong \mathcal{N}_{K/E} \subset E^\times$, hence $\widehat{K^\times/K_E^\times} \cong \widehat{\mathcal{N}_{K/E}}$. Now suppose that χ_K corresponds to the character $\chi_{K/E}$ of $\mathcal{N}_{K/E}$. Hence we can write $\chi_K \circ N_{K/F}^{-1} = \chi_{K/E}$. Thus the character $\chi_{K/E} : \mathcal{N}_{K/E} \to \mathbb{C}^\times$ is well defined. Since $E^\times$ is an abelian group and $\mathcal{N}_{K/E} \subset E^\times$ is a subgroup of finite index (by class field theory) $[K : E]$, we can extend $\chi_{K/E}$ to $E^\times$, and χ_K is of the form $\chi_K = \chi_E \circ N_{K/E}$ with $\chi_E|_{\mathcal{N}_{K/E}} = \chi_{K/E}$.

(ii) If condition (i) is satisfied, then this part is obvious. If χ_E is a solution of $\chi_K = \chi_E \circ N_{K/E}$, with $\chi_{K/E} := \chi_K \circ N_{K/E}^{-1} : \mathcal{N}_{K/E} \to \mathbb{C}^\times$, then certainly χ_E is an extension of the character $\chi_{K/E}$.

Conversely, if χ_E extends $\chi_{K/E}$, then it is a solution of $\chi_K = \chi_E \circ N_{K/E}$ with $\chi_K \circ N_{K/E}^{-1} = \chi_{K/E} : \mathcal{N}_{K/E} \to \mathbb{C}^\times$. □

Remark 5.1.3. Now take Heisenberg representation $\rho = \rho(X, \chi_K)$ of G_F. Let E/F be any extension corresponding to a maximal isotropic for X. In this Heisenberg setting, from Theorem 5.1.1(2), we know χ_K is a character of $K^\times/I_F K^\times$, and from the first commutative diagram on p. 302 of [11] we have $N_{K/E} : K_F^\times/I_F K^\times \to E_F^\times/I_F \mathcal{N}_{K/E}$. Thus in the Heisenberg setting, we have more information than Lemma 5.1.2(i), that χ_K is a character of

$$K^\times/K_E^\times I_F K^\times \xrightarrow{N_{K/E}} \mathcal{N}_{K/E}/I_F \mathcal{N}_{K/E} \subset E^\times/I_F \mathcal{N}_{K/E}, \tag{5.1.8}$$

and therefore $\chi_{K/F}$ is actually a character of $\mathcal{N}_{K/E}/I_F\mathcal{N}_{K/E}$, or in other words, it is a Gal(E/F)-invariant character of the Gal(E/F)-module $\mathcal{N}_{K/E} \subset E^\times$. And if χ_E is one of the solution of Lemma 5.1.2(ii), then the complete solutions is the set $\{\chi_E^\sigma \,|\, \sigma \in \mathrm{Gal(E/F)}\}$.

We know that $W(\chi_E, \psi \circ \mathrm{Tr_{K/E}})$ has the same value for all solutions χ_E of $\chi_E \circ N_{K/E} = \chi_K$, which means for all χ_E which extend the character $\chi_{K/E}$.

Moreover, from the above Lemma 5.1.2, we also can see that $\chi_E|_{\mathcal{N}_{K/E}} = \chi_K \circ N_{K/E}^{-1}$.

Let $\rho = \rho(X, \chi_K)$ be a Heisenberg representation of G_F. Let E/F be any extension corresponding to a maximal isotropic for X. Then by using the above Lemma 5.1.2, we have the following lemma.

Lemma 5.1.4. *Let $\rho = \rho(Z, \chi_\rho) = \rho(\mathrm{Gal(L/K)}, \chi_\mathrm{K})$ be a Heisenberg representation of a finite local Galois group $G = \mathrm{Gal(L/F)}$, where F is a non-archimedean local field. Let $H = \mathrm{Gal(L/E)}$ be a maximal isotropic for ρ. Then we obtain*

$$\rho = \mathrm{Ind_{E/F}}(\chi_\mathrm{E}^\sigma) \quad \textit{for all } \sigma \in \mathrm{Gal(E/F)}, \tag{5.1.9}$$

where $\chi_E : E^\times/I_F\mathcal{N}_{K/E} \to \mathbb{C}^\times$ with $\chi_K = \chi_E \circ N_{K/E}$.
Moreover, for a fixed base field E of a maximal isotropic for ρ, this construction of ρ is independent of the choice of this character χ_E.

Proof. From the group theoretical construction of Heisenberg representation (cf. see Section 2.6), we can write

$$\rho = \mathrm{Ind_H^G}(\chi_\mathrm{H}^\mathrm{g}), \quad \text{for all } g \in G/H, \tag{5.1.10}$$

where $\chi_H : H \to \mathbb{C}^\times$ is an extension of χ_ρ. From Remark 2.6.2 we know that all extensions of character χ_ρ are conjugate with respect to G/H, and they are different. If we fix H, then ρ is independent of the choice of character χ_H. For every extension of χ_ρ we will have same ρ. The assertion of the lemma is the arithmetic expression of this group theoretical facts, and which we will prove in the following.

By the given conditions, L/F is a finite Galois extension of the local field F and $G = \mathrm{Gal(L/F)}$, and $H = \mathrm{Gal}(L/E)$, $Z = \mathrm{Gal}(L/K)$ and $\{1\} = \mathrm{Gal}(L/L)$. Then by class field theory, equation (5.1.3), and the condition $X := \chi_K \circ [-,-]$, χ_ρ identifies with a character

$$\chi_K : K^\times/I_FK^\times \to \mathbb{C}^\times.$$

Moreover, for the Heisenberg representations we also have the following commutative diagram

$$\begin{array}{ccc} K_E^\times/I_EK^\times & \xrightarrow{\textit{inclusion}} & K_F^\times/I_FK^\times \\ \uparrow c & & \uparrow c \\ E^\times/\mathcal{N}_{K/E} \wedge E^\times/\mathcal{N}_{K/E} & \xrightarrow{N_{E/F}\wedge N_{E/F}} & F^\times/\mathcal{N}_{K/F} \wedge F^\times/\mathcal{N}_{K/F} \end{array} \tag{5.1.11}$$

where $N_{E/F}\wedge N_{E/F}(a\wedge b) = N_{E/F}(a)\wedge N_{E/F}(b)$ for all $a, b \in E^\times$, and the vertical isomorphisms in upward direction are given as the commutator maps (cf. equation (5.1.3)) in the Weil groups $W_{K/E}/I_EK^\times$ and $W_{K/F}/I_FK^\times$ respectively. Under the right vertical χ_K corresponds (cf. Theorem 5.1.1(3)) to the alternating character X which is trivial on $N_{E/F}\wedge N_{E/F}$, because H corresponding to $E^\times$ is isotropic. The commutative diagram now shows that χ_K must be trivial on the image of the upper horizontal, i.e., χ_K is trivial on the subgroups $K_E^\times$ for all maximal isotropic E. Hence χ_K is actually a character of $K^\times/K_E^\times$.

Then from Lemma 5.1.2 we can say that there exists a character $\chi_E : E^\times/I_F\mathcal{N}_{K/E} \to \mathbb{C}^\times$ such that $\chi_K = \chi_E \circ N_{K/E}$. And this χ_E is determined by the character χ_H. Since χ_E is trivial on $I_F\mathcal{N}_{K/E}$, for $\sigma \in G/H = \mathrm{Gal}(E/F)$ we have $\chi_E^\sigma \circ N_{K/E} = \chi_E \circ N_{K/E} = \chi_K$ because $\chi_E^{\sigma-1} \circ N_{K/E} \equiv 1$.

Therefore instead of $\rho = \mathrm{Ind}_H^G(\chi_H^g)$ for all $g \in G/H$, we obtain

$$\rho = \mathrm{Ind_{E/F}}(\chi_{\mathrm{E}}^\sigma),\ \text{for all } \sigma \in \mathrm{Gal(E/F)},$$

independently of the choice of χ_E.

□

Remark 5.1.5. Moreover, we have the exact sequence

$$K^\times/I_FK^\times \xrightarrow{N_{K/E}} E^\times/I_F\mathcal{N}_{K/E} \xrightarrow{N_{E/F}} F^\times/\mathcal{N}_{K/F}, \tag{5.1.12}$$

which is only exact in the middle term. For the dual groups this gives

$$\widehat{K^\times/I_FK^\times} \xleftarrow{N^*_{K/E}} \widehat{E^\times/I_F\mathcal{N}_{K/E}} \xleftarrow{N^*_{E/F}} \widehat{F^\times/\mathcal{N}_{K/F}}. \tag{5.1.13}$$

But $N^*_{K/E}(\chi_E^{\sigma-1}) = \chi_E^{\sigma-1} \circ N_{K/E} \equiv 1$, and therefore the exactness of sequence (5.1.13) yields

$$\chi_E^{\sigma-1} = \chi_F \circ N_{E/F}, \quad \text{for some } \chi_F \in \widehat{F^\times/\mathcal{N}_{K/F}}, \tag{5.1.14}$$

For our (arithmetic) determinant computation of Heisenberg representation ρ of G_F, we need the following lemma regarding transfer map.

Lemma 5.1.6. *Let $\rho = \rho(Z, \chi_\rho)$ be a Heisenberg representation of a group G and assume that $H/Z \subset G/Z$ is a maximal isotropic for ρ. Then transfer map $T_{(G/Z)/(H/Z)} \equiv 1$ is the trivial map.*

Proof. In general, if H is a central subgroup[1] of finite index $n = [G : H]$ of a group G, then by Theorem 5.6 on p. 154 of [22] we have $T_{G/H}(g) = g^n$. If G is abelian, then center $Z(G) = G$. Hence every subgroup of G is central subgroup. Now if we take G as an abelian group and H is a subgroup of finite index, then we can write $T_{G/H}(g) = g^{[G:H]}$.

Now we come to the Heisenberg setting. We know that G/Z is abelian, hence $H/Z \subset G/Z$ is a central subgroup. Then we have $T_{(G/Z)/(H/Z)}(g) = g^{[G/Z:H/Z]} = g^d$, where d is the dimension of ρ. For the Heisenberg setting, we also know (cf. Lemma 4.1.3) that $G^d \subseteq Z$, hence $g^d \in Z$. This implies

$$T_{(G/Z)/(H/Z)}(g) = g^d = 1, \quad \text{the identity in } H/Z,$$

for all $g \in G$, hence $T_{(G/Z)/(H/Z)} \equiv 1$ is a trivial map. □

[1] A subgroup of a group which lies inside the center of the group, i.e., a subgroup H of G is central if $H \subseteq Z(G)$.

By using the above Lemma 5.1.2 and Lemma 5.1.6, in the following, we give the arithmetic description of Proposition 4.2.2.

Proposition 5.1.7. *Let $\rho = \rho(Z, \chi_\rho) = \rho(G_K, \chi_K)$ be a Heisenberg representation of the absolute Galois group G_F. Let E be a base field of a maximal isotropic for ρ. Then $F^\times \subseteq \mathcal{N}_{K/E}$, and*

$$\det(\rho)(x) = \Delta_{E/F}(x) \cdot \chi_K \circ N_{K/E}^{-1}(x) \quad \textit{for all } x \in F^\times, \tag{5.1.15}$$

where, for all $x \in F^\times$,

$$\Delta_{E/F}(x) = \begin{cases} 1 & \textit{when } \mathrm{rk}_2(\mathrm{Gal(E/F)}) \neq 1 \\ \omega_{E'/F}(x) & \textit{when } \mathrm{rk}_2(\mathrm{Gal(E/F)}) = 1, \end{cases} \tag{5.1.16}$$

where E'/F is a uniquely determined quadratic subextension in E/F, and $\omega_{E'/F}$ is the character of $F^\times$ which corresponds to E'/F by class field theory.

Proof. From the given condition, we can write $G/Z = \mathrm{Gal(K/F)} \supset \mathrm{H/Z} = \mathrm{Gal(K/E)}$. Here both G/Z and H/Z are abelian, then from class field theory we have the following commutative diagram

$$\begin{array}{ccc} F^\times/\mathcal{N}_{K/F} & \xrightarrow{\textit{inclusion}} & E^\times/\mathcal{N}_{K/E} \\ \downarrow{\scriptstyle \theta_{K/F}} & & \downarrow{\scriptstyle \theta_{K/E}} \\ \mathrm{Gal(K/F)} & \xrightarrow{T_{(G/Z)/(H/Z)}} & \mathrm{Gal(K/E)} \end{array} \tag{5.1.17}$$

Here $\theta_{K/F}$, $\theta_{K/F}$ are the isomorphism (Artin reciprocity) maps and $T_{(G/Z)/(H/Z)}$ is transfer map. From Lemma 5.1.6, we have $T_{(G/Z)/(H/Z)} \equiv 1$. Therefore from the above diagram (5.1.17) we can say $F^\times \subseteq \mathcal{N}_{K/E}$, i.e., all elements [2] from the base field F are norms with respect to the extension K/E.

Now identify $\chi_\rho = \chi_K : K^\times/I_F K^\times \to \mathbb{C}^\times$. Then the map

$$x \in F^\times \mapsto \chi_K \circ N_{K/E}^{-1}(x)$$

is well-defined character of $F^\times$.

Now by Gallagher's Theorem 2.8.2 (arithmetic side) (cf. equation (3.2.4)) we can write for all $x \in F^\times$,

$$\det(\rho)(x) = \Delta_{E/F}(x) \cdot \chi_E(x) = \Delta_{E/F}(x) \cdot \chi_K(N_{K/E}^{-1}(x)), \tag{5.1.18}$$

since $F^\times \subseteq \mathcal{N}_{K/E}$, and $\chi_E|_{\mathcal{N}_{K/E}} = \chi_K \circ N_{K/E}^{-1}$.

Furthermore, since E/F is an abelian extension, $\mathrm{Gal(E/F)} \cong \widehat{\mathrm{Gal(E/F)}}$, and from Miller's Theorem 2.7.1, we can write (cf. equation (3.3.7))

$$\Delta_{E/F} = \begin{cases} 1 & \text{when } \mathrm{rk}_2(\mathrm{Gal(E/F)}) \neq 1 \\ \omega_{E'/F}(x) & \text{when } \mathrm{rk}_2(\mathrm{Gal(E/F)}) = 1, \end{cases}$$

[2]This condition $F^\times \subseteq \mathcal{N}_{K/E}$ implies that for every $x \in F^\times$ must have a preimage under the $N_{K/E}$, but the preimage is not unique.

where E'/F is a uniquely determined quadratic subextension in E/F, and $\omega_{E'/F}$ is the character of $F^\times$ which corresponds to E'/F by class field theory.

□

5.1.1 Heisenberg representations of G_F of dimensions prime to p

Let $F/\mathbb{Q}_p$ be a non-archimedean local field, and G_F be the absolute Galois group of F. In this subsection we construct all Heisenberg representations of G_F of dimensions prime to p. Studying the construction of this type (i.e., dimension prime to p) Heisenberg representations are important for our next section.

Definition 5.1.8 (U-isotropic). Let F be a non-archimedean local field. Let $X : FF^\times \to \mathbb{C}^\times$ be an alternating character with the property

$$X(\varepsilon_1, \varepsilon_2) = 1, \qquad \text{for all } \varepsilon_1, \varepsilon_2 \in U_F.$$

In other words, X is a character of $FF^\times/U_F \wedge U_F$. Then X is said to be the U-isotropic. These X are easy to classify:

Lemma 5.1.9. *Fix a uniformizer π_F and write $U := U_F$. Then we obtain an isomorphism*

$$\widehat{U} \cong \widehat{FF^\times/U \wedge U}, \quad \eta \mapsto X_\eta, \quad \eta_X \leftarrow X$$

between characters of U and U-isotropic alternating characters as follows:

$$X_\eta(\pi_F^a\varepsilon_1, \pi_F^b\varepsilon_2) := \eta(\varepsilon_1)^b \cdot \eta(\varepsilon_2)^{-a}, \quad \eta_X(\varepsilon) := X(\varepsilon, \pi_F), \tag{5.1.19}$$

where $a, b \in \mathbb{Z}$, $\varepsilon, \varepsilon_1, \varepsilon_2 \in U$, and $\eta : U \to \mathbb{C}^\times$. Then

$$\mathrm{Rad(X_\eta)} =< \mathrm{\pi_F^{\#\eta}} > \times \mathrm{Ker}(\eta) =< (\mathrm{\pi_F}\varepsilon)^{\#\eta} > \times \mathrm{Ker}(\eta),$$

does not depend on the choice of π_F, where $\#\eta$ is the order of the character η, hence

$$F^\times/\mathrm{Rad(X_\eta)} \cong < \mathrm{\pi_F} > / < \mathrm{\pi_F^{\#\eta}} > \times \mathrm{U/Ker}(\eta) \cong \mathbb{Z}_{\#\eta} \times \mathbb{Z}_{\#\eta}.$$

Therefore all Heisenberg representations of type $\rho = \rho(X_\eta, \chi)$ have dimension $\dim(\rho) = \#\eta$.

Proof. To prove $\widehat{U} \cong \widehat{FF^\times/U \wedge U}$, we have to show that $\eta_{X_\eta} = \eta$ and $X_{\eta_X} = X_\eta$, and that the inverse map $X \mapsto \eta_X$ does not depend on the choice of π_F.

From the above definition of η_X, we can write:

$$\eta_{X_\eta}(\varepsilon) = X_\eta(\epsilon, \pi_F) = \eta(\varepsilon)^1 \cdot \eta(1)^0 = \eta(\varepsilon),$$

for all $\varepsilon \in U$, hence $\eta_{X_\eta} = \eta$.

Similarly, from the above definition of X, we have:

$$\begin{aligned} X_{\eta_X}(\pi_F^a\varepsilon_1, \pi_F^b\varepsilon) &= \eta_X(\varepsilon_1)^b \cdot \eta_X(\varepsilon_2)^{-a} = X(\varepsilon_1, \pi_F)^b \cdot X(\varepsilon_2, \pi_F)^{-a} \\ &= X(\varepsilon_1, \pi_F)^b \cdot X(\pi_F, \varepsilon_2)^a = X(\varepsilon_1, \pi_F^b) \cdot X(\pi_F^a, \varepsilon_2) \\ &= X(\pi_F^a\varepsilon_1, \pi_F^b\varepsilon). \end{aligned}$$

This shows that $X_{\eta_X} = X$.

Now we choose a uniformizer $\pi_F\varepsilon$, where $\varepsilon \in U$, instead of choosing π_F. Then we can write

$$\begin{aligned} X_\eta((\pi_F\varepsilon)^a\varepsilon_1, (\pi_F\varepsilon)^b\varepsilon_2) &= X_\eta(\pi_F^a(\varepsilon^a\varepsilon_1), \pi_F^b(\varepsilon^b\varepsilon_2)) \\ &= \eta(\varepsilon^a\varepsilon_1)^b \cdot \eta(\varepsilon^b\varepsilon_2)^{-a} \\ &= \eta(\varepsilon_1)^b \cdot \eta(\varepsilon_2)^{-a} \cdot \eta(\varepsilon^{ab-ab}) \\ &= \eta(\varepsilon_1)^b \cdot \eta(\varepsilon_2)^{-a} = X(\pi_F^a\varepsilon_1, \pi_F^b\varepsilon_2). \end{aligned}$$

This shows that X_η does not depend on the choice of the uniformizer π_F. Similarly since $\eta_X(\varepsilon) := X(\varepsilon, \pi_F)$, it is clear that η_X is also does not depend on the choice of the uniformizer π_F.

