STUDY OF QUALITATIVE BEHAVIOUR OF SOLUTIONS OF A CLASS OF NONLINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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This is to certify that I, RAKHEE BASU, have carried out the research embodied in the present thesis entitled **STUDY OF QUALITATIVE BEHAVIOUR OF SOLUTIONS OF A CLASS OF NONLINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS** for the full period prescribed under Ph.D. ordinance of the University.

I declare that, to the best of my knowledge, no part of this thesis was earlier submitted for the award of research degree of any university.

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To every single person I met

(whether still with me or departed)

whose care, help, faith and work

always motivate me at every walk of my life.

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- Rakhee Basu

List of Symbols

$\mathbb{R}$	the set of all real numbers
$\mathbb{Q}$	the set of all rational numbers
$\mathbb{N}$	the set of all natural numbers
$\lvert \quad \rvert$	modulus
$\lVert \quad \rVert$	norm
$BC([t_0, \infty), \mathbb{R})$	set of all bounded continuous function defined on $[t_0, \infty)$.
$C^n(A, B)$	set of all n-times continuously differentiable function from the set A to B.
$\not\equiv$	not equivalent.
$\sum$	summation.
sgn	sign function.
$x^{(n)}(t)$	n^{th} derivative of $'x'$ w.r.t $'t'$.
$x^n(t)$	n^{th} power of $'x'$ w.r.t $'t'$.
NDDE	Neutral Delay Differential Equation.

Abstract

This thesis deals with the qualitative behaviour of solutions of a class of nonlinear neutral delay differential equations with positive and negative coefficients. We have established some sufficient conditions under which all solutions of a neutral delay differential equation either oscillates or converges to zero as $t \to \infty$. This thesis consists of six chapters :

Chapter 1 deals with the Introduction of neutral delay differential equations and it's applications in various fields.

In Chapter 2, the oscillatory and asymptotic behaviour of solutions of a second order nonlinear neutral delay differential equation of the form

$$(r_1(t)(y(t)+p_1(t)y(\tau(t)))')' + r_2(t)(y(t)+p_2(t)y(\sigma(t)))' + p(t)G(y(\alpha(t))) - q(t)H(y(\beta(t))) = 0 \qquad (E_1)$$

and

$$(r_1(t)(y(t)+p_1(t)y(\tau(t)))')' + r_2(t)(y(t)+p_2(t)y(\sigma(t)))' + p(t)G(y(\alpha(t))) - q(t)H(y(\beta(t))) = f(t) \qquad (E_2)$$

have been studied for various ranges of $p_1(t)$ and $p_2(t)$.

In Third Chapter, Section 3.1, we have studied the oscillatory and asymptotic behaviour of solutions of a third order nonlinear neutral delay differential equation of the form

$$(a(t)(b(t)(y(t)+p(t)y(\sigma(t)))')')' + q(t)G(y(\alpha(t))) - h(t)H(y(\beta(t))) = 0, \qquad (E_3)$$

by using Riccati Transformation technique. In Section 3.3, we generalize the results obtained in the Section 3.1 to a higher(n^{th}, n is odd, $n \geqslant 3$) order NDDE of the form

$$(a(t)(b(t)(y(t)+p(t)y(\sigma(t)))')')^{(n-2)} + q(t)G(y(\alpha(t))) - h(t)H(y(\beta(t))) = 0. \qquad (E_4)$$

In Chapter 4, Section 4.2, the oscillatory and asymptotic behaviour of solutions of a fourth order homogeneous NDDE of the form

$$(r(t)(y(t)+p(t)y(t-\tau))'')'' + q(t)G(y(t-\alpha)) - h(t)H(y(t-\beta)) = 0 \qquad (E_5)$$

and

$$(r(t)(y(t)+p(t)y(t-\tau))'')''+q(t)G(y(t-\alpha))-h(t)H(y(t-\beta))=f(t) \qquad (E_6)$$

have been studied under the assumption $\int_0^\infty \frac{t}{r(t)}dt = \infty$. Moreover, sufficient conditions are obtained for the existence of bounded positive solution of (E_6) for the range $0 \leqslant p(t) \leqslant p_1 < 1$ by using Schauder's fixed point theorem. The Section 4.4 deals with the oscillatory and asymptotic behaviour of solutions of (E_5) and (E_6) under the assumption $\int_0^\infty \frac{t}{r(t)}dt < \infty$. Further, using Krasnosel'skii fixed point theorem, sufficient conditions are obtained for the existence of bounded positive solution of (E_6) for the range $1 < b_1 \leqslant p(t) \leqslant b_2 < \frac{{b_1}^2}{2}$.

In Chapter 5, a fourth order NDDE with quasi-derivative of the form

$$L_4(y(t)+p(t)y(t-\tau))+q(t)G(y(t-\alpha))-h(t)H(y(t-\beta))=0 \qquad (E_7)$$

and

$$L_4(y(t)+p(t)y(t-\tau))+q(t)G(y(t-\alpha))-h(t)H(y(t-\beta))=f(t) \qquad (E_8)$$

have been studied under the assumption $\int_0^\infty \frac{1}{r_n(t)}dt = \infty$ and $\int_0^\infty \frac{1}{r_n(t)}dt < \infty$ for $n = 1, 2, 3$.

In Chapter 6, a higher(n^{th}, n is even, $n \geqslant 4$) order NDDE of the form

$$(r(t)(y(t)+p(t)y(t-\tau))'')^{(n-2)}+q(t)G(y(t-\alpha))-h(t)H(y(t-\beta))=0 \qquad (E_9)$$

and

$$(r(t)(y(t)+p(t)y(t-\tau))'')^{(n-2)}+q(t)G(y(t-\alpha))-h(t)H(y(t-\beta))=f(t) \qquad (E_{10})$$

have been studied under the assumption $\int_0^\infty \frac{1}{r(t)}dt = \infty$ and $\int_0^\infty \frac{1}{r(t)}dt < \infty$.

Publications related to this thesis :

1. A. K. Tripathy, S. Panigrahi and R. Basu, Oscillatory and asymptotic behaviour of fourth order nonlinear neutral delay differential equations, Fasciculi Mathematici, **52** (2014), 155-174.

2. A. K. Tripathy, S. Panigrahi and R. Basu, Oscillation results for fourth order nonlinear neutral differential equations with positive and negative coefficients, Nonliear Oscillations, **15** (2012), 539-555 (Springer).

3. S. Panigrahi and R. Basu, Oscillation results for fourth order nonlinear mixed neutral differential equations with quasi-derivatives, Functional Differential Equations, **19** (2012), 189-205.

4. S. Panigrahi and R. Basu, Oscillation results for third order nonlinear mixed neutral differential equations (Accepted in Mathematica Slovaca), Springer.

5. A. K. Tripathy, S. Panigrahi and R. Basu, Oscillation results for fourth order nonlinear neutral differential equations with positive and negative coefficients, Journal of Mathematical Sciences, **194** (2013), 453-471.

6. S. Panigrahi and R. Basu, Oscillation of nonlinear neutral differential equations of fourth order with several delays, Annals of the Academy of Romanian Scientists, **4** (2012), 59-89.

7. R. Basu and S. Panigrahi, Oscillation results for higher order nonlinear neutral differential equations (Communicated).

8. S. Panigrahi and R. Basu, Oscillation results for nonlinear neutral delay differential equations of higher order with positive and negative coefficients (Communicated).

9. S. Panigrahi and R. Basu, Oscillation results for higher order nonlinear mixed neutral differential equations (Communicated).

10. S. Panigrahi and R. Basu, Oscillation results for a second order nonlinear neutral delay differential equations (Communicated).

Contents

Chapter 1

Introduction

1.1 Neutral Differential Equations

Delay differential equations, differential integral equations and functional differential equations have been studied for at least 200 years. During last 50 years, the theory of functional differential equations has been developed extensively and applied to many problems like viscoelasticity, mechanics, nuclear reactors, distributed networks, heat flows, interaction of species, microbiology, physiology etc.

Delay differential equation is very important because small delays can produce large effect on a system. Consider the first order differential equation

$$y'(t) + 2y'(t) = -y(t).$$

It can be proved ([43], [53]) that the trivial solution of this equation is asymptotically stable. However, the trivial solution of

$$y'(t) + 2y'(t-\tau) = -y(t), \tau > 0$$

is unstable.

A neutral differential equation is a differential equation in which the highest-order derivative of the unknown function appears with and without deviations. For example,

$$F(t, y(t), y(t-\tau_1), ..., y(t-\tau_m), y'(t), y'(t-\sigma_1), ..., y'(t-\sigma_k), ...,$$
$$y^{(n)}(t), y^{(n)}(t-r_1), ..., y^{(n)}(t-r_l)) = 0, \tag{1.1}$$

where $\tau_1, ..., \tau_m, \sigma_1, ..., \sigma_k, r_1, ..., r_l$ are nonzero real numbers, $\partial F/\partial y^{(n)}(t) \neq 0$ and $\partial F/\partial y^{(n)}(t - r_i) \neq 0$ for at least one $i = \{1, ..., l\}$. If (1.1) is solvable for $y^{(n)}(t)$, then it may be written as

$$\begin{aligned} y^{(n)}(t) &= f(t, y(t), y(t-\tau_1), ..., y(t-\tau_m), y'(t), y'(t-\sigma_1), ..., y'(t-\sigma_k), ..., \\ &\quad y^{(n)}(t), y^{(n)}(t-r_1), ..., y^{(n)}(t-r_l)). \end{aligned} \tag{1.2}$$

Equation (1.1) is said to be a **Neutral Delay Differential Equation** (NDDE) if the deviations $\tau_1, ..., \tau_m, \sigma_1, ..., \sigma_k, r_1, ..., r_l$ are positive. If these deviations are negative, then (1.1) is said to be a **Neutral Advanced Differential Equation.**

In this work, we deal with the oscillatory and asymptotic behaviour of solutions of n^{th} order ($n \geqslant 2$) NDDEs which are particular case of (1.2), where deviations are all positive. In general, the deviations in (1.1) may be functions of t or even functions of t and the unknown function $y(t)$. However, such equations are very complex to handle.

Neutral differential equations are encountered in problems dealing with electrical networks containing lossless transmission lines. Such network arise, for example, in high speed computers where lossless transmission lines are used to interconnect switching circuits [see[20], [79]]. Second order NDDE occur in the study of vibrating masses attached to an elastic bar and also (as in the Euler equation) in some variational problems in Calculus of Variations [see [43], pp. 4-7].

