The Algebra of Differential Operators on Triangular Algebras and Some Remarks on the Algebra of Quantum Differential Operators on $k[X^{\pm 1}]$

Thesis submitted to the

University of Hyderabad

for the Degree of

DOCTOR OF PHILOSOPHY

by

KOTESWARA RAO. G

SCHOOL OF MATHEMATICS AND STATISTICS

UNIVERSITY OF HYDERABAD

HYDERABAD - 500 046

2016

CERTIFICATE

School of Mathematics and Statistics

University of Hyderabad

Hyderabad- 500 046

This is to certify that I, KOTESWARA RAO. G, have carried out the research embodied in the present thesis entitled **The Algebra of Differential Operators on Triangular Algebras and Some Remarks on the Algebra of Quantum Differential Operators on** $k[X^{\pm 1}]$ for the full period prescribed under Ph.D. ordinance of the University.

I declare that, to the best of my knowledge, no part of this thesis was earlier submitted for the award of research degree of any university.

Koteswara Rao. G

Candidate (Reg. No. 10MMPP03)

Prof. B. Sri Padmavathi	Dr. M. Sumanth Datt
(Dean of the School)	Supervisor

Dedicated to My parents

ACKNOWLEDGEMENT

I am very thankful to my supervisor Dr. M. Sumanth Datt for his continuous help throughout my research work with much patience and also for spending time for me.

My sincere thanks to Prof. B. Sri Padmavathi, School of Mathematics and Statistics and Prof. S. Kumaresan, former Dean, for extending all necessary facilities for my research work.

My special thanks to Prof. Uma N Iyer for her valuable support during my research work and her encouragement.

I thank all the other faculty members and the office staff of the School for their support. They have been very helpful, supportive and cooperative.

I am very thankful to my family members and friends for their adorable support throughout my studies. I am very much grateful to them.

I am very thankful to Council for Scientific and Industrial Research (CSIR, Govt. of India), for the financial assistance (as JRF and SRF).

- Koteswara Rao

Contents

Introduction

In this thesis, we study mainly (i) the algebra of differential operators on upper triangular algebras and (ii) the algebra of quantum differential operators on Laurent polynomial ring $k[X, X^{-1}]$, (sometimes we also write $k[X^{\pm}]$ for $k[X, X^{-1}]$), over a field of characteristic zero. This thesis consists four chapters, Chapter 1, Preliminary. Chapter 2, the algebra of differential operators on triangular algebras. Chapter 3, quantum differential operators on $k[X^{\pm 1}]$. Chapter 4, deals with Jordan problem of α- simple rings, $k[X]$, $k(X)[Y, Z]$.

In Chapter 1, we give basic definitions and we recall some of the well known results in the area of Weyl algebras and Quantum Weyl algebras.

In Chapter 2, we describe the algebra $D_k(U_n)$ of differential operators on the algebra U_n of upper triangular $n \times n$ matrix over a field k of characterestic zero. We give generators and relations for $D_k(U_n)$. We characterise all two-sided, left, right ideals and the maximal ideals of $D(U_n)$.

Chapter 3 has been devoted to the algebra of quantum differential operators on $k[X, X^{-1}]$. The algebra of quantum differential operators has been studied by Uma N. Iyer and David A. Jordan in [UD]. In their work they gave a basis for $D_q(k[X])$, generators and relations. We give an another basis for $D_q(k[X])$ and we show that $D_q(k[X, X^{-1}])$ is isomorphic to its opposite algebra $D_q(k[X, X^{-1}])^o$ by giving an isomorphism.

In Chapter 4, we define α- simple ring and we show that $k[X]$, $k(X)[Y, Z]$ are α-simple rings by following [S]. We consider the problem of Jordan for α-simple rings in [J]. We give an elementary proof to this problem for $k[X]$ and also we have some partial results for the α-simple ring $k(X)[Y, Z]$.

Chapter 1

Preliminaries

In this chapter, we give basic definitions and we recall some of the well known results. We refer [SC], [LR], [GB], [MR], [J], etc for terminology and notations.

1.1 Algebra of differential operators

Definition 1.1.1. Let k be a field of characterestic 0 and A be a finitely generated k-algebra. Let End(A) denote the k-algebra of all k-linear endomorphisms on A. For each $a \in A$, we let $\lambda_a, \rho_a \in$ End(A) be given by $\lambda_a(s) = as$ and $\rho_a(s) = sa$ for all $s \in A$. The algebra End(A) is an A-bimodule with $a.\varphi = \lambda_a\varphi$ and $\varphi.b = \varphi\rho_b$ for $a, b \in A$ and $\varphi \in$ End(A). For $\varphi \in \mathrm{End}(A)$ and $a \in A$, let $[\varphi, a] = [\varphi, \lambda_a] = \varphi\lambda_a - \lambda_a\varphi$. Let $Z_0 = \{\varphi \in \mathrm{End}(A) \mid [\varphi, a] = 0, \text{for all } a \in A\}$. Then Z_0 is an additive subgroup of $\mathrm{End}(A)$. Let $D_k^0(A)$ be the A-bimodule generated by Z_0 and it is a k-

algebra. By induction, for $m \in \mathbb{Z}_+$, let $D_k^m(A)$ be the A-bimodule generated by $Z_m = \{\varphi \in \text{End(A)} \mid [\varphi, a] \in D_k^{m-1}(A), \text{for all } a \in A\}$.

For $i, j \in \mathbb{Z}_+$, we have $D_k^i(A).D_k^j(A) \subset D_k^{i+j}(A)$. We see this as follows: the commutator $[\varphi_1, \varphi_2] = \varphi_1\varphi_2 - \varphi_2\varphi_1$, for $\varphi_1, \varphi_2 \in \text{End}_k(A)$. Let $\varphi_1 \in D_k^i(A)$, $\varphi_2 \in D_k^j(A)$, then $\varphi_1\varphi_2 \in D_k^{i+j}(A)$. We will prove it by induction on $i+j$. If $i+j=0$, then clearly $\varphi_1\varphi_2 \in D_k^{i+j}(A)$. Assume that for $m+n<l$, $\varphi_1\varphi_2 \in D_k^{n+m}(A)$. Let $m+n=l$ and $a \in A$, we have that $[\varphi_1\varphi_2, a] = \varphi_1[\varphi_2, a] + [\varphi_1, a]\varphi_2$ and $[\varphi_2, a] \in D_k^{m-1}(A)$, $[\varphi_1, a] \in D_k^{n-1}(A)$. Thus, by induction $\varphi_1[\varphi_2, a], [\varphi_1, a]\varphi_2 \in D_k^{n+m-1}(A)$. Hence $[\varphi_1\varphi_2, a] \in D_k^{n+m-1}(A)$. Therefore $\varphi_1\varphi_2 \in D_k^{n+m}(A)$. *Let $D_k(A) = \cup_{i\geq 0} D_k^i(A)$, this is a k-algebra and is called the algebra of differential operators on A.* And we say that elements of $D_k^m(A)$, for $m \geq 0$, are the differential operators of degree $\leq m$.

The elements of $D_k^0(A)$ are called the inner differential operators on A and the ring $D_k^0(A)$ is called the ring of inner differential operators.

If φ is a derivation on A, then $\varphi(rs) = r\varphi(s) + \varphi(r)s$. Thus, $[\varphi, r] = \lambda_{\varphi(r)}$ for $r \in A$. Hence, a derivation is a first order differential operator.
If A is commutative, Z_m is already an A-bimodule and hence $\varphi \in \text{End}_k(A)$ is a differential operator of order m if and only if $[\cdots[[\varphi, r_0], r_1], \cdots, r_m] = 0$ for all $r_i \in A$, where $[\varphi, r] = (\varphi r) - (r\varphi)$ which is the classical definition of an algebraic differential operator on a commutative k-algebra.

Lemma 1.1.2. The set $\{\lambda_a, \rho_a \mid a \in A\}$ generates $D_k^0(A)$.

Proof. If $\varphi \in Z_0$, then $\varphi\lambda_a = \lambda_a\varphi$, for all $a \in A$. Now consider,

$\varphi(a) = \varphi(a.1) = \varphi(\lambda_a(1)) = (\varphi\lambda_a)(1) = (\lambda_a\varphi)(1) = \lambda_a(\varphi(1)) = a\varphi(1) = \rho_{\varphi(1)}(a)$. Therefore $\varphi = \rho_{\varphi(1)}$. □

1.2 Weyl algebras

Let $A = k[X_1, \cdots, X_n]$ be the commutative polynomial ring in n indeterminates $X_1, \cdots, X_n$ over the field k. The set $\mathrm{End(A)} = \mathrm{End_k(A)}$ of all linear transformations is a k-algebra with respect to the usual addition and composition of linear transformations.

For each i, $1 \leq i \leq n$, let λ_{X_i} be the operator on A such that $\lambda_{X_i}(f) = X_i f$, for $f \in A$. Similarly, for each i and $1 \leq i \leq n$, let ∂_i be the operator on A such that $\partial_i(f) = \partial f/\partial X_i$, for $f \in A$ (formal partial derivative of f with respect to the variable X_i). Clearly $\lambda_{X_i}, \partial_i \in \mathrm{End(A)}$. *The subalgebra A_n of* $\mathrm{End}_k(A)$ *generated by $\lambda_{X_i}, \partial_i$ is is called n-th Weyl algebra.*

For $f \in A$, we have

$$\begin{aligned}
\partial_i\lambda_{X_i}(f) &= \partial_i(\lambda_{X_i}(f)) \\
&= \partial_i(X_i f) \\
&= X_i\partial_i f + f\partial_i X_i \\
&= X_i\partial_i f + f \\
&= (\lambda_{X_i}\partial_i + 1)(f)
\end{aligned}$$

Therefore, $\partial_i\lambda_{X_i} = \lambda_{X_i}\partial_i + 1$, where 1 stands for the identity operator.

For $1 \leq i, j \leq n$, we have $[\partial_i, \lambda_{X_j}] = \delta_{ij}1$ and $[\partial_i, \partial_j] = [\lambda_{X_i}, \lambda_{X_j}] = 0$ for $i \neq j$.

For $\alpha = (\alpha_1, \cdots, \alpha_n) \in {\mathbb{Z}_+}^n$, let $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Define $\mid \alpha \mid = \alpha_1 + \cdots + \alpha_n$ and $\mid \alpha \mid$ is called the length of α. Similarly we define $\alpha! = \alpha_1! \cdots \alpha_n!$ and we call it the factorial of α. By definition, $\mid \alpha \mid$ is the degree of X^α.

Lemma 1.2.1. Let $\alpha, \beta \in {\mathbb{Z}_+}^n$ and assume that $\mid \alpha \mid \leq \mid \beta \mid$. Then $\partial^\beta(X^\alpha) = \beta!$ if $\alpha = \beta$, and zero otherwise .

Proof. Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$.

Case 1. Suppose that $\alpha = \beta$, then $\alpha_i = \beta_i$, for all i. We have, for any r and $1 \leq r \leq n$,

$$\partial_i^{\beta_r} = \begin{cases} 0 & \text{for} \quad i \neq r \\ \beta_r! & \text{for} \quad i = r \end{cases}$$

and for $i \neq j$, $1 \leq i, j \leq n$,

$$\partial_i^{\beta_i} \partial_j^{\beta_j} = \partial_j^{\beta_j} \partial_i^{\beta_i}$$

Hence $\partial^\beta(X^\alpha) = \beta_1! \cdots \beta_n!$.

Case 2. Suppose $\alpha \neq \beta$. Since $\mid \alpha \mid \leq \mid \beta \mid$, there exist i such that $\alpha_i < \beta_i$. Then we have $\partial^{\beta_i}(X^{\alpha_i}) = 0$, and therefore $\partial^\beta(X^\alpha) = 0$.

□

Proposition 1.2.2. *The set* $B = \{X^\alpha \partial^\beta \mid \alpha, \beta \in {\mathbb{Z}_+}^n\}$ *is a basis of the vector space* A_n *over* k.

Proof. We have, for $m \in \mathbb{Z}_+$,

$$\partial_i X_j^m - X_j^m \partial_i = \begin{cases} m X_i^{m-1}, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

Therefore, for any $f \in A$ and ∂_i, $\partial_i f - f\partial_i = \partial f/\partial X_i$. i.e $\partial_i f = f\partial_i + \partial f/\partial X_i$. Therefore any monomial of A_n can be expressed as a finite linear combination of the elements of B.

Consider a finite linear combination of elements of B, say $D = \sum c_{\alpha\beta} X^\alpha \partial^\beta$. It is enough to show that if some $c_{\alpha\beta} \neq 0$ then $D \neq 0$. For this we show that, there exist $f \in A$ such that $D(f) \neq 0$.

Let $\gamma \in {\mathbb{Z}_+}^n$ such that $c_{\alpha\gamma} \neq 0$ for some index α, but $c_{\alpha\beta} = 0$, for all indices β such that $\mid \beta \mid < \mid \gamma \mid$. Then $D(X^\gamma) = \gamma! \sum_\alpha c_{\alpha\gamma} X^\gamma \neq 0$. Therefore $f = X^\gamma$ and $D(f) \neq 0$. □

We call this as the canonical basis. Any element of A_n is a finite linear combination of elements of B and this expression is called the canonical form.

Definition 1.2.3. (**degree of an operator**)

Let $D \in A_n$. The degree of D is the largest length of the multi-indices $(\alpha, \beta) \in {\mathbb{Z}_+}^n \times {\mathbb{Z}_+}^n$ for which $X^\alpha \partial^\beta$ appears with non-zero coefficient in the canonical form of D and it is denoted by $\deg(D)$. We define the degree of the zero polynomial as $-\infty$.

Theorem 1.2.4. *For $D, D' \in A_n$, we have the following.*

1. $\deg(D + D') \leq \max\{\deg(D), \deg(D')\}$.

2. $\deg(DD') = \deg(D) + \deg(D')$.

3. $\deg[D, D'] \leq \deg(D) + \deg(D') - 2$.

Proof. If $D, D' \in A_n$ are written in canonical form, then $D + D'$ is also in the canonical form. Therefore $\deg(D, D') \leq \max\{\deg(D), \deg(D')\}$. We prove (2) and (3) at the same time, by induction on $\deg(D) + \deg(D')$. If either $\deg(D)$ or $\deg(D')$ is zero, then the result is obvious. Suppose that $\deg(D), \deg(D') \geq 1$ and assume that the result is true for $\deg(D) + \deg(D') < l,\ l \in \mathbb{N}$. Let $D,\ D' \in A_n$ with $\deg(D) + \deg(D') = l$. From (1), it is enough to prove the result for D, D' are monomials. Suppose first that $D = \partial^\beta$ and $D' = X^\alpha$ with $\mid \alpha \mid + \mid \beta \mid = k$. If $\beta_i \neq 0$, then

$$[\partial^\beta, X^\alpha] = \partial_i[\partial^{\beta - e_i}, X^\alpha] + [\partial_i, X^\alpha]\partial^{\beta - e_i}$$

where $e_i = (0, \cdots, 1^{i^{th}}, \cdots, 0) \in \mathbb{N}$. By induction, $\deg[\partial^{\beta-e_i}, X^\alpha] \leq \mid \alpha \mid + \mid \beta \mid -3$ and $[\partial_i, X^\alpha] \leq \mid \alpha \mid -1$. Again by induction, $\deg(\partial_i[\partial^{\beta-e_i}, X^\alpha])$ and $\deg([\partial_i, X^\alpha]\partial^{\beta-e_i})$ are $\leq \mid \alpha \mid + \mid \beta \mid -2$. Therefore $\deg[\partial^\beta, X^\alpha] \leq \mid \alpha \mid + \mid \beta \mid -2$. But $\partial^\beta X^\alpha = [\partial^\beta, X^\alpha] + X^\alpha\partial^\beta$. Since $\deg(X^\alpha\partial^\beta) = \mid \alpha \mid + \mid \beta \mid$ and $[\partial^\beta, X^\alpha] \leq \mid \alpha \mid + \mid \beta \mid -2$, therefore

$$\deg(\partial^\beta X^\alpha) = \deg(X^\alpha\partial^\beta) = \mid \alpha \mid + \mid \beta \mid .$$

Now let $D = X^\sigma \partial^\beta$ and $D' = X^\alpha \partial^\eta$.

If $|\alpha| = |\beta| = 0$, the result is obvious. Suppose that atleast one among $\alpha,\ \beta$ is not zero. We have $\partial^\beta X^\alpha = X^\alpha \partial^\beta + P$, where $P = [\partial^\beta, X^\alpha]$ and degree of P is $\leq |\alpha| + |\beta| - 2$. Then $DD' = X^{\sigma+\alpha}\partial^{\beta+\eta} + X^\sigma P \partial^\eta$. By induction, $\deg(X^\sigma P \partial^\eta) \leq \deg(D) + \deg(D') - 2$. Hence

$$\deg(DD') = \deg(X^{\sigma+\alpha}\partial^{\beta+\eta}) = \deg(D) + \deg(D').$$

By the above, we have

$$DD' = X^{\sigma+\alpha}\partial^{\beta+\eta} + Q_1,$$

where $\deg(Q_1) \leq \deg(D) + \deg(D') - 2$,

and

$$D'D = X^{\sigma+\alpha}\partial^{\beta+\eta} + Q_2,$$

where $\deg(Q_2) \leq \deg(D) + \deg(D') - 2$. Therefore $[D, D'] = Q_1 - Q_2$ and hence $\deg[D, D'] \leq \deg(D) + \deg(D') - 2$. □

Corollary 1.2.5. *The algebra A_n is a domain.*

Lemma 1.2.6. Let $\alpha, \beta \in {\mathbb{Z}_+}^n$, $|\alpha| + |\beta| = l > 0$ and $\beta_i \neq 0$. Then

$$\deg[X_i, X^\alpha \partial^\beta] = l - 1.$$

Proof. Let us first find $\partial_i^{\beta_i} X_i$. Since $\partial_i^{\beta_i} X_i = \partial_i^{\beta_i - 1}(\partial_i X_i)$ and $\partial_i X_i = 1 + X_i \partial_i$, we get $\partial_i^{\beta_i} X_i = \partial_i^{\beta_i-1} + \partial_i^{\beta_i-1} X_i \partial_i$. By repeating this, we will have

$$\partial_i^{\beta_i} X_i = (\partial_i^{\beta_i-1} + \partial_i^{\beta_i-2} + \cdots + 1) + X_i \partial_i^{\beta_i}.$$

Now consider,

$$\begin{aligned} X^{\alpha}\partial_{\beta}X_i &= X^{\alpha}\partial_1^{\beta_1}\cdots\partial_i^{\beta_i}X_i\cdots\partial_n^{\beta_n} \\ &= X^{\alpha}\partial_1^{\beta_1}\cdots\partial_{i-1}^{\beta_{i-1}}[(\partial_i^{\beta_i-1}+\partial_i^{\beta_i-2}+\cdots+1)+X_i\partial_i^{\beta_i}]\partial_{i+1}^{\beta_{i+1}}\cdots\partial_n^{\beta_n} \end{aligned}$$

therefore

$$[X_i, X^{\alpha}\partial^{\beta}] = -X^{\alpha}\partial_1^{\beta_1}\cdots\partial_{i-1}^{\beta_{i-1}}[(\partial_i^{\beta_i-1}+\partial_i^{\beta_i-2}+\cdots+1)]\partial_{i+1}^{\beta_{i+1}}\cdots\partial_n^{\beta_n}$$

and hence $deg[X_i, X^{\alpha}\partial^{\beta}] = l-1$. □

Theorem 1.2.7. *The algebra A_n is simple.*

Proof. Let I be a non-zero two-sided ideal of A_n and $D \neq 0$ be an element of smallest degree in I. If D has degree 0, it is constant, and $I = A_n$. Assume that D has degree $l > 0$.

