A Study on Location and Zero-Free Regions for Polynomials with Complex Coefficients

THESIS

IS SUBMITTED TO THE

UNIVERSITY OF HYDERABAD

FOR THE AWARD OF THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN MATHEMATICS

by

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2017

CERTIFICATE

Date:3^{rd} may 2017

This is to certify that the thesis entitled **A STYDY ON LOCATION AND ZERO FREE REGIONS FOR POLYNOMIALS WITH COMPLEX COEFFICIENTS** submitted by C.GANGADHAR bearing number 11MMPP03 in partial fulfillment of the requirements for award of Doctor of Philosophy in the School of Mathematics and Statistics is a bonafied work carried out by him under my supervision and guidance.

This thesis is free from plagiarism and has not been submitted previously in part or in full to this or any other university or institution for award of any degree or diploma.

Parts of this thesis have been:

A.Published in the following publications:

1.Advances in Dynamical Systems and Applications.(ISSN 0973-5321),Chapter-4

2.Inernational J. of Pure and Engg.Mathematics (ISSN 2348-3881), Chapter-4.

B.Presented in the following conference.

1.81^{st} Annual conference of THE INDIAN MATHEMATICAL SOCIETY.(National).

Further, the student has passed the following courses towards fullfillment of course work required for Ph.D.

Course code	Name	Credits	
MM 801	Advance Anlysis	05	Pass
MM 826	Algebra	05	Pass
MM 827	Goemetric Function Theory	05	Pass

Supervisor

Dean of the School.

Dedicated to

My Teachers and

My Friends.

ACKNOWLEDGEMENT

I would like to express my deepest gratitude to my supervisor Prof.G.L.Reddy, School of Mathematics and Statistics, University of Hyderabad for his patience and continuous help in my research work.He keeps happyness always in his face.I am very fortunate to work with him.

My sincere thanks to Prof.B.Sri Padmavati, Dean,Scool of Mathematics and Statistics and former Deans Prof.T.Amarnath,Prof.S.Kumaresan for extending all necessay facilities for my research work.

I thank all the faculty members and the office staff of the school for their support.

I thank all my teachers.Special thanks to Pratap Reddy.

My heartful thanks to Dr.P.Ramulu, Sampath Lonka,M.Archana, Bhargav Sharma and all other research scholars who are pursuing their research work at the Scool of Mathematics and Statistics.

I would like to sincearly thank the UGC, Gov. of India, for the financial support (as JRF and SRF), for carrying out this research work.

- C.Gangadhar

Abstract

This thesis is divided into four chapters.

Chapter-1 is an introduction to the thesis.In this Chapter we have given brief back ground of zeros of polynomials, some existing generalized results of Enstrom -Kakeya theorem and definitions.

Chapter-2 we extended the results for location of zeros of polynomials with restricted complex coefficients where the real and imazinary parts of coefficients are monotonically increasing, and also we extended the results where the real and imazinary parts of coefficients of polynomial are increasing(or decreasing) upto certain stage and decreasing(or increasing) after that stage.In another section of this chapter we extended the results for zeros of polynomial whose real and imazinary parts of coefficients are alternatively increasing and decreasing upto some stage, after that stage increasing (or decreasing).

Chapter-3 is discussed about few results for zero free regions which are generalizations of Enström-Kakeya theorem.We found zero free regions for polynomials where the coefficients satisfy certain conditions.

Chapter-4 deals with the location and zero free region for polar derivative of polynomials.In this chapter we have found location and zero free regions for polar derivative of polynomial, where the real and imazinary parts of coefficients of the polynomial satisfies conditions like in Chapter-2 and Chapter-3.

Publications related to this thesis

1. C.Gangadhar, P.Ramulu and G.L. Reddy, Zero-free region for polar derivatives of polynomials with restricted coefficients, International J. Of Pure and Engg. Mathematics,Vol.4, No.III,(2016) 67-74

2. C.Gangadhar, G.L.Reddy and P.Ramulu, Zero-free region for polar derivatives of polynomials, Universal Journal of Mathematics and Mathematical sciences,Vol.9 Numbers (3-4)(2016),93-103.

3. C.Gangadhar, P.Ramulu and G.L.Reddy,On the zero free region for polar derivatives of polynomials with restricted coefficients(communicated).

4. C.Gangadhar and G.L.Reddy, on the location of zeros of polynomials with different complex coefficients(communicated).

5. G.L.Reddy, P.Ramulu and C.Gangadhar,On the zeros of polar derivatives of polynomials, Journal of Research in Applied Mathematics, Vol. 2, Issue 4(2015), 7-10.

6. G.L.Reddy, C.Gangadhar and P.Ramulu, Location of zeros of polynomials with complex coefficients(communicated).

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8. P.Ramulu, G.L.Reddy and C.Gangadhar, Zero free region for polynomials with restricted real coefficients, International Research Journal of Pure Algebra-5(5) (2015), 54-62.

9. P.Ramulu, C.Gangadhar and G.L.Reddy, Zero free region for polar derivatives of polynomials with special coefficients, Advances in Dynamical systems and Applications,Vol.11, Number 1(2016),35-39.

10. P.Ramulu, G.L.Reddy and C Gangadhar, On the zeros of polynomials with complex coefficients(communicated).

11. P.Ramulu, C.Gangadhar and G.L.Reddy, Zero free region for polynomials with special complex coefficients.International Journal of Mathematical Archive(accepted).

Relevant Mathematics Subject Classification: 30C10, 30C15.

Contents

Chapter 1

Introduction

1.1 Role of zeros of polynomials in extending number systems

Algebra started mainly for finding zeros of polynomials.To find zeros of algebraic equations like $x + 1 = 1,\ x + 2 = 2$, we extended natural number system ($\mathbb{N}$) to whole number system($\mathbb{W}$), to solve algebraic equations like $x + 1 = 0,\ x + 2 = 1$, we extended whole number system($\mathbb{W}$) to integers($\mathbb{Z}$). In the same way to solve equations like $2x - 1 = 0,\ 5x = -7$, we inroduced rational numbers($\mathbb{Q}$), to solve equatios like $x^2 = 2,\ x^3 = 2,\ 4x^5 = 7$ we introduced irrational numbers.At the end we defined set of real numbers($\mathbb{R}$) as the union of set of rational numbers and set of irrational numbers.To find zeros of the polynomial $x^2 + 1$, we introduce complex numbers, now complex number field is algebraically closed by fundamental theorem of algebra.

1.2 Formula to find zeros of a polynomial

Formula to find zeros of a polynomial means expressing the zeros interms of their coefficients, we can express zeros interms of their coefficients for $1,^{st}\ 2,^{nd}\ 3,^{rd}\ 4^{th}$ degree polynomials, but French mathematician Évariste Galois proved that, for the polynomials from 5^{th} degree onwards we cannot express zeros interms of their coefficients.

1.3 Background of zeros of polynomials

The Fundamental Theorem of Algebra (FTA) states:

Every polynomial equation of degree n with complex coefficients has n roots in the complex numbers.

In fact there are many equivalent formulations: for example that every real polynomial can be expressed as the product of real linear and real quadratic factors.

Early studies of equations by Al-Khwarizmi(about 780-850 ,Iraq) only allowed positive real roots and the FTA was not relevant. Girolamo Cardano(1501-1576,Italian) was the first to realise that one could work with quantities more general than the real numbers. This discovery was made in the course of studying a formula which gave the roots of a cubic equation. The formula when applied to the equation $x^3 = 15x + 4$ gave an answer involving $\sqrt{-121}$ yet Cardan knew that the equation had x = 4 as a solution. He was able to manipulate with his 'complex numbers' to obtain the right answer yet he in no way understood his own mathematics. Rafael Bombelli(Italian), in his Algebra, published in 1572, was to produce a proper set of rules for manipulating these 'complex numbers'. Descartes in 1637 says that one can 'imagine' for every equation of degree n, n roots but these imagined roots do not correspond to any real quantity.

François Viète(1540-1603,Frech) gave equations of degree n with n roots but the first claim that there are always n solutions was made by a Flemish mathematician Albert Girard in 1629 in L'invention en algèbre . However he does not assert that solutions are of the form a + bi, a, b real, so allows the possibility that solutions come from a larger number field than C. In fact this was to become the whole problem of the FTA for many years since mathematicians accepted Albert Girard's assertion as self-evident. They believed that a polynomial equation of degree n must have n roots, the problem was, they believed, to show that these roots were of the form a + bi, a, b real. Now Harriot knew that a polynomial which vanishes at t has a root x - t but this did not become well known until stated by Descartes in 1637 in La gèomètrie, so Albert Girard did not have much of the background to understand the problem properly.

A 'proof' that the FTA was false was given by Leibniz in 1702 when he asserted that $x^4 + t^4$ could never be written as a product of two real quadratic factors. His mistake came in not realising that $\sqrt{i}$ could be written in the form a + bi, a, b real. Euler, in a 1742 correspondence with Nicolaus(II) Bernoulli and Goldbach, showed that the Leibniz counterexample was false.

D'Alembert in 1746 made the first serious attempt at a proof of the FTA. For a polynomial f he takes a real b, c so that f(b) = c. Now he shows that there are complex numbers z_1 *and* w_1 so that
$|z_1| < |c|, |w_1| < |c|$.
He then iterates the process to converge on a zeros of f. His proof has several weaknesses. Firstly, he uses a lemma without proof which was proved in 1851 by Puiseau, but whose proof uses the FTA! Secondly, he did not have the necessary knowledge to use a compactness argument to give the final convergence. Despite this, the ideas in this proof are important. Euler was soon able to prove that every real polynomial of degree $n,\ n \leq 6$ had exactly n complex roots. In 1749 he attempted a proof of the general case, so he tried to proof the FTA for real polynomials:
Every polynomial of the nth degree with real coefficients has precisely n zeros in $\mathbb{C}$.
His proof in Recherches sur les racines imaginaires des èquations is based on decomposing a monic polynomial of degree 2^n into the product of two monic polynomials of degree $m = 2^{n-1}$.Then since an arbitrary polynomial can be converted to a monic polynomial by multiplying by ax^k for some k the theorem would follow by iterating the decomposition. Now Euler knew a fact which went back to Cardan in Ars Magna, or earlier, that a transformation could be applied to remove the second largest degree term of a polynomial. Hence he assumed that
$x^{2m} + Ax^{2m-2} + Bx^{2m-3} + ... = (x^m + tx^{m-1} + gx^{m-2} + ...)(x^m - tx^{m-1} + hx^{m-2} + ...)$
and then multiplied up and compared coefficients. This Euler claim led to g, h, ... being rational functions of A, B, ..., t. All this was carried out in detail for n = 4, but the general case is only a sketch.

In 1772 Lagrange raised objections to Euler's proof. He objected that Euler's rational functions could lead to 0/0. Lagrange used his knowledge of permutations of roots to fill all the gaps in Euler's proof except that he was still assuming that the polynomial equation of degree n must have n roots of some kind so he could work with them and deduce properties, like eventually that they had the form a + bi, a, b real.

Laplace, in 1795, tried to prove the FTA using a completely different approach using the discriminant of a polynomial. His proof was very elegant and its only 'problem' was that again the existence of roots was assumed.

Gauss is usually credited with the first proof of the FTA. In his doctoral thesis of 1799 he presented his first proof and also his objections to the other proofs. He is undoubtedly the first to spot the fundamental flaw in the earlier proofs, to which we have referred many times above, namely the fact that they were assuming the existence of roots and then trying to deduce properties of them. Of Euler's proof Gauss says

... if one carries out operations with these impossible roots, as though they really existed, and says for example, the sum of all roots of the equation $x^m + ax^{m-1} + bx^{m-2} + ... = 0$ is equal to -a even though some of them may be impossible (which really means: even if some are non-existent and therefore missing), then I can only say that I thoroughly disapprove of this type of argument.

Gauss himself does not claim to give the first proper proof. He merely calls his proof new but says, for example of d'Alembert's proof, that despite his objections a rigorous proof could be constructed on the same basis.
Gauss's proof of 1799 is topological in nature and has some rather serious gaps. It does not meet our present day standards required for a rigorous proof.

In 1814 the Swiss accountant Jean Robert Argand published a proof of the FTA which may be the simplest of all the proofs. His proof is based on d'Alembert's 1746 idea. Argand had already sketched the idea in a paper published two years earlier Essai sur une manière de reprèsenter les quantitiès imaginaires dans les constructions gèometriques. In this paper he interpreted i as a rotation of the plane through 90° so giving rise to the Argand plane or Argand diagram as a geometrical representation of complex numbers. Now in the later paper Rèflexions sur la nouvelle thèorie d'analyse Argand simplifies d'Alembert's idea using a general theorem on the existence of a minimum of a continuous function.

In 1820 Cauchy was to devote a whole chapter of Cours d'analyse to Argand's proof (although it will come as no surprise to anyone who has studied Cauchy's work to learn that he fails to mention Argand !) This proof only fails to be rigorous because the general concept of a lower bound had not been developed at that time. The Argand proof was to attain fame when it was given by Chrystal in his Algebra textbook in 1886. Chrystal's book was very influential.

Two years after Argand's proof appeared Gauss published in 1816 a second proof of the FTA. Gauss uses Euler's approach but instead of operating with roots which may not exist, Gauss operates with indeterminates. This proof is complete and correct.

A third proof by Gauss also in 1816 is, like the first, topological in nature. Gauss introduced in 1831 the term 'complex number'. The term 'conjugate' had been introduced by Cauchy in 1821.

Gauss's criticisms of the Lagrange-Laplace proofs did not seem to find immediate favour in France. Lagrange's 1808 2nd Edition of his treatise on equations makes no mention of Gauss's new proof or criticisms. Even the 1828 Edition, edited by Poinsot, still expresses complete satisfaction with the Lagrange-Laplace proofs and no mention of the Gauss criticisms.

In 1849 (on the 50th anniversary of his first proof!) Gauss produced the first proof that a polynomial equation of degree n with complex coefficients has n complex roots. The proof is similar to the first proof given by Gauss. However it adds little since it is straightforward to deduce the result for complex coefficients from the result about polynomials with real coefficients.

It is worth noting that despite Gauss's insistence that one could not assume the existence of roots which were then to be proved reals he did believe, as did everyone at that time, that there existed a whole hierarchy of imaginary quantities of which complex numbers were the simplest. Gauss called them a shadow of shadows.

It was in searching for such generalisations of the complex numbers that Hamilton discovered the quaternions around 1843, but of course the quaternions are not a commutative system. The first proof that the only commutative algebraic field containing R was given by Weierstrass in his lectures of 1863. It was published in Hankel's book

Theorie der complexen Zahlensysteme.

Of course the proofs described above all become valid once one has the modern result that there is a splitting field for every polynomial. Frobenius, at the celebrations in Basle for the bicentenary of Euler's birth said:-

Euler gave the most algebraic of the proofs of the existence of the roots of an equation, the one which is based on the proposition that every real equation of odd degree has a real root. I regard it as unjust to ascribe this proof exclusively to Gauss, who merely added the finishing touches.

The Argand proof is only an existence proof and it does not in any way allow the roots to be constructed. Weierstrass noted in 1859 made a start towards a constructive proof but it was not until 1940 that a constructive variant of the Argand proof was given by Hellmuth Kneser. This proof was further simplified in 1981 by Martin Kneser, Hellmuth Kneser's son.One can refere ragarding above histrory [10,15,22,26,27,28,38,41].

1.4 Recalling of existing results

Complex number field($\mathbb{C}$) is algebraically closed by the following theorem

Theorem 1.4.1. *(Fundamental Theorem of Algebra):Given any positive integer* $n \geq 1$ *and any choice of complex numbers* $a_0, a_1, ..., a_n$ *such that* $a_n \neq 0$*, the polynomial equation* $a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 = 0$ *—(1)*
has at least one solution $z \in \mathbb{C}$*.*

This is a remarkable statement.No analogous result holds for guaranteeing that a real solution exists to Equation (1) if we restrict the coefficients $a_0, a_1, ..., a_n$ to be real. E.g.,there does not exist a real number x satisfying an equation as simple equation $x^2 + 1 = 0$. Similarly,the consideration of polynomial equations having integer (resp. rational) coefficients quickly forces us to consider solutions that cannot possibly be integers (resp. rational numbers).Thus,the complex numbers are special in this respect.Now a days we have many proofs for Fundamental Theorem of Algebra-we have algebraic proof, topological proof, geometric proof etc.

The study of the zeros of polynomials dates from about the time when the geometric representation of complex numbers was introduced into mathematics. The first contributions of the subject were Gauss and Cauchy.Cauchy also added much of the value to the subject and derived for the moduli of the zeros of a polynomial more exact bounds than those given by Gauss.Since the days of and Cauchy many mathematicians have contributed to further growth of the subject.Here,we first mention two class result of Cauchy [4], concerning the bounds for the moduli of the zeros of a polynomial.

Theorem 1.4.2. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n, then all the zeros of* $P(z)$ *lie in* $|z| \leq r$*, where r is unique positive root of the equation*

$$|a_n|z^n - (|a_{n-1}|z^{n-1} + ... + |a_1|z + |a_0|) = 0.$$

Theorem 1.4.3. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n,*
if $M = \max_{1\leq j\leq n} |\frac{a_j}{a_n}|$ *then all the zeros of* $P(z)$ *lie in*

$$|z| \leq 1 + M.$$

There are several developed and gereralized results for above Theorems (for reference see the results [20,24 and 25]).

The above stated results and the results given under the reference are concerning the bounds for moduli of the zeros of a polynomial with complex coefficients.Now the following results are some of the important results on the location of zeros of polynomials in complex plane with restricted coefficients. First,we mention the following elegant result which is commonly known as Enestrom-Kakeya Theorem in the theory of location of zeros of polynomials.

Theorem 1.4.4. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n such that*

$$a_n \geq a_{n-1} \geq .... \geq a_1 \geq a_0 > 0$$

then all the zeros of $P(z)$ *lie in* $|z| \leq 1$.

Theorem 1.4.4 was proved by Enestrom [9], independently by Kakeya [21] and Hurwitz[18]. If we apply Theorem 1.4.4 to the polynomial $P(tz)$, the following more general result immediate.

Theorem 1.4.5. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n such that for* $t>0$,

$$t^n a_n \geq t^{n-1} a_{n-1} \geq .... \geq t a_1 \geq a_0 > 0$$

then all the zeros of $P(z)$ *lie in* $|z| \leq t$.

In the literature ([2,3,7,8,16,17,19]) there exist various generaliztions and refinements of Theorem 1.4.4.Here, we mention a few of them .Joyal, Labelle and Rahman [20] extended Theorem 1.4.4 to the class of polynomials whose coefficients are monotonic but not necessarily non-negetive by proving following result.

Theorem 1.4.6. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n such that*

$$a_n \geq a_{n-1} \geq .... \geq a_1 \geq a_0$$

then all the zeros of $P(z)$ *lie in* $|z| \leq \frac{1}{|a_n|}\{a_n - a_0 + |a_0|\}$.
If $a_i > 0$, *for* $i = 0, 1, .., n$ *then it reduces to Theorem 1.4.4.*

Aziz and Zerger[3] relaxed the hypothesis of Enstrom-Kakeya Theorem and proved the following extention of Theorem 1.4.6.

Theorem 1.4.7. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n such that for some* $k \geq 1$,

$$k a_n \geq a_{n-1} \geq .... \geq a_1 \geq a_0 > 0$$

then all the zeros of $P(z)$ *lie in* $|z + k - 1| \leq \frac{k a_n - a_0 + |a_0|}{|a_n|}$.

Govil and Rahman[17] generalized Enestrom-Kakeya Theorem to the polynomials with complex coefficients by considering the moduli of the coefficients to be monotonically increasing and proved the following result.

Theorem 1.4.8. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n such that for some* $a>0$,

$$|a_n| \geq a|a_{n-1}| \geq a^2|a_{n-2}| \geq .... \geq a^{n-1}|a_1| \geq a^n|a_0|$$

then all the zeros of $P(z)$ *lie in the disc* $|z| \leq \frac{k_1}{a}$, *where* k_1 *is the greatest positive root of the equation* $X^{n+1} - 2X^n + 1 = 0$.

In the same research paper they also proved the following result.

Theorem 1.4.9. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with complex coefficients such that for some real* β,

$$|arg\ a_j - \beta| \leq \alpha \leq \frac{\pi}{2}\ ,\ for\ j=0,1,...,n$$

and

$$|a_n| \geq |a_{n-1}| \geq ... \geq |a_1| \geq |a_0|$$

then $P(z)$ *has all its zeros in the circle*

$$|z| \leq (cos\alpha + sin\alpha) + \frac{2sin\alpha}{|a_n|} \sum_{j=0}^{n-1} |a_j|.$$

Theorem 1.4.10. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n.If Re* $a_k = \alpha_k$, $Im\ a_k = \beta_k\ for\ 0 \leq k \leq n\ and$

$$\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_1 \geq \alpha_0 > 0\ ,$$

then all the zeros of $P(z)$ *has all its zeros in the disc*

$$|z| \leq 1 + \frac{1}{\alpha_n}\{2\sum_{k=0}^{n-1} |\beta_k| + |\beta_n|\}.$$

As an extension of Theorem 1.4.5, Dewan and Bidhkam [7] proved the following result.

Theorem 1.4.11. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n such that for* $t > 0$, *and* $0 < k < n$,

$$t^n a_n \leq t^{n-1} a_{n-1} \leq ... \leq t^k a_k \geq t^{k-1} a_{k-1} \geq .... \geq t a_1 \geq a_0,$$

then all the zeros of $P(z)$ *lie in the circle*

$$|z| \leq \frac{t}{|a_n|}\{\left(\frac{2a_k}{t^{n-k}} - a_n\right) + \frac{1}{t^n}(|a_0| - a_0)\}.$$

Later Shah and Liman [37] extended Theorem 1.4.7 to the class of polynomials with complex coefficients and generalized the result of Govil and Rehman[17] by proving the following two results.

Theorem 1.4.12. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with complex coefficients.If Re* $a_j = \alpha_j$, $Im\ a_j = \beta_j\ for\ j = 0, 1, ..., n,\ a_n \neq 0$ *and for some* $k \geq 1$,

$$k\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_1 \geq \alpha_0,$$

$$\beta_n \geq \beta_{n-1} \geq ... \geq \beta_1 \geq \beta_0,$$

then all the zeros of $P(z)$ *lie in* $\left|z - \frac{\alpha_n}{a_n}(k-1)\right| \leq \frac{1}{a_n}\{\alpha_n - \alpha_0 + |\alpha_0| + \alpha_n\}$.

Theorem 1.4.13. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that for some real β,*

$$|arg\ a_j - \beta| \leq \alpha \leq \tfrac{\pi}{2},\ j = 0, 1, 2..., n$$

and for some $k \geq 1$,

$$k|a_n| \geq |a_{n-1}| \geq .... \geq |a_0|,$$

then all the zeros of $P(z)$ lie in

$$|z + k - 1| \leq \tfrac{1}{|a_n|}\Big\{(k|a_n| - a_0)(sin\alpha + cos\alpha) + |a_0| + 2sin\alpha \textstyle\sum_{j=0}^{n-1} |a_j|\Big\}.$$

There are several extensions and generalizations of Enestrom-Kakeya theorem in literature in various ways. For reference see [1],[5],[6],[23],[39],[40].

1.5 Regarding the work done

In the first chapter we started with importance of zeros of a polynomial in extending number systems, expressing zeros of polynomials interms of their coefficients and briefly explained about theory of location of zeros of polynomials.We stated Enstrom -Kakeya theorem for restricted positive real coefficients and given that how the result of this theorem is generalized in locating the zeros of a polynomial with restricted coefficients. We generalized most of the results for restricted complex coefficients.We have given some existing results and definitions.

Second chapter is divided into five sections.In section 2.1 an introduction to the known results and results generalized by A.Joyal, G.Labelle and Q.I.Rehaman were given. In section 2.2 i.e. On the zeros of polynomials with complex coefficients,we extended the results for restricted complex coefficients where real and imazinary parts of coefficients are monotonically increasing .In section 2.3 i.e Location of zeros of polynomials with complex coefficients, we extended the results where real parts of coefficients of polynomial are increasing(or decreasing) upto certain stage and decreasing(or increasing) after that stage as well as imazinary parts of coefficients of polynomial are increasing(or decreasing) upto certain stage and decreasing(or increasing) after that stage. In section 2.4 i.e Location of zeros of polynomials with restricted coefficients, results for zeros of polynomial whose real parts of coefficients are alternatively increasing and decreasing upto some stage, after that stage increasing (or decreasing), as well as imazinary parts of coefficients are alternatively increasing and decreasing upto some stage,and after that stage increasing (or decreasing). In section 2.5, i.e On the location of zeros of polynomials with different complex coefficients, we tried to find location of zeros of polynomials with real parts of coefficients are alternatively increasing and decreasing as well as imazinary parts of coefficients are alternatively increasing and decreasing.

Third chapter is divided into three sections,section 3.1 is discussed about few results for zero free region which are generalizations of Enstrom-Kakeya theorem. In section 3.2 i.e, zero free region for polynomials with restricted real coefficients,we have found a region where polynomials with real and imazinary parts of coefficients are alternatively increasing(or decreasing) and decreasing(or increasing) upto certain stage ,after that stage increasing(or decreasing) do not vanish . In section 3.3 i.e, zero free region for polynomials with special complex coefficients,we found a zero free region for complex polynomials with real and imazinary parts of coefficients are increasing(or decreasing) upto certain stage, after that stage decreasing (or increasing).

Fourth chapter is divided into six sections,in section 4.1 definition of polar derivtive and already exitsting results which is useful to prove the results on polar derivatives are given.In section 4.2 i.e,on the zeros of polar derivatives of polynomials,we found location of zeros of polar derivative of polynomials with real coefficients, where coefficients related on certain conditions.In section 4.3 i.e, zero free region for polar derivative of polynomials with special coefficients, we have found zero free region for polar derivative of real polynomial where coefficients are related with some condition upto certain stage, after that stage related with some other condition. In section 4.4 i.e, zero free region for polar derivative of polynomials ,we have found similar results as we did in section 4.3. In section 4.5 i.e, zero free region for polar derivative of polynomials with restricted coefficients,by putting separate conditions on coefficients of polynomials whenever degree is even or odd,we locate the zeros of this restricted polynomials in certain region. In section 4.6 i.e, On the zero free regions for polar derivative of polynomials with restricted coefficients,we tried to find zero free region where the coefficients are intially decreasing (or increasing) upto certain stage, after that stage alternatively increasing and decreasing.

Chapter 2

On The Location of Zeros of Polynomials With Complex Coefficients

2.1 Introduction and some known results

The estimation of the location of zeros of a polynomial is a long standing classical problem. It is an interesting area of research for engineers as well as mathematicians and many results on the same topic are available in literature. A.Joyal , G. Labelle and Q. I. Rahman [20] obtained the following generalization, by considering the coefficients to be real, instead of being only positive.

Theorem 2.1.1. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n such that* $a_0 \leq a_1 \leq ... \leq a_{n-1} \leq a_n$*. Then all the zeros of P(z) lie in* $|z| \leq \frac{1}{a_n}\big[a_n - a_0 + |a_0|\big]$.

In the literature some attempts have been made to extend and generalize the Eneström-Kakeya theorem. Aziz and Zargar [3] relaxed the hypothesis of Eneström-Kakeya theorem in different ways and proved the following results:

Theorem 2.1.2. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n such that for some*

$$k \geq 1,\ 0 < a_0 \leq a_1 \leq ... \leq a_{n-1} \leq k a_n.$$

Then all the zeros of P(z) lie in $|z + k - 1| \leq k$.

Theorem 2.1.3. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n such that for some* $k \geq 1,\ a_0 \leq a_1 \leq ... \leq a_{n-1} \leq k a_n$. *Then all the zeros of* $P(z)$ *lie in*

$$|z + k - 1| \leq \frac{1}{|a_n|}\big[k a_n - a_0 + |a_0|\big].$$

In this chapter many results, W M Shah and A Liman [37], were proved by weakening the hypotheses of Theorems 1.2.1 and 2.1.2 and by considering a larger class of polynomials. We have obtained some extensions of the classical results concerning the Eneström-Kakeya theorem related to analytic functions. Besides, we considerably improve the bounds in some cases by relaxing the hypotheses in several ways.

2.2 On the zeros of polynomials with complex coefficients

Theorem 2.2.1. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree n with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $k \geq 1, k_1 \geq 1, l \geq 1, l_1 \geq 1, \delta \geq 0, \eta \geq 0, a_m \neq 0, b_m \neq 0,$

$$a_0 - \delta \leq a_1 \leq ... \leq a_{m-1} \leq k_1 a_m \leq a_{m+1} \leq ... \leq a_{n-1} \leq k a_n \quad \textit{and}$$

$$b_0 - \eta \leq b_1 \leq ... \leq b_{m-1} \leq l_1 b_m \leq b_{m+1} \leq ... \leq b_{n-1} \leq l b_n$$

then all the zeros of P(z) lie in

$$|z| \leq \tfrac{1}{|\alpha_n|}\big[k(a_n + |a_n|) + l(b_n + |b_n|) + |a_0| + |b_0| + 2(k_1 - 1)|a_m| + 2(l_1 - 1)|b_m| - (a_0 + b_0 + |a_n| + |b_n|) + 2\delta + 2\eta\big].$$

Proof. Let $P(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + ... + \alpha_{m-1} z^{m-1} + \alpha_m z^m + \alpha_{m+1} z^{m+1} + ... + \alpha_1 z + \alpha_0$ be a polynomial of degree n.

Then consider the polynomial $Q(z) = (1 - z)P(z)$ so that

$Q(z) = -\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + ... + (\alpha_{m+1} - \alpha_m)z^{m+1} + (\alpha_m - \alpha_{m-1})z^m + ... + (\alpha_1 - \alpha_0)z + \alpha_0.$

$= -\alpha_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_{m+1} - a_m)z^{m+1} + (a_m - a_{m-1})z^m + ... + (a_1 - a_0)z + a_0 + i\big\{(b_n - b_{n-1})z^n + ... + (b_{m+1} - b_m)z^{m+1} + (b_m - b_{m-1})z^m + ... + (b_1 - b_0)z + b_0\big\}$.Also if $|z|>1$ then $\frac{1}{|z|^{n-i}}<1$ for $i = 0, 1, 2, ..., n-1$.

Now $|Q(z)| \geq |\alpha_n||z|^{n+1} - \Big\{\big(|a_n - a_{n-1}||z|^n + ... + |a_{m+1} - a_m||z|^{m+1} + |a_m - a_{m-1}||z|^m + ... + |a_1 - a_0||z| + a_0\big) + \big(|b_n - b_{n-1}||z|^n + ... + |b_{m+1} - b_m||z|^{m+1} + |b_m - b_{m-1}||z|^m + ... + |b_1 - b_0||z| + b_0\big)\Big\}$

$\geq |\alpha_n||z|^n\Bigg[|z| - \frac{1}{|\alpha_n|}\Bigg\{\big(|a_n - a_{n-1}| + \frac{|a_{n-1}-a_{n-2}|}{|z|} + ... + \frac{|a_{m+1}-a_m|}{|z|^{n-m-1}} + \frac{|a_m-a_{m-1}|}{|z|^{n-m}} + ... + \frac{|a_1-a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\big) + \big(|b_n - b_{n-1}| + \frac{|b_{n-1}-b_{n-2}|}{|z|} + ... + \frac{|b_{m+1}-b_m|}{|z|^{n-m-1}} + \frac{|b_m-b_{m-1}|}{|z|^{n-m}} + ... + \frac{|b_1-b_0|}{|z|^{n-1}} + \frac{|b_0|}{|z|^n}\big)\Bigg\}\Bigg]$

$\geq |\alpha_n||z|^n\Bigg[|z| - \frac{1}{|\alpha_n|}\Bigg\{\big(|ka_n - a_{n-1} - ka_n + a_n| + |a_{n-1} - a_{n-2}| + ... + |a_{m+1} - k_1a_m + k_1a_m - a_m| + |a_m - k_1a_m + k_1a_m - a_{m-1}| + ... + |a_1 + \delta - a_0 - \delta| + |a_0|\big) + \big(|lb_n - b_{n-1} - lb_n + b_n| + |b_{n-1} - b_{n-2}| + ... + |b_{m+1} - l_1b_m + l_1b_m - b_m| + |b_m - l_1b_m + l_1b_m - b_{m-1}| + ... + |b_1 + \eta - b_0 - \eta| + |b_0|\big)\Bigg\}\Bigg]$

$\geq |\alpha_n||z|^n\Bigg[|z| - \frac{1}{|\alpha_n|}\Bigg\{\big[(ka_n - a_{n-1}) + (k-1)|a_n| + (a_{n-1} - a_{n-2}) + ... + (a_{m+1} - k_1a_m) + (k_1 - 1)|a_m| + (k_1a_m - a_{m-1}) + (k_1 - 1)|a_m| + .. + (a_1 + \delta - a_0) + \delta + |a_0|\big] + \big[(lb_n - b_{n-1}) + (l-1)|b_n| + (b_{n-1} - b_{n-2}) + ... + (b_{m+1} - l_1b_m) + (l_1 - 1)|b_m| + (l_1b_m - b_{m-1}) + (l_1 - 1)|b_m| + .. + (b_1 + \eta - b_0) + \eta + |b_0|\big]\Bigg\}\Bigg]$

$= |\alpha_n||z|^n\Bigg[|z| - \frac{1}{|\alpha_n|}\Bigg\{k(a_n + |a_n|) + l(b_n + |b_n|) + |a_0| + |b_0| + 2(k_1 - 1)|a_m| + 2(l_1 - 1)|b_m| - (a_0 + b_0 + |a_n| + |b_n|) + 2\delta + 2\eta\Bigg\}\Bigg]$

>0 provided

$|z|>\frac{1}{|\alpha_n|}\Bigg[k(a_n + |a_n|) + l(b_n + |b_n|) + |a_0| + |b_0| + 2(k_1 - 1)|a_m| + 2(l_1 - 1)|b_m| - (a_0 + b_0 + |a_n| + |b_n|) + 2\delta + 2\eta\Bigg]$.

Thish shows that $Q(z)>0$ provided

$|z|>\frac{1}{|\alpha_n|}\Bigg[k(a_n + |a_n|) + l(b_n + |b_n|) + |a_0| + |b_0| + 2(k_1 - 1)|a_m| + 2(l_1 - 1)|b_m| - (a_0 + b_0 + |a_n| + |b_n|) + 2\delta + 2\eta\Bigg]$.

Hence all the zeros of $Q(z)$ with $|z|>1$ lie in

$|z| \leq \frac{1}{|\alpha_n|}\Bigg[\big[k(a_n+|a_n|)+l(b_n+|b_n|)+|a_0|+|b_0|+2(k_1-1)|a_m|+2(l_1-1)|b_m|-(a_0+b_0+|a_n|+|b_n|)+2\delta+2\eta\Bigg]$.

But those zeros of $Q(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality.

Since all the zeros of $P(z)$ are also the zeros of $Q(z)$ lie in the circle defined by the above inequality and this completes the proof of the Theorem. □

Corollary 2.2.2. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $k \geq 1, k_1 \geq 1, \delta \geq 0, \eta \geq 0, a_m \neq 0, b_m \neq 0,$

$$a_0 - \delta \leq a_1 \leq ... \leq a_{m-1} \leq k_1 a_m \leq a_{m+1} \leq ... \leq a_{n-1} \leq k a_n \quad \textit{and}$$

$$b_0 - \eta \leq b_1 \leq ... \leq b_{m-1} \leq k_1 b_m \leq b_{m+1} \leq ... \leq b_{n-1} \leq k b_n$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\big[k(a_n+b_n+|a_n|+|b_n|)+|a_0|+|b_0|+2k_1(|a_m|+|b_m|)-(a_0+b_0+|a_n|+|b_n|) +2(\delta+\eta-|a_m|-|b_m|)\big].$$

Corollary 2.2.3. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $k \geq 1, \delta \geq 0,$

$$a_0 - \delta \leq a_1 \leq ... \leq a_{m-1} \leq a_m \leq a_{m+1} \leq ... \leq a_{n-1} \leq k a_n \quad \textit{and}$$

$$b_0 - \delta \leq b_1 \leq ... \leq b_{m-1} \leq b_m \leq b_{m+1} \leq ... \leq b_{n-1} \leq k b_n$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\big[k(a_n+b_n+|a_n|+|b_n|)+|a_0|+|b_0|-(a_0+b_0+|a_n|+|b_n|)+4\delta\big].$$

Corollary 2.2.4. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $k \geq 1, k_1 \geq 1, \delta \geq 0, a_m \neq 0$

$$a_0 - \delta \leq a_1 \leq ... \leq a_{m-1} \leq k_1 a_m \leq a_{m+1} \leq ... \leq a_{n-1} \leq k a_n \quad \textit{and}$$

$$b_0 \leq b_1 \leq ... \leq b_{m-1} \leq b_m \leq b_{m+1} \leq ... \leq b_{n-1} \leq b_n$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\big[k(a_n + |a_n|) + b_n + |a_0| + |b_0| + 2(k_1 - 1)|a_m| - (a_0 + b_0 + |a_n|) + 2\delta\big].$$

Corollary 2.2.5. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that*

$$a_0 \leq a_1 \leq ... \leq a_{m-1} \leq a_m \leq a_{m+1} \leq ... \leq a_{n-1} \leq a_n \quad and$$

$$b_0 \leq b_1 \leq ... \leq b_{m-1} \leq b_m \leq b_{m+1} \leq ... \leq b_{n-1} \leq b_n$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\big[a_n + b_n + |a_0| + |b_0| - (a_0 + b_0)\big].$$

Corollary 2.2.6. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i > 0$ *and* $Im(\alpha_i) = b_i > 0$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $k \geq 1, k_1 \geq 1, \delta \geq 0, \eta \geq 0, a_m \neq 0, b_m \neq 0,$

$$a_0 - \delta \leq a_1 \leq ... \leq a_{m-1} \leq k_1 a_m \leq a_{m+1} \leq ... \leq a_{n-1} \leq k a_n \quad and$$

$$b_0 - \eta \leq b_1 \leq ... \leq b_{m-1} \leq k_1 b_m \leq b_{m+1} \leq ... \leq b_{n-1} \leq k b_n$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\big[(2k-1)(a_n + b_n) + (2k_1 - 1)(a_m + b_m) + 2\delta + 2\eta\big].$$

Remark 2.2.7. By taking $k = l, k_1 = l_1$ in Theorem 2.2.1, then it reduces to Corollary 2.2.2

Remark 2.2.8. By taking $l = k, l_1 = k_1 = 1$ and $\delta = \eta$ in Theorem 2.2.1, then it reduces to Corollary 2.2.3

Remark 2.2.9. By taking $l = l_1 = 1$ and $\eta = 0$ in Theorem 2.2.1, then it reduces to Corollary 2.2.4

Remark 2.2.10. By taking $l = k_1 = l_1 = k = 1$ and $\delta = \eta = 0$ in Theorem 2.2.1, then it reduces to Corollary 2.2.5

Remark 2.2.11. By taking $b_i>0$ *and* $a_i>0$ for $i = 0, 1, 2, ..., n$ in Corollary 2.2.2, then it reduces to Corollary 2.2.6

Theorem 2.2.12. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree n with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $0<r \leq 1, 0<s \leq 1, 0<x \leq 1, 0<y \leq 1, \delta \geq 0, \eta \geq 0, a_m \neq 0, b_m \neq 0,$

$$ra_n \leq a_{n-1} \leq ... \leq a_{m+1} \leq sa_m \leq a_{m-1} \leq ... \leq a_1 \leq a_0 + \delta \quad and$$

$$xb_n \leq b_{n-1} \leq ... \leq b_{m+1} \leq yb_m \leq b_{m-1} \leq ... \leq b_1 \leq b_0 + \eta$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Big[a_0 + b_0 + |a_0| + |b_0| + |a_n| + |b_n| - r(a_n + |a_n|) - x(b_n + |b_n|)$$

$$+2\big[|a_m| + |b_m| - s|a_m| - y|b_m| + \delta + \eta\big]\Big].$$

Proof. Let $P(z) = \alpha_n z^n + \alpha_{n-1}z^{n-1} + ... + \alpha_{m-1}z^{m-1} + \alpha_m z^m + \alpha_{m+1}z^{m+1} + ... + \alpha_1 z + \alpha_0$ be a polynomial of degree n.