The study of oscillatory/nonoscillatory and asymptotic behaviour of solution of first order NDDE with positive and negative coefficients

$$(y(t) + p(t)y(t-\tau))' + q(t)G(y(t-\alpha)) - h(t)H(y(t-\beta)) = f(t) \tag{1.3}$$

differes from that of n^{th} order ($n \geqslant 2$) NDDE

$$(y(t) + p(t)y(t-\tau))^{(n)} + q(t)G(y(t-\alpha)) - h(t)H(y(t-\beta)) = f(t), \tag{1.4}$$

where $f, p, q, h \in C([t_0, \infty), \mathbb{R})$, $t_0 \geqslant 0$, $G \in C(\mathbb{R}, \mathbb{R})$, $\tau > 0$ and $\alpha, \beta \geqslant 0$. Let $\rho = \max\{\tau, \alpha, \beta\}$. By a solution of (1.4) (or (1.3)) on $[T, \infty)$, for some $T \geqslant t_0$, we mean a real valued continuous function y on $[T - \rho, \infty)$ such that $y(t) + p(t)y(t - \tau)$

is n-times continuously differentiable on $[T, \infty)$ and equation (1.4)(or (1.3)) is satisfied for $t \geqslant T$. We note that T depends on the solution y. Let $T \geqslant t_0$ be a given initial point and $\phi \in C([T-\rho, T], \mathbb{R})$ be a given initial function and let z_k for $k = 0, 1, ..., n-1$ be given initial constants such that $(y(t) + p(t)\phi(t-\tau))^{(n)}$ exists for $t \geqslant T$ and for any continuous function y. By using method of steps it is possible to show that the equation

$$(y(t) + p(t)y(t-\tau))^{(n)} + q(t)G(y(t-\alpha)) - h(t)H(y(t-\beta)) = 0$$

admits a unique solution $y \in C([T-\rho, \infty), \mathbb{R})$ such that

$$y(t) = \phi(t), \quad t \in [T-\rho, T]$$

and

$$[y(t) + p(t)\phi(t-\tau)]_{t=T}^{(k)} = z_k, \quad k = 0, 1, ..., n-1.$$

For the study of problem of existence and uniqueness and also continuous dependence of solutions of NDDEs (1.3) and (1.4), one can see [15, 28, 30, 43] and references cited therein. By a solution we mean a solution which exists in a certain ray $[T, \infty)$.

A solution $y(t)$ of (1.3) or (1.4) is said to be oscillatory if it has arbitrary large zeros, that is, for every $t_0 \geqslant 0$ there exists a $t_1 > t_0$ such that $y(t_1) = 0$; otherwise, it is called nonoscillatory.

It is well known that behaviour of solutions of NDDEs exibits features which are not true for nonneutral delay differential equations. There are examples see ([20, 22, 43, 79, 80]) of NDDEs whose characteristic equation have roots which are simple and which lie on the negative half-plane and yet the equation has unbounded solutions. Such a behaviour is not possible in the case of corresponding nonneutral delay differential equations. This fact is clear from the following example: All bounded solutions of delay-differential equation

$$y''(t) + 4e^{(4t-6)}y^3(t-1) = 0$$

are oscillatory (see Theorem 1, [23]) but the corresponding neutral delay differential equation

$$[y(t) - 2e^{-2}y(t-1)]'' + 4e^{(4t-6)}y^3(t-1) = 0$$

admits a bounded non-oscillatory solution $y(t) = e^{-2t}$. For this reason the study of NDDEs has lot of complications which are unfamilier to non-neutral delay differential equations. A delay differential equation is a particular case of NDDE.

In [36], [61], [75], [92], Erbe, Lin, Sahiner and Yan studied a second order neutral equation of the form

$$[y(t) - p(t)y(t-\tau)]'' + q(t)f(y(t-\sigma)) = 0, \quad t \geq 0, \tag{1.5}$$

where $p, q \in C((0,\infty),(-\infty,\infty))$, $0 \leq p(t) \leq 1$, $q(t) \geq 0$, $f \in C((-\infty,\infty),(-\infty,\infty))$, $yf(y) > 0, y \neq 0$ and τ, $\sigma > 0$. However, in [82] Tanaka has proved that equation (1.5) is oscillatory if and only if the second-order delay differential equation

$$y(t) + q(t)f(S(t-\sigma)y(t-\sigma)) = 0, t \geq 0$$

is oscillatory, where $S(t)$ is a positive function formulated in terms of $p(t)$ and τ. For the case that $p(t) \geq 0$, we only find a paper [91], in which the author attempted to extend the known results in ([7], [16]) of (1.5) with $p(t) \equiv 0$ to a neutral equation of the form (1.5) with $p(t) \equiv p \in (0,1)$. In [63] Manjojlović, Shoukaku, Tanigawa and Yoshida have studied the oscillatory behaviour of the differential equations with positive and negative coefficients of the form

$$\left[r(t)\left[y(t) \pm \sum_{i=1}^{l} h_i(t)y(t-\rho_i)\right]'\right]' + \sum_{i=1}^{m} p_i(t)y(t-\delta_i) - \sum_{i=1}^{n} q_i(t)y(t-\sigma_i) = 0. \tag{1.6}$$

Later on in [50], Karpuz et.al studied the oscillation criteria of a second order NDDE with variable delay of the form

$$\left[y(t) \pm \sum_{i\in R} r_i(t)y(\alpha_i(t))\right]'' + \sum_{i\in P} p_i(t)y(\beta_i(t)) - \sum_{i\in Q} q_i(t)y(\gamma_i(t)) = f(t) \tag{1.7}$$

for $t \geq 0$, where R, P, Q are bounded beginning segments of positive integer, $r_i \in C([t_0,\infty),\mathbb{R}^+), p_i, q_i \in C([t_0,\infty),\mathbb{R}^+), \alpha_i, \beta_i, \gamma_i \in C([t_0,\infty),\mathbb{R})$ are delay functions and

f is a real valued continuous function. Later in [12], Bai modified equation (1.7) as

$$[y(t)+\sum_{i\in R}c_i(t)y(\alpha_i(t))]''+r(t)[y(t)+\sum_{i\in R}c_i(t)y(\alpha_i(t))]'+\sum_{i\in P}p_i(t)y(\beta_i(t)) \\ -\sum_{i\in Q}q_i(t)y(\gamma_i(t))=0 \quad (1.8)$$

and

$$[y(t)+\sum_{i\in R}c_i(t)y(\alpha_i(t))]''+r(t)[y(t)+\sum_{i\in R}c_i(t)y(\alpha_i(t))]'+\sum_{i\in P}p_i(t)y(\beta_i(t)) \\ -\sum_{i\in Q}q_i(t)y(\gamma_i(t))=f(t), \quad (1.9)$$

where

(C_1) R, P, Q are bounded starting segments of positive integers; that is, $R = 1, 2, ..., R_0$, $P = 1, 2, ..., P_0, Q = 1, 2, ..., Q_0, R_0, P_0, Q_0 \in N$;

(C_2) $c_i \in C([t_0, \infty, \mathbb{R}^+)$ for all $i \in R, p_i \in C([t_0, \infty), \mathbb{R}^+)$ for all $i \in P$ and $q_i \in C([t_0, \infty), \mathbb{R}^+)$ for all $i \in Q$;

(C_3) $\alpha_i \in D([t_0, \infty))$ with $\liminf_{t\to\infty} \alpha_i'(t) > 0$ for all $i \in R$, $\beta_i \in D([t_0, \infty))$ for all $i \in P$ and $\gamma_i \in D([t_0, \infty))$ for all $i \in Q$;

(C_4) $r \in C^1([t_0, \infty), \mathbb{R}^+)$ and $r'(t) \leq 0$;

(C_5) $f \in C^1([t_0, \infty), \mathbb{R})$ and that there exists a function $F \in C^2((t_0, \infty), \mathbb{R}^+)$ which satisfies $F'' = f$ and $\lim_{t\to\infty} F(t) = 0$ with $D([t_0, \infty))$ equipped with functions satisfying the following properties;

(*i*) $h \in C^1([t_0, \infty), \mathbb{R})$ is strictly increasing and $\lim_{t\to\infty} h(t) = \infty$;

(*ii*) $h(t) \leq t$ holds for all $t \geq t_0$.

We may note that in (1.8) and (1.9), first two terms inside the derivatives are same, that is, $[y(t) + \sum_{i\in R} c_i(t)y(\alpha_i(t))]$. We may also note that, equations (1.7)-(1.9) are linear neutral delay differential equations with positive and negative coefficients. In ([12, 50, 63]), the authors employed the same technique to study the qualitative behaviour of solutions of the equations (1.6)-(1.9). In Chapter 2, we investigated the oscillatory and asymptotic behaviour of solutions of a second order NDDE with positive and negative coefficients of the form (E_1) and (E_2) with a new technique for the ranges

$p_1 \in C([t_0,\infty),\mathbb{R})$, $p_2 \in C([t_0,\infty),\mathbb{R}^+)$.

In recent years, the oscillatory and asymptotic behaviour of solutions of differential equations of third order and their applications have been receiving intensive attention till date. In fact, there are several monographs and hundreds of research papers have been appeared for ordinary and functional differential equations of third order in the literature (see [4, 8, 10, 11, 25, 33, 39, 38, 44, 77, 85, 84]) and references cited therein. In [85], Tiryaki and Aktas are concerned with the oscillation of third order nonlinear delay differential equation of the form

$$(r_2(t)(r_1(t)y')')' + p(t)y' + q(t)f(y(g(t))) = 0,$$

where $q \in C([a,\infty),\mathbb{R})$, $r_2, p \in C^1([a,\infty),\mathbb{R})$, $a \geq 0$ is a constant such that $r_1, r_2, q > 0$, $p > 0, g \in C^1([a,\infty),\mathbb{R})$ satisfies $0 < g(t) \leq t, g'(t) \geq 0$ and $g(t) \to \infty$ as $t \to \infty$ and $f \in C(\mathbb{R},\mathbb{R})$ satisfies $\frac{f(u)}{u} \geq K > 0$ for $u \neq 0$. By using a generalized Riccati transformation and integral averaging technique, authors establish some new sufficient conditions which ensure that any solution of this equation oscillates or converges to zero as $t \to \infty$. In [11], Baculíková and Dźurina have studied the oscillatory and asymptotic behaviour of solutions of the third order nonlinear delay differential equation

$$[a(t)(y''(t))^\gamma]' + q(t)f(y[\tau(t)]) = 0,$$

by applying suitable comparison theorems. They obtained new criterion for oscillation and certain asymptotic behaviours of nonoscillatory solutions of last equation with the assumptions

(C_6) $a(t), q(t) \in C([t_0,\infty)), a(t), q(t)$ are positive, $\tau(t) \in C([t_0,\infty)), \tau(t) \leq t, \lim_{t\to\infty} \tau(t) = \infty$, $\int_{t_0}^{\infty} a^{-\frac{1}{\gamma}}(t)dt = \infty$;

(C_7) γ is the quotient of odd positive integers;

(C_8) $f(y) \in C(-\infty,\infty), yf(y) > 0, f'(y) \geq 0$ for $y \neq 0$ and $-f(-xy) \geq f(xy) \geq f(x)f(y)$ for $xy > 0$. In [10], Baculíková and Dźurina have studied the asymptotic properties of the third order delay differential equation

$$[a(t)([y(t) \pm p(t)y(\delta(t))]'')^\gamma]' + q(t)y^\gamma(\tau(t)) = 0,$$

where $a(t), q(t), p(t)$ are positive function, $\gamma > 0$ is a quotient of odd positive integers and $\tau(t) \leq t$, $\delta(t) \leq t$, $\lim_{t\to\infty} \tau(t) = \infty$, $\lim_{t\to\infty} \delta(t) = \infty$ and $\int_{t_0}^{\infty} a^{-\frac{1}{\gamma}}(t)dt = \infty$. In this paper, authors established some sufficient conditions which ensure that all nonoscillatory solutions of the last equation converges to zero. Recently, in [60], Li, Zhang and Xing have studied the oscillatory and asymptotic behaviour of solutions of

$$(a(t)(b(t)(y(t) + p(t)y(\sigma(t)))')')' + q(t)y(\tau(t)) = 0 \tag{1.10}$$

for $0 \leq p(t) \leq p < 1$, with the following assumptions

$$(C_9) \quad \int_{t_0}^{\infty} \frac{1}{a(t)}dt = \infty, \qquad \int_{t_0}^{\infty} \frac{1}{b(t)}dt = \infty,$$

$$(C_{10}) \quad \int_{t_0}^{\infty} \frac{1}{a(t)}dt < \infty, \qquad \int_{t_0}^{\infty} \frac{1}{b(t)}dt = \infty,$$

$$(C_{11}) \quad \int_{t_0}^{\infty} \frac{1}{a(t)}dt < \infty, \qquad \int_{t_0}^{\infty} \frac{1}{b(t)}dt < \infty,$$

where $a, b, q \in C([t_0, \infty), \mathbb{R}^+)$, $p, \sigma, \tau \in C([t_0, \infty), \mathbb{R})$, $\sigma(t) \leqslant t$, $\tau(t) \leqslant t$, $\lim_{t\to\infty} \sigma(t) = \infty, \lim_{t\to\infty} \tau(t) = \infty$. Moreover, they fails to investigate the oscillatory and asymptotic behavior of (1.10) under the following assumption

$$(C_{12}) \quad \int_{t_0}^{\infty} \frac{1}{a(t)}dt = \infty, \qquad \int_{t_0}^{\infty} \frac{1}{b(t)}dt < \infty,$$

for the range $0 \leq p(t) \leq p < 1$. It seems that no work has been done for a third order neutral delay differential equation with positive and negative coefficients. In Chapter 3, we have established some sufficient conditions which guarantee the oscillatory and asymptotic behavior of solutions of the functional differential equations (E_3) under the assumption $(C_9), (C_{10}), (C_{11})$ and (C_{12}) for the ranges $0 \leqslant p(t) \leqslant p_1 < 1$ and $-1 < p_2 \leqslant p(t) \leqslant 0$.