Suppose that $(\alpha, \beta) \in {\mathbb{Z}_+}^n \times {\mathbb{Z}_+}^n$ and $|\alpha|+|\beta| = l$. If $X^{\alpha}\partial^{\beta}$ is a term in D with non-zero coefficient and $\beta_i \neq 0$, then $[X_i, D] \neq 0$ and by lemma, $[X_i, D]$ has degree $l-1$. Since I is two-sided ideal of A_n, $[X_i, D] = X_iD - DX_i \in I$, which is a contradiction to the assumption that the degree of D is the smallest. Thus $\beta = (0, \cdots, 0)$. Since $l > 0$, we have $\alpha_i \neq 0$, for some $i = 1, 2, \cdots, n$. Hence $[\partial_i, D]$ is a non-zero element of I of degree $l-1$, and once again we have a contradiction. □

Corollary 1.2.8. *Every endomorphism of A_n is injective.*

If $D \in A_n$ has inverse. Then there exists $D' \in A_n$ such that $DD' = 1$. But we have $\deg(D) + \deg(D') = 0$, which implies that $\deg(D) = 0$. Therefore the only elements of A_n that have an inverse are the constants.

A commutative simple ring must be a field but it is not true of non-commutative rings. The Weyl algebra is simple ring but it is very far from being even a division ring. Since the constants only have inverse in A_n, every non-constant operator generates a non-trivial left ideal of A_n. Therefore, A_n is not a division ring even though it does not have any non-trivial two-sided ideal.

1.3 Algebra of quantum differential operators

Let $\mathbb{Q}$ be the field of rational numbers. Suppose k is a field extension over $\mathbb{Q}$ and $q \in k$ be a transcendental element over $\mathbb{Q}$. Let A be a k-algebra which is graded by an abelian group G. That is $A = \bigoplus_{a \in G} A_a$ and $A_a A_b \subseteq A_{a+b}$.

We say that a k-linear endomorphism φ of A is homogeneous of degree $a \in G$ if for every $b \in G$, $\varphi(A_b) \subseteq A_{a+b}$. A k-linear homomorphism φ is said to be graded if it is sum of homogeneous endomorphims. The k-vector space of all graded endomorphisms of A will be denoted by $\mathrm{grHom}_k(A, A)$.

Let G be an abelian group and k be a field. A bicharacter on a group G over a field k is a group homomorphism β from $G \times G$ to $k^\times$. To each bicharacter β, we will associate a family of automorphisms of $\mathrm{grHom}_k(A, A)$ as follows: for each $a \in G$, define $\sigma_a : \mathrm{grHom}(A, A) \mapsto \mathrm{grHom}(A, A)$ such that $\sigma_a(\varphi) = \beta(a, b)\varphi$ for any homogeneous φ of degree b. This extends linearly to all of $\mathrm{grHom}_k(A, A)$. Now for $a \in G$, and for any graded endomorphisms φ_1, φ_2, we define a-twisted bracket,

$$[\varphi_1, \varphi_2]_a = \varphi_1\varphi_2 - \sigma_a(\varphi_2)\varphi_1$$

The algebra of quantum differential operators on A:

Let $M = \mathrm{grHom}(A, A)$ denote the k-vector space of all G-graded endomorphisms on A. For each $r \in A$, we have two endomorphisms λ_r and ρ_r of A given by left multiplication by r and right multiplication by r respectively. If r is homogeneous of degree a, then λ_r, ρ_r are also homogeneous of degree a and hence $\lambda_r, \rho_r \in \mathrm{grHom}(A, A)$. The k-algebra $\mathrm{grHom}(A, A)$ is an A-bimodule with $a.\varphi.b = \lambda_a\varphi\rho_b$ for a,$b \in A$ and $\varphi \in \mathrm{grHom}(A, A)$. Let $h(A)$, $h(M)$ denote the set of homogeneous elements of A and M respectively. For each $a \in A$, define σ_a by $\sigma_a(m) = \beta(a, d_m)m$ for $m \in h(M)$ of degree d_m, and extend σ_a linearly from $h(M)$ to M. For $\varphi \in \mathrm{grHom}(A, A)$ and $r \in A$, let $[\varphi, r]_a = [\varphi, \lambda_r] = \varphi\lambda_r - \sigma_a(\lambda_r)\varphi$. Let $Z_{q,0} = \mathrm{span}_k\{\varphi \in h(M) \mid \exists a \in G \text{ such that } [\varphi, r]_a = 0, \text{for all } r \in A\}$. Then $Z_{q,0}$ is an additive subgroup of $\mathrm{grHom}(A, A)$. Let $D^0_q(A)$ be the A-bimodule generated by $Z_{q,0}$. Then $D^0_q(A)$ is k-algebra. By induction, for $m \in \mathbb{Z}_+$, let $D^m_q(A)$ be the A-bimodule generated by $Z_{q,m} = \mathrm{span}_k\{\varphi \in h(M) \mid \exists a \in G \text{ such that } [\varphi, r]_a \in D^{m-1}_q(A), \text{ for all } r \in A\}$. We have $D^i_q(A)D^j_q(A) \subseteq D^{i+j}_q(A)$, for $i, j \in \mathbb{Z}_+$. Hence each $D^i_q(A)$ is a $D^0_q(A)$-module.

Let $D_q(A) = \cup_{i\geq 0} D^i_q(A)$. Then $D_q(A)$ is called the algebra of quantum differential operators on A.

Lemma 1.3.1. $D^0_q(A)$ is a k-algebra generated by the set

$$\{\lambda_r\rho_s\sigma_a \mid a \in G \text{ and } r, s \in A\}.$$

Proof. Let $\varphi \in Z^0_q$. Then, $[\varphi, \lambda_r]_a = 0$ for some $a \in G$ and for all $r \in A$. Therefore, we have $\varphi\lambda_r = \sigma_a(\lambda_r)\varphi$. Now for any $s \in A$,

$$\varphi(s) = \varphi(s.1) = \varphi\lambda_s(1) = \sigma_a(\lambda_s)\varphi(1) = \rho_{\varphi(1)}\sigma_a(s).$$

Hence, we have $\varphi = \rho_{\varphi(1)}\sigma_a$.

Let $s \in h(A)$ and $a \in G$. Since $\rho_s\lambda_r = \lambda_r\rho_s$ for any r, we have $[\rho_s\sigma_a, \lambda_r]_a = \rho_s[\sigma_a, \lambda_r]_a$. But $[\sigma_a, \lambda_r]_a = 0$. Hence $\rho_s\sigma_a \in Z^0_q$.

Therefore Z^0_q is generated as k-module by $\{\rho_s\sigma_a \mid s \text{ is homogeneous}\}$.

Since $D^0_q(A)$ is A-bimodule, $D^0_q(A)$ is spanned by $\{\lambda_r\rho_s\sigma_a\lambda_t\}$. We have $\lambda_t\sigma_a(m) = \beta(a, d_m)tm$ and $\sigma_a\lambda_t = \beta(a, d_{tm})tm$, so that $\lambda_t\sigma_a = c\sigma_a\lambda_t$ for some scalar c. Also $\rho_s\lambda_t = \lambda_t\rho_s$. Therefore $D^0_q(A)$ is generated by $\{\lambda_r\rho_s\sigma_a \mid a \in G \text{ and } r, s \in A\}$. □

Corollary 1.3.2. *The module $D^{n+1}_q(A)$ is generated over $D^0_q(A)$ by*

$$W_{q,n+1} = \{\varphi \in h(M) \mid \text{ for any } r \in A, [\varphi, \lambda_r] \in D^n_q(A)\}.$$

Proof. Clearly $W_{q,n+1} \subset Z_{q,n+1}$. Let $\varphi \in Z_{q,n+1}$. Then $[\varphi, \lambda_r]_a \in D^n_q(A)$ for some $a \in G$ and all $r \in A$. For any $b \in G$, we have

$$\begin{aligned}
[\varphi\sigma_b, \lambda_r]_{a+b} &= \varphi\sigma_b\lambda_r - \sigma_{a+b}(\lambda_r)\varphi\sigma_b \\
&= \varphi\sigma_b(\lambda_r)\sigma_b - \sigma_a(\sigma_b(\lambda_r))\varphi\sigma_b \\
&= [\varphi, \sigma_b(\lambda_r)]_a\sigma_b.
\end{aligned}$$

Since $[\varphi, \sigma_b(\lambda_r)]_a \in D_q^n(A)$, $[\varphi\sigma_b, \lambda_r]_{a+b} \in D_q^n(A)$. Since G is an abelian group, we may put $b = -a$. Then $[\varphi\sigma_{-a}, \lambda_r] \in D_q^n(A)$ for any $r \in A$. Hence $\varphi\sigma_{-a} \in W_{q,n+1}$. Therefore every element in $Z_{q,n+1}$ is in the span of $W_{q,n+1}$ over $D_q^0(A)$. □

Definition 1.3.3. Let k be a field and A be a k-algebra generated by two elements x and y. Any monomial of the form x^j or $x^j y x^t y^l$, for $j, t, l \in \mathbb{Z}^+$, is called a special monomial.

Let $k[X, X^{-1}]$ be the Laurent polynomial ring over k. For any $p \in k^\times$ and $m \in \mathbb{Z}^+$, let $[m]_p = \sum_{i=0}^{m-1} p^i$ and $[-m]_p = -p^{-m}[m]_p$. For $n \in \mathbb{Z}$, define $\varphi_p(X^n) = [n]_p X^{n-1}$. Then $\varphi_p, \varphi_q \in \mathrm{End}_k(k[X, X^{-1}])$, for $p, q \in k^\times$. Let $A_{p,q}$ be k-subalgebra of $\mathrm{End}_k(k[X, X^{-1}])$ generated by φ_p and φ_q. For $n \geq 0$, let $A_n = \{\varphi \in A_{p,q} \mid \varphi(X^m) \in kX^{m-n} \text{ for all } m \in \mathbb{Z}\}$. Then $A_{p,q} = \oplus_{n\geq 0} A_n$ is $\mathbb{N}$-graded. The degree of φ_p and the degree of φ_q is 1.

Proposition 1.3.4. *(See [JA]). The special monomials in φ_p and φ_q form a basis for $A_{p,q}$.*

Proposition 1.3.5. *Let $n \geq 0$, let F_n be the k-subspace spanned by the monomials $\varphi_p^i \varphi_q^j$ with $i + j = n$. Then $XF_{n+1} \cap A_n = 0$.*

1.4 Quantum Weyl algebras

Let $A = k[X]$ be the commutative polynomial ring in one variable X over the field k and $q \in k^{\times}$. The set of all linear transformations $\mathrm{End}(A)$ is a k-algebra with respect to the usual addition and composition of linear transformations. let λ_X, ∂ be the operators on A such that $\lambda_X(f) = Xf$ and $\partial(f) = \frac{\partial f}{\partial X}$, for any $f \in A$. Then *the algebra* $A_1^q = A\langle \lambda_X, \partial \mid \partial\lambda_X - q\lambda_X\partial = 1\rangle$ *generated by* λ_X, ∂ *subject to the relation* $\partial\lambda_X - q\lambda_X\partial = 1$, *where* 1 *stands for the identity operator, is a subalgebra of* $End_k(A)$ *and is called the quantum Weyl algebra.*

Definition 1.4.1. Let α be an automorphism of a ring R. Then the skew polynomial ring over R is a ring $A = R[X; \alpha]$ such that

1. R is a subring of A,
2. X is an element of A and $Xr = \alpha(r)X$ for all $r \in R$,
3. A is a free left R-module with basis $\{1, X, X^2, \cdots\}$.

Definition 1.4.2. Let α be an automorphism of a ring R. Then the skew-Laurent polynomial ring over R is a ring $A = R[X^{\pm 1}; \alpha]$ such that

1. R is a subring of A,
2. X is an invertible of A and $Xr = \alpha(r)X$ for all $r \in R$,
3. A is a free left R-module with basis $\{1, X^{\pm 1}, X^{\pm 2}, \cdots\}$.

Definition 1.4.3. Let k be a field and $q \in k^{\times}$. Then the quantum torus, denoted by $\mathcal{O}_q((k^{\times})^2)$, is a k-algebra generated by X, X^{-1}, Y, Y^{-1} subject to the relation $XY = qYX$.

Let $k[Y^{\pm}]$ be a Laurent polynomial ring and α be a k-algebra automorphism of $k[Y^{\pm}]$ such that $\alpha(Y) = qY$. Then $\mathcal{O}_q((k^{\times})^2) = k[Y^{\pm 1}][X^{\pm 1}; \alpha]$.

As $AX, AX^2, \cdots$ are nontrivial ideals of the skew polynomial ring $A = R[X; \alpha]$, there is no chance for A to be simple. But for a skew-Laurent polynomial ring $B = R[X^{\pm 1}; \alpha]$ to be simple, it has to satisfy some conditions as given in the theorem below. We state the theorem here without proof. For the proof, refer [GB].

Theorem 1.4.4. *Let $B = R[X^{\pm 1}; \alpha]$, where α is an automorphism of R. Then B is a simple ring if and only if R is an α-simple ring and no positive power of α is an inner automorphism of R.*

Corollary 1.4.5. *Let k be a field and $q \in k^{\times}$. Then $\mathcal{O}_q((k^{\times})^2)$ is a simple ring if and only if q is not a root of unity.*

Proposition 1.4.6. *(See [GB]). Let $A = R[X, \alpha, \delta]$ be a skew polynomial ring. Suppose that T is a ring, $\phi : R \to T$ is a ring homomorphism and $Y \in T$ is an element such that $Y\phi(r) = \phi\alpha(r)Y + \phi\delta(r)$, for all $r \in R$. Then there is a unique ring homomorphism $\psi : A \to T$ such that $\psi \mid_R = \phi$ and $\psi(X) = Y$.*

Definition 1.4.7. Let R be a ring and S be a multiplicatively closed set in R with unit element. Then S is said to be a left Ore set if it satisfies the conditions:

(i) If $rs = ts$ for $r, t \in R$ and $s \in S$, then there exists $s' \in S$ such that $s'r = s't$.

(ii) For any $r \in R$ and $s \in S$, there exists $r' \in R$, $s' \in S$ such that $s'r = r's$.

Now we have the following result (see [LR]).

Theorem 1.4.8. *Let A be a k-algebra and A' be the localization of A at a left Ore set. Then $A' \otimes_A D_q(A) \cong D_q(A')$.*

Definition 1.4.9. A normal element of a ring R is any element $r \in R$ such that $rR = Rr$.

Definition 1.4.10. The opposite of a given ring $(R, +, .)$ is the ring $(R, +, *)$ whose multiplication $*$ is defined by $a * b = b.a$ and addition is the same. We denote the opposite of the ring R by R^o.

Chapter 2

Differential operators on triangular algebras

2.1 Introduction

Let U_n be the algebra of upper triangular $n \times n$ matrices over a field k of characteristic zero. In this chapter, we describe the algebra $D_k(U_n)$ of differential operators on U_n. We give generators and relations for $D_k(U_n)$. Also we characterise all two-sided, left, right ideals and the maximal ideals of $D(U_n)$.

2.2 Differential Operators on the triangular algebra

Let R, S be two k-algebras and B be an (R, S)-bimodule. Let

$$T(R, B, S) = \left\{ \begin{pmatrix} r & b \\ 0 & s \end{pmatrix} \mid r \in R, b \in B, s \in S \right\},$$

and additions, subtractions, and multiplications in $T(R, B, S)$ are as in the matrix algebra. Let $T = T(R, B, S)$. By definition, We have

$$R \to \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}, \quad S \to \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}.$$

We define the operators $\lambda_r, \rho_r, \lambda_s, \rho_s, \lambda_b, \rho_b$ in $D_k^0(T)$ as follows:

For $r \in R$, $\lambda_r\left(\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\right) = \begin{pmatrix} rx & ry \\ 0 & 0 \end{pmatrix}$, $\rho_r\left(\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\right) = \begin{pmatrix} xr & 0 \\ 0 & 0 \end{pmatrix}$,

for $s \in S$, $\lambda_s\left(\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & sz \end{pmatrix}$, $\rho_s\left(\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\right) = \begin{pmatrix} 0 & ys \\ 0 & zs \end{pmatrix}$,

and for $b \in B$, $\lambda_b\left(\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\right) = \begin{pmatrix} 0 & bz \\ 0 & 0 \end{pmatrix}$, $\rho_b\left(\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\right) = \begin{pmatrix} 0 & xb \\ 0 & 0 \end{pmatrix}$.

Proposition 2.2.1. *The algebra $D_k^0(T)$ is generated by the set*

$$\{\lambda_r, \rho_r\}_{r \in R} \cup \{\lambda_s, \rho_s\}_{s \in S} \cup \{\lambda_b, \rho_b\}_{b \in B}.$$

Proof. Follows from Lemma 1.1.2. □

Remark 2.2.2. The following relations hold:

$\lambda_r\rho_r = \rho_r\lambda_r;$	$\lambda_r\lambda_s = \lambda_s\lambda_r = 0;$
$\lambda_r\rho_s = \rho_s\lambda_r$ on $B;$	$\lambda_r\rho_s = \rho_s\lambda_r = 0$ on $R \oplus S;$
$\lambda_r\rho_b = \rho_b\lambda_r$ on $R;$	$\lambda_r\rho_b = \rho_b\lambda_r = 0$ on $B \oplus S;$
$\lambda_r\lambda_b = \lambda_{rb};$	$\rho_r\lambda_s = 0 = \lambda_s\rho_r;$
$\lambda_s\rho_s = \rho_s\lambda_s;$	$\lambda_s\lambda_b = 0;$
$\rho_r\rho_s = \rho_s\rho_r = 0;$	$\rho_r\lambda_b = \lambda_b\rho_r = 0;$
$\rho_r\rho_b = \rho_b\rho_r = 0;$	$\lambda_s\rho_s = \rho_s\lambda_s;$
$\lambda_s\lambda_b = \lambda_b\lambda_s = 0;$	$\lambda_s\rho_b = 0;$
$\rho_s\lambda_b = \lambda_b\rho_s$ on $S;$	$\rho_s\lambda_b = \lambda_b\rho_s = 0$ on $R \oplus B;$
$\rho_s\rho_b = \rho_{bs};$	$\rho_b\rho_s = 0;$
$\lambda_b\rho_{b'} = 0 = \rho_{b'}\lambda_b.$	

Notations 2.2.1. By $t = (r, b, s) \in T$ we mean the element $t = \begin{pmatrix} r & b \\ 0 & s \end{pmatrix}$.

For any k-algebra A, by $\mathrm{Der}_k(A)$ we mean the k-vector space of derivations on A.

Lemma 2.2.3. For $\varphi \in \mathrm{Hom}(T, T)$, let $\varphi(t) = (\varphi_1(t), \varphi_2(t), \varphi_3(t))$ where $t = (r, b, s) \in T$. Then $\varphi \in \mathrm{Der}_k(T)$ if and only if

$$\varphi_1(t_1t_2) = \varphi_1(t_1)r_2 + r_1\varphi_1(t_2),$$

$$\varphi_2(t_1t_2) = \varphi_1(t_1)b_2 + r_1\varphi_2(t_2) + \varphi_2(t_1)s_2 + b_1\varphi_3(t_2),$$

$$\varphi_3(t_1t_2) = \varphi_3(t_1)s_2 + s_1\varphi_3(t_2)$$

for every $t_1 = (r_1, b_1, s_1), t_2 = (r_2, b_2, s_2) \in T$.

Proof.

$$\varphi(t_1t_2) = \begin{pmatrix} \varphi_1(t_1t_2) & \varphi_2(t_1t_2) \\ 0 & \varphi_3(t_1t_2) \end{pmatrix}$$

and we have, $t_1\varphi(t_2) + \varphi(t_1)t_2$

$$= \begin{pmatrix} r_1 & b_1 \\ 0 & s_1 \end{pmatrix} \begin{pmatrix} \varphi_1(t_2) & \varphi_2(t_2) \\ 0 & \varphi_3(t_2) \end{pmatrix} + \begin{pmatrix} \varphi_1(t_1) & \varphi_2(t_1) \\ 0 & \varphi_3(t_1) \end{pmatrix} \begin{pmatrix} r_2 & b_2 \\ 0 & s_2 \end{pmatrix}$$

$$= \begin{pmatrix} r_1\varphi_1(t_2) + \varphi_1(t_1)r_2 & r_1\varphi_2(t_2) + b_1\varphi_3(t_2) + \varphi_1(t_1)b_2 + \varphi_2(t_1)s_2 \\ 0 & s_1\varphi_3(t_2) + \varphi_3(t_1)s_2 \end{pmatrix}.$$

Hence the lemma. □

Remark 2.2.4.