Then consider the polynomial $Q(z) = (1 - z)P(z)$ so that

$Q(z) = -\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + ... + (\alpha_{m+1} - \alpha_m)z^{m+1} + (\alpha_m - \alpha_{m-1})z^m + ... + (\alpha_1 - \alpha_0)z + \alpha_0.$

$= -\alpha_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_{m+1} - a_m)z^{m+1} + (a_m - a_{m-1})z^m + ... + (a_1 - a_0)z + a_0 + + i\big\{(b_n - b_{n-1})z^n + ... + (b_{m+1} - b_m)z^{m+1} + (b_m - b_{m-1})z^m + ... + (b_1 - b_0)z + b_0\big\}.$

Also if $|z|>1$ then $\frac{1}{|z|^{n-i}}<1$ for $i = 0, 1, 2, ..., n - 1$.

Now $|Q(z)| \geq |\alpha_n||z|^{n+1} - \Big\{\big(|a_n - a_{n-1}||z|^n + ... + |a_{m+1} - a_m||z|^{m+1} + |a_m - a_{m-1}||z|^m + ... + |a_1 - a_0||z| + a_0\big) + \big(|b_n - b_{n-1}||z|^n + ... + |b_{m+1} - b_m||z|^{m+1} + |b_m - b_{m-1}||z|^m + ... + |b_1 - b_0||z| + b_0\big)\Big\}$

$\geq |\alpha_n||z|^n\Big[|z| - \frac{1}{|\alpha_n|}\Big\{\big(|a_n - a_{n-1}| + \frac{|a_{n-1}-a_{n-2}|}{|z|} + ... + \frac{|a_{m+1}-a_m|}{|z|^{n-m-1}} + \frac{|a_m-a_{m-1}|}{|z|^{n-m}} + ... + \frac{|a_1-a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\big) + \big(|b_n - b_{n-1}| + \frac{|b_{n-1}-b_{n-2}|}{|z|} + ... + \frac{|b_{m+1}-b_m|}{|z|^{n-m-1}} + \frac{|b_m-b_{m-1}|}{|z|^{n-m}} + ... + \frac{|b_1-b_0|}{|z|^{n-1}} + \frac{|b_0|}{|z|^n}\big)\Big\}\Big]$

$\geq |\alpha_n||z|^n\Bigg[|z|-\frac{1}{|\alpha_n|}\Bigg\{\big(|ra_n-a_{n-1}-ra_n+a_n|+|a_{n-1}-a_{n-2}|+...+|a_{m+1}-sa_m+sa_m-a_m|+|a_m-sa_m+sa_m-a_{m-1}|+...+|a_1+\delta-a_0-\delta|+|a_0|\big)+\big(|xb_n-b_{n-1}-xb_n+b_n|+|b_{n-1}-b_{n-2}|+...+|b_{m+1}-yb_m+yb_m-b_m|+|b_m-yb_m+yb_m-b_{m-1}|+...+|b_1+\eta-b_0-\eta|+|b_0|\big)\Bigg\}\Bigg]$

$\geq |\alpha_n||z|^n\Bigg[|z|-\frac{1}{|\alpha_n|}\Bigg\{\big[(a_{n-1}-ra_n)+(1-r)|a_n|+(a_{n-2}-a_{n-1})+...+(sa_m-a_{m+1})+(1-s)|a_m|+(a_{m-1}-sa_m)+(1-s)|a_m|+..+(a_0+\delta-a_1)+\delta+|a_0|\big]+\big[(b_{n-1}-xb_n)+(1-x)|b_n|+(b_{n-2}-b_{n-1})+...+(yb_m-b_{m+1})+(1-y)|b_m|+(b_{m-1}-yb_m)+(1-y)|b_m|+..+(b_0+\eta-b_1)+\eta+|b_0|\big]\Bigg\}\Bigg]$

$=|\alpha_n||z|^n\Bigg[|z|-\frac{1}{|\alpha_n|}\Bigg\{a_0+b_0+|a_0|+|b_0|+|a_n|+|b_n|-r(a_n+|a_n|)$

$-x(b_n+|b_n|)+2\big[|a_m|+|b_m|-s|a_m|-y|b_m|+\delta+\eta\big]\Bigg\}\Bigg]$

>0

provided $|z|>\frac{1}{|\alpha_n|}\Bigg[a_0+b_0+|a_0|+|b_0|+|a_n|+|b_n|-r(a_n+|a_n|)-x(b_n+|b_n|)+2\big[|a_m|+|b_m|-s|a_m|-y|b_m|+\delta+\eta\big]\Bigg]$.

Thish shows that $Q(z)>0$ provided

$|z|>\frac{1}{|\alpha_n|}\Bigg[a_0+b_0+|a_0|+|b_0|+|a_n|+|b_n|-r(a_n+|a_n|)-x(b_n+|b_n|)+2\big[|a_m|+|b_m|-s|a_m|-y|b_m|+\delta+\eta\big]\Bigg]$

Hence all the zeros of $Q(z)$ with $|z|>1$ lie in

$|z|\leq\frac{1}{|\alpha_n|}\Bigg[a_0+b_0+|a_0|+|b_0|+|a_n|+|b_n|-r(a_n+|a_n|)-x(b_n+|b_n|)+2\big[|a_m|+|b_m|-s|a_m|-y|b_m|+\delta+\eta\big]\Bigg]$.

But those zeros of $Q(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of $P(z)$ are also the zeros of $Q(z)$ lie in the circle defined by the above inequality and this completes the proof of the Theorem. □

Corollary 2.2.13. *Let* $P(z)=\sum_{i=0}^{n}\alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i)=a_i$ *and* $Im(\alpha_i)=b_i$ *for* $i=0,1,2,...,n$ *such that for some* $0<r\leq 1, 0<s\leq 1, \delta\geq 0, \eta\geq 0, a_m\neq 0, b_m\neq 0,$

$$ra_n\leq a_{n-1}\leq ...\leq a_{m+1}\leq sa_m\leq a_{m-1}\leq ...\leq a_1\leq a_0+\delta \quad \text{and}$$

$$rb_n \leq b_{n-1} \leq ... \leq b_{m+1} \leq sb_m \leq b_{m-1} \leq ... \leq b_1 \leq b_0 + \eta$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Big[a_0+b_0+|a_0|+|b_0|+|a_n|+|b_n|-r(a_n+|a_n|+b_n+|b_n|)+2\big[|a_m|+|b_m|-s(|a_m|+|b_m|)+\delta+\eta\big]\Big].$$

Corollary 2.2.14. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $0<r \leq 1, \delta \geq 0,$

$$ra_n \leq a_{n-1} \leq ... \leq a_{m+1} \leq a_m \leq a_{m-1} \leq ... \leq a_1 \leq a_0 + \delta \quad \textit{and}$$

$$rb_n \leq b_{n-1} \leq ... \leq b_{m+1} \leq b_m \leq b_{m-1} \leq ... \leq b_1 \leq b_0 + \delta$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Big[a_0 + b_0 + |a_0| + |b_0| + |a_n| + |b_n| - r(a_n + b_n + |a_n| + |b_n|) + 4\delta\Big].$$

Corollary 2.2.15. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $0<r \leq 1, 0<s \leq 1, \delta \geq 0, a_m \neq 0,$

$$ra_n \leq a_{n-1} \leq ... \leq a_{m+1} \leq sa_m \leq a_{m-1} \leq ... \leq a_1 \leq a_0 + \delta \quad \textit{and}$$

$$b_n \leq b_{n-1} \leq ... \leq b_{m+1} \leq b_m \leq b_{m-1} \leq ... \leq b_1 \leq b_0$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Big[a_0 + b_0 + |a_0| + |b_0| + |a_n| + |b_n| - r(a_n + |a_n|) - b_n + 2\big[|a_m| - s|a_m| + \delta\big]\Big].$$

Corollary 2.2.16. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that*

$$a_n \leq a_{n-1} \leq ... \leq a_{m+1} \leq a_m \leq a_{m-1} \leq ... \leq a_1 \leq a_0 \quad \textit{and}$$

$$b_n \leq b_{n-1} \leq ... \leq b_{m+1} \leq b_m \leq b_{m-1} \leq ... \leq b_1 \leq b_0$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Big[a_0 + b_0 + |a_0| + |b_0| - a_n - b_n\Big].$$

Corollary 2.2.17. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i > 0$ *and* $Im(\alpha_i) = b_i > 0$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $0 < r \leq 1, 0 < s \leq 1, 0 < x \leq 1, 0 < y \leq 1, \delta \geq 0, \eta \geq 0, a_m \neq 0, b_m \neq 0,$

$$ra_n \leq a_{n-1} \leq ... \leq a_{m+1} \leq sa_m \leq a_{m-1} \leq ... \leq a_1 \leq a_0 + \delta \quad and$$

$$xb_n \leq b_{n-1} \leq ... \leq b_{m+1} \leq yb_m \leq b_{m-1} \leq ... \leq b_1 \leq b_0 + \eta$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Bigg[a_0 + b_0 + |a_0| + |b_0| + (1-2r)a_n + (1-2x)b_n + 2\big[(1-s)a_m + (1-y)b_m + \delta + \eta\big]\Bigg].$$

Remark 2.2.18. By taking $r = x, s = y$ in Theorem 2.2.12, then it reduces to Corollary 2.2.13.

Remark 2.2.19. By taking $x = r, s = y = 1$ and $\delta = \eta$ in Theorem 2.2.12, then it reduces to Corollary 2.2.14.

Remark 2.2.20. By taking $x = y = 1$ and $\eta = 0$ in Theorem 2.2.12, then it reduces to Corollary 2.2.15.

Remark 2.2.21. By taking $r = s = x = y = 1$ and $\delta = \eta = 0$ in Theorem 2.2.12, then it reduces to Corollary 2.2.16.

Remark 2.2.22. By taking $b_i > 0$ *and* $a_i > 0$ for $i = 0, 1, 2, ..., n$ in Theorem 2.2.12, then it reduces to Corollary 2.2.17.

The results of section 2.2 have appeared in [32]

2.3 Location of zeros of polynomials with complex coefficients

Theorem 2.3.1. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $k \geq 1,\ l \geq 1,\ 0 < r \leq 1,\ 0 < s \leq 1,\ \delta \geq 0,\ \eta \geq 0,\ a_m \neq 0,\ b_m \neq 0,$

$$a_0 - \delta \leq a_1 \leq ... \leq a_{m-1} \leq ka_m \geq a_{m+1} \geq ... \geq a_{n-1} \geq ra_n \quad and$$

$$b_0-\eta \le b_1 \le ... \le b_{m-1} \le lb_m \ge b_{m+1} \ge ... \ge b_{n-1} \ge sb_n$$

then all the zeros of P(z) lie in

$$|z| \le \frac{1}{|\alpha_n|}\Big[(|a_n|+|b_n|)+k(a_m+|a_m|)+l(b_m+|b_m|)+|a_0|+|b_0|+2(k-1)|a_m| + 2(l-1)|b_m| - \big[a_0+b_0+r(a_n+|a_n|)+s(b_n+|b_n|)\big]+2\delta+2\eta\Big].$$

Proof. Let $P(z)=\alpha_n z^n+\alpha_{n-1}z^{n-1}+...+\alpha_{m-1}z^{m-1}+\alpha_m z^m+\alpha_{m+1}z^{m+1}+...+\alpha_1 z+\alpha_0$ be a polynomial of degree n. Then consider the polynomial

$$\begin{aligned}Q(z) =&(1-z)P(z)\\ =&-\alpha_n z^{n+1}+(\alpha_n-\alpha_{n-1})z^n+...+(\alpha_{m+1}-\alpha_m)z^{m+1}+(\alpha_m-\alpha_{m-1})z^m\\ &+...+(\alpha_1-\alpha_0)z+\alpha_0.\\ =&-\alpha_n z^{n+1}+(a_n-a_{n-1})z^n+...+(a_{m+1}-a_m)z^{m+1}+(a_m-a_{m-1})z^m\\ &+...+(a_1-a_0)z+a_0\\ &+i\big\{(b_n-b_{n-1})z^n+...+(b_{m+1}-b_m)z^{m+1}+(b_m-b_{m-1})z^m+...+(b_1-b_0)z+b_0\big\}.\end{aligned}$$

Also if $|z|>1$ *then* $\frac{1}{|z|^{n-i}}<1$ *for* $i=0,1,2,...,n-1$. *Now*

$$\begin{aligned}|Q(z)| \ge&|\alpha_n||z|^{n+1}-\Big\{\big(|a_n-a_{n-1}||z|^n+...+|a_{m+1}-a_m||z|^{m+1}+|a_m-a_{m-1}||z|^m\\ &+...+|a_1-a_0||z|+a_0\big)+\big(|b_n-b_{n-1}||z|^n+...+|b_{m+1}-b_m||z|^{m+1}\\ &+|b_m-b_{m-1}||z|^m+...+|b_1-b_0||z|+b_0\big)\Big\}\\ \ge& |\alpha_n||z|^n\Big[|z|-\frac{1}{|\alpha_n|}\Big\{\big(|a_n-a_{n-1}|+\frac{|a_{n-1}-a_{n-2}|}{|z|}+...+\frac{|a_{m+1}-a_m|}{|z|^{n-m-1}}\\ &+\frac{|a_m-a_{m-1}|}{|z|^{n-m}}+...+\frac{|a_1-a_0|}{|z|^{n-1}}+\frac{|a_0|}{|z|^n}\big)\\ &+\big(|b_n-b_{n-1}|+\frac{|b_{n-1}-b_{n-2}|}{|z|}+...+\frac{|b_{m+1}-b_m|}{|z|^{n-m-1}}\\ &+\frac{|b_m-b_{m-1}|}{|z|^{n-m}}+...+\frac{|b_1-b_0|}{|z|^{n-1}}+\frac{|b_0|}{|z|^n}\big)\Big\}\Big]\\ \ge& |\alpha_n||z|^n\Big[|z|-\frac{1}{|\alpha_n|}\Big\{\big(|ra_n-a_{n-1}-ra_n+a_n|+|a_{n-1}-a_{n-2}|+...+|a_{m+1}-ka_m+ka_m-a_m|\\ &+|a_m-ka_m+ka_m-a_{m-1}|+...+|a_1+\delta-a_0-\delta|+|a_0|\big)\\ &+\big(|sb_n-b_{n-1}-sb_n+b_n|+|b_{n-1}-b_{n-2}|+...+|b_{m+1}-lb_m+lb_m-b_m|\end{aligned}$$

$$+|b_m - lb_m + lb_m - b_{m-1}| + ... + |b_1 + \eta - b_0 - \eta| + |b_0|)\Big\}\Big]$$

$$\geq |\alpha_n||z|^n\Big[|z| - \frac{1}{|\alpha_n|}\Big\{\big[(a_{n-1} - ra_n) + (1-r)|a_n| + (a_{n-2} - a_{n-1}) + ... + (ka_m - a_{m+1})$$

$$+ (k-1)|a_m| + (ka_m - a_{m-1}) + (k-1)|a_m| + .. + (a_1 + \delta - a_0) + \delta + |a_0|\big]$$

$$+\big[(b_{n-1} - sb_n) + (1-s)|b_n| + (b_{n-2} - b_{n-1}) + ... + (lb_m - b_{m+1}) + (l-1)|b_m|$$

$$+ (lb_m - b_{m-1}) + (l-1)|b_m| + .. + (b_1 + \eta - b_0) + \eta + |b_0|\big]\Big\}\Big]$$

$$= |\alpha_n||z|^n\Big[|z| - \frac{1}{|\alpha_n|}\Big\{(|a_n| + |b_n|) + k(a_m + |a_m|) + l(b_m + |b_m|) + |a_0| + |b_0| + 2(k-1)|a_m|$$

$$+ 2(l-1)|b_m| - \big[a_0 + b_0 + r(a_n + |a_n|) + s(b_n + |b_n|)\big] + 2\delta + 2\eta\Big\}\Big] > 0$$

$$if\ |z| > \frac{1}{|\alpha_n|}\Big[(|a_n| + |b_n|) + k(a_m + |a_m|) + l(b_m + |b_m|) + |a_0| + |b_0| + 2(k-1)|a_m|$$

$$+ 2(l-1)|b_m| - \big[a_0 + b_0 + r(a_n + |a_n|) + s(b_n + |b_n|)\big] + 2\delta + 2\eta\Big].$$

This shows that if $|z|> 1,\ Q(z)>0$

$$provided\ |z| > \frac{1}{|\alpha_n|}\Big[(|a_n| + |b_n|) + k(a_m + |a_m|) + l(b_m + |b_m|) + |a_0| + |b_0| + 2(k-1)|a_m|$$

$$+ 2(l-1)|b_m| - \big[a_0 + b_0 + r(a_n + |a_n|) + s(b_n + |b_n|)\big] + 2\delta + 2\eta\Big].$$

Hence all the zeros of $Q(z)$ with $|z|>1$ lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Big[(|a_n| + |b_n|) + k(a_m + |a_m|) + l(b_m + |b_m|) + |a_0| + |b_0| + 2(k-1)|a_m|$$

$$+ 2(l-1)|b_m| - \big[a_0 + b_0 + r(a_n + |a_n|) + s(b_n + |b_n|)\big] + 2\delta + 2\eta\Big].$$

But those zeros of $Q(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of $P(z)$ are also the zeros of $Q(z)$ lie in the circle defined by the above inequality and this completes the proof of the Theorem. □

Corollary 2.3.2. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $k \geq 1,\ 0< r \leq 1,\ \delta \geq 0,\ a_m \neq 0,\ b_m \neq 0,$

$$a_0 - \delta \leq a_1 \leq ... \leq a_{m-1} \leq ka_m \geq a_{m+1} \geq ... \geq a_{n-1} \geq ra_n \quad and$$

$$b_0 - \delta \leq b_1 \leq ... \leq b_{m-1} \leq kb_m \geq b_{m+1} \geq ... \geq b_{n-1} \geq rb_n$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Bigg[(|a_n| + |b_n|) + k(a_m + b_m + |a_m| + |b_m|) + |a_0| + |b_0| + 2(k-1)(|a_m| + |b_m|) - \big[a_0 + b_0 + r(a_n + b_n + |a_n| + |b_n|)\big] + 4\delta\Bigg].$$

Corollary 2.3.3. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that*

$$a_0 \leq a_1 \leq ... \leq a_{m-1} \leq a_m \geq a_{m+1} \geq ... \geq a_{n-1} \geq a_n \quad and$$

$$b_0 \leq b_1 \leq ... \leq b_{m-1} \leq b_m \geq b_{m+1} \geq ... \geq b_{n-1} \geq b_n$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Bigg[|a_0| + |b_0| + a_m + b_m + |a_m| + |b_m| - [a_0 + b_0 + a_n + b_n]\Bigg].$$

Corollary 2.3.4. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i > 0$ *and* $Im(\alpha_i) = b_i > 0$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $k \geq 1,\ l \geq 1,\ 0 < r \leq 1,\ 0 < s \leq 1,\ \delta \geq 0,\ \eta \geq 0,$

$$a_0 - \delta \leq a_1 \leq ... \leq a_{m-1} \leq ka_m \geq a_{m+1} \geq ... \geq a_{n-1} \geq ra_n \quad and$$

$$b_0 - \eta \leq b_1 \leq ... \leq b_{m-1} \leq lb_m \geq b_{m+1} \geq ... \geq b_{n-1} \geq sb_n$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Bigg[(2r+1)a_n + (2l+1)b_n + (4k-1)a_m + (4s-1)b_m + 2\delta + 2\eta\Bigg].$$

Corollary 2.3.5. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $k \geq 1,\ l \geq 1,\ 0 < r \leq 1,\ 0 < s \leq 1,\ \delta \geq 0,\ \eta \geq 0,\ a_m \neq 0,\ b_m \neq 0,$

$$a_0 - \delta \leq a_1 \leq ... \leq a_{m-1} \leq ka_m \geq a_{m+1} \geq ... \geq a_{n-1} \geq ra_n \quad and$$

$$b_0 - \eta \leq b_1 \leq ... \leq b_{m-1} \leq b_m \leq b_{m+1} \leq ... \leq b_{n-1} \leq lb_n$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Big[|a_n| + l(b_n + |b_n|) + k(a_m + |a_m|) + |a_0| + |b_0|$$
$$+ 2(k-1)|a_m| - \big[a_0 + b_0 + r(a_n + |a_n|)\big] + 2\delta + 2\eta\Big].$$

Remark 2.3.6. By taking $l = k,\ s = r\ and\ \delta = \eta$ in Theorem 2.3.1, it reduces to Corollary 2.3.2

Remark 2.3.7. By taking $l = k = 1,\ s = r = 1\ and\ \delta = \eta = 0$ in Theorem 2.3.1, it reduces to Corollary 2.3.3

Remark 2.3.8. By taking $a_i> 0 and\ b_i> 0$ in Theorem 2.3.1, it reduces to Corollary 2.3.4

Remark 2.3.9. By taking $l = 1,\ s = 1\ and\ \eta = 0$ in Theorem 2.3.1, it reduces to Corollary 2.3.5

Theorem 2.3.10. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree n with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $k \geq 1,\ l \geq 1,\ 0< r \leq 1,\ 0< s \leq 1,\ \delta \geq 0,\ \eta \geq 0,\ a_m \neq 0,\ b_m \neq 0,$

$$a_0 + \delta \geq a_1 \geq ... \geq a_{m-1} \geq ra_m \leq a_{m+1} \leq ... \leq a_{n-1} \leq ka_n \quad and$$

$$b_0 + \eta \geq b_1 \geq ... \geq b_{m-1} \geq sb_m \leq b_{m+1} \leq ... \leq b_{n-1} \leq lb_n$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Big[k(a_n + |a_n|) + l(b_n + |b_n|) + |a_0| + |b_0| + a_0 + b_0 + 2(1-r)|a_m|$$
$$+ 2(1-s)|b_m| - \big[(|a_n| + |b_n|) + 2ra_m + 2sb_m\big] + 2\delta + 2\eta\Big].$$

Proof. Let $P(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + ... + \alpha_{m-1} z^{m-1} + \alpha_m z^m + \alpha_{m+1} z^{m+1} + ... + \alpha_1 z + \alpha_0$ be a polynomial of degree n. Then consider the polynomial

$$\begin{aligned} Q(z) =& (1-z)P(z) \\ =& -\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + ... + (\alpha_{m+1} - \alpha_m)z^{m+1} + (\alpha_m - \alpha_{m-1})z^m \\ &+ ... + (\alpha_1 - \alpha_0)z + \alpha_0. \\ =& -\alpha_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_{m+1} - a_m)z^{m+1} + (a_m - a_{m-1})z^m + ... \end{aligned}$$

$$+ (a_1 - a_0)z + a_0 + i\{(b_n - b_{n-1})z^n + ...$$

$$+ (b_{m+1} - b_m)z^{m+1} + (b_m - b_{m-1})z^m + ... + (b_1 - b_0)z + b_0\}.$$

$$Also\ if\ |z|>1\ then\ \frac{1}{|z|^{n-i}}<1\ for\ i = 0, 1, 2, ..., n-1.\ Now$$

$$|Q(z)| \geq |\alpha_n||z|^{n+1} - \Big\{\big(|a_n - a_{n-1}||z|^n + ... + |a_{m+1} - a_m||z|^{m+1} + |a_m - a_{m-1}||z|^m$$

$$+ ... + |a_1 - a_0||z| + a_0\big) + \big(|b_n - b_{n-1}||z|^n + ...+$$

$$|b_{m+1} - b_m||z|^{m+1} + |b_m - b_{m-1}||z|^m + ... + |b_1 - b_0||z| + b_0\big)\Big\}$$

$$\geq |\alpha_n||z|^n\Bigg[|z| - \frac{1}{|\alpha_n|}\Big\{\Big(|a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} + ... + \frac{|a_{m+1} - a_m|}{|z|^{n-m-1}}$$

$$+ \frac{|a_m - a_{m-1}|}{|z|^{n-m}} + ... + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\Big)$$

$$+ \Big(|b_n - b_{n-1}| + \frac{|b_{n-1} - b_{n-2}|}{|z|} + ... + \frac{|b_{m+1} - b_m|}{|z|^{n-m-1}}$$

$$+ \frac{|b_m - b_{m-1}|}{|z|^{n-m}} + ... + \frac{|b_1 - b_0|}{|z|^{n-1}} + \frac{|b_0|}{|z|^n}\Big)\Big\}\Bigg]$$

$$\geq |\alpha_n||z|^n\Bigg[|z| - \frac{1}{|\alpha_n|}\Big\{\big(|ka_n - a_{n-1} - ka_n + a_n| + |a_{n-1} - a_{n-2}| + ... + |a_{m+1} - ra_m + ra_m - a_m|$$

$$+ |a_m - ra_m + ra_m - a_{m-1}| + ... + |a_1 - \delta - a_0 + \delta| + |a_0|\big)$$

$$+ \big(|lb_n - b_{n-1} - lb_n + b_n| + |b_{n-1} - b_{n-2}| + ... + |b_{m+1} - sb_m + sb_m - b_m|$$

$$+ |b_m - lb_m + sb_m - b_{m-1}| + ... + |b_1 - \eta - b_0 + \eta| + |b_0|\big)\Big\}\Bigg]$$

$$\geq |\alpha_n||z|^n\Bigg[|z| - \frac{1}{|\alpha_n|}\Big\{\big[(ka_n - a_{n-1}) + (k-1)|a_n| + (a_{n-1} - a_{n-2}) + ... + (a_{m+1} - ra_m)$$

$$+ (1-r)|a_m| + (a_{m-1} - ra_m) + (1-r)|a_m| + .. + (a_0 - a_1 + \delta) + \delta + |a_0|\big]$$

$$+ \big[(lb_n - b_{n-1}) + (l-1)|b_n| + (b_{n-1} - b_{n-2}) + ... + (b_{m+1} - sb_m) + (1-s)|b_m|$$

$$+ (b_{m-1} - sb_m) + (1-s)|b_m| + .. + (b_0 - b_1 + \eta) + \eta + |b_0|\big]\Big\}\Bigg]$$

$$= |\alpha_n||z|^n\Bigg[|z| - \frac{1}{|\alpha_n|}\Big\{k(a_n + |a_n|) + l(b_n + |b_n|) + |a_0| + |b_0| + a_0 + b_0 + 2(1-r)|a_m|$$

$$+ 2(1-s)|b_m| - \big[(|a_n| + |b_n|) + 2ra_m + 2sb_m\big] + 2\delta + 2\eta\Big\}\Bigg] > 0$$

$$if\ |z|>\frac{1}{|\alpha_n|}\Big[k(a_n + |a_n|) + l(b_n + |b_n|) + |a_0| + |b_0| + a_0 + b_0 + 2(1-r)|a_m|$$

$$+ 2(1-s)|b_m| - \big[(|a_n| + |b_n|) + 2ra_m + 2sb_m\big] + 2\delta + 2\eta\Big].$$

This shows that if $|z|>1,\ Q(z)>0$ *provided*

$$|z| > \frac{1}{|\alpha_n|}\Bigg[k(a_n+|a_n|)+l(b_n+|b_n|)+|a_0|+|b_0|+a_0+b_0+2(1-r)|a_m| + 2(1-s)|b_m| - \big[(|a_n|+|b_n|)+2ra_m+2sb_m\big]+2\delta+2\eta\Bigg].$$

Hence all the zeros of $Q(z)$ with $|z|>1$ lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Bigg[k(a_n+|a_n|)+l(b_n+|b_n|)+|a_0|+|b_0|+a_0+b_0+2(1-r)|a_m| + 2(1-s)|b_m| - \big[(|a_n|+|b_n|)+2ra_m+2sb_m\big]+2\delta+2\eta\Bigg].$$

But those zeros of $Q(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of $P(z)$ are also the zeros of $Q(z)$ lie in the circle defined by the above inequality and this completes the proof of the Theorem. □

Corollary 2.3.11. *Let* $P(z)=\sum_{i=0}^{n}\alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i)=a_i$ *and* $Im(\alpha_i)=b_i$ *for* $i=0,1,2,...,n$ *such that for some* $k\geq 1,\ 0< r\leq 1,\ \delta\geq 0,\ \ a_m\neq 0,\ b_m\neq 0,$

$$a_0+\delta \geq a_1 \geq ... \geq a_{m-1} \geq ra_m \leq a_{m+1} \leq ... \leq a_{n-1} \leq ka_n \quad and$$

$$b_0+\delta \geq b_1 \geq ... \geq b_{m-1} \geq rb_m \leq b_{m+1} \leq ... \leq b_{n-1} \leq kb_n$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Bigg[k(a_n+b_n+|a_n|+|b_n|)+|a_0|+|b_0|+a_0+b_0 + 2(1-r)[|a_m||b_m|] - \big[(|a_n|+|b_n|)+2r(a_m+b_m)\big]+4\delta\Bigg].$$

Corollary 2.3.12. *Let* $P(z)=\sum_{i=0}^{n}\alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i)=a_i$ *and* $Im(\alpha_i)=b_i$ *for* $i=0,1,2,...,n$ *such that*

$$a_0 \geq a_1 \geq ... \geq a_{m-1} \geq a_m \leq a_{m+1} \leq ... \leq a_{n-1} \leq a_n \quad and$$

$$b_0 \geq b_1 \geq ... \geq b_{m-1} \geq b_m \leq b_{m+1} \leq ... \leq b_{n-1} \leq b_n$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Bigg[(a_n+b_n)-2[a_m+b_m]+|a_0|+|b_0|+a_0+b_0\Bigg].$$

Corollary 2.3.13. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree n with* $Re(\alpha_i) = a_i > 0$, *and* $Im(\alpha_i) = b_i > 0$, *for* $i = 0, 1, 2, ..., n$ *such that for some* $k \geq 1,\ l \geq 1,\ 0 < r \leq 1,\ 0 < s \leq 1,\ \delta \geq 0,\ \eta \geq 0,\ a_m \neq 0,\ b_m \neq 0,$

$$a_0 + \delta \geq a_1 \geq ... \geq a_{m-1} \geq ra_m \leq a_{m+1} \leq ... \leq a_{n-1} \leq ka_n \quad and$$

$$b_0 + \eta \geq b_1 \geq ... \geq b_{m-1} \geq sb_m \leq b_{m+1} \leq ... \leq b_{n-1} \leq lb_n$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Big[2[ka_n + lb_n a_0 + b_0] + 2(1-2r)a_m + 2(1-2s)b_m - (a_n + b_n) + 2\delta + 2\eta\Big].$$

Corollary 2.3.14. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree n with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $k \geq 1,\ 0 < r \leq 1,\ \delta \geq 0,\ \eta \geq 0,\ a_m \neq 0,$

$$a_0 + \delta \geq a_1 \geq ... \geq a_{m-1} \geq ra_m \leq a_{m+1} \leq ... \leq a_{n-1} \leq ka_n \quad and$$

$$b_0 \geq b_1 \geq ... \geq b_{m-1} \geq b_m \leq b_{m+1} \leq ... \leq b_{n-1} \leq b_n$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Big[k(a_n + |a_n|) + b_n + |a_0| + |b_0| + a_0 + b_0 + 2(1-r)|a_m| - \big[|a_n| + 2ra_m + 2b_m\big] + 2\delta\Big].$$

Remark 2.3.15. By taking $l = k,\ s = r$ *and* $\delta = \eta$ in Theorem 2.3.10, it reduces to Corollary 2.3.11

Remark 2.3.16. By taking $l = k = 1,\ s = r = 1$ *and* $\delta = \eta = 0$ in Theorem 2.3.10, it reduces to Corollary 2.3.12

Remark 2.3.17. By taking $a_i > 0$ *and* $b_i > 0$ in Theorem 2.3.10, it reduces to Corollary 2.3.13

Remark 2.3.18. By taking $l = 1,\ s = 1$ *and* $\eta = 0$ in Theorem 2.3.10, it reduces to Corollary 2.3.14

By rearranig coefficients in the above Theorem 2.3.1 and Theorem 2.3.10 we get the follwoing Theorem 2.3.19 and Theorem 2.3.20

Theorem 2.3.19. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $k \geq 1,\ l \geq 1,\ 0< r \leq 1,\ 0< s \leq 1,\ \delta \geq 0,\ \eta \geq 0,\ a_m \neq 0,\ b_m \neq 0,$

$$a_0 - \delta \leq a_1 \leq ... \leq a_{m-1} \leq ka_m \geq a_{m+1} \geq ... \geq a_{n-1} \geq ra_n \quad and$$

$$b_n \leq b_{n-1} \leq ... \leq b_1 \leq b_0$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Big[|a_0| + |b_0| + |a_n| + k(a_m + |a_m|) + 2(k-1)|a_m| - \big[a_0 + r(a_n + |a_n|) + b_n\big] + 2\delta\Big].$$

Theorem 2.3.20. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $k \geq 1,\ 0< r \leq 1,\ \delta \geq 0,\ \eta \geq 0,\ a_m \neq 0,$

$$a_0 + \delta \geq a_1 \geq ... \geq a_{m-1} \geq ra_m \leq a_{m+1} \leq ... \leq a_{n-1} \leq ka_n \quad and$$

$$b_0 \leq b_1 \leq ... \leq b_{m-1} \leq b_m \leq b_{m+1} \leq ... \leq b_{n-1} \leq b_n$$

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Big[k(a_n + |a_n|) + b_n + |a_0| + |b_0| + a_0 + 2(1-r)|a_m| - [|a_n| + 2ra_m + b_0] + 2\delta\Big].$$

The results of section 2.3 have appeared in [36].

2.4 Location of zeros of polynomials with restricted coefficients

Theorem 2.4.1. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq 2$ *with real coefficients such that for some* $k \geq 1, \delta{>}0,\ ka_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq ... \geq a_{n-m+1} \leq a_{n-m} \geq a_{n-m-1} \geq ... \geq a_4 \geq a_3 \geq a_2 \geq a_1 \geq a_0 - \delta$ *if both n and (n-m) are even or odd (OR)*
$ka_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq ... \leq a_{n-m+1} \geq a_{n-m} \geq a_{n-m-1} \geq ... \geq a_4 \geq a_3 \geq a_2 \geq a_1 \geq a_0 - \delta$ *if n is even and (n-m) is odd (or) if n is odd and (n-m) is even*

then all the zeros of P(z) lie in

$|z+k-1| \leq \frac{1}{|a_n|}\{2\delta + ka_n + |a_0| - a_0 + 2([a_{n-2}+a_{n-4}+...+a_{n-m+2}+a_{n-m}]$
$-[a_{n-1}+a_{n-3}+...+a_{n-m+3}+a_{n-m+1}])\}$ *if both n and (n-m) are even or odd (OR)*
$|z+k-1| \leq \frac{1}{|a_n|}\{2\delta + ka_n + |a_0| - a_0 + 2([a_{n-2}+a_{n-4}+...+a_{n-m+3}+a_{n-m+1}]$
$-[a_{n-1}+a_{n-3}+...+a_{n-m+4}+a_{n-m+2}])\}$
if n is even and (n-m) is odd (or) if n is odd and (n-m) is even.