By the definition of the radical of X_η, we have:

$$\mathrm{Rad(X_\eta)} = \{\pi_\mathrm{F}^\mathrm{a}\varepsilon \in \mathrm{F}^\times \mid \mathrm{X}_\eta(\pi_\mathrm{F}^\mathrm{a}\varepsilon, \pi_\mathrm{F}^\mathrm{b}\varepsilon') = \eta(\varepsilon)^\mathrm{b} \cdot \eta(\varepsilon')^{-\mathrm{a}} = 1\},$$

for all $b \in \mathbb{Z}$, and $\varepsilon' \in U$.

Now if we fix a uniformizer $\pi_F\varepsilon''$, where $\varepsilon'' \in U$ instead of π_F, we can write:

$$\mathrm{Rad(X_\eta)} = \{(\pi_\mathrm{F}\varepsilon'')^\mathrm{a}\varepsilon \in \mathrm{F}^\times \mid \mathrm{X}_\eta((\pi_\mathrm{F}\varepsilon'')^\mathrm{a}\varepsilon, (\pi_\mathrm{F}\varepsilon'')^\mathrm{b}\varepsilon') = \eta(\varepsilon''^\mathrm{a}\varepsilon)^\mathrm{b}\cdot\eta(\varepsilon''^\mathrm{b}\varepsilon')^{-\mathrm{a}} = \eta(\varepsilon)^\mathrm{b}\cdot\eta(\varepsilon')^{-\mathrm{a}} = 1\},$$

This gives $\mathrm{Rad(X_\eta)} =< \pi_\mathrm{F}^{\#\eta} > \times\mathrm{Ker}(\eta) =< (\pi_\mathrm{F}\varepsilon)^{\#\eta} > \times\mathrm{Ker}(\eta)$, hence

$$F^\times/\mathrm{Rad(X_\eta)} \cong < \pi_\mathrm{F} > / < \pi_\mathrm{F}^{\#\eta} > \times\mathrm{U}/\mathrm{Ker}(\eta) \cong \mathbb{Z}_{\#\eta} \times \mathbb{Z}_{\#\eta}.$$

Then all Heisenberg representations of type $\rho = \rho(X_\eta, \chi)$ have dimension

$$\dim(\rho) = \sqrt{[\mathrm{F}^\times : \mathrm{Rad(X_\eta)}]} = \#\eta.$$

□

From the above Lemma 5.1.9 we know that the dimension of a U-isotropic Heisenberg representation $\rho = \rho(X_\eta, \chi)$ of G_F is $\dim(\rho) = \#\eta$, and $F^\times/\mathrm{Rad(X_\eta)} \cong \mathbb{Z}_{\#\eta} \times \mathbb{Z}_{\#\eta}$, a direct product of two cyclic (bicyclic) groups of the same order $\#\eta$. In general, if $A = \mathbb{Z}_m \times \mathbb{Z}_m$ is a bicyclic group of order m^2, then by the following lemma we can compute total number of elements of order m in A, and number of cyclic complementary subgroup of a fixed cyclic subgroup of order m.

Lemma 5.1.10. *Let $A \cong \mathbb{Z}_m \times \mathbb{Z}_m$ be a bicyclic abelian group of order m^2. Then:*

1. *Then number $\psi(m)$ of cyclic subgroups $B \subset A$ of order m is a multiplicative arithmetic function (i.e., $\psi(mn) = \psi(m)\psi(n)$ if $gcd(m,n) = 1$).*

2. *Explicitly we have*

$$\psi(m) = m \cdot \prod_{p|m}(1 + \frac{1}{p}). \tag{5.1.20}$$

And the number of elements of order m in A is:

$$\varphi(m)\cdot\psi(m) = m^2\cdot\prod_{p|m}(1-\frac{1}{p^2}). \tag{5.1.21}$$

Here p is a prime divisor of m and $\varphi(n)$ is the Euler's totient function of n.

3. *Let $B\subset A$ be cyclic of order m. Then B has always a complementary subgroup $B'\subset A$ such that $A = B\times B'$, and B' is again cyclic of order m. And for B fixed, the number of all different complementary subgroups B' is $= m$.*

Proof. To prove these assertions we need to recall the fact: If G is a finite cyclic group of order m, then number of generators of G is $\varphi(m) = m\prod_{p|m}(1-\frac{1}{p})$.
(1). By the given condition $A\cong\mathbb{Z}_m\times\mathbb{Z}_m$ and $\psi(m)$ is the number of cyclic subgroup of A of order m. Then it is clear that ψ is an arithmetic function with $\psi(1) = 1\neq 0$, hence ψ is not **additive**. Now take $m\geqslant 2$, and the prime factorization of m is: $m = \prod_{i=1}^{k}p_i^{a_i}$. To prove this, first we should start with $m = p^n$, hence $A\cong\mathbb{Z}_{p^n}\times\mathbb{Z}_{p^n}$. Then number of subgroup of A of order p^n is:

$$\psi(p^n) = \frac{2\varphi(p^n)p^n-\varphi(p^n)^2}{\varphi(p^n)} = 2p^n-\varphi(p^n) = p^n(2-1+\frac{1}{p}) = p^n(1+\frac{1}{p}).$$

Now take $m = p^nq^r$, where p, q are both prime with $gcd(p,q) = 1$. We also know that $\mathbb{Z}_{p^nq^r}\times\mathbb{Z}_{p^nq^r}\cong\mathbb{Z}_{p^n}\times\mathbb{Z}_{p^n}\times\mathbb{Z}_{q^r}\times\mathbb{Z}_{q^r}$. This gives $\psi(p^nq^r) = \psi(p^n)\cdot\psi(q^r)$. By the similar method we can show that $\psi(m) = \prod_{i=i}^{k}\psi(p_i^{a_i})$, where $m = \prod_{i=1}^{k}p_i^{a_i}$. This condition implies that ψ is a multiplicative arithmetic function.

(2). Since ψ is multiplicative arithmetic function, we have

$$\begin{aligned}\psi(m) &= \prod_{i=1}^{k}\psi(p_i^{a_i}) = \prod_{i=1}^{k}p_i^{a_i}(1+\frac{1}{p_i}) \quad \text{since } \psi(p^n) = p^n(1+\frac{1}{p}),\\ &= p_1^{a_1}\cdots p_k^{a_k}\prod_{i=1}^{k}(1+\frac{1}{p_i}) = m\cdot\prod_{p|m}(1+\frac{1}{p}).\end{aligned}$$

We also know that number of generator of a finite cyclic group of order m is $\varphi(m)$, hence number of elements of order m is $\varphi(m)$. Then the number of elements of order m in A is:

$$\varphi(m)\cdot\psi(m) = m\cdot\prod_{p|m}(1-\frac{1}{p})\cdot m\prod_{p|m}(1+\frac{1}{p}) = m^2\cdot\prod_{p|m}(1-\frac{1}{p^2}).$$

(3). Let $B\subset A$ be a cyclic subgroup of order m. Since A is abelian and bicyclic of order m^2, B has always a complementary subgroup $B'\subset A$ such that $A = B\times B'$, and B' is again cyclic (because A is cyclic, hence A/B and $|A/B| = m$) of order m.

To prove the last part of (3), we start with $m = p^n$. Here B is a cyclic subgroup of A of order p^n, hence $B = <(a,e)>$, where $\#a = p^n$, and e is the identity of B'. Since B has complementary cyclic subgroup, namely B', of order p^n. we can choose $B' =<(b,c)>$, where $B \cap B' = (e,e)$. This gives that c is a generator of B', and b could be any element in $\mathbb{Z}_{p^n}$. Thus total number $\psi_{B'}(p^n)$ of all different complementary subgroups B' is:

$$\psi_{B'}(p^n) = \frac{p^n \varphi(p^n)}{\varphi(p^n)} = p^n = m.$$

Now if we take $m = p^n q^r$, where q is a different prime from p. Then by same method we can see that $\psi_{B'}(p^n q^r) = \psi_{B'}(p^n) \cdot \psi_{B'}(q^r) = p^n q^r = m$. Thus for arbitrary m we can conclude that $\psi_{B'}(m) = m$.

□

In the following lemma, we give an equivalent condition for U-isotropic Heisenberg representation.

Lemma 5.1.11. *Let G_F be the absolute Galois group of a non-archimedean local field F. For a Heisenberg representation $\rho = \rho(Z, \chi_\rho) = \rho(X, \chi_K)$ the following are equivalent:*

1. *The alternating character X is U-isotropic.*
2. *Let E/F be the maximal unramified subextension in K/F. Then* $\mathrm{Gal(K/E)}$ *is maximal isotropic for X.*
3. *$\rho = \mathrm{Ind_{E/F}(\chi_E)}$ can be induced from a character χ_E of $E^\times$ (where E is as in (2)).*

Proof. This proof follows from the above Lemma 5.1.9.
First, assume that X is U-isotropic, i.e., $X \in \widehat{FF^\times/U} \wedge U$. We also know that $\widehat{U} \cong \widehat{FF^\times/U} \wedge U$. Then X corresponds a character of U, namely $X \mapsto \eta_X$. Then from Lemma 5.1.9 we have $F^\times/\mathrm{Rad(X)} \cong \mathbb{Z}_{\#\eta_\mathrm{X}} \times \mathbb{Z}_{\#\eta_\mathrm{X}}$, i.e., product of two cyclic groups of same order.

Since K/F is the abelian bicyclic extension which corresponds to Rad(X), we can write:

$$\mathcal{N}_{K/F} = \mathrm{Rad(X)}, \qquad \mathrm{Gal(K/F)} \cong \mathrm{F^\times/Rad(X)}.$$

Let E/F be the maximal unramified subextension in K/F. Then $[E:F] = \#\eta_K$ because the order of maximal cyclic subgroup of Gal(K/F) is $\#\eta_X$. Then $f_{E/F} = \#\eta_X$, hence $f_{K/F} = e_{K/F} = \#\eta_X$ because $f_{K/F} \cdot e_{K/F} = [K:F] = \#\eta_X^2$ and Gal(K/F) is not cyclic group.

Now we have to prove that the extension E/F corresponds to a maximal isotropic for X. Let H/Z be a maximal isotropic for X, hence $[G_F/Z : H/Z] = \#\eta_X$, hence $H/Z = \mathrm{Gal(K/E)}$, i.e., the maximal unramified subextension E/F in K/F corresponds to a maximal isotropic subgroup, hence

$$\rho(X, \chi_K) = \mathrm{Ind_{E/F}(\chi_E)}, \text{ for } \chi_E \circ N_{K/E} = \chi_K.$$

Finally, since E/F is unramified and the extension E corresponds a maximal isotropic subgroup for X, we have $U_F \subset \mathcal{N}_{E/F}$, hence $U_F \subset \mathcal{N}_{K/F}$ and $X|_{U\times U} = 1$ because $U_F \subset F^\times \subset \mathcal{N}_{K/E}$. This shows that X is U-isotropic. □

Corollary 5.1.12. *The U-isotropic Heisenberg representation $\rho = \rho(X_\eta, \chi)$ can never be wild because it is induced from an unramified extension E/F, but the dimension $\dim(\rho(\mathrm{X}_\eta, \chi)) = \#\eta$ can be a power of p.*
The representations ρ of dimension prime to p are precisely given as $\rho = \rho(X_\eta, \chi)$ for characters η of U/U^1.

Proof. This is clear from the above lemma 5.1.9 and the fact $|U/U^1| = q_F - 1$. □

Remark 5.1.13 (**Arithmetic description of representations** $\rho(X_\eta, \chi)$**:**). We let $K_\eta|F$ be the abelian bicyclic extension which corresponds to $\mathrm{Rad}(\mathrm{X}_\eta)$:

$$\mathcal{N}_{K_\eta/F} = \mathrm{Rad}(\mathrm{X}_\eta), \qquad \mathrm{Gal}(\mathrm{K}_\eta/\mathrm{F}) \cong \mathrm{F}^\times/\mathrm{Rad}(\mathrm{X}_\eta).$$

Then we have $f_{K_\eta|F} = e_{K_\eta|F} = \#\eta$ and the maximal unramified subextension $E/F \subset K_\eta/F$ corresponds to a maximal isotropic subgroup, hence

$$\rho(X_\eta, \chi) = \mathrm{Ind}_{\mathrm{E/F}}(\chi_{\mathrm{E}}), \quad \text{for } \chi_{\mathrm{E}} \circ \mathrm{N}_{\mathrm{K}_\eta/\mathrm{E}} = \chi.$$

We recall here that $\chi : K_\eta^\times/I_F K_\eta^\times \to \mathbb{C}^\times$ is a character such that (cf. Theorem 5.1.1(3))

$$\chi|_{(K_\eta^\times)_F} \leftrightarrow X_\eta, \quad \text{with respect to } (K_\eta^\times)_F/I_F K_\eta^\times \cong F^\times/\mathrm{Rad}(\mathrm{X}_\eta) \wedge \mathrm{F}^\times/\mathrm{Rad}(\mathrm{X}_\eta).$$

In particular, we see that $(K_\eta^\times)_F/I_F K_\eta^\times$ is cyclic of order $\#\eta$ and $\chi|_{(K_\eta^\times)_F}$ must be a faithful character of that cyclic group.

In the following lemma we see the explicit description of the representation $\rho = \rho(X_\eta, \chi)$.

Lemma 5.1.14 (Explicit Lemma). *Let $\rho = \rho(X_\eta, \chi_K)$ be a U-isotropic Heisenberg representation of the absolute Galois group G_F of a local field $F/\mathbb{Q}_p$. Let $K = K_\eta$ and let E/F be the maximal unramified subextension in K/F. Then:*

1. *The norm map induces an isomorphism:*

$$N_{K/E} : K_F^\times/I_F K^\times \xrightarrow{\sim} I_F E^\times/I_F \mathcal{N}_{K/E}.$$

2. *Let $c_{K/F} : F^\times/\mathrm{Rad}(\mathrm{X}_\eta) \wedge \mathrm{F}^\times/\mathrm{Rad}(\mathrm{X}_\eta) \cong \mathrm{K}_\mathrm{F}^\times/\mathrm{I}_\mathrm{F}\mathrm{K}^\times$ be the isomorphism which is induced by the commutator in the relative Weil-group $W_{K/F}$. Then for units $\varepsilon \in U_F$ we explicitly have:*

$$c_{K/F}(\varepsilon \wedge \pi_F) = N_{K/E}^{-1}(N_{E/F}^{-1}(\varepsilon)^{1-\varphi_{E/F}}),$$

where $\varphi_{E/F}$ is the Frobenius automorphism for E/F and where N^{-1} means to take a preimage of the norm map.

3. *The restriction $\chi_K|_{K_F^\times}$ is characterized by:*

$$\chi_K \circ c_{K/F}(\varepsilon \wedge \pi_F) = X_\eta(\varepsilon, \pi_F) = \eta(\varepsilon),$$

for all $\varepsilon \in U_F$, where $c_{K/F}(\varepsilon \wedge \pi_F)$ is explicitly given via (2).

Proof. **(1).** By the given conditions we have: $K = K_\eta$, and K/F is the bicyclic extension with $\mathrm{Rad}(\mathrm{X}_\eta) = \mathcal{N}_{\mathrm{K/F}}$, and E/F is the maximal unramified subextension in K/F. So K/E and E/F both are cyclic, hence

$$E_F^\times = I_F E^\times, \qquad K_E^\times = I_E K^\times.$$

From the diagram (3.6.1) on p. 41 of [13], we have

$$N_{K/E} : K_F^\times / I_F K^\times \xrightarrow{\sim} E_F^\times / I_F \mathcal{N}_{K/E}.$$

We also know that $E_F^\times = I_F E^\times$. Thus the norm map $N_{K/E}$ induces an isomorphism:

$$N_{K/E} : K_F^\times / I_F K^\times \cong I_F E^\times / I_F \mathcal{N}_{K/E}.$$

(2). By the given conditions, $c_{K/F}$ is the isomorphism which is induced by the commutator in the relative Weil-group $W_{K/F}$ (cf. the map (5.1.3). Here $\mathrm{Rad}(\mathrm{X}_\eta) = \mathcal{N}_{\mathrm{K/F}} =: \mathrm{N}$. Then from Proposition 1(iii) of [14] on p. 128, we have

$$c_{K/F} : N \wedge F^\times / N \wedge N \xrightarrow{\sim} I_F K^\times / I_F K_F^\times$$

as an isomorphism by the map:

$$c_{K/F}(x \wedge y) = N_{K/F}^{-1}(x)^{1-\phi_F(y)},$$

where $\phi_F(y) \in \mathrm{Gal(K/F)}$ for $y \in F^\times$ by class field theory. If $y = \pi_F$, then by class field theory (cf. [31], p. 20, Theorem 1.1(a)), we can write $\phi_F(\pi_F)|_E = \varphi_{E/F}$, where $\varphi_{E/F}$ is the Frobenius automorphism for E/F.

Now we come to our special case. Since E/F is unramified, we have $U_F \subset \mathcal{N}_{E/F}$, and we obtain (cf. [13], pp. 46-47 of Section 4.4 and the diagram on p. 302 of [11]):

$$N_{K/E} \circ c_{K/F}(\varepsilon \wedge \pi_F) = N_{E/F}^{-1}(\varepsilon)^{1-\varphi_{E/F}}. \tag{5.1.22}$$

We also know (see the first two lines under the upper diagram on p. 302 of [11]) that $E_F^\times \subseteq \mathcal{N}_{K/E}$. Here

$$N_{E/F}^{-1}(\varepsilon)^{1-\varphi_{E/F}} \in I_F E^\times / I_F \mathcal{N}_{K/E} = E_F^\times / I_F \mathcal{N}_{K/E},$$

because E/F is cyclic, hence $E_F^\times = I_F E^\times$. Therefore from equation (5.1.22) we can conclude:

$$c_{K/F}(\varepsilon \wedge \pi_F) = N_{K/E}^{-1}(N_{E/F}^{-1}(\varepsilon)^{1-\varphi_{E/F}}).$$

(3.) We know that the $c_{K/F}(\varepsilon \wedge \pi_F) \in K_F^\times$ and $\chi_K : K^\times / I_F K^\times \to \mathbb{C}^\times$. Then we can write

$$\begin{aligned}
\chi_K \circ c_{K/F}(\varepsilon \wedge \pi_F) &= \chi_K(N_{K/E}^{-1}(N_{E/F}^{-1}(\varepsilon)^{1-\varphi_{E/F}}) \\
&= \chi_E \circ N_{K/E}(N_{K/E}^{-1}(N_{E/F}^{-1}(\varepsilon)^{1-\varphi_{E/F}}), \quad \text{since } \chi_K = \chi_E \circ N_{K/E} \\
&= \chi_E(N_{E/F}^{-1}(\varepsilon)^{1-\varphi_{E/F}}) = X_\eta(\varepsilon, \pi_F) \\
&= \eta(\varepsilon).
\end{aligned}$$

This is true for all $\varepsilon \in U_F$. Therefore we can conclude that $\chi_K|_{K_F^\times} = \eta$. □

Example 5.1.15 (Explicit description of Heisenberg representations of dimension prime to p). Let $F/\mathbb{Q}_p$ be a local field, and G_F be the absolute Galois group of F. Let $\rho = \rho(X, \chi_K)$ be a Heisenberg representation of G_F of dimension m prime to p. Then from Corollary 5.1.12 the alternating character $X = X_\eta$ is U-isotropic for a character $\eta : U_F/U_F^1 \to \mathbb{C}^\times$. Here from Lemma 5.1.9 we can say $m = \sqrt{[F^\times : \mathrm{Rad}(X_\eta)]} = \#\eta$ divides $q_F - 1$.