- Let $\psi \in \mathrm{Der}_k(R)$ (respectively, $\psi \in \mathrm{Der}_k(S)$). If the image of ψ is contained in the annihilator of B as a left R-module (respectively, as a right S-module), then ψ can be extended to a k-linear derivation on T denoted by $\overline{\psi}$ defined as $\overline{\psi}(r, b, s) = (\psi(r), 0, 0)$ (respectively, $\overline{\psi}(r, b, s) = (0, 0, \psi(s))$) for $(r, b, s) \in T$.

- Let $\psi \in \mathrm{Hom}_{R\otimes S^o}(B, B)$; that is, ψ be (R, S)-linear map on B. Then, $\overline{\psi} \in \mathrm{Der}_k(T)$ where $\overline{\psi}(r, b, s) = (0, \psi(b), 0)$ for $(r, b, s) \in T$.

Notations 2.2.2. Any $\varphi \in \mathrm{Hom}(T,T)$ can be viewed as a matrix $\varphi = (\varphi_{i,j})_{1\le i,j\le 3}$ written

$$\varphi = \begin{matrix} \\ R \\ B \\ S \end{matrix} \begin{matrix} \begin{matrix} R & B & S \end{matrix} \\ \begin{pmatrix} \varphi_{1,1} & \varphi_{1,2} & \varphi_{1,3} \\ \varphi_{2,1} & \varphi_{2,2} & \varphi_{2,3} \\ \varphi_{3,1} & \varphi_{3,2} & \varphi_{3,3} \end{pmatrix} \end{matrix} \quad \text{where}$$

$$\varphi_{1,1} \in \mathrm{Hom}(R,R), \varphi_{1,2} \in \mathrm{Hom}(B,R), \ldots, \varphi_{3,3} \in \mathrm{Hom}(S,S).$$

The algebra $D(R)$ is the algebra of differential operators on R (here R is a left R-module); the algebra $D(S)$ is the algebra of differential operators on S where S is a left S-module; the algebra $D(B,B)$ is the algebra of differential operators on the left R-module B; the left R-module $D(R,B)$ is the left module of differential operators from R to B both viewed as left R-modules. Let A be the (R,S) bimodule generated by the set $\{\eta\lambda_b\xi \mid \eta \in D(B,B), b \in B, \xi \in D(S)\}$.

Proposition 2.2.5. *When* $\varphi \in D(T)$, $\varphi_{1,2} = \varphi_{1,3} = \varphi_{3,1} = \varphi_{3,2} = 0$.

Proof. Let W be the algebra of homomorphisms $\varphi \in \mathrm{Hom}(T,T)$ with $\varphi_{1,2} = \varphi_{1,3} = \varphi_{3,1} = \varphi_{3,2} = 0$. It is enough to show that if $[\varphi, t] \in W$ for all $t \in T$, then $\varphi \in W$. If $[\varphi, r] \in A$ for every $r \in R$, then $\varphi_{1,3} = \varphi_{3,1} = \varphi_{3,2} = 0$. Further, if $[\varphi, b] \in W$, then $\varphi_{1,2}\lambda_b : S \to S$ is the zero map for every $b \in B$. Thus, $\varphi_{1,2}\lambda_b(1) = \varphi_{1,2}(b) = 0$ for every $b \in B$. That is, $\varphi_{1,2} = 0$. □

Proposition 2.2.6. *(1)* $\varphi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \varphi_{2,3} \\ 0 & 0 & 0 \end{pmatrix} \in D(T)$ *if and only if* $\varphi_{2,3} \in A$.

(2) We have the following inclusions:

$D(R) \hookrightarrow D(T)$ *as left* R*-modules*; $D(S) \hookrightarrow D(T)$ *as left* S*-modules*;

$$\varphi_{1,1} \mapsto \begin{pmatrix} \varphi_{1,1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \varphi_{3,3} \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \varphi_{3,3} \end{pmatrix}$$

$D(B,B) \hookrightarrow D(T)$ *as left* R*-modules*; $D(R,B) \hookrightarrow D(T)$ *as left* R*-modules*;

$$\varphi_{2,2} \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & \varphi_{2,2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \varphi_{1,2} \mapsto \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{1,2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where for any $b \in B$, $\varphi_{2,2}\lambda_b - \lambda_b\varphi_{3,3} \in A$.

Proof. We first note the following commutators for $r \in R, s \in S$:

$$[\varphi, r] = \begin{pmatrix} [\varphi_{1,1}, r] & 0 & 0 \\ [\varphi_{2,1}, r] & [\varphi_{2,2}, r] & -r\varphi_{2,3} \\ 0 & 0 & 0 \end{pmatrix}; \quad [\varphi, s] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \varphi_{2,3}s \\ 0 & 0 & [\varphi_{3,3}, s] \end{pmatrix}; \tag{2.2.0.1}$$

$$[\varphi, b] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \varphi_{2,2}\lambda_b - \lambda_b\varphi_{3,3} \\ 0 & 0 & 0 \end{pmatrix}.$$

For item (1), note that for $\varphi \in D^0(T)$, since $D^0(T)$ is generated by multiplication morphisms, we have $\varphi_{2,3} = \sum \eta_i \lambda_{b_i} \xi_i$ for $\eta_i \in D^0(R), b_i \in B, \xi_i \in D^0(S)$. For general $n \geq 0$, we need to see that if $\varphi \in D^n(T)$ then $\varphi_{2,3} \in A$ with $\varphi_{2,3} = \sum_i \eta_i \lambda_{b_i} \xi_i$ for $\eta_i \in D^p(R), \xi_i \in D^q(S), p + q \leq n$. This follows from the commutator relations (2.2.0.1). □

Theorem 2.2.7. *$\varphi \in D^n(T)$ if and only if $\varphi_{1,1} \in D^n(R)$, $\varphi_{2,2} \in D^n(B, B)$, $\varphi_{3,3} \in D^n(S)$, $\varphi_{2,1} \in D^n(R, B)$, and $\varphi_{2,3} \in A$ such that $\varphi_{2,3} = \sum_i \eta_i \lambda_{b_i} \xi_i$ for $\eta_i \in D^p(R), \xi_i \in D^q(S), p + q \leq n$.*

Proof. By Proposition 2.2.6 it is enough to show that if $\varphi \in D^n(T)$ then the $\varphi_{i,j}$ have the stated properties. We see this by induction on $n \geq 0$, with $n = 0$ being the base case and using commutator equations (2.2.0.1). For general n it is enough to note that if for every $s \in S$, $\varphi_{2,3} s = \sum \eta_i \lambda_{b_i} \xi_i$ for some η_i, b_i, ξ_i dependent on s, with $\eta_i \in D^{n-1}(B, B), b_i \in B, \xi_i \in D^{n-1}(S)$ then for $s = 1$ we have $\varphi_{2,3} = \sum \eta_i \lambda_{b_i} \xi_i$ for some fixed η_i, b_i, ξ_i. □

2.3 Differential Operators on the algebra of upper triangular matrices

The algebra $U_n(k)$ is finite dimensional with basis $\{E_{ij}\}_{1\leq i\leq j\leq n}$ consisting of upper elementary matrices. For $1 \leq i \leq l \leq n$ and $1 \leq j \leq m \leq n$, let E_{il}^{jm} be the linear map on $U_n(k)$ defined by

$$E_{il}^{jm}(E_{ef}) = \begin{cases} E_{il}, & if\ (e,f) = (j,m) \\ 0, & if\ (e,f) \neq (j,m) \end{cases}$$

where $1 \leq e \leq f \leq n$.

That is, E_{il}^{jm} is the linear map on $U_n(k)$ which takes basis vector E_{jm} to E_{il} and every other basis vector to 0.

Let $a = (a_{ef})_{n\times n} \in U_n(k)$. Then, for $e > f$, $a_{ef} = 0$. As $a = \sum_{1\leq e\leq f\leq n} a_{ef}E_{ef}$, we have $E_{il}^{jm}(a) = a_{jm}E_{il}$.

Now consider

$$\begin{aligned} E_{il}^{jm}(a) &= a_{jm}E_{il} \\ &= E_{jm}aE_{il} \\ &= \lambda_{E_{ij}}\rho_{E_{ml}}(a) \end{aligned}$$

Hence $E_{il}^{jm} = \lambda_{E_{ij}}\rho_{E_{ml}} \in D_k^0(U_n(k))$, for $1 \leq i \leq j \leq m \leq l \leq n$.

Theorem 2.3.1. *Every differential operator on $U_n(k)$ is inner and is a linear combination of $\{E_{il}^{jm}\}_{1\leq i\leq j\leq m\leq l}$. That is,*

$$D_k(U_n(k)) = D_k^0(U_n(k)) = \text{ Span}\{E_{il}^{jm} \mid 1 \leq i \leq j \leq m \leq l \leq n\}.$$

Proof. Since $D_k^0(U_n(k))$ is generated by left and right multiplication homomorphisms, the second equality fallows.

We show that $D_k^1(U_n(k)) \subset \text{ Span}\{E_{il}^{jm} \mid 1 \leq i \leq j \leq m \leq l \leq n\}$.

Suppose $\varphi \in \text{Hom}(U_n(k), U_n(k))$ is such that $[\varphi, E_{ij}] \in D_k^0(U_n(k))$ and suppose $\varphi \notin \text{ Span}\{E_{il}^{jm} \mid 1 \leq i \leq j \leq m \leq l \leq n\}$. Write $\varphi = \sum \alpha_{il}^{jm} E_{il}^{jm}$ for $\alpha_{il}^{jm} \in k$.

Case $\alpha_{il}^{jm} \neq 0$ for some $i > j$: In this case, $[\varphi, E_{jj}] \in D_k^0(U_n(k))$ has a non-zero summand $\alpha_{il}^{jm} E_{il}^{jm}$ with $i > j$ which is a contradiction.

Case $\alpha_{il}^{jm} \neq 0$ for some $l < m$: Here, in case $i < j$, $[\varphi, E_{ii}] \in D_k^0(U_n(k))$ has a non-zero summand $\alpha_{il}^{jm} E_{il}^{jm}$ with $l < m$. Hence, assume that $i = j$. This gives rise to two subcases. When $i = l < m$, we see that $[\varphi, E_{ij}]$ has a non-zero summand $\alpha_{ii}^{im} E_{ii}^{mm}$. Lastly, if $i < l < m$, then $[\varphi, E_{il}]$ has a non-zero summand $\alpha_{il}^{im} E_{il}^{lm}$.

All these contradictions show that $\varphi \in D_k^1(U_n(k))$ implies that $\varphi \in D_k^0(U_n(k))$. We have thus proved the theorem. □

Lemma 2.3.2.

$$E_{il}^{jm}E_{i'l'}^{j'm'} = \begin{cases} E_{il}^{j'm'}, & \text{if } (j,m) = (i',l'), \\ 0, & \text{if } (j,m) \neq (i',l'). \end{cases}$$

Proof. Let $0 \neq a = (a_{ef}) \in U_n(k)$, then we have $a = \sum_{1 \leq e \leq f \leq n} a_{ef}E_{ef}$.
If $(j,m) = (i',l')$, then

$$\begin{aligned} E_{il}^{jm}E_{i'l'}^{j'm'}(a) &= E_{il}^{jm}E_{jm}^{j'm'}(a) \\ &= E_{il}^{jm}(a_{j'm'}E_{jm}) \\ &= a_{j'm'}E_{il} \\ &= E_{il}^{j'm'}(a) \end{aligned}$$

Now suppose $(j,m) \neq (i',l')$, then

$$E_{il}^{jm}E_{i'l'}^{j'm'}(a) = E_{il}^{jm}(a_{j'm'}E_{i'l'}) = a_{j'm'}E_{il}^{jm}(E_{i'l'}) = 0.$$

□

Remark 2.3.3.

1. We have $\lambda_{E_{ij}}^2 = 0$ when $i \neq j$, it shows that $D(U)$ is not a domain.

2. Unlike most other finite dimensional cases, the algebra of differential operators, $D(U)$, is not simple. In fact, we describe all the proper ideals of $D(U)$ in the following section.

2.4 Ideals of $D(U_n)$

Lemma 2.4.1. Let I be a two-sided ideal of $D(U)$ and $\sum a_{il}^{jm} E_{il}^{jm} \in I$. If $a_{il}^{jm} \neq 0$, then $E_{il}^{jm} \in I$.

Proof. Suppose $\sum a_{il}^{jm} E_{il}^{jm} \in I$ and $a_{ts}^{ur} \neq 0$. Now $E_{ts}^{ts} \sum a_{il}^{jm} E_{il}^{jm}$ $= \sum a_{ts}^{jm} E_{ts}^{jm} \in I$. Again consider $(\sum a_{ts}^{jm} E_{ts}^{jm}) E_{ur}^{ur} = a_{ts}^{ur} E_{ts}^{ur} \in I$. □

Proposition 2.4.2. *The left or right ideal of $D(U)$ generated by the set $\{E_{ij}^{ij} | 1 \leq i \leq j \leq n\}$ is $D(U)$.*

Proof. For any $1 \leq i \leq j \leq m \leq l \leq n$, we have $E_{il}^{jm} = E_{il}^{il} E_{il}^{jm}$ and $E_{il}^{jm} = E_{il}^{jm} E_{jm}^{jm}$. □

Corollary 2.4.3. *Any two sided ideal of $D(U)$ generated by the set $\{E_{ij}^{ij} | 1 \leq i \leq j \leq n\}$ is $D(U)$.*

Proposition 2.4.4. *Fix t, u such that $1 \leq t \leq u \leq n$ and let S be a subset of $\{E_{il}^{jm} \mid 1 \leq i \leq j \leq m \leq l \leq n\}$ not containing E_{tu}^{tu}. Then the two sided ideal generated by S is a proper ideal.*

Proof. We have

$$E_{il}^{jm} E_{i'l'}^{j'm'} = \begin{cases} E_{il}^{j'm'}, & \text{if } (j, m) = (i', l') \\ 0, & \text{otherwise} \end{cases}$$

In other words, $E_{tu}^{tu} = E_{tu}^{j_1 m_1} E_{j_1 m_1}^{j_2 m_2} \cdots E_{j_r m_r}^{tu}$ if and only if $(j_1, m_1) = (j_2, m_2) = \ldots = (j_r, m_r) = (t, u)$. That is,

$E_{tu}^{j_1m_1} = E_{j_1m_1}^{j_2m_2} = E_{j_2m_2}^{j_3m_3} = \ldots = E_{j_rm_r}^{tu} = E_{tu}^{tu}$. Now, let I be the two-sided ideal generated by S. Suppose $E_{tu}^{tu} \in I$, Then

$$E_{tu}^{tu} = \sum_{1 \leq i \leq j \leq m \leq l \leq n} a_{(i,j,m,l)} E_{il}^{jm}$$

for some E_{il}^{jm} expressed as products of elements from S. But the set $\{E_{il}^{jm} \mid 1 \leq i \leq j \leq m \leq l \leq n\}$ is linearly independent. Thus, E_{tu}^{tu} can be expressed as a product of elements from S. The only way this can happen is that $E_{tu}^{tu} \in S$, which contradicts the hypothesis. Hence I is proper two-sided ideal. □

Now we shall give characterization for left ideals of $D(U)$.

Remark 2.4.5. Suppose I is a left ideal of $D(U)$ and $\sum_{(i,j,m,l)} a_{il}^{jm} E_{il}^{jm} \in I$, then for any fixed (i,l) we see that

$$\sum_{(j,m)} a_{il}^{jm} E_{il}^{jm} = E_{il}^{il} (\sum_{(i,j,m,l)} a_{il}^{jm} E_{il}^{jm}) \in I.$$

Lemma 2.4.6. If I is a maximal left ideal of $D(U)$ and $\sum_{(ijml)} a_{il}^{jm} E_{il}^{jm} \in I$, then $E_{il}^{jm} \in I$ for $a_{il}^{jm} \neq 0$.

Proof. Let I be a maximal ideal and $\sum_{(ijml)} a_{il}^{jm} E_{il}^{jm} \in I$. By Remark 2.4.5 we may assume that $\varphi = \sum_{s=1}^{r} a_s E_{il}^{j_sm_s} \in I$ for some fixed (i,l) and $a_s \neq 0$ for all s and distinct $E_{j_sm_s}^{j_sm_s}$. We want to prove that every $E_{il}^{j_sm_s} \in I$.

If $r = 1$, then $E_{il}^{j_1m_1} \in I$. So assume that $r \geq 2$. If $E_{j_sm_s}^{j_sm_s} \in I$ for any $s, 1 \leq s \leq r$, then $\varphi E_{j_sm_s}^{j_sm_s} = a_s E_{il}^{j_sm_s} \in I$ and $\varphi - a_s E_{il}^{j_sm_s} \in I$ with fewer terms, and proceed by induction on r.

Now assume that $E_{j_s,m_s}^{j_s m_s} \notin I$ for any s, $1 \leq s \leq r$. Since $r \geq 2$, we may assume that $i < j_1$ or $l > m_1$. Suppose $E_{il}^{j_1 m_1} \notin I$.

Then by maximality of I, we have $D(U) = I + D(U)E_{il}^{j_1 m_1}$. Therefore, $E_{j_1 m_1}^{j_1 m_1} = \eta + \psi E_{il}^{j_1 m_1}$ for some $\eta \in I$ and $\psi \in D(U)$. Since $i < j_1$ or $m_1 < l$, we have $E_{j_1 m_1}^{j_1 m_1} \notin D(U)E_{il}^{j_1 m_1}$. Thus, $(E_{j_1 m_1}^{j_1 m_1} - \psi E_{il}^{j_1 m_1}) = \eta \in I$ with $\eta \neq 0$. Now note that $E_{j_1 m_1}^{j_1 m_1} = E_{j_1 m_1}^{j_1 m_1}\eta \in I$, which contradicts our assumption. Therefore, $E_{il}^{j_1 m_1} \in I$. Thus, $\varphi - a_1 E_{il}^{j_1 m_1}$ has fewer than r terms and now we proceed by induction on r. □

Notations 2.4.1. For $1 \leq t \leq u \leq n$, let S_{tu} be the left ideal generated by the set $\{E_{ij}^{ij}, E_{(t-1)u}^{tu}, E_{t(u+1)}^{tu} \mid 1 \leq i \leq j \leq n, (i,j) \neq (t,u)\}$.

Theorem 2.4.7. *Fix t, u, where $1 < t \leq u < n$. Then $\{S_{tu} \mid 1 \leq t \leq u \leq n\}$ is the set of all maximal left ideal of $D(U)$. Hence we have $\frac{n(n+1)}{2}$ distinct maximal left ideals.*

Proof. It is clear that S_{tu} is a proper left ideal of $D(U)$ and we have $E_{tu}^{tu} \notin S_{tu}$.

Now we show that S_{tu} is a maximal left ideal. Consider the pair (j,m) where $(j,m) \neq (t,u)$. Then $E_{il}^{jm} = E_{il}^{jm}E_{jm}^{jm} \in S_{tu}$ for $i \leq j$ and $l \geq m$.

Similar arguments show that $E_{il}^{tu} \in S_{tu}$ for $(i,l) \neq (t,u)$, $i \leq t$ and $l \geq u$. In other words, S_{tu} is of codimension 1 in $D(U)$. Hence it is a maximal left ideal.

Now suppose that I is any maximal left ideal. Then by Lemma 2.4.6 we see that I contains the basic vectors E_{il}^{jm} whenever $\sum_{(ijml), a_{il}^{jm} \neq 0} a_{il}^{jm} E_{il}^{jm} \in I$. In other words, $I \subset S_{tu}$ for some $t, u, 1 \leq t, u \leq n$. Since I is maximal,

$I = S_{tu}$. We have thus proved the theorem. □

Let $\mathbf{S}$ be the collection of subspaces of $D(U)$. Let $\mathbf{I}_l$, $\mathbf{I}_r$ and $\mathbf{I}$ be the collection of left, right and two-sided ideals of $D(U)$ respectively.