In [83] R. D. Terry and P. K. Wong have studied the oscillatory and nonoscillatory behaviour of solutions of the nonlinear fourth order delay differential equation of the form

$$(r(t)y''(t))'' + y_\tau(t)f(t, y_\tau(t)) = 0, t \geq t_0,$$

where $y_\tau(t) = y[t - \tau(t)]$, $0 \leq \tau(t) \leq T$ and $0 < m \leq r(t) \leq M$. The function $f(t, u)$ is assumed to satisfy the following three hypotheses :

(C_{13}) $f(t,u)$ is a continuous real-valued function on $[0,\, \infty) \times R$, $R = (-\infty,\, \infty)$;

(C_{14}) for each fixed $t \in [0,\, \infty)$, $f(t,\text{u}) < f(t,v)$ for $0 < u < v$;

(C_{15}) for each fixed $t \in [0,\, \infty)$, $f(t,u) > 0$ and $f(t,u) = f(t,-u)$ for $u \neq 0$. Kusano and Naito [54, 55] have studied the fourth order nonlinear differential equation of the form

$$[r(t)y'']'' + yF(y^2,t) = 0,$$

where the following conditions are always assumed to hold :

(C_{16}) $r(t)$ is continuous and positive for $t \geq 0$;

(C_{17}) $yF(y^2,t)$ is continuous for $|y| < \infty, t \geq 0$, and $F(z,t)$ is positive for $z > 0, t \geq 0$ under the assumptions

$$(C_{18}) \quad \int_0^\infty \tfrac{t}{r(t)} dt = \infty$$

and

$$(C_{19}) \quad \int_0^\infty \tfrac{t}{r(t)} dt < \infty.$$

Later in [68, 69], Parhi and Tripathy have studied the oscillatory behaviour of solutions of a class of fourth order nonlinear neutral differential equations of the form

$$(r(t)(y(t) + p(t)y(t-\tau))'')'' + q(t)G(y(t-\sigma)) = 0 \tag{1.11}$$

and its associated forced equation

$$(r(t)(y(t) + p(t)y(t-\tau))'')'' + q(t)G(y(t-\sigma)) = f(t), \tag{1.12}$$

where $r \in C([0,\infty),(0,\infty))$, $p \in C([0,\infty),\mathbb{R})$, $q \in C([0,\infty),[0,\infty))$, $G \in C(\mathbb{R},\mathbb{R})$ is nondecreasing and $uG(u) > 0$ for $u \neq 0$, $\tau > 0$ and $\sigma > 0$, $f \in C([0,\infty),\mathbb{R})$ under the assumption (C_{18}) and (C_{19}). Different ranges of $p(t)$ and different forcing functions are considered here. In recent papers [72] and [73], Parhi and Rath have discussed oscillation and asymptotic behaviour of solutions of n^{th} order neutral differential equations of the form

$$[y(t) + p(t)y(t-\tau)]^{(n)} + q(t)G(y(t-\sigma)) = 0 \tag{1.13}$$

and

$$[y(t) + p(t)y(t-\tau)]^{(n)} + q(t)G(y(t-\sigma)) = f(t). \tag{1.14}$$

Equations (1.11) and (1.12) can not be termed as particular case of the (1.13) and (1.14) in view of (C_{18}) and (C_{19}). In Chapter 4, we have obtained some sufficient conditions under which every solution of the fourth order nonlinear neutral delay differential equations with positive and negative coefficients of the form (E_5) and (E_6) either oscillates or tends to zero as $t \to \infty$ under the assumption $\int_0^\infty \frac{t}{r(t)}dt = \infty$ or $\int_0^\infty \frac{t}{r(t)}dt < \infty$. Moreover, using Schauder's fixed point theorem and Krasnosel'skii's fixed point theorem existence of positive bounded solution of (E_6) has been obtained.

In Chapter 5, we would like to find some sufficient conditions under which every solution of the fourth order nonlinear neutral delay differential equations with quasi derivatives of the form (E_7) and (E_8) either oscillates or tends to zero as $t \to \infty$ with the assumptions

$$\int_0^\infty \frac{1}{r_n(t)}dt = \infty$$

and

$$\int_0^\infty \frac{1}{r_n(t)}dt < \infty$$

for $n = 1, 2, 3$. In Chapter 6, we established some sufficient conditions under which every solution of n^{th} (for n is even; $n \geq 4$) order nonlinear neutral delay differential equations with positive and negative coefficients of the form (E_9) and (E_{10}) either oscillates or tends to zero as $t \to \infty$ under the assumptions

$$\int_0^\infty \frac{1}{r(t)}dt = \infty$$

and

$$\int_0^\infty \frac{1}{r(t)}dt < \infty.$$

Moreover, using Banach fixed point theorem existence of positive bounded solution of (E_{10}) has been obtained. We need the following definitions and well known fixed point theorems for our use in the sequel.

Definition 1.1.1. (**Superlinear and Sublinear Functions**): A real valued function $G \in C(\mathbb{R}, \mathbb{R})$ is superlinear if it satisfies one of the following properties:

$$\liminf_{|u|\to\infty} \frac{G(u)}{u} \geq \alpha > 0 \tag{1.15}$$

or

$$\int_{\pm c}^{\pm\infty} \frac{du}{G(u)} < \infty, \tag{1.16}$$

for every $c > 0$

or

$$\frac{|G(u_1)|}{|u_1|} \leq \frac{|G(u_2)|}{|u_2|} \tag{1.17}$$

for $|u_1| < |u_2|$, $u_1 u_2 > 0$

or

$$\lim_{u\to 0} \left(\frac{u}{G(u)} \right) = \infty \tag{1.18}$$

or there exists a number $\gamma > 1$ such that

$$\frac{|G(u_1)|}{|u_1|^\gamma} \leq \frac{|G(u_2)|}{|u_2|^\gamma} \tag{1.19}$$

for $|u_1| < |u_2|$, $u_1 u_2 > 0$.

A real valued function $G \in C(\mathbb{R}, \mathbb{R})$ is sublinear if it satisfies one of the following properties:

$$\liminf_{u\to 0} \frac{G(u)}{u} \geq \beta > 0 \tag{1.20}$$

or

$$\int_{0}^{\pm c} \frac{du}{G(u)} < \infty, \tag{1.21}$$

for every $c > 0$

or

$$\frac{|G(u_1)|}{|u_1|} \geq \frac{|G(u_2)|}{|u_2|} \tag{1.22}$$

for $|u_1| < |u_2|$, $u_1u_2 > 0$

or

$$\lim_{u\to 0}\left(\frac{u}{G(u)}\right) = k < \infty \tag{1.23}$$

or there exists a number $\delta, 0 < \delta < 1$ such that

$$\frac{|G(u_1)|}{|u_1|^\delta} \geq \frac{|G(u_2)|}{|u_2|^\delta} \tag{1.24}$$

for $|u_1| < |u_2|$, $u_1u_2 > 0$.

Theorem 1.1.2. ***(Schauder's fixed point theorem)*** (see [42]) *Let M be a closed, convex and non-empty subset of a Banach space X. Let $T : M \to M$ be a continuous function such that TM is a relatively compact subset of X. Then T has at least one fixed point in M. That is, there exists an $x \in M$ such that $Tx = x$.*

Theorem 1.1.3. ***(Krasnosel'skii's fixed point theorem)*** (see [49, Lemma 3]) *Let S be a bounded, convex and closed subset of the Banach space X. Suppose there exists two operators A, $B : S \to X$ such that*

(i) $Ax + By \in S$ for all $x, y \in S$,

(ii) A is a contraction mapping,

(iii) B is completely continuous.

Then $A + B$ has a fixed point in S.

Theorem 1.1.4. ***(Banach fixed point theorem)*** (see [42]) *A contraction mapping on a complete metric space has exactly one fixed point.*

1.2 A Brief Overview of Thesis

In Chapter 2, we have studied the oscillatory and asymptotic behaviour of solutions of a second order homogeneous NDDE (E_1) and associated forced equation (E_2). This chapter consists of four sections.

The first Section 2.1 deals with the introduction. Section 2.2 deals with the oscillatory and asymptotic behaviour of solutions of homogeneous equation (E_1). We use two Lemmas namely Lemma 2.2.1 and Lemma 2.2.2 for the purpose of the theorems. Under certain sufficient conditions, in Theorem 2.2.3, we obtain every bounded solution of (E_1) either oscillates or tends to zero as $t \to \infty$ for the range $0 \le p_1(t) \le p_1 < \infty$, $0 \le p_2(t) \le p_2 < \infty$, where p_1 and p_2 are real numbers. In Section 2.3, in Theorems 2.3.1 and 2.3.3 we discussed the oscillatory behaviour of bounded solutions of the forced equation (E_2) for the ranges $0 \le p_1(t) \le p_1 < \infty$, $0 \le p_2(t) \le p_2 < \infty$ and $-\infty < p_3 \le p_1(t) \le p_4 \le 0$, $0 \le p_2(t) \le p_2 < \infty$ under the assumption $\liminf_{t\to\infty} F(t) = -\infty, \limsup_{t\to\infty} F(t) = \infty$.

Section 2.4 deals with the conclusion.

In Chapter 3, we have studied the oscillatory and asymptotic behaviour of solutions of a third and odd higher order homogeneous NDDE (E_3) and (E_4) respectively. This chapter consists of five sections.

Section 3.1 deals with precise introduction and motivation of the problem.

In Section 3.2, sufficient conditions are obtained for oscillatory and asymptotic behavior of solutions of (E_3). In this section we have devoloped four basic lemmas namely Lemmas 3.2.1, 3.2.2, 3.2.3 and 3.2.4 which will be used to obtain the main results in this section. In Theorems 3.2.5 and 3.2.13, we have studied the oscillatory and asymptotic behaviour of solutions of (E_3) for the ranges $0 \leqslant p(t) \leqslant p_1 < 1$ and $-1 < p_2 \leqslant p(t) \leqslant 0$ under the assumptions $\int_{t_0}^{\infty} \frac{1}{a(t)} dt = \infty, \int_{t_0}^{\infty} \frac{1}{b(t)} dt = \infty$.