Since U_F^1 is a pro-p-group and $gcd(m,p) = 1$, we have $(U_F^1)^m = U_F^1 \subset F^{\times m}$, and therefore

$$F^\times/F^{\times m} \cong \mathbb{Z}_m \times \mathbb{Z}_m,$$

is a bicyclic group of order m^2. So by class field theory there is precisely one extension K/F such that $\mathrm{Gal(K/F)} \cong \mathrm{\mathbb{Z}_m \times \mathbb{Z}_m}$ and the norm group $\mathcal{N}_{K/F} := N_{K/F}(K^\times) = F^{\times m}$.

We know that U_F/U_F^1 is a cyclic group of order $q_F - 1$, hence $\widehat{U_F/U_F^1} \cong U_F/U_F^1$. By the given condition $m|(q_F - 1)$, hence U_F/U_F^1 has exactly one subgroup of order m. Then number of elements of order m in U_F/U_F^1 is $\varphi(m)$, the Euler's φ-function of m. In this setting, we have $\eta \in \widehat{U_F/U_F^1} \cong \widehat{FF^\times/U_F^1} \wedge U_F^1$ with $\#\eta = m$. This implies that up to 1-dimensional character twist there are $\varphi(m)$ representations corresponding to X_η where $\eta : U_F/U_F^1 \to \mathbb{C}^\times$ is of order m. According to Corollary 1.2 of [11], all dimension-m-Heisenberg representations of $G_F = \mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F})$ are given as

$$\rho = \rho(X_\eta, \chi_K), \tag{1H}$$

where $\chi_K : K^\times/I_F K^\times \to \mathbb{C}^\times$ is a character such that the restriction of χ_K to the subgroup $K_F^\times$ corresponds to X_η under the map (5.1.3), and

$$F^\times/F^{\times m} \wedge F^\times/F^{\times m} \cong K_F^\times/I_F K^\times, \tag{2H}$$

which is given via the commutator in the relative Weil-group $W_{K/F}$ (for details arithmetic description of Heisenberg representations of a Galois group, see [11], pp. 301-304). The condition (2H) corresponds to (5.1.3). Here the above Explicit Lemma 5.1.14 comes in.

Here due to our assumption both sides of (2H) are groups of order m. And if one choice $\chi_K = \chi_0$ has been fixed, then all other χ_K are given as

$$\chi_K = (\chi_F \circ N_{K/F}) \cdot \chi_0, \tag{5.1.23}$$

for arbitrary characters of $F^\times$. For an optimal choice $\chi_K = \chi_0$, and order of χ_0 we need the following lemma.

Lemma 5.1.16. *Let K/F be the extension of $F/\mathbb{Q}_p$ for which* $\mathrm{Gal(K/F)} = \mathrm{\mathbb{Z}_m \times \mathbb{Z}_m}$. *The $K_F^\times$ and $I_F K^\times$ are as above. Then the sequence*

$$1 \to U_K^1 K_F^\times/U_K^1 I_F K^\times \to U_K/U_K^1 I_F K^\times \xrightarrow{N_{K/F}} U_F/U_F^1 \to U_F/U_F \cap F^{\times m} \to 1 \tag{5.1.24}$$

is exact, and the outer terms are both of order m, hence inner terms are both cyclic of order $q_F - 1$.

Proof. The sequence is exact because $F^{\times m} = N_{K/F}(K^\times)$ is the group of norms, and $F^\times/F^{\times m} \cong \mathbb{Z}_m \times \mathbb{Z}_m$ implies that the right hand term[3] is of order m. By our assumption the order of $K_F^\times/I_F K^\times$ is m. Now we consider the exact sequence

$$1 \to U_K^1 \cap K_F^\times/U_K^1 \cap I_F K^\times \to K_F^\times/I_F K^\times \to U_K^1 K_F^\times/U_K^1 I_F K^\times \to 1. \tag{5.1.25}$$

Since the middle term has order m, the left term must have order 1, because U_K^1 is a pro-p-group and $gcd(m,p) = 1$. Hence the right term is also of order m. So the outer terms of the sequence (5.1.24) have both order m, hence the inner terms must have the same order $q_F - 1 = [U_F : U_F^1]$, and they are cyclic, because the groups U_F/U_F^1 and U_K/U_K^1 are both cyclic. □

We now are in a position to choose $\chi_K = \chi_0$ as follows:

1. we take χ_0 as a character of $K^\times/U_K^1 I_F K^\times$,
2. we take it on $U_K^1 K_F^\times/U_K^1 I_F K^\times$ as it is prescribed by the above Explicit Lemma 5.1.14, in particular, χ_0 restricted to that subgroup (which is cyclic of order m) will be faithful.
3. we take it trivial on all primary components of the cyclic group $U_K/U_K^1 I_F K^\times$ which are not p_i-primary, where $m = \prod_{i=1}^n p_i^{a_i}$.
4. we take it trivial for a fixed prime element π_K.

Under the above optimal choice of χ_0, we have

Lemma 5.1.17. *Denote $\nu_p(n) :=$ as the highest power of p for which $p^{\nu_p(n)}|n$. The character χ_0 must be a character of order*

$$m_{q_F-1} := \prod_{l|m} l^{\nu_l(q_F-1)},$$

which we will call the m-primary part of $q_F - 1$, so it determines a cyclic extension L/K of degree m_{q_F-1} which is totally tamely ramified, and we can consider the Heisenberg representation $\rho = (X, \chi_0)$ of $G_F = \mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F})$ is a representation of $\mathrm{Gal}(\mathrm{L}/\mathrm{F})$, which is of order $m^2 \cdot m_{q_F-1}$.

Proof. By the given conditions, $m|q_F - 1$. Therefore we can write

$$q_F - 1 = \prod_{l|m} l^{\nu_l(q_F-1)} \cdot \prod_{p|q_F-1,\, p\nmid m} p^{\nu_p(q_F-1)} = m_{q_F-1} \cdot \prod_{p|q_F-1,\, p\nmid m} p^{\nu_p(q_F-1)},$$

[3]Since $gcd(m,p) = 1$, we have

$$U_F \cdot F^{\times m} = (<\zeta> \times U_F^1)(<\pi_F^m> \times <\zeta^m> \times U_F^1) = <\pi_F^m> \times <\zeta> \times U_F^1,$$

where ζ is a $(q_F - 1)$-st root of unity. Then

$$U_F/U_F \cap F^{\times m} = U_F \cdot F^{\times m}/F^{\times m} = <\pi_F^m> \times <\zeta> \times U_F^1/<\pi_F^m> \times <\zeta^m> \times U_F^1 \cong \mathbb{Z}_m.$$

Hence $|U_F/U_F \cap F^{\times m}| = m$.

where l, p are prime, and $m_{q_F-1} = \prod_{l|m} l^{\nu_l(q_F-1)}$.

From the construction of χ_0, $\pi_K \in \mathrm{Ker}(\chi_0)$, hence the order of χ_0 comes from the restriction to U_K. Then the order of χ_0 is m_{q_F-1}, because from Lemma 5.1.16, the order of $U_K/U_K^1 I_F K$ is $q_F - 1$. Since order of χ_0 is m_{q_F-1}, by class field theory χ_0 determines a cyclic extension L/K of degree m_{q_F-1}, hence

$$N_{L/K}(L^\times) = \mathrm{Ker}(\chi_0) = \mathrm{Ker}(\rho).$$

This means G_L is the kernel of $\rho(X, \chi_0)$, hence $\rho(X, \chi_0)$ is actually a representation of $G_F/G_L \cong \mathrm{Gal(L/F)}$.

Since G_L is normal subgroup of G_F, hence L/F is a normal extension of degree $[L : F] = [L : K] \cdot [K : F] = m_{q_F-1} \cdot m^2$. Thus Gal(L/F) is of order $m^2 \cdot m_{q_F-1}$.

Moreover, since $[L : K] = m_{q_F-1}$ and $gcd(m, p) = 1$, L/K is tame. By construction we have a prime $\pi_K \in \mathrm{Ker}(\chi_0) = \mathrm{N_{L/K}(L^\times)}$, hence L/K is totally ramified extension.

□

Lemma 5.1.18. *(Here L, K, and F are the same as in Lemma 5.1.17) Let F^{ab}/F be the maximal abelian extension. Then we have*

$$L \supset L \cap F^{ab} \supset K \supset F, \quad \{1\} \subset G' \subset Z(G) \subset G = \mathrm{Gal(L/F)},$$

where $[L : L \cap F^{ab}] = |G| = m$ and $[L : K] = |Z(G)| = m_{q_F-1}$.

Proof. Let F^{ab}/F be the maximal abelian extension. Then we have

$$L \supset L \cap F^{ab} \supset K \supset F.$$

Here $L \cap F^{ab}/F$ is the maximal abelian in L/F. Then from Galois theory we can conclude

$$\mathrm{Gal(L/L \cap F^{ab}) = [Gal(L/F), Gal(L/F)] =: G'}.$$

Since $\mathrm{Gal(L/F) = G_F/Ker(\rho)}$, and $[[G_F, G_F], G_F] \subseteq \mathrm{Ker}(\rho)$, from relation (5.1.3) we have

$$G' = [G_F, G_F]/\mathrm{Ker}(\rho) \cap \mathrm{[G_F, G_F] = [G_F, G_F]/[[G_F, G_F], G_F] \cong K_F^\times/I_F K^\times}.$$

Again from sequence 5.1.25 we have $|U_K^1 K_F^\times / U_K^1 I_F K^\times| = |K_F^\times / I_F K^\times| = m$. Hence $|G'| = m$.

From the Heisenberg property of ρ, we have $[[G_F, G_F], G_F] \subseteq \mathrm{Ker}(\rho)$, hence $\mathrm{Gal(L/F)} = \mathrm{G_F/Ker}(\rho)$ is a two-step nilpotent group. This gives $[G', G] = 1$, hence $G' \subseteq Z := Z(G)$. Thus G/Z is abelian.

Moreover, here Z is the scalar group of ρ, hence the dimension of ρ is:

$$\dim(\rho) = \sqrt{\mathrm{[G : Z]}} = \mathrm{m}$$

Therefore the order of Z is m_{q_F-1} and $Z = \mathrm{Gal(L/K)}$.

□

Remark 5.1.19 (**Special case:** $m = 2$**, hence** $p \neq 2$). Now if we take $m = 2$, hence $p \neq 2$, and choose χ_0 as the above optimal choice, then we will have $m_{q_F-1} = 2_{q_F-1} =$ 2-primary factor of the number $q_F - 1$, and Gal(L/F) is a 2-group of order $4 \cdot 2_{q_F-1}$.

When $q_F \equiv -1 \pmod 4$, q_F is of the form $q_F = 4l - 1$, where $l \geqslant 1$. So we can write $q_F - 1 = 2(2l - 1)$. Since $2l - 1$ is always odd, therefore when $q_F \equiv -1 \pmod 4$, the order of χ_0 is $2_{q_F-1} = 2$. Then Gal(L/F) will be of order 8 if and only if $q_F \equiv -1 \pmod 4$, i.e., if and only if $i \notin F$. And if $q_F \equiv 1 \pmod 4$, then similarly, we can write $q_F - 1 = 4m$ for some integer $m \geqslant 1$, hence $2_{q_F-1} \geqslant 4$. Therefore when $q_F \equiv 1 \pmod 4$, the order of Gal(L/F) will be at least 16.

5.1.2 Artin conductors, Swan conductors, and the dimensions of Heisenberg representations

Definition 5.1.20 (Artin and Swan conductor). Let G be a finite group and $R(G)$ be the complex representation ring of G. For any two representations $\rho_1, \rho_2 \in R(G)$ with characters χ_1, χ_2 respectively, we have the Schur's inner product:

$$< \rho_1, \rho_2 >_G = < \chi_1, \chi_2 >_G := \frac{1}{|G|} \sum_{g \in G} \chi_1(g) \cdot \overline{\chi_2(g)}.$$

Let K/F be a finite Galois group with Galois group $G := \mathrm{Gal(K/F)}$. For an element $g \in G$ different from identity 1, we define the positive integer (cf. [27], Chapter IV, p. 62)

$$i_G(g) := \inf\{\nu_{\mathrm{K}}(\mathrm{x} - \mathrm{g}(\mathrm{x}))|\ \mathrm{x} \in \mathrm{O_K}\}.$$

By using this non-negative (when $g \neq 1$) integer $i_G(g)$ we define a function $a_G : G \to \mathbb{Z}$ as follows:

$$a_G(g) = -f_{K/F} \cdot i_G(g) \text{ when } g \neq 1, \text{ and } a_G(1) = f_{K/F} \textstyle\sum_{g \neq 1} i_G(g).$$

Thus from this definition we can see that $\sum_{g \in G} a_G(g) = 0$, hence $< a_G, 1_G >= 0$. It can be proved (cf. [27], p. 99, Theorem 1) that the function a_G is the character of a linear representation of G, and that corresponding linear representation is called the **Artin representation** A_G of G.

Similarly, for a nontrivial $g \neq 1 \in G$, we define (cf. [48], p. 247)

$$s_G(g) = \inf\{\nu_{\mathrm{K}}(1 - \mathrm{g}(\mathrm{x})\mathrm{x}^{-1})|\ \mathrm{x} \in \mathrm{K}^\times\}, \qquad \mathrm{s_G}(1) = -\sum_{\mathrm{g} \neq 1} \mathrm{s_G}(\mathrm{g}).$$

And we can define a function $\mathrm{sw_G} : \mathrm{G} \to \mathbb{Z}$ as follows:

$$\mathrm{sw_G}(\mathrm{g}) = -\mathrm{f_{K/F}} \cdot \mathrm{s_G}(\mathrm{g})$$

It can also be shown that $\mathrm{sw_G}$ is a character of a linear representation of G, and that corresponding representation is called the **Swan representation** SW_G of G.

From [28], p. 160 , we have the relation between the Artin and Swan representations (cf. [48], p. 248, equation (6.1.9))

$$SW_G = A_G + \mathrm{Ind}_{\mathrm{G_0}}^{\mathrm{G}}(1) - \mathrm{Ind}_{\{1\}}^{\mathrm{G}}(1), \tag{5.1.26}$$

G_0 is the 0-th ramification group (i.e., inertia group) of G.

Now we are in a position to define the Artin and Swan conductor of a representation $\rho \in R(G)$. The Artin conductor of a representation $\rho \in R(G)$ is defined by

$$a_F(\rho) := < A_G, \rho >_G = < a_G, \chi >_G,$$

where χ is the character of the representation ρ. Similarly, for the representation ρ, the Swan conductor is:

$$\mathrm{sw_F}(\rho) := < \mathrm{SW_G}, \rho >_{\mathrm{G}} = < \mathrm{sw_G}, \chi >_{\mathrm{G}} .$$

For more details about Artin and Swan conductor, see Chapter 6 of [48] and Chapter VI of [27].

From equation (5.1.26) we obtain

$$a_F(\rho) = \mathrm{sw_F}(\rho) + \dim(\rho) - < 1, \rho >_{\mathrm{G_0}} . \tag{5.1.27}$$

Moreover, from Corollary of Proposition 4 on p. 101 of [27], for an induced representation $\rho := \mathrm{Ind}_{\mathrm{Gal(K/E)}}^{\mathrm{Gal(K/F)}}(\rho_{\mathrm{E}}) = \mathrm{Ind_{E/F}}(\rho_{\mathrm{E}})$, we have

$$a_F(\rho) = f_{E/F} \cdot \left(d_{E/F} \cdot \dim(\rho_{\mathrm{E}}) + \mathrm{a_E}(\rho_{\mathrm{E}})\right) . \tag{5.1.28}$$

We apply this formula (5.1.28) for $\rho_E = \chi_E$ of dimension 1 and then conversely

$$a(\chi_E) = \frac{a_F(\rho)}{f_{E/F}} - d_{E/F}.$$

So if we know $a_F(\rho)$ then we can compute $a(\chi_E)$.

Let $\{G^i\}$, where $i \geqslant 0, \in \mathbb{Q}$ be the ramification subgroups (in the upper numbering) of a local Galois group G. Now let ρ be an irreducible representation of G. For this irreducible ρ we define

$$j(\rho) := \max\{\mathrm{i} \mid \rho|_{\mathrm{G^i}} \not\equiv 1\}.$$

Now if ρ is an irreducible representation of G, then $\rho|_I \not\equiv 1$, where $I = G^0 = G_0$ is the inertia subgroup of G. Thus from the definition of $j(\rho)$ we can say, if ρ is irreducible, then we always have $j(\rho) \geqslant 0$, i.e., ρ is nontrivial on the inertia group G_0. Then from the definitions of Swan and Artin conductors, and equation (5.1.27), when ρ is irreducible, we have the following relations

$$\mathrm{sw_F}(\rho) = \dim(\rho) \cdot \mathrm{j}(\rho), \qquad \mathrm{a_F}(\rho) = \dim(\rho) \cdot (\mathrm{j}(\rho) + 1). \tag{5.1.29}$$

From the Theorem of Hasse-Arf (cf. [27], p. 76), if $\dim(\rho) = 1$, i.e., ρ is a character of $G/[G,G]$, we can say that $j(\rho)$ must be an integer, then $\mathrm{sw_F}(\rho) = \mathrm{j}(\rho), \mathrm{a_F}(\rho) = \mathrm{j}(\rho) + 1.$

Moreover, by class field theory, ρ corresponds to a linear character χ_F, hence for linear character χ_F, we can write

$$j(\chi_F) := \max\{\mathrm{i} \mid \chi_{\mathrm{F}}|_{\mathrm{U_F^i}} \not\equiv 1\},$$

because under class field theory (under Artin isomorphism) the upper numbering in the filtration of $\mathrm{Gal(F_{ab}/F)}$ is compatible with the filtration (descending chain) of the group of units U_F.

From equation (5.1.29), it is easy to see that for higher dimensional ρ, we have $\mathrm{sw_F}(\rho), \mathrm{a_F}(\rho)$ multiples of $\dim(\rho)$ if and only if $j(\rho)$ is an integer.