Theorem 2.4.8. *(a) Let $\Phi_l : \mathbf{I}_l \longrightarrow \mathbf{S}$, $\Phi_l(I) = I$, be the inclusion map. Then*

$$\Phi_l(\mathbf{I}) = \{W \in \mathbf{S} \mid \text{if } E_{il}^{jm} \in W, \text{ then } E_{i'l'}^{jm} \in W \text{ for } 1 \leq i' \leq j \leq m \leq l' \leq n\}.$$

(b) Let $\Phi_r : \mathbf{I}_r \longrightarrow \mathbf{S}$, $\Phi_l(I) = I$, be the inclusion map. Then

$$\Phi_r(\mathbf{I}) = \{W \in \mathbf{S} \mid \text{if } E_{il}^{jm} \in W, \text{ then } E_{il}^{j'm'} \in W \text{ for } 1 \leq i \leq j' \leq m' \leq l \leq n\}.$$

(c) Let $\Phi_l : \mathbf{I} \longrightarrow \mathbf{S}$, $\Phi_l(I) = I$, be the inclusion map. Then $\Phi(\mathbf{I}) = \{W \in \mathbf{S} \mid$ if $E_{il}^{jm} \in W$, then $E_{il}^{j'm'}, E_{i'l'}^{jm} \in W$ for $1 \leq i' \leq j \leq m \leq l' \leq n$ and $1 \leq j \leq j' \leq m' \leq m \leq n\}$.

Proof. Suppose $W \in \mathbf{I}_l$. Then, for $E_{i'l'}^{il} \in D_k(U)$ and $E_{il}^{jm} \in W$, $E_{i'l'}^{il} E_{il}^{jm} = E_{i'l'}^{jm} \in W$. Hence $W \in \Phi_l(\mathbf{I}_l)$. Conversely, suppose $W \in \mathbf{S}$ such that, whenever $E_{il}^{jm} \in W$, we have $E_{i'l'}^{jm} \in W$, for all $1 \leq i' \leq j \leq m \leq l' \leq n$. Then clearly it is a left ideal. Similar arguments hold for Φ_r and Φ. □

Chapter 3

Quantum Differential Operators on $k[X^{\pm 1}]$

3.1 Introduction

Let k be a field extension over the field $\mathbb{Q}$ of rational numbers and $q \in k$ be a transcendental over $\mathbb{Q}$. In this chapter, we define the algebra of quantum differential operators $D_q(k[X])$ on $k[X]$ with respect to q. We follow, [UD], [JA], [LR], to give generators and relations. Apart from the basis given in [UD], we give another basis. By [LR], the quantum differential operators on $k[X]$ can be extended to the quantum differential operators on $k[X, X^{-1}]$. We show that $D_q(k[X, X^{-1}])$ is isomorphic to its opposite algebra $D_q(k[X, X^{-1}])^o$ by giving an isomorphism.

3.2 Generators for the algebra of quantum differential operators on $k[X]$

For any $n \in \mathbb{N}$ and $m \in k^{\times}$, we define $[n]_m = 1 + m + m^2 + \cdots + m^{n-1}$. If $m \neq 1$ then $[n]_m = \frac{m^n - 1}{m-1}$ and if $m = 1$ then $[n]_m = n$.

Let $A = k[X]$ be the k-algebra of polynomials in the variable X. Then the algebra A is a $\mathbb{Z}$-graded ring with $\deg(X) = 1$. The map β: $\mathbb{Z} \times \mathbb{Z} \to k^{\times}$ defined by $\beta(n, m) = q^{nm}$ is a bicharacter. For each $i \in \mathbb{Z}$, define ∂^{β^i} by $\partial^{\beta^i}(X^n) = [n]_{q^i} X^{n-1}$, then ∂^{β^i} is a graded homomorphism from A to A. For $i \in \mathbb{Z}$, we write ∂_i for ∂^{β^i} and for $i = 0$, $\partial_0 = \partial$, the usual derivative.

Lemma 3.2.1. Suppose $\varphi \in D_q(A)$ is a homogeneous element. If $[\varphi, X] \in D_q^n(A)$, then $\varphi \in W_{q,n+1}$.

Proof. We have $[\varphi, X^m] = [\varphi, X]X^{m-1} + X[\varphi, X^{m-1}]$ for any $m > 0$. Inductively, we have that if $[\varphi, X] \in D_q^n(A)$ then $[\varphi, X^{m-1}] \in D_q^n(A)$. Hence $[\varphi, X^m] \in D_q^n(A)$. Therefore for any $r \in A$, $[\varphi, r] \in D_q^n(A)$. By Corollary 1.3.2, $\varphi \in W_{q,n+1}$. □

Lemma 3.2.2. For $\varphi \in D_q(A)$, $\exists\ i_1, i_2, \cdots, i_n \in \mathbb{Z}$, for some $n > 0$, such that

$$[\cdots[[\varphi, X]_{i_1}, X]_{i_2}, \cdots, X]_{i_n} = 0.$$

Proof. If $\varphi \in D_q^m$, then

$$\varphi = \varphi_1 + \varphi_2 + \cdots + \varphi_l$$

for homogeneous $\varphi_j \in Z_{q,m}$. To each φ_j, there exists $b_j \in \mathbb{Z}$ such that $[\varphi_j, X]_{b_j} \in D_q^{m-1}(A)$. Hence,

$$[\cdots[[\varphi, X]_{b_1}, X]_{b_2}, \cdots, X]_{b_m} \in D_q^{m-1}(A)$$

Therefore by induction, $[\cdots[[\varphi, X]_{i_1}, X]_{i_2}, \cdots, X]_{i_n} = 0$. □

Lemma 3.2.3. For any $i \in \mathbb{Z}$, the operator ∂_i is in $D_q^1(A)$.

Proof. To prove this lemma, we shall show that $[\partial_i, X] \in D_q^0(A)$. For $m \in \mathbb{Z}^+$, consider,

$$\begin{aligned} [\partial_i, X](X^m) &= \partial_i(X^{m+1}) - X\partial_i(X^m) \\ &= [m]_{q^i} X^m - X[m-1]_{q^i} X^{m-1} \\ &= q^{im} X^m \\ &= \beta(i, m) X^m \\ &= \sigma_i(X^m) \end{aligned}$$

Therefore, for any $r \in A$, we have $[\partial_i, X](r) = \sigma_i(r)$. Hence $[\partial_i, X] = \sigma_i \in D_q^0(A)$. □

For $m \in \mathbb{Z}_+$, consider the free module $\mathbb{Z}^m$ of rank m. For any $I = (i_1, i_2, \cdots, i_m) \in \mathbb{Z}^n$, we define $\partial_I = \partial_{i_1}\partial_{i_2}\cdots\partial_{i_m}$. Here we write $\mid I \mid = m$.

Corollary 3.2.4. *For any $n \geq m$, we have $\partial_I \in D_q^n(A)$.*

Proof. We know that $D_q^r(A)D_q^s(A) \subseteq D_q^{r+s}(A)$ and $\partial_{i_j} \in D_q^1(A),\ \forall i_1, \cdots, i_m$. Hence $\partial_I \in D_q^n(A)$. □

Lemma 3.2.5. For $\varphi_1, \varphi_2 \in D_q(A)$ and $i \in \mathbb{Z}$, We have

$$[[\varphi_1, \varphi_2]_i, X] = [[\varphi_1, X], \varphi_2]_i + \varphi_1[\varphi_2, X] - [\sigma_i(\varphi_2), X]\varphi_1.$$

Proof. Since $[[\varphi_1, \varphi_2]_i, X] = (\varphi_1\varphi_2 - \sigma_i(\varphi_2)\varphi_1)X - X(\varphi_1\varphi_2 - \sigma_i(\varphi_2)\varphi_1)$ and $[\sigma_i(\varphi_2), X]\varphi_1 = \sigma_i(\varphi_2)X\varphi_1 - X\sigma_i(\varphi_2)\varphi_1$. Therefore we have,

$$\begin{aligned}
[[\varphi_1, \varphi_2]_i, X] + [\sigma_i(\varphi_2), X]\varphi_1 &= \varphi_1\varphi_2X - \sigma_i(\varphi_2)\varphi_1X - X\varphi_1\varphi_2 + X\sigma_i(\varphi_2)\varphi_1 \\
&\quad +\sigma_i(\varphi_2)X\varphi_1 - X\sigma_i(\varphi_2)\varphi_1 \\
&= \varphi_1\varphi_2X - \sigma_i(\varphi_2)\varphi_1X - X\varphi_1\varphi_2 + \sigma_i(\varphi_2)X\varphi_1
\end{aligned}$$

And as $[[\varphi_1, X], \varphi_2]_i = (\varphi_1X - X\varphi_1)\varphi_2 - \sigma_i(\varphi_2)(\varphi_1X - X\varphi_1)$, and $\varphi_1[\varphi_2, X] = \varphi_1(\varphi_2X - X\varphi_2)$. We have

$$\begin{aligned}
[[\varphi_1, X], \varphi_2]_i + \varphi_1[\varphi_2, X] &= \varphi_1X\varphi_2 - X\varphi_1\varphi_2 - \sigma_i(\varphi_2)\varphi_1X \\
&\quad +\sigma_i(\varphi_2)X\varphi_1 + \varphi_1\varphi_2X - \varphi_1X\varphi_2 \\
&= -X\varphi_1\varphi_2 - \sigma_i(\varphi_2)\varphi_1X + \sigma_i(\varphi_2)X\varphi_1 + \varphi_1\varphi_2X.
\end{aligned}$$

Hence $[[\varphi_1, \varphi_2]_i, X] + [\sigma_i(\varphi_2), X]\varphi_1 = [[\varphi_1, X], \varphi_2]_i + \varphi_1[\varphi_2, X]$. □

Lemma 3.2.6. Let P be a quantum differential operator in ∂_i's, $i \in \mathbb{Z}$. For any $j \in \mathbb{Z}$, there exists a quantum differential operator Q in ∂_i's, $i \in \mathbb{Z}$ such that $[Q, X] = P\sigma_j$.

Proof. First we prove this lemma when P is a monomial in ∂_i's. Suppose that

$P = \partial_{i_n}\partial_{i_{n-1}}\cdots\partial_{i_2}\partial_{i_1}$. We shall prove the lemma by induction on n.

For $n = 1$, we have $P = 1$. Now, take $Q = \partial_j$. Then we have $[Q, X] = \sigma_j = P\sigma_j$.

Now, let us suppose the statement holds for monomials of degree $n-1$. Let

$$\begin{aligned} S_1 &= \partial_{i_n}\partial_{i_{n-1}}\cdots\partial_{i_2}, \\ S_r &= \partial_{i_{r-1}}\partial_{i_{r-2}}\cdots\partial_1\partial_n\partial_{n-1}\cdots\partial_{i_{r+1}} \quad \text{for } 2 \le r \le n-1, \\ S_n &= \partial_{i_{n-1}}\partial_{i_{n-2}}\cdots\partial_{i_1}. \end{aligned}$$

For any $r, 1 \le r \le n$, $S_r\partial_{i_r}$ is a monomial formed by P with a cyclic permutation of i_r's. We have $P = S_1\partial_{i_1} = \partial_{i_n}S_n$, and for $1 < r < n$, we have $S_r\partial_{i_r} = \partial_{i_{r-1}}S_{r-1}$.

For $s_1, s_2, \cdots, s_n \in \mathbb{Z}$, we have

$$\begin{aligned} [S_1, \partial_{i_1}]_{s_1} + q^{-s_1}[S_2, \partial_{i_2}]_{s_2} + q^{-s_1-s_2}[S_3, \partial_{i_3}]_{s_3} + & \\ \cdots + q^{-s_1-\cdots-s_{n-1}}[S_n, \partial_{i_n}]_{s_n} &= S_1\partial_{i_1} - q^{-\sum_{i=1}^n s_i}\partial_{i_n}S_n \\ &= (1 - q^{-\sum_{i=1}^n s_i})P. \end{aligned}$$

Since each S_r, for $1 \le r \le n$, is of degree n, by the induction hypothesis, there exists homogeneous monomial T_r in ∂_i's of degree n such that $[T_r, X] = S_r\sigma_j$ for some j. Now for r, $1 \le r \le n$, let $s_r = -ni_r$. Consider $T_r[\partial_{i_r}, X]$. We know that $T_r[\partial_{i_r}, X] = T_r\sigma_{i_r}$. Since $T_r\sigma_{i_r} = q^{ni_r}\sigma_{i_r}T_r$, we have $T_r[\partial_{i_r}, X] = [\sigma_{-ni_r}(\partial_{i_r}), X]T_r$. That is

$$T_r[\partial_{i_r}, X] - [\sigma_{s_r}(\partial_{i_r}), X]T_r = 0.$$

From this and from Lemma 3.2.5, we get

$$[[T_r, \partial_{i_r}]_{s_r}, X] = [S_r\sigma_j, \partial_{i_r}]_{s_r}$$

Let $Q' = [T_1, \partial_{i_1}]_{s_1} + q^{-s_1}[T_2, \partial_{i_2}]_{s_2} + q^{-s_1-s_2}[T_3, \partial_{i_3}]_{s_3} + \cdots + q^{-(s_1+\cdots+s_{n-1})}[T_n, \partial_{i_n}]_{s_n}$. Then,

$$[Q', X] = [S_1\sigma_j, \partial_{i_1}]_{s_1} + q^{-s_1}[S_2\sigma_j, \partial_{i_2}]_{s_2} + q^{-(s_1+s_2)}[S_3\sigma_j, \partial_{i_3}]_{s_3} + \cdots$$
$$+ q^{-(s_1+\cdots+s_{n-1})}[S_n\sigma_j, \partial_{i_n}]_{s_n}.$$

But $[S_r\sigma_j, \partial_{i_r}]_{s_r} = q^{-j}[S_r, \partial_{i_r}]_{s_r-j}\sigma_j$, therefore

$$[Q', X] = q^{-j}([S_1, \partial_{i_1}]_{s_1-j} + q^{-s_1}[S_2, \partial_{i_2}]_{s_2-j} + q^{-(s_1+s_2)}[S_3, \partial_{i_3}]_{s_3-j} + \cdots$$
$$+ q^{-(s_1+\cdots+s_{n-1})}[S_n, \partial_{i_n}]_{s_n-j})\sigma_j,$$

which is equal to $q^{-j}(1 - q^{-\sum(s_r-j)})P\sigma_j$.

We choose s_r so that $\sum s_r = -n\sum i_r$. So $-\sum(s_r - j) = n(j + \sum i_r)$. We show that if $j = -\sum_{i=1}^n i_r$ then every $i_r = 0$. Suppose $j = -\sum_{r=1}^n i_r$ and $i_n \neq 0$. Put $P' = \partial_{-i_n}\partial_{i_{n-1}} \cdots \partial_{i_2}\partial_{i_1}$. By (3.2.1), $\partial_{i_n} = \sigma_{i_n}\partial_{-i_n}$. Thus we have

$$P\sigma_j = \sigma_{i_n}\partial_{-i_n}\partial_{i_{n-1}} \cdots \partial_{i_2}\partial_{i_1}\sigma_j = cP'\sigma_{i_n+j}$$

for some non zero constant c. If $[Q', X] = P'\sigma_{i_n+j}$ then $[cQ', X] = P\sigma_j$. Since $i_n > 0$, we have $-(-i_n + i_{n-1} + \cdots + i_1) = 2i_n - \sum i_r \neq i_n + j$.

By assumption on j, either all $i_r = 0$ or $c = q^{-j}(1 - q^{n(j+\sum i_r)})$ is non-zero. Case(1) If some $i_r \neq 0$, then $[c^{-1}Q', X] = P$.

Case(2) If $i_r = 0$, then $P = \partial^n$. In this case, $Q = \frac{1}{n+1}\partial^{n+1}$.

Let $P = m_1 + m_2 + \cdots + m_s$ and, for $r, 1 \leq r \leq s$, m_r be a monomial in $\{\partial_i \mid i \in \mathbb{Z}\}$. Then there exists differential operators M_r such that $[M_r, X] = m_r\sigma_j$. Let $Q = M_1 + M_2 + \cdots + M_s$. Then $[Q, X] = P\sigma_j$. Therefore the result holds for $n = 0$. □

Corollary 3.2.7. *For any* $f \in A$*, we have* $[fQ, X] = fP\sigma_j$.

Recall that, for $I \in \mathbb{Z}^m$, we write $\mid I \mid = m$, for $m \in \mathbb{N}$.

Theorem 3.2.8. *The* k*-algebra* $D_q(A)$ *is generated by the set* $\{\lambda_X = X,\ \partial_{-1},\ \partial,\ \partial_1\}$.

Proof. Let R_0 be the k-module generated by $\{\sigma_i \mid i \in \mathbb{Z}\}$ and, for $n \in \mathbb{N}$, R_n be the k-module generated by $\{\partial_I\sigma_i \mid \ \mid I \mid \leq n \text{ and } i \in \mathbb{Z}\}$. Let $R = \cup_{n\geq 0} R_n$. For $1 \leq r \leq m$, we have $\sigma_i\partial_{i_r} = q^{-i}\partial_{i_r}\sigma_i$. So that we have $\sigma_i\partial_I = q^{-i|I|}\partial_I\sigma_i$. And also, from Lemma 1.3.2, $D_q^0(A)$ is a k-algebra generated by $\{\lambda_r\rho_s\sigma_a \mid a \in \mathbb{Z} \text{ and } r, s \in A\}$. Therefore, we have $D_q^0 R_n D_q^0 = AR_nA$.

To prove this theorem, we show that $AR_nA = D_q^n$.
We shall prove this by induction on n. Since $R_0 = k < \sigma_i \mid i \in \mathbb{Z} >$, we have $AR_0A = D_q^0$. Therefore it is clear for $n = 1$.

Assume that the induction hypothesis upto $n-1$, that is $AR_{n-1}A = D_q^{n-1}$. By Corollary 3.2.4, we have $\partial_I \in D_q^n$, therefore $R_n \subset D_q^n$. So, we have to show that $D_q^n \subset AR_nA$. From Corollary 1.3.2, we have $D_q^n = D_q^0 W_{q,n} D_q^0$. Since $D_q^n = D_q^0 W_{q,n} D_q^0$ and $AR_nA = D_q^0 R_n D_q^0$, it is enough to show that

$W_{q,n} \subseteq AR_nA$.

Let $\varphi \in W_{q,n}$. Then $[\varphi, X] \in D_q^{n-1} = AR_{n-1}A$. Therefore $[\varphi, X] = \sum f_jP_j\sigma_{b_j}g_j$ where $f_j, g_j \in A, P_j = \partial_{I_j}$ with $\mid I_j \mid\leq n-1$, and $b_j \in \mathbb{Z}$. From Lemma 3.2.6, there is $Q_j \in R_n$ such that $[Q_j, X] = P_j\sigma_{b_j}$. Put $Q = \sum f_jQ_jg_j$. Then $Q \in AR_nA$. Since $f_jX = Xf_j$ and $g_jX = Xg_j$,

$$\begin{aligned}
[Q, X] &= QX - XQ \\
&= \sum f_j[Q_jX - XQ_j]g_j \\
&= \sum f_j[Q_j, X]g_j \\
&= \sum f_jP_j\sigma_{b_j}g_j.
\end{aligned}$$

Hence $[\varphi - Q, X] = [\varphi, X] - [Q, X] = 0$, so that $\varphi - Q = \mu \in D_q^0 \subset AR_nA$. Hence, $\varphi = Q + \mu$, and so $\varphi \in AR_nA$.

Therefore $R = \cup R_n$ generates $D_q(A)$ over A. By the construction of R_n, it is clear that R is generated as a k-algebra by $\{\partial_i \mid i \in \mathbb{Z}\} \cup \{\sigma_i \mid i \in \mathbb{Z}\}$.

For any positive integer i, we have $\partial_i(X^n) = [n]_{q^i}X^{n-1}$. Therefore,

$$\partial_i = (\frac{1-q}{1-q^i})\partial_1[1 + \sigma_1 + \cdots + \sigma_{i-1}]$$

$$\partial_{-i} = (\frac{1-q}{1-q^{-i}})\partial_{-1}[1 + \sigma_{-1} + \cdots + \sigma_{1-i}]$$

Hence, each $\partial_i \in$ span$_k$ $(\{\sigma_i \mid i \in \mathbb{Z}\} \cup \{\partial_{-1}, \partial, \partial_1\})$.

And also $\sigma_{-1} = \partial_{-1}X - X\partial_{-1}$, $\sigma_0 = 1 = \partial X - X\partial$ and $\sigma_1 = \partial_1X - X\partial_1$. For any positive integer i, $\sigma_{-i} = (\sigma_{-1})^i$ and $\sigma_i = (\sigma_1)^i$. Hence, each σ_i is generated by X, $\partial_{-1}, \partial, \partial_1$ over k. $\square$

3.3 Relations

In this section we give full set of defining relations for $D_q(A)$, for $A = k[X]$.