Theorems 3.2.7 and 3.2.15 deal with the oscillatory and asymptotic behaviour of solutions of (E_3) for the ranges $0 \leqslant p(t) \leqslant p_1 < 1$ and $-1 < p_2 \leqslant p(t) \leqslant 0$ under the assumption $\int_{t_0}^{\infty} \frac{1}{a(t)} dt < \infty, \int_{t_0}^{\infty} \frac{1}{b(t)} dt = \infty$.

In Theorems 3.2.9 and 3.2.17, we have studied the oscillatory and asymptotic behaviour of solutions of (E_3) for the ranges $0 \leqslant p(t) \leqslant p_1 < 1$ and $-1 < p_2 \leqslant p(t) \leqslant 0$ under the assumption $\int_{t_0}^{\infty} \frac{1}{a(t)} dt < \infty, \int_{t_0}^{\infty} \frac{1}{b(t)} dt < \infty$.
In Theorems 3.2.11 and 3.2.19, we have studied the oscillatory and asymptotic behaviour of solutions of (E_3) for the ranges $0 \leqslant p(t) \leqslant p_1 < 1$ and $-1 < p_2 \leqslant p(t) \leqslant 0$ under the assumption $\int_{t_0}^{\infty} \frac{1}{a(t)} dt = \infty, \int_{t_0}^{\infty} \frac{1}{b(t)} dt < \infty$.
In Section 3.3, we are concerned with the qualitative behaviour of solutions of (E_4). We have used Lemma 3.4.1 for the purpose of the theorems. In Theorem 3.4.2 and Theorem 3.4.8, we have studied the oscillatory and asymptotic behaviour of solutions of (E_4) for the ranges $0 \leqslant p(t) \leqslant p_1 < 1$ and $-1 < p_2 \leqslant p(t) \leqslant 0$ under the assumption $\int_{t_0}^{\infty} \frac{1}{a(t)} dt = \infty, \int_{t_0}^{\infty} \frac{1}{b(t)} dt = \infty$.
In Theorems 3.4.4 and 3.4.10, we have studied the oscillatory and asymptotic behaviour of solutions of (E_4) for the ranges $0 \leqslant p(t) \leqslant p_1 < 1$ and $-1 < p_2 \leqslant p(t) \leqslant 0$ under the assumption $\int_{t_0}^{\infty} \frac{1}{a(t)} dt < \infty, \int_{t_0}^{\infty} \frac{1}{b(t)} dt = \infty$.
In Theorems 3.4.5 and 3.4.11, we have studied the oscillatory and asymptotic behaviour of solutions of (E_4) for the ranges $0 \leqslant p(t) \leqslant p_1 < 1$ and $-1 < p_2 \leqslant p(t) \leqslant 0$ under the assumption $\int_{t_0}^{\infty} \frac{1}{a(t)} dt < \infty, \int_{t_0}^{\infty} \frac{1}{b(t)} dt < \infty$.
In Theorems 3.4.6 and 3.4.12, we have studied the oscillatory and asymptotic behaviour of solutions of (E_4) for the ranges $0 \leqslant p(t) \leqslant p_1 < 1$ and $-1 < p_2 \leqslant p(t) \leqslant 0$ under the assumption $\int_{t_0}^{\infty} \frac{1}{a(t)} dt = \infty, \int_{t_0}^{\infty} \frac{1}{b(t)} dt < \infty$. The results obtain in all these sections generalizes the work of [60].
Section 3.5 deals with the conclusion.

Chapter 4 deals with the oscillatory and asymptotic behaviour of solutions of a fourth order homogeneous NDDE (E_5) and associated forced equation (E_6). This chapter mainly consists of seven sections.
Section 4.1 deals with precise introduction and motivation of the problem.
In Section 4.2, sufficient conditions are obtained for oscillatory and asymptotic behaviour of all solutions of (E_5) under the assumption $\int_0^{\infty} \frac{t}{r(t)} dt = \infty$. We have introduced four lemmas namely Lemmas 4.2.1, 4.2.2, 4.2.3 and 4.2.4 which will be used to

obtain the main results in this section. In Theorems 4.2.5, 4.2.8 and 4.2.10, we have studied the oscillatory and asymptotic behaviour of solutions of the equations (E_5) for the range $0 \leq p(t) \leq p_1 < 1$ or $1 < p_2 \leq p(t) \leq p_3 < \infty$ whereas in Theorems 4.2.14 and 4.2.16, we discussed the oscillatory and asymptotic behaviour of solutions of the equations (E_5) for the range $-1 < p_4 \leq p(t) \leq 0$ and $-\infty < p_5 \leq p(t) \leq p_6 < -1$.
In Section 4.3, in Theorems 4.3.1 we have discussed the oscillatory behaviour of solutions of the forced equation (E_6) for the range $0 \leq p(t) \leq p_1 < \infty$ but in Theorem 4.3.3 only bounded solutions of equation (E_6) has been studied for the range $-1 < p(t) \leq 0$ under the assumptions $\liminf_{t\to\infty} F(t) = -\infty, \limsup_{t\to\infty} F(t) = \infty$. Again in Theorem 4.3.4 the oscillatory behaviour of unbounded solutions of equation (E_6) has been studied for $0 \leq p(t) \leq p_1 < \infty$ under the assumption $-\infty < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \infty$. Finally, in the Theorem 4.3.6, existence of positive bounded solutions of equation (E_6) has been studied using Schauder's fixed point theorem.
In Section 4.4, we would like to study the oscillatory and asymptotic behaviour of solutions of $(E_5)/(E_6)$ under the assumption $\int_0^\infty \frac{t}{r(t)} dt < \infty$. We have devoloped four lemmas namely Lemmas 4.4.1, 4.4.2, 4.4.3, and 4.4.4 which will be used to obtain the main results in this section.
In Section 4.5, in Theorems 4.5.3 and 4.5.6, we have studied the oscillatory and asymptotic behaviour of solutions of (E_5) for the range $0 \leq p(t) \leq p_1 < 1$ or $1 < p_2 \leq p(t) \leq p_3 < \infty$, whereas in Theorems 4.5.8 and 4.5.9, we discussed the the oscillatory and asymptotic behaviour of solutions of (E_5) range $-1 < p_4 \leq p(t) \leq 0$ and $-\infty < p(t) \leq 0$ respectively.
In Section 4.6, in Theorem 4.6.1, we have discussed the oscillatory behaviour of solutions of equation (E_6) for the range $0 \leq p(t) \leq p_1 < \infty$, whereas in Theorem 4.6.3 oscillatory behaviour of bounded solutions of (E_6) are studied for the range $-1 < p(t) \leq 0$ under the assumption $\liminf_{t\to\infty} F(t) = -\infty, \limsup_{t\to\infty} F(t) = \infty$. In Theorem 4.6.4 oscillatory behaviour of unbounded solutions of (E_6) are studied for the range $-1 < p_4 \leq p(t) \leq 0$ under the assumption $-\infty < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \infty$.
Moreover, in Theorem 4.6.5 existence of positive bounded solutions of equation (E_6) has been studied using Krasnosel'skii's fixed point theorem.

Section 4.7 deals with the conclusion for this chapter.

Chapter 5 deals with the oscillatory and asymptotic behaviour of solutions of a fourth order homogeneous NDDE with quasi-derivative (E_7) and associated forced equation (E_8). This chapter mainly consists of seven sections.

Section 5.1 deals with precise introduction and motivation of the problem.

In Section 5.2, sufficient conditions are obtained for oscillatory and asymptotic behaviour of all solutions of (E_7) under the assumption $\int_0^\infty \frac{1}{r_n(t)}dt = \infty$. We have introduced two lemmas namely Lemmas 5.2.1 and 5.2.2 which will be used to obtain the main results in this section. In Theorems 5.2.3, 5.2.5 and 5.2.7, we have studied the oscillatory and asymptotic behaviour of solutions of (E_7) for the range $0 \le p(t) \le p_1 < 1$ or $1 < p_2 \le p(t) \le p_3 < \infty$, whereas in Theorems 5.2.10 and 5.2.13, we discussed the oscillatory and asymptotic behaviour of solutions of (E_7) for the ranges $-1 < p_4 \le p(t) \le 0$ and $-\infty < p(t) < -1$.

In Section 5.3, in Theorem 5.3.1 we have discussed the oscillatory behaviour of solutions of the forced equation (E_8) for the range $0 \le p(t) \le p_1 < \infty$ but in Theorem 5.3.3 oscillatory behaviour of bounded solutions of (E_8) have been studied for the range $-1 < p(t) \le 0$ under the assumption $\liminf_{t\to\infty} F(t) = -\infty, \limsup_{t\to\infty} F(t) = \infty$. Again in Theorem 5.3.4 the oscillatory behaviour of unbounded solutions of equation (E_8) has been studied for $0 \le p(t) \le p_1 < \infty$ under the assumption $-\infty < \liminf_{t\to\infty} F(t) < 0$ $< \limsup_{t\to\infty} F(t) < \infty$.

In Section 5.4, we would like to study the oscillatory and asymptotic behaviour of solutions of $(E_7)/(E_8)$ under the assumption $\int_0^\infty \frac{1}{r_n(t)}dt < \infty$.

In Section 5.5, in Theorems 5.5.1 we have studied the oscillatory behaviour of bounded solutions of (E_7) for the range $0 \le p(t) \le p_1 < 1$ or $1 < p_2 \le p(t) \le p_3 < \infty$ whereas in Theorems 5.5.3 and 5.5.4, we have discussed the oscillatory and asymptotic behaviour of bounded solutions of (E_7) for the range $-\infty < p_4 \le p(t) \le p_5 < -1$ and $-1 < p_6 \le p(t) \le 0$ respectively.

In Section 5.6, in Theorem 5.6.1 we have discussed the oscillatory behaviour of solutions of equation (E_8) for the range $0 \le p(t) \le p_1 < \infty$.

Section 5.7 deals with the conclusion.

Chapter 6 deals with the oscillatory and asymptotic behaviour of solutions of a higher order homogeneous NDDE (E_9) and associated forced equation (E_{10}).This chapter mainly consists of seven sections.

Section 6.1 deals with precise introduction.

In Section 6.2, sufficient conditions are obtained for oscillatory and asymptotic behaviour of all solutions of (E_9) under the assumption $\int_0^\infty \frac{1}{r(t)}dt = \infty$. In Theorem 6.2.2, we have studied the oscillatory and asymptotic behaviour of bounded solutions of equation (E_9) for the range $0 \leqslant p(t) \leqslant p_1 < 1$ or $1 < p_2 \leqslant p(t) \leqslant p_3 < \infty$ and in Theorems 6.2.4 and 6.2.6 we have discussed the oscillatory and asymptotic behaviour of the bounded solution of (E_9) for the range $-\infty < p_4 \leqslant p(t) \leqslant p_5 < -1$ and $-1 < p_6 \leqslant p(t) \leqslant 0$ respectively. In Theorem 6.2.7 we have studied the oscillatory and asymptotic behaviour of all solutions of equation (E_9) for the range $0 \leqslant p(t) \leqslant p_1 < 1$ or $1 < p_2 \leqslant p(t) \leqslant p_3 < \infty$ and in Theorem 6.2.8 we have studied the oscillatory and asymptotic behaviour of all solutions of equation (E_9) for the range $-1 < p_6 \leqslant p(t) \leqslant 0$.