Proof. Let $P(z) = a_nz^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + a_{n-3}z^{n-3} + ... + a_{n-m}z^{n-m} + ... + a_3z^3 + a_2z^2 + a_1z + a_0$ be a polynomial of degree $n \geq 2$.
Then consider the polynomial $Q(z) = (1-z)P(z)$ so that
$Q(z) = -a_nz^n(z+k-1) + (ka_n - a_{n-1})z^n + (a_{n-1}-a_{n-2})z^{n-1} + ... + (a_{n-m+1} - a_{n-m})z^{n-m+1} + (a_{n-m}-a_{n-m-1})z^{n-m} + (a_{n-m-1}-a_{n-m-2})z^{n-m-1} + ... + (a_3-a_2)z^3 + (a_2-a_1)z^2 + (a_1-a_0)z + a_0$
Also if $|z|>1$ then $\frac{1}{|z|^{n-i}}<1$ for $i = 1,2,...,n-1$
Now $|Q(z)| \geq |a_n||z+k-1|^n - \{|ka_n - a_{n-1}||z|^n + |a_{n-1}-a_{n-2}||z|^{n-1} + ... + |a_{n-m+1} - a_{n-m}||z|^{n-m+1} + |a_{n-m}-a_{n-m-1}||z|^{n-m} + |a_{n-m-1}-a_{n-m-2}||z|^{n-m-1} + ... + |a_3-a_2||z|^3 + |a_2-a_1||z|^2 + |a_1-a_0|z + |a_0|\}$
$\geq |a_n||z|^n \Bigg[|z+k-1| - \frac{1}{|a_n|}\{k|a_n-a_{n-1}| + \frac{|a_{n-1}-a_{n-2}|}{|z|} + \frac{|a_{n-2}-a_{n-3}|}{|z|^2} + \frac{|a_{n-3}-a_{n-4}|}{|z|^3} + ... + \frac{|a_{n-m+1}-a_{n-m}|}{|z|^{m-1}} + \frac{|a_{n-m}-a_{n-m-1}|}{|z|^m} + \frac{|a_{n-m-1}-a_{n-m-2}|}{|z|^{m+1}} + ... + \frac{|a_3-a_2|}{|z|^{n-3}} + \frac{|a_2-a_1|}{|z|^{n-2}} + \frac{|a_1-a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\}\Bigg]$
$\geq |a_n||z|^n \Bigg[|z+k-1| - \frac{1}{|a_n|}\{k|a_n-a_{n-1}| + |a_{n-1}-a_{n-2}| + |a_{n-2}-a_{n-3}| + |a_{n-3}-a_{n-4}| + ... + |a_{n-m+1}-a_{n-m}| + |a_{n-m}-a_{n-m-1}| + |a_{n-m-1}-a_{n-m-2}| + .. + |a_3-a_2| + |a_2-a_1| + |a_1-\delta-a_0+\delta| + |a_0|\}\Bigg]$
$\geq |a_n||z|^n \Bigg[|z+k-1| - \frac{1}{|a_n|}\{(ka_n-a_{n-1}) + (a_{n-2}-a_{n-1}) + (a_{n-2}-a_{n-3}) + (a_{n-4}-a_{n-3}) + ... + (a_{n-m}-a_{n-m+1}) + (a_{n-m}-a_{n-m-1}) + (a_{n-m-1}-a_{n-m-2}) + ... + (a_3-a_2) + (a_2-a_1) + (a_1+\delta-a_0) + \delta + |a_0|\}\Bigg]$
if both n and (n-m) are even or odd
$= |a_n||z|^n \Bigg[|z+k-1| - \frac{1}{|a_n|}\{2\delta + ka_n + |a_0| - a_0 + 2([a_{n-2}+a_{n-4}+...+a_{n-m+2}+a_{n-m}] - [a_{n-1}+a_{n-3}+...+a_{n-m+3}+a_{n-m+1}])\}\Bigg] >0$
if $|z+k-1| > \frac{1}{|a_n|}\{2\delta + ka_n + |a_0| - a_0 + 2([a_{n-2}+a_{n-4}+...+a_{n-m+2}+a_{n-m}]$

$- [a_{n-1} + a_{n-3} + ... + a_{n-m+3} + a_{n-m+1}]\big)\big\}.$

This shows that for $|z|>1$, then $Q(z)>0$

if $|z+k-1|>\frac{1}{|a_n|}\big\{2\delta + ka_n + |a_0| - a_0 + 2\big([a_{n-2} + a_{n-4} + ... + a_{n-m+2} + a_{n-m}]$
$- [a_{n-1} + a_{n-3} + ... + a_{n-m+3} + a_{n-m+1}]\big)\big\}.$

Hence all the zeros of $Q(z)$ with $|z|>1$ lie in

$|z+k-1| \leq \frac{1}{|a_n|}\big\{2\delta + ka_n + |a_0| - a_0 + 2\big([a_{n-2} + a_{n-4} + ... + a_{n-m+2} + a_{n-m}] - [a_{n-1} + a_{n-3} + ... + a_{n-m+3} + a_{n-m+1}]\big)\big\}.$ if both n and (n-m) even or odd

But those zeros of $Q(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of $P(z)$ are also the zeros of $Q(z)$ lie in the circle defined by the above inequality and this completes the proof of the Theorem , if both n and (n-m) even or odd.

Similarly we can also prove for n is even and (n-m) is odd (or) if n is odd and (n-m) is even degree polynomials.For this we can rearrange the terms.

That is if n is even and (n-m) is odd (or) if n is odd and (n-m) is even then all the zeros of $P(z)$ lie in

$|z+k-1| \leq \frac{1}{|a_n|}\big\{2\delta + ka_n + |a_0| - a_0 + 2\big([a_{n-2} + a_{n-4} + ... + a_{n-m+3} + a_{n-m+1}]$
$- [a_{n-1} + a_{n-3} + ... + a_{n-m+4} + a_{n-m+2}]\big)\big\}.$ □

Corollary 2.4.2. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq 2$ *with real coefficients such that*

$$a_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq ... \leq a_{n-m+1} \geq a_{n-m} \geq a_{n-m-1} \geq ... \geq a_3 \geq a_2 \geq a_1 \geq a_0$$

if both n and (n-m) are even or odd (OR)

$$a_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq ... \geq a_{n-m+1} \leq a_{n-m} \geq a_{n-m-1} \geq ... \geq a_3 \geq a_2 \geq a_1 \geq a_0$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even
then all the zeros of P(z) lie in

$|z| \leq \frac{1}{|a_n|}\big\{|a_0| - a_0 - a_n + 2\big([a_{n-1} + a_{n-3} + ... + a_{n-m+3} + a_{n-m+1}]$
$- [a_{n-2} + a_{n-4} + ... + a_{n-m+4} + a_{n-m+2}]\big)\big\}$ *if both n and (n-m) are even or odd (OR)*
$|z| \leq \frac{1}{|a_n|}\big\{|a_0| - a_0 - a_n + 2\big([a_{n-1} + a_{n-3} + ... + a_{n-m+2} + a_{n-m}]$
$- [a_{n-2} + a_{n-4} + ... + a_{n-m+3} + a_{n-m+1}]\big)\big\}$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even.

Corollary 2.4.3. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq 2$ *with positive real coefficients such that for some* $0<r \leq 1, \delta>0$,

$$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq ... \leq a_{n-m+1} \geq a_{n-m} \geq a_{n-m-1} \geq ... \geq a_3 \geq a_2 \geq a_1 \geq a_0 - \delta$$

if both n and (n-m) are even or odd (OR)

$$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq ... \geq a_{n-m+1} \leq a_{n-m} \geq a_{n-m-1} \geq ... \geq a_3 \geq a_2 \geq a_1 \geq a_0 - \delta$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even
then all the zeros of P(z) lie in
$|z| \leq \frac{1}{|a_n|}\{2([a_{n-1} + a_{n-3} + ... + a_{n-m+3} + a_{n-m+1}]$
$- [a_{n-2} + a_{n-4} + ... + a_{n-m+4} + a_{n-m+2}] - a_n)\}$ *if both n and (n-m) are even or odd (OR)*
$|z| \leq \frac{1}{|a_n|}\{2([a_{n-1} + a_{n-3} + ... + a_{n-m+2} + a_{n-m}]$
$- [a_{n-2} + a_{n-4} + ... + a_{n-m+3} + a_{n-m+1}] - a_n)\}$
if n is even and (n-m) is odd (or) if n is odd and (n-m) is even.

Remark 2.4.4. By taking $k = 1, \delta = 0$ in Theoem 2.4.1, then it reduces to Corollary 2.4.2

Remark 2.4.5. By taking $a_i>0$, for $i = 0, 1, 2, ..., n$ in Theorem 2.4.1,then it reduces to Corollary 2.4.3

Theorem 2.4.6. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq 2$ *with real coefficients such that for some* $0<r \leq 1, \delta>0$,

$$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq ... \leq a_{n-m+1} \geq a_{n-m} \geq a_{n-m-1} \geq ... \geq a_3 \geq a_2 \geq a_1 \geq a_0 - \delta$$

if both n and (n-m) are even or odd (OR)

$$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq ... \geq a_{n-m+1} \leq a_{n-m} \geq a_{n-m-1} \geq ... \geq a_3 \geq a_2 \geq a_1 \geq a_0 - \delta$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then all the zeros of P(z) lie in

$|z| \leq \frac{1}{|a_n|}\Big\{|a_0| - a_0 - a_n + 2\Big([a_{n-1} + a_{n-3} + ... + a_{n-m+3} + a_{n-m+1}]$

$- [a_{n-2} + a_{n-4} + ... + a_{n-m+4} + a_{n-m+2}]\Big)\Big\}$ *if both n and (n-m) are even or odd (OR)*

$|z| \leq \frac{1}{|a_n|}\Big\{|a_0| - a_0 - a_n + 2\Big([a_{n-1} + a_{n-3} + ... + a_{n-m+2} + a_{n-m}]$

$- [a_{n-2} + a_{n-4} + ... + a_{n-m+3} + a_{n-m+1}]\Big)\Big\}$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even.

Proof. Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + a_{n-3}z^{n-3} + ... + a_{n-m}z^{n-m} + ... + a_3 z^3 + a_2 z^2 + a_1 z + a_0$ be a polynomial of degree $n \geq 2$.

Then consider the polynomial $Q(z) = (1 - z)P(z)$ so that

$Q(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + ... + (a_{n-m+1} - a_{n-m})z^{n-m+1} + (a_{n-m} - a_{n-m-1})z^{n-m} + (a_{n-m-1} - a_{n-m-2})z^{n-m-1} + ... + (a_3 - a_2)z^3 + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0.$

Also if $|z|>1$ then $\frac{1}{|z|^{n-i}}<1$ for $i = 0, 1, 2, ..., n - 1$. Now

$|Q(z)| \geq |a_n||z|^{n+1} - \Big\{|a_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} + ... + |a_{n-m+1} - a_{n-m}||z|^{n-m+1} + |a_{n-m} - a_{n-m-1}||z|^{n-m} + |a_{n-m-1} - a_{n-m-2}||z|^{n-m-1} + ... + |a_3 - a_2||z|^3 + |a_2 - a_1||z|^2 + |a_1 - a_0|z + |a_0|\Big\}$

$\geq |a_n||z|^n \Bigg[|z| - \frac{1}{|a_n|}\Big\{|a_n - a_{n-1}| + \frac{|a_{n-1}-a_{n-2}|}{|z|} + \frac{|a_{n-2}-a_{n-3}|}{|z|^2} + \frac{|a_{n-3}-a_{n-4}|}{|z|^3} + ... + \frac{|a_{n-m+1}-a_{n-m}|}{|z|^{m-1}} + \frac{|a_{n-m}-a_{n-m-1}|}{|z|^m} + \frac{|a_{n-m-1}-a_{n-m-2}|}{|z|^{m+1}} + ... + \frac{|a_3-a_2|}{|z|^{n-3}} + \frac{|a_2-a_1|}{|z|^{n-2}} + \frac{|a_1-a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\Big\}\Bigg]$

$\geq |a_n||z|^n \Bigg[|z| - \frac{1}{|a_n|}\{|ra_n - a_{n-1} - ra_n + a_n| + |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}| + |a_{n-3} - a_{n-4}| + ... + |a_{n-m+1} - a_{n-m}| + |a_{n-m} - a_{n-m-1}| + |a_{n-m-1} - a_{n-m-2}| + ... + |a_3 - a_2| + |a_2 - a_1| + |a_1 + \delta - a_0 - \delta| + |a_0|\Big\}\Bigg]$

$\geq |a_n||z|^n \Bigg[|z| - \frac{1}{|a_n|}\{(a_{n-1} - ra_n) + (1 - r)|a_n| + (a_{n-1} - a_{n-2}) + (a_{n-3} - a_{n-2}) + (a_{n-3} - a_{n-4}) + ... + (a_{n-m+1} - a_{n-m+2}) + (a_{n-m+1} - a_{n-m}) + (a_{n-m} - a_{n-m-1}) + (a_{n-m-1} - a_{n-m-2}) + ... + (a_3 - a_2) + (a_2 - a_1) + (a_1 - a_0 + \delta) + \delta + |a_0|\Big\}\Bigg]$ if both n and (n-m) are even or odd

$= |a_n||z|^n \Bigg[|z| - \frac{1}{|a_n|}\Big\{|a_n| + |a_0| - a_0 - r(|a_n| + a_n) + 2\delta + 2\Big([a_{n-1} + a_{n-3} + ... + a_{n-m+3} + a_{n-m+1}] - [a_{n-2} + a_{n-4} + ... + a_{n-m+4} + a_{n-m+2}]\Big)\Big\}\Bigg] >0$

if $|z|>\frac{1}{|a_n|}\big\{|a_n|+|a_0|-a_0-r(|a_n|+a_n)+2\delta+2\big([a_{n-1}+a_{n-3}+...+a_{n-m+3}+a_{n-m+1}]$
$-[a_{n-2}+a_{n-4}+...+a_{n-m+4}+a_{n-m+2}]\big)\big\}$.

This shows that for $|z|>1$, then $Q(z)>0$
if $|z|>\frac{1}{|a_n|}\big\{|a_n|+|a_0|-a_0-r(|a_n|+a_n)+2\delta+2\big([a_{n-1}+a_{n-3}+...+a_{n-m+3}+a_{n-m+1}]$
$-[a_{n-2}+a_{n-4}+...+a_{n-m+4}+a_{n-m+2}]\big)\big\}$.

Hence all the zeros of $Q(z)$ with $|z|>1$ lie in
$|z|\leq\frac{1}{|a_n|}\big\{|a_n|+|a_0|-a_0-r(|a_n|+a_n)+2\delta+2\big([a_{n-1}+a_{n-3}+...+a_{n-m+3}+a_{n-m+1}]-$
$[a_{n-2}+a_{n-4}+...+a_{n-m+4}+a_{n-m+2}]\big)\big\}$. if both n and (n-m) even or odd

But those zeros of $Q(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of $P(z)$ are also the zeros of $Q(z)$ lie in the circle defined by the above inequality and this completes the proof of the Theorem , if both n and (n-m) are even or odd.

Similarly we can also prove for n is even and (n-m) is odd (or) if n is odd and (n-m) is even degree polynomials.For this we can rearrange the terms. That is if n is even and (n-m) is odd (or) if n is odd and (n-m) is even then all the zeros $P(z)$ lie in
$|z|\leq\frac{1}{|a_n|}\big\{|a_n|+|a_0|-a_0-r(|a_n|+a_n)+2\delta+2\big([a_{n-1}+a_{n-3}+...+a_{n-m+2}+a_{n-m}]$
$-[a_{n-2}+a_{n-4}+...+a_{n-m+3}+a_{n-m+1}]\big)\big\}$. □

Corollary 2.4.7. *Let* $P(z)=\sum_{i=0}^{n}a_iz^i$ *be a polynomial of degree* $n\geq 2$ *with real coefficients such that*

$$a_n\leq a_{n-1}\geq a_{n-2}\leq a_{n-3}\geq a_{n-4}\leq...\leq a_{n-m+1}\geq a_{n-m}\geq a_{n-m-1}\geq...\geq a_4\geq a_3\geq a_2 \geq a_1\geq a_0$$

if both n and (n-m) are even or odd
(OR)

$$a_n\leq a_{n-1}\geq a_{n-2}\leq a_{n-3}\geq a_{n-4}\leq...\geq a_{n-m+1}\leq a_{n-m}\geq a_{n-m-1}\geq...\geq a_4\geq a_3\geq a_2 \geq a_1\geq a_0$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even
then all the zeros of P(z) lie in

$|z| \leq \frac{1}{|a_n|}\big\{|a_0| - a_0 - a_n + 2\big([a_{n-1} + a_{n-3} + ... + a_{n-m+3} + a_{n-m+1}]$
$- [a_{n-2} + a_{n-4} + ... + a_{n-m+4} + a_{n-m+2}]\big)\big\}$ *if both n and (n-m) are even or odd (OR)*
$|z| \leq \frac{1}{|a_n|}\big\{|a_0| - a_0 - a_n + 2\big([a_{n-1} + a_{n-3} + ... + a_{n-m+2} + a_{n-m}]$
$- [a_{n-2} + a_{n-4} + ... + a_{n-m+3} + a_{n-m+1}]\big)\big\}$
if n is even and (n-m) is odd (or) if n is odd and (n-m) is even.

Corollary 2.4.8. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq 2$ *with positive real coefficients such that for some* $0<r \leq 1, \delta>0$*,*

$$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq ... \leq a_{n-m+1} \geq a_{n-m} \geq a_{n-m-1} \geq ....$$

$$\geq a_4 \geq a_3 \geq a_2 \geq a_1 \geq a_0 - \delta$$

if both n and (n-m) are even or odd
(OR)

$$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq ... \geq a_{n-m+1} \leq a_{n-m} \geq a_{n-m-1} \geq ....$$

$$\geq a_4 \geq a_3 \geq a_2 \geq a_1 \geq a_0 - \delta$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even
then all the zeros of P(z) lie in
$|z| \leq \frac{1}{|a_n|}\big\{2\big([a_{n-1} + a_{n-3} + ... + a_{n-m+3} + a_{n-m+1}]$
$- [a_{n-2} + a_{n-4} + ... + a_{n-m+4} + a_{n-m+2}] - a_n\big)\big\}$ *if both n and (n-m) are even or odd*
(OR)
$|z| \leq \frac{1}{|a_n|}\big\{2\big([a_{n-1} + a_{n-3} + ... + a_{n-m+2} + a_{n-m}]$
$- [a_{n-2} + a_{n-4} + ... + a_{n-m+3} + a_{n-m+1}] - a_n\big)\big\}$
if n is even and (n-m) is odd (or) if n is odd and (n-m) is even.

Remark 2.4.9. By taking $r = 1, \delta = 0$ in Theoem 2.4.6, then it reduces to Corollary 2.4.7

Remark 2.4.10. By taking $a_i>0$, for $i = 0, 1, 2, ..., n$ in Theorem 2.4.6,then it reduces to Corollary 2.4.8

Theorem 2.4.11. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq 2$ *with real coefficients such that for some* $k \geq 1, \delta{>}0$,

$$ka_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq ... \geq a_{n-m+1} \leq a_{n-m} \leq a_{n-m-1} \leq ....$$

$$\leq a_4 \leq a_3 \leq a_2 \leq a_1 \leq a_0 + \delta$$

if both n and (n-m) are even or odd

(OR)

$$ka_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq ... \leq a_{n-m+1} \geq a_{n-m} \leq a_{n-m-1} \leq ....$$

$$\leq a_4 \leq a_3 \leq a_2 \leq a_1 \leq a_0 + \delta$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then all the zeros of P(z) lie in

$|z + k - 1| \leq \frac{1}{|a_n|}\big\{2\delta + ka_n + |a_0| + a_0 + 2\big([a_{n-2} + a_{n-4} + ... + a_{n-m+4} + a_{n-m+2}]$

$- [a_{n-1} + a_{n-3} + ... + a_{n-m+3} + a_{n-m+1}]\big)\big\}$ *if both n and (n-m) are even or odd (OR)*

$|z + k - 1| \leq \frac{1}{|a_n|}\big\{2\delta + ka_n + |a_0| + a_0 + 2\big([a_{n-2} + a_{n-4} + ... + a_{n-m+3} + a_{n-m+1}]$

$- [a_{n-1} + a_{n-3} + ... + a_{n-m+2} + a_{n-m}]\big)\big\}$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even.

Proof. Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + a_{n-3}z^{n-3} + ... + a_{n-m}z^{n-m} + ... + a_3 z^3 + a_2 z^2 + a_1 z + a_0$ be a polynomial of degree $n \geq 2$.

Then consider the polynomial $Q(z) = (1 - z)P(z)$ so that

$Q(z) = -a_n z^n(z + k - 1) + (ka_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + ... + (a_{n-m+1} - a_{n-m})z^{n-m+1} +$

$(a_{n-m} - a_{n-m-1})z^{n-m} + (a_{n-m-1} - a_{n-m-2})z^{n-m-1} + ... + (a_3 - a_2)z^3 + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0$

Also if $|z|{>}1$ then $\frac{1}{|z|^{n-i}}{<}1$ for $i = 0, 1, 2, ..., n - 1$.

Now $|Q(z)| \geq |a_n||z + k - 1|^n - \big\{|ka_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} + ... + |a_{n-m+1} - a_{n-m}||z|^{n-m+1} + |a_{n-m} - a_{n-m-1}||z|^{n-m} + |a_{n-m-1} - a_{n-m-2}||z|^{n-m-1} + ... + |a_3 - a_2||z|^3 + |a_2 - a_1||z|^2 + |a_1 - a_0|z + |a_0|\big\}$

$\geq |a_n||z|^n \Bigg[|z + k - 1| - \frac{1}{|a_n|}\big\{k|a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} + \frac{|a_{n-2} - a_{n-3}|}{|z|^2} + \frac{|a_{n-3} - a_{n-4}|}{|z|^3} + ... +$

$\frac{|a_{n-m+1}-a_{n-m}|}{|z|^{m-1}} + \frac{|a_{n-m}-a_{n-m-1}|}{|z|^{m}} + \frac{|a_{n-m-1}-a_{n-m-2}|}{|z|^{m+1}} + ... + \frac{|a_3-a_2|}{|z|^{n-3}} + \frac{|a_2-a_1|}{|z|^{n-2}} + \frac{|a_1-a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\Big\}\Bigg]$

$\geq |a_n||z|^n\Bigg[|z+k-1| - \frac{1}{|a_n|}\Big\{k|a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}| + |a_{n-3} - a_{n-4}| +$
$... + |a_{n-m+1} - a_{n-m}| + |a_{n-m} - a_{n-m-1}| + |a_{n-m-1} - a_{n-m-2}| + .. + |a_3 - a_2| + |a_2 - a_1| +$
$|a_1 - \delta - a_0 + \delta| + |a_0|\Big\}\Bigg]$

$\geq |a_n||z|^n\Bigg[|z+k-1| - \frac{1}{|a_n|}\Big\{(ka_n - a_{n-1}) + (a_{n-2} - a_{n-1}) + (a_{n-2} - a_{n-3}) + (a_{n-4} - a_{n-3})$
$+ ... + (a_{n-m+2} - a_{n-m+1}) + (a_{n-m} - a_{n-m+1}) + (a_{n-m-1} - a_{n-m}) + (a_{n-m-2} - a_{n-m-1})$
$+ ... + (a_2 - a_3) + (a_1 - a_2) + (a_0 + \delta - a_1) + \delta + |a_0|\Big\}\Bigg]$

if both n and (n-m) are even or odd

$= |a_n||z|^n\Bigg[|z+k-1| - \frac{1}{|a_n|}\Big\{2\delta + ka_n + |a_0| + a_0 + 2\big([a_{n-2} + a_{n-4} + ... + a_{n-m+4} + a_{n-m+2}]$
$- [a_{n-1} + a_{n-3} + ... + a_{n-m+3} + a_{n-m+1}]\big)\Big\}\Bigg]$

>0

if $|z + k - 1| > \frac{1}{|a_n|}\Big\{2\delta + ka_n + |a_0| + a_0 + 2\big([a_{n-2} + a_{n-4} + ... + a_{n-m+4} + a_{n-m+2}]$
$- [a_{n-1} + a_{n-3} + ... + a_{n-m+3} + a_{n-m+1}]\big)\Big\}$.

Thish shows that for $|z|>1$, then $Q(z)>0$

if $|z + k - 1| > \frac{1}{|a_n|}\Big\{2\delta + ka_n + |a_0| - a_0 + 2([a_{n-2} + a_{n-4} + ... + a_{n-m+4} + a_{n-m+2}]$
$- [a_{n-1} + a_{n-3} + ... + a_{n-m+3} + a_{n-m+1}]\big)\Big\}$.

Hence all the zeros of $Q(z)$ with $|z|>1$ lie in

$|z + k - 1| \leq \frac{1}{|a_n|}\Big\{2\delta + ka_n + |a_0| + a_0 + 2([a_{n-2} + a_{n-4} + ... + a_{n-m+4} + a_{n-m+2}]$
$- [a_{n-1} + a_{n-3} + ... + a_{n-m+3} + a_{n-m+1}]\big)\Big\}$. if both n and (n-m) even or odd

But those zeros of $Q(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of $P(z)$ are also the zeros of $Q(z)$ lie in the circle defined by the above inequality and this completes the proof of the Theorem, if both n and (n-m) are even or odd.

Similarly we can also prove for n is even and (n-m) is odd (or) if n is odd and (n-m) is even degree polynomials.For this we can rearrange the terms. That is if n is even and (n-m) is odd (or) if n is odd and (n-m) is even then all the zeros $P(z)$ lie in

$|z+k-1| \leq \frac{1}{|a_n|}\big\{2\delta + ka_n + |a_0| + a_0 + 2\big([a_{n-2} + a_{n-4} + ... + a_{n-m+3} + a_{n-m+1}]$
$- [a_{n-1} + a_{n-3} + ... + a_{n-m+2} + a_{n-m}]\big)\big\}.$

□

Corollary 2.4.12. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq 2$ *with real coefficients such that*

$$a_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq ... \geq a_{n-m+1} \leq a_{n-m} \leq a_{n-m-1} \leq ... \leq a_4 \leq a_3$$

$$\leq a_2 \leq a_1 \leq a_0$$

if both n and (n-m) are even or odd
(OR)

$$a_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq ... \leq a_{n-m+1} \geq a_{n-m} \leq a_{n-m-1} \leq ... \leq a_4 \leq a_3$$

$$\leq a_2 \leq a_1 \leq a_0$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even
then all the zeros of P(z) lie in
$|z| \leq \frac{1}{|a_n|}\big\{|a_0| + a_0 + a_n + 2\big([a_{n-2} + a_{n-4} + ... + a_{n-m+4} + a_{n-m+2}]$
$- [a_{n-1} + a_{n-3} + ... + a_{n-m+3} + a_{n-m+1}]\big)\big\}$ *if both n and (n-m) are even or odd (OR)*
$|z| \leq \frac{1}{|a_n|}\big\{|a_0| + a_0 + a_n + 2\big([a_{n-2} + a_{n-4} + ... + a_{n-m+3} + a_{n-m+1}]$
$- [a_{n-1} + a_{n-3} + ... + a_{n-m+2} + a_{n-m}]\big)\big\}$
if n is even and (n-m) is odd (or) if n is odd and (n-m) is even.

Corollary 2.4.13. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq 2$ *with positive real coefficients such that for some* $k \geq 1, \delta{>}0$,

$$ka_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq ... \geq a_{n-m+1} \leq a_{n-m} \leq a_{n-m-1} \leq ....$$

$$\leq a_4 \leq a_3 \leq a_2 \leq a_1 \leq a_0 + \delta$$

if both n and (n-m) are even or odd
(OR)

$$ka_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq ... \leq a_{n-m+1} \geq a_{n-m} \leq a_{n-m-1} \leq ....$$

$$\leq a_4 \leq a_3 \leq a_2 \leq a_1 \leq a_0 + \delta$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then all the zeros of P(z) lie in

$|z+k-1| \leq \frac{1}{|a_n|}\big\{2\delta + ka_n + 2\big(a_0 + [a_{n-2} + a_{n-4} + ... + a_{n-m+4} + a_{n-m+2}]$

$- [a_{n-1} + a_{n-3} + ... + a_{n-m+3} + a_{n-m+1}]\big)\big\}$ *if both n and (n-m) are even or odd (OR)*

$|z+k-1| \leq \frac{1}{|a_n|}\big\{2\delta + ka_n + 2\big(a_0 + [a_{n-2} + a_{n-4} + ... + a_{n-m+3} + a_{n-m+1}]$

$- [a_{n-1} + a_{n-3} + ... + a_{n-m+2} + a_{n-m}]\big)\big\}$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even.

Remark 2.4.14. By taking $k = 1, \delta = 0$ in Theorem 2.4.11, then it reduces to Corollary 2.4.12

Remark 2.4.15. By taking $a_i>0$, for $i = 0, 1, 2, ..., n$ in Theorem 2.4.11,then it reduces to Corollary 2.4.13

Theorem 2.4.16. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq 2$ *with real coefficients such that for some* $0<r \leq 1, \delta>0$,

$$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq ... \leq a_{n-m+1} \geq a_{n-m} \leq a_{n-m-1} \leq ....$$

$$\leq a_4 \leq a_3 \leq a_2 \leq a_1 \leq a_0 + \delta$$

if both n and (n-m) are even or odd

(OR)

$$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq ... \geq a_{n-m+1} \leq a_{n-m} \leq a_{n-m-1} \leq ....$$

$$\leq a_4 \leq a_3 \leq a_2 \leq a_1 \leq a_0 + \delta$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then all the zeros of P(z) lie in

$|z| \leq \frac{1}{|a_n|}\big\{|a_n| + |a_0| + a_0 - r(|a_n| + a_n) + 2\delta + 2\big([a_{n-1} + a_{n-3} + ... + a_{n-m+3} + a_{n-m+1}]$

$- [a_{n-2} + a_{n-4} + ... + a_{n-m+2} + a_{n-m}]\big)\big\}$ *if both n and (n-m) are even or odd (OR)*

$|z| \leq \frac{1}{|a_n|}\big\{|a_n| + |a_0| + a_0 - r(|a_n| + a_n) + 2\delta + 2\big([a_{n-1} + a_{n-3} + ... + a_{n-m+4} + a_{n-m+2}]$

$-\,[a_{n-2}+a_{n-4}+...+a_{n-m+3}+a_{n-m+1}]\big)\Big\}$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even.

Proof. Let $P(z)=a_nz^n+a_{n-1}z^{n-1}+a_{n-2}z^{n-2}+a_{n-3}z^{n-3}+...+a_{n-m}z^{n-m}+...+a_3z^3+a_2z^2+a_1z+a_0$ be a polynomial of degree $n\geq 2$.

Then consider the polynomial $Q(z)=(1-z)P(z)$ so that

$Q(z)=-a_nz^{n+1}+(a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+...+(a_{n-m+1}-a_{n-m})z^{n-m+1}+(a_{n-m}-a_{n-m-1})z^{n-m}+(a_{n-m-1}-a_{n-m-2})z^{n-m-1}+...+(a_3-a_2)z^3+(a_2-a_1)z^2+(a_1-a_0)z+a_0$

Also if $|z|>1$ then $\frac{1}{|z|^{n-i}}<1$ for $i=0,1,2,...,n-1$

Now $|Q(z)|\geq |a_n||z|^{n+1}-\Big\{|a_n-a_{n-1}||z|^n+|a_{n-1}-a_{n-2}||z|^{n-1}+...+|a_{n-m+1}-a_{n-m}||z|^{n-m+1}$

$+|a_{n-m}-a_{n-m-1}||z|^{n-m}+|a_{n-m-1}-a_{n-m-2}||z|^{n-m-1}+...+|a_3-a_2||z|^3+|a_2-a_1||z|^2+|a_1-a_0|z+|a_0|\Big\}$

$\geq |a_n||z|^n\Bigg[|z|-\frac{1}{|a_n|}\Big\{|a_n-a_{n-1}|+\frac{|a_{n-1}-a_{n-2}|}{|z|}+\frac{|a_{n-2}-a_{n-3}|}{|z|^2}+\frac{|a_{n-3}-a_{n-4}|}{|z|^3}+...+\frac{|a_{n-m+1}-a_{n-m}|}{|z|^{m-1}}+\frac{|a_{n-m}-a_{n-m-1}|}{|z|^m}+\frac{|a_{n-m-1}-a_{n-m-2}|}{|z|^{m+1}}+...+\frac{|a_3-a_2|}{|z|^{n-3}}+\frac{|a_2-a_1|}{|z|^{n-2}}+\frac{|a_1-a_0|}{|z|^{n-1}}+\frac{|a_0|}{|z|^n}\Big\}\Bigg]$

$\geq |a_n||z|^n\Bigg[|z|-\frac{1}{|a_n|}\Big\{|ra_n-a_{n-1}-ra_n+a_n|+|a_{n-1}-a_{n-2}|+|a_{n-2}-a_{n-3}|+|a_{n-3}-a_{n-4}|+...+|a_{n-m+1}-a_{n-m}|+|a_{n-m}-a_{n-m-1}|+|a_{n-m-1}-a_{n-m-2}|+..+|a_3-a_2|+|a_2-a_1|+|a_1-\delta-a_0+\delta|+|a_0|\Big\}\Bigg]$

$\geq |a_n||z|^n\Bigg[|z|-\frac{1}{|a_n|}\Big\{(a_{n-1}-ra_n)+(1-r)|a_n|+(a_{n-1}-a_{n-2})+(a_{n-3}-a_{n-2})+(a_{n-3}-a_{n-4})+...+(a_{n-m+1}-a_{n-m+2})+(a_{n-m+1}-a_{n-m})+(a_{n-m-1}-a_{n-m})+(a_{n-m-2}-a_{n-m-1})+...+(a_2-a_3)+(a_1-a_2)+(a_0+\delta-a_1)+\delta+|a_0|\Big\}\Bigg]$ if both n and (n-m) are even or odd

$=|a_n||z|^n\Bigg[|z|-\frac{1}{|a_n|}\Big\{|a_n|+|a_0|+a_0-r(|a_n|+a_n)+2\delta+2\big([a_{n-1}+a_{n-3}+...+a_{n-m+3}+a_{n-m+1}]-[a_{n-2}+a_{n-4}+...+a_{n-m+2}+a_{n-m}]\big)\Big\}\Bigg]$

>0

if $|z|>\frac{1}{|a_n|}\Big\{|a_n|+|a_0|+a_0-r(|a_n|+a_n)+2\delta+2\big([a_{n-1}+a_{n-3}+...+a_{n-m+3}+a_{n-m+1}]-[a_{n-2}+a_{n-4}+...+a_{n-m+2}+a_{n-m}]\big)\Big\}$.

This shows that for $|z|>1$, then $Q(z)>0$

if $|z|>\frac{1}{|a_n|}\big\{|a_n|+|a_0|+a_0-r(|a_n|+a_n)+2\delta+2\big([a_{n-1}+a_{n-3}+...+a_{n-m+3}+a_{n-m+1}]$
$-[a_{n-2}+a_{n-4}+...+a_{n-m+2}+a_{n-m}]\big)\big\}$.
Hence all the zeros of $Q(z)$ with $|z|>1$ lie in
$|z|\leq\frac{1}{|a_n|}\big\{|a_n|+|a_0|+a_0-r(|a_n|+a_n)+2\delta+2\big([a_{n-1}+a_{n-3}+...+a_{n-m+3}+a_{n-m+1}]$
$-[a_{n-2}+a_{n-4}+...+a_{n-m+2}+a_{n-m}]\big)\big\}$.
if both n and (n-m) are even or odd

But those zeros of $Q(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of $P(z)$ are also the zeros of $Q(z)$ lie in the circle defined by the above inequality and this completes the proof of the Theorem, if both n and (n-m) are even or odd.

Similarly we can also prove for n is even and (n-m) is odd (or) if n is odd and (n-m) is even degree polynomials.For this we can rearrange the terms. That is if n is even and (n-m) is odd (or) if n is odd and (n-m) is even then all the zeros $P(z)$ lie in
$|z|\leq\frac{1}{|a_n|}\big\{|a_n|+|a_0|+a_0-r(|a_n|+a_n)+2\delta+2\big([a_{n-1}+a_{n-3}+...+a_{n-m+4}+a_{n-m+2}]$
$-[a_{n-2}+a_{n-4}+...+a_{n-m+3}+a_{n-m+1}]\big)\big\}$.

□

Corollary 2.4.17. *Let* $P(z)=\sum_{i=0}^{n}a_iz^i$ *be a polynomial of degree* $n\geq 2$ *with real coefficients such that*

$$a_n\leq a_{n-1}\geq a_{n-2}\leq a_{n-3}\geq a_{n-4}\leq ...\leq a_{n-m+1}\geq a_{n-m}\leq a_{n-m-1}\leq ...\leq a_4\leq a_3$$
$$\leq a_2\leq a_1\leq a_0$$

if both n and (n-m) are even or odd
(OR)

$$a_n\leq a_{n-1}\geq a_{n-2}\leq a_{n-3}\geq a_{n-4}\leq ...\geq a_{n-m+1}\leq a_{n-m}\leq a_{n-m-1}\leq ...\leq a_4\leq a_3$$
$$\leq a_2\leq a_1\leq a_0$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even
then all the zeros of P(z) lie in

$|z| \leq \frac{1}{|a_n|}\{|a_0| + a_0 - a_n + 2([a_{n-1} + a_{n-3} + ... + a_{n-m+3} + a_{n-m+1}]$
$- [a_{n-2} + a_{n-4} + ... + a_{n-m+2} + a_{n-m}])\}$ *if both n and (n-m) are even or odd (OR)*
$|z| \leq \frac{1}{|a_n|}\{|a_0| + a_0 - a_n + 2([a_{n-1} + a_{n-3} + ... + a_{n-m+4} + a_{n-m+2}]$
$- [a_{n-2} + a_{n-4} + ... + a_{n-m+3} + a_{n-m+1}])\}$
if n is even and (n-m) is odd (or) if n is odd and (n-m) is even.

Corollary 2.4.18. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq 2$ *with positive real coefficients such that for some* $0<r \leq 1, \delta>0$,

$$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq ... \leq a_{n-m+1} \geq a_{n-m} \leq a_{n-m-1} \leq ....$$

$$\leq a_4 \leq a_3 \leq a_2 \leq a_1 \leq a_0 + \delta$$

if both n and (n-m) are even or odd
(OR)

$$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq ... \geq a_{n-m+1} \leq a_{n-m} \leq a_{n-m-1} \leq ....$$

$$\leq a_4 \leq a_3 \leq a_2 \leq a_1 \leq a_0 + \delta$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even
then all the zeros of P(z) lie in
$|z| \leq \frac{1}{|a_n|}\{(1-2r)a_n + 2\delta + 2(a_0 + [a_{n-1} + a_{n-3} + ... + a_{n-m+3} + a_{n-m+1}]$
$- [a_{n-2} + a_{n-4} + ... + a_{n-m+2} + a_{n-m}])\}$ *if both n and (n-m) are even or odd (OR)*
$|z| \leq \frac{1}{|a_n|}\{(1-2r)a_n + 2\delta + 2(a_0 + [a_{n-1} + a_{n-3} + ... + a_{n-m+4} + a_{n-m+2}]$
$- [a_{n-2} + a_{n-4} + ... + a_{n-m+3} + a_{n-m+1}])\}$
if n is even and (n-m) is odd (or) if n is odd and (n-m) is even.

Remark 2.4.19. By taking $r = 1, \delta = 0$ in Theorem 2.4.16, then it reduces to Corollary 2.4.17

Remark 2.4.20. By taking $a_i>0$, for $i = 0, 1, 2, ..., n$ in Theorem 2.4.16,then it reduces to Corollary 2.4.18

The results of section 2.4 have appeared in [29].