For $a \in \{-1, 0, 1\}$, we have

$$\partial_a X - q^a X \partial_a = 1 \tag{3.3.0.1}$$

and for any $a, b \in \{-1, 0, 1\}$, $a \neq b$, we have

$$\partial_a X \partial_b = \partial_b X \partial_a, \tag{3.3.0.2}$$

$$\partial_{-1}\partial_1 = q\partial_1\partial_{-1}. \tag{3.3.0.3}$$

We shall show that these relations form a full set of defining relations for $D_q(A)$. Let E be the k-algebra generated by $\{\lambda_X = X, \partial_1, \partial, \partial_{-1}\}$ and I be the two sided ideal generated by $\{\partial_a X - q^a X\partial_a - 1, \partial_a X\partial_b - \partial_b X\partial_a, \partial_{-1}\partial_1 - q\partial_1\partial_{-1} \mid a, b \in \{-1, 0, 1\}\}$. Let $\bar{E}$ be the quotient E/I. Then there exists a surjective homomorphism $\theta : \bar{E} \to D_q(A)$. We shall show that this θ is an isomorphism.

Lemma 3.3.1. For $a, b \in \{1, 0, -1\}$, we have the following relations in $D_q(A)$ as well as in $\bar{E}$.

$$X(q^a\partial_a\partial_b - q^b\partial_b\partial_a) = \partial_a - \partial_b. \tag{3.3.0.4}$$

$$(q^a\partial_a\partial_b - q^b\partial_b\partial_a)X = q^a\partial_a - q^b\partial_b. \tag{3.3.0.5}$$

$$\partial_a X^n - q^{an}X^n\partial_a = [n]_{q^a}X^{n-1}. \tag{3.3.0.6}$$

$$\partial_a X - q^{an}X\partial_a^n = [n]_{q^a}\partial^{n-1}. \tag{3.3.0.7}$$

$$(q-1)X\partial_1\partial_{-1} = \partial_1 - \partial_{-1}. \tag{3.3.0.8}$$

$$q\partial_1 - \partial_{-1} = (q-1)\partial_1 X\partial_{-1}. \tag{3.3.0.9}$$

$$(q-1)\partial_1\partial_{-1}X = q\partial_1 - q^{-1}\partial_{-1}. \tag{3.3.0.10}$$

Proof. We will prove (3.3.0.4), (3.3.0.5) by using (3.3.0.2) and (3.3.0.1). To prove (3.3.0.4), consider

$$\begin{aligned} X(q^a\partial_a\partial_b - q^b\partial_b\partial_a) &= (q^aX\partial_a)\partial_b - (q^bX\partial_b)\partial_a \\ &= (\partial_aX-1)\partial_b - (\partial_bX-1)\partial_a \\ &= \partial_aX\partial_b - \partial_b - \partial_bX\partial_a + \partial_a \\ &= \partial_a - \partial_b. \end{aligned}$$

To prove (3.3.0.5), consider

$$\begin{aligned} (q^a\partial_a\partial_b - q^b\partial_b\partial_a)X &= q^a\partial_a(\partial_bX) - q^b\partial_b(\partial_aX) \\ &= q^a\partial_a(1+q^bX\partial_b) - q^b\partial_b(1+q^aX\partial_a) \\ &= q^a\partial_a - q^b\partial_b. \end{aligned}$$

We prove (3.3.0.6) by induction on n. From (3.3.0.1), we have $\partial_aX = 1 + q^aX\partial_a$, therefore it is true for $n=1$. Suppose that this is true for r, $1 \le r \le n-1$. Therefore, for $r = n-1$, we have $\partial_aX^{n-1} - q^{a(n-1)}X^{n-1}\partial_a$ $= [n-1]_{q^a}X^{n-2}$. Now we will prove it for $r = n$. We have

$$\partial_aX^{n-1} - q^{a(n-1)}X^{n-1}\partial_a = [n-1]_{q^a}X^{n-2}.$$

Multiplying the above equation with X from the right side,

$$\partial_a X^n - q^{a(n-1)} X^{n-1} \partial_a X = [n-1]_{q^a} X^{n-1}.$$

Since $\partial_a X = 1 + q^a X \partial_a$, $\partial_a X^n - q^{a(n-1)} X^{n-1}(1 + q^a X \partial_a) = [n-1]_{q^a} X^{n-1}$. which implies, $\partial_a X^n - q^{an} X^n \partial_a = q^{a(n-1)} X^{n-1} + [n-1]_{q^a} X^{n-1}$.

Therefore,

$$\partial_a X^n - q^{an} X^n \partial_a = [n]_{q^a} X^{n-1}.$$

Similarly, (3.3.0.7).

From (3.3.0.4), for $a = 1$, $b = -1$, $X(q\partial_1\partial_{-1} - q^{-1}\partial_{-1}\partial_1) = \partial_1 - \partial_{-1}$ and from (3.3.0.3), $\partial_{-1}\partial_1 = q\partial_1\partial_{-1}$. Therefore we have

$(q-1)X\partial_1\partial_{-1} = \partial_1\partial_{-1}$, which is (3.3.0.8).

To prove (3.3.0.9), consider $(q-1)\partial_1 X \partial_{-1}$. But $\partial_1 X = 1 + qX\partial_1$ from (3.3.0.1) for $a = 1$, therefore $(q-1)\partial_1 X \partial_{-1} = (q-1)(1 + qX\partial_1)\partial_{-1} = (q-1)\partial_{-1} + q(q-1)X\partial_1\partial_{-1}$. And from (3.3.0.8), we have $(q-1)X\partial_1\partial_{-1} = \partial_1 - \partial_{-1}$, hence $q\partial_1 - \partial_{-1} = (q-1)\partial_1 X \partial_{-1}$.

From (3.3.0.5), for $a = 1$, $b = -1$, $(q\partial_1\partial_{-1} - q^{-1}\partial_{-1}\partial_1)X = q\partial_1 - q^{-1}\partial_{-1}$ and from (3.3.0.3), $\partial_{-1}\partial_1 = q\partial_1\partial_{-1}$. Hence $(q-1)\partial_1\partial_{-1}X = q\partial_1 - q^{-1}\partial_{-1}$, which is (3.3.0.10). □

Proposition 3.3.2. *If* $\partial_a X - q^a X \partial_a = 1$, $\partial_a X \partial_b = \partial_b X \partial_a$, $q\partial_1 - \partial_{-1} = (q-1)\partial_1 X \partial_{-1}$, *for* $a,\ b \in \{-1,\ 0,\ 1\}$, *then* $\partial_{-1}\partial_1 = q\partial_1\partial_{-1}$.

Proof. From (3.3.0.9), $\partial_{-1} = q\partial_1 - (q-1)\partial_1 X \partial_{-1}$ and from (3.3.0.2), $\partial_{-1} X \partial_1 = \partial_1 X \partial_{-1}$ for $a = -1,\ b = 1$. Therefore $\partial_{-1} = q\partial_1 - (q-1)\partial_{-1} X \partial_1$. From

(3.3.0.1), for $a = -1$, we have $\partial_{-1}X = 1 + q^{-1}X\partial_{-1}$. And by substituting this in $\partial_{-1} = q\partial_1 - (q-1)\partial_{-1}X\partial_1$, we get

$$\partial_{-1} = (1 + (q^{-1} - 1)X\partial_{-1})\partial_1 \tag{3.3.0.11}$$

Then, multiplying with ∂_1 from left side and applying (3.3.0.2), with $a = 1$ and $b = -1$,

$$\partial_1\partial_{-1} = (\partial_1 + (q^{-1} - 1)\partial_1 X\partial_{-1})\partial_1 = (1 + (q^{-1} - 1)\partial_{-1}X)\partial_1^2. \tag{3.3.0.12}$$

Now apply (3.3.0.1) with $a = -1$ and (3.3.0.11), to get (3.3.0.3). □

In this proposition, we derived the relation (3.3.0.3) by using the relations (3.3.0.1), (3.3.0.2) and (3.3.0.9). Therefore we can replace (3.3.0.3) by (3.3.0.9).

Let $\chi_a = 1 + (q^a - 1)X\partial_a$ for $a = 1, \; -1$.

$$\begin{aligned}
\chi_a &= 1 + (q^a - 1)X\partial_a \\
&= \partial_a X - X\partial_a \qquad \text{(by (3.3.0.1))} \\
&= \partial_a X + q^{-a}(1 - \partial_a X) \qquad \text{(by (3.3.0.1))} \\
&= q^{-a}(1 + (q^a - 1)\partial_a X).
\end{aligned}$$

Lemma 3.3.3. (i) χ_1, χ_{-1} are inverses of each other in $D_q(A)$ and in $\bar{E}$.

(ii) We have the following relations:

$$\partial_{-1} = \chi_{-1}\partial_1, \tag{3.3.0.13}$$

$$\partial_a\chi_b = q^b\chi_b\partial_a \quad a = -1,\ 0,\ 1,\ b = 1,\ -1. \tag{3.3.0.14}$$

Proof. (i) We know that $\chi_1 = 1 + (q-1)X\partial_1$ and $\chi_{-1} = 1 + (q^{-1}-1)X\partial_{-1}$.

$$\begin{aligned}\chi_1\chi_{-1} &= (1 + (q-1)X\partial_1)(1 + (q^{-1}-1)X\partial_{-1}) \\ &= 1 + (q^{-1}-1)X\partial_{-1} + (q-1)X\partial_1 + (q-1)(q^{-1}-1)X\partial_1 X\partial_{-1}.\end{aligned}$$

From (3.3.0.1), $\partial_1 X = 1 + qX\partial_1$,

$$\begin{aligned}\chi_1\chi_{-1} = 1 + (q^{-1}-1)X\partial_{-1} + (q-1)X\partial_1 + (q-1)(q^{-1}-1)X\partial_{-1} \\ + q(q-1)(q^{-1}-1)X(X\partial_1\partial_{-1}).\end{aligned}$$

From (3.3.0.8), $(q-1)X(\partial_1\partial_{-1}) = \partial_1 - \partial_{-1}$, we have

$$\begin{aligned}\chi_1\chi_{-1} = 1 + (q^{-1}-1)X\partial_{-1} + (q-1)X\partial_1 + (q-1)(q^{-1}-1)X\partial_{-1} \\ + q(q^{-1}-1)X(\partial_1 - \partial_{-1}) = 1.\end{aligned}$$

Similarly, $\chi_{-1}\chi_1 = 1$.

(ii) From (3.3.0.8), $\partial_{-1} = \partial_1 - (q-1)X\partial_1\partial_{-1}$ and from (3.3.0.3),

$\partial_1\partial_{-1} = q^{-1}\partial_{-1}\partial_1$.

Therefore, we have $\partial_{-1} = \partial_1 - q^{-1}(q-1)X\partial_{-1}\partial_1 = \chi_{-1}\partial_1$.

We know that $\partial_a \chi_b = \partial_a + (q^b - 1)\partial_a X \partial_b$ and, from (3.3.0.1), $\partial_a X \partial_b = \partial_b X \partial_a$. Therefore

$$\begin{aligned}
\partial_a \chi_b &= (1 + (q^b - 1)\partial_b X)\partial_a \\
&= [1 + (q^b - 1)(1 + q^b X \partial_b)]\partial_a \quad \text{from (3.3.0.1)} \\
&= q^b[1 + (q^b - 1)X\partial_b]\partial_a \\
&= q^b \chi_b \partial_a.
\end{aligned}$$

□

Lemma 3.3.4. For $a = 1, -1$, the following homogeneous relations hold in $D_q(A)$ and in $\bar{E}$.

$$q^{2a}\partial_a^2\partial + \partial^2\partial_a - (q^a + 1)\partial\partial_a\partial + \partial\partial_a^2 - 2q^a\partial_a\partial\partial_a = 0 \tag{3.3.0.15}$$

Proof. By (3.3.0.4), we have

$$\begin{aligned}
(\partial\partial_a - q^a\partial_a\partial)X(\partial\partial_a - q^a\partial_a\partial) &= (\partial\partial_a - q^a\partial_a\partial)(\partial - \partial_a) \\
&= \partial\partial_a\partial - q^a\partial_a\partial^2 - \partial\partial_a^2 + q^a\partial_a\partial\partial_a.
\end{aligned}$$

And, by (3.3.0.5),

$$\begin{aligned}
(\partial\partial_a - q^a\partial_a\partial)X(\partial\partial_a - q^a\partial_a\partial) &= (\partial - q^a\partial_a)(\partial\partial_a - q^a\partial_a\partial) \\
&= \partial^2\partial_a - q^a\partial\partial_a\partial - q^a\partial_a\partial\partial_a + q^{2a}\partial_a\partial.
\end{aligned}$$

Therefore, we will have

$$q^{2a}\partial_a^2\partial + \partial^2\partial_a - (q^a + 1)\partial\partial_a\partial + \partial\partial_a^2 - 2q^a\partial_a\partial\partial_a = 0$$

□

As $\chi_1 = 1 + (q-1)X\partial_1$, it is clear that

$$\chi_1 X = qX\chi_1. \tag{3.3.0.16}$$

Definition 3.3.5. Any monomial in ∂_1 and ∂ of the form ∂_1^j or $\partial_1^j \partial \partial_1^t \partial^l$ where $j,\ t,\ l \in \mathbb{Z}^+$, is called a special monomial.

Thus the special monomial is either a standard monomial $\partial_1^d \partial^e$, d, $e \geq 0$, or can be obtained from such a monomial by moving one occurence of ∂ to the left of a power of ∂_1. By Proposition 1.3.4, this special monomials in ∂ and ∂_1 form a basis for the subalgebra of $D_q(A)$ generated by ∂ and ∂_1. Hence the number of special monomials of total degree n in ∂ and ∂_1 is $\frac{1}{2}(n^2 + n + 2)$, and $n+1$ of these are standard.

Lemma 3.3.6. $\chi_{-1}\partial^n = q^n \partial^n \chi_{-1}$.

Proof. We prove the lemma by induction on n. From (3.3.0.14), $\partial\chi_{-1} = q^{-1}\chi_{-1}\partial$. Therefore it is true for $n = 1$. Suppose that it is true for all r, $1 \leq r \leq n-1$. Then, for $r = n-1$, we have

$$\chi_{-1}\partial^{n-1} = q^{n-1}\partial^{n-1}\chi_{-1}.$$

Multiplying with ∂ from right side,

$$\chi_{-1}\partial^n = q^{n-1}\partial^{n-1}(\chi_{-1}\partial).$$

From (3.3.0.14), for $a = 0$ and $b = -1$,

$$\chi_{-1}\partial^n = q^n \partial^n \chi_{-1}.$$

$\square$

Lemma 3.3.7. We have the following relations in both $D_q(A)$ and $\bar{E}$:

$$\partial_{-1}\partial\partial_1 = q^2\partial_1\partial\partial_{-1} \tag{3.3.0.17}$$

$$\partial_{-1}\partial^n\partial_1 = q^{n+1}\partial_1\partial^n\partial_{-1} \quad \text{for} \quad n \geq 0 \tag{3.3.0.18}$$

$$\partial\partial_{-1} = \partial\partial_1 - q\partial_1\partial + (q-1)\partial_1\partial_{-1} + q^{-1}\partial_{-1}\partial \tag{3.3.0.19}$$

$$\partial\partial_1\partial_{-1} = q^{-1}\partial_1\partial_{-1}\partial + q^{-1}\partial\partial_1^2 - q\partial_1^2\partial - 2(1-q)\partial_1^2\partial_{-1} \tag{3.3.0.20}$$

Proof. By multiplying (3.3.0.13) with $\partial\partial_1$, $\partial_{-1}\partial\partial_1 = \chi_{-1}\partial_1\partial\partial_1$. Applying (3.3.0.14) for $a = 1$ and $b = -1$, and again for $a = 0$ and $b = -1$, $\partial_{-1}\partial\partial_1 = q^2\partial_1\partial\chi_{-1}\partial_1$. Now from (3.3.0.13), $\partial_{-1} = \chi_{-1}\partial_1$, we have $\partial_{-1}\partial\partial_1 = q^2\partial_1\partial\partial_{-1}$ which is (3.3.0.17).

(3.3.0.18) is generalization of (3.3.0.17). Consider

$$\begin{aligned}
\partial_{-1}\partial^n\partial_1 &= \chi_{-1}\partial_1\partial^n\partial_1 && \text{(from (3.3.0.13))}\\
&= q\partial_1\chi_{-1}\partial^n\partial_1 && \text{((3.3.0.14) for } a=1 \text{ and } b=-1)\\
&= q^{n+1}\partial_1\partial^n\chi_{-1}\partial_1 && \text{(from Lemma 3.3.6)}\\
&= q^{n+1}\partial_1\partial^n\partial_{-1}. && \text{(from (3.3.0.13))}
\end{aligned}$$

By multiplying from left side (3.3.0.8) with ∂ and then applying (3.3.0.3) and (3.3.0.2),

$$(1-q^{-1})\partial_{-1}X\partial\partial_1 = \partial\partial_1 - \partial\partial_{-1}. \tag{3.3.0.21}$$

By multiplying from right side (3.3.0.10) with ∂ and applying (3.3.0.3) and (3.3.0.2),

$$(1-q^{-1})\partial_{-1}\partial X\partial_1 = q\partial_1\partial - q^{-1}\partial_{-1}\partial. \tag{3.3.0.22}$$

By subtracting (3.3.0.21) from (3.3.0.22), we have

$$\partial\partial_{-1} = \partial\partial_1 - q\partial_1\partial + q^{-1}\partial_{-1}\partial + (1-q^{-1})\partial_{-1}(\partial X - X\partial)\partial_1.$$

But $\partial X - X\partial = 1$ and, from (3.3.0.3), $\partial_{-1}\partial_1 = \partial_1\partial_{-1}$, which yields (3.3.0.19).

From (3.3.0.3), we have $\partial_1\partial_{-1} = q^{-1}\partial_{-1}\partial_1$. By multiplying this with ∂ from left side, $\partial\partial_1\partial_{-1} = q^{-1}\partial\partial_{-1}\partial_1$. But, from (3.3.0.19), $\partial\partial_{-1} = \partial\partial_1 - q\partial_1\partial + (q-1)\partial_1\partial_{-1} + q^{-1}\partial_{-1}\partial$. Therefore,

$$\begin{aligned}\partial\partial_1\partial_{-1} &= q^{-1}[\partial\partial_1 - q\partial_1\partial + (q-1)\partial_1\partial_{-1} + q^{-1}\partial_{-1}\partial]\partial_1\\ &= q^{-1}\partial\partial_1^2 - \partial_1\partial\partial_1 + (1-q^{-1})\partial_1\partial_{-1}\partial_1 + q^{-2}\partial_{-1}\partial\partial_1.\end{aligned}$$

From (3.3.0.19), we have $\partial\partial_1 = \partial\partial_{-1} + q\partial_1\partial - (q-1)\partial_1\partial_{-1} - q^{-1}\partial_{-1}\partial_1$. Therefore

$$\begin{aligned}\partial\partial_1\partial_{-1} &= q^{-1}\partial\partial_1^2 - \partial_1[\partial\partial_{-1} + q\partial_1\partial - (q-1)\partial_1\partial_{-1} - q^{-1}\partial_{-1}\partial_1]\\ &\quad +(1-q^{-1})\partial_1\partial_{-1}\partial_1 + q^{-2}\partial_{-1}\partial\partial_1\\ &= q^{-1}\partial\partial_1^2 - \partial_1\partial\partial_{-1} - q\partial_1^2\partial + (q-1)\partial_1^2\partial_{-1} + q^{-1}\partial_1\partial_{-1}\partial_1\\ &\quad +(1-q^{-1})\partial_1\partial_{-1}\partial_1 + q^{-2}\partial_{-1}\partial\partial_1\\ &= q^{-1}\partial\partial_1^2 - q\partial_1^2\partial - 2(1-q)\partial_1^2\partial_{-1} + q^{-1}\partial_1\partial_{-1}\partial\\ &\qquad\qquad \text{(from, (3.3.0.3) and (3.3.0.17)),}\end{aligned}$$

which is Equation (3.3.0.20). □

From (3.3.0.19) and (3.3.0.3), we will have

$$\partial\partial_1 - q\partial_1\partial + q\partial_1\partial_{-1} - q^{-1}\partial_{-1}\partial_1 + q^{-1}\partial_{-1}\partial - \partial\partial_{-1} = 0. \qquad (3.3.0.23)$$

3.4 An isomorphism of $D_q(k[X^{\pm 1}])$, $D_q(k[X^{\pm 1}])^o$

Let H be the subalgebra of $D_q(A)$, where $A = k[X]$, generated by ∂, ∂_1 and ∂_{-1}. The operators ∂, ∂_1 and ∂_{-1} as on $k[X]$ each reduce degree in X by one, H is graded with ∂, ∂_1 and ∂_{-1} having degree 1. By defining $\partial_a(X^{-n}) = [-n]_{q^a}X^{-n-1} = -q^{-na}[n]_{q^a}X^{-n-1}$, for $a \in \{-1,\ 0,\ 1\}$, the actions of ∂, ∂_1 and ∂_{-1} on $k[X]$ can be extended to actions on $k[X, X^{-1}]$. Let S be the subalgebra of $\mathrm{End}_k(k[X,\ X^{-1}])$ generated by ∂, ∂_1, ∂_{-1} and X^{-1}. Then $H \hookrightarrow S$. And let T be the subalgebra of S generated by ∂_1, ∂_{-1} and X^{-1}.