Again in Section 6.3, in Theorems 6.3.3 we have discussed the oscillatory behaviour of the unbounded solutions of the forced equation (E_{10}) for the range $0 \leqslant p(t) \leqslant p_1 < \infty$ under the assumption $-\infty < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \infty$. In Theorem 6.3.1 the oscillatory behaviour of solutions of equation (E_{10}) has been studied for the range $0 \leqslant p(t) \leqslant p_1 < \infty$ and in Theorems 6.3.4 the oscillatory behaviour of the bounded solution has been studied for the range $-\infty < p_4 \leqslant p(t) \leqslant p_5 \leqslant 0$ under the assumption $\liminf_{t\to\infty} F(t) = -\infty$ and $\limsup_{t\to\infty} F(t) = \infty$. Moreover, in Theorem 6.3.6 the existence of positive bounded solutions of equation (E_{10}) has been studied by using Banach fixed point theorem.

In this Section 6.4, we are concerned with the oscillatory and asymptotic behavior of solutions of the higher order nonlinear neutral delay differential equations of the form (E_9) and (E_{10}) under the assumption $\int_0^\infty \frac{1}{r(t)}dt < \infty$.

In Section 6.5, in Theorems 6.5.1 we have studied the oscillatory and asymptotic behavior of bounded solutions of (E_9) for the range $0 \leqslant p(t) \leqslant p_1 < 1$ or $1 < p_2 \leqslant p(t) \leqslant$

$p_3 < \infty$ and in Theorems 6.5.3 and 6.5.4 we have discussed the behaviour of solutions of (E_9) for the ranges $-\infty < p_4 \leqslant p(t) \leqslant p_5 < -1$ and $-1 < p_6 \leqslant p(t) \leqslant 0$ under the assumption $\int_0^\infty \frac{1}{r(t)} dt < \infty$.

Again in Section 6.6, in Theorem 6.6.1 the oscillatory behaviour of bounded solutions of equation (E_{10}) has been studied for $0 \leqslant p(t) \leqslant p_1 < \infty$ and in Theorems 6.6.3 the oscillatory behaviour of bounded solution are obtained for $-\infty < p_4 \leqslant p(t) \leqslant p_5 \leqslant 0$ under the assumption $\liminf_{t\to\infty} F(t) = -\infty, \limsup_{t\to\infty} F(t) = \infty$.
Section 6.7 deals with the conclusion.

1.3 Applications of Delay Differential Equations

Delay differential equations (DDEs) are important in many areas of engineering and science. The aim of this section is to bring together contributions from the leading experts on the theory and applications of functional and delay differential equations.

Calculus on delay differential equations can be applied in a variety of important fields as in biology (see [31], [47]), in Economy and in general in the field of the inequalities (see [18]), in control theory (see [13]), variational calculus (see [6]), multiobjective optimization (see[62]) and so on. The theory of delay differential equations give rise to plenty of applications (see [3], [5], [18], [19], [20], [81]). For example, neutral differential equations arises many areas of applied mathematics, such as population dynamics (see[37]), circuit theory (see [14]) and so on. Some more neutral delay differential equations appear in modelling of the network containing lossless transmission lines (as in high-speed computers in which the lossless transmission lines are used to interconnect switching circuits) (see [21], [79]), in the study of vibrating mass attached to an elastic bar, as Euler type of equations in some variational problems, in the study of automatic control and in nueromechanical systems in which inertia plays a major role (see [17], [30], [43], [74]). In (see [1], [2]), several authors from diverse fields have illustrated the importance of the qualitative as well as quantitative study of the differential equations, in applications of the quantum calculus, in physics (see [48], [81]), economics (see [5],

[62]). Similar examples concerning insect population, where all the adults die before the babies are born can be found in [18]. Interest in functional differential equations is growing due to the development in science and technology and the varied applications in many areas. For examples, equations involving delay and those involving advance and a combination of both arise in nerve conduction (Life Sciences), organizational behaviour (Social Sciences), signal processing pantograph equations (Mechanical Engineering), to mention a few (see [17], [30], [43], [74]). Study of such equations has been an active area of research for many researchers.

Applications of second, third and fourth order delay differential equations:

Second and Third order neutral delay differential equations have application in many fields. In ([29] p. 234), it has been noticed that any system involving a feedback control will almost certainly involve delays. Hence the equation of motion governs the system is of the form

$$my''(t) + by'(t) + qy'(t-r) + ky(t) = 0.$$

Consider a problem where the interaction of two electrons or charged particles has been studied by equations of the form

$$\begin{aligned} y_1''(t) &= f_1(y_1(t) - y_2(t - r_{21}(t)), y_1'(t), y_2'(t - r_{21}(t))) \\ y_2''(t) &= f_2(y_2(t) - y_1(t - r_{12}(t)), y_2'(t), y_1'(t - r_{12}(t))). \end{aligned}$$

Another example encountered by Hale [43] in his study of vibrating massess attached to a elastic bar is

$$\begin{aligned} x''(t) + w_1^2 x(t) &= \epsilon f_1(x(t), x'(t), y(t), y'(t)) + \gamma_1 y''(t-r) \\ y''(t) + w_1^2 y(t) &= \epsilon f_2(x(t), x'(t), y(t), y'(t)) + \gamma_2 x''(t-r). \end{aligned}$$

Hodgekin and Huxley devoloped a mathematical model for the propagation of electric pulses in the nerve of a squid. The reduced version of this model was found by Nagumo [64], he suggested a relatively third-order differential equation of the form

$$y'''(x) - cy''(x) + f'(y)y'(x) - (b/c)y(x) = 0, f' = \frac{df}{dy},$$

as a model exibiting many of the features of Hodgekin-Huxley equations, where f is a cubic function. Vreeke and Sandquist [90] proposed the systems of differential equations(equivalent to third order differential equation) to describe two temperature feedback nuclear reactor problem as

$$\frac{dx_1(t)}{dt} = x_1(\gamma_1(1-x_2) + \gamma_2(1-x_3))$$
$$\frac{dx_2(t)}{dt} = \gamma_3(x_1 - x_2)$$
$$\frac{dx_2(t)}{dt} = \gamma_4(x_1 - x_3),$$

where x_1 is normalized neutron density, x_2 and x_3 are normalized temperatures, x_2 being associated with fuel and x_3 with the moderator or coolant, γ_3 and γ_4 are positive heat transfer coefficients, γ_1 and γ_2 are normalized effective neutron lifetime parameters associated with the temprature feedbacks. The expression $\rho = \gamma_1(1-x_2) + \gamma_2(1-x_3)$ in the first equation is called the reactivity and is a measure of multiplication factor of the neutrons in the fission reactor.

Fourth order (NDDE) which has application in the study of mathematical models of deflection of beams. These beams, which appear in many structures, deflect under their own weight or under the influence of some external forces. In [41], Gupta explained the bending of an elastic beams with simply-supported ends under an external force $e(x)$ is described by the boundary-value problem

$$\frac{d^4u}{dx^4} = e(x), \quad 0 < x < 1,$$

under the assumption $u(0) = u(1) = u''(0) = u''(1) = 0$. Moreover, Usami [89] proved that the linear boundary value problem

$$\frac{d^4u}{dx^4} + g(x)u = e(x), \quad 0 < x < 1,$$

under the assumption $u(0) = u(1) = u''(0) = u''(1) = 0$, $g(x), e(x)$ are given real-valued continuous function on $[0, 1]$, has exactly one solution.

Chapter 2

Oscillation Results for a Second Order Nonlinear Neutral Delay Differential Equations

2.1 Introduction

Recently Lin [61] has studied some sufficient conditions for oscillatory and non-oscillatory behaviour of solutions of a second order nonlinear neutral differential equation of the form

$$[y(t) - p(t)y(t-\tau)]'' + q(t)f(y(t-\sigma)) = 0, \qquad t \geq 0,$$

where $p, q \in C((0,\infty), (-\infty,\infty))$, $0 \leq p(t) \leq 1$, $q(t) \geq 0$, $f \in C((-\infty,\infty), (-\infty,\infty))$, $xf(x) > 0, x \neq 0$ and τ, $\sigma > 0$. Later, Shi-Wang [78] have studied equations of the form

$$(a(t)(y(t) + p(t)y(t-\tau))')' + q(t)f(y(t), y[\sigma(t)]) = 0, \qquad t \geq t_0 > 0$$

by using a generalized Riccati transformation, introducing so-called H-Method (i.e; by using parameter functions and integral averaging technique), where $a, p, q \in C([t_0,\infty), \mathbb{R}^+)$, $\tau > 0$. Moreover, Zayed-El-Moneam [93] have studied oscillatory behavior of solutions of the following functional differential equation

$$(a(t)y'(t))' + \delta_1 p(t)y'(t) + \delta_2 q(t)f(y(g(t))) = 0, \quad t \geq t_0 > 0,$$

where $\delta_1 = \pm 1$, $\delta_2 = \pm 1$, p, q, $g : [t_0, \infty) \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ are continuous and $a(t)$, $p(t)$, $q(t) > 0$. But the problem is still left if $q(t)$ changes sign. In particular, if $q(t) = q^+(t) - q^-(t)$, where $q^+(t) = \max\{0, q(t)\}$ and $q^-(t) = \max\{0, -q(t)\}$. Later Karpuz et. al [50] have studied oscillatory and asymptotic behaviour of all solutions of a class of neutral delay differential equations of second-order with several positive and negative coefficients of the form

$$\left[y(t) \pm \sum_{i \in R} r_i(t)y(\alpha_i(t))\right]'' + \sum_{i \in P} p_i(t)y(\beta_i(t)) - \sum_{i \in Q} q_i(t)y(\gamma_i(t)) = f(t)$$

for $t \geq 0$, where R, P, Q are bounded beginning segments of positive integers, $r_i \in C([t_0, \infty), \mathbb{R}^+), p_i, q_i \in C([t_0, \infty), \mathbb{R}^+), \alpha_i, \beta_i, \gamma_i \in C([t_0, \infty), \mathbb{R})$ are delay functions and f is a continuous function.

In this section, oscillatory and asymptotic behavior of solutions of a second order nonlinear neutral delay differential equations of the form

$$\begin{aligned}(r_1(t)(y(t) + p_1(t)y(\tau(t)))')' + r_2(t)(y(t) + p_2(t)y(\sigma(t)))' \\ +p(t)G(y(\alpha(t))) - q(t)H(y(\beta(t))) = 0 \qquad (2.1)\end{aligned}$$

and

$$\begin{aligned}(r_1(t)(y(t) + p_1(t)y(\tau(t)))')' + r_2(t)(y(t) + p_2(t)y(\sigma(t)))' \\ +p(t)G(y(\alpha(t))) - q(t)H(y(\beta(t))) = f(t) \qquad (2.2)\end{aligned}$$

have been studied with a new technique, where $r_1 \in C^1([t_0, \infty), (0, \infty))$, $r_2 \in C^1([t_0, \infty), [0, \infty))$, $p_1, p_2, \tau, \sigma \in C([t_0, \infty), \mathbb{R})$, $p \in C([t_0, \infty), (0, \infty))$, $q \in C([t_0, \infty), [0, \infty))$, $\alpha, \beta \in C([t_0, \infty), \mathbb{R})$, $t_0 \geq 0$, $\tau(t) \leq t$, $\sigma(t) \leq t$, $\alpha(t) \leq t$, $\beta(t) \leq t$, $\lim_{t \to \infty} \tau(t) = \infty, \lim_{t \to \infty} \sigma(t) = \infty, \lim_{t \to \infty} \alpha(t) = \infty$, $\lim_{t \to \infty} \beta(t) = \infty$, $G, H \in C(\mathbb{R}, \mathbb{R})$ are functions with $uG(u) > 0$, $vH(v) > 0$ for $u, v \neq 0$ and H is bounded function. The oscillation criteria obtained in this chapter are essentially new.