2.5 On the location of zeros of polynomials with different complex coefficients

Theorem 2.5.1. *Let* $P(z) = \sum_{j=0}^{n} \alpha_j z^j$ *be a polynomial with complex coefficients of degree* $n \geq 2$ *with* $Re(\alpha_j) = a_j$ *and* $Im(\alpha_j) = b_j$ *for* $j = 0, 1, 2, ..., n$ *such that*

$a_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq ... \leq a_4 \geq a_3 \leq a_2 \geq a_1 \leq a_0$ *and*

$b_n \geq b_{n-1} \leq b_{n-2} \geq b_{n-3} \leq b_{n-4} \geq ... \leq b_4 \geq b_3 \leq b_2 \geq b_1 \leq b_0$ *if n is even*

OR

$a_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq ... \geq a_4 \leq a_3 \geq a_2 \leq a_1 \geq a_0$ *and*

$b_n \geq b_{n-1} \leq b_{n-2} \geq b_{n-3} \leq b_{n-4} \geq ... \geq b_4 \leq b_3 \leq b_2 \leq b_1 \geq b_0$ *if n is odd*

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Big[|a_0| + a_0 + a_n + 2([a_{n-2} + a_{n-4} + ... + a_4 + a_2] - [a_{n-1} + a_{n-3} + ...a_3 + a_1]) + |b_0| + b_0 + b_n + 2([b_{n-2} + b_{n-4} + ... + b_4 + b_2] - [b_{n-1} + b_{n-3}... + b_3 + b_1])\Big] \text{ if } n \text{ is even}$$

OR

$$|z| \leq \frac{1}{|\alpha_n|}\Big[|a_0| - a_0 + a_n + 2([a_{n-2} + a_{n-4} + ... + a_3 + a_1] - [a_{n-1} + a_{n-3} + ...a_4 + a_2]) + |b_0| - b_0 + b_n + 2([b_{n-2} + b_{n-4} + ... + b_3 + b_1] - [b_{n-1} + b_{n-3}... + b_4 + b_2])\Big] \text{ if } n \text{ is odd}$$

Proof. Let $P(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + ... + \alpha_3 z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0$ be a polynomial with complex coefficients of degree n≥ 2 $with\ \alpha_j = a_j + ib_j for\ j = 0, 1, 2, ..., n$

Then consider the polynomial

$$\begin{aligned} Q(z) =& (1 - z)P(z) \\ =& -\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + (\alpha_{n-2} - \alpha_{n-3})z^{n-2} + ... \\ & + (\alpha_1 - \alpha_0)z + \alpha_0. \\ =& -\alpha_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + (a_{n-2} - a_{n-3})z^{n-2} + ... \\ & + (a_1 - a_0)z + a_0 + i\big\{(b_n - b_{n-1})z^n + (b_{n-1} - b_{n-2})z^{n-1} + (b_{n-2} - b_{n-3})z^{n-2} + ... \\ & + (b_1 - b_0)z + b_0\big\}. \end{aligned}$$

$$If\ |z|>1\ then\ \frac{1}{|z|^{n-i}}<1\ for\ i = 0, 1, 2, ..., n - 1.$$

$$Now\ |Q(z)| \geq |\alpha_n||z|^{n+1} - \{|a_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} + |a_{n-2} - a_{n-3}||z|^{n-2} + ...$$
$$+ |a_3 - a_2||z|^3 + |a_2 - a_1||z|^2 + |a_1 - a_0||z| + |a_0| + |b_n - b_{n-1}||z|^n$$
$$+ |b_{n-1} - b_{n-2}||z|^{n-1}$$
$$+ |b_{n-2} - b_{n-3}||z|^{n-2} + ... + |b_3 - b_2||z|^3 + |b_2 - b_1||z|^2 + |b_1 - b_0||z| + |b_0|\}$$
$$\geq |\alpha_n||z|^n \Bigg[|z| - \frac{1}{|\alpha_n|}\{|a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} + \frac{|a_{n-2} - a_{n-3}|}{|z|^2} + ... + \frac{|a_3 - a_2|}{|z|^{n-3}}$$
$$+ \frac{|a_2 - a_1|}{|z|^{n-2}} + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} + |b_n - b_{n-1}| + \frac{|b_{n-1} - b_{n-2}|}{|z|^1} + \frac{|b_{n-2} - b_{n-3}|}{|z|^2} + ...$$
$$+ \frac{|b_3 - b_2|}{|z|^{n-3}} + \frac{|b_2 - b_1|}{|z|^{n-2}} + \frac{|b_1 - b_0|}{|z|^n} + \frac{|b_0|}{|z|^n}\}\Bigg]$$
$$\geq |\alpha_n||z|^n \Bigg[|z| - \frac{1}{|\alpha_n|}\{|a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}| + ... + |a_3 - a_2|$$
$$+ |a_2 - a_1| + |a_1 - a_0| + |a_0| + |b_n - b_{n-1}| + |b_{n-1} - b_{n-2}| + |b_{n-2} - b_{n-3}| + ...$$
$$+ |b_3 - b_2| + |b_2 - b_1| + |b_1 - b_0| + |b_0|\}\Bigg]$$
$$\geq |\alpha_n||z|^n \Bigg[|z| - \frac{1}{|\alpha_n|}\{(a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + (a_{n-2} - a_{n-3}) + ... + (a_2 - a_3)$$
$$+ (a_2 - a_1) + (a_0 - a_1) + |a_0| + (b_n - b_{n-1}) + (b_{n-1} - b_{n-2}) + (b_{n-2} - b_{n-3}) + ...$$
$$+ (b_2 - b_3) + (b_2 - b_1) + (b_0 - b_1) + |b_0|\}\Bigg]\ if\ n\ is\ even\ ,by\ hypothesis$$
$$= |\alpha_n||z|^n \Bigg[|z| - \frac{1}{|\alpha_n|}\{|a_0| + a_0 + a_n + 2([a_{n-2} + a_{n-4} + ... + a_4 + a_2]$$
$$- [a_{n-1} + a_{n-3} + ...a_3 + a_1]) + |b_0| + b_0 + b_n + 2([b_{n-2} + b_{n-4} + ... + b_4 + b_2]$$
$$- [b_{n-1} + b_{n-3}... + b_3 + b_1])\}\Bigg] > 0$$
$$if\ |z| > \{|a_0| + a_0 + a_n + 2([a_{n-2} + a_{n-4} + ... + a_4 + a_2] - [a_{n-1} + a_{n-3} + ...a_3 + a_1]) + |b_0|$$
$$+ b_0 + b_n + 2([b_{n-2} + b_{n-4} + ... + b_4 + b_2] - [b_{n-1} + b_{n-3}... + b_3 + b_1])\}$$

This shows that if $|z|>1$ *then* $|Q(z)|>0$ *whenever*

$$|z| > \frac{1}{|\alpha_n|}\{|a_0| + a_0 + a_n + 2([a_{n-2} + a_{n-4} + ... + a_4 + a_2] - [a_{n-1} + a_{n-3} + ...a_3 + a_1])$$
$$+ |b_0| + b_0 + b_n + 2([b_{n-2} + b_{n-4} + ... + b_4 + b_2] - [b_{n-1} + b_{n-3}... + b_3 + b_1])\}.$$

Hence all the zeros of $Q(z)$ *with* $|z|>1$ *lie in*

$$|z| < \frac{1}{|\alpha_n|}\{|a_0| + a_0 + a_n + 2([a_{n-2} + a_{n-4} + ... + a_4 + a_2] - [a_{n-1} + a_{n-3} + ...a_3 + a_1]) + |b_0|$$
$$+ b_0 + b_n + 2([b_{n-2} + b_{n-4} + ... + b_4 + b_2] - [b_{n-1} + b_{n-3}... + b_3 + b_1])\}\ if\ n\ is\ even$$

But the zeros of $Q(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality.Since all the zeros of Q(z) lie in the circle defined by the above inequality , we conclude that the proof of the Theorem is complete if n is even.

Similarly we can also prove for odd degree polynomials.For this we can rearrange the terms of the given polynomial and compute as above. That is if n is odd the all then zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Bigg[|a_0| - a_0 + a_n + 2([a_{n-2} + a_{n-4} + ... + a_3 + a_1] - [a_{n-1} + a_{n-3} + ...a_4 + a_2]) + |b_0| - b_0 + b_n + 2([b_{n-2} + b_{n-4} + ... + b_3 + b_1] - [b_{n-1} + b_{n-3}... + b_4 + b_2])\Bigg]$$

This completes the proof of the Theorem. □

Corollary 2.5.2. *Let $P(z) = \sum_{j=0}^{n} \alpha_j z^j$ be a polynomial with complex coefficients of degree $n \geq 2$ with $Re(\alpha_j) = a_j \geq 0$ and $Im(\alpha_j) = b_j \geq 0$ for $j = 0, 1, 2, ..., n$ such that*

$a_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq ... \leq a_4 \geq a_3 \leq a_2 \geq a_1 \leq a_0$ *and*

$b_n \geq b_{n-1} \leq b_{n-2} \geq b_{n-3} \leq b_{n-4} \geq ... \leq b_4 \geq b_3 \leq b_2 \geq b_1 \leq b_0$ *if n is even*

OR

$a_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq ... \geq a_4 \leq a_3 \geq a_2 \leq a_1 \geq a_0$ *and*

$b_n \geq b_{n-1} \leq b_{n-2} \geq b_{n-3} \leq b_{n-4} \geq ... \geq b_4 \leq b_3 \leq b_2 \leq b_1 \geq b_0$ *if n is odd*

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Bigg[a_n + 2([a_{n-2} + a_{n-4} + ... + a_4 + a_2 + a_0] - [a_{n-1} + a_{n-3} + ...a_3 + a_1]) + b_n + 2([b_{n-2} + b_{n-4} + ... + b_4 + b_2 + b_0] - [b_{n-1} + b_{n-3}... + b_3 + b_1])\Bigg] \text{ if } n \text{ is even}$$

OR

$$|z| \leq \frac{1}{|\alpha_n|}\Bigg[a_n + 2([a_{n-2} + a_{n-4} + ... + a_3 + a_1] - [a_{n-1} + a_{n-3} + ...a_4 + a_2]) + b_n + 2([b_{n-2} + b_{n-4} + ... + b_3 + b_1] - [b_{n-1} + b_{n-3}... + b_4 + b_2])\Bigg] \text{ if } n \text{ is odd}$$

Remark 2.5.3. By taking $a_j \geq 0, b_j \geq 0$ for $j = 0, 1, 2, ..., n$ in Theorem 2.5.1, it reduces to Corollary 2.5.2.

Theorem 2.5.4. *Let $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ be a polynomial with complex coefficients of degree $n \geq 2$ with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ for $i = 0, 1, 2, 3, ..., n$ such that for some $k \geq 1, \delta \geq 0$,*

$ka_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} .... \leq a_4 \geq a_3 \leq a_2 \geq a_1 \leq a_0 + \delta$ *and*

$kb_n \geq b_{n-1} \leq b_{n-2} \geq b_{n-3} \leq b_{n-4} .... \leq b_4 \geq b_3 \leq b_2 \geq b_1 \leq b_0 + \delta$ *if n is even*

OR

$ka_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} .... \geq a_4 \leq a_3 \geq a_2 \leq a_1 \geq a_0 - \delta$ *and*

$kb_n \geq b_{n-1} \leq b_{n-2} \geq b_{n-3} \leq b_{n-4} .... \geq b_4 \leq b_3 \geq b_2 \leq b_1 \geq b_0 - \delta$ *if n is odd*

then all the zeros of the polynomial $P(z)$ *lie in*

$|z+k-1| \leq \frac{1}{|\alpha_n|}\{ka_n+|a_0|+a_0+2\delta+2[(a_{n-2}+a_{n-4+}+...+a_4+a_2)-(a_{n-1}+a_{n-3}+...+ a_3+a_1)]+kb_n+|b_0|+b_0+2\delta+2[(b_{n-2}+b_{n-4+}+...+b_4+b_2)-(b_{n-1}+b_{n-3}+...+b_3+b_1)]\}$

if n is even

OR

$|z+k-1| \leq \frac{1}{|\alpha_n|}\{ka_n+|a_0|-a_0+2\delta+2[(a_{n-2}+a_{n-4+}+...+a_3+a_1)-(a_{n-1}+a_{n-3}+...+ a_4+a_2)]+kb_n+|b_0|-b_0+2\delta+2[(b_{n-2}+b_{n-4+}+...+b_3+b_1)-(b_{n-1}+b_{n-3}+...+b_4+b_2)]\}$

if n is odd.

Proof. Let $P(z) = \alpha_n z^n + \alpha_{n-1}z^{n-1} + ... + \alpha_3 z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0$ be a polynomial with complex coefficients of degree n≥ 2 $with\ \alpha_j = a_j + ib_j for\ j = 0, 1, 2, ..., n$

Then consider the polynomial

$$\begin{aligned}
Q(z) &= (1-z)P(z) \\
&= -\alpha_n z^n(z+k-1) + (k\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + (\alpha_{n-2} - \alpha_{n-3})z^{n-2} + ... \\
&\qquad + (\alpha_1 - \alpha_0)z + \alpha_0. \\
&= -\alpha_n z^n(z+k-1) + (ka_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + (a_{n-2} - a_{n-3})z^{n-2} + ... \\
&\qquad + (a_1 - a_0)z + a_0 + i\big\{(kb_n - b_{n-1})z^n + (b_{n-1} - b_{n-2})z^{n-1} + (b_{n-2} - b_{n-3})z^{n-2} \\
&\qquad + ... + (b_1 - b_0)z + b_0\big\}.
\end{aligned}$$

$If\ |z|>1\ then\ \dfrac{1}{|z|^{n-1}} < for\ i = 0, 1, 2, ..., n-1$

$$\begin{aligned}
Now\ |Q(z)| &\geq |\alpha_n||z|^n|z+k-1| - \big\{|ka_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} \\
&\quad + |a_{n-2} - a_{n-3}||z|^{n-2} + ... + |a_1 - a_0||z| + |a_0| + |kb_n - b_{n-1}||z|^n \\
&\quad + |b_{n-1} - b_{n-2}||z|^{n-1} + |b_{n-2} - b_{n-3}|z|^{n-2} + ... + |b_1 - b_0||z| + |b_0|\big\} \\
&\geq |\alpha_n||z|^n\Big[|z+k-1| - \big\{|ka_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} + \frac{|a_{n-2} - a_{n-3}|}{|z|^2} + ...
\end{aligned}$$

$$+\frac{|a_1-a_0|}{|z|^{n-1}}+\frac{|a_0|}{|z|^n}+|kb_n-b_{n-1}|+\frac{|b_{n-1}-b_{n-2}|}{|z|}+\frac{|b_{n-2}-b_{n-3}|}{|z|^2}+...$$

$$+\frac{|b_1-b_0|}{|z|^{n-1}}+\frac{|b_0|}{|z|^n}\}\Big]$$

$$\geq |\alpha_n||z|^n\Big[|z+k-1|-\big\{|ka_n-a_{n-1}|+|a_{n-1}-a_{n-2}|+|a_{n-2}-a_{n-3}|+......$$

$$+|a_1-\delta-a_0+\delta|+|a_0|+|kb_n-b_{n-1}|+|b_{n-1}-b_{n-2}|+|b_{n-2}-b_{n-3}|+.....$$

$$+|b_1-\delta-b_0+\delta|+|b_0|\big\}\Big]$$

$$\geq |\alpha_n||z|^n\Big[|z+k-1|-\big\{(ka_n-a_{n-1})+(a_{n-1}-a_{n-2})+(a_{n-2}-a_{n-3})+....$$

$$+(a_0+\delta-a_)+\delta+|a_0|+(kb_n-b_{n-1})+(b_{n-1}-b_{n-2})+(b_{n-2}-b_{n-3})+...$$

$$+(b_0+\delta-b_1)+\delta+|b_0|\big\}\Big]\ if\ n\ is\ even,(by\ hypothesis)$$

$$=|\alpha_n||z|^n\Big[|z+k-1|-\frac{1}{|\alpha_n|}\{ka_n+|a_0|+a_0+2\delta+2[(a_{n-2}+a_{n-4}+...$$

$$+a_4+a_2)-(a_{n-1}+a_{n-3}+....+a_3+a_1)]+kb_n+|b_0|+b_0+2\delta$$

$$+2[(b_{n-2}+b_{n-4}+...+b_4+b_2)-(b_{n-1}+b_{n-3}+....+b_3+b_1)]\}\Big]>0$$

$$if\ |z+k-1|>\frac{1}{|\alpha_n|}\{ka_n+|a_0|+a_0+2\delta+2[(a_{n-2}+a_{n-4}+...+a_4+a_2)$$

$$-(a_{n-1}+a_{n-3}+....+a_3+a_1)]+kb_n+|b_0|+b_0$$

$$+2\delta+2[(b_{n-2}+b_{n-4}+...+b_4+b_2)-(b_{n-1}+b_{n-3}+....+b_3+b_1)]\}$$

$$This\ shows\ that\ if\ |z|>1\ then\ |Q(z)|>0$$

$$provided\ |z+k-1|>\frac{1}{|\alpha_n|}\{ka_n+|a_0|+a_0+2\delta+2[(a_{n-2}+a_{n-4}+...+a_4+a_2)$$

$$-(a_{n-1}+a_{n-3}+....+a_3+a_1)]+kb_n+|b_0|+b_0+2\delta$$

$$+2[(b_{n-2}+b_{n-4}+...+b_4+b_2)-(b_{n-1}+b_{n-3}+....+b_3+b_1)]\}$$

$$Hence\ all\ the\ zeros\ of\ Q(z)\ with\ |z|>1\ lie\ in$$

$$|z+k-1|<\frac{1}{|\alpha_n|}\{ka_n+|a_0|+a_0+2\delta+2[(a_{n-2}+a_{n-4}+...+a_4+a_2)$$

$$-(a_{n-1}+a_{n-3}+....+a_3+a_1)]+kb_n+|b_0|+b_0+2\delta$$

$$+2[(b_{n-2}+b_{n-4}+...+b_4+b_2)-(b_{n-1}+b_{n-3}+....+b_3+b_1)]\}$$

$$if\ n\ is\ even$$

But those zeros of Q(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of P(z) are also the zeros of Q(z) lie in the circle defined by the above inequality, we conclude that the proof of the Theorem is complete, if n is even.

Similarly we can also prove for odd degree polynomials. For this we can rearrange the terms of the given polynomial and compute as above.That is if n is odd then all the zeros of P(z) lie in

$|z+k-1| \leq \frac{1}{|\alpha_n|}\{ka_n+|a_0|-a_0+2\delta+2[(a_{n-2}+a_{n-4+}+...+a_3+a_1)-(a_{n-1}+a_{n-3}+...+a_4+a_2)]+kb_n+|b_0|-b_0+2\delta+2[(b_{n-2}+b_{n-4+}+...+b_3+b_1)-(b_{n-1}+b_{n-3}+...+b_4+b_2)]\}$

This completes the proof of the Theorem. □

Corollary 2.5.5. *Let $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ be a polynomial with complex coefficients of degree $n \geq 2$ with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ for $i = 0, 1, 2, 3, ..., n$ such that for some $k \geq 1$,*

$ka_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} .... \leq a_4 \geq a_3 \leq a_2 \geq a_1 \leq a_0$ *and*

$kb_n \geq b_{n-1} \leq b_{n-2} \geq b_{n-3} \leq b_{n-4} .... \leq b_4 \geq b_3 \leq b_2 \geq b_1 \leq b_0$ *if n is even*

OR

$ka_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} .... \geq a_4 \leq a_3 \geq a_2 \leq a_1 \geq a_0$ *and*

$kb_n \geq b_{n-1} \leq b_{n-2} \geq b_{n-3} \leq b_{n-4} .... \geq b_4 \leq b_3 \geq b_2 \leq b_1 \geq b_0$ *if n is odd*

then all the zeros of the polynomial $P(z)$ lie in

$|z + k - 1| \leq \frac{1}{|\alpha_n|}\{ka_n + |a_0| + a_0 + 2[(a_{n-2} + a_{n-4+} + ... + a_4 + a_2) - (a_{n-1} + a_{n-3} + ... + a_3 + a_1)] + kb_n + |b_0| + b_0 + 2[(b_{n-2} + b_{n-4+} + ... + b_4 + b_2) - (b_{n-1} + b_{n-3} + ... + b_3 + b_1)]\}$

if n is even

OR

$|z + k - 1| \leq \frac{1}{|\alpha_n|}\{ka_n + |a_0| - a_0 + 2[(a_{n-2} + a_{n-4+} + ... + a_3 + a_1) - (a_{n-1} + a_{n-3} + ... + a_4 + a_2)] + kb_n + |b_0| - b_0 + 2[(b_{n-2} + b_{n-4+} + ... + b_3 + b_1) - (b_{n-1} + b_{n-3} + ... + b_4 + b_2)]\}$

if n is odd.

Remark 2.5.6. By taking $\delta = 0$ in Theorem 2.5.4, it reduces to Corollary 2.5.5.

Theorem 2.5.7. *Let $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ be a polynomial with complex coefficients of degree $n \geq 2$ with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ for $i = 0, 1, 2, ..., n$ such that*

$a_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq ... \geq a_4 \leq a_3 \geq a_2 \leq a_1 \geq a_0,$

$b_n \leq b_{n-1} \geq b_{n-2} \leq b_{n-3} \geq b_{n-4} \leq ... \geq b_4 \leq b_3 \geq b_2 \leq b_1 \geq b_0$ *if n is even*

OR

$a_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq ... \leq a_4 \geq a_3 \leq a_2 \geq a_1 \leq a_0,$

$b_n \leq b_{n-1} \geq b_{n-2} \leq b_{n-3} \geq b_{n-4} \leq ... \leq b_4 \geq b_3 \leq b_2 \geq b_1 \leq b_0$ *if n is odd*

then all the zeros of P(z) lie in

$|z| \leq \frac{1}{|\alpha_n|}\big[|a_0| - a_0 - a_n + 2([a_{n-1} + a_{n-3} + ... + a_3 + a_1] - [a_{n-2} + a_{n-4} + ... + a_4 + a_2]) + |b_0| - b_0 - b_n + 2([b_{n-1} + b_{n-3} + ... + b_3 + b_1] - [b_{n-2} + b_{n-4} + ... + b_4 + b_2])\big]$ *if n is even*

OR

$|z| \leq \frac{1}{|\alpha_n|}\big[|a_0| + a_0 - a_n + 2([a_{n-2} + a_{n-4} + ... + a_3 + a_1] - [a_{n-1} + a_{n-3} + ... + a_4 + a_2]) + |b_0| + b_0 - b_n + 2([b_{n-2} + b_{n-4} + ... + b_3 + b_1] - [b_{n-1} + b_{n-3} + ... + b_4 + b_2])\big]$ *if n is odd.*

Proof.

Let $P(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + ... + \alpha_3 z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0$ *be a polynomial of degree* $n \geq 2$ *with* $\alpha_j = a_j + ib_j$ *for* $j = 0, 1, 2, ..., n$. *Then consider the polynomial*

$$\begin{aligned}
Q(z) =& (1 - z)P(z)\\
=& -\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + (\alpha_{n-2} - \alpha_{n-3})z^{n-2} + ...\\
&+ (\alpha_1 - \alpha_0)z + \alpha_0.\\
=& -\alpha_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + (a_{n-2} - a_{n-3})z^{n-2} + ...\\
&+ (a_1 - a_0)z + a_0 + i\big\{(b_n - b_{n-1})z^n + (b_{n-1} - b_{n-2})z^{n-1} + (b_{n-2} - b_{n-3})z^{n-2} + ...\\
&+ (b_1 - b_0)z + b_0\big\}.
\end{aligned}$$

If $|z| < 1$ *then* $\dfrac{1}{|z|^{n-i}} > 1$ *for* $i = 0, 1, 2, ..., n - 1.$

$$\begin{aligned}
Now\ |Q(z)| \geq& |\alpha_n||z|^{n+1} - \{|a_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} + |a_{n-2} - a_{n-3}||z|^{n-2} + ...\\
&+ |a_3 - a_2||z|^3 + |a_2 - a_1||z|^2 + |a_1 - a_0||z| + |a_0| + |b_n - b_{n-1}||z|^n\\
&+ |b_{n-1} - b_{n-2}||z|^{n-1} + |b_{n-2} - b_{n-3}||z|^{n-2} + ... + |b_3 - b_2||z|^3 + |b_2 - b_1||z|^2\\
&+ |b_1 - b_0||z| + |b_0|\}
\end{aligned}$$

$$\geq |\alpha_n||z|^n\Bigg[|z| - \frac{1}{|\alpha_n|}\{|a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} + \frac{|a_{n-2} - a_{n-3}|}{|z|^2} + ...$$
$$+ \frac{|a_3 - a_2|}{|z|^{n-3}} + \frac{|a_2 - a_1|}{|z|^{n-2}} + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}$$
$$+ |b_n - b_{n-1}| + \frac{|b_{n-1} - b_{n-2}|}{|z|} + \frac{|b_{n-2} - b_{n-3}|}{|z|^2} + ... + \frac{|b_3 - b_2|}{|z|^{n-3}} + \frac{|b_2 - b_1|}{|z|^{n-2}} + \frac{|b_1 - b_0|}{|z|^{n-1}}$$
$$+ \frac{|b_0|}{|z|^n}\}\Bigg]$$
$$\geq |\alpha_n||z|^n\Bigg[|z| - \frac{1}{|\alpha_n|}\{|a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}| + ... + |a_3 - a_2|$$
$$+ |a_2 - a_1| + |a_1 - a_0| + |a_0| + |b_n - b_{n-1}| + |b_{n-1} - b_{n-2}| + |b_{n-2} - b_{n-3}| + ...$$
$$+ |b_3 - b_2| + |b_2 - b_1| + |b_1 - b_0| + |b_0|\}\Bigg]$$
$$\geq |\alpha_n||z|^n\Bigg[|z| - \frac{1}{|\alpha_n|}\{(a_{n-1} - a_n) + (a_{n-1} - a_{n-2}) + (a_{n-3} - a_{n-2}) + ...$$
$$+ (a_3 - a_2) + (a_1 - a_2) + (a_1 - a_0) + |a_0| + (b_{n-1} - b_n) + (b_{n-1} - b_{n-2}) + (b_{n-3} - b_{n-2})$$
$$+ .... + (b_3 - b_2) + (b_1 - b_2) + (b_1 - b_0) + |b_0|\}\Bigg]$$

if n is even ,(*by hypothesis*)

$$=|\alpha_n||z|^n\Bigg[|z| - \frac{1}{|\alpha_n|}\{|a_0| - a_0 - a_n + 2([a_{n-1} + a_{n-3} + ... + a_3 + a_1]$$
$$- [a_{n-2} + a_{n-4} + ... + a_4 + a_2]) + |b_0| - b_0 - b_n + 2([b_{n-1} + b_{n-3} + ... + b_3 + b_1]$$
$$- [b_{n-2} + b_{n-4} + ... + b_4 + b_2])\}\Bigg] > 0$$
$$if |z| > \frac{1}{|\alpha_n|}\{|a_0| - a_0 - a_n + 2([a_{n-1} + a_{n-3} + ... + a_3 + a_1] - [a_{n-2} + a_{n-4} + ... + a_4 + a_2])$$
$$+ |b_0| - b_0 - b_n + 2([b_{n-1} + b_{n-3} + ... + b_3 + b_1] - [b_{n-2} + b_{n-4} + ... + b_4 + b_2])\}$$

This shows that if $|z|>1$ *then* $|Q(z)|>0$

$$if |z| > \frac{1}{|\alpha_n|}\{|a_0| - a_0 - a_n + 2([a_{n-1} + a_{n-3} + ... + a_3 + a_1] - [a_{n-2} + a_{n-4} + ... + a_4 + a_2])$$
$$+ |b_0| - b_0 - b_n + 2([b_{n-1} + b_{n-3} + ... + b_3 + b_1] - [b_{n-2} + b_{n-4} + ... + b_4 + b_2])\}$$

Hence all the zeros of $Q(z)$ *with* $|z|>1$ *lie in*

$$|z| < \frac{1}{|\alpha_n|}\{|a_0| - a_0 - a_n + 2([a_{n-1} + a_{n-3} + ... + a_3 + a_1] - [a_{n-2} + a_{n-4} + ... + a_4 + a_2])$$
$$+ |b_0| - b_0 - b_n + 2([b_{n-1} + b_{n-3} + ... + b_3 + b_1] - [b_{n-2} + b_{n-4} + ... + b_4 + b_2])\}$$

if n is even.

But those zeros of Q(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of P(z) are also the zeros of Q(z) lie in the circle defined by the above inquality, we conclude that the proof of the Theorem is complete, if n is even.

Similarly we can also prove for odd degree polynomials. For this we can rearrange the terms of the given polynomial and compute as above.That is if n is odd then all the zeros of P(z) lie in

$|z| \leq \frac{1}{|\alpha_n|}\{|a_0| + a_0 - a_n + 2([a_{n-2} + a_{n-4} + ... + a_3 + a_1] - [a_{n-1} + a_{n-3} + ... + a_4 + a_2]) + |b_0| + b_0 - b_n + 2([b_{n-2} + b_{n-4} + ... + b_3 + b_1] - [b_{n-1} + b_{n-3} + ... + b_4 + b_2])\}$

This completes the proof of the Theorem. □

Corollary 2.5.8. *Let $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ be a polynomial with complex coefficients of degree $n \geq 2$ with $Re(\alpha_i) = a_i \geq 0$ and $Im(\alpha_i) = b_i \geq 0$ for $i = 0, 1, 2, ..., n$ such that*

$a_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq ... \geq a_4 \leq a_3 \geq a_2 \leq a_1 \geq a_0,$

$b_n \leq b_{n-1} \geq b_{n-2} \leq b_{n-3} \geq b_{n-4} \leq ... \geq b_4 \leq b_3 \geq b_2 \leq b_1 \geq b_0$ *if n is even*

OR

$a_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq ... \leq a_4 \geq a_3 \leq a_2 \geq a_1 \leq a_0,$

$b_n \leq b_{n-1} \geq b_{n-2} \leq b_{n-3} \geq b_{n-4} \leq ... \leq b_4 \geq b_3 \leq b_2 \geq b_1 \leq b_0$ *if n is odd*

then all the zeros of $P(z)$ lie in

$|z| \leq \frac{1}{|\alpha_n|}\big[2([a_{n-1} + a_{n-3} + ... + a_3 + a_1] - [a_{n-2} + a_{n-4} + ... + a_4 + a_2]) - a_n + +2([b_{n-1} + b_{n-3} + ... + b_3 + b_1] - [b_{n-2} + b_{n-4} + ... + b_4 + b_2]) - b_n\big]$ *if n is even*

OR

$|z| \leq \frac{1}{|\alpha_n|}\big[2([a_{n-2} + a_{n-4} + ... + a_3 + a_1 + a_0] - [a_{n-1} + a_{n-3} + ... + a_4 + a_2]) - a_n + +2([b_{n-2} + b_{n-4} + ... + b_3 + b_1 + b_0] - [b_{n-1} + b_{n-3} + ... + b_4 + b_2]) - b_n\big]$ *if n is odd.*

Remark 2.5.9. By taking $Re(\alpha_i) = a_i \geq 0, Im(\alpha_i) = b_i \geq 0 \ for \ i = 0, 1, 2, .., n$ in Theorem 2.5.7, it reduces to Corollary 2.5.8

Theorem 2.5.10. *Let $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ be a polynomial with complex coefficients of degree $n \geq 2$ with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ for $i = 0, 1, 2, ..., n$ such that for* $0 \leq r, s \leq 1, \delta > 0, \eta > 0$

$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq ... \geq a_4 \leq a_3 \geq a_2 \leq a_1 \geq a_0 - \delta,$

$sb_n \leq b_{n-1} \geq b_{n-2} \leq b_{n-3} \geq b_{n-4} \leq ... \geq b_4 \leq b_3 \geq b_2 \leq b_1 \geq b_0 - \eta$ *if n is even*

OR

$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq ... \leq a_4 \geq a_3 \leq a_2 \geq a_1 \leq a_0 + \delta,$

$sb_n \leq b_{n-1} \geq b_{n-2} \leq b_{n-3} \geq b_{n-4} \leq ... \leq b_4 \geq b_3 \leq b_2 \geq b_1 \leq b_0 + \eta$ *if n is odd*

then all the zeros of the polynomial $P(z)$ *lie in*

$|z| \leq \frac{1}{|\alpha_n|}\{2\delta + |a_n| + |a_0| - a_0 - r(|a_n| + a_n) + 2[(a_{n-1} + a_{n-3} + ... + a_3 + a_1) - (a_{n-2} + a_{n-4} + ... + a_4 + a_2)] + 2\eta + |b_n| + |b_0| - b_0 - s(|b_n| + b_n) + 2[(b_{n-1} + b_{n-3} + ... + b_3 + b_1) - (b_{n-2} + b_{n-4} + ... + b_4 + b_2)]\}$ *if n is even*

OR

$|z| \leq \frac{1}{|\alpha_n|}\{2\delta + |a_n| + |a_0| + a_0 - r(|a_n| + a_n) + 2[(a_{n-1} + a_{n-3} + ... + a_4 + a_2) - (a_{n-2} + a_{n-4} + ... + a_3 + a_1)] + 2\eta + |b_n| + |b_0| + b_0 - s(|b_n| + b_n) + 2[(b_{n-1} + b_{n-3} + ... + b_4 + b_2) - (b_{n-2} + b_{n-4} + ... + b_3 + b_1)]\}$ *if n is odd.*

Proof. Let $P(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + ... + \alpha_3 z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0$ be a polynomial with complex coefficients of degree n≥ 2 $with\ \alpha_j = a_j + ib_j for\ j = 0, 1, 2, ..., n$

Then consider the polynomial

$$\begin{aligned}
Q(z) &= (1 - z)P(z) \\
&= -\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + (\alpha_{n-2} - \alpha_{n-3})z^{n-2} + ... + (\alpha_1 - \alpha_0)z \\
&\quad + \alpha_0. \\
&= -\alpha_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + (a_{n-2} - a_{n-3})z^{n-2} + ... \\
&\quad + (a_1 - a_0)z + a_0 + i\Big\{(b_n - b_{n-1})z^n + (b_{n-1} - b_{n-2})z^{n-1} + (b_{n-2} - b_{n-3})z^{n-2} + ... \\
&\quad + (b_1 - b_0)z + b_0\Big\}.
\end{aligned}$$

$$If\ |z|>1\ then\ \frac{1}{|z|^{n-i}}<1\ for\ i = 0, 1, 2, ..., n-1.$$

$$\begin{aligned}
Now\ |Q(z)| \geq\ & |\alpha_n||z|^{n+1} - \{|a_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} + |a_{n-2} - a_{n-3}||z|^{n-2} + ... \\
& + |a_3 - a_2||z|^3 + |a_2 - a_1||z|^2 + |a_1 - a_0||z| + |a_0| + |b_n - b_{n-1}||z|^n \\
& + |b_{n-1} - b_{n-2}||z|^{n-1} + |b_{n-2} - b_{n-3}||z|^{n-2} + ... + |b_3 - b_2||z|^3 \\
& + |b_2 - b_1||z|^2 + |b_1 - b_0||z| + |b_0|\}
\end{aligned}$$

$$\geq |\alpha_n||z|^n\Bigg[|z| - \frac{1}{|\alpha_n|}\{|a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} + \frac{|a_{n-2} - a_{n-3}|}{|z|^2} + ...$$

$$+\frac{|a_3 - a_2|}{|z|^{n-3}} + \frac{|a_2 - a_1|}{|z|^{n-2}} + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}$$

$$+|b_n - b_{n-1}| + \frac{|b_{n-1} - b_{n-2}|}{|z|} + \frac{|b_{n-2} - b_{n-3}|}{|z|^2} + ... + \frac{|b_3 - b_2|}{|z|^{n-3}} + \frac{|b_2 - b_1|}{|z|^{n-2}}$$

$$+\frac{|b_1 - b_0|}{|z|^{n-1}} + \frac{|b_0|}{|z|^n}\}\Bigg]$$

$$\geq |\alpha_n||z|^n\Bigg[|z| - \frac{1}{|\alpha_n|}\{|ra_n - a_{n-1} - ra_n + a_n| + |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}|$$

$$+|a_{n-3} - a_{n-4}| + ... + |a_3 - a_2| + |a_2 - a_1| + |a_1 + \delta - a_0 - \delta| + |a_0|$$

$$+|sb_n - b_{n-1} - sb_n + b_n| + |b_{n-1} - b_{n-2}| + |b_{n-2} - b_{n-3}| + |b_{n-3} - b_{n-4}| + ...$$

$$+|b_3 - b_2| + |b_2 - b_1| + |b_1 + \eta - b_0 - \eta| + |b_0|\}\Bigg]$$

$$\geq |\alpha_n||z|^n\Bigg[|z| - \frac{1}{|\alpha_n|}\{(a_{n-1} - ra_n) - (1 - r)|a_n| + (a_{n-1} - a_{n-2}) + (a_{n-3} - a_{n-2})$$

$$+(a_{n-3} - a_{n-4}) + ... + (a_3 - a_2) + (a_1 - a_2) + (a_1 - a_0 + \delta) + \delta + |a_0| + (b_{n-1} - b_n)$$

$$-(1 - s)|b_n| + (b_{n-1} - b_{n-2}) + (b_{n-3} - b_{n-2} + (b_{n-3} - b_{n-4}) + ... + (b_3 - b_2)$$

$$+(b_1 - b_2) + (b_1 - b_0 + \eta) + \eta + |b_0|\}\Bigg] if\ n\ is\ n\ is\ even\ , (by\ hypothesis)$$

$$= |\alpha_n||z|^n - \Bigg[|z| - \frac{1}{|\alpha_n|}\{2\delta + |a_n| - |a_0| - a_0 - r(|a_n| + a_n) + 2([a_{n-1} + a_{n-3} + ... + a_3 + a_1]$$

$$-[a_{n-2} + a_{n-4} + ... + a_4 + a_2]) + 2\eta + |b_n| - |b_0| - b_0 - r(|b_n| + b_n)$$

$$+2([b_{n-1} + b_{n-3} + ... + b_3 + b_1] - [b_{n-2} + b_{n-4} + ... + b_4 + b_2])\}\Bigg] > 0,\ if\ n\ is\ even.$$

$$if\ |z| > \frac{1}{|\alpha_n|}\{2\delta + |a_n| - |a_0| - a_0 - r(|a_n| + a_n) + 2([a_{n-1} + a_{n-3} + ... + a_3 + a_1]$$

$$-[a_{n-2} + a_{n-4} + ... + a_4 + a_2]) + 2\eta + |b_n| - |b_0| - b_0 - r(|b_n| + b_n)$$

$$+2([b_{n-1} + b_{n-3} + ... + b_3 + b_1] - [b_{n-2} + b_{n-4} + ... + b_4 + b_2])\}$$

This shows that for $|z| > 1$ *, then* $|Q(z)| > 0$

$$if\ |z| > \frac{1}{|\alpha_n|}\{2\delta + |a_n| - |a_0| - a_0 - r(|a_n| + a_n) + 2([a_{n-1} + a_{n-3} + ... + a_3 + a_1]$$

$$-[a_{n-2} + a_{n-4} + ... + a_4 + a_2]) + 2\eta + |b_n| - |b_0| - b_0 - r(|b_n| + b_n)$$

$$+2([b_{n-1} + b_{n-3} + ... + b_3 + b_1] - [b_{n-2} + b_{n-4} + ... + b_4 + b_2])\}$$

Hence all the zeros of $Q(z)$ *with* $|z| > 1$ *lie in*

$$|z| < \frac{1}{|\alpha_n|}\{2\delta + |a_n| - |a_0| - a_0 - r(|a_n| + a_n) + 2([a_{n-1} + a_{n-3} + ... + a_3 + a_1]$$

$$-[a_{n-2} + a_{n-4} + ... + a_4 + a_2]) + 2\eta + |b_n| - |b_0| - b_0 - r(|b_n| + b_n)$$

$+2([b_{n-1}+b_{n-3}+...+b_3+b_1]-[b_{n-2}+b_{n-4}+...+b_4+b_2])\}$ if n is even.