Now we come to our Heisenberg representations. For each $X \in \widehat{FF^\times}$ we define

$$j(X) := \begin{cases} 0 & \text{when } X \text{ is trivial} \\ \max\{\mathrm{i} \mid \mathrm{X}|_{\mathrm{UU^i}} \not\equiv 1\} & \text{when } X \text{ is nontrivial,} \end{cases} \tag{5.1.30}$$

where $UU^i \subseteq FF^\times$ is a subgroup which under (5.1.2) corresponds

$$G_F^i \cap [G_F, G_F]/G_F^i \cap [[G_F, G_F], G_F] \subseteq [G_F, G_F]/[[G_F, G_F], G_F].$$

Let $\rho = \rho(X_\rho, \chi_K)$ be the **minimal conductor** (i.e., a representation with the smallest Artin conductor) Heisenberg representation for X_ρ of the absolute Galois group G_F. From Theorem 3 on p. 125 of [14], we have

$$\mathrm{sw_F}(\rho) = \dim(\rho) \cdot \mathrm{j(X_\rho)} = \sqrt{[\mathrm{F^\times : Rad(X_\rho)}]} \cdot \mathrm{j(X_\rho)}. \tag{5.1.31}$$

Let $\rho_0 = \rho_0(X, \chi_0)$ be a minimal representation corresponding X, then all other Heisenberg representations of dimension $\dim(\rho)$ are of the form $\rho = \chi_F \otimes \rho_0 = (X, (\chi_F \circ N_{K/F})\chi_0)$, where $\chi_F : F^\times \to \mathbb{C}^\times$. Then we have (cf. [11], p. 305, equation (5))

$$\mathrm{sw_F}(\rho) = \mathrm{sw_F}(\chi_\mathrm{F} \otimes \rho_0) = \sqrt{[\mathrm{F^\times : Rad(X)}]} \cdot \max\{\mathrm{j(\chi_F), j(X)}\}. \tag{5.1.32}$$

For minimal conductor U-isotopic Heisenberg representation we have the following proposition.

Proposition 5.1.21. *Let $\rho = \rho(X_\eta, \chi_K)$ be a U-isotropic Heisenberg representation of G_F of minimal conductor. Then we have the following conductor relation*

$$j(X_\eta) = j(\eta),\ \mathrm{sw_F}(\rho) = \dim(\rho) \cdot \mathrm{j(X_\eta)} = \#\eta \cdot \mathrm{j}(\eta),$$
$$a_F(\rho) = \mathrm{sw_F}(\rho) + \dim(\rho) = \#\eta(\mathrm{j}(\eta) + 1) = \#\eta \cdot \mathrm{a_F}(\eta).$$

Proof. From [14], on p. 126, Proposition 4(i) and Proposition 5(ii), and $U \wedge U = U^1 \wedge U^1$, we see the injection $U^i \wedge F^\times \subseteq UU^i$ induces a natural isomorphism

$$U^i \wedge < \pi_F > \cong UU^i/UU^i \cap (U \wedge U)$$

for all $i \geqslant 0$.

Now let $j(X_\eta) = n-1$, hence $X_\eta|_{UU^n} = 1$ but $X_\eta|_{UU^{n-1}} \neq 1$. This gives $X_\eta|_{U^n \wedge <\pi_F>} = 1$ but $X_\eta|_{U^{n-1} \wedge <\pi_F>} \neq 1$. Now from equation (5.1.19) we can conclude that $\eta(x) = 1$ for all $x \in U^n$ but $\eta(x) \neq 1$ for $x \in U^{n-1}$. Hence

$$j(\eta) = n - 1 = j(X_\eta).$$

Again from the definition of $j(\chi)$, where χ is a character of $F^\times$, we can see that $j(\chi) = a(\chi) - 1$, i.e., $a(\chi) = j(\chi) + 1$.

From equation (5.1.31) we obtain:

$$\mathrm{sw_F}(\rho) = \dim(\rho) \cdot \mathrm{j}(\mathrm{X}_\eta) = \#\eta \cdot \mathrm{j}(\eta),$$

since $\dim(\rho) = \#\eta$ and $j(X_\eta) = j(\eta)$. Finally, from equation (5.1.27) for ρ (here $< 1, \rho >_{G_0} = 0$), we have

$$a_F(\rho) = \mathrm{sw_F}(\rho) + \dim(\rho) = \#\eta \cdot \mathrm{j}(\eta) + \#\eta = \#\eta \cdot (\mathrm{j}(\eta) + 1) = \#\eta \cdot \mathrm{a_F}(\eta). \tag{5.1.33}$$

□

By using the equation (5.1.28) in our Heisenberg setting, we have the following proposition.

Proposition 5.1.22. *Let $\rho = \rho(Z, \chi_\rho) = \rho(X, \chi_K)$ be a Heisenberg representation of the absolute Galois group G_F of a field $F/\mathbb{Q}_p$ of dimension m. Let E/F be any subextension in K/F corresponding to a maximal isotropic subgroup for X. Then*

$$a_F(\rho) = a_F(\mathrm{Ind_{E/F}}(\chi_\mathrm{E})), \qquad \mathrm{m} \cdot \mathrm{a_F}(\rho) = \mathrm{a_F}(\mathrm{Ind_{K/F}}(\chi_\mathrm{K})).$$

As a consequence we have

$$a(\chi_K) = e_{K/E} \cdot a(\chi_E) - d_{K/E}.$$

Proof. We know that $\rho = \mathrm{Ind_{E/F}}(\chi_\mathrm{E})$ and $m \cdot \rho = \mathrm{Ind_{K/F}}(\chi_\mathrm{K})$. By the definition of Artin conductor we can write

$$a_F(\dim(\rho) \cdot \rho) = \dim(\rho) \cdot \mathrm{a_F}(\rho) = \mathrm{m} \cdot \mathrm{a_F}(\mathrm{Ind_{E/F}}(\chi_\mathrm{E})).$$

Since $K/E/F$ is a tower of Galois extensions with $[K : E] = m = e_{K/E} f_{K/E}$, we have the transitivity relation of different (cf. [27], p. 51, Proposition 8)

$$\mathcal{D}_{K/F} = \mathcal{D}_{K/E} \cdot \mathcal{D}_{E/F}.$$

Now from the definition of different of a Galois extension, and taking K-valuation we obtain:

$$d_{K/F} = d_{K/E} + e_{K/E} \cdot d_{E/F}. \tag{5.1.34}$$

Now by using equation (5.1.28) we have:

$$m \cdot a_F(\mathrm{Ind_{E/F}}(\chi_\mathrm{E})) = \mathrm{m} \cdot \mathrm{f_{E/F}}\left(\mathrm{d_{E/F}} + \mathrm{a}(\chi_\mathrm{E})\right) = \mathrm{m} \cdot \mathrm{f_{E/F}} \cdot \mathrm{d_{E/F}} + \mathrm{e_{K/E}} \cdot \mathrm{f_{K/F}} \cdot \mathrm{a}(\chi_\mathrm{E}), \tag{5.1.35}$$

and

$$a_F(\mathrm{Ind_{K/F}}(\chi_\mathrm{K})) = \mathrm{f_{K/F}} \cdot \left(\mathrm{d_{K/F}} + \mathrm{a}(\chi_\mathrm{K})\right) = \mathrm{f_{K/F}} \cdot \mathrm{d_{K/F}} + \mathrm{f_{K/F}} \cdot \mathrm{a}(\chi_\mathrm{K}). \tag{5.1.36}$$

By using equation (5.1.34), from equations (5.1.35), (5.1.36), we have

$$a(\chi_K) = e_{K/E} \cdot a(\chi_E) - d_{K/E}$$

□

Now by combining Proposition 5.1.22 with Proposition 5.1.21, we get the following result.

Lemma 5.1.23. *Let $\rho = \rho(X_\eta, \chi_K)$ be a U-isotopic Heisenberg representation of the absolute Galois group G_F of a non-archimedean local field F. Let $K = K_\eta$ correspond to the radical of X_η, and let E_1/F be the maximal unramified subextension, and E/F be any maximal cyclic and totally ramified subextension in K/F. Let m denote the order of η. Then ρ is induced by χ_{E_1} or by χ_E respectively, and we have*

1. $a_E(\chi_E) = m \cdot a(\eta) - d_{E/F}$,
2. $a_{E_1}(\chi_{E_1}) = a(\eta)$,
3. *and for the character $\chi_K \in \widehat{K^\times}$,*

$$a_K(\chi_K) = m \cdot a(\eta) - d_{K/F}.$$

Moreover, $a_E(\chi_E) = a_K(\chi_K)$.

Proof. Proof of these assertions follows from equation (5.1.28) and Proposition 5.1.21. When $\rho = \mathrm{Ind_{E/F}}(\chi_\mathrm{E})$, where E/F is a maximal cyclic and totally ramified subextension in K/F, from equation (5.1.28) we have

$$\begin{aligned} a_F(\rho) &= m \cdot a(\eta) \quad \text{using Proposition 5.1.21,} \\ &= f_{E/F} \cdot \left(d_{E/F} \cdot 1 + a_E(\chi_E)\right), \quad \text{since } \rho = \mathrm{Ind_{E/F}}(\chi_\mathrm{E}) \\ &= 1 \cdot \left(d_{E/F} + a_E(\chi_E)\right). \end{aligned}$$

because E/F is totally ramified, hence $f_{E/F} = 1$. This implies $a_E(\chi_E) = m \cdot a(\eta) - d_{E/F}$.

Similarly, when $\rho = \mathrm{Ind_{E_1/F}}(\chi_{\mathrm{E_1}})$, where E_1/F is the maximal unramified subextension in K/F, hence $f_{E_1/F} = m$ and $d_{E_1/F} = 0$, by using equation (5.1.28) we obtain $a_{E_1}(\chi_{E_1}) = a(\eta)$.

Again from Proposition 5.1.22 we have

$$a_K(\chi_K) = m \cdot a(\chi_{E_1}) - d_{K/E_1} = m \cdot a(\eta) - d_{K/F}.$$

Finally, since E/F is a maximal cyclic totally ramified implies K/E is unramified and therefore

$$d_{E/F} = d_{K/F}, \quad \text{and hence } a_E(\chi_E) = a_K(\chi_K).$$

□

Remark 5.1.24. Assume that we are in the dimension $m = \#\eta$ prime to p case. Then from Corollary 5.1.12, η must be a character of U/U^1 (for $U = U_F$), hence

$$a(\eta) = 1 \qquad a_F(\rho_0) = m.$$

Therefore in this case the minimal conductor of ρ is m, hence it is equal to the dimension of ρ.

From the above Lemma 5.1.23, in this case we have

$$a_{E_1}(\chi_{E_1}) = a(\eta) = 1.$$

And $K/F, E/F$ are tamely ramified of ramification exponent $e_{K/F} = m$, hence

$$a_E(\chi_E) = a_K(\chi_K) = m \cdot a(\eta) - d_{K/F} = m - (e_{K/F} - 1) = m - (m-1) = 1.$$

Thus we can conclude that in this case all three characters (i.e., χ_{E_1}, χ_E, and χ_K) are of conductor 1.

In the general case $a_{E_1}(\chi_{E_1}) = a(\eta)$ and

$$a_E(\chi_E) = a_K(\chi_K) = m \cdot a(\eta) - d,$$

where $d = d_{E/F} = d_{K/F}$, conductors will be different.

In general, if $\rho = \rho_0 \otimes \chi_F$, where ρ_0 is a finite dimensional minimal conductor representation of G_F, and $\chi_F \in \widehat{F^\times}$, then we have the following result.

Lemma 5.1.25. *Let ρ_0 be a finite dimensional representation of G_F of minimal conductor. Then we have*

$$a_F(\rho) = \dim(\rho_0) \cdot \mathrm{a_F}(\chi_\mathrm{F}), \tag{5.1.37}$$

where $\rho = \rho_0 \otimes \chi_F = \rho(X_\eta, (\chi_F \circ N_{K/F})\chi_0)$ and $\chi_F \in \widehat{F^\times}$ with $a(\chi_F) > \frac{a(\rho_0)}{\dim(\rho)}$.

Proof. From equation (5.1.29) we have $a_F(\rho_0) = \dim(\rho_0) \cdot (1 + \mathrm{j}(\rho_0))$. By the given condition ρ_0 is of minimal conductor. So for representation $\rho = \rho_0 \otimes \chi_F$, we have

$$\begin{aligned} a_F(\rho) = a_F(\rho_0 \otimes \chi_F) &= \dim(\rho_0) \cdot (1 + \max\{\mathrm{j}(\rho_0), \mathrm{j}(\chi_\mathrm{F})\}) \\ &= \dim(\rho_0) \cdot \max\{1 + \mathrm{j}(\chi_\mathrm{F}), 1 + \mathrm{j}(\rho_0)\} \\ &= \dim(\rho_0) \cdot \max\{\mathrm{a}(\chi_\mathrm{F}), 1 + \mathrm{j}(\rho_0)\} \\ &= \dim(\rho_0) \cdot \mathrm{a_F}(\chi_\mathrm{F}), \end{aligned}$$

because by the given condition

$$a(\chi_F) > \frac{a(\rho_0)}{\dim(\rho_0)} = \frac{\dim(\rho_0) \cdot (1 + \mathrm{j}(\rho_0))}{\dim(\rho_0)} = 1 + j(\rho_0).$$

□

Proposition 5.1.26. *Let $\rho = \rho(X, \chi_K)$ be a Heisenberg representation dimension m of the absolute Galois group G_F of a non-archimedean local field F. Then $m | a_F(\rho)$ if and only if: X is U-isotropic, or (if X is not U-isotropic) $a_F(\rho)$ is with respect to X not the minimal conductor.*

Proof. From the above Lemma 5.1.25 we know that if ρ is not minimal, then $a_F(\rho)$ is always a multiple of the dimension m. So now we just have to check for minimal conductors. In the U-isotropic case the minimal conductor is multiple of the dimension (cf. Proposition 5.1.21).

Finally, suppose that X is not U-isotropic, i.e., $X|_{U\wedge U} = X|_{U^1\wedge U^1} \not\equiv 1$, because $U \wedge U = U^1 \wedge U^1$ (see the Remark on p. 126 of [14]). We also know that $UU^i = (UU^i \cap U^1 \wedge U^1) \times (U^i \wedge <\pi_F>)$ (cf. [14], p. 126, Proposition 5(ii)). In Proposition 5 of [14], we observe that all the jumps v in the filtration $\{UU^i \cap (U^1 \wedge U^1)\}, i \in \mathbb{R}_+$ are not **integers with** $v > 1$. This shows that $j(X)$ is also not an integer, hence $a_F(\rho_0)$ is not multiple of the dimension. This implies the conductor $a_F(\rho)$ is not minimal.

□

Let $\rho = \rho(X, \chi_K)$ be a Heisenberg representation of the absolute Galois group G_F. Then from equation (5.1.6), we have

$$\dim(\rho) = \sqrt{[\mathrm{K}:\mathrm{F}]} = \sqrt{[\mathrm{F}^\times : \mathcal{N}_{\mathrm{K/F}}]},$$

when $\mathcal{N}_{K/F} = \mathrm{Rad(X)}$.

Lemma 5.1.27. *Let $\rho = (Z_\rho, \chi) = \rho(X_\rho, \chi)$ be a Heisenberg representation of the absolute Galois group G_F of a non-archimedean local field $F/\mathbb{Q}_p$. Then following are equivalent:*

1. $\dim(\rho)$ *is prime to* p.

2. $\dim(\rho)$ *is a divisor of* $q_F - 1$.

3. *The alternating character X_ρ is U-isotropic and $X_\rho = X_\eta$ for a character η of U_F/U_F^1.*

Proof. From Corollary 5.1.12 we know that all Heisenberg representations of dimensions prime to p, are U-isotropic representations of the form $\rho = \rho(X_\eta, \chi)$, where $\eta : U_F/U_F^1 \to \mathbb{C}^\times$, and the dimensions $\dim(\rho) = \#\eta$.

Thus if $\dim(\rho)$ is prime to p, then $\dim(\rho) = \#\eta$ is a divisor of $q_F - 1$. And if $\dim(\rho)$ is a divisor of $q_F - 1$, then $gcd(p, \dim(\rho)) = 1$. Then from Corollary 5.1.12, the alternating character X_ρ is U-isotropic and $X_\rho = X_\eta$ for a character $\eta \in \widehat{U_F/U_F^1}$.

Finally, if $\rho = \rho(X_\rho, \chi_K) = \rho(X_\rho, \chi_K)$ be a Heisenberg representation of G_F for a character η of U_F/U_F^1, then from Corollary 5.1.12, we know that $\dim(\rho)$ is prime to p. □

For giving invariant formula of $W(\rho)$, we need to know the explicit dimension formula of ρ. In the following theorem we give the general dimension formula of a Heisenberg representation.

Theorem 5.1.28 (Dimension). *Let $F/\mathbb{Q}_p$ be a local field and G_F be the absolute Galois group of F. If ρ is a Heisenberg representation of G_F, then $\dim(\rho) = \mathrm{p}^{\mathrm{n}} \cdot \mathrm{d}'$, where $n \geqslant 0$ is an integer and where the prime to p factor d' must divide $q_F - 1$.*

Proof. By the definition of Heisenberg representation ρ we have the relation

$$[[G_F, G_F], G_F] \subseteq \mathrm{Ker}(\rho).$$

Then we can consider ρ as a representation of $G := G_F/[[G_F, G_F], G_F]$. Since $[x, g] \in [[G_F, G_F], G_F]$ for all $x \in [G_F, G_F]$ and $g \in G_F$, we have $[G, G] = [G_F, G_F]/[[G_F, G_F], G_F] \subseteq Z(G)$, hence G is a two-step nilpotent group.

We know that each nilpotent group is isomorphic to the direct product of its Sylow subgroups. Therefore we can write

$$G = G_p \times G_{p'},$$

where G_p is the Sylow p-subgroup, and $G_{p'}$ is the direct product of all other Sylow subgroups. Therefore each irreducible representation ρ has the form $\rho = \rho_p \otimes \rho_{p'}$, where ρ_p and $\rho_{p'}$ are irreducible representations of G_p and $G_{p'}$ respectively.

We also know that finite p-groups are nilpotent groups, and direct product of a finite number of nilpotent groups is again a nilpotent group. So G_p and $G_{p'}$ are both two-step nilpotent group because G is a two-step nilpotent group. Therefore the representations ρ_p and $\rho_{p'}$ are both Heisenberg representations of G_p and $G_{p'}$ respectively.

Now to prove our assertion, we have to show that $\dim(\rho_{\mathrm{p}})$ can be an arbitrary power of p, whereas $\dim(\rho_{\mathrm{p}'})$ must divide $q_F - 1$. Since ρ_p is an **irreducible** representation of p-group G_p, so the dimension of ρ_p is some p-power.

Again from the construction of $\rho_{p'}$ we can say that $\dim(\rho_{\mathrm{p}'})$ is **prime** to p. Then from Lemma 5.1.27 $\dim(\rho_{\mathrm{p}'})$ is a divisor of $q_F - 1$.

This completes the proof.

□

Remark 5.1.29. **(1).** Let V_F be the wild ramification subgroup of G_F. We can show that $\rho|_{V_F}$ is irreducible if and only if $Z_\rho = G_K \subset G_F$ corresponds to an abelian extension K/F which is totally ramified and wildly ramified[4] (cf. [11], p. 305). If $N := N_{K/F}(K^\times)$ is the subgroup of norms, then this means that $N \cdot U_F^1 = F^\times$, in other words,

$$F^\times/N = N \cdot U_F^1/N = U_F^1/N \cap U_F^1,$$

where N can be also considered as the radical of X_ρ. So we can consider the alternating character X_ρ on the principal units $U_F^1 \subset F^\times$. Then

$$\dim(\rho) = \sqrt{[\mathrm{F}^\times : \mathrm{N}]} = \sqrt{[\mathrm{U}_\mathrm{F}^1 : \mathrm{N} \cap \mathrm{U}_\mathrm{F}^1]},$$

is a power of p, because U_F^1 is a pro-p-group.