The solution of (2.1)/(2.2) means a continuous function y which is defined for

$t \geq t_{-1}$ such that $(y(t) + p_1(t)y(\tau(t)))$ is continuously differentiable, $(r_1(t)(y(t) + p_1(t)y(\tau(t)))')$ is continuously differentiable, $(y(t) + p_2(t)y(\sigma(t)))$ is continuously differentiable and (2.1)/(2.2) are satisfied for all $t \geq t_0$, where $t_{-1} = \min\{\alpha(t_0), \beta(t_0), \tau(t_0), \sigma(t_0)\}$. We restrict our attention to those solutions $y(t)$ of equations (2.1)/(2.2) which exist on $[t_0, \infty)$ with $\sup\{|y(t)|; t \geq T\} > 0$ for every $T \geq t_0$. It has been assumed that equations (2.1) and (2.2) posseses such a solution. A solution of (2.1)/(2.2) is said to be oscillatory if it has arbitrarily large zeros on $[t_0, \infty)$; otherwise, it is called non-oscillatory.

2.2 Oscillation Criteria for Homogeneous Equation

In this section, sufficient conditions are obtained for oscillatory and asymptotic behavior of the solutions of (2.1). We will need the following lemmas for our use in the sequel.

Lemma 2.2.1. [50] Let t_1 be such that $g(t_1) \geq t_0$. Suppose $g \in D([t_1, \infty))$ with $\limsup_{t\to\infty} g'(t) < \infty$ and $f \in C([t_0, \infty), \mathbb{R}^+)$, where $D([t_1, \infty))$ equipped with functions satisfying the following properties;

(*i*) $h \in C^1([t_1, \infty), \mathbb{R})$ is strictly increasing and $\lim_{t\to\infty} h(t) = \infty$;

(*ii*) $h(t) \leq t$ holds for all $t \geq t_1$.

If $fog \in L^1([t_1, \infty))$ holds, then $f \in L^1([t_0, \infty))$, where $L^1([t_0, \infty)) = \{f \in C([t_0, \infty), \mathbb{R}^+);$ $\int_{t_0}^{\infty} f(t)dt < \infty\}$.

Lemma 2.2.2. [50] Let t_1 be such that $g(t_1) \geq t_0$. Suppose $g \in D([t_1, \infty))$ with $\liminf_{t\to\infty} g'(t) > 0$, $fog \in C([t_1, \infty), \mathbb{R}^+)$ and $f \in C([t_0, \infty), \mathbb{R}^+)$. If $f \in L^1([t_0, \infty))$ holds, then $fog \in L^1([t_1, \infty))$.

Theorem 2.2.3. *Let* $0 \leq p_1(t) \leq p_1 < \infty$, $0 \leq p_2(t) \leq p_2 < \infty$ *and*

(H_1) $r_1'(t) \leq 0$, $r_2'(t) \leq 0$ *and* $\liminf_{t\to\infty} r_1(t) > 0$*;*

(H_2) $G(u) \geq k_1 u$ *for some* $k_1 > 0$, $u \neq 0$, $G(-u) = -G(u)$ *and* $H(-u) = -H(u)$, $u \in \mathbb{R}$*;*

(H_3) $\int_{t_0}^{\infty} \int_t^{\infty} q(v)dvdt < \infty$;

(H_4) $\liminf_{t\to\infty} p(t) > 0$;

(H_5) *there exists* $T_1 > t_0$ *such that* $\alpha(T_1) \geq t_0$, $\alpha \in D([T_1, \infty))$ *and* $\limsup_{t\to\infty} \alpha'(t) < \infty$;

(H_6) *there exists* $T_2 > t_0$ *such that* $\tau(T_2) \geq t_0$, $\sigma(T_2) \geq t_0$, $\tau, \sigma \in D([T_2, \infty))$, $\liminf_{t\to\infty} \tau'(t) > 0$ *and* $\liminf_{t\to\infty} \sigma'(t) > 0$

hold, then every bounded solution of (2.1) *either oscillates or tends to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a non-oscillatory bounded solution of (2.1) on $[t_0, \infty)$, $t_0 \geq 0$, say $y(t)$ is an eventually positive solution. (The proof in the case $y(t) < 0$ eventually is similar and will be omitted.) Then there exists $t_1 > t_0$ such that $y(\alpha(t)) > 0$, $y(\beta(t)) > 0$, $y(\tau(t)) > 0$, $y(\sigma(t)) > 0$ for $t \geq t_1 > t_0$.
Set

$$w_1(t) = y(t) + p_1(t)y(\tau(t)), \tag{2.3}$$

$$w_2(t) = y(t) + p_2(t)y(\sigma(t)). \tag{2.4}$$

Note that $w_1(t) > 0$ and $w_2(t) > 0$ for $t \geq t_1$. Define

$$z(t) = r_1(t)w_1(t) - \int_{t_1}^{t} r_1'(u)w_1(u)du + \int_{t_1}^{t} r_2(u)w_2(u)du \tag{2.5}$$

for $t \geq t_1$ and

$$k(t) = \int_t^{\infty} \int_s^{\infty} q(v)H(y(\beta(v)))dvds. \tag{2.6}$$

Note that condition (H_3) and the fact that H is bounded function implies that $k(t)$ exists for all t. We may also note that $z(t) > 0$ for $t \geq t_1$. Differentiating (2.5) with respect to $'t'$, we obtain

$$z'(t) = r_1(t)w_1'(t) + r_2(t)w_2(t).$$

Further differentiating the preceeding equation and using (H_1), we obtain

$$\begin{aligned} z''(t) &= (r_1(t)w_1'(t))' + r_2'(t)w_2(t) + r_2(t)w_2'(t) \\ &\leq (r_1(t)w_1'(t))' + r_2(t)w_2'(t) \end{aligned}$$

for $t \geq t_1$. Using (2.6), the last inequality becomes

$$z''(t) - k''(t) \leq (r_1(t)w_1'(t))' + r_2(t)w_2'(t) - q(t)H(y(\beta(t))).$$

Further using (2.1) in the last inequality, we obtain

$$w''(t) \leq -p(t)G(y(\alpha(t))) \leq 0(\not\equiv 0) \tag{2.7}$$

for $t \geq t_1$, where

$$w(t) = z(t) - k(t), \tag{2.8}$$

which implies that $w'(t)$ and $w(t)$ are monotonic functions and of constant sign for some $t \geq t_2 > t_1$. Therefore $w'(t) > 0$ or $w'(t) < 0$ for $t \geq t_2$.

Case I. Let $w'(t) < 0$ for $t \geq t_2$. Integrating (2.7) twice consecutively from t_2 to t, we obtain

$$w(t) \leq w(t_2) + w'(t_2)(t - t_2),$$

which implies $\lim_{t\to\infty} w(t) = -\infty$. Hence $w(t) < 0$ for large t. Thus, $z(t) < k(t)$ which implies $z(t)$ and hence $w(t)$ is bounded, a contradiction.

Case II. Next assume that $w'(t) > 0$ for $t \geq t_2$. Integrating (2.7) from t_2 to t and using (H_2), we obtain

$$\begin{aligned} \infty > w'(t_2) > w'(t_2) - w'(t) &\geq \int_{t_2}^{t} p(v)G(y(\alpha(v)))dv \\ &\geq k_1 \int_{t_2}^{t} p(v)y(\alpha(v))dv. \end{aligned}$$

By using (H_4), we obtain $p(t) > k_2 > 0$ for $t \geq t_3 > t_2$. Hence using this in the last inequality, we obtain

$$\infty > k_1k_2 \int_{t_3}^{t} y(\alpha(v))dv.$$

Choose $T_1 \geq t_3$ such that $\alpha(T_1) \geq t_0$. By using (H_5) and Lemma 2.2.1, we have from the last inequality $y \in L^1([t_0, \infty))$. Since $y(t) > 0$ and $y \in L^1([t_0, \infty))$, then by using (H_6) and Lemma 2.2.2, $w_1, w_2 \in L^1([T_2, \infty))$ for some $T_2 > T_1$ such that $\tau(T_2), \sigma(t_2) \geq t_0$. Moreover, $y(t)$ is bounded, it implies $w_1(t)$ is bounded and hence by (H_1) and the fact that $w_2 \in L^1([T_2, \infty))$, $z(t)$ is bounded. Hence $w(t)$ is bounded. Now $w'(t) > 0$ implies $w(t) > 0$ or < 0 for $t \geq t_4 > T_2$.

Subcase (i): If $w(t) > 0$ for $t \geq t_4$, then using the fact that $w(t)$ is bounded we obtain $\lim_{t\to\infty} w(t) < \infty$. Hence, $\lim_{t\to\infty} z(t) < \infty$. Let us define

$$\phi_1(t) = -\int_{t_1}^{t} r_1'(u) w_1(u) du,$$

$$\phi_2(t) = \int_{t_1}^{t} r_2(u) w_2(u) du.$$

Thus, $\phi_1'(t) \geq 0, \phi_2'(t) > 0$. By (H_1) and using the fact that $w_2 \in L^1([T_2, \infty))$, we obtain

$$\int_{t_1}^{t} (-r_1'(u)) w_1(u) du \leq M_1[r_1(t_1) - r_1(t)] \leq r_1(t_1) M_1$$

and

$$\int_{t_1}^{t} r_2(u) w_2(u) du \leq r_2(t_1) \int_{t_1}^{t} w_2(u) du,$$

where M_1 is the bound of $w_1(t)$. Thus, $\phi_1(t)$, $\phi_2(t)$ are both bounded and monotonic functions. Hence, $\lim_{t\to\infty} \phi_1(t) < \infty$ and $\lim_{t\to\infty} \phi_2(t) < \infty$ which implies $\lim_{t\to\infty} r_1(t) w_1(t) < \infty$. Since $r_1 w_1 \in L^1([T_2, \infty))$, then $\lim_{t\to\infty} r_1(t) w_1(t) = 0$. For every $\epsilon > 0$, there exists $t_5 > t_4$ such that

$$r_1(t) w_1(t) < \epsilon \tag{2.9}$$

for $t \geq t_5$. Since $\liminf_{t\to\infty} r_1(t) > 0$, then there exists $k_3 > 0$ such that

$$r_1(t) > k_3 \tag{2.10}$$

for $t \geq t_6 > t_5$. Using (2.10) in (2.9), we get

$$k_3 w_1(t) < r_1(t) w_1(t) < \epsilon$$

for $t \geq t_6$. Thus, $\lim_{t\to\infty} w_1(t) = 0$. Since, $y(t) \leq w_1(t)$, then $\lim_{t\to\infty} y(t) = 0$.

Subcase (ii): If $w(t) < 0$ for $t \geq t_4$, then $\lim_{t\to\infty} w(t)$ exists. Thus, $\lim_{t\to\infty} z(t)$ exists. Proceeding same as *Subcase (i)* we obtain $\lim_{t\to\infty} y(t) = 0$. Hence proof of the theorem is complete. □

Example 2.2.4. Consider

$$((1+e^{-t})(y(t)+y(t-\pi))')' + (1+e^{-t})(y(t)+2y(t-2\pi))' + 3y\left(t-\frac{\pi}{2}\right)$$

$$-3e^{-t}(1+\cos^2 t)\frac{y\left(t-\frac{3\pi}{2}\right)}{1+y^2\left(t-\frac{3\pi}{2}\right)} = 0 \tag{2.11}$$

for $t \geq 7$. Clearly, all the conditions of Theorem 2.2.3 are satisfied. Hence every bounded solution of (2.11) either oscillates or tends to zero as $t \to \infty$. Indeed, $y(t) = \sin t$ is an oscillatory solution of (2.11).