But those zeros of Q(z) whose modulus is less than or equal to 1 already satisfy the above ineauality. Since all the zeros of P(z) are also the zeros of Q(z) lie in the circle defined by the above inequality, we conclude that the proof of the Theorem is complete, if n is even.

Similarly we can also prove for odd degree polynomials. For this we can rearrange the terms of the given polynomial and compute as above.That is if n is odd then all the zeros of P(z) lie in

$|z| \leq \frac{1}{|\alpha_n|}\{2\delta+|a_n|+|a_0|+a_0-r(|a_n|+a_n)+2[(a_{n-1}+a_{n-3}+....+a_4+a_2)-(a_{n-2}+a_{n-4}+...+a_3+a_1)]+2\eta+|b_n|+|b_0|+b_0-s(|b_n|+b_n+2[(b_{n-1}+b_{n-3}+....+b_4+b_2)-(b_{n-2}+b_{n-4}+...+b_3+b_1)]\}$ This completes the proof of the Theorem. □

Corollary 2.5.11. *Let $P(z)=\sum_{i=0}^{n}\alpha_i z^i$ be a polynomial of degreen ≥ 2 with $Re(\alpha_i)=a_i$ and $Im(\alpha_i)=b_i$ for $i=0,1,2,...,n$ such that for $0\leq r,s\leq 1$*

$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq ... \geq a_4 \leq a_3 \geq a_2 \leq a_1 \geq a_0,$

$sb_n \leq b_{n-1} \geq b_{n-2} \leq b_{n-3} \geq b_{n-4} \leq ... \geq b_4 \leq b_3 \geq b_2 \leq b_1 \geq b_0$ *if n is even*

OR

$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq ... \leq a_4 \geq a_3 \leq a_2 \geq a_1 \leq a_0,$

$sb_n \leq b_{n-1} \geq b_{n-2} \leq b_{n-3} \geq b_{n-4} \leq ... \leq b_4 \geq b_3 \leq b_2 \geq b_1 \leq b_0$ *if n is odd*

then all the zeros of the polynomial P(z) lie in

$|z| \leq \frac{1}{|\alpha_n|}\{|a_n|+|a_0|-a_0-r(|a_n|+a_n)+2[(a_{n-1}+a_{n-3}+....+a_3+a_1)-(a_{n-2}+a_{n-4}+...+a_4+a_2)]+|b_n|+|b_0|-b_0-s(|b_n|+b_n)+2[(b_{n-1}+b_{n-3}+....+b_3+b_1)-(b_{n-2}+b_{n-4}+...+b_4+b_2)]\}$ *if n is even*

OR

$|z| \leq \frac{1}{|\alpha_n|}\{|a_n|+|a_0|+a_0-r(|a_n|+a_n)+2[(a_{n-1}+a_{n-3}+....+a_4+a_2)-(a_{n-2}+a_{n-4}+...+a_3+a_1)]+|b_n|+|b_0|+b_0-s(|b_n|+b_n)+2[(b_{n-1}+b_{n-3}+....+b_4+b_2)-(b_{n-2}+b_{n-4}+...+b_3+b_1)]\}$ *if n is odd.*

Corollary 2.5.12. *Let $P(z)=\sum_{i=0}^{n}\alpha_i z^i$ be a polynomial of degreen ≥ 2 with $Re(\alpha_i)=a_i\geq 0$ and $Im(\alpha_i)=b_i\geq 0$ for $i=0,1,2,...,n$ such that for $0\leq r,s\leq 1$*

$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq ... \geq a_4 \leq a_3 \geq a_2 \leq a_1 \geq a_0,$

$sb_n \leq b_{n-1} \geq b_{n-2} \leq b_{n-3} \geq b_{n-4} \leq ... \geq b_4 \leq b_3 \geq b_2 \leq b_1 \geq b_0$ *if n is even*

OR

$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq ... \leq a_4 \geq a_3 \leq a_2 \geq a_1 \leq a_0,$

$sb_n \leq b_{n-1} \geq b_{n-2} \leq b_{n-3} \geq b_{n-4} \leq ... \leq b_4 \geq b_3 \leq b_2 \geq b_1 \leq b_0$ *if n is odd*

then all the zeros of the polynomial $P(z)$ *lie in*

$|z| \leq \frac{1}{|\alpha_n|}\{a_n + 2[(a_{n-1} + a_{n-3} + ... + a_3 + a_1) - (ra_n + a_{n-2} + a_{n-4} + ... + a_4 + a_2)] + b_n + +2[(b_{n-1} + b_{n-3} + ... + b_3 + b_1) - (sb_n + b_{n-2} + b_{n-4} + ... + b_4 + b_2)]\}$ *if n is even*

OR

$|z| \leq \frac{1}{|\alpha_n|}\{a_n + 2[(a_{n-1} + a_{n-3} + ... + a_4 + a_2 + a_0) - (2ra_n + a_{n-2} + a_{n-4} + ... + a_3 + a_1)] + b_n + 2[(b_{n-1} + b_{n-3} + ... + b_4 + b_2 + b_0) - (2sb_o + b_{n-2} + b_{n-4} + ... + b_3 + b_1)]\}$ *if n is odd*

Remark 2.5.13. By taking $\delta = 0$ *and* $\eta = 0$ in Theorem 2.5.10, it reduces to Corollary 2.5.11.

Remark 2.5.14. By taking $\delta = 0, \eta = 0$ and $a_j b_j \geq 0$ for j=0,1,..n. in Theorem 2.5.10, it reduces to Corollary 2.5.12.

Chapter 3

Zero-free regions for complex polynomials

3.1 Introduction to the chapter

In second chapter we have found Locaton of Zeros of polynomials by restricting the coefficients. But we do not have a general formula for locating zeros of polynomials.For some of the polynomials by restricting the coefficients we can find a region where zeros are not contained in, by using the same idea that we used for locating the zeros of polynomial in perticular region.There are many results on zero free regions.In Enstrom-Kakeya theorem,by replacing polynomial $z^n P(\frac{1}{z})$, the following result is immediate

Theorem 3.1.1. *Let* $P(z) = \sum_{k=0}^{n} b_k z^k$ *be a polynomial of degree* n *such that* $b_0 \geq b_1 \geq ..... \geq b_{n-1} \geq b_n \geq 0$, *then* $P(z)$ *does not vanish in* $|z| \leq 1$.

Later B.A.Zargar [42] generalized the above and proved the following resuls.

Theorem 3.1.2. *Let* $P(z) = \sum_{j=0}^{n} b_j z^j$ *be a polynomial of degree* n *such that for some* $k \geq 1$, $kb_0 \geq b_1 \geq ..... \geq b_{n-1} \geq b_n \geq 0$, *then* $P(z)$ *does not vanish in the disk*

$$|z| \leq \frac{1}{2k-1}.$$

Theorem 3.1.3. *Let* $P(z) = \sum_{j=0}^{n} b_j z^j$ *be a polynomial of degree n such that for some real number* ρ, $0 \leq \rho < b_n, 0 < b_n - \rho \leq b_{n-1} \leq .... \leq b_1 \leq b_0$, *then* $P(z)$ *does not vanish in the disc*

$$|z| \leq \frac{1}{1+\frac{2\rho}{b_0}}.$$

Theorem 3.1.4. *Let* $P(z) = \sum_{j=0}^{n} b_j z^j$ *be a polynomial of degree n such that for some* $k \geq 1$, $0 < b_0 \leq b_1 \leq .... \leq b_{n-1} \leq kb_n$, *then* $P(z)$ *does not vanish in the disc*

$$|z| \leq \frac{b_0}{2kb_n - b_0}.$$

Theorem 3.1.5. *Let* $P(z) = \sum_{j=0}^{n} b_j z^j$ *be a polynomial of degree n such that for some* ρ, $0 < b_0 \leq b_1 \leq .... \leq b_{n-1} \leq b_n + \rho$, *then* $P(z)$ *does not vanish in the disc*

$$|z| \leq \frac{b_0}{2(b_n+\rho) - b_0}.$$

Here we prove the following results on zero free region for polynomials with restricted coefficients.

3.2 Zero free region for polynomials with restricted real coefficients

Theorem 3.2.1. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degee* $n \geq m \geq 2$ *with real coefficients such that* $\rho \geq 0, k \geq 1$ *and*

$ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq .... \geq a_{n-m-1} \leq a_{n-m} \geq a_{n-m+1} \geq .... \geq a_{n-2} \geq a_{n-1} \geq a_n - \rho$ *if both n and (n-m) are even or odd*

(or)

$ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq .... \leq a_{n-m-1} \geq a_{n-m} \geq a_{n-m+1} \geq .... \geq a_{n-2} \geq a_{n-1} \geq a_n - \rho$ *if one of the n, (n-m) is even and other is odd then*

(i) P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{k(|a_0| + a_0) + |a_n|) - (|a_0| + a_n) + 2\rho + s_1}$$

if both n and (n-m) are even or odd

where $s_1 = 2[(a_2+a_4+a_6+...+a_{n-m-2}+a_{n-m})-(a_1+a_3+a_5+...+a_{n-m-3}+a_{n-m-1})]$

(ii)P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{k(|a_0|+a_0)+|a_n|)-(|a_0|+a_n)+2\rho+s_2}$$

if one of n and (n-m) is even and other one is odd,

where $s_2 = 2[(a_2+a_4+a_6+...+a_{n-m-3}+a_{n-m-1})-(a_1+a_3+a_5+...+a_{n-m-4}+a_{n-m-2})]$.

Proof. Let$P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$ be a polynomial of degee n.

Let us consider the polynomials $J(z) = z^n P(\frac{1}{z})$ *and* $R(z) = (z-1)J(z)$ so that

$R(z) = (z-1)(a_0 z^n + a_1 z^{n-1} + ... + a_{m-1} z^{n-m+1} + a_m z^{n-m} + a_{m+1} z^{n-m+1} + ... + a_{n-1} z + a_n)$

$\implies R(z) = a_0 z^{n+1} - \{(a_0-a_1)z^n + (a_1-a_2)z^{n-1} + ... + (a_{m-1}-a_m)z^{n-m+1} + (a_m - a_{m-1})z^{n-m} + ... + (a_{n-1}-a_n)z + a_n\}$

Also if $|z| > 1$ *then* $\frac{1}{|z|^{n-i}} < 1$ *for* $i = 0, 1, 2, ..., n-1$

Now

$|R(z)| \geq |a_0||z|^{n+1} - \{|a_0-a_1||z^n| + |a_1-a_2||z^{n-1}| + ... + |a_{m-1}-a_m||z^{n-m+1}| + |a_m - a_{m-1}||z^{n-m} + ... + |a_{n-1}-a_n|| + |a_n|\}$

$\implies |R(z)| \geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{|a_0-a_1| + \frac{|a_1-a_2|}{|z|} + \frac{|a_2-a_3|}{|z|^2} + ... + \frac{|a_{m-1}-a_m|}{|z|^{m-1}} + \frac{|a_m-a_{m-1}|}{|z|^m} + ... + \frac{|a_{n-3}-a_{n-2}|}{|z|^{n-3}} + \frac{|a_{n-2}-a_{n-1}|}{|z|^{n-2}} + \frac{|a_{n-1}-a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n}\}]$

$\implies |R(z)| \geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{|ka_0 - a_1 - ka_0 + a_0| + |a_1-a_2| + |a_2-a_3| + |a_3-a_4| + ... + |a_{m-1}-a_m| + |a_m - a_{m+1}| + ... + |a_{n-3}-a_{n-2}| + |a_{n-2}-a_{n-1}| + |a_{n-1}+\rho-a_n-\rho| + |a_n|\}]$

$\implies |R(z)| \geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{(ka_0-a_1) + (k-1)|a_0| + (a-2-a_1) + (a_2-a_3) + ... + (a_{n-m} - a_{n-m-1}) + (a_{n-m}-a_{n-m+1}) + ... + (a_{n-3}-a_{n-2}) + (a_{n-2}-a_{n-1}) + (a_{n-1}+\rho-a_n) + \rho + |a_n|\}]$

if both n and n-m are even or odd

$\implies |R(z)| \geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{(k(|a_0|+a_0) + |a_n| - (|a_0|+a_n) + 2\rho + s_1\}]$

where $s_1 = 2[(a_2+a_4+a_6+...+a_{n-m-2}+a_{n-m})-(a_1+a_3+a_5+...+a_{n-m-3}+a_{n-m-1})]$

$\implies |R(z)| \geq 0 \; if \; |z| > \frac{1}{|a_0|}\{(k(|a_0|+a_0) + |a_n| - (|a_0|+a_n) + 2\rho + s_1\}$.

This shows that all the zeros of $R(z)$ whose modulus is greater than 1 lie in the closed disk

$|z| < \frac{1}{|a_0|}\{(k(|a_0|+a_0) + |a_n| - (|a_0|+a_n) + 2\rho + s_1\}$.

But those zeros of $R(z)$ whose modulus is less than or equal to 1 already lie in the

above disk.Therefore, it follow that all the zeros of $R(z)$ and hence $J(z)$ lie in
$|z| < \frac{1}{|a_0|}\{(k(|a_0| + a_0) + |a_n| - (|a_0| + a_n) + 2\rho + s_1\}$.
Since $P(z) = z^n J(\frac{1}{z})$ it follows by replacing z *by* $\frac{1}{z}$, all the zeros of $P(z)$ lie in
$|z| > \frac{|a_0|}{k(|a_0|+a_0)+|a_n|-(|a_0|+a_n)+2\rho+s_1}$ if both n and n-m are even or odd.
Hence P(z) does not vanish in the disk $|z| < \frac{|a_0|}{k(|a_0|+a_0)+|a_n|)-(|a_0|+a_n)+2\rho+s_1}$ if both n and (n-m) are even or odd,
where $s_1 = 2[(a_2+a_4+a_6+...+a_{n-m-2}+a_{n-m})-(a_1+a_3+a_5+...+a_{n-m-3}+a_{n-m-1})]$.
Similarly we can also prove if one of n, n-m is odd and other is even.For this we can rearrange the terms of the given polynomial and compute as above .That is P(z) does not vanish in the disk
$|z| < \frac{|a_0|}{k(|a_0|+a_0)+|a_n|)-(|a_0|+a_n)+2\rho+s_2}$ if one of n and (n-m) is even and other one is odd,
where $s_2 = 2[(a_2+a_4+a_6+...+a_{n-m-3}+a_{n-m-1})-(a_1+a_3+a_5+...+a_{n-m-4}+a_{n-m-2})]$.
This completes the proof of the Theorem . □

Corollary 3.2.2. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degee* $n \geq m \geq 2$ *with positive real coefficients such that* $\rho \geq 0, k \geq 1$ *and*
$ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq .... \geq a_{n-m-1} \leq a_{n-m} \geq a_{n-m+1} \geq .... \geq a_{n-2} \geq a_{n-1} \geq a_n - \rho$ *if both n and (n-m) are even or odd*

(or)

$ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq .... \leq a_{n-m-1} \geq a_{n-m} \geq a_{n-m+1} \geq .... \geq a_{n-2} \geq a_{n-1} \geq a_n - \rho$ *if one of the n ,(n-m) is even and other is odd then*
(i) P(z) does not vanish in the disk

$$|z| < \frac{a_0}{(2k-1)a_0 + 2\rho + s_1}$$

if both n and (n-m) are even or odd
where $s_1 = 2[(a_2+a_4+a_6+...+a_{n-m-2}+a_{n-m})-(a_1+a_3+a_5+...+a_{n-m-3}+a_{n-m-1})]$
(ii)P(z) does not vanish in the disk

$$|z| < \frac{a_0}{(2k-1)a_0 + 2\rho + s_2}$$

if one of n and (n-m) is even and other one is odd

where $s_2 = 2[(a_2+a_4+a_6+...+a_{n-m-3}+a_{n-m-1})-(a_1+a_3+a_5+...+a_{n-m-4}+a_{n-m-2})]$.

Remark 3.2.3. By taking $a_i > 0\ for\ i = 0, 1, 2, ...n$ in the Theorem 3.2.1 , then it reduces to Corollary 3.2.2.

Theorem 3.2.4. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degee* $n \geq m \geq 2$ *with real coefficients such that* $\rho \geq 0, 0 < r \leq 0$ *and*

$ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq .... \leq a_{n-m-1} \geq a_{n-m} \geq a_{n-m+1} \geq .... \geq a_{n-2} \geq a_{n-1} \geq a_n - \rho$ *if both n and (n-m) are even or odd*

(or)

$ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq .... \geq a_{n-m-1} \leq a_{n-m} \geq a_{n-m+1} \geq .... \geq a_{n-2} \geq a_{n-1} \geq a_n - \rho$ *if one of the n ,(n-m) is even and other is odd then*

(i) P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{(|a_n| + a_0) - |a_n| - r(|a_0| + a_0) + 2\rho + T_1}$$

if both n and (n-m) are even or odd

where $T_1 = 2[(a_1 + a_3 + ... + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + a_6 + ... + a_{n-m-4} + a_{n-m-2})]$

(ii)P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{(|a_n| + a_0) - |a_n| - r(|a_0| + a_0) + 2\rho + T_2}$$

if one of n and (n-m) is even and other one is odd

where $T_2 = 2[(a_1 + a_3 + ... + a_{n-m-2} + a_{n-m}) - (a_2 + a_+ a_4 + ... + a_{n-m-3} + a_{n-m-1})]$.

Proof. Let$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ be a polynomial of degee n.

Let us consider the polynomials $J(z) = z^n P(\frac{1}{z})$ *and* $R(z) = (z - 1)J(z)$ so that

$R(z) = (z-1)(a_0 z^n + a_1 z^{n-1} + ... + a_{m-1} z^{n-m+1} + a_m z^{n-m} + a_{m+1} z^{n-m+1} + ... + a_{n-1} z + a_n)$

$\Longrightarrow R(z) = a_0 z^{n+1} - \{(a_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + ... + (a_{m-1} - a_m)z^{n-m+1} + (a_m - a_{m-1})z^{n-m} ... + (a_{n-1} - a_n)z + a_n\}$

Also if $|z| > 1\ then\ \frac{1}{|z|^{n-i}} < 1\ for\ i = 0, 1, 2, ..., n - 1$

Now

$|R(z)| \geq |a_0||z|^{n+1} - \{|a_0 - a_1||z^n| + |a_1 - a_2||z^{n-1}| + ... + |a_{m-1} - a_m||z^{n-m+1}| + |a_m - a_{m-1}||z^{n-m} + ... + |a_{n-1} - a_n|| + |a_n|\}$

$\implies |R(z)| \geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{|a_0 - a_1| + \frac{|a_1-a_2|}{|z|} + \frac{|a_2-a_3|}{|z|^2} + ... + \frac{|a_{m-1}-a_m|}{|z|^{m-1}} + \frac{|a_m-a_{m-1}|}{|z|^m|} + ... + \frac{|a_{n-3}-a_{n-2}|}{|z|^{n-3}} + \frac{|a_{n-2}-a_{n-1}|}{|z|^{n-2}} + \frac{|a_{n-1}-a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n}\}]$

$\implies |R(z)| \geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{|ra_0 - a_1 - ra_0 + a_0| + |a_1 - a_2| + |a_2 - a_3| + |a_3 - a_4| + ... + |a_{m-1} - a_m| + |a_m - a_{m+1}| + ... + |a_{n-3} - a_{n-2}| + |a_{n-2} - a_{n-1}| + |a_{n-1} + \rho - a_n - \rho| + |a_n|\}]$

$\implies |R(z)| \geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{(a_1 - ra_0) + (1-r)|a_0| + (a_1 - a_2) + (a_3 - a_2) + ... + (a_{n-m-1} - a_{n-m}) + (a_{n-m} - a_{n-m+1} + ... + (a_{n-3} - a_{n-2}) + (a_{n-2} - a_{n-1}) + (a_{n-1} + \rho - a_n) + \rho + |a_n|\}]$

if both n and n-m are even or odd

$\implies |R(z)| \geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{|a_n| + |a_0| - a_n - r(a_0 + |a_0|) + 2\rho + T_1\}]$

where $T_1 = 2[(a_1 + a_3 + ... + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + ... + a_{n-m-4} + a_{n-m-2})]$

$\implies |R(z)| > 0$ if $|z| > \frac{1}{|a_0|}\{|a_n| + |a_0| - a_n - r(a_0 + |a_0|) + 2\rho + T_1]$.

This shows that all the zeros of $R(z)$ whose modulus is greater than one lie in the closed disk

$|z| < \frac{1}{|a_0|}\{|a_n| + |a_0| - a_n - r(a_0 + |a_0|) + 2\rho + T_1]$

But those zeros of $R(z)$ whose modulus is less than or equal to 1 already lie in the above disk.Therefore , it follows that all the zeros of $R(z)$ and hence $J(z)$ lie in

$|z| < \frac{1}{|a_0|}\{|a_n| + |a_0| - a_n - r(a_0 + |a_0|) + 2\rho + T_1]$

Since $P(z) = z^n J(\frac{1}{z})$ it follows by replacing $z\ by\ \frac{1}{z}$, all the zeros of $P(z)$ lie in

$|z| \geq \frac{|a_0|}{|a_n|+|a_0|-a_n-r(a_0+|a_0|)+2\rho+T_1}$

if both n and n-m are even of odd.

Hence $P(z)$ does not vanish in the disk

$|z| \leq \frac{|a_0|}{|a_n|+|a_0|-a_n-r(a_0+|a_0|)+2\rho+T_1}$ if both n and n-m are even or odd

where $T_1 = 2[(a_1 + a_3 + ... + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + ... + a_{n-m-4} + a_{n-m-2})]$

Similarly we can also prove if one of n, n-m is even and other one is odd.For this we can rearrange the terms of the given polynomial and compute as above .That is P(z) does not vanish in the disk

$|z| < \frac{|a_0|}{(|a_n|+a_0)-|a_n|-r(|a_0|+a_0)+2\rho+T_2}$ if one of n and (n-m) is even and other one is odd

where $T_2 = 2[(a_1 + a_3 + ... + a_{n-m-2} + a_{n-m}) - (a_2 + a_+ a_4 + ... + a_{n-m-3} + a_{n-m-1})]$

This completes the proof of the Theorem.

□

Corollary 3.2.5. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degee* $n \geq m \geq 2$ *with positive real coefficients such that* $\rho \geq 0, 0 < r \leq 0$ *and*
$ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq .... \leq a_{n-m-1} \geq a_{n-m} \geq a_{n-m+1} \geq .... \geq a_{n-2} \geq a_{n-1} \geq a_n - \rho$ *if both n and (n-m) are even or odd*

(or)

$ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq .... \geq a_{n-m-1} \leq a_{n-m} \geq a_{n-m+1} \geq .... \geq a_{n-2} \geq a_{n-1} \geq a_n - \rho$ *if one of the n ,(n-m) is even and other is odd then*
(i) P(z) does not vanish in the disk

$$|z| < \frac{a_0}{(1-2r)a_0 + 2\rho + T_1}$$

if both n and (n-m) are even or odd
where $T_1 = 2[(a_1 + a_3 + ... + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + a_6 + ... + a_{n-m-4} + a_{n-m-2})]$
(ii)P(z) does not vanish in the disk

$$|z| < \frac{a_0}{(1-2r)a_0 + 2\rho + T_2}$$

if one of n and (n-m) is even and other one is odd
where $T_2 = 2[(a_1 + a_3 + ... + a_{n-m-2} + a_{n-m}) - (a_2 + a_4 + ... + a_{n-m-3} + a_{n-m-1})]$.

Remark 3.2.6. By taking $a_i > 0\ for\ i = 0, 1, 2, ..., n$ in Theorem 3.2.4 ,then it reduces to Corollary 3.2.5.

Theorem 3.2.7. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degee* $n \geq m \geq 2$ *with real coefficients such that* $\rho \geq 0, k \geq 1$ *and*
$ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq .... \geq a_{n-m-1} \leq a_{n-m} \leq a_{n-m+1} \leq .... \leq a_{n-2} \leq a_{n-1} \leq a_n + \rho$ *if both n and (n-m) are even or odd*

(or)

$ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq .... \leq a_{n-m-1} \geq a_{n-m} \leq a_{n-m+1} \leq .... \leq a_{n-2} \leq a_{n-1} \leq a_n + \rho$ *if one of the n ,(n-m) is even and other is odd then*

(i) P(z) does not vanish in the disk

$$|z + k - 1| < \frac{|a_0|}{ka_0 + |a_n| + a_n + 2\rho + U_1}$$

if both n and (n-m) are even or odd

where $U_1 = 2[(a_2+a_4+a_6+...+a_{n-m-4}+a_{n-m-2})-(a_1+a_3+a_5+...+a_{n-m-3}+a_{n-m-1})]$

(ii)P(z) does not vanish in the disk

$$|z + k - 1| < \frac{|a_0|}{ka_0 + |a_n| + a_n + 2\rho + U_2}$$

if one of n and (n-m) is even and other one is odd

where $U_2 = 2[(a_2+a_4+a_6+...+a_{n-m-3}+a_{n-m-1})-(a_1+a_3+a_5+...+a_{n-m-2}+a_{n-m})]$.

Proof. :

Let$P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$ be a polynomial of degee n.

Let us consider the polynomials $J(z) = z^n P(\frac{1}{z})$ *and* $R(z) = (z-1)J(z)$ so that

$R(z) = (z-1)(a_0 z^n + a_1 z^{n-1} + ... + a_{m-1}z^{n-m+1} + a_m z^{n-m} + a_{m+1}z^{n-m+1} + ... + a_{n-1}z + a_n)$

$R(z) = a_0 z^n(z + k - 1) - \{(ka_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + ... + (a_{m-1} - a_m)z^{n-m+1} + (a_m - a_{m+1})z^{n-m} + ... + (a_{n-1} - a_n)z + a_n\}$.

Also if $|z| > 1$ *then* $\frac{1}{|z|^{n-i}} < 1$ *for* $i = 0, 1, 2, ..., n-1$

Now

$|R(z)| \geq |a_0||z|^n|z+k-1| - \{|ka_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + ... + |a_{m-1} - a_m||z|^{n-m+1} + |a_m - a_{m+1}||z|^{n-m} + ... + |a_{n-1} - a_n|z| + |a_n|\}$

$\implies R(z) \geq |a_0||z|^n[|z + k - 1| - \frac{1}{|a_0|}\{|ka_0 - a_1| + \frac{|a_1-a_2|}{|z|} + \frac{|a_2-a_3|}{|z|^2} + \frac{|a_3-a_4|}{|z|^3} + ... + \frac{|a_{m-1}-a_m|}{|z|^{m-1}} + \frac{|a_m-a_{m+1}|}{|z|^m} + ... + \frac{|a_{n-3}-a_{n-2}|}{|z|^{n-3}} + \frac{|a_{n-2}-a_{n-1}|}{|z|^{n-2}} + \frac{|a_{n-1}-a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z^n|}]$

$\implies |R(z)| \geq |a_0||z|^n[|z + k - 1| - \frac{1}{|a_0|}\{|ka_0 - a_1| + |a_1 - a_2| + |a_2 - a_3| + |a_3 - a_4|... + |a_{m-1} - a_m| + |a_m - a_{m+1}| + ... + |a_{n-3} - a_{n-2}| + a_{n-2} - a_{n-1}| + |a_{n-1} - \rho - a_n + \rho| + |a_n|\}$

$\implies |R(z)| \geq |a_0||z|^n[|z+k-1| - \frac{1}{|a_0|}\{(ka_0 - a_1) + (a_2 - a_1) + (a_2 - a_3) + ... + (a_{n-m} - a_{n-m-1}) + (a_{n-m+1} - a_{n-m}) + ... + (a_{n-2} - a_{n-3}) + (a_{n-1} - a_{n-2}) + (a_n + \rho - a_{n-1}) + \rho + |a_n|\}]$

if both n and n-m are even or odd

$\implies |R(z)| \geq |a_0||z|^n[|z+k-1| - \frac{1}{|a_0|}\{ka_0 + |a_n| + a_n + 2\rho + U_1\}$
where $U_1 = 2[(a_2+a_4+a_6+...+a_{n-m-4}+a_{n-m-2})-(a_1+a_3+a_5+...+a_{n-m-3}+a_{n-m-1})]$
$\implies |R(z)| > 0 \; if \; |z+k-1| > \frac{1}{|a_0|}\{ka_0 + |a_n| + a_n + 2\rho + U_1\}$
This shows that all the zeros of $R(z)$ whose modulus is greater than one lie in the disk
$|z+k-1| < \frac{1}{|a_0|}\{ka_0 + |a_n| + a_n + 2\rho + U_1\}$.
But those zeros of $R(z)$ whose modulus is less than or equal to 1 already lie in the above disk.Therefore , it follows that all the zeros of $R(z)$ and hence $J(z)$ lie in
$|z+k-1| < \frac{1}{|a_0|}\{ka_0 + |a_n| + a_n + 2\rho + U_1\}$.
Since $P(z) = z^n J(\frac{1}{z})$ it followed by replacing z *by* $\frac{1}{z}$, all the zeros of $P(z)$ lie in
$|z+k-1| \geq \frac{|a_0|}{ka_0+|a_n|+a_n+2\rho+U_1}$ if both n and (n-m) are even odd.
Hence $P(z)$ does not vanish in the disk
$|z+k-1| < \frac{|a_0|}{ka_0+|a_n|+a_n+2\rho+U_1}$ if both n and (n-m) are even odd
where $U_1 = 2[(a_2+a_4+a_6+...+a_{n-m-4}+a_{n-m-2})-(a_1+a_3+a_5+...+a_{n-m-3}+a_{n-m-1})]$.
Similarly we can also prove if one of n, n-m is even and other one is odd.For this we can rearrange the terms of the given polynomial and compute as above .That is P(z) does not vanish in the disk
$|z+k-1| < \frac{|a_0|}{ka_0+|a_n|+a_n+2\rho+U_2}$ if one of n and (n-m) is even and other one is odd
where $U_2 = 2[(a_2+a_4+a_6+...+a_{n-m-3}+a_{n-m-1})-(a_1+a_3+a_5+...+a_{n-m-2}+a_{n-m})]$
This completes the proof of the Theorem.

□

Corollary 3.2.8. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degee* $n \geq m \geq 2$ *with positive real coefficients such that* $\rho \geq 0, k \geq 1$ *and*
$ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq .... \geq a_{n-m-1} \leq a_{n-m} \leq a_{n-m+1} \leq .... \leq a_{n-2} \leq a_{n-1} \leq a_n + \rho$ *if both n and (n-m) are even or odd*

(or)

$ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq .... \leq a_{n-m-1} \geq a_{n-m} \leq a_{n-m+1} \leq .... \leq a_{n-2} \leq a_{n-1} \leq a_n + \rho$ *if one of the n ,(n-m) is even and other is odd then*
(i) P(z) does not vanish in the disk

$$|z+k-1| < \frac{a_0}{ka_0+2a_n|+2\rho+U_1}$$

if both n and (n-m) are even or odd

where $U_1 = 2[(a_2+a_4+a_6+...+a_{n-m-2}+a_{n-m})-(a_1+a_3+a_5+...+a_{n-m-3}+a_{n-m-1})]$

(ii)P(z) does not vanish in the disk

$$|z+k-1| < \frac{a_0}{ka_0+2a_n+2\rho+U_2}$$

if one of n and (n-m) is even and other one is odd

where $U_2 = 2[(a_2+a_4+a_6+...+a_{n-m-3}+a_{n-m-1})-(a_1+a_3+a_5+...+a_{n-m-2}+a_{n-m})]$.

Remark 3.2.9. By taking $a_i > 0\ for\ i = 0,1,2,....,n$ in Theorem 3.2.7 ,then it reduces to Corollary 3.2.8.

Theorem 3.2.10. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degee* $n \geq m \geq 2$ *with real coefficients such that* $\rho \geq 0, 0 < r \leq 1$ *and*

$ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \geq .... \leq a_{n-m-1} \geq a_{n-m} \leq a_{n-m+1} \leq .... \leq a_{n-2} \leq a_{n-1} \leq a_n+\rho$ *if both n and (n-m) are even or odd*

(or)

$ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq .... \geq a_{n-m-1} \leq a_{n-m} \leq a_{n-m+1} \leq .... \leq a_{n-2} \leq a_{n-1} \leq a_n+\rho$ *if one of the n ,(n-m) is even and other is odd then*

(i) P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{|a_0|+|a_n|-r(|a_0|+a_0)+V_1}$$

if both n and (n-m) are even or odd

where $V_1 = 2[(a_1+a_3+...+a_{n-m-3}+a_{n-m-1})-(a_2+a_4+...+a_{n-m-2}+a_{n-m})]$

(ii)P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{|a_0|+|a_n|-r(|a_0|+a_0)+V_2}$$

if one of n and (n-m) is even and other one is odd

where $V_2 = 2[(a_1+a_3+...+a_{n-m-4}+a_{n-m-2})-(a_2+a_4+...+a_{n-m-3}+a_{n-m-1})]$.

Proof. Let$P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$ be a polynomial of degee n.
Let us consider the polynomials $J(z) = z^n P(\frac{1}{z})$ *and* $R(z) = (z-1)J(z)$ so that
$R(z) = (z-1)(a_0 z^n + a_1 z^{n-1} + ... + a_{m-1}z^{n-m+1} + a_m z^{n-m} + a_{m+1}z^{n-m+1} + ... + a_{n-1}z + a_n)$
$\implies R(z) = a_0 z^{n+1} - \{(a_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + ... + (a_{m-1} - a_m)z^{n-m+1} + (a_m - a_{m-1})z^{n-m} ... + (a_{n-1} - a_n)z + a_n\}$
Also if $|z| > 1$ *then* $\frac{1}{|z|^{n-i}} < 1$ *for* $i = 0, 1, 2, ..., n-1$
Now
$|R(z)| \geq |a_0||z|^{n+1} - \{|a_0 - a_1||z^n| + |a_1 - a_2||z^{n-1}| + ... + |a_{m-1} - a_m||z^{n-m+1}| + |a_m - a_{m-1}||z^{n-m} + ... + |a_{n-1} - a_n|| + |a_n|\}$
$\implies |R(z)| \geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{|a_0 - a_1| + \frac{|a_1-a_2|}{|z|} + \frac{|a_2-a_3|}{|z|^2} + \frac{|a_3-a_4|}{|z|^3} + ... + \frac{|a_{m-1}-a_m|}{|z|^{m-1}} + \frac{|a_m-a_{m+1}|}{|z|^m} + ... + \frac{|a_{n-3}-a_{n-2}|}{|z|^{n-3}} + \frac{|a_{n-2}-a_{n-1}|}{|z|^{n-2}} + \frac{|a_{n-1}-a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z^n|}]$
$\implies |R(z)| \geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{|ra_0 - a_1 - ra_0 - a_0| + |a_1 - a_2| + |a_2 - a_3| + |a_3 - a_4| + ... + |a_{m-1} - a_m| + |a_m - a_{m+1}| + ... + |a_{n-3} - a_{n-2}| + |a_{n-2} - a_{n-1}| + |a_{n-1} - a_n| + |a_n|\}]$
$\implies |R(z)| \geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{(a_1 - ra_0) + (1-r)|a_0| + (a_1 - a_2) + (a_3 - a_2) + ... + (a_{n-m-1} - a_{n-m}) + (a_{n-m+1} - a_{n-m}) + ... + (a_{n-2} - a_{n-3}) + (a_{n-1} - a_{n-2}) + (a_n - a_{n-1} + |a_n|)\}]$ if both n and (n-m) are even or odd
$\implies |R(z)| \geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{|a_0 + ||a_n| + a_n - r(|a - 0| + a_0) + V_1\}$
where $V_1 = 2[(a_1 + a_3 + ... + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + ... + a_{n-m-2} + a_{n-m})]$
$\implies |R(z)| > 0$ *if* $|z| > \frac{1}{|a_0|}\{|a_0| + |a_n| + a_n - r(|a_0| + a_0) + V_1\}$
This shows that all the zeros of $R(z)$ whose modulus is greater than one lie in the closed disk
$|z| \leq \frac{1}{|a_0|}\{|a_0| + |a_n| + a_n - r(|a_0| + a_0) + V_1\}$
But those zeros of $R(z)$ whose modulus is less than or equal to 1 already lie in the above disk.Therefore , it follows that all the zeros of $R(z)$ and hence $J(z)$ lie in
$|z| \leq \frac{1}{|a_0|}\{|a_0| + |a_n| + a_n - r(|a_0| + a_0) + V_1\}$.
Since $P(z) = z^n J(\frac{1}{z})$ it follows by replacing z *by* $\frac{1}{z}$, all the zeros of $P(z)$ lie in
$|z| \geq \frac{1}{|a_0|}\{|a_0| + |a_n| + a_n - r(|a_0| + a_0) + V_1\}$ if both n and (n-m) are even or odd.
Hence $P(z)$ does not vanish in the disk
$|z| < \frac{|a_0|}{|a_0| + |a_n| - r(|a_0| + a_0) + V_1}$ if both n and (n-m) are even or odd,
where $V_1 = 2[(a_1 + a_3 + ... + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + ... + a_{n-m-2} + a_{n-m})]$.

Similarly we can also prove if one of n, n-m is even and other one is odd.For this we can rearrange the terms of the given polynomial and compute as above. That is P(z) does not vanish in the disk

$|z| < \frac{|a_0|}{|a_0|+|a_n|-r(|a_0|+a_0)+V_2}$ if one of n and (n-m) is even and other one is odd

where $V_2 = 2[(a_1 + a_3 + ... + a_{n-m-4} + a_{n-m-2}) - (a_2 + a_4 + ... + a_{n-m-3} + a_{n-m-1})]$

This completes the proof of the Theorem. □

Corollary 3.2.11. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degee $n \geq m \geq 2$ with positive real coefficients such that $\rho \geq 0, 0 < r \leq 1$ and*

$ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \geq .... \leq a_{n-m-1} \geq a_{n-m} \leq a_{n-m+1} \leq .... \leq a_{n-2} \leq a_{n-1} \leq a_n + \rho$ *if both n and (n-m) are even or odd*

(or)

$ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq .... \geq a_{n-m-1} \leq a_{n-m} \leq a_{n-m+1} \leq .... \leq a_{n-2} \leq a_{n-1} \leq a_n + \rho$ *if one of the n ,(n-m) is even and other is odd then*

(i) P(z) does not vanish in the disk

$$|z| < \frac{a_0}{2a_n + (1-2r)a_0 + V_1}$$

if both n and (n-m) are even or odd

where $V_1 = 2[(a_1 + a_3 + ... + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + ... + a_{n-m-2} + a_{n-m})]$

(ii)P(z) does not vanish in the disk

$$|z| < \frac{a_0}{2a_n + (1-2r)a_0 + V_2}$$

if one of n and (n-m) is even and other one is odd

where $V_2 = 2[(a_1 + a_3 + ... + a_{n-m-4} + a_{n-m-2}) - (a_2 + a_4 + ... + a_{n-m-3} + a_{n-m-1})]$.