[4]Group theoretically, if $\rho|_{V_F} = \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G_F}}(\chi_{\mathrm{H}})|_{\mathrm{V_F}}$ is irreducible, then from Section 7.4 of [28], we can say $G_F = H \cdot V_F$. Here $H = G_L$, where L is a certain extension of F, and $V_F = G_{F_{mt}}$ where F_{mt}/F is the maximal tame extension of F. Therefore $G_F = H \cdot V_F$ is equivalent to $F = L \cap F_{mt}$ that means the extension L/F must be totally ramified and wildly ramified, and $[G_F : H] = [L : F] = |V_F|$. We know that the wild ramification subgroup V_F is a pro-p-group (cf. [21], p. 106). Then $\dim(\rho)$ is a power of p.

Here we observe: If $\rho = \rho(X, \chi_K)$ with $\rho|_{V_F}$ stays irreducible, then $\dim(\rho) = \mathrm{p^n}$, $n \geqslant 1$ and K/F is a totally and **wildly** ramified. But there is a **big** class of Heisenberg representations ρ such that $\dim(\rho) = \mathrm{p^n}$ is a p-power, but which are not wild representations (see the Definition 5.1.8 of U-isotropic).

(2). Let $\rho = \rho(X, \chi_K)$ be a Heisenberg representation of the absolute Galois group G_F of dimension $d > 1$ which is prime to p. Then from above Lemma 5.1.27, we have $d | q_F - 1$. For this representation ρ, here K/F must be tame if $\mathrm{Rad(X)} = \mathcal{N}_{\mathrm{K/F}}$ (cf. [21], p. 115).

5.2 Invariant formula for $W(\rho)$

Lemma 5.2.1. *Let $\rho = \rho(Z, \chi_Z)$ be a Heisenberg representation of the local Galois group $G = \mathrm{Gal}(L/F)$ of odd dimension. Let H be a maximal isotropic subgroup for ρ and $\chi_H \in \widehat{H}$ with $\chi_H|_Z = \chi_Z$ then:*

$$W(\rho) = W(\chi_H), \quad W(\rho)^{\dim(\rho)} = W(\chi_Z), \tag{5.2.1}$$

and

$$W(\chi_H)^{[H:Z]} = W(\chi_Z). \tag{5.2.2}$$

Proof. From the construction of Heisenberg representation ρ of G we have

$$\rho = \mathrm{Ind_H^G(\chi_H)}, \quad \dim(\rho) \cdot \rho = \mathrm{Ind_Z^G(\chi_Z)}.$$

This implies that $W(\rho) = \lambda_H^G \cdot W(\chi_H)$ and $W(\rho)^{\dim(\rho)} = \lambda_Z^G \cdot W(\chi_Z)$.

Since $\dim(\rho)$ is odd we may apply now Lemma 3.2.2, and we obtain

$$\lambda_H^G = \lambda_Z^G = 1.$$

So, we have $W(\rho) = \lambda_H^G(W) \cdot W(\chi_H) = W(\chi_H)$. Similarly, we have $W(\rho)^{\dim(\rho)} = W(\chi_Z)$.

Moreover, it is easy to see[5] that $W(\mathrm{Ind_Z^H}(\chi_Z)) = \mathrm{W(\chi_H)^{[H:Z]}}$. By the given condition, $[H : Z] = \dim(\rho)$ is odd, hence $\lambda_Z^H = 1$, then we have

$$W(\chi_H)^{[H:Z]} = W(\mathrm{Ind_Z^H}(\chi_Z)) = \mathrm{W}(\chi_Z). \tag{5.2.3}$$

□

[5] We have

$$d \cdot \rho = \mathrm{Ind}_Z^G \chi_Z = \mathrm{Ind}_H^G(\mathrm{Ind}_Z^H \chi_Z),$$

and $\mathrm{Ind}_Z^H \chi_Z$ of dimension $d = [H : Z]$. Therefore:

$$W(\rho)^d = (\lambda_H^G)^d \cdot W(\mathrm{Ind}_Z^H \chi_Z).$$

On the other hand $W(\rho) = \lambda_H^G \cdot W(\chi_H)$ implies

$$W(\rho)^d = (\lambda_H^G)^d \cdot W(\chi_H)^d.$$

Now comparing these two expressions for $W(\rho)^d$ we see that

$$W(\chi_H)^d = W(\mathrm{Ind}_Z^H \chi_Z).$$

Remark 5.2.2. Related to $G \supset H \supset Z$ we have the base fields $F \subset E \subset K$, and χ_Z is the restriction of χ_H. In arithmetic terms this means:

$$\chi_K = \chi_E \circ N_{K/E}.$$

So in arithmetic terms of $W(\mathrm{Ind}_Z^G(\chi_Z)) = \mathrm{W}(\mathrm{Ind}_H^G(\chi_H))^{[G:H]}$ is as follows:

$$W(\mathrm{Ind}_{K/F}(\chi_K), \psi) = \mathrm{W}(\mathrm{Ind}_{E/F}(\chi_E), \psi)^{[K:E]}.$$

Then we can conclude that

$$\lambda_{K/E} \cdot W(\chi_K, \psi_K) = W(\chi_E, \psi_E)^{[K:F]}.$$

If the dimension $\dim(\rho) = [\mathrm{K} : \mathrm{E}]$ is odd, we have $\lambda_{K/E} = 1$, because K/E is Galois. Then we obtain

$$W(\chi_E, \psi_E)^{[K:E]} = W(\chi_E \circ N_{K/E}, \psi_E \circ \mathrm{Tr}_{K/E}). \tag{5.2.4}$$

The formula (5.2.4) is known as a **Davenport-Hasse** relation.

Corollary 5.2.3. *Let $\rho = \rho(Z, \chi_Z)$ be a Heisenberg representation of a local Galois group G. Let $\dim(\rho) = \mathrm{d}$ be odd. Let the order of $W(\chi_Z)$ be n (i.e., $W(\chi_Z)^n = 1$). If d is prime to n, then $d^{\varphi(n)} \equiv 1 \mod n$, and*

$$W(\rho) = W(\chi_Z)^{\frac{1}{d}} = W(\chi_Z)^{d^{\varphi(n)-1}},$$

where $\varphi(n)$ is the Euler's totient function of n.

Proof. By our assumption, here d and n are coprime. Therefore from **Euler's theorem** we can write

$$d^{\varphi(n)} \equiv 1 \mod n.$$

This implies $d^{\varphi(n)} - 1$ is a multiple of n.

Here d is odd, then from the above Lemma 5.2.1 we have $W(\rho)^d = W(\chi_Z)$. So we obtain

$$W(\rho) = W(\chi_Z)^{\frac{1}{d}} = W(\chi_Z)^{d^{\varphi(n)-1}},$$

since $d^{\varphi(n)} - 1$ is a multiple of n, and by assumption $W(\chi_Z)^n = 1$. □

We observe that when $\dim(\rho) = \mathrm{d}$ is odd, if we take second part of the equation (5.2.1), we have $W(\rho) = W(\chi_Z)^{\frac{1}{d}}$, but it is not well-defined in general. Here we have to make precise which root $W(\chi_Z)^{\frac{1}{d}}$ really occurs. That is why, giving invariant formula of $W(\rho)$ using λ-functions computation is difficult. In the following theorem we give an invariant formula of local constant for Heisenberg representation.

Theorem 5.2.4. *Let $\rho = \rho(X, \chi_K)$ be a Heisenberg representation of the absolute Galois group G_F of a local field $F/\mathbb{Q}_p$ of dimension d. Let ψ_F be the canonical additive character of F and $\psi_K := \psi_F \circ \mathrm{Tr}_{K/F}$. Denote μ_{p^∞} as the group of roots of unity of p-power order and μ_n as the group of n-th roots of unity, where $n > 1$ is an integer.*

1. *When the dimension d is odd, we have*

$$W(\rho) \equiv W(\chi_\rho)' \mod \mu_d,$$

where $W(\chi_\rho)'$ *is any d-th root of* $W(\chi_K, \psi_K)$.

2. *When the dimension d is even, we have*

$$W(\rho) \equiv W(\chi_\rho)' \mod \mu_{d'},$$

where $d' = \mathrm{lcm(4, d)}$.

Proof. **(1).** We know that the lambda functions are always fourth roots of unity. In particular, when degree of the Galois extension K/F is odd, from Theorem 3.2.3 we have $\lambda_{K/F} = 1$. For proving our assertions we will use these facts about λ-functions.

We know that $\dim(\rho)\cdot\rho = \mathrm{Ind_{K/F}(\chi_K)}$, where by class field theory $\chi_K \leftrightarrow \chi_\rho$ is a character of $K^\times$. When d is odd, we can write

$$W(\rho)^d = \lambda_{K/F}\cdot W(\chi_K, \psi_K) = W(\chi_K, \psi_K).$$

Now let $W(\chi_\rho)'$ be any d-th root of $W(\chi_K, \psi_K)$. Then we have

$$W(\rho)^d = W(\chi_\rho)'^d,$$

hence $\frac{W(\rho)}{W(\chi_\rho)}$ is a d-th root of unity. Therefore we have

$$W(\rho) \equiv W(\chi_\rho)' \mod \mu_d.$$

(2). Similarly, we can give invariant formula for even degree Heisenberg representations. When the dimension d of ρ is even, we have

$$W(\rho)^d = \lambda_{K/F}\cdot W(\chi_K, \psi_K) \equiv W(\chi_K, \psi_K) \mod \mu_4, \tag{5.2.5}$$

because $\lambda_{K/F}$ is a fourth root of unity. Now let $W(\chi_\rho)'$ be any d-th root of $W(\chi_K, \psi_K)$, hence $W(\chi_K, \psi_K) = W(\chi_\rho)'^d$. Then from equation (5.2.5) we have

$$\left(\frac{W(\rho)}{W(\chi_\rho)'}\right)^d \equiv 1 \mod \mu_4.$$

Therefore we can conclude that

$$W(\rho) \equiv W(\chi_\rho)' \mod \mu_{d'}, \tag{5.2.6}$$

where $d' = \mathrm{lcm(4, d)}$.

□

When dimension of a Heisenberg representation $\rho = \rho(X, \chi_K)$ of G_F is prime to p, then from Lemma 5.1.27 we can say that $X = X_\eta$ is U-isotropic with $\eta : U_F/U_F^1 \to \mathbb{C}^\times$. Again from Remark 5.1.24 we observe that $a(\chi_K) = 1$ when ρ is of minimal conductor. In the following lemma for minimal conductor ρ with dimension prime to p, we show that $W(\rho)$ is a root of unity.

Lemma 5.2.5. *Let $\rho = \rho(X, \chi_K)$ be a minimal conductor Heisenberg representation with respect to X of the absolute Galois group G_F of a non-archimedean local field $F/\mathbb{Q}_p$. If dimension* $\dim(\rho)$ *is prime to p, then $W(\rho)$ is always a root of unity.*

Proof. Assume that $\dim(\rho) = \mathrm{d}$ and $gcd(g, p) = 1$. Then from Lemma 5.1.27, we can say that $\rho = \rho(X, \chi_K) = \rho(X_\eta, \chi_K)$ is a U-isotropic with $a(\eta) = 1$. Since ρ is of minimal conductor, from Remark 5.1.24 we have $a(\chi_K) = 1$.

From equation (5.1.5) we also know that:

$$d \cdot \rho = \mathrm{Ind}_{\mathrm{K/F}}(\chi_{\mathrm{K}}).$$

Then we can write

$$\begin{aligned} W(\rho)^d &= \lambda_{K/F} \cdot W(\chi_K) \\ &= \lambda_{K/F} \cdot q_K^{-\frac{1}{2}} \sum_{x \in U_K/U_K^1} \chi_K^{-1}(x/c)\psi_K(x/c) \\ &= \lambda_{K/F} \cdot q_K^{-\frac{1}{2}} \tau(\chi_K), \end{aligned} \tag{5.2.7}$$

where $c = \pi_K^{1+n(\psi_K)}$, $\psi_K = \psi_F \circ \mathrm{Tr}_{\mathrm{K/F}}$, the canonical character of K and

$$\tau(\chi_K) = \sum_{x \in U_K/U_K^1} \chi_K^{-1}(x/c)\psi_K(x/c). \tag{5.2.8}$$

Since $U_K/U_K^1 \cong k_K^\times$, $a(\chi_K) = 1$, and $n(\frac{1}{c} \cdot \psi_K) = -1$, we can consider $\tau(\chi_K)$ as a classical Gauss sum of χ_K. We also know that $|\tau(\chi_K)| = q_K^{\frac{1}{2}}$ (cf. [26], p. 30, Proposition 2.2(i)).

Moreover, here we have $f_{K/F} = e_{K/F} = d$, hence $f_{K/\mathbb{Q}_p} \geqslant d$. So here we have $q_K = p^{f_{K/\mathbb{Q}_p}} \geqslant p^d$. Then from Theorem 1.6.2 on p. 33 of [3], we can write

$$\tau(\chi_K) = q_K^{\frac{1}{2}} \cdot \gamma,$$

where γ is a certain root of unity.

We also know that $\lambda_{K/F}^4 = 1$, then from the equation (5.2.7) we obtain:

$$W(\rho)^{4dn} = \gamma^{4n} = 1, \tag{5.2.9}$$

where n is the order of γ.

This completes the proof. ☐

Remark 5.2.6. As to the computation of $W(\rho) = W(\rho(X, \chi_K))$ we also can precisely say what an unramified twist will do by the formula of local constant of unramified character twist (cf. [33], p. 15, (3.4.5)). Let $\omega_{K,s}$ be an unramified character of $K^\times$ such that $\omega_{K,s}|_{F^\times} = \omega_{F,s}$, then we have

$$\omega_{F,s} \otimes \rho(X, \chi_K) = \rho(X, \omega_{K,s} \cdot \chi_K), \quad W(\rho(X, \omega_{K,s} \cdot \chi_K)) = \omega_{F,s}(c_{\rho,\psi}) \cdot W(\rho(X, \chi_K)). \tag{5.2.10}$$

Therefore the question: Is $W(\rho)$ a root of unity or not?, is completely under control if we do unramified twists. In particular, unramified twists of finite order cannot change the answer.

In the following theorem we give an invariant formula for $W(\rho, \psi)$, where $\rho = \rho(X, \chi_K)$ is a minimal conductor Heisenberg representation of the absolute Galois group G_F of a local field $F/\mathbb{Q}_p$ of dimension m which is prime to p.

Theorem 5.2.7. *Let $\rho = \rho(X, \chi_K)$ be a minimal conductor Heisenberg representation of the absolute Galois group G_F of a non-archimedean local field $F/\mathbb{Q}_p$ of dimension m with $gcd(m, p) = 1$. Let ψ be a nontrivial additive character of F. Then*

$$W(\rho, \psi) = R(\psi, c) \cdot L(\psi, c), \tag{5.2.11}$$

where

$$R(\psi, c) := \lambda_{E/F}(\psi)\Delta_{E/F}(c),$$

is a fourth root of unity that depends on $c \in F^\times$ with $\nu_F(c) = 1 + n(\psi)$ but not on the totally ramified cyclic subextension E/F in K/F, and

$$L(\psi, c) := \det(\rho)(c) q_F^{-\frac{1}{2}} \sum_{x \in k_F^\times} (\chi_K \circ N_{E_1/F}^{-1})^{-1}(x) \cdot (c^{-1}\psi)(mx),$$

where E_1/F is the unramified extension of F of degree m.

Before proving this Theorem 5.2.7 we need to prove the following lemma.

Lemma 5.2.8. *(With the notation of the above theorem)*

1. *Let E/F be any totally ramified cyclic extension of degree m inside K/F. Then:*

$$\Delta_{E/F}(\epsilon) =: \Delta(\epsilon), \qquad \text{for all } \epsilon \in U_F,$$

   does not depend on E if we restrict to units of F.

2. *We have $L(\psi, \epsilon c) = \Delta(\epsilon) L(\psi, c)$, and therefore changing c by unit we see that*

$$\Delta_{E/F}(\epsilon c) \cdot L(\psi, \epsilon c) = \Delta(\epsilon)^2 \Delta_{E/F}(c) \cdot L(\psi, c) = \Delta_{E/F}(c) L(\psi, c).$$

3. *We also have the transformation rule $R(\psi, \epsilon c) = \Delta(\epsilon) \cdot R(\psi, c)$.*

Proof. **(1).** Denote $G := \mathrm{Gal(E/F)}$. From equation (3.3.7) and by class field theory we know that

$$\Delta_{E/F} = \begin{cases} \omega_{E'/F} & \text{when } \mathrm{rk_2(G)} = 1 \\ 1 & \text{when } \mathrm{rk_2(G)} = 0, \end{cases} \tag{5.2.12}$$

where E'/F is a uniquely determined quadratic extension inside E/F, and $\omega_{E'/F}$ is the quadratic character of $F^\times$ which corresponds to the extension E'/F by class field theory.

When m is odd, i.e., $\mathrm{rk_2(G)} = 0$, hence $\Delta_{E/F} \cong 1$. So for odd case, the assertion (1) is obvious.

When m is even, we choose two different totally ramified cyclic subextensions, namely $L_1/F,\ L_2/F$, in K/F of degree m. Then we can write for all $\epsilon \in U_F$,

$$\Delta_{L_1/F}(\epsilon) = \omega_{E'/F}(\epsilon) = \eta(\epsilon)\cdot\omega_{E'/F}(\epsilon) = \omega_{E'/F}(\epsilon) = \Delta_{L_2/F}(\epsilon),$$

where η is the unramified quadratic character of $F^\times$. This proves that $\Delta_{E/F}$ does not depend on E if we restrict to U_F.