By D_X, we denote the subalgebra of $End_k(k[X, X^{-1}])$ generated by ∂, ∂_1, ∂_{-1}, X and X^{-1}. As $\chi_a = 1 + (q^a - 1)X\partial_a$, $a = 1,\ -1$, D_X is generated by X, X^{-1}, ∂, χ_1 and χ_{-1}. By (3.3.0.14), we have $\chi_1\partial = q^{-1}\partial\chi_1$ and, from (3.3.0.16), $\chi_1 X = qX\chi_1$. Therefore, D_X is a homomorphic image of the skew polynomial ring $(A_1)_X[Y^{\pm 1}, \alpha]$, with $Y \mapsto \chi_1$, where A_1 is the first Weyl algebra generated by ∂ and X, $(A_1)_X$ is its localization at the powers of X and α is the k-automorphism of $(A_1)_X$ such that $\alpha(X) = qX$ and $\alpha(\partial) = q^{-1}\partial$. Here $(A_1)_X$ is simple with group of units $\{\lambda X^i : i \in \mathbb{Z},\ \lambda \in k^\times\}$ and α is not inner. From Theorem 1.4.4, $(A_1)_X[Y^{\pm 1}, \alpha]$ is simple. And whence $D_X \simeq (A_1)_X[Y^{\pm}, \alpha]$. Therefore D_X and its subalgebras $D_q(A)$, H, S and T, are domains and that D_X is simple.

D_X is the localization of $D_q(A)$ at powers of X which, as a consequence of (3.3.0.1), form an ore set in $D_q(A)$. Also D_X is the localization of S at the powers of X^{-1}.

Let $\mathcal{B}_1 = \{\partial_1^j \partial_{-1}^t \partial^l : j,\ t,\ l \geq 0\} \cup \{\partial_1^j \partial \partial_1^t \partial^l : j,\ l \geq 0,\ t > 0\}$. Thus $\mathcal{B}_1$ consists of the standard monomials $\partial_1^j \partial_{-1}^t \partial^l$ in ∂_1, ∂_{-1} and ∂ and the non-standard special monomials in ∂_1 and ∂.

Let $\mathcal{B}_2 = \mathcal{B}_1 \cup \{X^j \partial_1^t \partial^l : j > 0,\ t,\ l \geq 0\} \cup \{X^j \partial_{-1}^t \partial^l : j > 0,\ t,\ l \geq 0\}$ and let $\mathcal{B}_3$ be the corresponding set of monomials in $\bar{E}$. Thus in addition to the monomials in $\mathcal{B}_1$, $\mathcal{B}_2$ contains those standard monomials in X, ∂_1 and ∂ or in X, ∂_{-1} and ∂ having a positive power of X.

Let $\delta_0 = -\partial$, $\delta_1 = -q^{-1}\partial_{-1}$, $\delta_{-1} = -q\partial_1$. Let $\mathcal{B'}_1 = \{\delta_1^j \delta_{-1}^t \delta_0^l \mid j,\ t,\ l \geq 0\} \cup \{\delta_1^j \delta_0 \delta_1^t \delta_0^l \mid j,\ l \geq 0,\ t > 0\}$. Then we have $\mathcal{B'}_1 = \{(-1)^{j+t+l} q^{-j+t} \partial_{-1}^j \partial_1^t \partial_0^l \mid j,\ t,\ l \geq 0\} \cup \{(-1)^{j+t+2} q^{-j-t} \partial_{-1}^j \partial_0 \partial_{-1}^t \partial_0^l \mid j,\ l \geq 0,\ t > 0\}$. Let $\mathcal{B'}_2 = \mathcal{B'}_1 \cup \{X^j \delta_1^t \delta_0^l \mid j > 0,\ t,\ l \geq 0\} \cup \{X^j \delta_{-1}^t \delta_0^l \mid j > 0,\ t,\ l \geq 0\}$, then $\mathcal{B'}_2 = \mathcal{B'}_1 \cup \{(-1)^{t+l} q^{-t} X^j \partial_{-1}^t \partial^l \mid j > 0,\ t,\ l \geq 0\} \cup \{(-1)^{t+l} q^t X^j \partial_1^t \partial^l \mid j > 0,\ t,\ l \geq 0\}$. Let H' be a subalgebra of $D_q(A)$ generated by the set $\{\delta_0,\ \delta_1,\ \delta_{-1}\}$. Let S' be the subalgebra $\mathrm{End}_k(k[X,\ X^{-1}])$ generated by $\{\delta_0,\ \delta_1,\ \delta_{-1},\ X^{-1}\}$ and T' be the subalgebra of S' generated by $\{\delta_1,\ \delta_{-1},\ X^{-1}\}$.

Let $\chi'_a = 1 + (q^a - 1)X\delta_a$, for $a = 1,\ -1$. Then we have

$$\chi'_1 = 1 + (q-1)X\delta_1 = 1 + (q-1)X(-q^{-1}\partial_{-1}) = 1 + (q^{-1} - 1)X\partial_{-1} = \chi_{-1},$$

$$\chi'_{-1} = 1 + (q^{-1} - 1)X\delta_{-1} = 1 + (q-1)X(-q\partial_1) = 1 + (q-1)X\partial_1 = \chi_1.$$

Definition 3.4.1. Let $h = f_1 f_2 \cdots f_r$ be a monomial of degree r in ∂_a and ∂_b. The complexity of h, comp(h), with respect to ∂_a and ∂_b is the number of pairs $(i,\ j)$ with $1 \leq i < j \leq r$ such that $f_i = \partial_b$ and $f_j = \partial_a$.

The complexity of stardard monomials is 0.

Lemma 3.4.2. Let M be the graded subalgebra generated by any two operators ∂_a and ∂_b, for $a,\ b \in \{1,\ 0,\ -1\}$. And let N be the subspace of M spanned by the standard monomials in ∂_a and ∂_b, then $XM_{n+1} \subseteq XN_{n+1} \oplus M_n$, for $n \geq 0$.

Proof. Suppose M is the graded subalgebra generated by ∂_a, ∂_b and N be the subspace of M spanned by the standard monomials in ∂_a and ∂_b. By Proposition 1.3.5, $XN_{n+1} \cap M_n = 0$.

Now we have to show that $XM_{n+1} \subseteq XN_{n+1} \oplus M_n$,for $n \geq 0$. We will prove it by induction on n. This is clear for $n = 0$ and from the identity (3.3.0.4), it is also true for $n = 1$.

Now we will prove it for the general case. Let h be a monomial of degree $n + 1$ with comp$(h) > 0$. So $h = \partial_a^i \partial_b^j \partial_a^l h'$ for some $i \geq 0$, $j > 0$ and $l > 0$, and for some monomial h'. Then,

$$\begin{aligned}
Xh &= X\partial_a^i\partial_b^j\partial_a^l h' \\
&= q^{-ai}q^{-bj}\partial_a^i\partial_b^j X\partial_a^l h' - q^{-ai}q^{-bj}[j]_{q^b}\partial_a^i\partial_b^{j-1}\partial_a^l h' - q^{-ai}[i]_{q^a}\partial_a^{i-1}\partial_b^j\partial_a^l h' \\
&\qquad\qquad \text{by (3.3.0.7)} \\
&\equiv q^{-ai}q^{-bj}\partial_a^i\partial_b^j X\partial_a^l h'(\mod M_n) \\
&\equiv q^{-ai}q^{-bj}\partial_a^i\partial_b^{j-1}\partial_a X\partial_b\partial_a^{l-1}h'(\mod M_n) \qquad \text{by (3.3.0.2))} \\
&\equiv q^a q^{-b} X\partial_a^i\partial_b^{j-1}\partial_a\partial_b\partial_a^{l-1}h'(\mod M_n) \qquad \text{by (3.3.0.7))}
\end{aligned}$$

so that, $\mathrm{comp}(h) > \mathrm{comp}(\partial_a^i\partial_b^{j-1}\partial_a\partial_b\partial_a^{l-1})$. Therefore, by induction on $\mathrm{comp}(h)$, $XM_{n+1} \subseteq XN_{n+1} \oplus M_n$. □

Note that if $U_n = XM_{n+1} \cap M_n$, then $XM_{n+1} = XN_{n+1} \oplus U_n$.

Lemma 3.4.3. H, $D_q(A)$, $\bar{E}$ are spanned by $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ respectively.

Proof. (i) Let F_n, $n \geq 0$, be the subspace of H_n spanned by $H_n \cap \mathcal{B}_1$. Now we show that $H_n = F_n$ by induction on n. Certainly it is true for $n = 0, 1$. Let $n > 1$ and suppose that $H_{n-1} = F_{n-1}$. We have to show that $gH_{n-1} \subseteq F_n$, for all $g \in \{\partial_{-1},\ \partial,\ \partial_1\}$. It is enough to show that $gh \in F_n$ if $h = \partial_1^j\partial_{-1}^t\partial^l \in H_{n-1}$ is standard or if $h = \partial_1^j\partial\partial_1^t\partial^l \in H_{n-1}$ is special but not standard ($t \neq 0$). Clearly it is true for $g = \partial_1$.

Consider the case, $g = \partial_{-1}$. Suppose $h = \partial_1^j\partial_{-1}^t\partial^l$. If $j = 0$, then $\partial_{-1}h = \partial_{-1}^{t+1}\partial^l$ is standard and otherwise, by (3.3.0.3),

$$\partial_{-1}h = \partial_{-1}\partial_1^j\partial_{-1}^t\partial^l = q^j\partial_1(\partial_1^{j-1}\partial_{-1}^{t+1}\partial^l) \in G_{n-1} \subseteq F_n.$$

Let $h = \partial_1^j \partial \partial_1^t \partial^l$, $t \neq 0$, be special but not standard. If $j = 0$ then, by (3.3.0.17), $\partial_{-1} h = \partial_{-1}(\partial \partial_1^t \partial^l) = q^2 \partial_1(\partial \partial_{-1} \partial_1^{t-1} \partial^l) \in \partial_1 H_{n-1} \subseteq F_n$, since $\partial \partial_{-1} \partial_1^{t-1} \partial^l$ is a monomial in ∂, ∂_{-1}, ∂_1 of degree $(n-1)$. If $j \neq 0$ then, by (3.3.0.3), $\partial_{-1} h = q \partial_1 (\partial_{-1} \partial_1^{j-1} \partial \partial_1^t \partial^l) \in \partial_1 H_{n-1} \subseteq F_n$.

Finally consider the case, where $g = \partial$. By Proposition 1.3.4, the special monomials in ∂, ∂_1 form a k-basis for the k-algebra generated by ∂, ∂_1. If h is special then ∂h is a linear combination of special monomials. Now we have to show that $\partial \partial_1^j \partial_{-1}^t \partial^l \in F_n$, for $j, t, l \geq 0$. If $t = 0$, $\partial \partial_1^j \partial^l$ is special. Let $t > 0$ and suppose, inductively, that $\partial \partial_1^j \partial_{-1}^d \partial^l \in F_n$ whenever $d < t$.

If $j = 0$ then, by (3.3.0.19),

$$\begin{aligned} \partial \partial_{-1}^t \partial^l &= (\partial \partial_1 - q \partial_1 \partial + (q-1) \partial_1 \partial_{-1} + q^{-1} \partial_{-1} \partial) \partial_{-1}^{t-1} \partial^l \\ &\in \partial_{-1} H_{n-1} + \partial_1 H_{n-1} + t \partial \partial_1 \partial_{-1}^{t-1} \partial^l \subseteq F_n. \end{aligned}$$

If $j \neq 0$ then, by (3.3.0.3) and (3.3.0.20),

$$\begin{aligned} \partial \partial_1^j \partial_{-1}^t \partial^l &= \partial \partial_1 (\partial_1^{j-1} \partial_{-1}) \partial_{-1}^{t-1} \partial^l \\ &= \partial \partial_1 (q^{-(j-1)} \partial_{-1} \partial_1^{j-1}) \partial_{-1}^{t-1} \partial^l \\ &= q^{-(j-1)} (q^{-1} \partial_1 \partial_{-1} \partial + q^{-1} \partial \partial_1^2 - q \partial_1^2 \partial - 2(1-q) \partial_1^2 \partial_{-1}) \partial_1^{j-1} \partial_{-1}^{t-1} \partial^l \\ &\in \partial_1 H_{n-1} + t \partial \partial_1^{j+1} \partial_{-1}^{t-1} \partial^l \subseteq F_n. \end{aligned}$$

Therefore $H_n = F_n$, and hence $\mathcal{B}_1$ spans H.

(ii) Let D^n, $n \geq 0$, be the subspace of $D_q(A)$ spanned by the monomials of degree $\leq n$ in $X, \partial, \partial_1, \partial_{-1}$. Thus $\{D^n : n \geq 0\}$ is a filtration of $D_q(A)$ and $D^n = D^1 D^{n-1}$. Let W be the subspace spanned by the standard monomials $\partial_1^t\partial^l$ in ∂_1, ∂ and the standard monomials $\partial_{-1}^t\partial^l$ in ∂_{-1}, ∂. Let $K = \sum_{n\geq 1}^{\infty} X^n W + H$. We will show that $D_q(A) = K$. It is enough to show that $mK \subseteq K$ for all $m \in \{X,\ \partial,\ \partial_1,\ \partial_{-1}\}$, then it will follow, inductively, that $D^n \subseteq K$ for all n.

First consider the case where $h = X$. It is clear that $X(\sum_{n\geq 1}^{\infty} X^n W) \subseteq K$, so it suffices to show that $XH \subseteq XW + H$. Let M be the graded subalgebra generated by ∂_1, ∂ and let N be the subspace of M spanned by the standard monomials in ∂_1 and ∂. Then, for $n \geq 0$, $XM_{n+1} \subseteq XN_{n+1} \oplus M_n$, whence $XM \subseteq XN \oplus M$. If m is a special in ∂_1 and ∂ then $m \in M$, and also we have $N \subseteq W$, so $Xm \in XN \oplus M \subseteq XW + H$. Now let $m = \partial_1^j\partial_{-1}^t\partial^l$ be a standard monomial. If $t = 0$ then m is special and $Xm \in XW + H$. Replacing q by q^{-1}, the same is true if $j = 0$. Now assume that $j \neq 0$ and $t \neq 0$. By (3.3.0.3) and (3.3.0.8),

$$\begin{aligned} Xm &= X\partial_1^j\partial_{-1}^t\partial^l \\ &= q^{-(j-1)}X\partial_1\partial_{-1}\partial_1^{j-1}\partial_{-1}^{t-1}\partial^l \\ &= \frac{q^{1-j}}{q-1}(\partial_1 - \partial_{-1})\partial_1^{j-1}\partial_{-1}^{t-1}\partial^l \in H \subseteq K. \end{aligned}$$

Thus $XH \subseteq XW + H$, from which it follows that $XK \subseteq K + XH \subseteq K$ and $X^n K \subseteq K$ for all $n \geq 1$.

If $h \in \{\partial,\ \partial_1,\ \partial_{-1}\}$, it is enough to show that $hX^nW \subseteq K$ for all n. By

(3.3.0.6),

$$\begin{aligned} hX^nW &= \partial_a X^n W, \text{ for } a \in \{-1, 0, 1\} \\ &= (q^{an} X^n \partial_a + [n]_{q^a} X^{n-1})W \\ &\subseteq X^n hW + X^{n-1}W \\ &\subseteq X^n H + X^{n-1}H \subseteq X^n K + X^{n-1}K \subseteq K. \end{aligned}$$

Therefore $D_q(A) = \sum_{n\geq 1}^{\infty} X^n W + H$ and from (1), the result follows.

(iii) In proof (ii) we used only the relation derived from (3.3.0.1), (3.3.0.2) and (3.3.0.3). So it is valid for $\bar{E}$ also. □

The next step is to show that the elements of $\mathcal{B}_1$ are linearly independent.

Lemma 3.4.4. Let $V_n = XH_{n+1} \cap H_n$. Then, for $a \in \{1, \ -1\}$,

$$n\partial^{n-1}\partial_a \equiv [n]_{q^a}\partial^n \mod V_n.$$

Proof. We prove it by induction on n. From (3.3.0.4), we have $X(\partial\partial_a - q^t\partial_a\partial) = \partial - \partial_a$, therefore it is true for $n = 1$. Let $n > 1$ and suppose that $(n-1)\partial^{n-2}\partial_a \equiv [n-1]_{q^a}\partial^{n-1} \mod V_{n-1}$. For $'n'$ case, premultiplying throughout (3.3.0.14) by ∂^{n-1}, we have $\partial^n - \partial^{n-1}\partial_a = \partial^{n-1}X(\partial\partial_a - q^a\partial_a\partial)$, and by (3.3.0.7), $\partial^n - \partial^{n-1}\partial_a = (X\partial^{n-1} + (n-1)\partial^{n-2})(\partial\partial_a - q^a\partial_a\partial)$ which gives $(\partial^n - \partial^{n-1}\partial_a) - ((n-1)\partial^{n-1}\partial_a - (n-1)q^a\partial^{n-2}\partial_a\partial) = X\partial^{n-1}(\partial\partial_a - q^a\partial_a\partial) \in (XH_{n+1} \cap H_n)$. Therefore,

$$\partial^n - \partial^{n-1}\partial_a \equiv (n-1)\partial^{n-1}\partial_a - (n-1)q^a\partial^{n-2}\partial_a\partial \mod V_n$$

And by induction hypothesis for $n-1$,

$$\partial^n - \partial^{n-1}\partial_a \equiv (n-1)\partial^{n-1}\partial_a - q^a[n-1]_{q^a}\partial^n \mod V_n$$

Hence $n\partial^{n-1}\partial_a \equiv (1+q^a[n-1]_{q^a})\partial^n \mod V_n$, which implies,

$$n\partial^{n-1}\partial_a \equiv [n]_{q^a}\partial^n \mod V_n.$$

□

Lemma 3.4.5. Let $V_n = XH_{n+1} \cap H_n$ and $[n]_{q^a}! = [n]_{q^a}[n-1]_{q^a}\cdots[1]_{q^a}$. Then, for $a \in \{1,-1\}$, $n!\partial_a^n \equiv [n]_{q^a}!\partial^n \mod V_n$.

Proof. We prove it by induction on n. From (3.3.0.4), it is true for $n=1$. Now let $n>1$ and suppose that $(n-1)!\partial_a^{n-1} \equiv [n-1]_{q^a}!\partial^{n-1} \mod V_n$. Then by post multiplying throughout this with $n\partial_a$, we have $n!\partial_a^n \equiv n[n-1]_{q^a}!\partial^{n-1}\partial_a \mod V_n$. But by Lemma 3.4.4, we have $n\partial^{n-1}\partial_a \equiv [n]_{q^a}\partial^n \mod V_n$. Hence $n!\partial_a^n \equiv [n]_{q^a}!\partial^n \mod V_n$, for $a \in \{1,\ -1\}$. □

Lemma 3.4.6. Let $V_n = XH_{n+1} \cap H_n$. Then $\dim V_n = \dim H_n - 1$.