Remark 3.2.12. By taking $a_i > 0 \ for \ i = 0, 1, 2, ..., n$ in Theorem 3.2.10, then it reduces to Corollary 3.2.11.

The results of this section appeared in [30].

3.3 Zero free region for polynomials with special complex coefficients

Theorem 3.3.1. *Let $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ be a polynomial of degree n with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ for $i = 0, 1, 2, ..., n$ such that for some $k \geq 1,\ \xi \geq 0, a_m \neq 0$, $a_n - \xi \leq a_{n-1} \leq .... \leq a_{m+1} \leq ka_m \geq a_{m-1} \geq ... \geq a_1 \geq a_0$ and for some $t \geq 1$, $\eta \geq 0, b_m \neq 0$ $b_n - \eta \leq b_{n-1} \leq, ..., \leq b_{m+1} \leq tb_m \geq b_{m-1} \geq .... \geq b_1 \geq b_0$ then $P(z)$ does not vanish in the disk*

$$|z|<\frac{|\alpha_0|}{2[k(|a_m| + a_m) + t(|b_m| + b_m) - |a_m| - |b_m| + \xi + \eta] + |a_n| + |b_n| - (a_0 + b_0 + a_n + b_n)}$$

Proof. Let $P(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + ... + \alpha_{m+1} z^{m+1} + \alpha_m z^m + \alpha_{m-1} z^{m-1} ... + \alpha_1 z + \alpha_0$ be a polynomial of degree n and consider the polynomials $J(z) = z^n P(\frac{1}{z})$ *and* $R(z) = (z-1)J(z)$ so that

$R(z) = (z-1)(\alpha_0 z^n + \alpha_1 z^{n-1} + ... + \alpha_{m-1} z^{n-(m-1)} + \alpha_m z^{n-m} + \alpha_{m+1} z^{n-(m+1)} + .. + \alpha_{n-1} z + \alpha_n)$

$\Rightarrow R(z) = \alpha_0 z^{n+1} - \{(\alpha_0 - \alpha_1)z^n + (\alpha_1 - \alpha_2)z^{n-1} + ... + (\alpha_{m-1} - \alpha_m)z^{n-m+1} + (\alpha_m - \alpha_{m+1})z^{n-m} + ... + (\alpha_{n-1} - \alpha_n)z + \alpha_n\}$

$\Rightarrow R(z) = \alpha_0 z^{n+1} - \{(a_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + ... + (a_{m-1} - a_m)z^{n-m+1} + (a_m - a_{m+1})z^{n-m} + ... + (a_{n-1} - a_n)z + a_n\} - i\{(b_0 - b_1)z^n + (b_1 - b_2)z^{n-1} + ... + (b_{m-1} - b_m)z^{n-m+1} + (b_m - b_{m+1})z^{n-m} + ... + (b_{n-1} - b_n)z + b_n\}$

Also if $|z|>1 then\ \frac{1}{|z|^{n-i}}\ for\ i = 0, 1, 2, .., n-1$. Now $|R(z)|>|\alpha_0||z|^{n+1} - \{|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + ... + |a_{m-1} - a_m||z|^{n-m+1} + |a_m - a_{m+1}||z|^{n-m} + ... + |a_{n-1} - a_n||z| + |a_n|\} + \{|b_0 - b_1||z|^n + |b_1 - b_2||z|^{n-1} + ... + |b_{m-1} - b_m||z|^{n-m+1} + |b_m - b_{m+1}||z|^{n-m} + ... + |b_{n-1} - b_n||z| + |b_n|\}$

$\Rightarrow |R(z)|>|\alpha_0||z|^n[|z| - \frac{1}{|\alpha_0|}\{(|a_0 - a_1| + \frac{|a_1 - a_2|}{|z|} + \frac{|a_2 - a_3|}{|z|^2} + ... + \frac{|a_{m-1} - a_m|}{|z|^{m-1}} + \frac{a_m - a_{m+1}}{|z|^m} + ... + \frac{|a_{n-2} - a_{n-1}|}{|z|^{n-2}} + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n}) + (|b_0 - b_1| + \frac{|b_1 - b_2|}{|z|} + \frac{|b_2 - b_3|}{|z|^2} + ... + \frac{|b_{m-1} - b_m|}{|z|^{m-1}} + \frac{b_m - b_{m+1}}{|z|^m} + ... + \frac{|b_{n-2} - b_{n-1}|}{|z|^{n-2}} + \frac{|b_{n-1} - b_n|}{|z|^{n-1}} + \frac{|b_n|}{|z|^n})\}]$

$\Rightarrow |R(z)|>|\alpha_0||z|^n[|z| - \frac{1}{|\alpha_0|}\{(|a_0 - a_1| + |a_1 - a_2| + ... + |a_{m-1} - ka_m + ka_m - a_m| + |a_m - ka_m + ka_m - a_{m-1}| + ... + |a_{n-2} - a_{n-1}| + ... + |a_{n-1} - \xi + \xi - a_n| + |a_n|) + (|b_0 - b_1| + |b_1 - b_2| + ... + |b_{m-1} - kb_m + kb_m - b_m| + |b_m - kb_m + kb_m - b_{m-1}| + ... + |b_{n-2} -$

$b_{n-1}| + ... + |b_{n-1} - \eta + \eta - b_n| + |b_n|)\}]$

$\Rightarrow |R(z)|>|\alpha_0||z|^n[|z| - \frac{1}{|\alpha_0|}\{((a_0 - a_1) + (a_2 - a_1) + (a_3 - a_2) + ... + (ka_m - a_{m-1} + 2(k-1)|a_m| + (ka_m - a_{m+1} + ... + (a_{n-2} - a_{n-1}) + (a_{n-1} + \xi - a_n) + \xi + |a_n|)) + (b_0 - b_1) + (b_2 - b_1) + (b_3 - b_2) + ... + (tb_m - b_{m-1} + 2(t-1)|b_m| + (tb_m - b_{m+1}) + ... + (b_{n-2} - b_{n-1}) + (b_{n-1} + \eta - b_n) + \eta + |b_n|)\}]$

$\Rightarrow |R(z)|>|\alpha_0||z|^n[|z| - \frac{1}{|\alpha_0|}\{2[k(|a_m| + a_m) + t(|b_m| + b_m) - |a_m| - |b_m| + \xi + \eta] + |a_n| + |b_n| - (a_0 + b_0 + a_n + b_n)\}]>0$

$If|z|>\frac{1}{|\alpha_0|}\{2[k(|a_m|+a_m)+t(|b_m|+b_m)-|a_m|-|b_m|+\xi+\eta]+|a_n|+|b_n|-(a_0+b_0+a_n+b_n)\}]$.

This shows that all the zeros of $R(z)$ whose modulus is greater tha 1 lie in the disk $|z|<\frac{1}{|\alpha_0|}\{2[k(|a_m|+a_m)+t(|b_m|+b_m)-|a_m|-|b_m|+\xi+\eta]+|a_n|+|b_n|-(a_0+b_0+a_n+b_n)\}]$.

But those zeros of $R(z)$ whose modulus is less than or equal to 1 already lie in the above disk.

Therefore,it follows that all the zeros of $R(z)$ and hence $J(z)$ lie in $|z|<\frac{1}{|\alpha_0|}\{2[k(|a_m|+a_m)+t(|b_m|+b_m)-|a_m|-|b_m|+\xi+\eta]+|a_n|+|b_n|-(a_0+b_0+a_n+b_n)\}]$

Since $P(z) = z^n J(\frac{1}{z})$ it fallows, by replacing z by $\frac{1}{z}$, we get that all the zeros of $P(z)$ lie in

$$|z|>\frac{|\alpha_0|}{2[k(|a_m| + a_m) + t(|b_m| + b_m) - |a_m| - |b_m| + \xi + \eta] + |a_n| + |b_n| - (a_0 + b_0 + a_n + b_n)}.$$

Hence $P(z)$ does not vanish in the disk

$$|z|<\frac{|\alpha_0|}{2[k(|a_m| + a_m) + t(|b_m| + b_m) - |a_m| - |b_m| + \xi + \eta] + |a_n| + |b_n| - (a_0 + b_0 + a_n + b_n)}.$$

This completes the proof of the Theorem. □

Corollary 3.3.2. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $k \geq 1,\ \xi \geq 0, a_m \neq 0,$ $0 < a_n - \xi \leq a_{n-1} \leq .... \leq a_{m+1} \leq ka_m \geq a_{m-1} \geq ... \geq a_1 \geq a_0 > 0$ *and for some* $t \geq 1, \eta \geq 0, b_m \neq 0$ $0 < b_n - \eta \leq b_{n-1} \leq, ..., \leq b_{m+1} \leq tb_m \geq b_{m-1} \geq .... \geq b_1 \geq b_0 > 0$ *then* $P(z)$ *does not vanish in the disk*

$$|z|<\frac{|\alpha_0|}{2[(2K-1)a_m + (2t-1)b_m + \xi + \eta] - (a_0 + b_0)}.$$

Corollary 3.3.3. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $k \geq 1,\ \xi \geq 0, a_m \neq 0,$
$a_n - \xi \leq a_{n-1} \leq .... \leq a_{m+1} \leq ka_m \geq a_{m-1} \geq ... \geq a_1 \geq a_0$ *and for some* $b_m \neq 0$
$b_n \leq b_{n-1} \leq, ..., \leq b_{m+1} \leq b_m \geq b_{m-1} \geq .... \geq b_1 \geq b_0$
then $P(z)$ *does not vanish in the disk*

$$|z| < \frac{|\alpha_0|}{2[k(|a_m| + a_m) + b_m - |a_m| + \xi] + |a_n| + |b_n| - (a_0 + b_0 + a_n + b_n)}$$

Corollary 3.3.4. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $a_m \neq 0,$
$a_n \leq a_{n-1} \leq .... \leq a_{m+1} \leq a_m \geq a_{m-1} \geq ... \geq a_1 \geq a_0$ *and for some* $t \geq 1, \eta \geq 0, b_m \neq 0$
$b_n - \eta \leq b_{n-1} \leq, ..., \leq b_{m+1} \leq tb_{b_m} \geq b_{m-1} \geq .... \geq b_1 \geq b_0$
then $P(z)$ *does not vanish in the disk*

$$|z| < \frac{|\alpha_0|}{2[a_m + t(|b_m| + b_m) - |b_m| + \eta] + |a_n| + |b_n| - (a_0 + b_0 + a_n + b_n)}$$

Corollary 3.3.5. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $k \geq 1,\ \xi \geq 0, a_m \neq 0,$
$a_n - \xi \leq a_{n-1} \leq .... \leq a_{m+1} \leq ka_m \geq a_{m-1} \geq ... \geq a_1 \geq a_0$ *and for* $b_m \neq 0$
$b_n - \xi \leq b_{n-1} \leq, ..., \leq b_{m+1} \leq kb_m \geq b_{m-1} \geq .... \geq b_1 \geq b_0$
then $P(z)$ *does not vanish in the disk*

$$|z| < \frac{|\alpha_0|}{2[k(|a_m| + a_m + |b_m| + b_m) - |a_m| - |b_m| + 2\xi] + |a_n| + |b_n| - (a_0 + b_0 + a_n + b_n)}$$

Corollary 3.3.6. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $a_m \neq 0,$
$a_n \leq a_{n-1} \leq .... \leq a_{m+1} \leq a_m \geq a_{m-1} \geq ... \geq a_1 \geq a_0$ *and for some* $b_m \neq 0$
$b_n \leq b_{n-1} \leq, ..., \leq b_{m+1} \leq b_m \geq b_{m-1} \geq .... \geq b_1 \geq b_0$
then $P(z)$ *does not vanish in the disk*

$$|z| < \frac{|\alpha_0|}{2[(a_m + b_m)] + |a_n| + |b_n| - (a_0 + b_0 + a_n + b_n)}$$

Corollary 3.3.7. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree n with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $a_m \neq 0$,
$0 < a_n \leq a_{n-1} \leq .... \leq a_{m+1} \leq a_m \geq a_{m-1} \geq ... \geq a_1 \geq a_0 > 0$ *and for some* $b_m \neq 0$
$0 < b_n \leq b_{n-1} \leq, ..., \leq b_{m+1} \leq b_m \geq b_{m-1} \geq .... \geq b_1 \geq b_0 > 0$
then $P(z)$ *does not vanish in the disk*

$$|z| < \frac{|\alpha_0|}{2[(a_m + b_m)] - (a_0 + b_0)}.$$

Remark 3.3.8. By taking $a_i > 0$ *and* $b_i > 0$ *for* $i = 0, 2, ...n$ in Theorem 3.3.1, it reduces to Corollary3.3.2.

Remark 3.3.9. By taking $\eta = 0$ *and* $t = 1$ in Theorem 3.3.1, it reduces to Corollary 3.3.3.

Remark 3.3.10. By taking $\xi = 0$ *and* $k = 1$in Theorem 3.3.1, it reduces to Corollary 3.3.4.

Remark 3.3.11. By taking $\xi = \eta$ *and* $t = k$ in Theorem 3.3.1, it reduces to Corollary 3.3.5.

Remark 3.3.12. By taking $\xi = \eta = 0$ *and* $t = k = 1$ in Theorem 3.3.1, it reduces to Corollary 3.3.6.

Remark 3.3.13. By taking $\xi = \eta = 0$ *and* $t = k = 1$ *and* $a_i > 0, b_i > 0$ for i=0,1,2,...n in Theorem 3.3.1, it reduces to Corollary 3.3.7.

Theorem 3.3.14. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree n with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $0 < \tau \leq 1, k \geq 1, \xi \geq 0, a_m \neq 0$
$a_n + \xi \geq a_{n-1} \geq .... \geq a_{m+1} \geq \tau a_m \leq a_{m-1} \leq, ..., \leq a_1 \leq k a_0$ *and for some* $0 < \mu \leq 1$, $t \geq 1, \eta \geq 0, b_m \neq 0,$ $b_n + \eta \geq b_{n-1} \geq, ..., \geq b_{m+1} \geq \mu b_m \leq b_{m-1} \leq .... \leq b_1 \leq t b_0$
then $P(z)$ *does not vanish in the disk*

$$|z| < \frac{\alpha_0}{k(|a_0| + a_0) + t(|b_0| + b_0) + X + |a_n| + |b_n| + a_n + b_n - (|a_0| + |b_0|),}$$

where $X = 2[|a_m| + |b_m| + \xi + \eta - \tau(|a_m| + a_m) - \mu(|b_m| + b_m)]$

Proof. Proof of the Theorem is similar to the proof of Theorem 3.3.1. □

Corollary 3.3.15. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $0 < \tau \leq 1, k \geq 1, \xi \geq 0, a_m \neq 0,\ 0 < a_n + \xi \geq a_{n-1} \geq .... \geq a_{m+1} \geq \tau a_m \leq a_{m-1} \leq, ..., \leq a_1 \leq ka_0 > 0$ *and for some* $0 < \mu \leq 1\ , t \geq 1 \eta \geq 0, b_m \neq 0,\ 0 < b_n + \eta \geq b_{n-1} \geq, ..., \geq b_{m+1} \geq \mu b_m \leq b_{m-1} \leq .... \leq b_1 \leq tb_0 > 0$ *then* $P(z)$ *does not vanish in the disk*

$$|z| < \frac{|\alpha_0|}{2[ka_0 + tb_0 - \tau a_m - \mu b_m + a_n + b_n + \xi + \eta] + [a_n + b_m - a_0 - b_0]}$$

.

Corollary 3.3.16. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $0 < \tau \leq 1, k \geq 1, \xi \geq 0, a_m \neq 0$ $a_n + \xi \geq a_{n-1} \geq .... \geq a_{m+1} \geq \tau a_m \leq a_{m-1} \leq, ..., \leq a_1 \leq ka_0$ *and for some* $b_m \neq 0,\ b_n \geq b_{n-1} \geq, ..., \geq b_{m+1} \geq b_m \leq b_{m-1} \leq .... \leq b_1 \leq b_0$ *then* $P(z)$ *does not vanish in the disk*

$$|z| < \frac{|\alpha_0|}{k(|a_0| + a_0) + b_0 + 2[|a_m| + \xi - \tau(|a_m| + a_m) - b_m] + |a_n| + |b_n| + a_n + b_n - |a_0|}$$

.

Corollary 3.3.17. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $a_m \neq 0$ $a_n \geq a_{n-1} \geq .... \geq a_{m+1} \geq a_m \leq a_{m-1} \leq, ..., \leq a_1 \leq a_0$ *and for some* $0 < \mu \leq 1\ , t \geq 1, \eta \geq 0, b_m \neq 0,\ b_n + \eta \geq b_{n-1} \geq, ..., \geq b_{m+1} \geq \mu b_m \leq b_{m-1} \leq .... \leq b_1 \leq tb_0$ *then* $P(z)$ *does not vanish in the disk*

$$|z| < \frac{|\alpha_0|}{t(|b_0| + b_0) + 2[|b_n| + \eta - a_m - \mu(|b_m| + b_m)] + |a_n| + |b_n| + a_n + b_n - |b_0|}$$

.

Corollary 3.3.18. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $0 < \tau \leq 1, k \geq 1, \xi \geq 0, a_m \neq 0$ $a_n + \xi \geq a_{n-1} \geq .... \geq a_{m+1} \geq \tau a_m \leq a_{m-1} \leq, ..., \leq a_1 \leq ka_0$ *and for some*$b_m \neq$

$0,\ b_n+\xi \geq b_{n-1} \geq, ..., \geq b_{m+1} \geq \tau b_m \leq b_{m-1} \leq .... \leq b_1 \leq kb_0$ *then* $P(z)$ *does not vanish in the disk*

$$|z|<\frac{\alpha_0}{k(|a_0|+a_0+|b_0|+b_0)+X+|a_n|+|b_n|+a_n+b_n-(|a_0|+|b_0|),}$$

where $X=2[|a_m|+|b_m|+2\xi-\tau(|a_m|+a_m+|b_m|+b_m)]$.

Corollary 3.3.19. *Let* $P(z)=\sum_{i=0}^{n}\alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i)=a_i$ *and* $Im(\alpha_i)=b_i$ *for* $i=0,1,2,...,n$ *such that for some* $a_m\neq 0$ $a_n \geq a_{n-1} \geq .... \geq a_{m+1} \geq a_m \leq a_{m-1} \leq, ..., \leq a_1 \leq a_0$ *and* for $b_m \neq 0,\ b_n \geq b_{n-1} \geq, ..., \geq b_{m+1} \geq b_m \leq b_{m-1} \leq .... \leq b_1 \leq b_0$ *then* $P(z)$ *does not vanish in the disk*

$$|z|<\frac{|\alpha_0|}{(|a_0|+|b_0|+a_0+b_0)-(a_m+b_m)+|a_n|+|b_n|+a_n+b_n-(|a_0|+|b_0|)}$$

.

Corollary 3.3.20. *Let* $P(z)=\sum_{i=0}^{n}\alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i)=a_i$ *and* $Im(\alpha_i)=b_i$ *for* $i=0,1,2,...,n$ *such that for some* $a_m\neq 0$ $0<a_n \geq a_{n-1} \geq .... \geq a_{m+1} \geq a_m \leq a_{m-1} \leq, ..., \leq a_1 \leq a_0$ *and* for $b_m \neq 0,\ 0<b_n+ \geq b_{n-1} \geq, ..., \geq b_{m+1} \geq b_m \leq b_{m-1} \leq .... \leq b_1 \leq b_0>0$ *then* $P(z)$ *does not vanish in the disk*

$$|z|<\frac{|\alpha_0|}{(a_0+b_0)-(a_m+b_m)+|a_n|+|b_n|+2(a_n+b_n)}$$

.

Remark 3.3.21. By taking $a_i>0$ *and* $b_i>0$ for $i=0,1,2,..,n$ in Theorem 3.3.14, it reduces to Corollary 3.3.15.

Remark 3.3.22. By taking $\eta=0$ *and* $t=\mu=1$ in Theorem 3.3.14, it reduces to Corollary 3.3.16.

Remark 3.3.23. By taking $\xi=0$ *and* $k=\tau=1$ in Theorem 3.3.14, it reduces to Corollary 3.3.17.

Remark 3.3.24. By taking $\eta = \xi, \mu = \tau$ *and* $t = k$ in Theorem 3.3.14, it reduces to Corollary 3.3.18.

Remark 3.3.25. By taking $\eta = \xi = 0$ *and* $\mu = \tau = t = k = 1$ in Theorem 3.3.14, it reduces to Corollary 3.3.19.

Remark 3.3.26. By taking $\eta = \xi = 0$ *and* $\mu = \tau = k = t = 1$ *and* $a_i>0, b_i>0$ *for* $i = 0, 1, 2, .., n$ in Theorem 3.3.14, it reduces to Corollary 3.3.20.

Theorem 3.3.27. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $k \geq 0, \xi \geq 0, a_m \neq 0$ $a_n - \xi \leq a_{n-1} \leq .... \leq a_{m+1} \leq ka_m \geq a_{m-1} \geq ... \geq a_1 \geq a_0$ *and for some* $0<\mu \leq 1$, $t \geq 1, \eta \geq 0, b_m \neq 0$, $b_n + \eta \geq b_{n-1} \geq, ..., \geq b_{m+1} \geq \mu b_m \leq b_{m-1} \leq .... \leq b_1 \leq tb_0$ *then all the zeros of* $P(z)$ *lie in the disk*

$$|z|<\frac{|\alpha_0|}{t(|b_0| + b_0) + (a_0 + |b_0| + X_1) + |a_n| + |b_n| - a_n + b_n}$$

Where $X_1 = 2[k(|a_m| + a_m) - |a_m| + |b_m| - \mu(|b_m| + b_m) + \xi + \eta]$.

Proof. Proof of the Theorem is similar to the proof of Theorem 3.3.1. □

Theorem 3.3.28. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* n *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $0<\tau \leq 1, k \geq 1, \xi \geq 0, a_m \neq 0$ $a_n + \xi \geq a_{n-1} \geq .... \geq a_{m+1} \geq \tau a_m \leq a_{m-1} \leq, ..., \leq a_1 \leq ka_0$ *and for some* $t \geq 1, \eta \geq 0, b_m \neq 0$ $b_n - \eta \leq b_{n-1} \leq, ..., \leq b_{m+1} \leq tb_m \geq b_{m-1} \geq .... \geq b_1 \geq b_0$ *then* $P(z)$ *does not vanish in the disk*

$$|z|<\frac{|\alpha_0|}{k(|a_0| + a_0) - (|a_0| + b_0) + X_2 + a_n - b_n + |a_n| + |b_n|}$$

Where $X_2 = 2[|a_m| - |b_m| - \mu(|a_m| + a_m) + t(|b_m| + b_m) + \xi + \eta]$.

Proof. Proof of the Theorem is similar to the proof of Theorem 3.3.1 □

The results of this section appeared in [33].

Chapter 4

Zero-free regions for the derivatives of complex polynomials

4.1 Introduction of results

If $P(z)$ is polynomial of degree n then the Polar derivative of $P(z)$ with respect to α is defined by $D_\alpha P(z) = nP(z) + (\alpha - z)P^{'}(z)$, α is a complex number.There are many results on zeros of polar derivatives of polynomial.

The following Results are due to P.Ramulu and G.L.Reddy[34].

Theorem 4.1.1. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be polynomial of degree n with real coefficients and* $D_\alpha P(z)$ *be the polar derivative of* $P(z)$ *with respect to a real number* α *such that* $na_0 \leq (n-1)a_1 \leq (n-2)a_2 \leq .... \leq 3a_{n-3} \leq 2a_{n-2} \leq a_{n-1},$ if $\alpha = 0$ *then all the zeros of* $D_0P(z)$ *lie in* $|z| \leq \frac{1}{|a_{n-1}|}[a_{n-1} - na_0 + |na_0|]$.

Theorem 4.1.2. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be polynomial of degree n with real coefficients and* $D_\alpha P(z)$ *be the polar derivative of* $P(z)$ *with respect to a real number* α *such that* $na_0 \geq (n-1)a_1 \geq (n-2)a_2 \geq .... \geq 3a_{n-3} \geq 2a_{n-2} \geq a_{n-1},$ if $\alpha = 0$ *then all the zeros of* $D_0P(z)$ *lie in* $|z| \leq \frac{1}{|a_{n-1}|}[|na_0| + na_0 - a_{n-1}]$.

4.2 On the zeros of polar derivatives of polynomials

Theorem 4.2.1. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n and $D_\alpha P(z)$ be the polar derivative of $P(z)$ with respect to a real number α such that $[i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1} \geq (i+1)\alpha a_{i+1} + (n-i)a_i$ for $i = 0, 1, 2, ..., n-2$ then all the zeros of $D_\alpha P(z)$ lie in*

$$|z| \leq \frac{1}{|n\alpha a_n + a_{n-1}|}[n\alpha a_n + a_{n-1} - (\alpha a_1 + na_0) + |\alpha_+ na_0|].$$

Proof. Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_2 z^2 + a_1 z + a_0$ be a polynomial of degree n, then

$D_\alpha P(z) = [n\alpha a_n + a_{n-1}]z^{n-1} + [(n-1)\alpha a_{n-1} + 2a_{n-2}]z^{n-2} + [(n-2)\alpha a_{n-2} + 3a_{n-3}]z^{n-3} + ... + [3\alpha a_3 + (n-2)a_2]z^2 + [2\alpha a_2 + (n-1)a_1]z + [\alpha a_1 + na_0]$.

Let us consider the polynomial $Q(z) = (1-z)D_\alpha P(z)$ *so that*

$Q(z) = (1-z)([n\alpha a_n + a_{n-1}]z^{n-1} + [(n-1)\alpha a_{n-1} + 2a_{n-2}]z^{n-2} + [(n-2)\alpha a_{n-2} + 3a_{n-3}]z^{n-3} + ... + [3\alpha a_3 + (n-2)a_2]z^2 + [2\alpha a_2 + (n-1)a_1]z + [\alpha a_1 + na_0])$

$\implies Q(z) = -[n\alpha a_n + a_{n-1}]z^n + \{[n\alpha a_n + (1+\alpha-n\alpha)a_{n-1} - 2a_{n-2}]z^{n-1} + [(n-1)\alpha a_{n-1} + (2+2\alpha-n\alpha)a_{n-2} - 3a_{n-3}]z^{n-2} ...... + [3\alpha a_3 + (n-2-2\alpha)a_2(n-1)a_1]z^2 + [2\alpha a_2 + (n-1-\alpha)a_1 - na_0]z + [\alpha a_1 + na_0]\}$.

Also if $|z|>1$ *then* $\frac{1}{|z|^{n-i}}<1$ *for* $i = 0, 1, 2, .., n-2$. Now

$|Q(z)| \geq |n\alpha a_n + a_{n-1}||z|^{n-1} - \{|n\alpha a_n + (1+\alpha-n\alpha)a_{n-1} - 2a_{n-2}||z|^{n-1} + |(n-1)\alpha a_{n-1} + (2+2\alpha-n\alpha)a_{n-2} - 3a_{n-3}||z|^{n-2} + ... + |3\alpha a_3 + (n-2-2\alpha)a_2 - (n-1)a_1||z|^2 + |2\alpha a_2 + (n-1-\alpha)a_1 - na_0||z| + |\alpha a_1 + na_0|\}$

$\implies |Q(z)| \geq |n\alpha a_n + a_{n-1}||z|^{n-1}[|z| - \frac{1}{|n\alpha a_n + a_{n-1}|}\{|n\alpha a_n + (1+\alpha-n\alpha)a_{n-1} - 2a_{n-2}| + \frac{|(n-1)\alpha a_{n-1} + (2+2\alpha-n\alpha-3a_{n-3})|}{|z|} + ... + \frac{|3\alpha a_3 + (n-2-2\alpha)a_2 - (n-1)a_1|}{|z|^3} + \frac{|2\alpha a_2 + (n-1-\alpha)a_1 - na_0|}{|z|^{n-2}} + \frac{|\alpha a_1 + na_0|}{|z|^{n-1}}\}]$

$\implies |Q(z)| \geq |n\alpha a_n + a_{n-1}||z|^{n-1}[|z| - \frac{1}{|n\alpha a_n + a_{n-1}|}\{|n\alpha a_n + (1+\alpha-n\alpha)a_{n-1} - 2a_{n-2}| + |(n-1)\alpha a_{n-1} + (2+2\alpha-n\alpha-3a_{n-3})| + ... + |3\alpha a_3 + (n-2-2\alpha)a_2 - (n-1)a_1| + |2\alpha a_2 + (n-1-\alpha)a_1 - na_0| + |\alpha_1 + na_0|\}]$

$\implies |Q(z)| \geq |n\alpha a_n + a_{n-1}||z|^{n-1}[|z| - \frac{1}{|n\alpha a_n + a_{n-1}|}\{(n\alpha a_n + (1+\alpha-n\alpha)a_{n-1} - 2a_{n-2} + ((n-1)\alpha a_{n-1} + (2+2\alpha-n\alpha-3a_{n-3}) + ... + (3\alpha a_3 + (n-2-2\alpha)a_2 - (n-1)a_1) + (2\alpha a_2 + (n-1-\alpha)a_1 - na_0) + |\alpha_1 + na_0|\}]$

$\implies |Q(z)| \geq |n\alpha a_n + a_{n-1}||z|^{n-1}[|z| - \frac{1}{|n\alpha a_n + a_{n-1}|}\{n\alpha a_n + a_{n-1} - (\alpha a_1 + na_0)$

$+ |\alpha a_1 + na_0|\}]>0$

$if\ |z|>\frac{1}{|n\alpha a_n+a_{n-1}|}\{n\alpha a_n + a_{n-1} - (\alpha a_1 + na_0) + |\alpha a_1 + na_0|\}$

This shows that $|Q(z)|>0$,

provided $|z|>\frac{1}{|n\alpha a_n+a_{n-1}|}\{n\alpha a_n + a_{n-1} - (\alpha a_1 + na_0) + |\alpha a_1 + na_0|\}$.

Hence all the zeros of $Q(z)$ *with* $|z|>1$ lie in

$|z| \leq \frac{1}{|n\alpha a_n+a_{n-1}|}[n\alpha a_n + a_{n-1} - (\alpha a_1 + na_0) + |\alpha_+ na_0|]$.

But those zeros of $Q(z)$ whose modulus is less than or equal to 1 already satisfy above inequality.Since all the zeros of $D_\alpha P(z)$ are also the zeros of $Q(z)$ which lie in the circle defined by the above inequality and this completes the proof of the Theorem.

□

Corollary 4.2.2. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with positive real coefficients and* $D_\alpha P(z)$ *be a polar derivative of* $P(z)$ *with respect to a real number* α *such that*

$[i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1} \geq (i+1)\alpha a_{i+1} + (n-i)a_i$ *for* $i = 0, 1, 2, ..., n-2$ *then all the zeros of* $D_\alpha P(z)$ *lie in* $|z| \leq 1$.

Remark 4.2.3. By taking $a_i>0\ for\ i = 0, 1, 2, ..., n-1$ in the Theorem 4.2.1, then it reduces to Corollary 4.2.2.

Theorem 4.2.4. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n and* $D_\alpha P(z)$ *be the polar derivative of* $P(z)$ *with respect to a real number* α *such that*

$[i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1} \leq (i+1)\alpha a_{i+1} + (n-i)a_i$ *for* $i = 0, 1, 2, ..., n-2$ *then all the zeros of* $D_\alpha P(z)$ *lie in*

$|z| \leq \frac{1}{|n\alpha a_n+a_{n-1}|}[|\alpha a_1 + na_0| + (\alpha a_1 + na_0) - n\alpha a_n - a_{n-1}]$.

Proof. Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_2 z^2 + a_1 z + a_0$ be a polynomial of degree n, then

$D_\alpha P(z) = [n\alpha a_n + a_{n-1}]z^{n-1} + [(n-1)\alpha a_{n-1} + 2a_{n-2}]z^{n-2} + [(n-2)\alpha a_{n-2} + 3a_{n-3}]z^{n-3} + ... + [3\alpha a_3 + (n-2)a_2]z^2 + [2\alpha a_2 + (n-1)a_1]z + [\alpha a_1 + na_0]$.

Let us consider the polynomial $Q(z) = (1-z)D_\alpha P(z)$ *so that*

$Q(z) = (1-z)([n\alpha a_n + a_{n-1}]z^{n-1} + [(n-1)\alpha a_{n-1} + 2a_{n-2}]z^{n-2} + [(n-2)\alpha a_{n-2} + 3a_{n-3}]z^{n-3} + ... + [3\alpha a_3 + (n-2)a_2]z^2 + [2\alpha a_2 + (n-1)a_1]z + [\alpha a_1 + na_0])$

$\implies Q(z) = -[n\alpha a_n + a_{n-1}]z^n + \{[n\alpha a_n + (1+\alpha-n\alpha)a_{n-1} - 2a_{n-2}]z^{n-1} + [(n-1)\alpha a_{n-1} + (2+2\alpha-n\alpha)a_{n-2} - 3a_{n-3}]z^{n-2}...... + [3\alpha a_3 + (n-2-2\alpha)a_2(n-1)a_1]z^2 + [2\alpha a_2 + (n-1-\alpha)a_1 - na_0]z + [\alpha a_1 + na_0]\}$

Also if $|z|>1$ $then$ $\frac{1}{|z|^{n-i}}<1$ for $i = 0, 1, 2, .., n-2$. Now

$|Q(z)| \geq |n\alpha a_n + a_{n-1}||z|^{n-1} - \{|n\alpha a_n + (1+\alpha-n\alpha)a_{n-1} - 2a_{n-2}||z|^{n-1} + |(n-1)\alpha a_{n-1} + (2+2\alpha-n\alpha)a_{n-2} - 3a_{n-3}||z|^{n-2} + ... + |3\alpha a_3 + (n-2-2\alpha)a_2 - (n-1)a_1||z|^2 + |2\alpha a_2 + (n-1-\alpha)a_1 - na_0||z| + |\alpha a_1 + na_0|\}$

$\implies |Q(z)| \geq |n\alpha a_n + a_{n-1}||z|^{n-1}[|z| - \frac{1}{|n\alpha a_n + a_{n-1}|}\{|n\alpha a_n + (1+\alpha-n\alpha)a_{n-1} - 2a_{n-2}| + \frac{|(n-1)\alpha a_{n-1} + (2+2\alpha-n\alpha-3a_{n-3})|}{|z|} + ... + \frac{|3\alpha a_3 + (n-2-2\alpha)a_2 - (n-1)a_1|}{|z|^3} + \frac{|2\alpha a_2 + (n-1-\alpha)a_1 - na_0|}{|z|^{n-2}} + \frac{|\alpha a_1 + na_0|}{|z|^{n-1}}\}]$

$\implies |Q(z)| \geq |n\alpha a_n + a_{n-1}||z|^{n-1}[|z| - \frac{1}{|n\alpha a_n + a_{n-1}|}\{|n\alpha a_n + (1+\alpha-n\alpha)a_{n-1} - 2a_{n-2}| + |(n-1)\alpha a_{n-1} + (2+2\alpha-n\alpha-3a_{n-3})| + ... + |3\alpha a_3 + (n-2-2\alpha)a_2 - (n-1)a_1| + |2\alpha a_2 + (n-1-\alpha)a_1 - na_0| + |\alpha_1 + na_0|\}]$

$\implies |Q(z)| \geq |n\alpha a_n + a_{n-1}||z|^{n-1}[|z| - \frac{1}{|n\alpha a_n + a_{n-1}|}\{[2a_{n-2} - (1+\alpha-n\alpha)a_{n-1} - n\alpha a_n] + [3a_{n-3} - (2+2\alpha-n\alpha)a_{n-2} - (n-1)\alpha a_{n-1}] + ...[(n-1)a_1 - (n-2-2\alpha)a_2 - 3\alpha a_3] + [na_0 - (n-1-\alpha)a_1 - 2\alpha a_2] + |\alpha a_1 + na_0|]\}$

$\implies |Q(z)| \geq |n\alpha a_n + a_{n-1}||z|^{n-1}[|z| - \frac{1}{|n\alpha a_n + a_{n-1}|}\{|\alpha a_1 + na_0| + (\alpha a_1 + na_0) - n\alpha a_n - a_{n-1}\}]>0$

if $|z|>\frac{1}{|n\alpha a_n + a_{n-1}|}\{|\alpha a_1 + na_0| + (\alpha a_1 + na_0) - n\alpha a_n - a_{n-1}\}$.

This shows that $|Q(z)|>0$

provided $|z|>\frac{1}{|n\alpha a_n + a_{n-1}|}\{|\alpha a_1 + na_0| + (\alpha a_1 + na_0) - n\alpha a_n - a_{n-1}\}$.

Hence all the zeros of $Q(z)$ $with$ $|z|>1$ lie in

$|z| \leq \frac{1}{|n\alpha a_n + a_{n-1}|}[|\alpha a_1 + na_0| + (\alpha a_1 + na_0) - n\alpha a_n - a_{n-1}]$.

But those zeros of $Q(z)$ whose modulus is less than or equal to 1 already satisfy above inequality.Since all the zeros of polar derivative $D_\alpha P(z)$ are also in the zeros of $Q(z)$ lie in the circle defined by the above inequality and this completes the proof of the Theorem. □

Corollary 4.2.5. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n such that* $0<[i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1} \leq (i+1)\alpha a_{i+1} + (n-i)a_i$ *for* $i = 0, 1, 2, ..., n-2$, *then all the zeros of* $D_\alpha P(z)$ *lie in*

$|z| \leq \frac{1}{n\alpha a_n + a_{n-1}}[2(\alpha a_1 + na_0) - n\alpha_n - a_{n-1}]$.