2.3 Oscillation Criteria for Non-Homogeneous Equation

Theorem 2.3.1. *Suppose* $0 \leq p_1(t) \leq p_1 < \infty$, $0 \leq p_2(t) \leq p_2 < \infty$ *holds. If* $(H_1) - (H_3)$ *and*

(H_7) $r_2 \in L^1([t_0, \infty))$

(H_8) *there exists* $F \in C^2([0,\infty), \mathbb{R})$ *such that* $F(t)$ *changes sign with* $\limsup_{t\to\infty} F(t) = +\infty$, $\liminf_{t\to\infty} F(t) = -\infty$ *and* $F''(t) = f(t)$

hold, then every bounded solution of (2.2) *oscillates.*

Proof. Let $y(t)$ be a bounded non-oscillatory solution of (2.2) on $[t_0, \infty)$, $t_0 \geq 0$, say $y(t)$ is an eventually positive solution. (The proof in the case $y(t) < 0$ eventually is similar and will be omitted.) Then there exists $t_1 > t_0$ such that $y(\alpha(t)) > 0$,

$y(\beta(t)) > 0$, $y(\tau(t)) > 0$, $y(\sigma(t)) > 0$ for $t \geq t_1 > t_0$. Set $w_1(t), w_2(t), z(t)$ and $k(t)$ as in (2.3), (2.4), (2.5), (2.6) respectively. Differentiating $z(t)$ twice consecutively and in view of (H_1), we obtain

$$\begin{aligned} z''(t) &= (r_1(t)w_1'(t))' + r_2'(t)w_2(t) + r_2(t)w_2'(t) \\ &\leq (r_1(t)w_1'(t))' + r_2(t)w_2'(t) \end{aligned}$$

for $t \geq t_1$. Using (2.6) and (H_8), the last inequality implies

$$z''(t) - k''(t) - F''(t) \leq (r_1(t)w_1'(t))' + r_2(t)w_2'(t) - q(t)H(y(\beta(t))) - f(t)$$

for $t \geq t_1$. Further by using (2.2) in the preceeding inequality, we obtain

$$v''(t) \leq -p(t)G(y(\alpha(t))) \leq 0 (\not\equiv 0) \tag{2.12}$$

for $t \geq t_1$, where

$$v(t) = z(t) - k(t) - F(t), \tag{2.13}$$

which implies that $v'(t)$ and $v(t)$ are monotonic functions and of constant sign for some $t \geq t_2 > t_1$. Therefore, $v'(t) > 0$ or $v'(t) < 0$ for $t \geq t_2$. Since $y(t)$ is bounded, then $w_1(t)$, $w_2(t)$ are bounded. Thus, using (H_1) and (H_7) we obtain that $z(t)$ is also a bounded.

Case I. Let $v'(t) < 0$ for $t \geq t_2$. Integrating (2.12) twice consecutively from t_2 to t, we obtain

$$v(t) \leq v(t_2) + v'(t_2)(t - t_2),$$

which implies

$$\lim_{t\to\infty} v(t) = -\infty.$$

Hence $v(t) < 0$ for large t, which implies $z(t) - k(t) < F(t)$. Thus, (H_8) implies $\liminf_{t\to\infty} z(t) = -\infty$, a contradiction to the fact that $z(t) > 0$.

Case II. If $v'(t) > 0$ for $t \geq t_2$, then either $v(t) > 0$ or $v(t) < 0$ for $t \geq t_3 > t_2$.

Subcase (i): If $v(t) < 0$ for $t \geq t_3$, then $-\infty < \lim_{t\to\infty} v(t) \leq 0$. Hence $z(t) = v(t) + k(t) + F(t)$ and (H_8) implies $\liminf_{t\to\infty} z(t) = -\infty$, a contradiction to the fact that $z(t) > 0$.

Subcase (ii): If $v(t) > 0$ for $t \geq t_3$, then (2.13) implies $0 < v(t) + k(t) = z(t) - F(t)$ which implies

$$z(t) > F(t)$$

for $t \geq t_4 > t_3$. Thus, using (H_8) we get a contradiction due to the boundedness of $z(t)$. Hence proof of the theorem is complete. □

Example 2.3.2. Consider

$$\left((1+e^{-t})(y(t)+y(t-\pi))'\right)' + e^{-t}(y(t)+2y(t-2\pi))' + e^t y\left(t-\frac{\pi}{2}\right)$$

$$-3e^{-t}(1+\cos^2 t)\frac{y\left(t-\frac{3\pi}{2}\right)}{1+y^2\left(t-\frac{3\pi}{2}\right)} = -e^t cost \tag{2.14}$$

for $t \geq 7$.

If $F(t) = -\frac{e^t sint}{2}$, then $F''(t) = f(t)$, $\limsup_{t\to\infty} F(t) = \infty$ and $\liminf_{t\to\infty} F(t) = -\infty$. Moreover, the conditions $(H_1)-(H_3)$, (H_7) and (H_8) of the Theorem 2.3.1 are satisfied. Hence, every bounded solution of (2.14) oscillates. Indeed, $y(t) = sint$ is a bounded oscillatory solution of (2.14).

Theorem 2.3.3. *Suppose* $-\infty < p_3 \leq p_1(t) \leq p_4 \leq 0$, $0 \leq p_2(t) \leq p_2 < \infty$. *If* $(H_1) - (H_3)$, (H_7) *and* (H_8) *hold, then every bounded solution of* (2.2) *oscillates.*

Proof. Proceeding same as Theorem 2.3.1, we obtain $v'(t)$ and $v(t)$ are monotonic functions and of constant sign for some $t \geq t_2 > t_1$. Therefore, either $v'(t) > 0$ or $v'(t) < 0$ for $t \geq t_2$.

*C*ase I. Let $v'(t) < 0$ for $t \geq t_2$. Proceeding same as in *C*ase I of Theorem 2.3.1, we get $\liminf_{t\to\infty} z(t) = -\infty$, a contradiction to the fact that $z(t)$ is bounded.

*C*ase II. If $v'(t) > 0$ for $t \geq t_2$, then either $v(t) > 0$ or $v(t) < 0$ for $t \geq t_3 > t_2$.

Subcase (i): If $v(t) < 0$ for $t \geq t_3$, then proceeding same as in *Subcase (i)* of *C*ase II of Theorem 2.3.1, we get $\liminf_{t\to\infty} z(t) = -\infty$, a contradiction to the fact that $z(t)$ is bounded.

Subcase (ii): If $v(t) > 0$ for $t \geq t_3$, then proceeding same as in *Subcase (ii)* of *C*ase II of Theorem 2.3.1, we get a contradiction due to the boundedness of $z(t)$. Hence proof of the theorem is complete. □

2.4 Conclusion

This chapter deals with the oscillatory and asymptotic behaviour of bounded solutions of a second order NDDE with positive and negative coefficients of the form (2.1) and (2.2). For homogeneous equation (2.1), we established some sufficient conditions under which every bounded solution of (2.1) either oscillates or tends to zero as $t \to \infty$ for the range $0 \leq p_1(t) \leq p_1 < \infty$, $0 \leq p_2(t) \leq p_2 < \infty$ with a completely new technique.

The oscillatory behaviour of bounded solution of non-homogeneous equation (2.2) has been studied for $p_1(t) \in (-\infty, \infty)$, $0 \leq p_2(t) \leq p_2 < \infty$ under the assumption $\limsup_{t\to\infty} F(t) = +\infty$, $\liminf_{t\to\infty} F(t) = -\infty$.

It would be interesting to study the oscillatory and asymptotic behaviour of solutions of (2.1) for the following ranges:

(a) $-\infty < p_1(t) \leq 0$ and $0 \leq p_2(t) < \infty$

(b) $0 \leq p_1(t) < \infty$ and $-\infty < p_2(t) \leq 0$

(c) $-\infty < p_1(t) \leq 0$ and $-\infty < p_2(t) \leq 0$.

Similarly, the problem concerning nonhomogeneous equation (2.2) is still open for the range:

(d) $p_1(t) \in (-\infty, \infty)$ and $-\infty < p_2(t) < 0$.

Chapter 3

Oscillation Results for n^{th} Order Nonlinear Neutral Delay Differential Equations with $n \geq 3$, n is Odd

3.1 Third Order NDDE

We are concerned with the oscillatory and asymptotic behavior of all solutions of the third order nonlinear neutral delay differential equations of the form

$$(a(t)(b(t)(y(t)+p(t)y(\sigma(t)))')')' + q(t)G(y(\alpha(t))) - h(t)H(y(\beta(t))) = 0, \tag{3.1}$$

where $a, b, q \in C([t_0,\infty),\mathbb{R}^+)$, $\mathbb{R}^+ = (0,\infty)$, $h \in C([t_0,\infty),[0,\infty))$, $p, \sigma,\ \alpha,\ \beta \in C([t_0,\infty),\mathbb{R})$, $\sigma(t) \leqslant t$, $\alpha(t) \leqslant t$, $\beta(t) \leqslant t$, $\lim_{t\to\infty}\sigma(t) = \infty, \lim_{t\to\infty}\alpha(t) = \infty$, $\lim_{t\to\infty}\beta(t) = \infty$, G and $H \in C(\mathbb{R},\mathbb{R})$ with $uG(u) > 0$, $vH(v) > 0$ for u, $v \neq 0$, H is bounded, G is non-decreasing under the assumption

$$\int_{t_0}^{\infty} \frac{1}{b(t)} \int_{t}^{\infty} \frac{1}{a(s)} \int_{s}^{\infty} h(u)dudsdt < \infty. \tag{H_1}$$

Hartman and Wintner [45], Hanan [44], and Erbe [35] studied the linear third order differential equation of the form

$$y'''(t) + q(t)y(t) = 0.$$

Džurina and Kotorová [33], Tiryaki and Aktas [85] investigated the oscillatory and asymptotic behavior of the solution of third order trinomial differential equations with delayed argument of the form

$$y'''(t) + p(t)y'(t) + g(t)y(\tau(t)) = 0.$$

Baculíková and Džurina [9, 11], Candan and Dahiya [24], Grace et.al [38], Saker and Džurina [77] examined the oscillatory behavior of

$$[a(t)(y''(t))^{\gamma}]' + q(t)f(y([\tau(t)])) = 0.$$

Baculíková and Džurina [8, 10], Thandapani and Li [84] investigated the oscillation of

$$[a(t)(b(t)(y(t) + p(t)y(\sigma(t)))')']' + q(t)y(\tau(t)) = 0$$

under the assumption

$$b(t) = 1, \; \int_{t_0}^{\infty} \frac{1}{a(t)} dt = \infty, \; a'(t) \geqslant 0.$$

Graef et. al [39] and Candan and Dahiya [25] considered the oscillation of third order neutral differential equations with constant delay of the form

$$[a(t)(b(t)(y(t) + py(t-\sigma))')']' + q(t)y(t-\tau) = 0, \; 0 \leqslant p < 1.$$