Proof. Let $f = \partial^{d_1}\partial_1^{d_2}\partial_{-1}^{d_3}\cdots$ be any monomial of degree n in ∂, ∂_1 and ∂_{-1}. By Lemma 3.4.5, for each positive integer d_i, we have $\partial_a^{d_i} - \frac{[d_i]_{q^a}!}{q_i!}\partial^{d_i} \in V_{d_i} (\subseteq V_n)$ for $a \in \{-1,\ 1\}$. So that $f \equiv \lambda\partial^n \mod V_n$ for some $\lambda \in k^\times$. Hence, $\dim V_n \geq H_n - 1$. By the action of ∂^n on $k[X]$, $\partial^n X^n = n! \neq 0$, where as $XH_{n+1}X^n = 0$. Therefore $\partial^n \notin V_n$, hence $\dim V_n = \dim H_n - 1$. □

Lemma 3.4.7. Let B be the k-algebra generated by X_1, X_2 and X_3 subject to the relations

$$X_2X_1 = q^2X_1X_2, X_3X_1 = qX_1X_3 \text{ and } X_3X_2 = q^{-1}X_2X_3.$$

Then $c := X_3^2 - qX_1X_2$ and $d := X_1X_2$ are normal in B.

Proof. Since $X_2X_1 = q^2X_1X_2$ and $X_3X_1 = qX_1X_3$,

$$\begin{aligned} cX_1 &= (X_3^2 - qX_1X_2)X_1 \\ &= q^2X_1(X_3^2 - qX_1X_2) \\ &= q^2X_1c. \end{aligned}$$

And also we have $X_3X_2 = q^{-1}X_2X_3$ and $X_1X_2 = q^{-2}X_2X_1$,

$$cX_2 = (X_3^2 - qX_1X_2)X_2 = q^{-2}X_2(X_3^2 - qX_1X_2) = q^{-2}X_2c.$$

Now as $X_2X_3 = qX_3X_2$ and $qX_1X_3 = X_3X_1$, we get

$$cX_3 = (X_3^2 - qX_1X_2)X_3 = X_3(X_3^2 - qX_1X_2) = X_3c.$$

Therefore the left ideal generated by c is the same as the right ideal generated by c. Hence c is a normal element in B. Similarly, for d, we have

$$dX_1 = (X_1X_2)X_1 = X_1(q^2X_1X_2) = q^2X_1d$$

$$dX_2 = (X_1X_2)X_2 = (q^{-2}X_2X_1)X_2 = q^{-2}X_2d$$

$$dX_3 = X_1(X_2X_3) = X_1(qX_3X_2) = qq^{-1}X_3X_1X_2 = X_3d$$

Hence d is a normal element in B. □

Recall that S is the subalgebra of $\mathrm{End}_k(k[X,\ X^{-1}])$ genereted by ∂, ∂_1, ∂_{-1} and X^{-1}, and T is the subalgebra of S generated by ∂_1, ∂_{-1} and X^{-1}. And also S' is the subalgebra of $\mathrm{End}_k(k[X,\ X^{-1}])$ generated by δ_0, δ_1, δ_{-1}

and X^{-1}, and T' is the subalgebra of S' generated by δ_1, δ_{-1} and X^{-1}. Note that X^{-1} is normal in S, S'.

Theorem 3.4.8. *If B, c and d are as in the above lemma. Then*

(i) $T \simeq B/cB$.

(ii) there exists a k-automorphism α of B/cB and an α-derivation δ of A/cA such that $S \simeq (B/cB)[Z, \alpha, \delta]$ and $S/X^{-1}S \simeq B/dB$.

And also

(iii) $T \simeq B/cB$, $S' \simeq (B/cB)[Z, \alpha, \delta]$ and $S'/X^{-1}S' \simeq B/dB$.

Proof. We prove this theorem by following [UD] closely.

(i) Let C be the subalgebra of B generated by X_1 and, X_2, and let C_ℓ be the localization of C at the powers of X_1 and X_2. By Corollary 1.4.5, C_ℓ is simple. Let β be the k-automorphism of C (resp. C_ℓ) such that $\beta(X_1) = qX_1$ and $\beta(X_2) = q^{-1}X_2$. Then the algebra B is the skew polynomial ring $C[X_3, \beta]$. Similarly the localization B_ℓ at the powers of X_1, X_2 and X_3 is the skew Laurent polynomial ring $C_\ell[X_3^{\pm 1}, \beta]$.

Let $\bar{B} = B/cB$ and $\bar{B}_\ell = B_\ell/cB_\ell$. Every element of $\bar{B}$ is of the form $a\bar{X}_3 + b$, where a, $b \in C_\ell$, $\bar{X}_3$ is the image of X_3 in $\bar{B}_\ell$, and this expression is unique. Let I be a non-zero ideal of $\bar{B}_\ell$ and let $\tau(I) = \{f \in C_\ell \mid f\bar{X}_3 + g \in I$ for some $g \in C_\ell\}$. Then $\tau(I)$ is a non-zero ideal of C_ℓ. Since C_ℓ is simple, we have $\tau(I) = C_\ell$. Now $1 \in C_\ell$, $\bar{X}_3 + g \in I$ for some $g \in C_\ell$. Again we have, $I \cap C_\ell = (0)$ or $I \cap C_\ell = C_\ell$. If suppose, $I \cap C_\ell = (0)$. Let $r = qX_1X_2 \in C_\ell$. Then we have $(\bar{X}_3 + g)\bar{X}_3 = r + g\bar{X}_3 \in I$ and $g(\bar{X}_3 + g) = g\bar{X}_3 + g^2 \in I$. By uniqueness, we have $r - g^2 \in I \cap C_\ell = (0)$. So either $r = g^2$ or $I = \bar{B}_l$. But it is clear, by degree argument, that r has no square root in C_ℓ. Therefore $I \cap C_\ell = C_\ell$, which implies that $I = \bar{B}_l$. And hence $\bar{B}_l$ is simple.

Now let $\vartheta_1 = X^{-1}\chi_1 = X^{-1} + (q-1)\partial_1$ and $\vartheta_{-1} = X^{-1}\chi_{-1} = X^{-1} + (q^{-1}-1)\partial_{-1}$. Then ϑ_1, ϑ_{-1} and X^{-1} generate T while ϑ_1,ϑ_{-1}, ∂ and X^{-1} generate S. The actions of ϑ_1 and ϑ_{-1} on $k[X]$ are given by

$$\vartheta_1 : X^n \mapsto q^n X^{n-1}, \vartheta_{-1} : X^n \mapsto q^{-n} X^{n-1}.$$

And also we have,

$$\vartheta_1\vartheta_{-1} = q^2\vartheta_{-1}\vartheta_1, X^{-1}\vartheta_1 = q\vartheta_1 X^{-1},$$

$$X^{-1}\vartheta_{-1} = q^{-1}\vartheta_{-1}X^{-1} \text{ and } q\vartheta_1\vartheta_{-1} = X^{-2}.$$

Therefore we have a surjective k-homomorphism $\theta : \bar{B} \to T$ such that $\theta(X_1) = \vartheta_1$, $\theta(X_2) = \vartheta_{-1}$ and $\theta(\bar{X}_3) = X^{-1}$, so that $T \simeq \bar{B}/\ker\theta$. By the simplcity of $\bar{B}_l$, either $\ker\theta = 0$ or $X_1^j X_2^k X_3^l \in \ker\theta$ for some $j, k, l \geq 0$. But T is a domain. Clearly $X_i \notin \ker\theta$, for $i = 1, 2, 3$, so $\ker\theta = 0$ and hence $T \simeq \bar{B}$.

(ii) From (3.3.0.1),(3.3.0.14) and (3.3.0.16) or by actions on $k[X]$, we have the following relations between ∂ and the generators of T :

$$\partial\vartheta_1 = q\vartheta_1\partial - X^{-1}\vartheta_1, \qquad (3.4.0.24)$$

$$\partial\vartheta_{-1} = q^{-1}\vartheta_{-1}\partial - X^{-1}\vartheta_{-1}, \qquad (3.4.0.25)$$

$$\partial X^{-1} = -X^{-1}\partial - X^{-2}. \qquad (3.4.0.26)$$

There is a k-automorphism σ of B such that $\sigma(X_1) = qX_1$, $\sigma(X_2) = q^{-1}X_2$ and $\sigma(X_3) = X_3$. And also there is a left σ-derivation η such that $\eta(X_1) = -X_3X_1$, $\eta(X_2) = -X_3X_2$ and $\eta(X_3) = -X^2$, and that $\sigma(c) = c$ and $\eta(c) = -2X_3c \in cB$. Consequently σ and η induce, respectively, a k-automorphism α of $\bar{B}$ and an α-derivation δ of $\bar{B}$. By (3.4.0.24), (3.4.0.25) and (3.4.0.26) and by Proposition 1.4.6, there is a k-homomorphism $\phi : \bar{B}[Z;\ \alpha,\ \delta] \twoheadrightarrow S$ such that $\phi(X_1) = \vartheta_1$, $\phi(X_2) = \vartheta_{-1}$, $\phi(\bar{X}_3) = X^{-1}$ and $\phi(Z) = \partial$.

The element $\bar{X}_3$ is normal in the domain $\bar{B}[Z;\alpha,\delta]$. Let L be the localization of $\bar{B}[Z;\ \alpha,\ \delta]$ at the powers of $\bar{X}_3$. As $\bar{X}_3^2 = qX_1X_2$, X_1 and X_2 are invertible in L and it is generated by Z, $\bar{X}_3$, $Y := \bar{X}_3^{-1}$, $Y_1 = \bar{X}_3^{-1}X_1$, $Y_2 := \bar{X}_3^{-1}X_2 = Y_1^{-1}$. Since $\partial Y - Y\partial = 1$, ∂ and Y generates the first Weyl algebra A_1. Also $Y_1\partial = q^{-1}\partial Y_1$ and $Y_1Y = qYY_1$ so L is isomorphic to the simple ring $A_1[Y_1^{\pm 1};\ \psi]$ where $\psi(\partial) = q^{-1}\partial$ and $\psi(Y) = qY$. Consequently either $\ker\phi = 0$ or, as $\bar{B}[Z;\ \alpha,\ \delta]$ is a domain, $\bar{X}_3 \in \ker\phi$. But, by the action of X^{-1} on $k[X]$, we have $\phi(\bar{X}_3) \neq 0$. So $\bar{X}_3 \notin \ker\phi$, which gives $\ker\phi = 0$. Therefore, $\phi : \bar{B}[Z;\alpha,\delta] \to S$ is an isomorphism. Note that $L \simeq D_X$.

The algebra $\bar{B}[Z;\alpha,\delta]/\bar{X}_3\bar{B}[Z;\alpha,\delta]$ is generated by (the image of) X_1, X_2 and Z subject to the relations

$$ZX_1 = qX_1Z,\ ZX_2 = q^{-1}X_2Z \text{ and } X_1X_2 = 0.$$

So $\bar{B}[Z;\alpha,\delta]/\bar{X}_3\bar{B}[Z;\alpha,\delta] \simeq B/dB$. Also $S/X^{-1}S \simeq \bar{B}[Z;\alpha,\delta]/\bar{X}_3\bar{B}[Z;\alpha,\delta] \simeq B/dB$.

For Proof of (iii), let $\vartheta'_1 = X^{-1}\chi'_1$ and $\vartheta'_{-1} = X^{-1}\chi'_{-1}$. Since $\chi'_1 = \chi_{-1}$ and

$\chi'_{-1} = \chi_1$, we have $\vartheta'_1 = \vartheta_{-1}$ and $\vartheta'_{-1} = \vartheta_1$, so that here θ, ϕ will change as follows:

$\theta : \bar{B} \to T'$ such that $\theta(X_1) = \vartheta'_1 = \vartheta_{-1}$, $\theta(X_2) = \vartheta'_{-1} = \vartheta_1$ and $\theta(\bar{X}_3) = X^{-1}$. And $\phi : \bar{A}[Z;\ \alpha,\ \delta] \twoheadrightarrow S'$ such that $\phi(X_1) = \vartheta'_1 = \vartheta_{-1}$, $\phi(X_2) = \vartheta'_{-1} = \vartheta_1$, $\phi(\bar{X}_3) = X^{-1}$ and $\phi(Z) = -\partial$. □

Theorem 3.4.9. *(i) For* $n \geq 1$, $\dim H_n = n^2 + n + 1$ *and* $\mathcal{B}_1$ *is a basis for* H.

(ii) $\mathcal{B}_2$ *is a basis for* $D_q(A)$.

Proof. (i) The number of monomials of degree n in the spanning set $\mathcal{B}_1$ for H is $n^2 + n + 1 = \frac{1}{2}[(n+2)(n+1) + n(n+1)]$. The number of standard monomials in ∂_1, ∂_{-1} and ∂ of degree n is $\frac{1}{2}(n+2)(n+1)$. The number of non-standard monomials in ∂_1, ∂_{-1} and ∂ of degree n is $\frac{1}{2}n(n-1)$. To prove the theorem, it is enough to show that $\dim_k H_n = n^2 + n + 1$.

Let W be the k-subspace of H spanned by the standard monomials in ∂_1, ∂ and the standard monomials in ∂_1, ∂. Let $W_n = W \cap H_n$. Let ψ be the composition of the homomorphisms

$$H \hookrightarrow S \twoheadrightarrow S/X^{-1}S \simeq B/dB$$

where B and d as in Lemma 3.4.8. Thus $\psi(\partial_1) = \overline{X_1}$, $\psi(\partial_{-1}) = \overline{X_2}$ and $\psi(\partial) = \overline{X_3}$. Also we have $\psi(\partial_1\partial_{-1}) = 0$. The standard monomials in $\psi(\partial_1)$ and $\psi(\partial)$, and the standard monomials in $\psi(\partial_{-1})$ and $\psi(\partial)$ form a basis for B/dB. Hence the standard monomials in ∂_1 and ∂, and the standard monomials in ∂_{-1}, ∂ are linearly independent. Therfore $\dim W_n = 2n+1$, for $n \geq 0$. And also that the restriction of ψ to W is injective. As $\ker(\psi) = \psi(H \cap X^{-1}S) = (0)$, we have $W \cap X^{-1}S = (0)$, whence $W \cap X^{-1}H = (0)$ and $XW \cap S = (0)$. Therefore the sum $J = \sum_{n\geq 1}^{\infty} X^n W + H$ is direct sum. To see this, let $J_m = \sum_{n\geq 1}^{m} X^n W + H$ for $m \geq 1$. Since $XW \cap S = (0)$, the sum $J_1 = XW + H$ is direct. Suppose that $m > 1$ and the sum J_{m-1} is direct. Let $X^m w \in X^m W \cap J_{m-1}$, then $X^m w \in J_{m-1} \subseteq S$. Therefore $w \in W \cap X^{-1}S = (0)$, and the sum for J_k is direct. Hence the sum for J is a direct sum.

From Lemma 3.4.2, $XH_{n+1} = XW_{n+1} \oplus V_n$, where $V_n = XH_{n+1} \cap H_n$. Let $d_n = \dim H_n$. Then, by Lemma 3.4.6, $\dim_k(V_n) = d_n - 1$. As X is invertible in $\operatorname{End} k[X^{\pm 1}]$, $\dim_k XH_{n+1} = d_{n+1}$ and $\dim_k XW_{n+1} = W_{n+1} = 2n+3$. Therefore $d_{n+1} = d_n + 2n + 2$, with $d_0 = 1$. Hence we have $d_n = n^2 + n + 1$. Thus $\dim_k H_n = n^2 + n + 1$.
(ii) We have $D_q(A) = J = \sum_{n\geq 1}^{\infty} X^n W + H$ and $\mathcal{B}_1$ is a basis for H. And W has a basis $\mathcal{W}$ consisting of the monomials that are standard in ∂_1 and ∂ or in ∂_{-1} and ∂. As X is regular element in $\operatorname{End}_k(k[X])$, each $X^n W$, $n \geq 1$, has a basis $X^n \mathcal{W}$. And the sum $J = \sum_{n\geq 1}^{\infty} X^n W + H$ is a direct sum. Therefore $\mathcal{B}_2$ is a basis for $D_q(A)$. □

Similarly $\mathcal{B}'_2$ is also basis for $D_q(A)$.

Corollary 3.4.10. *The surjective homomorphism* $\theta : \bar{E} \to D_q(A)$ *is an isomorphism.*

Proof. Since $\mathcal{B}_2$ is linearly independent, the spanning set $\mathcal{B}_3$ of $\bar{E}$ is linearly independent. Therefore, θ is an isomorphism. $\square$

By Theorem 1.4.8, we have $D_q(k[X, X^{-1}]) \cong k[X, X^{-1}] \otimes D_q(k[X])$. Therefore the set $\{\lambda_X,\ \lambda_{X^{-1}},\ \partial,\ \partial_1,\ \partial_{-1}\}$ will be a generating set for $D_q(k[X, X^{-1}])$ and for $a,\ b \in \{-1,\ 0,\ 1\}$, the relations $\partial_a X - q^a X \partial_a = 1$, $\partial_a X \partial_b = \partial_b X \partial_a$, $\partial_{-1}\partial_1 = q\partial_1\partial_{-1}$, $\lambda_X \lambda_{X^{-1}} = \lambda_{X^{-1}}\lambda_X = 1$ form a full set of defining relations. An alternative set of generators is $\{\lambda_X,\ \lambda_{X^{-1}},\ \partial,\ \chi_1,\ \chi_{-1}\}$.

Theorem 3.4.11. *The algebra $D_q(k[X, X^{-1}])$ is isomorphic to its opposite algebra $D_q(k[X, X^{-1}])^o$.*

Proof. Define $\Phi : D_q(k[X, X^{-1}]) \to D_q(k[X, X^{-1}])^o$, by using generators, as follows:

$$\Phi(\lambda_X) = (\lambda_X),\ \Phi(\lambda_{X^{-1}}) = (\lambda_{X^{-1}}),\ \Phi(\partial) = -(\partial),$$

$$\Phi(\partial_1) = -q^{-1}(\partial_{-1}),\ \Phi(\partial_{-1}) = -q(\partial_1).$$

Then this Φ gives an isomorphism. In terms of the alternative set of generators, we can define Φ as,

$$\Phi(\lambda_X) = (\lambda_X),\ \Phi(\lambda_{X^{-1}}) = (\lambda_{X^{-1}}),\ \Phi(\partial) = -(\partial),$$

$$\Phi(\chi_1) = q^{-1}(\chi_{-1}),\ \Phi(\chi_{-1}) = q(\chi_1).$$

And it is clear that, for $a,\ b = \{-1,\ 0,\ 1\}$, $\Phi(\partial_a X - q^a X \partial_a - 1) = 0$, $\Phi(\partial_a X \partial_b - \partial_b X \partial_a) = 0$ and $\Phi(\partial_{-1}\partial_1 - q\partial_1\partial_{-1}) = 0$. $\square$

Chapter 4

Jordan problem of α-simple rings

4.1 Introduction

Let A be a commutative ring with unity and α be an automorphism of A, or more generally an injective endomorphism of A. We say that the ring A is an α -simple ring if for any ideal I of A, if $\alpha(I) \subset I$, then either $I = 0$ or $I = A$.

Suppose A is a α-simple ring. In [J], Jordan posed a problem, for a maximal ideal M of A and for $0 \neq u \in A$, whether $|\{n \in \mathbb{N} | \alpha^n(u) \in M\}|$ is finite. It has been proved in [S], (a) $k[X]$ with $X \mapsto X + 1$ and (b) $k(X)[Y, Z]$ with $X \mapsto X + 1$, $Y \mapsto XY + 1$, $Z \mapsto XZ + Y$ are α- simple. For completeness, we show (a), (b) are α-simple by following [S]. In this chapter, we give an elementary proof for the α-simple ring $k[X]$ and also we have a partial result for the α-simple ring $k(X)[Y, Z]$.

4.2 The α-simple ring $A = k[X]$

Lemma 4.2.1. Let D be an α-simple domain. Then $S = \{a \in D \mid \alpha(a) = a\}$ is a subfield of D.