Remark 4.2.6. By taking $a_j > 0\ for\ j = 0, 1, 2, 3, ..., n-1$ in the Theorem 4.2.4, then it reduces to Corollary 4.2.5

4.3 Zero free region for polar derivative of polynomials with special coefficients

Theorem 4.3.1. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients and let $D_\alpha P(z)$ be the polar derivative of $P(z)$ with respect to a real number α such that $n\alpha a_n + a_{n-1} \neq 0$,*

$$(i+1)\alpha a_{i+1} + (n-i)a_i \geq [i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1}\ for\ i = 0, 1, ..., m-1$$

and $[i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1} \geq\ (i+1)\alpha a_{i+1} + (n-i)a_i\ for\ i = m, m+1, ..., n-2.$

Then $D_\alpha P(z)$ does not vanish in the disk

$$|z| < \frac{1}{|na_0 + \alpha a_1|}\Bigg[(na_0 + \alpha a_1) - 2[(n-m)a_m + (m+1)\alpha a_{m+1}] + (n\alpha a_n + a_{n-1}) + |n\alpha a_n + a_{n-1}|\Bigg].$$

Proof.

Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_{m-1}z^{m-1} + a_m z^m + a_{m-1}z^{m-1} + ... + a_1 z + a_0$ *be a polynomial of degree n with real coefficients. Then the polar derivative*

$$\begin{aligned} D_\alpha P(z) = {} & [n\alpha a_n + a_{n-1}]z^{n-1} + [(n-1)\alpha a_{n-1} + 2a_{n-2}]z^{n-2} + [(n-2)\alpha a_{n-2} + 3a_{n-3}]z^{n-3} \\ & + ... + [(m+2)\alpha a_{m+2} + (n-m-1)a_{m+1}]z^{m+1} + [(m+1)\alpha a_{m+1} + (n-m)a_m]z^m \\ & + [m\alpha a_m + (n-m+1)a_{m-1}]z^{m-1} + ... + [3\alpha a_3 + (n-2)a_2]z^2 \\ & + [2\alpha a_2 + (n-1)a_1]z + [\alpha a_1 + na_0]. \end{aligned}$$

Now consider the polynomials $J(z) = z^{n-1}D_\alpha P(\frac{1}{z})$ *and* $R(z) = (z-1)J(z)$ *so that*

$$\begin{aligned} R(z) = {} & (z-1)\Bigg([n\alpha a_n + a_{n-1}] + [(n-1)\alpha a_{n-1} + 2a_{n-2}]z + [(n-2)\alpha a_{n-2} + 3a_{n-3}]z^2 + ... \\ & + [(m+2)\alpha a_{m+2} + (n-m-1)a_{m+1}]z^{n-m-2} + [(m+1)\alpha a_{m+1} + (n-m)a_m]z^{n-m-1} \\ & + [m\alpha a_m + (n-m+1)a_{m-1}]z^{n-m} + [(m-1)\alpha a_{m-1} + (n-m+2)a_{m-2}]z^{n-m+1} \\ & + ... + [3\alpha a_3 + (n-2)a_2]z^{n-3} + [2\alpha a_2 + (n-1)a_1]z^{n-2} + [\alpha a_1 + na_0]z^{n-1}\Bigg). \end{aligned}$$

$$
\begin{aligned}
=&[na_0+\alpha a_1]z^n-\Bigg(\Big[na_0+\{\alpha-(n-1)\}a_1-2\alpha a_2\Big]z^{n-1}\\
&+\Big[(n-1)a_1+\{2\alpha-(n-2)\}a_2-3\alpha a_3\Big]z^{n-2}\\
&+...+\Big[(n-m)a_m+\{(m+1)\alpha-(n-m-1)\}a_{m+1}-(m+2)\alpha a_{m+2}\Big]z^{n-m-1}\\
&+\Big[(n-m+1)a_{m-1}+\{m\alpha-(n-m)\}a_m-(m+1)\alpha a_{m+1}\Big]z^{n-m}\\
&+\Big[(n-m+2)a_{m-2}+\{(m-1)\alpha-(n-m+1)\}a_{m-1}-m\alpha a_m\Big]z^{n-m+1}\\
&+...+\Big[3a_{n-3}+\{(n-2)\alpha-2\}a_{n-2}-(n-1)\alpha a_{n-1}\Big]z^2\\
&+\Big[2a_{n-2}+\{(n-1)\alpha-1\}a_{n-1}-n\alpha a_n\Big]z+\Big[n\alpha a_n+a_{n-1}\Big]\Bigg)
\end{aligned}
$$

Also if $|z|>1$ *then* $\dfrac{1}{|z|^{n-i}}<1$ *for* $i=0,1,2,...,n-2$. *Now*

$$
\begin{aligned}
|R(z)|\geq&\Big|na_0+\alpha a_1\Big|\Big|z\Big|^n-\Bigg(\Big|na_0+\{\alpha-(n-1)\}a_1-2\alpha a_2\Big|\Big|z\Big|^{n-1}\\
&+\Big|(n-1)a_1+\{2\alpha-(n-2)\}a_2-3\alpha a_3\Big|\Big|z\Big|^{n-2}\\
&+...+\Big|(n-m)a_m+\{(m+1)\alpha-(n-m-1)\}a_{m+1}-(m+2)\alpha a_{m+2}\Big|\Big|z\Big|^{n-m-1}\\
&+\Big|(n-m+1)a_{m-1}+\{m\alpha-(n-m)\}a_m-(m+1)\alpha a_{m+1}\Big|\Big|z\Big|^{n-m}\\
&+\Big|(n-m+2)a_{m-2}+\{(m-1)\alpha-(n-m+1)\}a_{m-1}-m\alpha a_m\Big|\Big|z\Big|^{n-m+1}\\
&+...+\Big|3a_{n-3}+\{(n-2)\alpha-2\}a_{n-2}-(n-1)\alpha a_{n-1}\Big|\Big|z\Big|^2\\
&+\Big|2a_{n-2}+\{(n-1)\alpha-1\}a_{n-1}-n\alpha a_n\Big|\Big|z\Big|+\Big|n\alpha a_n+a_{n-1}\Big|\Bigg)\\
\geq&|na_0+\alpha a_1||z|^{n-1}\Bigg[|z|-\frac{1}{|na_0+\alpha a_1|}\Bigg\{|na_0+\{\alpha-(n-1)\}a_1-2\alpha a_2|\\
&+\frac{|(n-1)a_1+\{2\alpha-(n-2)\}a_2-3\alpha a_3|}{|z|}\\
&+...+\frac{|(n-m-1)a_{m+1}+\{(m+2)\alpha-(n-m-2)\}a_{m+2}-(m+3)\alpha a_{m+3}|}{|z|^{m-1}}\\
&+\frac{|(n-m)a_m+\{(m+1)\alpha-(n-m-1)\}a_{m+1}-(m+2)\alpha a_{m+2}|}{|z|^m}\\
&\frac{|(n-m+1)a_{m-1}+\{m\alpha-(n-m)\}a_m-(m+1)\alpha a_{m+1}|}{|z|^{m+1}}\\
&+...+\frac{|3a_{n-3}+\{(n-2)\alpha-2\}a_{n-2}-(n-1)\alpha a_{n-1}|}{|z|^{n-3}}\\
&+\frac{|2a_{n-2}+\{(n-1)\alpha-1\}a_{n-1}-n\alpha a_n|}{|z|^{n-2}}+\frac{|n\alpha a_n+a_{n-1}|}{|z|^{n-1}}\Bigg\}\Bigg]
\end{aligned}
$$

$$\geq |na_0 + \alpha a_1||z|^{n-1}\Bigg[|z| - \frac{1}{|na_0 + \alpha a_1|}\Bigg\{|na_0 + \{\alpha - (n-1)\}a_1 - 2\alpha a_2|$$
$$+ |(n-1)a_1 + \{2\alpha - (n-2)\}a_2 - 3\alpha a_3|$$
$$+ ... + |(n-m-1)a_{m+1} + \{(m+2)\alpha - (n-m-2)\}a_{m+2} - (m+3)\alpha a_{m+3}|$$
$$+ |(n-m)a_m + \{(m+1)\alpha - (n-m-1)\}a_{m+1} - (m+2)\alpha a_{m+2}|$$
$$+ |(n-m+1)a_{m-1} + \{m\alpha - (n-m)\}a_m - (m+1)\alpha a_{m+1}|$$
$$+ ... + |3a_{n-3} + \{(n-2)\alpha - 2\}a_{n-2} - (n-1)\alpha a_{n-1}|$$
$$+ |2a_{n-2} + \{(n-1)\alpha - 1\}a_{n-1} - n\alpha a_n| + |n\alpha a_n + a_{n-1}|\Bigg\}\Bigg]$$
$$\geq |na_0 + \alpha a_1||z|^{n-1}\Bigg[|z| - \frac{1}{|na_0 + \alpha a_1|}\Bigg\{[na_0 + \{\alpha - (n-1)\}a_1 - 2\alpha a_2]$$
$$+ [(n-1)a_1 + \{2\alpha - (n-2)\}a_2 - 3\alpha a_3]$$
$$+ [(n-m+1)a_{m-1} + \{m\alpha - (n-m)\}a_m - (m+1)\alpha a_{m+1}]$$
$$+ [(m+2)\alpha a_{m+2} - \{(m+1)\alpha - (n-m-1)\}a_{m+1} - (n-m)a_m]$$
$$+ ... + [(n-1)\alpha a_{n-1} - \{(n-2)\alpha - 2\}a_{n-2} - 3a_{n-3}]$$
$$+ [n\alpha a_n - \{(n-1)\alpha - 1\}a_{n-1} - 2a_{n-2}] + |n\alpha a_n + a_{n-1}|\Bigg\}\Bigg]$$

$$= |na_0 + \alpha a_1||z|^{n-1}\Bigg[|z| - \frac{1}{|na_0 + \alpha a_1|}\Bigg\{(na_0 + \alpha a_1) - 2[(n-m)a_m + (m+1)\alpha a_{m+1}]$$
$$+ (n\alpha a_n + a_{n-1}) + |n\alpha a_n + a_{n-1}|\Bigg\}\Bigg] > 0$$

$$if\ |z| > \frac{1}{|na_0 + \alpha a_1|}\big[(na_0 + \alpha a_1) - 2[(n-m)a_m + (m+1)\alpha a_{m+1}] + (n\alpha a_n + a_{n-1})$$
$$+ |n\alpha a_n + a_{n-1}|\big].$$

This shows that all the zeros of R(z) whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|na_0 + \alpha a_1|}\big[(na_0+\alpha a_1)-2[(n-m)a_m+(m+1)\alpha a_{m+1}]+(n\alpha a_n+a_{n-1})+|n\alpha a_n+a_{n-1}|\big].$$

But those zeros of R(z) whose modulus is less than or equal to 1 already lie in the above disk.

Therefore, it follows that all the zeros of R(z) and hence J(z) lie in

$$|z| \leq \frac{1}{|na_0+\alpha a_1|}\big[(na_0+\alpha a_1)-2[(n-m)a_m+(m+1)\alpha a_{m+1}]+(n\alpha a_n+a_{n-1})+|n\alpha a_n+a_{n-1}|\big].$$

Since $D_\alpha P(z) = z^n J(\frac{1}{z})$ it follows, by replacing z by $\frac{1}{z}$,

we get that all the zeros of $D_\alpha P(z)$ lie in

$$|z| \geq \frac{1}{|na_0+\alpha a_1|}\big[(na_0+\alpha a_1)-2[(n-m)a_m+(m+1)\alpha a_{m+1}]+(n\alpha a_n+a_{n-1})+|n\alpha a_n+a_{n-1}|\big].$$

Hence $D_\alpha P(z)$ does not vanish in the disk

$$|z| < \frac{1}{|na_0+\alpha a_1|}\big[(na_0+\alpha a_1)-2[(n-m)a_m+(m+1)\alpha a_{m+1}]+(n\alpha a_n+a_{n-1})+|n\alpha a_n+a_{n-1}|\big].$$

This completes the proof of the Theorem.

□

Corollary 4.3.2. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that $a_{n-1} \neq 0$, $(n-i)a_i \geq [n-(i+1)]a_{i+1}$ for $i = 0, 1, ..., m-1$ and $[n-(i+1)]a_{i+1} \geq (n-i)a_i$ for $i = m, m+1, ..., n-2$. Then $D_\alpha P(z)$ does not vanish in the disk*

$$|z| < \frac{1}{|na_0|}\Bigg[na_0 - 2(n-m)a_m + a_{n-1} + |a_{n-1}|\Bigg].$$

Corollary 4.3.3. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with positive real coefficients and $D_\alpha P(z)$ be the polar derivative of $P(z)$ with respect to a real number α such that $n\alpha a_n + a_{n-1} \neq 0$,*

$$(i+1)\alpha a_{i+1} + (n-i)a_i \geq [i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1} \ for \ i = 0, 1, ..., m-1.$$

and $[i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1} \geq (i+1)\alpha a_{i+1} + (n-i)a_i$ for $i = m, m+1, ..., n-2$. Then $D_\alpha P(z)$ does not vanish in the disk

$$|z| < \frac{1}{na_0+\alpha a_1}\Bigg[(na_0+\alpha a_1) - 2[(n-m)a_m + (m+1)\alpha a_{m+1}] + 2(n\alpha a_n + a_{n-1})\Bigg].$$

Remark 4.3.4. By taking $\alpha = 0$ in Theorem 4.3.1, it reduces to Corollary 4.3.2

Remark 4.3.5. By taking $(i+1)\alpha a_{i+1}+(n-i)a_i>0$ for $i = 0, 1, 2, ..., n-2$ in Theorem 4.3.1, it reduces to Corollary 4.3.3

Theorem 4.3.6. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients and let $D_\alpha P(z)$ be the polar derivative of $P(z)$ with respect to a real number α such that $n\alpha a_n + a_{n-1} \neq 0$,*

$$(i+1)\alpha a_{i+1} + (n-i)a_i \leq [i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1} \ for \ i = 0, 1, ..., m-1$$

and $[i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1} \leq (i+1)\alpha a_{i+1} + (n-i)a_i \ for \ i = m, m+1, ..., n-2.$

Then $D_\alpha P(z)$ does not vanish in the disk

$$|z| < \frac{1}{|na_0 + \alpha a_1|}\Bigg[|n\alpha a_n + a_{n-1}| + 2[(n-m)a_m + (m+1)\alpha a_{m+1}] - (na_0 + \alpha a_1 + n\alpha a_n + a_{n-1})\Bigg].$$

Proof of the Theorem is similar to proof of Theorem 4.3.1

Corollary 4.3.7. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that $a_{n-1} \neq 0$, $(n-i)a_i \leq [n-(i+1)]a_{i+1}$ for $i = 0, 1, ..., m-1$ and $[n-(i+1)]a_{i+1} \leq (n-i)a_i$ for $i = m, m+1, ..., n-2$. Then $D_\alpha P(z)$ does not vanish in the disk*

$$|z| < \frac{1}{|na_0|}\Bigg[|a_{n-1}| + 2(n-m)a_m - (na_0 + a_{n-1})\Bigg].$$

Corollary 4.3.8. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with positive real coefficients and $D_\alpha P(z)$ be the polar derivative of $P(z)$ with respect to a real number α such that $n\alpha a_n + a_{n-1} \neq 0$,*

$$(i+1)\alpha a_{i+1} + (n-i)a_i \leq [i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1} \ for \ i = 0, 1, ..., m-1$$

and $[i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1} \leq (i+1)\alpha a_{i+1} + (n-i)a_i \ for \ i = m, m+1, ..., n-2.$

Then $D_\alpha P(z)$ does not vanish in the disk

$$|z| < \frac{1}{na_0 + \alpha a_1}\Bigg[2[(n-m)a_m + (m+1)\alpha a_{m+1}] - (na_0 + \alpha a_1)\Bigg].$$

Remark 4.3.9. By taking $\alpha = 0$ in Theorem 4.3.6, it reduces to Corollary 4.3.7.

Remark 4.3.10. By taking $(i+1)\alpha a_{i+1} + (n-i)a_i > 0$ for $i = 0, 1, 2, ..., n-2$, in Theorem 4.3.6, it reduces to Corollary 4.3.8.

4.4 Zero free region for polar derivative of polynomials

Theorem 4.4.1. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with real coefficients and let* $D_\alpha P(z)$ *be the polar derivative of* $P(z)$ *with respect to a real number* α *such that* $n\alpha a_n + a_{n-1} \neq 0$,

$$and\ (i+1)\alpha a_{i+1} + (n-i)a_i \geq [i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1}\ for\ i = 0, 1, ..., n-2.$$

Then $D_\alpha P(z)$ *does not vanish in the disk*

$$|z| < \frac{|na_0 + \alpha a_1|}{(na_0 + \alpha a_1) - (n\alpha a_n + a_{n-1}) + |n\alpha a_n + a_{n-1}|}.$$

Proof.

Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$ *be a polynomial of degree n with real coefficients. Then the polar derivative*

$$\begin{aligned} D_\alpha P(z) =&[n\alpha a_n + a_{n-1}]z^{n-1} + [(n-1)\alpha a_{n-1} + 2a_{n-2}]z^{n-2} + [(n-2)\alpha a_{n-2} + 3a_{n-3}]z^{n-3} \\ &+ ... + [3\alpha a_3 + (n-2)a_2]z^2 + [2\alpha a_2 + (n-1)a_1]z + [\alpha a_1 + na_0]. \end{aligned}$$

Now consider the polynomials $J(z) = z^{n-1} D_\alpha P(\frac{1}{z})$ *and* $R(z) = (z-1)J(z)$ *so that*

$$\begin{aligned} R(z) =&(z-1)\Big([n\alpha a_n + a_{n-1}] + [(n-1)\alpha a_{n-1} + 2a_{n-2}]z + [(n-2)\alpha a_{n-2} + 3a_{n-3}]z^2 \\ &+ ... + [3\alpha a_3 + (n-2)a_2]z^{n-3} + [2\alpha a_2 + (n-1)a_1]z^{n-2} + [\alpha a_1 + na_0]z^{n-1}\Big). \\ =&[na_0 + \alpha a_1]z^n - \Big([na_0 + \{\alpha - (n-1)\}a_1 - 2\alpha a_2]z^{n-1} \\ &+ [(n-1)a_1 + \{2\alpha - (n-2)\}a_2 - 3\alpha a_3]z^{n-2} \\ &+ ... + [3a_{n-3} + \{(n-2)\alpha - 2\}a_{n-2} - (n-1)\alpha a_{n-1}]z^2 \\ &+ [2a_{n-2} + \{(n-1)\alpha - 1\}a_{n-1} - n\alpha a_n]z + [n\alpha a_n + a_{n-1}]\Big) \end{aligned}$$

Also if $|z| > 1$ *then* $\frac{1}{|z|^{n-i}} < 1$ *for* $i = 0, 1, 2, ..., n-2$. *Now*

$$\begin{aligned} |R(z)| \geq&|na_0 + \alpha a_1||z|^n - \Big(|na_0 + \{\alpha - (n-1)\}a_1 - 2\alpha a_2||z|^{n-1} \\ &+ |(n-1)a_1 + \{2\alpha - (n-2)\}a_2 - 3\alpha a_3||z|^{n-2} \end{aligned}$$

$$+ ... + |3a_{n-3} + \{(n-2)\alpha - 2\}a_{n-2} - (n-1)\alpha a_{n-1}||z|^2$$

$$+ |2a_{n-2} + \{(n-1)\alpha - 1\}a_{n-1} - n\alpha a_n||z| + |n\alpha a_n + a_{n-1}|\Big)$$

$$\geq |na_0 + \alpha a_1||z|^{n-1}\Bigg[|z| - \frac{1}{|na_0 + \alpha a_1|}\Bigg\{|na_0 + \{\alpha - (n-1)\}a_1 - 2\alpha a_2|$$

$$+ \frac{|(n-1)a_1 + \{2\alpha - (n-2)\}a_2 - 3\alpha a_3|}{|z|} + ...$$

$$+ \frac{|3a_{n-3} + \{(n-2)\alpha - 2\}a_{n-2} - (n-1)\alpha a_{n-1}|}{|z|^{n-3}}$$

$$+ \frac{|2a_{n-2} + \{(n-1)\alpha - 1\}a_{n-1} - n\alpha a_n|}{|z|^{n-2}} + \frac{|n\alpha a_n + a_{n-1}|}{|z|^{n-1}}\Bigg\}\Bigg]$$

$$\geq |na_0 + \alpha a_1||z|^{n-1}\Bigg[|z| - \frac{1}{|na_0 + \alpha a_1|}\Bigg\{|na_0 + \{\alpha - (n-1)\}a_1 - 2\alpha a_2|$$

$$+ |(n-1)a_1 + \{2\alpha - (n-2)\}a_2 - 3\alpha a_3| + ...$$

$$+ |3a_{n-3} + \{(n-2)\alpha - 2\}a_{n-2} - (n-1)\alpha a_{n-1}|$$

$$+ |2a_{n-2} + \{(n-1)\alpha - 1\}a_{n-1} - n\alpha a_n| + |n\alpha a_n + a_{n-1}|\Bigg\}\Bigg]$$

$$\geq |na_0 + \alpha a_1||z|^{n-1}\Bigg[|z| - \frac{1}{|na_0 + \alpha a_1|}\Bigg\{[na_0 + \{\alpha - (n-1)\}a_1 - 2\alpha a_2]$$

$$+ [(n-1)a_1 + \{2\alpha - (n-2)\}a_2 - 3\alpha a_3] + ...$$

$$+ [3a_{n-3} + \{(n-2)\alpha - 2\}a_{n-2} - (n-1)\alpha a_{n-1}]$$

$$+ [2a_{n-2} + \{(n-1)\alpha - 1\}a_{n-1} - n\alpha a_n] + |n\alpha a_n + a_{n-1}|\Bigg\}\Bigg]$$

$$= |na_0 + \alpha a_1||z|^{n-1}\Bigg[|z| - \frac{1}{|na_0 + \alpha a_1|}\Bigg\{(na_0 + \alpha a_1) - (n\alpha a_n + a_{n-1}) + |n\alpha a_n + a_{n-1}|\Bigg\}\Bigg]$$

$$> 0 \ if \ |z| > \frac{1}{|na_0 + \alpha a_1|}\big[(na_0 + \alpha a_1) - (n\alpha a_n + a_{n-1}) + |n\alpha a_n + a_{n-1}|\big].$$

This shows that all the zeros of R(z) whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|na_0 + \alpha a_1|}\big[(na_0 + \alpha a_1) - (n\alpha a_n + a_{n-1}) + |n\alpha a_n + a_{n-1}|\big].$$

But those zeros of R(z) whose modulus is less than or equal to 1 already lie in the above disk.

Therefore, it follows that all the zeros of R(z) and hence J(z) lie in

$$|z| \leq \frac{1}{|na_0 + \alpha a_1|}\big[(na_0 + \alpha a_1) - (n\alpha a_n + a_{n-1}) + |n\alpha a_n + a_{n-1}|\big].$$

Since $D_\alpha P(z) = z^n J(\frac{1}{z})$ it follows, by replacing z by $\frac{1}{z}$,

we get that all the zeros of $D_\alpha P(z)$ lie in

$$|z| \geq \frac{|na_0 + \alpha a_1|}{(na_0 + \alpha a_1) - (n\alpha a_n + a_{n-1}) + |n\alpha a_n + a_{n-1}|}.$$

Hence $D_\alpha P(z)$ does not vanish in the disk

$$|z| < \frac{|na_0 + \alpha a_1|}{(na_0 + \alpha a_1) - (n\alpha a_n + a_{n-1}) + |n\alpha a_n + a_{n-1}|}.$$

This completes the proof of the Theorem .

□

Corollary 4.4.2. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that $a_{n-1} \neq 0$, and $(n-i)a_i \geq [n-(i+1)]a_{i+1}$ for $i = 0, 1, ..., n-2$. Then $D_0 P(z)$ does not vanish in the disk*

$$|z| < \frac{|na_0|}{na_0 - a_{n-1} + |a_{n-1}|}$$

Corollary 4.4.3. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that*

$n\alpha a_n + a_{n-1} \neq 0,$

and $(i+1)\alpha a_{i+1} + (n-i)a_i \geq [i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1} > 0 \ for \ i = 0, 1, ..., n-2.$

Then $D_\alpha P(z)$ does not vanish in the disk $|z| < 1$.

Remark 4.4.4. By taking $\alpha = 0$ in Theorem 4.4.1, it reduces to Corollary 4.4.2.

Remark 4.4.5. By taking $(i+1)\alpha a_{i+1} + (n-i)a_i > 0$ for $i = 0, 1, 2, ..., n-2$, in Theorem 4.4.1, it reduces to Corollary 4.4.3.

Theorem 4.4.6. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that*

$n\alpha a_n + a_{n-1} \neq 0,$

and $(i+1)\alpha a_{i+1} + (n-i)a_i \leq [i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1} \ for \ i = 0, 1, ..., n-2.$

Then $D_\alpha P(z)$ does not vanish in the disk

$$|z| < \frac{|na_0 + \alpha a_1|}{|n\alpha a_n + a_{n-1}| + (n\alpha a_n + a_{n-1}) - (na_0 + \alpha a_1)}.$$

Proof.

Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$ *be a polynomial of degree n with real coefficients. Then the polar derivative*

$$D_\alpha P(z) = [n\alpha a_n + a_{n-1}]z^{n-1} + [(n-1)\alpha a_{n-1} + 2a_{n-2}]z^{n-2} + [(n-2)\alpha a_{n-2} + 3a_{n-3}]z^{n-3} + ... + [3\alpha a_3 + (n-2)a_2]z^2 + [2\alpha a_2 + (n-1)a_1]z + [\alpha a_1 + na_0].$$

Now consider the polynomials $J(z) = z^{n-1}D_\alpha P(\frac{1}{z})$ *and* $R(z) = (z-1)J(z)$ *so that*

$$\begin{aligned}R(z) =&(z-1)\big([n\alpha a_n + a_{n-1}] + [(n-1)\alpha a_{n-1} + 2a_{n-2}]z + [(n-2)\alpha a_{n-2} + 3a_{n-3}]z^2 \\ &+ ... + [3\alpha a_3 + (n-2)a_2]z^{n-3} + [2\alpha a_2 + (n-1)a_1]z^{n-2} + [\alpha a_1 + na_0]z^{n-1}\big). \\ =&[na_0 + \alpha a_1]z^n - \Bigg([na_0 + \{\alpha - (n-1)\}a_1 - 2\alpha a_2]z^{n-1} \\ &+ [(n-1)a_1 + \{2\alpha - (n-2)\}a_2 - 3\alpha a_3]z^{n-2} \\ &+ ... + [3a_{n-3} + \{(n-2)\alpha - 2\}a_{n-2} - (n-1)\alpha a_{n-1}]z^2 \\ &+ [2a_{n-2} + \{(n-1)\alpha - 1\}a_{n-1} - n\alpha a_n]z + [n\alpha a_n + a_{n-1}]\Bigg)\end{aligned}$$

Also if $|z| > 1$ *then* $\frac{1}{|z|^{n-i}} < 1$ *for* $i = 0, 1, 2, ..., n-2$. *Now*

$$\begin{aligned}|R(z)| \geq&|na_0 + \alpha a_1||z|^n - \Bigg(|na_0 + \{\alpha - (n-1)\}a_1 - 2\alpha a_2||z|^{n-1} \\ &+ |(n-1)a_1 + \{2\alpha - (n-2)\}a_2 - 3\alpha a_3||z|^{n-2} \\ &+ ... + |3a_{n-3} + \{(n-2)\alpha - 2\}a_{n-2} - (n-1)\alpha a_{n-1}||z|^2 \\ &+ |2a_{n-2} + \{(n-1)\alpha - 1\}a_{n-1} - n\alpha a_n||z| + |n\alpha a_n + a_{n-1}|\Bigg) \\ \geq&|na_0 + \alpha a_1||z|^{n-1}\Bigg[|z| - \frac{1}{|na_0 + \alpha a_1|}\Bigg\{|na_0 + \{\alpha - (n-1)\}a_1 - 2\alpha a_2| \\ &+ \frac{|(n-1)a_1 + \{2\alpha - (n-2)\}a_2 - 3\alpha a_3|}{|z|} + ... \\ &+ \frac{|3a_{n-3} + \{(n-2)\alpha - 2\}a_{n-2} - (n-1)\alpha a_{n-1}|}{|z|^{n-3}} \\ &+ \frac{|2a_{n-2} + \{(n-1)\alpha - 1\}a_{n-1} - n\alpha a_n|}{|z|^{n-2}} + \frac{|n\alpha a_n + a_{n-1}|}{|z|^{n-1}}\Bigg\}\Bigg]\end{aligned}$$

$$\begin{aligned}
\geq & |na_0+\alpha a_1||z|^{n-1}\Bigg[|z|-\frac{1}{|na_0+\alpha a_1|}\Bigg\{|na_0+\{\alpha-(n-1)\}a_1-2\alpha a_2| \\
& +|(n-1)a_1+\{2\alpha-(n-2)\}a_2-3\alpha a_3|+... \\
& +|3a_{n-3}+\{(n-2)\alpha-2\}a_{n-2}-(n-1)\alpha a_{n-1}| \\
& +|2a_{n-2}+\{(n-1)\alpha-1\}a_{n-1}-n\alpha a_n|+|n\alpha a_n+a_{n-1}|\Bigg\}\Bigg] \\
\geq & |na_0+\alpha a_1||z|^{n-1}\Bigg[|z|-\frac{1}{|na_0+\alpha a_1|}\Bigg\{[2\alpha a_2-\{\alpha-(n-1)\}a_1-na_0] \\
& +[3\alpha a_3+\{2\alpha-(n-2)\}a_2-(n-1)a_1]+... \\
& +[(n-1)\alpha a_{n-1}-\{(n-2)\alpha-2\}a_{n-2}-3a_{n-3}] \\
& +[n\alpha a_n-\{(n-1)\alpha-1\}a_{n-1}-2a_{n-2}]+|n\alpha a_n+a_{n-1}|\Bigg\}\Bigg] \\
= & |na_0+\alpha a_1||z|^{n-1}\Bigg[|z|-\frac{1}{|na_0+\alpha a_1|}\Bigg\{(na_0+\alpha a_1)-(n\alpha a_n+a_{n-1})+|n\alpha a_n+a_{n-1}|\Bigg\}\Bigg] \\
> & 0 \ if \ |z|>\frac{1}{|na_0+\alpha a_1|}\big[|n\alpha a_n+a_{n-1}|+(n\alpha a_n+a_{n-1})-(na_0+\alpha a_1)\big].
\end{aligned}$$

This shows that all the zeros of R(z) whose modulus is greater than 1 lie in the closed disk

$$|z|\leq\frac{1}{|na_0+\alpha a_1|}\big[|n\alpha a_n+a_{n-1}|+(n\alpha a_n+a_{n-1})-(na_0+\alpha a_1)\big].$$

But those zeros of R(z) whose modulus is less than or equal to 1 already lie in the above disk.

Therefore, it follows that all the zeros of R(z) and hence J(z) lie in

$$|z|\leq\frac{1}{|na_0+\alpha a_1|}\big[|n\alpha a_n+a_{n-1}|+(n\alpha a_n+a_{n-1})-(na_0+\alpha a_1)\big].$$

Since $D_\alpha P(z)=z^nJ(\frac{1}{z})$ it follows, by replacing z by $\frac{1}{z}$,

we get that all the zeros of $D_\alpha P(z)$ lie in

$$|z|\geq\frac{|na_0+\alpha a_1|}{|n\alpha a_n+a_{n-1}|+(n\alpha a_n+a_{n-1})-(na_0+\alpha a_1)}.$$

Hence $D_\alpha P(z)$ does not vanish in the disk

$$|z|<\frac{|na_0+\alpha a_1|}{|n\alpha a_n+a_{n-1}|+(n\alpha a_n+a_{n-1})-(na_0+\alpha a_1)}.$$

This completes the proof of the Theorem. □

Corollary 4.4.7. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that $a_{n-1} \neq 0$, and $(n-i)a_i \leq [n-(i+1)]a_{i+1}$ for $i = 0, 1, ..., n-2$. Then $D_0P(z)$ does not vanish in the disk*

$$|z| < \frac{|na_0|}{|a_{n-1}| + a_{n-1} - na_0}.$$

Corollary 4.4.8. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that $n\alpha a_n + a_{n-1} \neq 0$,*

and $0 < (i+1)\alpha a_{i+1} + (n-i)a_i \leq [i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1}$ for $i = 0, 1, ..., n-2$.

Then $D_\alpha P(z)$ does not vanish in the disk

$$|z| < \frac{na_0 + \alpha a_1}{2(n\alpha a_n + a_{n-1}) - (na_0 + \alpha a_1)}.$$

Remark 4.4.9. By taking $\alpha = 0$ in Theorem 4.4.6, it reduces to Corollary 4.4.7.

Remark 4.4.10. By taking $(i+1)\alpha a_{i+1} + (n-i)a_i > 0$ for $i = 0, 1, 2, ..., n-2$, in Theorem 4.4.6, it reduces to Corollary 4.4.8.

4.5 Zero free regions for polar derivative of polynomials with restricted coefficients

Theorem 4.5.1. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq 3$ with real coefficients and let $D_\alpha P(z)$ be the polar derivative of $P(z)$ with respect to a real number α such that $n\alpha a_n + a_{n-1} \neq 0$,*

$$n\alpha a_n + a_{n-1} \geq (n-1)\alpha a_{n-1} + 2a_{n-2} \leq (n-2)\alpha a_{n-2} + 3a_{n-3} \geq ... \leq 4\alpha a_4 + (n-3)a_3$$
$$\geq 3\alpha a_3 + (n-2)a_2 \leq 2\alpha a_2 + (n-1)a_1 \geq \alpha a_1 + na_0 \; if \; n \; is \; even, \; (OR)$$
$$n\alpha a_n + a_{n-1} \geq (n-1)\alpha a_{n-1} + 2a_{n-2} \leq (n-2)\alpha a_{n-2} + 3a_{n-3} \geq ... \geq 4\alpha a_4 + (n-3)a_3$$
$$\leq 3\alpha a_3 + (n-2)a_2 \geq 2\alpha a_2 + (n-1)a_1 \leq \alpha a_1 + na_0 \; if \; n \; is \; odd, \; then$$

(i) $D_\alpha P(z)$ does not vanish in the disk

$$|z| < \frac{1}{na_0 + \alpha a_1}\Big[|n\alpha a_n + a_{n-1}| + (n\alpha a_n + a_{n-1}) + X_1 - (na_0 + \alpha a_1)\Big] \; if \; n \; is \; even,$$

where $X_1 = 2\Big[\big\{[(n-2)\alpha a_{n-2} + 3a_{n-3}] + ... + [4\alpha a_4 + (n-3)a_3] + [2\alpha a_2 + (n-1)a_1]\big\}$

$$-\Big\{[(n-1)\alpha a_{n-1}+2a_{n-2}]+[(n-3)\alpha a_{n-3}+4a_{n-4}]+...+[3\alpha a_3+(n-2)a_2]\Big\}\Bigg].$$

(ii) $D_\alpha P(z)$ *does not vanish in the disk*

$$|z|<\frac{1}{na_0+\alpha a_1}\Big[|n\alpha a_n+a_{n-1}|+X_2+(na_0+\alpha a_1)\Big] \ if \ n \ is \ odd,$$

$$where \ X_2 = 2\Bigg[\Big\{[(n-2)\alpha a_{n-2}+3a_{n-3}]+...+[5\alpha a_5+(n-4)a_4]+[3\alpha a_3+(n-2)a_2]\Big\}$$
$$-\Big\{[(n-1)\alpha a_{n-1}+2a_{n-2}]+[(n-3)\alpha a_{n-3}+4a_{n-4}]+...+[4\alpha a_4+(n-3)a_3]$$
$$+[2\alpha a_2+(n-1)a_1]\Big\}\Bigg].$$

Proof.

$$Let \ P(z)=a_nz^n+a_{n-1}z^{n-1}+...+a_{m-1}z^{m-1}+a_mz^m+a_{m-1}z^{m-1}+...+a_1z+a_0$$

be a polynomial of degree n with real coefficients. Then the polar derivative

$$D_\alpha P(z)=[n\alpha a_n+a_{n-1}]z^{n-1}+[(n-1)\alpha a_{n-1}+2a_{n-2}]z^{n-2}+[(n-2)\alpha a_{n-2}+3a_{n-3}]z^{n-3}$$
$$+...+[4\alpha a_4+(n-3)a_3]z^3+[3\alpha a_3+(n-2)a_2]z^2+[2\alpha a_2+(n-1)a_1]z$$
$$+[\alpha a_1+na_0].$$

Now consider the polynomials $J(z)=z^{n-1}D_\alpha P(\frac{1}{z})$ *and* $R(z)=(z-1)J(z)$ *so that*

$$R(z)=(z-1)\Bigg([n\alpha a_n+a_{n-1}]+[(n-1)\alpha a_{n-1}+2a_{n-2}]z+[(n-2)\alpha a_{n-2}+3a_{n-3}]z^2$$
$$+...+[3\alpha a_3+(n-2)a_2]z^{n-3}+[2\alpha a_2+(n-1)a_1]z^{n-2}+[\alpha a_1+na_0]z^{n-1}\Bigg).$$

$$R(z)=[na_0+\alpha a_1]z^n-\Bigg(\Big[na_0+\{\alpha-(n-1)\}a_1-2\alpha a_2\Big]z^{n-1}$$
$$+\Big[(n-1)a_1+\{2\alpha-(n-2)\}a_2-3\alpha a_3\Big]z^{n-2}$$
$$+...+\Big[4a_{n-4}+\{(n-3)\alpha-3\}a_{n-3}-(n-2)\alpha a_{n-2}\Big]z^3$$
$$+\Big[3a_{n-3}+\{(n-2)\alpha-2\}a_{n-2}-(n-1)\alpha a_{n-1}\Big]z^2$$
$$+\Big[2a_{n-2}+\{(n-1)\alpha-1\}a_{n-1}-n\alpha a_n\Big]z+\Big[n\alpha a_n+a_{n-1}\Big]\Bigg)$$

Also if $|z|>1$ *then* $\frac{1}{|z|^{n-i}}<1$ *for* $i=0,1,2,...,n-2.$

$$Now \ |R(z)|\geq\Big|na_0+\alpha a_1\Big|\Big|z\Big|^n-\Bigg(\Big|na_0+\{\alpha-(n-1)\}a_1-2\alpha a_2\Big|\Big|z\Big|^{n-1}$$

$$+\left|(n-1)a_1+\{2\alpha-(n-2)\}a_2-3\alpha a_3\right|\left|z\right|^{n-2}+\left|(n-2)a_2+\{3\alpha-(n-3)\}a_3-4\alpha a_4\right|\left|z\right|^{n-3}$$
$$+\ldots+\left|4a_{n-4}+\{(n-3)\alpha-3\}a_{n-3}-(n-2)\alpha a_{n-2}\right|\left|z\right|^3$$
$$+\left|3a_{n-3}+\{(n-2)\alpha-2\}a_{n-2}-(n-1)\alpha a_{n-1}\right|\left|z\right|^2$$
$$\left.+\left|2a_{n-2}+\{(n-1)\alpha-1\}a_{n-1}-n\alpha a_n\right|\left|z\right|+\left|n\alpha a_n+a_{n-1}\right|\right)$$
$$\geq|na_0+\alpha a_1||z|^{n-1}\left[|z|-\frac{1}{|na_0+\alpha a_1|}\left\{|na_0+\{\alpha-(n-1)\}a_1-2\alpha a_2|\right.\right.$$
$$+\frac{|(n-1)a_1+\{2\alpha-(n-2)\}a_2-3\alpha a_3|}{|z|}+\frac{|(n-2)a_2+\{3\alpha-(n-3)\}a_3-4\alpha a_4|}{|z|^2}$$
$$+\ldots+\frac{|4a_{n-4}+\{(n-3)\alpha-3\}a_{n-3}-(n-2)\alpha a_{n-2}|}{|z|^{n-4}}$$
$$+\frac{|3a_{n-3}+\{(n-2)\alpha-2\}a_{n-2}-(n-1)\alpha a_{n-1}|}{|z|^{n-3}}$$
$$\left.\left.+\frac{|2a_{n-2}+\{(n-1)\alpha-1\}a_{n-1}-n\alpha a_n|}{|z|^{n-2}}+\frac{|n\alpha a_n+a_{n-1}|}{|z|^{n-1}}\right\}\right]$$
$$\geq|na_0+\alpha a_1||z|^{n-1}\left[|z|-\frac{1}{|na_0+\alpha a_1|}\left\{|na_0+\{\alpha-(n-1)\}a_1-2\alpha a_2|\right.\right.$$
$$+|(n-1)a_1+\{2\alpha-(n-2)\}a_2-3\alpha a_3|+|(n-2)a_2+\{3\alpha-(n-3)\}a_3-4\alpha a_4|$$
$$+\ldots+|4a_{n-4}+\{(n-3)\alpha-3\}a_{n-3}-(n-2)\alpha a_{n-2}|$$
$$+|3a_{n-3}+\{(n-2)\alpha-2\}a_{n-2}-(n-1)\alpha a_{n-1}|$$
$$\left.\left.+|2a_{n-2}+\{(n-1)\alpha-1\}a_{n-1}-n\alpha a_n|+|n\alpha a_n+a_{n-1}|\right\}\right]$$
$$\geq|na_0+\alpha a_1||z|^{n-1}\left[|z|-\frac{1}{|na_0+\alpha a_1|}\left\{[2\alpha a_2+\{(n-1)-\alpha\}a_1-na_0]\right.\right.$$
$$+[(n-1)a_1+\{2\alpha-(n-2)\}a_2-3\alpha a_3]+[4\alpha a_4+\{(n-3)-3\alpha\}a_3-(n-2)a_2]$$
$$+\ldots+[(n-2)\alpha a_{n-2}+\{3-(n-3)\alpha\}a_{n-3}-4a_{n-4}]$$
$$+[3a_{n-3}+\{(n-2)\alpha-2\}a_{n-2}-(n-1)\alpha a_{n-1}]$$
$$\left.\left.+[n\alpha a_n+\{1-(n-1)\alpha\}a_{n-1}-2a_{n-2}]+|n\alpha a_n+a_{n-1}|\right\}\right]\ if\ n\ is\ even,$$
$$=|na_0+\alpha a_1||z|^{n-1}\left[|z|-\frac{1}{|na_0+\alpha a_1|}\left\{|n\alpha a_n+a_{n-1}|+X_1-(na_0+\alpha a_1)\right\}\right]>0$$
$$if\ |z|>\frac{1}{|na_0+\alpha a_1|}\left[|n\alpha a_n+a_{n-1}|+(n\alpha a_n+a_{n-1})+X_1-(na_0+\alpha a_1)\right]$$
$$where\ X_1=2\Big[\big\{[(n-2)\alpha a_{n-2}+3a_{n-3}]+\ldots+[4\alpha a_4+(n-3)a_3]+[2\alpha a_2+(n-1)a_1]\big\}$$
$$-\big\{[(n-1)\alpha a_{n-1}+2a_{n-2}]+[(n-3)\alpha a_{n-3}+4a_{n-4}]+\ldots+[3\alpha a_3+(n-2)a_2]\big\}\Big]$$

if n is even.