(2). From Proposition 5.1.7 we know that $\det(\rho)(x) = \Delta_{E/F}(x)\cdot\chi_K\circ N_{K/E}^{-1}(x)$ for all $x \in F^\times$. Let E_1/F be the unramified subextension in K/F of degree m. Then we have $EE_1 = K$ and

$$N_{K/E}|_{E_1} = N_{E_1/F}, \qquad (E_1^\times)_F \subseteq K_E^\times \subset \mathrm{Ker}(\chi_{\mathrm{K}}).$$

Moreover, $U_F \subset \mathcal{N}_{E_1/F}$ and therefore we may write $N_{K/E}^{-1}(\epsilon) = N_{E_1/F}^{-1}(\epsilon)$ for all $\epsilon \in U_F$. Now we can write:

$$\begin{aligned} L(\psi,\epsilon c) &= \det(\rho)(\epsilon c) q_F^{-\frac{1}{2}} \sum_{x\in k_F^\times} (\chi_K\circ N_{E_1/F}^{-1})^{-1}(x)\cdot(c^{-1}\psi)(mx/\epsilon) \\ &= \Delta_{E/F}(\epsilon)\chi_K\circ N_{K/E}^{-1}(\epsilon)\det(\rho)(c) q_F^{-\frac{1}{2}} \sum_{x\in k_F^\times} (\chi_K\circ N_{E_1/F}^{-1})^{-1}(\epsilon x)\cdot(c^{-1}\psi)(mx) \\ &= \Delta(\epsilon)\chi_K\circ N_{E_1/F}^{-1}(\epsilon\epsilon^{-1})\det(\rho)(c) q_F^{-\frac{1}{2}} \sum_{x\in k_F^\times} (\chi_K\circ N_{E_1/F}^{-1})^{-1}(x)\cdot(c^{-1}\psi)(mx) \\ &= \Delta(\epsilon)\cdot\det(\rho)(c) q_F^{-\frac{1}{2}} \sum_{x\in k_F^\times} (\chi_K\circ N_{E_1/F}^{-1})^{-1}(x)\cdot(c^{-1}\psi)(mx) \\ &= \Delta(\epsilon)\cdot L(\psi,c). \end{aligned}$$

This implies that

$$\Delta_{E/F}(\epsilon c)\cdot L(\psi,\epsilon c) = \Delta(\epsilon)^2\Delta_{E/F}(c)\cdot L(\psi,c) = \Delta_{E/F}(c)L(\psi,c).$$

(3). By the definition of $R(\psi,c)$ we can write:

$$\begin{aligned} R(\psi,\epsilon c) &= \lambda_{E/F}(\psi)\Delta_{E/F}(\epsilon c) = \lambda_{E/F}(\psi)\Delta_{E/F}(\epsilon)\Delta_{E/F}(c) \\ &= \Delta(\epsilon)\lambda_{E/F}(\psi)\Delta_{E/F}(c) = \Delta(\epsilon)\cdot R(\psi,c). \end{aligned}$$

□

Now we are in a position to give a proof of Theorem 5.2.7 by using Lemma 5.2.8.

Proof of Theorem 5.2.7. By the given conditions: $\rho = \rho(X, \chi_K)$ is a minimal conductor Heisenberg representation of the absolute Galois group G_F of a local field $F/\mathbb{Q}_p$ of dimension m which is prime to p. This means we are in the situation: $\rho = \rho(X, \chi_K) = \rho(X_\eta, \chi_K)$, where η is a character of U_F/U_F^1, and $\dim(\rho) = \#\eta = \mathrm{m}$.

Since ρ is of minimal conductor, we have $a(\rho_0) = m$. Then from Remark 5.1.24 we have $a(\chi_K) = 1$.

Now we choose $E/F \subset K/F$ a totally ramified cyclic subextension of degree $[E : F] = m$, hence $k_E = k_F$ the same residue fields, and K/E is unramified of degree m. Then we can write $\rho = \mathrm{Ind_{E/F}(\chi_E)}$, and $a(\chi_E) = 1$. Again, from Proposition 5.1.7 we have

$$\det(\rho)(x) = \Delta_{E/F}(x) \cdot \chi_K \circ N_{K/E}^{-1}(x) \quad \text{for all } x \in F^\times.$$

Then for all $x \in F^\times$, we can write

$$\chi_K \circ N_{K/E}^{-1}(x) = \chi_E(x) = \Delta_{E/F}(x) \cdot \det(\rho)(x).$$

This is true for all subextensions[6] E/F in K/F which are cyclic of degree m.

Now we come to in our particular choice: $\rho = \mathrm{Ind_{E/F}(\chi_E)}$, with $a(\chi_E) = 1$ and E/F is totally ramified. We can write

$$\begin{aligned} W(\rho, \psi) &= W(\mathrm{Ind_{E/F}(\chi_E)}, \psi) = \lambda_{\mathrm{E/F}}(\psi) \cdot \mathrm{W}(\chi_{\mathrm{E}}, \psi \circ \mathrm{Tr_{E/F}}) \\ &= \lambda_{E/F}(\psi) \cdot q_E^{-\frac{1}{2}} \chi_E(c_E) \sum_{x \in U_E/U_E^1} \chi_E^{-1}(x)(c_E^{-1}\psi \circ \mathrm{Tr_{E/F}})(\mathrm{x}), \end{aligned}$$

where $v_E(c_E) = 1 + n(\psi \circ \mathrm{Tr_{E/F}}) = \mathrm{e_{E/F}}(1 + \mathrm{n}(\psi))$. This implies that we can choose $c_F \in F^\times$ such that $\nu_F(c_F = c_E) = 1 + n(\psi)$. Let E_1/F be the unramified subextension in K/F, then for each $\epsilon \in U_F$, we have $N_{K/E}^{-1}(\epsilon) = N_{E_1/F}^{-1}(\epsilon)$ where $N_{E_1/F} := N_{K/E}|_{E_1}$. Since E/F is totally ramified, we have $q_E = q_F$. And when $x \in F^\times$, we have $\mathrm{Tr_{E/F}(x) = mx}$.

Then the above formula rewrites:

$$\begin{aligned} W(\rho, \psi) &= \lambda_{E/F}(\psi) \cdot q_F^{-\frac{1}{2}} \chi_K \circ N_{K/E}^{-1}(c_F) \sum_{x \in k_F^\times} (\chi_K \circ N_{K/E}^{-1})^{-1}(x)(c_F^{-1}\psi)(mx) \\ &= \lambda_{E/F}(\psi) \cdot q_F^{-\frac{1}{2}} \Delta_{E/F}(c_F) \det(\rho)(c_F) \sum_{x \in k_F^\times} (\chi_K \circ N_{E_1/F}^{-1})^{-1}(x)(c_F^{-1}\psi)(mx) \\ &= \lambda_{E/F}(\psi)\Delta_{E/F}(c_F) \cdot \left(\det(\rho)(c_F) q_F^{-\frac{1}{2}} \sum_{x \in k_F^\times} (\chi_K \circ N_{E_1/F}^{-1})^{-1}(x)(c_F^{-1}\psi)(mx) \right) \\ &= R(\psi, c) \cdot L(\psi, c), \end{aligned}$$

[6]In K/F of type $\mathbb{Z}_m \times \mathbb{Z}_m$ any cyclic subextension E/F in K/F of degree m will correspond to a maximal isotropic subgroup. But we restrict to choosing E totally ramified or unramified.

where $c_F = c \in F^\times$ with $\nu_F(c) = 1 + n(\psi)$, $R(\psi, c) = \lambda_{E/F}(\psi)\Delta_{E/F}(c)$, and

$$L(\psi, c) = \det(\rho)(c_F) q_F^{-\frac{1}{2}} \sum_{x \in k_F^\times} (\chi_K \circ N_{E_1/F}^{-1})^{-1}(x)(c^{-1}\psi)(mx).$$

Now it is clear that $L(\psi, c)$ depends on c but not on the totally ramified cyclic extension E/F which we have chosen.

Again we know that $\lambda_{E/F}(\psi)$ is a fourth root of unity and $\Delta_{E/F}(c) \in \{\pm 1\}$. Therefore it is easy to see that $R(\psi, c)$ is a fourth root of unity. So to call our expression

$$W(\rho, \psi) = R(\psi, c) \cdot L(\psi, c)$$

is invariant, we are left to show $R(\psi, c)$ does not depend on the the totally ramified cyclic subextension E/F in K/F.

From equation (3.4.3) we can write here

$$R(\psi, c) = \lambda_{E/F}(\psi)\Delta_{E/F}(c) = \lambda_{E/F}(c\psi) = \lambda_{E/F}(\psi'),$$

where $\psi' = c\psi$, hence $n(\psi') = \nu_F(c) + n(\psi) = 1 + n(\psi) + n(\psi) = 2n(\psi) + 1$.

When $m(= [E : F])$ is odd, we have $\lambda_{E/F}(\psi') = 1$, hence $R(\psi, c) = \lambda_{E/F}(c\psi) = 1$. Thus in the odd case $R(\psi, c)$ is independent of the choice of the totally ramified subextension E/F in K/F.

When m is even, we have

$$\begin{aligned} R(\psi, c) &= \lambda_{E/F}(\psi') = \lambda_{E/E'}(\psi'') \cdot \lambda_{E'/F}^{[E:E'']} \\ &= \lambda_{E'/F}(\psi')^{\pm 1}, \end{aligned}$$

where $[E', F]$ is the 2-primary part of m, hence $[E : E']$ is odd. Here the sign only depends on m but not on E. So we can restrict to the case where $m = [E : F]$ is a power of 2. Let E_2/F be the unique quadratic subextension in E/F. Since E/F is a cyclic tame extension, from Corollary 3.3.11(1), we obtain:

$$\lambda_{E/F}(\psi') = \begin{cases} \lambda_{E_2/F}(\psi') & \text{if } [E : F] \neq 4 \\ \beta(-1) \cdot \lambda_{E_2/F}(\psi') & \text{if } [E : F] = 4, \end{cases} \tag{5.2.13}$$

where β is the character of $F^\times/\mathcal{N}_{E/F}$ of order 4.

Since here $n(\psi') = 2n(\psi) + 1$ is **odd**[7], from Remark 3.4.11 (see the table of Remark 3.4.11) we can tell that $\lambda_{E_2/F}(\psi')$ is invariant.

Finally, we have to see that $\beta(-1)$ does not depend on E if $[E : F] = 4$.

Since E/F is totally ramified of degree 4, we have $F^\times = U_F \cdot N$, hence $F^\times/N = U_F N/N = U_F/U_F \cap N \cong \mathbb{Z}_4$, where $N = N_{E/F}(E^\times)$. Again $U_F^1 \subset U_F$, and $U_F^1 \subset N$, hence $U_F^1 \subset N \cap U_F \subset$

[7]If $n(\psi')$ is even, then from the table of the Remark 3.4.11, $\lambda_{E_2/F}(\psi') = -\lambda_{E_2'/F}(\psi')$, where E_2'/F be the totally ramified quadratic extension different from E_2/F. Therefore $\lambda_{E/F}(\psi')$ depends on ψ'.

U_F. We know that U_F/U_F^1 is a cyclic group. Therefore $N\cap U_F$ is determined by its index in U_F, which does not depend on E. Hence, $U_F\cap N$ does not depend on E.

We also know that there are two characters of $U_F/U_F\cap N$ of order 4, and they are inverse to each other. Then

$$\beta(-1)=\beta(-1)^{-1}=\beta^{-1}(-1)$$

is the same in both cases. Since β is the character which corresponds to E/F by class field theory, we can say β is a character of $F^\times/U_F^1$, hence $a(\beta)=1$. It clearly shows that $\beta(-1)$ does not depend on E. So we can conclude that $R(\psi,c)$ does not depend on E.

Thus our expression $W(\rho,\psi)=R(\psi,c)\cdot L(\psi,c)$ does not depend on the choice of the totally ramified cyclic subextension E/F in K/F. Moreover, we notice that we have the transformation rules

$$R(\psi,\epsilon c)=\Delta(\epsilon)R(\psi,c),\qquad L(\psi,\epsilon c)=\Delta(\epsilon)L(\psi,c),$$

for all $\epsilon\in U_F$. Again $\Delta(\epsilon)^2=1$, hence the product $R(\psi,\epsilon c)\cdot L(\psi,\epsilon c)=R(\psi,c)\cdot L(\psi,c)=W(\rho,\psi)$ does not depend on the choice of c.

Therefore, finally, we can conclude our formula $W(\rho,\psi)=R(\psi,c)L(\psi,c)$ is an invariant expression.

□

Now let $\rho=\rho(X,\chi_K)$ be a Heisenberg representation of dimension prime to p but the conductor of ρ is **not** minimal. In the following theorem we give an invariant formula of $W(\rho,\psi)$.

Theorem 5.2.9. *Let $\rho=\rho(X_\rho,\chi_K)$ be a Heisenberg representation of the absolute Galois group G_F of a local field $F/\mathbb{Q}_p$ of dimension m prime to p. Let ψ be a nontrivial additive character of F. Suppose that the conductor of ρ is not minimal, $\rho=\rho_0\otimes\widetilde{\chi_F}$ and $a(\rho)=m\cdot a(\chi_F)$, where $\widetilde{\chi_F}:W_F\to\mathbb{C}^\times$ corresponds to $\chi_F:F^\times\to\mathbb{C}^\times$, and $h=a(\chi_F)\geqslant 2$.*
Case-1: *If m is odd, then*

1. *when $1+m(h-1)=2d$ is even, we have*

$$W(\rho,\psi)=\det(\rho)(c)\psi(mc^{-1}),$$

2. *when $1+m(h-1)=2d+1$ is odd, we have*

$$W(\rho,\psi)=\det(\rho)(c)\cdot H(\psi,c),$$

where

$$H(\psi,c)=q_F^{-\frac{1}{2}}\sum_{y\in U_F^{h'}/U_F^{h'+1}}(\chi_K|_{E_1}\circ N_{E_1/F}^{-1})^{-1}(y)(c^{-1}\psi)(my),$$

and $h'=[\frac{h}{2}]$, where $[x]$ denotes the largest integer $\leqslant x$.
Case-2: *If m is even, then*

1. *when h is odd, we have*

$$W(\rho,\psi)=R(\psi,c)\cdot\det(\rho)(c)\cdot H(\psi,c),$$

where $H(\psi,c)$ is the same as in Case-1(2).

2. *when h is even, we have*

$$W(\rho,\psi)=R(\psi,c)\cdot\det(\rho)(c)\cdot q_F^{\frac{1}{2}}\cdot\psi(c^{-1}m),$$

where $R(\psi,c)=\lambda_{E/F}(\psi)\cdot\Delta_{E/F}(c)$.
Here E_1/F is the maximal unramified subextension in K/F, and E/F is a totally ramified cyclic subextension in K/F and $c\in F^\times$ with $\nu_F(c)=h+n(\psi)$, and

$$\chi_F(1+x)=\psi(x/c),\qquad \textit{for all } x\in P_F^{h-h'}/P_F^h.$$

Proof. **Step-1:** Since by the given condition ρ is not minimal conductor, we can write

$$\rho=\rho_0\otimes\widetilde{\chi_F},\tag{5.2.14}$$

where $\rho_0=\rho_0(X,\chi_0)$ is a minimal conductor Heisenberg representation of dimension m, and $\widetilde{\chi_F}:W_F\to\mathbb{C}^\times$ corresponds to $\chi_F:F^\times\to\mathbb{C}^\times$ by class field theory.
Then we have $X_\rho=X_\eta$ for $\eta:U_F/U_F^1\to\mathbb{C}^\times$, $\#\eta=m$ and:

$$\rho_0=\mathrm{Ind_{E/F}}(\chi_{\mathrm{E},0})\qquad \rho=\mathrm{Ind_{E/F}}(\chi_{\mathrm{E}}),$$

where E/F is a cyclic totally ramified extension of degree m.
Again by the given condition, $\dim(\rho)=$ m prime to p, and the Artin conductor $a_F(\rho)=mh$ where $h\geqslant 2$, then from Lemma 5.1.23, we have $a(\chi_E)=mh-d_{E/F}=mh-m+1=1+m(h-1)$.

Because of (5.2.14) we may assume now that

$$\chi_E=\chi_{E,0}\cdot(\chi_F\circ N_{E/F}),\quad a(\chi_{E,0})=1,\quad a(\chi_E)=a(\chi_F\circ N_{E/F})=1+m(h-1).\tag{5.2.15}$$

From the first and second of the equalities (5.2.15) we deduce

$$\chi_E|_{U_E^1}=(\chi_F\circ N_{E/F})|_{U_E^1},\quad N_{E/F}(U_E^1)=U_F^1,\tag{5.2.16}$$

where the second equality holds because E/F is totally ramified, and it implies that conversely $\chi_E|_{U_E^1}$ determines $\chi_F|_{U_F^1}$.
Step-2: Now for $d\geqslant 1$ we put:

$$A_E:=U_E^d/U_E^{d+1},$$

which we consider as a Gal(E/F)-module. We also know[8] that $A_E/I_{E/F}A_E\cong A_E^{\mathrm{Gal(E/F)}}$, where $I_{E/F}A_E$ is the augmentation with respect to the extension E/F.

[8] Here Gal(E/F) is a cyclic group, and A_E is finite. Then the Herbrand quotient of A_E is 1 (cf. [27], p. 134, Proposition 8). This implies (cf. [27], §4 of Chapter VIII)

$$|H^0(\mathrm{Gal(E/F)},\mathrm{A_E})=\mathrm{A_E^{Gal(E/F)}}|=|\mathrm{H^1(Gal(E/F),A_E)=A_E/I_{E/F}A_E}|.$$

We also have the following exact sequence

$$1\to A_E^{\mathrm{Gal(E/F)}}\to A_E\xrightarrow{\sigma-1}I_{E/F}A_E\to 1.$$

Thus $A_E^{\mathrm{Gal(E/F)}}\cong A_E/I_{E/F}A_E$, and $|A_E^{\mathrm{Gal(E/F)}}|\cdot|I_{E/F}A_E|=|A_E|=q_E$.

We also know that for any finite extension E/F, we have

$$U_E^d \cap F^\times = \begin{cases} U_F^{\frac{d}{e_{E/F}}} & \text{if } e_{E/F} \text{ divides } d \\ U_F^{[\frac{d}{e_{E/F}}]+1} & \text{if } e_{E/F} \text{ does not divide } d. \end{cases} \tag{5.2.17}$$

Again we also have

$$A_E^{\mathrm{Gal(E/F)}} = U_E^n/U_E^{n+1} \cap F^\times = U_E^d \cap F^\times/U_E^{d+1} \cap F^\times.$$

Step-3: If $1 + m(h-1) = 2d+1$, then $\frac{d}{m} = \frac{h-1}{2}$. Let $h' := [\frac{h}{2}]$. If $A_E = U_E^d/U_E^{d+1}$, and h is odd, then we have:

$$U_E^d \cap F^\times/U_E^{d+1} \cap F^\times = U_F^{\frac{h-1}{2}}/U_F^{\frac{h-1}{2}+1} = U_F^{h'}/U_F^{h'+1},$$

and if h is even, hence 2 does not divide $h-1$, then we can write

$$U_E^d \cap F^\times/U_E^{d+1} \cap F^\times = U_F^{[\frac{h-1}{2}]+1}/U_F^{[\frac{h-1}{2}]+1} \cong \{1\}.$$

Since $A_E^{\mathrm{Gal(E/F)}} \cong A_E/I_{E/F}A_E$, we can uniquely write any element $x \in U_E^d/U_E^{d+1}$ as $x = yz$ where $y \in A_E^{\mathrm{Gal(E/F)}}$ and $z \in I_{E/F}A_E$. We also know that $U_E^d/U_E^{d+1} \cong k_E$, hence $|A_E| = |A_E^{\mathrm{Gal(E/F)}}|\cdot|I_{E/F}A_E| = q_E = q_F$. We also observe that when h is even, we have $A_E^{\mathrm{Gal(E/F)}} \cong \{1\}$, hence $|A_E| = |I_{E/F}A_E| = q_F$. And when h is odd, we have $A_E^{\mathrm{Gal(E/F)}} = U_F^{h'}/U_F^{h'+1}$, and hence $|A_E^{\mathrm{Gal(E/F)}}| = q_F$. So this implies $|I_{E/F}A_E| = 1$.