Proof. Let $0 \neq a \in S$. Then $\alpha(a) = a$ and $0 \subsetneq < a > \subseteq D$. Since D is α-simple, $< a >= D$. $1 \in < a >$, implies $1 = ab$ for some $b \in D$. Now $\alpha(ab) = \alpha(1) = 1 = ab$, implies $\alpha(a)\alpha(b) = ab$. As D is domain, $\alpha(b) = b$. Therefore there is b in S such that $ab = 1$. □

Lemma 4.2.2. Let D be a commutative domain of characteristic zero and α be an injective ring endomorphism on $R = D[X]$ such that $\alpha(D) \subseteq D$, and assume that D is α-simple. Suppose that $\alpha(X) = aX + b$; $a, \ b \in D$, a invertible in D. Then R is α-simple if and only if the equation $\alpha(\tau) = a\tau + b$ has no solution, $\tau \in D$.

Proof. Let I be a non-zero proper ideal of R with $\alpha(I) = I$. And let C denote the ideal of D consisting of all leading coefficients of all polynomials in I with minimum degree n together with 0. Since $I \neq D$, there exists $f \in I$ such that $\deg(f) > 0$. Let $\deg(f) = k$ and b_k be the leading coefficient in f. If $k \geq n$, then $b_k \in C$. Otherwise $X^n f \in I$, $b_k \in C$. Therefore C is non-zero ideal of D. Suppose $0 \neq b_m \in C$, there is $g = \sum_{i=0}^{m} b_i X^i \in I$.

$\alpha(g) = \sum_{i=0}^{m} \alpha(b_i)(aX+b)^i = \alpha(b_m)a^m X^m +$ terms with X of degree less than m. Since $\alpha(I) = I, \alpha(g) \in I$. Since a is invertible, $a^{-m}\alpha(f) \in I$. Hence $\alpha(b_m) \in C$. Therefore $\alpha(C) \subseteq C$. Since D is α-simple, $C = D$. Hence there is $h = \sum_{i=0}^{n} c_i X^i \in I$, $c_i \in D$, $c_n = 1$. Then $\alpha(h) - a^n h \in I$ and $\deg(\alpha(h) - a^n h) < n$. If $\alpha(h) - a^n h \neq 0$, which is a contradiction. Hence $\alpha(h) = a^n h$.

$$\begin{aligned}
\alpha(h) &= \sum_{i=0}^{n} \alpha(c_i)(aX+b)^i \\
&= \sum_{i=0}^{n} \alpha(c_i) \sum_{j=0}^{i} \binom{i}{j} a^j b^{i-j} X^i \\
&= \sum_{j=0}^{n} [\sum_{i=j}^{n} a^j (\alpha(c_i)) \binom{i}{j} b^{i-j}] X^j \\
&= \sum_{j=0}^{n} a^n c_j X^j.
\end{aligned}$$

We will have,

$$\sum_{i=j}^{n} (\alpha(c_i)) \binom{i}{j} b^{i-j} = a^{n-j} c_j, \ 0 \leq j \leq n. \qquad (4.2.0.1)$$

We have $char D = 0$, and by Lemma 4.2.1, $\frac{1}{n} \in D$. Substitute $j = n-1$ in (4.2.0.1), we will get $\alpha(\tau) = a\tau + b$ where $\tau = -\frac{1}{n} c_{n-1}$.

Conversely, if $\alpha(\tau) = a\tau + b$ for some $\tau \in D$. Then $I = R(X - \tau)$ such that $\alpha(I) = I$. □

Theorem 4.2.3. *Let k be a field of characteristic zero and let α be the k-automorphism on $k[X]$ given by $\alpha(X) = X + b$, $b \neq 0 \in k$. Then $k[X]$ is α-simple.*

Proof. Clearly k is α-simple. If $k[X]$ is not α-simple. Then there is $c \in k$ such that $\alpha(c) = c + b$ which implies that $b = 0$, a contradiction. □

Theorem 4.2.4. *Let k be the field of characteristic zero and let α be the k-automorphism on $k(X)[Y,Z]$ given by $\alpha(X) = X + 1,\ \alpha(Y) = XY + 1,\ \alpha(Z) = XZ + Y$. Then $k(X)[Y,Z]$ is α-simple.*

Proof. First, we will prove that $k(X)[Y]$ is α-simple. It is enough, by Lemma (4.2.2), to show that there is no $f(X) \in k(X)$ such that
$f(X+1) = Xf(X) + 1$. Suppose $f(X) = \frac{p(X)}{q(X)}$ where $p(X),\ q(X) \in k(X)$ are relatively prime and $q(X)$ is a monic polynomial. Then $f(X)$ satisfies $f(X+1) = Xf(X) + 1$ if and only if

$$q(X)[p(X+1) - q(X+1)] = Xp(X)q(X+1). \qquad (4.2.0.2)$$

Hence $q(X) \mid Xq(X+1)$. If $X \nmid q(X)$, then $q(X) \mid q(X+1)$ which implies that $q(X) \in k$. If $q(X) = Xq_1(X)$ then $q_1(X) \mid (X+1)q_1(X+1)$, if $(X+1) \nmid q_1(X)$ then $q_1 \in k$. If we continue like this, we will get $q(X) = X(X+1)\cdots(X+n)$. From (4.2.0.2), $(X+n+1) \mid p(X+1)$ or $(X+n) \mid p(X)$ which is contradiction to $p(X),\ q(X)$ are coprime. Therefore $k(X)[Y]$ is α-simple.
Now suppose that there is a polynomial $g(X,Y) \in k(X)[Y]$ and that satisfies

$$\alpha(g(X,Y)) = Xg(X,Y) + Y. \qquad (4.2.0.3)$$

Let $g(X,Y) = \sum_{i=0}^{n} a_i(X)Y^i,\ a_i(X) \in k(X)$, where $a_n(X) \neq 0$. If $n > 1$ then by comparing the leading coefficients of the polynomials in (4.2.0.3), we get $a_n(X+1)X^n = Xa_n$, but it is not possible in $k(X)$. Since $n \neq 0$, $n = 1$ and we have $Xa_1(X+1) = Xa_1(X)+1$. By using similar argument, which is used to prove that there is no $f(X) \in k(X)$ such that $f(X+1) = Xf(X)+1$ in the above paragraph, shows that $Xa_1(X+1) = Xa_1(X)+1$ is not possible. Therefore, by Theorem (4.2.2), $k(X)[Y, Z]$ is α-simple. □

Lemma 4.2.5. Suppose D is a commutative integral domain with unity and $b \in D$ and $b \neq 0$. Let $\alpha : D[X] \longrightarrow D[X]$, $\alpha(X) = X + b$. Then, for any non-zero polynomial $f(X) = a_0 + a_1X + a_2X^2 + \cdots + a_mX^m \in D[X]$ and $n \in \mathbb{N}$, we have $\alpha^n(f(X)) = f(nb) + \frac{f^{(1)}(nb)}{1!}X + \frac{f^{(2)}(nb)}{2!}X^2 + \cdots + \frac{f^{(m)}(nb)}{m!}X^m$.

Proof. We observe that for $r \in \mathbb{N}$, $\alpha^r(X) = X + rb$ and $\alpha^r(X^m) = (X + rb)^m$ for any $m \in \mathbb{N}$. We will prove it by induction on n. For $n = 1$, we have

$$\begin{aligned}
\alpha(f(X)) &= \alpha(a_0 + a_1X + a_2X^2 + \cdots + a_mX^m) \\
&= a_0 + a_1\alpha(X) + a_2\alpha(X^2) + \cdots + a_m\alpha(X^m) \\
&= a_0 + a_1(X + b) + a_2(X + b)^2 + \cdots + a_m(X + b)^m \\
&= f(b) + \frac{f^{(1)}(b)}{1!}X + \frac{f^{(2)}(b)}{2!}X^2 + \cdots + \frac{f^{(m)}(b)}{m!}X^m.
\end{aligned}$$

Now assume the induction hypothesis up to $n + 1$, $n \geq 2$, i.e. for any $r \leq n$, we have

$$\alpha^r(f(X)) = f(rb) + \frac{f^{(1)}(rb)}{1!}X + \frac{f^{(2)}(rb)}{2!}X^2 + \cdots + \frac{f^{(m)}(rb)}{m!}X^m.$$

We prove the result for $n + 1$:

$$\begin{aligned}
\alpha^{n+1}(f(X)) &= \alpha(\alpha^n(f(X))) \\
&= \alpha(f(nb) + \frac{f^{(1)}(nb)}{1!}X + \cdots + \frac{f^{(m)}(nb)}{m!}X^m) \\
&= f(nb) + \frac{f^{(1)}(nb)}{1!}(X+b) + \cdots + \frac{f^{(m)}(nb)}{m!}(X+b)^m \\
&= f((n+1)b) + \frac{f^{(1)}((n+1)b)}{1!}X + \cdots + \frac{f^{(m)}((n+1)b)}{m!}X^m.
\end{aligned}$$

□

Now let us assume that k is an algebraically closed field of characteristic zero and $A = k[X]$. Then A is α-simple ring with $\alpha(X) = X + b$ for $b \neq 0$ from Theorem 4.2.3.

Theorem 4.2.6. *For* $0 \neq u \in A$, $|\{n \in \mathbb{N} \mid \alpha^n(u) \in M\}|$ *is finite.*

Proof. We know that for any maximal ideal M of $k[X]$ there exist an $a \in k$ such that $M =< X - a >$. Let $u = f(X) \neq 0$ and $A = \{n \in \mathbb{N} \mid \alpha^n(u) \in M\}$. If $n \in A$, then $\alpha^n(f(X)) \in< X - a >$. Hence $\alpha^n f(X) = (X-a)g(X)$ for some $g(X) \in k[X]$. But we know that $\alpha^n(f(X)) = f(nb) + \frac{f^{(1)}(nb)}{1!}X + \frac{f^{(2)}(nb)}{2!}X^2 + \cdots + \frac{f^{(m)}(nb)}{m!}X^m$. If $\alpha^n(f(X)) \in< X - a >$, then $\alpha^n(f(a)) = 0$. This implies that $f(nb) + \frac{f^{(1)}(nb)}{1!}a + \frac{f^{(2)}(nb)}{2!}a^2 + \cdots + \frac{f^{(m)}(nb)}{m!}a^m = 0$, which is a polynomial of degree m in nb. Now since $\alpha^n(f(X)$ is a polynomial of degree m, it cannot have more that m distinct roots. Therefore $|\{n \in \mathbb{N} \mid \alpha^n(f) \in M\}|$ is finite. □

4.3 The α-simple ring $k(X)[Y, Z]$

We know that, from Theorem 4.2.4, $k(X)[Y, Z]$ is α-simple ring with $\alpha(X) = X + 1,\ \alpha(Y) = XY + 1,\ \alpha(Z) = XZ + Y$. In this section, we give a proof for the result that the cardinality $|\{n \in \mathbb{N} \mid \alpha^n(u) \in M\}|$ is finite, when the maximal ideal M is (Y, Z) and $0 \neq u \in k(X)[Y, Z]$.

Lemma 4.3.1. For $n \in \mathbb{N}$, we have $\alpha^n(X) = X + n$ and $\alpha^n(Y) = f_n(X)Y + g_n(X)$, where $f_n(X),\ g_n(X)$ are given as follows,

$$f_n(X) = (X + n - 1)(X + n - 2) \cdots (X + 1)X$$

$$g_n(X) = 1+(X+n-1)+(X+n-1)(X+n-2)+\cdots+(X+n-1)(X+n-2)\cdots(X+1).$$

By definition, we have the following recurrence relations,

$$f_n(X) = (X + n - 1)f_{n-1}(X) \text{ and } g_n(X) = (X + n - 1)g_{n-1}(X) + 1.$$

Proof. We will prove it by induction on n. Since $\alpha(Y) = XY + 1$, therefore we are through for $n = 1$ and $f_1(X) = X,\ g_1(X) = 1$. Let us assume that the induction hypothesis holds up to $n - 1$, $n \geq 2$, i.e. for $1 \leq r \leq n - 1$, $\alpha^r(Y) = f_r(X)Y + g_r(X)$ where $f_r(X) = (X + r - 1)(X + r - 2) \cdots (X + 1)X$, $g_r(X) = 1 + (X + r - 1) + (X + r - 1)(X + r - 2) + \cdots + (X + r - 1)(X + r - 2) \cdots (X + 1)$.

For $r = n$,

$$\begin{aligned}
\alpha^n(Y) &= \alpha^{n-1}(\alpha(Y)) \\
&= \alpha^{n-1}(XY+1) \\
&= \alpha^{n-1}(X)\alpha^{n-1}(Y)+1 \\
&= (X+n-1)(f_{n-1}(X)Y+g_{n-1}(X))+1 \\
&= (X+n-1)f_{n-1}(X)Y+[(X+n-1)g_{n-1}(X)+1] \\
&= f_n(X)Y+g_n(X).
\end{aligned}$$

□

Lemma 4.3.2. For $n \in \mathbb{N}$, $\alpha^n(Z) = p_n(X)Z + q_n(X)Y + r_n(X)$, where $p_n(X)$, $q_n(X)$, $r_n(X)$ are given by the following recurrence relations,

$$p_n(X) = (X+n-1)p_{n-1}$$

$$q_n(X) = f_{n-1}(X) + (X+n-1)q_{n-1}(X)$$

and

$$r_n(X) = g_{n-1}(X) + (X+n-1)r_{n-1}(X)$$

$f_n(X)$, $g_n(X)$ are as in the above lemma.

Proof. We will prove it by induction on n. Since $\alpha(Z) = Y + XZ$, therefore we are through for $n = 1$ and $P_1(X) = X$, $q_1(X) = 1$, $r_1(X) = 0$. Let us assume that the induction hypothesis holds up to n, $n \geq 2$, i.e., for $1 \leq m \leq n$, $\alpha^m(Z) = p_m(X)Z + q_m(X)Y + r_m(X)$ where $p_m(X) = (X+m-$

$1)(X+m-2)\cdots(X+1)X$ and $q_m = (X+m-1)(X+m-2)\cdots(X+1)+[(X+m-1)(X+m-2)\cdots(X+3)p_2(X)+\cdots+(X+m-1)(X+m-2)p_{m-3}(X)+(X+m-1)p_{m-2}(X)+p_{m-1}(X)]$. For $m = n+1$,

$$\begin{aligned}
\alpha^{n+1}(Z) &= \alpha^n(\alpha(Z)) \\
&= \alpha^n(Y+XZ) \\
&= \alpha^n(Y)+\alpha^n(X)\alpha^n(Z) \\
&= (f_n(X)Y+g_n(X))+(X+n)(p_n(X)Z+q_n(X)Y+r_n(X)) \\
&= (X+n)p_n(X)Z+(f_n(X)+(X+n)q_n(X))Y \\
&\quad +(g_n(X)+(X+n)r_n(X)) \\
&= p_{n+1}(X)Z+q_{n+1}(X)Y+r_{n+1}(X).
\end{aligned}$$

□

Lemma 4.3.3. The constant terms of $\alpha^n(Y^iZ^j)$, $\alpha^n(Y^pZ^q)$ are not equal, for all $(i,j) \neq (p,q)$, $n \in \mathbb{N}$ and $n > \max\{i+j,\ p+q\}$ (or $n > 1+|j-l|$).

Proof. By Lemma 4.3.1 and Lemma 4.3.2, we have the following,

$$\alpha^n(Y) = f_n(X)y + g_n(X)$$

$$\alpha^n(Z) = p_n(X)Z + q_n(X)Y + r_n(X).$$

Therefore, it is clear that the degrees of $g_n(X)$, $r_n(X)$ are $n-1$, $n-2$ respectively. We have $\alpha^n(Y^iZ^j) = \alpha^n(Y)^i\alpha^n(Z)^j$, the constant terms of

$\alpha^n(Y^iZ^j)$, $\alpha^n(Y^pZ^q)$ are $g_n(X)^i r_n(X)^j$, $g_n(X)^p r_n(X)^q$ and their degrees are $i(n-1)+j(n-2)$, $p(n-1)+q(n-2)$ respectively. Now we have two cases $i+j=p+q$ and $i+j\neq p+q$.

Case 1. Let $i+j=p+q$.

If $\deg(g_n(X)^i r_n(X)^j)=\deg(g_n(X)^p r_n(Y)^q)$, then $i(n-1)+j(n-2)=p(n-1)+q(n-2)$, which gives that $j=p$. As $i+j=p+q$, we will have $(i,j)=(p,q)$, which is contradiction to $(i,j)\neq(p,q)$ and hence $\deg(g_n(X)^i r_n(X)^j)\neq \deg(g_n(X)^p r_n(Y)^q)$.

Case 2. Let $i+j\neq p+q$ and let $n>1+|j-q|$.

If $\deg(g_n(X)^i r_n(X)^j)=\deg(g_n(X)^p r_n(Y)^q)$, then $i(n-1)+j(n-2)=p(n-1)+q(n-2)$, which implies that, $n=1+\frac{j-q}{[(i+j)-(p+q)]}\leq 1+|j-q|$, again it is a contradiction and hence $\deg(g_n(X)^i r_n(X)^j)\neq\deg(g_n(X)^p r_n(Y)^q)$.

□

Theorem 4.3.4. *Let $R=k(X)[Y,Z]$ be the α-simple ring and $0\neq u\in R$. Then, for the maximal ideal $M=(Y,Z)$, the cardinality $|\{n\in\mathbb{N}\mid \alpha^n(u)\in M\}|$ is finite.*

Proof. Follows from Lemma above. □

Bibliography

[SC] S.C. Coutinho, *A Primer of Algebraic D-modules*, London Mathematical Society student texts 33 (1995).

[LR] V.A. Lunts and A.L. Rosenberg, *Differential calculus on noncommutative rings*, Selecta Math.(N.S.)3 (1997), 335-359.

[GB] K.R. Goodearl and Jr. R.B. Warfield, *An Introduction to Noncommutative Noetherian Rings*, London Mathematical Society student texts 61 (2004).

[MR] J.C. McConnell and J.C. Robson, *Noncommutative Noetherian Rings*, American Mathematical Society, Vol 30 (2000).

[J] David A. Jordan, *Iterated skew polynomial rings and quantum groups*, Journal of Algebra, Volume 156 (1993), 194-218.

[JA] David A. Jordan, *The graded algebra generated by two eularian derivatives*, Algebras and Representation Theory 4 (2001), 249-275.

[S] Ahmad Shamsuddin, *Rings with automorphisms leaving no nontrivial proper ideals invariant*, Canad. Math. Bull., 25(4) (1982), 478-486.

[SA] Ahmad Shamsuddin, *On automorphisms of polynomial rings*, Bulletin of the London Mathematical Society, Volume 14 (1982), 407-409.

[UD] Uma N. Iyer and David A. Jordan, *Noetherian Algebras of Quantum Differential Operators*, Algebra and Representation Theory (2015), 1593-1622.

[UT] Uma N. Iyer and Timothy C. Mccune, *Quantum differential operators on* $\mathbb{K}[x]$, arXiv:math/0010041v3 (2002).

[VA] V.A. Lunts and A.L. Rosenberg, *Differential operators on noncommutative rings*, Sel. math., New ser. 3 (1997), 335-359.

[DR] D.R. Lane, *Fixed points of affine cremona transformation of the plane over an algebraically closed field*, Amer J.Math. Vol.97(1975), 707-732.

[BK] G.P. Barker and T.P. Kezlan, *Automorphisms of algebras of upper triangular matrices*, Arch. Math. 55(1990) 38-43.

[B] D. Benkovie, *Jordan derivations and antiderivations on triangular matrices*, Linear Algebras Appli. 397(2005) 235-244.

[CP] S.P. Coelho and C. Polcino Milies, *Derivations of upper triangular matrix rings* Linear Algebra Appli. 187 (1993), 263-267.

[I1] Uma N. Iyer, *Differential Operators on Azumaya algebras and Heisenberg algebras*, Comm. Algebra 29, (2001), No. 7, 3039-3053.

[I2] Uma N. Iyer, *Differential Operators on Holf algebras and some functorial properties*, Manuscripta Math. 109(2002), No. 1, 121-129

[I3] Uma N. Iyer, *Differential Operators on derivation rings*, J. Ramanujan Math. Soc. 20(2005), No. 1, 19-25.

[J] S. Jøndrup, *Automorphisms and derivations of upper triangular matrix rings*, Linear Algebra Appli. 221 (1995), 205-218.

[LB] D. Benkovič and Y. Li, *Jordan generalized derivations on triangular algebras*, Linear Multilinear Algebra 59 (2011), No. 8, 841-849.