This shows that all the zeros of R(z) whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{na_0 + \alpha a_1}\Big[|n\alpha a_n + a_{n-1}| + X_1 - (na_0 + \alpha a_1)\Big].$$

But those zeros of R(z) whose modulus is less than or equal to 1 already lie in the above disk.

Therefore, it follows that all the zeros of R(z) and hence J(z) lie in

$$|z| \leq \frac{1}{na_0 + \alpha a_1}\Big[|n\alpha a_n + a_{n-1}| + X_1 - (na_0 + \alpha a_1)\Big].$$

Since $D_\alpha P(z) = z^n J(\frac{1}{z})$ it follows, by replacing z by $\frac{1}{z}$,

we get that all the zeros of $D_\alpha P(z)$ lie in

$$|z| \geq \frac{1}{na_0 + \alpha a_1}\Big[|n\alpha a_n + a_{n-1}| + X_1 - (na_0 + \alpha a_1)\Big].$$

Hence $D_\alpha P(z)$ does not vanish in the disk

$$|z| < \frac{1}{na_0 + \alpha a_1}\Big[|n\alpha a_n + a_{n-1}| + (n\alpha a_n + a_{n-1}) + X_1 - (na_0 + \alpha a_1)\Big],$$

$$where\ X_1 = 2\Big[\big\{[(n-2)\alpha a_{n-2} + 3a_{n-3}] + ... + [4\alpha a_4 + (n-3)a_3] + [2\alpha a_2 + (n-1)a_1]\big\} - \big\{[(n-1)\alpha a_{n-1} + 2a_{n-2}] + [(n-3)\alpha a_{n-3} + 4a_{n-4}] + ... + [3\alpha a_3 + (n-2)a_2]\big\}\Big]$$

if n is even.

Similarly we can also prove for odd degree polynomials. For this we can rearrange the terms of the given polynomial and compute as above. That is if n is odd then $D_\alpha P(z)$ does not vanish in the disk

$$|z| < \frac{1}{na_0 + \alpha a_1}\Big[|n\alpha a_n + a_{n-1}| + (n\alpha a_n + a_{n-1}) + X_2 + (na_0 + \alpha a_1)\Big],$$

$$where\ X_2 = 2\Big[\big\{[(n-2)\alpha a_{n-2} + 3a_{n-3}] + ... + [5\alpha a_5 + (n-4)a_4] + [3\alpha a_3 + (n-2)a_2]\big\} - \big\{[(n-1)\alpha a_{n-1} + 2a_{n-2}] + [(n-3)\alpha a_{n-3} + 4a_{n-4}] + ... + [2\alpha a_2 + (n-1)a_1]\big\}\Big]$$

if n is odd.

This completes the proof of the Theorem . □

Corollary 4.5.2. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq 3$ *with real coefficients and let* $D_0P(z)$ *be the polar derivative of* $P(z)$ *such that* $a_{n-1} \neq 0$,

$$a_{n-1} \geq 2a_{n-2} \leq 3a_{n-3} \geq ... \leq (n-3)a_3 \geq (n-2)a_2 \leq (n-1)a_1 \geq na_0 \text{ if } n \text{ is even,}$$

(OR)

$$a_{n-1} \geq 2a_{n-2} \leq 3a_{n-3} \geq ... \geq (n-3)a_3 \leq (n-2)a_2 \geq (n-1)a_1 \leq na_0 \text{ if } n \text{ is odd.}$$

then (i) $D_0P(z)$ *does not vanish in the disk*

$$|z| < \frac{1}{na_0}\big[|a_{n-1}| + a_{n-1} + X_3 - na_0\big] \text{ if } n \text{ is even,}$$

where $X_3 = 2\Bigg[\big\{3a_{n-3} + ... + (n-3)a_3 + (n-1)a_1\big\} - \big\{2a_{n-2} + 4a_{n-4} + ... + (n-2)a_2\big\}\Bigg].$

(ii) $D_0P(z)$ *does not vanish in the disk*

$$|z| < \frac{1}{na_0}\big[|a_{n-1}| + a_{n-1} + X_4 + na_0\big] \text{ if } n \text{ is odd,}$$

$$\text{where } X_4 = 2\Bigg[\big\{3a_{n-3} + ... + (n-4)a_4 + (n-2)a_2\big\} - \big\{2a_{n-2} + 4a_{n-4} + ... + (n-3)a_3 + (n-1)a_1\big\}\Bigg].$$

Remark 4.5.3. By taking $\alpha = 0$ in Theorem 4.5.1, it reduces to Corollary 4.5.2.

Theorem 4.5.4. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq 3$ *with real coefficients and let* $D_\alpha P(z)$ *be the polar derivative of* $P(z)$ *with respect to a real number* α *such that* $n\alpha a_n + a_{n-1} \neq 0$,

$$n\alpha a_n + a_{n-1} \leq (n-1)\alpha a_{n-1} + 2a_{n-2} \geq (n-2)\alpha a_{n-2} + 3a_{n-3} \leq ... \geq 4\alpha a_4 + (n-3)a_3 \leq 3\alpha a_3 + (n-2)a_2 \geq 2\alpha a_2 + (n-1)a_1 \leq \alpha a_1 + na_0 \text{ if } n \text{ is even,}$$

(OR)

$$n\alpha a_n + a_{n-1} \leq (n-1)\alpha a_{n-1} + 2a_{n-2} \geq (n-2)\alpha a_{n-2} + 3a_{n-3} \leq ... \leq 4\alpha a_4 + (n-3)a_3 \geq 3\alpha a_3 + (n-2)a_2 \leq 2\alpha a_2 + (n-1)a_1 \geq \alpha a_1 + na_0 \text{ if } n \text{ is odd.}$$

then (i) $D_\alpha P(z)$ *does not vanish in the disk*

$$|z| < \frac{1}{na_0 + \alpha a_1}\big[|n\alpha a_n + a_{n-1}| - (n\alpha a_n + a_{n-1}) - X_1 + (na_0 + \alpha a_1)\big] \text{ if } n \text{ is even,}$$

$$\begin{aligned}where\ X_1 =2\Big[&\big\{[(n-2)\alpha a_{n-2}+3a_{n-3}]+...+[4\alpha a_4+(n-3)a_3]+[2\alpha a_2+(n-1)a_1]\big\}\\ &-\big\{[(n-1)\alpha a_{n-1}+2a_{n-2}]+[(n-3)\alpha a_{n-3}+4a_{n-4}]+...+[3\alpha a_3+(n-2)a_2]\big\}\Big].\end{aligned}$$

(ii) $D_\alpha P(z)$ does not vanish in the disk

$$|z| < \frac{1}{na_0+\alpha a_1}\big[|n\alpha a_n+a_{n-1}|-(n\alpha a_n+a_{n-1})-X_2-(na_0+\alpha a_1)\big]\ if\ n\ is\ odd,$$

$$\begin{aligned}where\ X_2 =2\Big[&\big\{[(n-2)\alpha a_{n-2}+3a_{n-3}]+...+[5\alpha a_5+(n-4)a_4]+[3\alpha a_3+(n-2)a_2]\big\}\\ &-\big\{[(n-1)\alpha a_{n-1}+2a_{n-2}]+[(n-3)\alpha a_{n-3}+4a_{n-4}]+...+[4\alpha a_4+(n-3)a_3]\\ &+[2\alpha a_2+(n-1)a_1]\big\}\Big].\end{aligned}$$

Proof. Proof of the Theorem is similar to proof of the Theorem 4.5.1 □

Corollary 4.5.5. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq 3$ with real coefficients and let $D_0P(z)$ be the polar derivative of $P(z)$ such that $a_{n-1} \neq 0$,*

$$a_{n-1} \leq 2a_{n-2} \geq 3a_{n-3} \leq ... \geq (n-3)a_3 \leq (n-2)a_2 \geq (n-1)a_1 \leq na_0\ if\ n\ is\ even,$$

(OR)

$$a_{n-1} \geq 2a_{n-2} \leq 3a_{n-3} \geq ... \leq (n-3)a_3 \geq (n-2)a_2 \leq (n-1)a_1 \geq na_0\ if\ n\ is\ odd.$$

then (i) $D_0P(z)$ does not vanish in the disk

$$|z| < \frac{1}{na_0}\big[|a_{n-1}|-a_{n-1}-X_3+na_0\big]\ if\ n\ is\ even,$$

$$\begin{aligned}where\ X_3 =2\Big[&\big\{3a_{n-3}+...+(n-3)a_3+(n-1)a_1\big\}\\ &-\big\{2a_{n-2}+4a_{n-4}+...+(n-2)a_2\big\}\Big].\end{aligned}$$

(ii) $D_0P(z)$ does not vanish in the disk

$$|z| < \frac{1}{na_0}\big[|a_{n-1}|-a_{n-1}-X_4-na_0\big]\ if\ n\ is\ odd,$$

$$\begin{aligned}where\ X_4 =2\Big[&\big\{3a_{n-3}+...+(n-4)a_4+(n-2)a_2\big\}\\ &-\big\{2a_{n-2}+4a_{n-4}+...+(n-3)a_3+(n-1)a_1\big\}\Big].\end{aligned}$$

Remark 4.5.6. By taking $(i+1)\alpha a_{i+1}+(n-i)a_i>0$ for $i = 0, 1, 2, ..., n-2$, in Theorem 4.5.4, it reduces to Corollary 4.5.5.

4.6 On the zero free regions for polar derivative of polynomials with restricted coefficients

Theorem 4.6.1. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq 3$ *with real coefficients and let* $D_\alpha P(z)$ *be the polar derivative of* $P(z)$ *with respect to a real number* α *such that* $n\alpha a_n + a_{n-1} \neq 0$, *and*

$$n\alpha a_n + a_{n-1} \geq (n-1)\alpha a_{n-1} + 2a_{n-2} \geq (n-2)\alpha a_{n-2} + 3a_{n-3} \geq ... \geq (m+1)\alpha a_{m+1}$$
$$+(n-m)a_m \geq m\alpha a_m + (n-m+1)a_{m-1} \leq (m-1)\alpha a_{m-1} + (n-m+2)a_{m-2} \geq ... \leq$$
$$4\alpha a_4 + (n-3)a_3 \geq 3\alpha a_3 + (n-2)a_2 \leq 2\alpha a_2 + (n-1)a_1 \geq \alpha a_1 + na_0,$$

then $D_\alpha P(z)$ *does not vanish in the disk*

$$|z| < \frac{1}{na_0 + \alpha a_1}\left[|n\alpha a_n + a_{n-1}| + (n\alpha a_n + a_{n-1}) + B_1 - (na_0 + \alpha a_1)\right]$$

$$\text{where } B_1 = 2\Bigg[\big\{[(m-1)\alpha a_{m-1} + (n-m+2)a_{m-2}] + ... + [4\alpha a_4 + (n-3)a_3]$$
$$+ [2\alpha a_2 + (n-1)a_1]\big\} - \big\{[m\alpha a_m + (n-m+1)a_{m-1}]$$
$$+ [(m-2)\alpha a_{m-2} + (n-m+3)a_{m-3}] + ... + [3\alpha a_3 + (n-2)a_2]\big\}\Bigg].$$

Proof.

Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_{m-1}z^{m-1} + a_m z^m + a_{m-1}z^{m-1} + ... + a_1 z + a_0$

be a polynomial of degree n with real coefficients. Then the polar derivative

$$D_\alpha P(z) = [n\alpha a_n + a_{n-1}]z^{n-1} + [(n-1)\alpha a_{n-1} + 2a_{n-2}]z^{n-2} + [(n-2)\alpha a_{n-2} + 3a_{n-3}]z^{n-3}$$
$$+ ... + [(m+1)\alpha a_{m+1} + (n-m)a_m]z^{m+1} + [m\alpha a_m + (n-m+1)a_{m-1}]z^m$$
$$+ [(m-1)\alpha a_{m-1} + (n-m+2)a_{m-2}]z^{m-1} + ... + [4\alpha a_4 + (n-3)a_3]z^3$$
$$+ [3\alpha a_3 + (n-2)a_2]z^2 + [2\alpha a_2 + (n-1)a_1]z + [\alpha a_1 + na_0].$$

Now consider the polynomials, $J(z) = z^{n-1}D_\alpha P(\frac{1}{z})$ *and* $R(z) = (z-1)J(z)$ *so that*

$$
\begin{aligned}
R(z) =&(z-1)\Bigg[[n\alpha a_n + a_{n-1}] + [(n-1)\alpha a_{n-1} + 2a_{n-2}]z + [(n-2)\alpha a_{n-2} + 3a_{n-3}]z^2 \\
&+ ... + [(m+1)\alpha a_{m+1} + (n-m)a_m]z^{n-m-2} + [m\alpha a_m + (n-m+1)a_{m-1}]z^{n-m-1} \\
&+ [(m-1)\alpha a_{m-1} + (n-m+2)a_{m-2}]z^{n-m} + ... + [3\alpha a_3 + (n-2)a_2]z^{n-3} \\
&+ [2\alpha a_2 + (n-1)a_1]z^{n-2} + [\alpha a_1 + na_0]z^{n-1}\Bigg].
\end{aligned}
$$

$$
\begin{aligned}
R(z) =&[na_0 + \alpha a_1]z^n - \Bigg[\big[na_0 + \{\alpha - (n-1)\}a_1 - 2\alpha a_2\big]z^{n-1} \\
&+ \big[(n-1)a_1 + \{2\alpha - (n-2)\}a_2 - 3\alpha a_3\big]z^{n-2} \\
&+ ... + \big[(n-m+2)a_{m-2} + \{(m-1)\alpha - (n-m+1)\}a_{m-1} - m\alpha a_m\big]z^{n-m+1} \\
&+ \big[(n-m+1)a_{m-1} + \{m\alpha - (n-m)\}a_m - (m+1)\alpha a_{m+1}\big]z^{n-m} \\
&+ \big[(n-m)a_m + \{(m+1)\alpha - (n-m-1)\}a_{m+1} - (m+2)\alpha a_{m+2}\big]z^{n-m-1} \\
&+ ... + \big[4a_{n-4} + \{(n-3)\alpha - 3\}a_{n-3} - (n-2)\alpha a_{n-2}\big]z^3 \\
&+ \big[3a_{n-3} + \{(n-2)\alpha - 2\}a_{n-2} - (n-1)\alpha a_{n-1}\big]z^2 \\
&+ \big[2a_{n-2} + \{(n-1)\alpha - 1\}a_{n-1} - n\alpha a_n\big]z + \big[n\alpha a_n + a_{n-1}\big]\Bigg]
\end{aligned}
$$

$$Also\ if\ |z| > 1\ then\ \frac{1}{|z|^{n-i}} < 1\ for\ i = 0, 1, 2, ..., n-2.$$

$$
\begin{aligned}
Now|R(z)| \geq&\big|na_0 + \alpha a_1\big|\big|z\big|^n - \Bigg[\big|na_0 + \{\alpha - (n-1)\}a_1 - 2\alpha a_2\big|\big|z\big|^{n-1} \\
&+ \big|(n-1)a_1 + \{2\alpha - (n-2)\}a_2 - 3\alpha a_3\big|\big|z\big|^{n-2} \\
&+ \big|(n-2)a_2 + \{3\alpha - (n-3)\}a_3 - 4\alpha a_4\big|\big|z\big|^{n-3} \\
&+ \big|(n-3)a_3 + \{4\alpha - (n-4)\}a_4 - 5\alpha a_5\big|\big|z\big|^{n-4} \\
&+ ... + \big|(n-m+2)a_{m-2} + \{(m-1)\alpha - (n-m+1)\}a_{m-1} - m\alpha a_m\big|\big|z\big|^{n-m+1} \\
&+ \big|(n-m+1)a_{m-1} + \{m\alpha - (n-m)\}a_m - (m+1)\alpha a_{m+1}\big|\big|z\big|^{n-m} \\
&+ \big|(n-m)a_m + \{(m+1)\alpha - (n-m-1)\}a_{m+1} - (m+2)\alpha a_{m+2}\big|\big|z\big|^{n-m-1} \\
&+ ... + \big|4a_{n-4} + \{(n-3)\alpha - 3\}a_{n-3} - (n-2)\alpha a_{n-2}\big|\big|z\big|^3 \\
&+ \big|3a_{n-3} + \{(n-2)\alpha - 2\}a_{n-2} - (n-1)\alpha a_{n-1}\big|\big|z\big|^2 \\
&+ \big|2a_{n-2} + \{(n-1)\alpha - 1\}a_{n-1} - n\alpha a_n\big|\big|z\big| + \big|n\alpha a_n + a_{n-1}\big|\Bigg] \\
\geq&|na_0 + \alpha a_1||z|^{n-1}\Bigg[|z| - \frac{1}{|na_0 + \alpha a_1|}\Bigg\{|na_0 + \{\alpha - (n-1)\}a_1 - 2\alpha a_2| \\
&+ \frac{|(n-1)a_1 + \{2\alpha - (n-2)\}a_2 - 3\alpha a_3|}{|z|} + \frac{|(n-2)a_2 + \{3\alpha - (n-3)\}a_3 - 4\alpha a_4|}{|z|^2}
\end{aligned}
$$

$$+\frac{|(n-3)a_3+\{4\alpha-(n-4)\}a_4-5\alpha a_5|}{|z|^{n-4}}+...+$$
$$\frac{|(n-m+2)a_{m-2}+\{(m-1)\alpha-(n-m+1)\}a_{m-1}-m\alpha a_m|}{|z|^{n-m+1}}$$
$$+\frac{|(n-m+1)a_{m-1}+\{m\alpha-(n-m)\}a_m-(m+1)\alpha a_{m+1}|}{|z|^{n-m}}$$
$$+\frac{|(n-m)a_m+\{(m+1)\alpha-(n-m-1)\}a_{m+1}-(m+2)\alpha a_{m+2}|}{|z|^{n-m-1}}+...+$$
$$\frac{|4a_{n-4}+\{(n-3)\alpha-3\}a_{n-3}-(n-2)\alpha a_{n-2}|}{|z|^{n-4}}$$
$$+\frac{|3a_{n-3}+\{(n-2)\alpha-2\}a_{n-2}-(n-1)\alpha a_{n-1}|}{|z|^{n-3}}$$
$$+\frac{|2a_{n-2}+\{(n-1)\alpha-1\}a_{n-1}-n\alpha a_n|}{|z|^{n-2}}+\frac{|n\alpha a_n+a_{n-1}|}{|z|^{n-1}}\Bigg\}\Bigg]$$
$$\geq|na_0+\alpha a_1||z|^{n-1}\Bigg[|z|-\frac{1}{|na_0+\alpha a_1|}\Bigg\{|na_0+\{\alpha-(n-1)\}a_1-2\alpha a_2|$$
$$+|(n-1)a_1+\{2\alpha-(n-2)\}a_2-3\alpha a_3|+|(n-2)a_2+\{3\alpha-(n-3)\}a_3-4\alpha a_4|$$
$$+|(n-3)a_3+\{4\alpha-(n-4)\}a_4-5\alpha a_5|+...+$$
$$|(n-m+2)a_{m-2}+\{(m-1)\alpha-(n-m+1)\}a_{m-1}-m\alpha a_m|$$
$$+|(n-m+1)a_{m-1}+\{m\alpha-(n-m)\}a_m-(m+1)\alpha a_{m+1}|$$
$$+|(n-m)a_m+\{(m+1)\alpha-(n-m-1)\}a_{m+1}-(m+2)\alpha a_{m+2}|+...+$$
$$|4a_{n-4}+\{(n-3)\alpha-3\}a_{n-3}-(n-2)\alpha a_{n-2}|+|3a_{n-3}+\{(n-2)\alpha-2\}a_{n-2}-(n-1)\alpha a_{n-1}|$$
$$+|2a_{n-2}+\{(n-1)\alpha-1\}a_{n-1}-n\alpha a_n|+|n\alpha a_n+a_{n-1}|\Bigg\}\Bigg]$$
$$\geq|na_0+\alpha a_1||z|^{n-1}\Bigg[|z|-\frac{1}{|na_0+\alpha a_1|}\Bigg\{[2\alpha a_2+\{(n-1)-\alpha\}a_1-na_0]$$
$$+[(n-1)a_1+\{2\alpha-(n-2)\}a_2-3\alpha a_3]+[4\alpha a_4+\{(n-3)-3\alpha\}a_3-(n-2)a_2]$$
$$+...+[(m-1)\alpha a_{m-1}-\{(n-m+1)-(m-1)\alpha\}a_{m-1}+(n-m+2)a_{m-2}]$$
$$+[(m+1)\alpha a_{m+1}+\{(n-m)-m\alpha\}a_m-(n-m+1)a_{m-1}]$$
$$+...+[(n-2)\alpha a_{n-2}+\{3-(n-3)\alpha\}a_{n-3}-4a_{n-4}]$$
$$+[(n-1)\alpha a_{n-1}+\{2-(n-2)\alpha\}a_{n-2}-3a_{n-3}]$$
$$+[n\alpha a_n+\{1-(n-1)\alpha\}a_{n-1}-2a_{n-2}]+|n\alpha a_n+a_{n-1}|\Bigg\}\Bigg]$$
$$=|na_0+\alpha a_1||z|^{n-1}\Bigg[|z|-\frac{1}{|na_0+\alpha a_1|}\Bigg\{|n\alpha a_n+a_{n-1}|+(n\alpha a_n+a_{n-1})$$
$$+B_1-(na_0+\alpha a_1)\Bigg\}\Bigg]>0$$

$$if\ |z| > \frac{1}{|na_0+\alpha a_1|}\Big[|n\alpha a_n + a_{n-1}| + (n\alpha a_n + a_{n-1}) + B_1 - (na_0+\alpha a_1)\Big],$$

$$where\ B_1 = 2\Bigg[\Big\{[(m-1)\alpha a_{m-1} + (n-m+2)a_{m-2}] + ... + [4\alpha a_4 + (n-3)a_3] + [2\alpha a_2 + (n-1)a_1]\Big\} - \Big\{[m\alpha a_m + (n-m+1)a_{m-1}] + [(m-2)\alpha a_{m-2} + (n-m+3)a_{m-3}] + ... + [3\alpha a_3 + (n-2)a_2]\Big\}\Bigg].$$

This shows that all the zeros of R(z) whose modulus is greater than 1 lie in the closed disk

$$|z| \le \frac{1}{na_0+\alpha a_1}\Big[|n\alpha a_n + a_{n-1}| + (n\alpha a_n + a_{n-1}) + B_1 - (na_0+\alpha a_1)\Big].$$

But those zeros of R(z) whose modulus is less than or equal to 1 already lie in the above disk.

Therefore, it follows that all the zeros of R(z) and hence J(z) lie in

$$|z| \le \frac{1}{na_0+\alpha a_1}\Big[|n\alpha a_n + a_{n-1}| + (n\alpha a_n + a_{n-1}) + B_1 - (na_0+\alpha a_1)\Big].$$

Since $D_\alpha P(z) = z^n J(\frac{1}{z})$ it follows, by replacing z by $\frac{1}{z}$,
we get that all the zeros of $D_\alpha P(z)$ lie in

$$|z| \ge \frac{1}{na_0+\alpha a_1}\Big[|n\alpha a_n + a_{n-1}| + (n\alpha a_n + a_{n-1}) + B_1 - (na_0+\alpha a_1)\Big].$$

Hence $D_\alpha P(z)$ does not vanish in the disk

$$|z| < \frac{1}{na_0+\alpha a_1}\Big[|n\alpha a_n + a_{n-1}| + (n\alpha a_n + a_{n-1}) + B_1 - (na_0+\alpha a_1)\Big],$$

$$where\ B_1 = 2\Bigg[\Big\{[(m-1)\alpha a_{m-1} + (n-m+2)a_{m-2}] + ... + [4\alpha a_4 + (n-3)a_3] + [2\alpha a_2 + (n-1)a_1]\Big\} - \Big\{[m\alpha a_m + (n-m+1)a_{m-1}] + [(m-2)\alpha a_{m-2} + (n-m+3)a_{m-3}] + ... + [3\alpha a_3 + (n-2)a_2]\Big\}\Bigg].$$

This completes the proof of the Theorem . □

Corollary 4.6.2. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq 3$ *with real coefficients and let* $D_o P(z)$ *be the polar derivative of* $P(z)$ *such that for* $a_{n-1} \neq 0$,

$$a_{n-1} \geq 2a_{n-2} \geq 3a_{n-3} \geq ... \geq (n-m-1)a_{m+1} \geq (n-m)a_m \geq (n-m+1)a_{m-1}$$
$$\leq (n-m+2)a_{m-2} \geq ... \leq (n-5)a_5 \geq (n-4)a_4 \leq (n-3)a_3 \geq (n-2)a_2 \leq (n-1)a_1 \geq na_0,$$

then $D_\alpha P(z)$ *does not vanish in the disk* $|z| < \dfrac{1}{na_0}\big[|a_{n-1}| + a_{n-1} + B_2 - na_0\big]$

where $B_2 = 2\Big[\big\{(n-m+2)a_{m-2} + (n-m+4)a_{m-4} + ... + (n-3)a_3 + (n-1)a_1\big\}$
$$- \big\{(n-m+1)a_{m-1} + (n-m+3)a_{m-3} + ... + (n-4)a_4 + (n-2)a_2\big\}\Big].$$

Remark 4.6.3. By taking $\alpha = 0$ in Theorem 4.6.1, it reduces to Corollary 4.6.2.

Theorem 4.6.4. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq 3$ *with real coefficients and let* $D_\alpha P(z)$ *be the polar derivative of* $P(z)$ *with respect to a real number* α *such that* $n\alpha a_n + a_{n-1} \neq 0$,

$$n\alpha a_n + a_{n-1} \geq (n-1)\alpha a_{n-1} + 2a_{n-2} \geq (n-2)\alpha a_{n-2} + 3a_{n-3} \geq ... \geq (m+1)\alpha a_{m+1} + (n-m)a_m$$
$$\leq m\alpha a_m + (n-m+1)a_{m-1} \geq (m-1)\alpha a_{m-1} + (n-m+2)a_{m-2} \leq ... \geq 4\alpha a_4 + (n-3)a_3$$
$$\leq 3\alpha a_3 + (n-2)a_2 \geq 2\alpha a_2 + (n-1)a_1 \leq \alpha a_1 + na_0,$$

then $D_\alpha P(z)$ *does not vanish in the disk*

$$|z| < \frac{1}{na_0 + \alpha a_1}\big[|n\alpha a_n + a_{n-1}| + (n\alpha a_n + a_{n-1}) + B_3 + (na_0 + \alpha a_1)\big],$$

where $B_3 = 2\Big[\big\{[m\alpha a_m + (n-m+1)a_{m-1}] + [(m-2)\alpha a_{m-2} + (n-m+3)a_{m-3}] + ... +$
$$[3\alpha a_3 + (n-2)a_2]\big\} - \big\{[(m+1)\alpha a_{m+1} + (n-m)a_m] + ... + [4\alpha a_4 + (n-3)a_3]$$
$$+ [2\alpha a_2 + (n-1)a_1]\big\}\Big].$$

Proof of the Theorem is similar to the proof of Theorem 4.6.1.

Corollary 4.6.5. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq 3$ *with real coefficients and let* $D_o P(z)$ *be the polar derivative of* $P(z)$ *such that for* $a_{n-1} \neq 0$,

$$a_{n-1} \geq 2a_{n-2} \geq 3a_{n-3} \geq ... \geq (n-m-1)a_{m+1} \geq (n-m)a_m \leq (n-m+1)a_{m-1} \geq$$
$$(n-m+2)a_{m-2} \leq ... \geq (n-5)a_5 \leq (n-4)a_4 \geq (n-3)a_3 \leq (n-2)a_2 \geq (n-1)a_1 \leq na_0,$$

then $D_\alpha P(z)$ *does not vanish in the disk* $|z| < \dfrac{1}{na_0}\big[|a_{n-1}| + a_{n-1} + B_4 + na_0\big]$,

$$\text{where } B_4 = 2\Big[\big\{(n-m+1)a_{m-1} + (n-m+3)a_{m-3} + ... + (n-4)a_4 + (n-2)a_2\big\} - \big\{(n-m)a_m + (n-m+2)a_{m-2} + ... + (n-3)a_3 + (n-1)a_1]\big\}\Big].$$

Remark 4.6.6. By taking $\alpha = 0$ in Theorem 4.6.4, it reduces to Corollary 4.6.5.

Theorem 4.6.7. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq 3$ *with real coefficients and let* $D_\alpha P(z)$ *be the polar derivative of* $P(z)$ *with respect to a real number* α *such that* $n\alpha a_n + a_{n-1} \neq 0,$ *and*

$$n\alpha a_n + a_{n-1} \leq (n-1)\alpha a_{n-1} + 2a_{n-2} \leq (n-2)\alpha a_{n-2} + 3a_{n-3} \leq ... \leq (m+1)\alpha a_{m+1} + (n-m)a_m$$
$$\leq m\alpha a_m + (n-m+1)a_{m-1} \geq (m-1)\alpha a_{m-1} + (n-m+2)a_{m-2} \leq ... \geq 4\alpha a_4 + (n-3)a_3$$
$$\leq 3\alpha a_3 + (n-2)a_2 \geq 2\alpha a_2 + (n-1)a_1 \leq \alpha a_1 + na_0,$$

then $D_\alpha P(z)$ *does not vanish in the disk*

$$|z| < \frac{1}{na_0 + \alpha a_1}\big[|n\alpha a_n + a_{n-1}| - (n\alpha a_n + a_{n-1}) + B_5 + (na_0 + \alpha a_1)\big],$$

$$\text{where } B_5 = 2\Big[\big\{[m\alpha a_m + (n-m+1)a_{m-1}] + [(m-2)\alpha a_{m-2} + (n-m+3)a_{m-3}] + ... + [3\alpha a_3 + (n-2)a_2]\big\} - \big\{[(m-1)\alpha a_{m-1} + (n-m+2)a_{m-2}] + ... + [4\alpha a_4 + (n-3)a_3] + [2\alpha a_2 + (n-1)a_1]\big\}\Big].$$

Proof of the Theorem is similar to the proof of Theorem 4.6.1.

Corollary 4.6.8. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq 3$ *with real coefficients and let* $D_o P(z)$ *be the polar derivative of* $P(z)$ *such that for* $a_{n-1} \neq 0$,

$$a_{n-1} \leq 2a_{n-2} \leq 3a_{n-3} \leq ... \leq (n-m-1)a_{m+1} \leq (n-m)a_m \leq (n-m+1)a_{m-1}$$
$$\geq (n-m+2)a_{m-2} \leq ... \geq (n-5)a_5 \leq (n-4)a_4 \geq (n-3)a_3 \leq (n-2)a_2 \geq (n-1)a_1 \leq na_0,$$

then $D_\alpha P(z)$ *does not vanish in the disk* $|z| < \dfrac{1}{na_0}\big[|a_{n-1}| - a_{n-1} + B_6 + na_0\big]$

$$\text{where } B_6 = 2\Big[\big\{(n-m+1)a_{m-1} + (n-m+3)a_{m-3} + ... + (n-4)a_4 + (n-2)a_2\big\} - \big\{(n-m+2)a_{m-2} + (n-m+4)a_{m-4} + ... + (n-3)a_3 + (n-1)a_1]\big\}\Big].$$

Remark 4.6.9. By taking $\alpha = 0$ in Theorem 4.6.7, it reduces to Corollary 4.6.8.

Theorem 4.6.10. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq 3$ with real coefficients and let $D_\alpha P(z)$ be the polar derivative of $P(z)$ with respect to a real number α such that $n\alpha a_n + a_{n-1} \neq 0$, and*

$$n\alpha a_n + a_{n-1} \leq (n-1)\alpha a_{n-1} + 2a_{n-2} \leq (n-2)\alpha a_{n-2} + 3a_{n-3} \leq ... \leq (m+1)\alpha a_{m+1}$$
$$+(n-m)a_m \geq m\alpha a_m + (n-m+1)a_{m-1} \leq (m-1)\alpha a_{m-1} + (n-m+2)a_{m-2} \geq ....$$
$$\leq 4\alpha a_4 + (n-3)a_3 \geq 3\alpha a_3 + (n-2)a_2 \leq 2\alpha a_2 + (n-1)a_1 \geq \alpha a_1 + na_0,$$

then $D_\alpha P(z)$ does not vanish in the disk

$$|z| < \frac{1}{na_0 + \alpha a_1}\big[|n\alpha a_n + a_{n-1}| - (n\alpha a_n + a_{n-1}) + B_7 - (na_0 + \alpha a_1)\big]$$

where $B_7 = 2\bigg[\big\{[(m+1)\alpha a_{m+1} + (n-m)a_m] + [(m-1)\alpha a_{m-1} + (n-m+2)a_{m-2}] + ...$

$$+[2\alpha a_2 + (n-1)a_1]\big\} - \big\{[m\alpha a_m + (n-m+1)a_{m-1}] + [(m-2)\alpha a_{m-2}$$
$$+(n-m+3)a_{m-3}] + ... + [3\alpha a_3 + (n-2)a_2]\big\}\bigg].$$

Proof of the Theorem is similar to the proof of Theorem 4.6.1.

Corollary 4.6.11. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq 3$ with real coefficients and let $D_o P(z)$ be the polar derivative of $P(z)$ such that for $a_{n-1} \neq 0$,*

$$a_{n-1} \leq 2a_{n-2} \leq 3a_{n-3} \leq ... \leq (n-m-1)a_{m+1} \leq (n-m)a_m \geq (n-m+1)a_{m-1}$$
$$\leq (n-m+2)a_{m-2} \geq ... \leq (n-5)a_5 \geq (n-4)a_4 \leq (n-3)a_3 \geq (n-2)a_2 \leq (n-1)a_1 \geq na_0,$$

then $D_\alpha P(z)$ does not vanish in the disk $|z| < \frac{1}{na_0}\big[|a_{n-1}| + a_{n-1} + B_8 - na_0\big]$

where $B_8 = 2\bigg[\big\{(n-m)a_m + (n-m+2)a_{m-2} + ... + (n-3)a_3 + (n-1)a_1\big\}$

$$- \big\{(n-m+1)a_{m-1} + (n-m+3)a_{m-3} + ... + (n-4)a_4 + (n-2)a_2\big\}\bigg].$$

Remark 4.6.12. By taking $\alpha = 0$ in Theorem 4.6.10, it reduces to Corollary 4.6.11

The results ragarding sections 4.2, 4.3, 4.4, 4.5 and 4.6 have appeared in [35],[31],[12],[11] and [13] respectively.

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Index