K/E_1 is totally ramified. Then we have the following commutative diagram

$$\begin{array}{ccc} U_K^d \cap E_1^\times/U_K^{d+1} \cap E_1^\times = A_K^{\mathrm{Gal(K/E_1)}} & \xrightarrow{\sim} & A_K/I_{K/E_1}A_K \\ \Big\downarrow N_{E_1/F} & & \Big\downarrow N_{K/E} \\ U_E^d \cap F^\times/U_E^{d+1} \cap F^\times = A_E^{\mathrm{Gal(E/F)}} & \xrightarrow{\sim} & A_E/I_{E/F}A_E \end{array} \tag{5.2.18}$$

Since $\chi_E|_{I_F\mathcal{N}_{K/E}} \equiv 1$, $\chi_K|_{K_{E_1}^\times} \equiv 1$, and $I_{K/E_1}A_K \subset K_{E_1}^\times$, the characters χ_K and χ_E are trivial on $I_{K/E_1}A_K$ and $I_{E/F}A_E$ respectively (cf. Lemma 5.1.4, Remark (5.1.5)). Therefore $\chi_E \circ N_{K/E} = \chi_K$ induces $\chi_E|_F \circ N_{E_1/F} = \chi_K|_{E_1}$, hence $\chi_E|_F = \chi_K|_{E_1} \circ N_{E_1/F}^{-1}$.

Now set:

$$S(\psi, c) := \sum_{x \in A_E} \chi_E^{-1}(x)(c^{-1}\psi)(\mathrm{Tr_{E/F}(x)}).$$

Then we can write

$$\begin{aligned}
S(\psi,c) &= \sum_{y\in A_E^{\mathrm{Gal(E/F)}},\, z\in I_{E/F}A_E} \chi_E^{-1}(yz)\cdot(c^{-1}\psi)(\mathrm{Tr_{E/F}(yz)})\\
&= \sum_{y\in A_E^{\mathrm{Gal(E/F)}}}\ \sum_{z\in I_{E/F}A_E} \chi_E^{-1}(yz)(c^{-1}\psi)(\mathrm{Tr_{E/F}(yz)})\\
&= |I_{E/F}A_E|\cdot \sum_{y\in A_E^{\mathrm{Gal(E/F)}}} \chi_E^{-1}(y)(c^{-1}\psi)(my)\\
&= |I_{E/F}A_E|\cdot \sum_{y\in A_E^{\mathrm{Gal(E/F)}}} (\chi_K|_{E_1}\circ N_{E_1/F}^{-1})^{-1}(y)(c^{-1}\psi)(my)\\
&= \begin{cases} \sum_{y\in U_F^{h'}/U_F^{h'+1}}(\chi_K|_{E_1}\circ N_{E_1/F}^{-1})^{-1}(y)(c^{-1}\psi)(my) & \text{when } h \text{ is odd}\\ q_F\cdot(c^{-1}\psi)(m)=q_F\cdot\psi(mc^{-1}) & \text{when } h \text{ is even,}\end{cases}
\end{aligned}$$

since $\chi_E(yz)=\chi_E(y)$ and $\mathrm{Tr_{E/F}(yz)=yTr_{E/F}(z)=ym}$.

Step-4: Again, we have $\rho=\mathrm{Ind_{E/F}(\chi_E)}$. Then

$$W(\rho,\psi)=W(\mathrm{Ind_{E/F}(\chi_E)},\psi)=\lambda_{\mathrm{E/F}}(\psi)\cdot\mathrm{W}(\chi_{\mathrm{E}},\psi\circ\mathrm{Tr_{E/F}}).$$

Case-1: Suppose that m is odd:

(1) When $a(\chi_E)=1+m(h-1)=2d$: In this situation, h must be even and we take $h=2h'$, hence $d=mh'-\frac{m-1}{2}$. Since $m(h'-1)<d\leqslant mh'$, we have $P_E^d\cap F=P_F^{h'}$. Now we choose $c\in F^\times$ such that

$$\chi_F(1+y)=\psi(c^{-1}y),\quad \text{for all } y\in P_F^{h-h'}/P_F^h, \tag{5.2.19}$$

hence $\nu_F(c)=a(\chi_F)+n(\psi)=h+n(\psi)$. Now if we take an element $y_E\in P_E^{a(\chi_E)-d}=P_E^d$, then $\mathrm{Tr_{E/F}(y_E)\in P_F^{h'}=P_F^{h-h'}}$ because $m(h'-1)<d\leqslant mh'=m(h-h')$. Since E/F is cyclic, from Proposition 1.1 on p. 68 of [21], we have:

$$N_{E/F}(1+y_E)=1+\mathrm{Tr_{E/F}(y_E)}+\mathrm{N_{E/F}(y_E)}+\mathrm{Tr_{E/F}}(\delta),$$

where $\nu_E(\delta)\geqslant 2d=a(\chi_E)$. Then for all $y_E\in P_E^{a(\chi_E)-d}/P_E^{a(\chi_E)}$, we can write

$$\begin{aligned}
\chi_E(1+y_E)&=\chi_F\circ N_{E/F}(1+y_E)=\chi_F(1+\mathrm{Tr_{E/F}(y_E)})\\
&=\psi(c^{-1}\mathrm{Tr_{E/F}(y_E)})=(\mathrm{c}^{-1}\psi_{\mathrm{E}})(\mathrm{y_E}),
\end{aligned}\tag{5.2.20}$$

because $N_{E/F}(y_E)+\mathrm{Tr_{E/F}}(\delta)\in\mathrm{P_F^h}$. This verifies that our choice of c is right for applying Lamprecht-Tate formula for $W(\chi_E,\psi_E)$.

Now we apply Lamprecht-Tate formula (cf. Theorem 6.1.1 and its Corollary) and we obtain:

$$W(\chi_E,\psi_E)=\chi_E(c)\cdot(c^{-1}\psi_E)(1)=\Delta_{E/F}(c)\det(\rho)(c)\psi(mc^{-1}).$$

Therefore

$$\begin{aligned}W(\rho,\psi)&=\lambda_{E/F}(\psi)\cdot W(\chi_E,\psi_E)\\&=\lambda_{E/F}(\psi)\cdot\Delta_{E/F}(c)\det(\rho)(c)\psi(mc^{-1})\\&=R(\psi,c)\cdot\det(\rho)(c)\cdot\psi(mc^{-1})\\&=\det(\rho)(c)\cdot\psi(mc^{-1}),\end{aligned}$$

where $R(\psi,c)=\lambda_{E/F}(\psi)\Delta_{E/F}(c)=\lambda_{E/F}(c\psi)=1$ because E/F is an odd degree Galois extension.

(2). When $a(\chi_E)=1+m(h-1)=2d+1$**:** Since m is odd, here h must be odd. Let $h':=[\frac{h}{2}]$. Then from Step-3 we have $A_E^{\mathrm{Gal(E/F)}}=U_F^{h'}/U_F^{h'+1}$. Now if we choose $c\in F^\times$ such that

$$\chi_F(1+y)=\psi(c^{-1}y),\qquad\text{for all } y\in P_F^{h-h'}/P_F^h.$$

Then this c also satisfies the following relation

$$\chi_E(1+y_E)=\psi_E(c^{-1}y_E),\qquad\text{for all } y_E\in P_E^{a(\chi_E)-d}/P_E^{a(\chi_E)},$$

because $d=\frac{m(h-1)}{2}$, and hence $m(h'-1)<d\leqslant mh'$. Then by Lamprecht-Tate (cf. Corollary 6.1.2(2)) formula we have

$$\begin{aligned}W(\chi_E,\psi_E)&=\chi_E(c)\psi_E(c^{-1})q_E^{-\frac12}\sum_{x\in P_E^d/P_E^{d+1}}\chi_E^{-1}(1+x)\cdot(c^{-1}\psi_E)(x)\\&=\chi_E(c)\cdot q_F^{-\frac12}\sum_{x\in U_E^d/U_E^{d+1}}\chi_E^{-1}(x)\cdot(c^{-1}\psi)(\mathrm{Tr_{E/F}(x)})\\&=\Delta_{E/F}(c)\det(\rho)(c)q_F^{-\frac12}\sum_{y\in U_F^{h'}/U_F^{h'+1}}(\chi_K|_{E_1}\circ N_{E_1/F}^{-1})^{-1}(y)(c^{-1}\psi)(my),\end{aligned}$$

because h is odd, and we use Step-3. Thus we obtain

$$\begin{aligned}W(\rho,\psi)&=W(\mathrm{Ind_{E/F}}(\chi_{\mathrm{E}}),\psi)=\lambda_{\mathrm{E/F}}(\psi)\cdot\mathrm{W}(\chi_{\mathrm{E}},\psi_{\mathrm{E}})\\&=\lambda_{E/F}(\psi)\cdot\Delta_{E/F}(c)\det(\rho)(c)q_F^{-\frac12}\sum_{y\in U_F^{h'}/U_F^{h'+1}}(\chi_K|_{E_1}\circ N_{E_1/F}^{-1})^{-1}(y)(c^{-1}\psi)(my).\\&=R(\psi,c)\cdot\det(\rho)(c)q_F^{-\frac12}\sum_{y\in U_F^{h'}/U_F^{h'+1}}(\chi_K|_{E_1}\circ N_{E_1/F}^{-1})^{-1}(y)(c^{-1}\psi)(my).\\&=\det(\rho)(c)q_F^{-\frac12}\sum_{y\in U_F^{h'}/U_F^{h'+1}}(\chi_K|_{E_1}\circ N_{E_1/F}^{-1})^{-1}(y)(c^{-1}\psi)(my),\end{aligned}$$

because m is odd, hence $R(\psi,c)=\lambda_{E/F}(c\psi)=1$.

Case-2: Suppose that m **is even.** If m is even, then $1+m(h-1)=2d+1$ is always an odd number and $d=\frac{m(h-1)}{2}$. But here h could be any number $\geqslant 2$, i.e., h is not fixed, and we put $h':=[\frac{h}{2}]$. This implies $m(h'-1)<d\leqslant mh'$ and $P_E^d\cap F=P_F^{h'}$. Now we take

$c \in F^\times$ such that (5.2.19) holds, and this again satisfies equation (5.2.20). Therefore we can use Lamprecht-Tate formula and we have two cases:

1. When h is odd, we are in the same situation of Case-1(2), and we have
$$W(\rho,\psi) = R(\psi,c)\cdot \det(\rho)(c)\cdot H(\psi,c).$$

2. When h is even, from Step-3 we know that $A_E^{\mathrm{Gal(E/F)}} \cong \{1\}$ and
$$\sum_{x\in A_E} \chi_E^{-1}(x)(c^{-1}\psi)(\mathrm{Tr_{E/F}(x)}) = q_F\cdot\psi(mc^{-1}).$$
Therefore in this situation we have
$$\begin{aligned}W(\rho,\psi) &= R(\psi,c)\cdot\det(\rho)(c)q_F^{-\frac{1}{2}}\cdot\sum_{x\in A_E}\chi_E^{-1}(x)(c^{-1}\psi)(\mathrm{Tr_{E/F}(x)})\\ &= R(\psi,c)\cdot\det(\rho)(c)q_F^{-\frac{1}{2}}\cdot q_F\cdot\psi(mc^{-1})\\ &= R(\psi,c)\cdot\det(\rho)(c)\cdot q_F^{\frac{1}{2}}\cdot\psi(mc^{-1}).\end{aligned}$$

Furthermore, in the proof of Theorem 5.2.7, we observe that $R(\psi,c)$ does not depend on E. Hence our above computations are invariant.

This completes the proof.

□

By using following lemma, without using λ-function we also can give invariant formula for $W(\rho)$, when $\dim(\rho)$ is prime to p, for sufficiently large conductor character χ_F.

Lemma 5.2.10 (Deligne-Henniart, [6], p. 190, Proposition 29.4(4)). *Let F be a non-archimedean local field and ψ be a nontrivial additive character of F. Let ρ be a finite dimensional representation of G_F. There is a sufficiently large integer m_ρ such that if χ_F is a character of $F^\times$ of conductor $a(\chi_F)\geqslant m_\rho$, then*
$$W(\rho\otimes\chi_F,\psi) = W(\chi_F,\psi)^{\dim(\rho)}\cdot\det(\rho)(c), \tag{5.2.21}$$
for any $c := c(\chi_F,\psi)\in F^\times$ such that $\chi_F(1+x) = \psi(c^{-1}x)$, $x\in P_F^{[\frac{a(\chi_F)}{2}]+1}$.

By using the above Lemma 5.2.10, we obtain the following theorem.

Theorem 5.2.11. *Let $\rho = \rho_0\otimes\widetilde{\chi_F}$ be a Heisenberg representation of G_F of dimension d with $gcd(d,p) = 1$, where $\rho_0 = \rho_0(X_\eta,\chi_0)$ is a minimal conductor Heisenberg representation. If $a(\chi_F)\geqslant m_\rho\geqslant 2$, a sufficiently large number which depends on ρ, then we have*
$$W(\rho,\psi) = W(\rho_0\otimes\widetilde{\chi_F}) = W(\chi_F,\psi)^d\cdot\det(\rho_0)(c), \tag{5.2.22}$$
where ψ is a nontrivial additive character of F, and $c := c(\chi_F,\psi)\in F^\times$, satisfies

$$\chi_F(1+x) = \psi(c^{-1}x) \text{ for all } x \in P_F^{[\frac{a(\chi_F)}{2}]+1}.$$

Proof. From Corollary 5.1.12 we know that all Heisenberg representation ρ of G_F of dimension prime to p are precisely given as $\rho = \rho(X_\eta, \chi)$ for characters η of U_F/U_F^1. Then from Remark 5.1.24 we have here $a_K(\chi_0) = 1$. This implies that we always can choose a character χ_0 of $K^\times$ with $a(\chi_0) = 1$ such that all other χ_K are given as

$$\chi_K = (\chi_F \circ N_{K/F}) \cdot \chi_0,$$

for arbitrary characters χ_F of $F^\times$. Therefore the whole set of Heisenberg (U-isotopic) representations of G_F of dimension prime to p is:

$$\rho_0 = \rho_0(G_K, \chi_0) \text{ and } \rho = \rho(G_K, \chi_K), \text{ where } \chi_K = (\chi_F \circ N_{K/F}) \cdot \chi_0, \text{ and } \chi_F \in \widehat{F^\times}.$$

We also know that there are d^2 characters of $F^\times/F^{\times d}$ such that $\rho_0 \otimes \widetilde{\chi} = \rho_0$ (cf. [11], p. 303, Proposition 1.4). So we always have:

$$\rho = \rho_0 \otimes \widetilde{\chi_F} = \rho_0 \otimes \widetilde{\chi\chi_F},$$

where $\chi \in \widehat{F^\times/F^{\times d}}$, and $\widetilde{\chi_F} : W_F \to \mathbb{C}^\times$ corresponds to χ_F by class field theory.

Let ζ be a $(q_F - 1)$-st root of unity. Since U_F^1 is a pro-p-group and $gcd(p, d) = 1$, we have

$$F^\times/F^{\times d} = <\pi_F> \times <\zeta> \times U_F^1 / <\pi_F^d> \times <\zeta>^d \times U_F^1 \cong \mathbb{Z}_d \times \mathbb{Z}_d, \tag{5.2.23}$$

that is, a direct product of two cyclic group of same order. Hence $F^\times/F^{\times d} \cong \widehat{F^\times/F^{\times d}}$. Since $F^{\times d} = <\pi_F^d> \times <\zeta>^d \times U_F^1$, and $F^\times/F^{\times d} \cong \mathbb{Z}_d \times \mathbb{Z}_d$, we have $a(\chi) \leqslant 1$ and $\#\chi$ is a divisor of d for all $\chi \in \widehat{F^\times/F^{\times d}}$. Now if we take a character χ_F of $F^\times$ conductor $\geqslant m_\rho \geqslant 2$, hence $a(\chi_F) \geqslant 2a(\chi)$ for all $\chi \in \widehat{F^\times/F^{\times d}}$. Then by using Deligne's formula (2.3.17) we have

$$W(\chi_F\chi, \psi)^d = \chi(c)^d \cdot W(\chi_F, \psi)^d = W(\chi_F, \psi)^d,$$

where $c \in F^\times$ with $\nu_F(c) = a(\chi_F) + n(\psi)$, satisfies

$$\chi_F(1+x) = \psi(c^{-1}x), \quad \text{for all } x \in F^\times \text{ with } 2\nu_F(x) \geqslant a(\chi).$$

Finally, by using Lemma 5.2.10 we can write

$$\begin{aligned} W(\rho, \psi) &= W(\rho_0 \otimes \widetilde{\chi_F\chi}, \psi) = W(\chi_F\chi, \psi)^{\dim(\rho_0)} \cdot \det(\rho_0)(c(\chi_F, \psi)) \\ &= W(\chi_F, \psi)^d \cdot \det(\rho_0)(c). \end{aligned}$$

□

5.3 Applications of Tate's root-of-unity criterion

Let K/F be a finite Galois extension of the non-archimedean local field F, and $\rho : \mathrm{Gal}(K/F) \to \mathrm{Aut}_{\mathbb{C}}(V)$ a representation of $\mathrm{Gal}(K/F)$ on a complex vector space V. Let $P(K/F)$ denote the first **wild** ramification group of K/F. Let V^P be the subspace of all elements of V fixed by $\rho(P(K/F))$. Then ρ induces a representation:

$$\rho^P : \mathrm{Gal}(K/F)/P(K/F) \to \mathrm{Aut}_{\mathbb{C}}(V^P).$$

Let $\overline{F}$ be an algebraic closure of the local field F, and $G_F = \mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F})$ be the absolute Galois group for $\overline{F}/F$. Let ρ be a representation of G_F.
Then by Tate, $W(\rho)/W(\rho^P)$ is a root of a unity (cf. [32], p. 112, Corollary 4).
Now let ρ be an irreducible representation G_F, then either $\rho^P = \rho$, in which case $\frac{W(\rho)}{W(\rho^P)} = 1$, or else $\rho^P = 0$, in this case from Tate's result we can say $W(\rho)$ is a root of unity. Equivalently: If $W(\rho)$ is not a root of unity then $\rho^P \neq 0$, hence $\rho^P = \rho$ because ρ is irreducible. This means that all vectors $v \in V$ of the representation space are fixed under P action on V.
In other words, if we consider ρ as a homomorphism $\rho : G_F \to \mathrm{Aut}_{\mathbb{C}}(\mathrm{V})$ then the elements from P are mapped to the identity, hence

$$\rho^P = \rho \text{ means } P \subset \mathrm{Ker}(\rho).$$

Therefore we can state the following lemma.

Lemma 5.3.1. *If ρ is an irreducible representation of G_F, such that the subgroup $P \subset G_F$, of wild ramification does* **not trivially** *act on the representation space V (this gives $\rho^P \neq \rho$, i.e., $\rho^P = 0$), then $W(\rho)$ is a root of unity.*

Before going to our next results we need to recall some facts from class field theory. Let F be a non-archimedean local field. Let F^{ab} be the maximal abelian extension of F and F_{nr} be the maximal unramified extension of F. Then by local class field theory there is a unique homomorphism

$$\theta_F : F^\times \to \mathrm{Gal}(\mathrm{F^{ab}}/\mathrm{F})$$

having certain properties (cf. [31], p. 20, Theorem 1.1). This local reciprocity map θ_F is continuous and injective with dense image. From class field theory we have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & v_F & & & \\
0 & \to & U_F & \to & F^\times & \to & \mathbb{Z} & \to & 0 \\
 & & \downarrow \theta_F & & \downarrow \theta_F & & \downarrow \mathrm{id} & & \\
0 & \to & I_F & \to & \mathrm{Gal}(\mathrm{F^{ab}}/\mathrm{F}) & \to & \widehat{\mathbb{Z}} & \to & 0,
\end{array}$$

where $I_F := \mathrm{Gal}(\mathrm{F^{ab}}/\mathrm{F_{nr}})$ is the inertia subgroup of $\mathrm{Gal}(\mathrm{F^{ab}}/\mathrm{F})$, and $\mathrm{Gal}(\mathrm{F_{nr}}/\mathrm{F})$ is identified with $\widehat{\mathbb{Z}}$ (cf. [29], p. 144). We also know that $\theta_F : U_F \to I_F$ is an isomorphism. Moreover, the descending chain

$$U_F \supset U_F^1 \supset U_F^2 \cdots$$

is mapped isomorphically by θ_F to the descending chain of ramification subgroups of $\mathrm{Gal(F^{ab}/F)}$ in the upper numbering.

Now let I be the inertia subgroup of G_F. Let P be the wild ramification subgroup of G_F. Then we have $G_F \supset I \supset P$. Parallel with this we have $F^\times \supset U_F \supset U_F^1$. Then we have

$$1 \to I/P \cdot [G_F, G_F] \to G_F/P \cdot [G_F, G_F] \to G_F/I \to 1, \tag{5.3.1}$$

and parallel

$$1 \to U_F/U_F^1 \to F^\times/U_F^1 \to F^\times/U_F \to 1. \tag{5.3.2}$$

Now by class field theory the left terms of sequences (5.3.1) and (5.3.2) are isomorphic, but for the right terms we have G_F/I is isomorphic to the total completion of $\mathbb{Z}$ (because here G_F/I is profinite group, hence compact). We also have $F^\times/U_F =< \pi_F > \times U_F/U_F \cong \mathbb{Z}$. Therefore sequence (5.3.2) is dense in (5.3.1) because $\mathbb{Z}$ is dense in the total completion $\widehat{\mathbb{Z}}$. But $\mathbb{Z}$ and $\widehat{\mathbb{Z}}$ have the same finite factor groups. **As a consequence $F^\times/U_F^1$ is also dense in $G_F/P \cdot [G_F, G_F]$.**

Let ρ be a Heisenberg representation of the absolute Galois group G_F. In the following proposition we show that if $W(\rho)$ is not a root of unity, then $\dim(\rho)|(\mathrm{q_F}-1)$, and $a_F(\rho)$ is not minimal.

Proposition 5.3.2. *Let $F/\mathbb{Q}_p$ be a local field and let $q_F = p^s$ be the order of its finite residue field. If $\rho = (Z_\rho, \chi_\rho) = \rho(X_\rho, \chi_K)$ is a Heisenberg representation of the absolute Galois group G_F such that $W(\rho)$ is not a root of unity, then $\dim(\rho)|(q_F-1)$ and $a_F(\rho)$ is not minimal.*

Proof. Let P denote the wild ramification subgroup of G_F. By Tate's root-of-unity criterion, we know that $\gamma := \frac{W(\rho)}{W(\rho^P)}$ is a root of unity. If $W(\rho)$ is not a root of unity, then $\rho = \rho^P$, otherwise $W(\rho)$ must be a root of unity. Again $\rho^P = \rho$ implies $P \subset \mathrm{Ker}(\rho) \subset \mathrm{Z}_\rho \subset \mathrm{G_F}$. So G_F/Z_ρ is a quotient of G_F/P, hence $F^\times/U_F^1$.

Moreover, from the dimension formula (5.1.6), we have

$$\dim(\rho) = \sqrt{[\mathrm{G_F} : \mathrm{Z}_\rho]} = \sqrt{[\mathrm{K} : \mathrm{F}]} = \sqrt{[\mathrm{F}^\times : \mathcal{N}_{\mathrm{K/F}}]},$$

where $Z_\rho = G_K$ and $\mathrm{Rad(X)} = \mathcal{N}_{\mathrm{K/F}}$, hence $F^\times/N$ is a quotient group of $F^\times/U_F^1$. Therefore the alternating character X_ρ induces an alternating character X on $F^\times/U_F^1$. We also know that $F^\times =< \pi_F > \times < \zeta > \times U_F^1$, where ζ is a root of unity of order $q_F - 1$. This implies $F^\times/U_F^1 =< \pi_F > \times < \zeta >$. So each element $x \in F^\times/U_F^1$ can be written as $x = \pi_F^a \cdot \zeta^b$, where $a, b \in \mathbb{Z}$. We now take $x_1 = \pi_F^{a_1}\zeta^{b_1}, x_2 = \pi_F^{a_2}\zeta^{b_2} \in F^\times/U_F^1$, where $a_i, b_i \in \mathbb{Z}(i = 1, 2)$, then

$$\begin{aligned} X(x_1, x_2) &= X(\pi_F^{a_1}\zeta^{b_1},\ \pi_F^{a_2}\zeta^{b_2}) \\ &= X(\pi_F^{a_1}, \zeta^{b_2}) \cdot X(\zeta^{b_1}, \pi_F^{a_2}) \\ &= \chi_\rho([\pi_F^{a_1}, \zeta^{b_2}]) \cdot \chi_\rho([\zeta^{b_1}, \pi_F^{a_2}]). \end{aligned}$$

But this implies $X^{q_F-1} \equiv 1$ because $\zeta^{q_F-1} = 1$, which means that X is actually an alternating character on $F^\times/(F^{\times(q_F-1)}U_F^1)$, and therefore G_F/G_K is actually a quotient of $F^\times/(F^{\times(q_F-1)}U_F^1)$. We also know that U_F^1 is a pro-p-group and therefore

$$U_F^1 = (U_F^1)^{q_F-1} \subset F^\times.$$

Thus the cardinality of $F^\times/(F^{\times(q_F-1)}U_F^1)$ is $(q_F-1)^2$ because

$$F^\times/(F^{\times(q_F-1)}U_F^1) \cong \mathbb{Z}/(q_F-1)\mathbb{Z}\times <\zeta> \cong \mathbb{Z}_{q_F-1}\times\mathbb{Z}_{q_F-1}.$$

Therefore $\dim(\rho)$ divides $q_F - 1$.

Since $\dim(\rho)|\mathrm{q_F}-1$, from Lemma 5.1.27 the alternating character X_ρ is U-isotropic and $X_\rho = X_\eta$ for a character $\eta : U_F/U_F^1 \to \mathbb{C}^\times$. Since $\rho = \rho(X_\eta, \chi_K)$ is U-isotropic, from Proposition 5.1.21, $a_F(\rho)$ is a multiple of $\dim(\rho)$. Moreover, by the given condition, $W(\rho)$ is not a root of unity, hence $a_F(\rho)$ is not minimal, otherwise if $a_F(\rho)$ is minimal, then from Lemma 5.2.5 $W(\rho)$ is a root of unity.

□

Chapter 6

Appendix

6.1 Lamprecht-Tate formula for $W(\chi)$

Theorem 6.1.1 (Lamprecht-Tate formula). *Let F be a non-archimedean local field. Let χ be a character of $F^\times$ of exponential Artin-conductor $a(\chi) = a_F(\chi)$ and let m be a natural number such that $2m \leqslant a(\chi)$. Let ψ_F be the canonical additive character of F. Then there exists $c \in F^\times$, $\nu_F(c) = a(\chi) + d_{F/\mathbb{Q}_p}$ such that*

$$\chi(1+y) = \psi_F(c^{-1}y) \qquad \text{for all } y \in P_F^{a(\chi)-m}, \tag{6.1.1}$$

and this will imply:

$$W(\chi) = W(\chi, c) = \chi(c) \cdot q_F^{-\frac{(a(\chi)-2m)}{2}} \sum_{x \in (1+P_F^m)/(1+P_F^{a(\chi)-m})} \chi^{-1}(x)\psi_F(c^{-1}x). \tag{6.1.2}$$

Remark: *Note that the assumption (6.1.1) is obviously fulfilled for $m = 0$ because then both sides are $= 1$, and the resulting formula for $m = 0$ is the original formula (2.3.4) for abelian local constant $W(\chi)$.*

Proof. We have seen already that for $m = 0$ everything is correct. In general, the assumption $2m \leqslant a(\chi)$ implies $2(a(\chi) - m) \geqslant a(\chi)$ and therefore

$$\chi(1+y)\chi(1+y') = \chi(1+y+y')$$

for $y, y' \in P_F^{a(\chi)-m}$. That is, $y \mapsto \chi(1+y)$ is a character of the additive group $P_F^{a(\chi)-m}$. This character extends to a character of F^+ and, by local additive duality, there is some $c \in F^\times$ such that

$$\chi(1+y) = \psi_F(c^{-1}y) = (c^{-1}\psi_F)(y), \quad \text{for all } y \in P_F^{a(\chi)-m}.$$

Now comparing the conductors of both sides we must have:

$$a(\chi) = -n(c^{-1}\psi_F) = \nu_F(c) - n(\psi_F),$$

hence $\nu_F(c) = a(\chi) + n(\psi_F) = a(\chi) + d_{F/\mathbb{Q}_p}$ is the right assumption for our formula.
Now we assume $m \geqslant 1$ (the case $m = 0$ we have checked already) and consider the filtration

$$O_F^\times \supseteq 1 + P_F^{a(\chi)-m} \supseteq 1 + P_F^{a(\chi)}.$$

Then we may represent $x \in O_F^\times/(1+P_F^{a(\chi)})$ as $x = z(1+y)$, where $y \in P_F^{a(\chi)-m}$ and z runs over the system of representatives for $O_F^\times/(1+P_F^{a(\chi)-m})$. Now computing $W(\chi)$ we have to consider the sum

$$\sum_{x \in O_F^\times/(1+P_F^{a(\chi)})} \chi^{-1}(x)\psi_F(c^{-1}x) = \sum_{z \in O_F^\times/(1+P_F^{a(\chi)-m})} \sum_{y \in P_F^{a(\chi)-m}/P_F^{a(\chi)}} \chi^{-1}(z(1+y))\psi_F(c^{-1}z(1+y)). \tag{6.1.3}$$

Now using (6.1.1) we obtain

$$\chi^{-1}(z(1+y)) = \chi^{-1}(z)\chi^{-1}(1+y) = \chi^{-1}(z)\chi(1-y) = \chi^{-1}(z)\psi_F(-c^{-1}y)$$

and therefore our double sum (6.1.3) rewrites as

$$\sum_{z \in O_F^\times/(1+P_F^{a(\chi)-m})} \chi^{-1}(z)\psi_F(c^{-1}z) \cdot \left(\sum_{y \in P_F^{a(\chi)-m}/P_F^{a(\chi)}} \psi_F(c^{-1}y(z-1)) \right).$$

But the inner sum is the sum on the additive group $P_F^{a(\chi)-m}/P_F^{a(\chi)}$ and $(c^{-1}(z-1))\psi_F$ is a character of that group. Hence this sum is equal to $[P_F^{a(\chi)-m} : P_F^{a(\chi)}] = q_F^m$ if the character is $\equiv 1$ and otherwise the sum will be zero. But:

$$n(c^{-1}(z-1)\psi_F) = \nu_F(c^{-1}(z-1)) + n(\psi_F) = -a(\chi) + \nu_F(z-1).$$

So the character $(c^{-1}(z-1))\psi_F$ is trivial on $P_F^{a(\chi)-\nu_F(z-1)}$, and therefore it will be $\equiv 1$ on $P_F^{a(\chi)-m}$ if and only if $\nu_F(z-1) \geqslant m$, i.e., $z = 1 + y' \in 1 + P_F^m$. Therefore our sum (6.1.3) rewrites as

$$\sum_{x \in O_F^\times/(1+P_F^{a(\chi)})} \chi^{-1}(x)\psi_F(c^{-1}x) = q_F^m \sum_{z \in (1+P_F^m)/(1+P_F^{a(\chi)-m})} \chi^{-1}(z)\psi_F(c^{-1}z). \tag{6.1.4}$$

And substituting this result into our original formula (2.3.4) we get

$$\begin{aligned} W(\chi) &= \chi(c) q_F^{-\frac{a(\chi)}{2}} \sum_{x \in O_F^\times/(1+P_F^{a(\chi)})} \chi^{-1}(x)\psi_F(c^{-1}x) \\ &= \chi(c) \cdot q_F^{-\frac{(a(\chi)-2m)}{2}} \sum_{x \in (1+P_F^m)/(1+P_F^{a(\chi)-m})} \chi^{-1}(x)\psi_F(c^{-1}x). \end{aligned}$$

□

Corollary 6.1.2. *Let χ be a character of $F^\times$. Let ψ be a nontrivial additive character of F.*

1. *When $a(\chi) = 2d\,(d \geqslant 1)$, we have*

$$W(\chi) = \chi(c)\psi(c^{-1}).$$

2. *When $a(\chi_\rho) = 2d+1\,(d \geqslant 1)$, we have*

$$W(\chi) = \chi(c)\psi(c^{-1}) \cdot q_F^{-\frac{1}{2}} \sum_{x \in P_F^d/P_F^{d+1}} \chi^{-1}(1+x)\psi(c^{-1}x).$$

3. **Deligne's twisting formula (2.3.17):** *If* $\alpha, \beta \in \widehat{F^\times}$ *with* $a(\alpha) \geqslant 2 \cdot a(\beta)$*, then*

$$W(\alpha\beta, \psi) = \beta(c) \cdot W(\alpha, \psi).$$

Here $c \in F^\times$ *with* F*-valuation* $\nu_F(c) = a(\chi) + n(\psi_F)$*, and in Case (1) and Case (2),* c *also satisfies*

$$\chi(1+x) = \psi(\tfrac{x}{c}) \text{ for all } x \in F^\times \text{ with } 2 \cdot \nu_F(x) \geqslant a(\chi).$$

And in case (3), we have $\alpha(1+x) = \psi(x/c)$ *for all* $\nu_F(x) \geqslant \frac{a(\alpha)}{2}$

Proof. From the above formula (6.1.2), the assertions are followed.
(1). When $a(\chi) = 2d$, where $d \geqslant 1$. In this case, we take $m = d$, and from equation (6.1.2) we obtain

$$W(\chi, \psi) = \chi(c) \cdot \sum_{x \in (1+P_F^d)/(1+P_F^d)} \chi^{-1}(x)\psi(c^{-1}x) = \chi(c) \cdot \psi(c^{-1}). \tag{6.1.5}$$

(2). When $a(\chi) = 2d+1$, where $d \geqslant 1$. In this case, we also take $m = d$, and then from equation (6.1.2) we obtain

$$\begin{aligned} W(\chi, \psi) &= \chi(c) \cdot q_F^{-\frac{1}{2}} \cdot \sum_{x \in (1+P_F^d)/(1+P_F^{d+1})} \chi^{-1}(x)\psi(c^{-1}x) \\ &= \chi(c) \cdot \psi(c^{-1}) \cdot q_F^{-\frac{1}{2}} \cdot \sum_{x \in P_F^d/P_F^{d+1}} \chi^{-1}(1+x)\psi(c^{-1}x). \end{aligned}$$

Here $c \in F^\times$ with $\nu_F(c) = a(\chi) + n(\psi)$, satisfies

$$\chi(1+x) = \psi(c^{-1}x) \text{ for all } x \in F^\times \text{ with } 2\nu_F(x) \geqslant a(\chi).$$

(3) Proof of Deligne's twisting formula (2.3.17): By the given condition, $a(\alpha) \geqslant 2a(\beta)$, hence we have $a(\alpha\beta) = a(\alpha)$. Now take $m = a(\beta)$, then from equation (6.1.2) we can write:

$$\begin{aligned} W(\alpha\beta, \psi) &= \alpha\beta(c) \cdot q_F^{-\frac{(a(\alpha)-2m)}{2}} \cdot \sum_{x \in (1+P_F^m)/(1+P_F^{a(\chi)-m})} (\alpha\beta)^{-1}(x)\psi(\frac{x}{c}) \\ &= \beta(c) \cdot \alpha(c) q_F^{-\frac{(a(\alpha)-2m)}{2}} \sum_{x \in (1+P_F^m)/(1+P_F^{a(\alpha)-m})} \alpha^{-1}(x)\psi(\frac{x}{c}) \\ &= \beta(c) \cdot W(\alpha, \psi), \end{aligned}$$

since $a(\beta) = m$, hence $\beta(x) = 1$ for all $x \in (1+P_F^m)/(1+P_F^{a(\alpha)-m})$.

□

Remark 6.1.3. Let μ_{p^∞} denote as the group of roots of unity of p-power order. Let F be a non-archimedean local field. Let ψ_F be an additive character of F. Since ψ_F is additive, its image lies in μ_{p^∞}. We also know that U_F^1 is a pro-p-group, hence $\chi(U_F^1) \subset \mu_{p^\infty}$.

1. When $a(\chi)$ is even, we have

$$W(\chi, \psi_F) = \chi(c) \cdot \psi_F(c^{-1}). \tag{6.1.6}$$

2. When $a(\chi) = 2d + 1$ is odd, we have

$$W(\chi, \psi_F) = \chi(c) \cdot \psi_F(c^{-1}) \cdot q_F^{-\frac{1}{2}} \sum_{x \in P_F^d / P_F^{d+1}} \chi^{-1}(1+x)\psi_F(x/c). \tag{6.1.7}$$

Now we give explicit formula for $W(\chi, \psi_F)$ modulo μ_{p^∞}. For this it is sufficient to know $c \in F^\times \mathrm{mod}\, 1 + \mathrm{P_F}$.

As for the correcting term it is $\equiv 1 \mod \mu_{p^\infty}$ in case $p = 2$. If $p \neq 2$ we compare the function $Q(x) := \chi_\rho^{-1}(1+x) \cdot \psi_F(x/c)$ and $H(x) := \psi_F(\frac{x^2}{2c})$ on P_F^d. It is easy to see (see the review of Henniart's artile [16] by E.-W. Zink) that $(1+x)(1+y) = (1+x+y)(1+\frac{xy}{1+x+y})$. Then we have

$$\frac{Q(x+y)}{Q(x)\cdot Q(y)} = \chi_\rho(1 + \frac{xy}{1+x+y}) = \psi_F(\frac{xy}{c}). \tag{6.1.8}$$

Similarly,

$$\frac{H(x+y)}{H(x)\cdot H(y)} = \psi_F(\frac{xy}{c}). \tag{6.1.9}$$

Then

$$\frac{Q(x+y)}{Q(x)\cdot Q(y)} = \frac{H(x+y)}{H(x)\cdot H(y)} = \psi_F(\frac{xy}{c}). \tag{6.1.10}$$

Therefore Q and H differ only by **an additive** character on P_F^d and then we can write:

1. $W(\chi, \psi_F) \equiv \chi(c) \mod \mu_{p^\infty}$ if $a(\chi)$ is even,

2. $W(\chi, \psi_F) \equiv \chi(c)G(c) \mod \mu_{p^\infty}$ if $a(\chi) = 2d+1$, where

$$G(c) := q_F^{-\frac{1}{2}} \cdot \sum_{x \in P_F^d / P_F^{d+1}} \psi_F(\frac{x^2}{2c})$$

depends only on $c \in F^\times \mod 1 + P_F$.

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