Recently, Li, Zhang and Xing in [60] have studied the oscillatory and asymptotic behaviour of solutions of neutral delay differential equation of the form

$$(a(t)(b(t)(y(t) + p(t)y(\sigma(t)))')')' + q(t)y(\tau(t)) = 0 \tag{3.2}$$

for $0 \leq p(t) \leq p < 1$, with the following assumptions

$$\int_{t_0}^{\infty} \frac{1}{a(t)} dt = \infty, \qquad \int_{t_0}^{\infty} \frac{1}{b(t)} dt = \infty, \tag{H_2}$$

$$\int_{t_0}^{\infty} \frac{1}{a(t)} dt < \infty, \qquad \int_{t_0}^{\infty} \frac{1}{b(t)} dt = \infty, \tag{H_3}$$

$$\int_{t_0}^{\infty} \frac{1}{a(t)} dt < \infty, \qquad \int_{t_0}^{\infty} \frac{1}{b(t)} dt < \infty, \tag{H_4}$$

where $a, b, q \in C([t_0, \infty), \mathbb{R}^+)$, $p, \sigma, \tau \in C([t_0, \infty), \mathbb{R})$, $\sigma(t) \leqslant t$, $\tau(t) \leqslant t$, $\lim_{t\to\infty} \sigma(t) = \infty, \lim_{t\to\infty} \tau(t) = \infty$. Moreover, they did not investigate the oscillatory and asymptotic behavior of solutions of (3.2) for the case

$$\int_{t_0}^{\infty} \frac{1}{a(t)} dt = \infty, \qquad \int_{t_0}^{\infty} \frac{1}{b(t)} dt < \infty, \tag{H_5}$$

for $0 \leq p(t) \leq p < 1$. It reveals that, if $q(t) < 0$, then also we can predict analogous results for oscillatory and asymptotic behaviour of solutions of (3.2). The problem is still left if $q(t)$ changes sign. In particular, if $q(t) = q^+(t) - q^-(t)$, where $q^+(t) = \max\{0, q(t)\}$ and $q^-(t) = \max\{0, -q(t)\}$, then equation (3.2) reduces to

$$(a(t)(b(t)(y(t) + p(t)y(\sigma(t)))')')' + q^+(t)y(\tau(t)) - q^-(t)y(\tau(t)) = 0, \tag{3.3}$$

which is a particular case of equation (3.1). Moreover, equation (3.2) is linear whereas equation (3.1) is highly nonlinear.

For the last decade, the study of the behavior of the solutions of functional differential/difference equations with positive and negative coefficients of first, second and higher order are major concerned of area of research. However, much attention has not given to oscillation results of a third order nonlinear delay differential equation with positive and negative coefficients. This fact is well understood due to the technical difficulties arising in the analysis. We refer the reader to some of the recent works in this direction (see [10, 60, 76]).

Keeping in view of the above fact, the motivation of the present work has come from the recent work of Li, Zhang and Xing [60]. We may note that a very few work is available in this direction. The objective of this chapter is to establish some sufficient conditions which guarantee the oscillatory and asymptotic behavior of solutions of the functional differential equations (3.1) under the assumptions $(H_1), (H_2), (H_3), (H_4)$ and (H_5) for the ranges $0 \leqslant p(t) \leqslant p_1 < 1$ and $-1 < p_2 \leqslant p(t) \leqslant 0$.

By a solution of (3.1) we understand a function $y(t) \in C([T_y, \infty))$, $T_y \geqslant t_0 \geqslant 0$ such that $y(t) + p(t)y(\sigma(t)) \in C^1([T_y, \infty))$, $b(t)(y(t) + p(t)y(\sigma(t)))' \in C^1([T_y, \infty))$, $a(t)(b(t)(y(t)+p(t)y(\sigma(t)))')' \in C^1([T_y, \infty))$ and satisfies (3.1) on $[T_y, \infty)$. We consider only those solutions $y(t)$ of (3.1) which satisfies $\sup\{|y(t)|; t \geqslant T\} > 0$ for every $T \geqslant T_y$. We assume that (3.1) has such a solution. A solution of (3.1) is said to be oscillatory if it has arbitrarily large zeros on $[T_y, \infty)$; otherwise, it is called nonoscillatory.

3.2 Oscillation Criteria for Homogeneous Equation

In this section, sufficient conditions are obtained for oscillatory and asymptotic behavior of all solutions of (3.1). We need the following lemmas for our use in the sequel.

Lemma 3.2.1. Let (H_2) hold. Let u be a continuously differentiable function on $[t_0, \infty)$ such that $b(t)u'(t)$ is continuously differentiable, $a(t)(b(t)u'(t))'$ is continuously differentiable and $(a(t)(b(t)u'(t))')' \leqslant 0 (\not\equiv 0)$ for large t. If $u(t) > 0$, then one of the cases (a) or (b) holds for large t and if $u(t) < 0$, then one of the cases (b) or (d) holds for large t, where

(a) $u'(t) > 0$ and $(b(t)u'(t))' > 0$,

(b) $u'(t) < 0$ and $(b(t)u'(t))' > 0$,

(c) $u'(t) > 0$ and $(b(t)u'(t))' < 0$,

(d) $u'(t) < 0$ and $(b(t)u'(t))' < 0$.

Proof. Since $(a(t)(b(t)u'(t))')' \leqslant 0$ for large t, consequently $u(t)$, $b(t)u'(t)$ and $a(t)(b(t)u'(t))'$ all are monotonic functions and of constant sign for $t \in [t_1, \infty)$, $t_1 \geqslant t_0$. Therefore, four cases $(a) - (d)$ are possible. Suppose $u(t) > 0$ for $t \geqslant t_1$. As (H_2) holds, it is enough to show cases $(c) - (d)$ are not possible. If case (c) holds, then integrating $(a(t)(b(t)u'(t))')' \leqslant 0$ twice successively from t_1 to t, we obtain

$$b(t)u'(t) \leqslant b(t_1)u'(t_1) + a(t_1)(b(t_1)u'(t_1))' \int_{t_1}^{t} \frac{ds}{a(s)}.$$

Using (H_2) we get $b(t)u'(t) \to -\infty$ as $t \to \infty$, implies $u'(t) < 0$ for $t \geqslant t_2 > t_1$, a contradiction. Further, if case (d) holds, integrating $(b(t)u'(t))' < 0$ twice successively from t_1 to t, we obtain

$$u(t) \leqslant u(t_1) + b(t_1)u'(t_1)\int_{t_1}^{t} \frac{ds}{b(s)}.$$

Consequently, $u(t) < 0$ for large t, a contradiction.

If $u(t) < 0$ for large t, then it is enough to show cases (a) and (c) are not possible. If case (a) holds, integrating $(b(t)u'(t))' > 0$ twice successively from t_1 to t, we obtain

$$u(t) \geqslant u(t_1) + b(t_1)u'(t_1)\int_{t_1}^{t} \frac{ds}{b(s)},$$

consequently, $u(t) > 0$ for large t, a contradiction. Again, if case (c) holds, then integrating $(a(t)(b(t)u'(t))')' \leqslant 0$ twice successively from t_1 to t, we obtain

$$b(t)u'(t) \leqslant b(t_1)u'(t_1) + a(t_1)(b(t_1)u'(t_1))'\int_{t_1}^{t} \frac{ds}{a(s)},$$

which implies $u'(t) < 0$ for large t, a contradiction. □

Lemma 3.2.2. Let (H_3) hold. Let u be a continuously differentiable function on $[t_0, \infty)$ such that $b(t)u'(t)$ is continuously differentiable, $a(t)(b(t)u'(t))'$ is continuously differentiable and $(a(t)(b(t)u'(t))')' \leqslant 0 (\not\equiv 0)$ for large t. If $u(t) > 0$, then one of the cases $(a) - (c)$ holds for large t and if $u(t) < 0$, then any one of the cases $(b) - (d)$ holds for large t.

Proof. Since $(a(t)(b(t)u'(t))')' \leqslant 0$ for large t, consequently $u(t), b(t)u'(t)$ and $a(t)(b(t)u'(t))'$ are monotonic functions and of constant sign for $t \in [t_1, \infty)$, $t_1 \geqslant t_0$. Therefore, four cases $(a) - (d)$ are possible. If $u(t) > 0$ for $t \geqslant t_1$, as (H_3) holds, then it is enough to show case (d) is not possible. If case (d) holds, then integrating $(a(t)(b(t)u'(t))')' \leqslant 0$ twice from t_1 to t, we obtain

$$\begin{aligned} b(t)u'(t) &\leqslant b(t_1)u'(t_1) + a(t_1)(b(t_1)u'(t_1))'\int_{t_1}^{t} \frac{ds}{a(s)} \\ &\leqslant b(t_1)u'(t_1). \end{aligned}$$

Integrating again from t_1 to t, we get

$$u(t) \leqslant u(t_1) + b(t_1)u'(t_1)\int_{t_1}^{t} \frac{ds}{b(s)},$$

which implies that $u(t) \to -\infty$ for large t, a contradiction.

If $u(t) < 0$ for $t \geqslant t_1$, depending upon (H_3), it is enough to show case (a) is not possible. Suppose case (a) holds, integrating $(b(t)u'(t))' > 0$ twice successively from t_1 to t, we obtain

$$u(t) \geqslant u(t_1) + b(t_1)u'(t_1)\int_{t_1}^{t} \frac{ds}{b(s)},$$

implies $u(t) > 0$ for large t, a contradiction. □

Lemma 3.2.3. Let (H_5) hold. Let u be a continuously differentiable function on $[t_0, \infty)$ such that $b(t)u'(t)$ is continuously differentiable, $a(t)(b(t)u'(t))'$ is continuously differentiable and $(a(t)(b(t)u'(t))')' \leqslant 0 (\not\equiv 0)$ for large t. If $u(t) > 0$ or $u(t) < 0$, then one of the cases $(a), (b), (d)$ holds for large t.

Proof. Since $(a(t)(b(t)u'(t))')' \leqslant 0$ for large t, consequently $u(t), b(t)u'(t)$ and $a(t)(b(t)u'(t))'$ are monotonic functions and of constant sign for $t \in [t_1, \infty)$, $t_1 \geqslant t_0$. Therefore, four cases $(a) - (d)$ are possible. As (H_5) holds, case (c) is not possible whether $u(t) > 0$ or < 0. If case (c) holds, then integrating $(a(t)(b(t)u'(t))')' \leqslant 0$ twice from t_1 to t, we obtain

$$b(t)u'(t) \leqslant b(t_1)u'(t_1) + a(t_1)(b(t_1)u'(t_1))' \int_{t_1}^{t} \frac{ds}{a(s)}.$$

Since $(b(t)u'(t))' < 0$ and using (H_5) we can get $u'(t) < 0$ eventually, a contradiction. □

Lemma 3.2.4. Let F, G, P $\in C([t_0, \infty), \mathbb{R})$ and $\tau \in C([t_0, \infty), \mathbb{R})$ with $\tau(t) \leqslant t$ and $\lim_{t\to\infty} \tau(t) = \infty$ be such that $F(t) = G(t) + P(t)G(\tau(t))$, for $t \geqslant t_1 > t_0$. Assume that there exists numbers P_1, P_2, P_3 and $P_4 \in \mathbb{R}$ such that $P(t)$ is one of the following ranges:

(1) $-\infty < P_1 \leqslant P(t) \leqslant 0$, (2) $0 \leqslant P(t) \leqslant P_2 < 1$, (3) $1 < P_3 \leqslant P(t) \leqslant P_4 < \infty$.

Suppose that $G(t) > 0$ for $t \geqslant t_0$, $\liminf_{t\to\infty} G(t) = 0$ and that $\lim_{t\to\infty} F(t) = L \in \mathbb{R}$ exists. Then $L = 0$.

Proof. Given $F(t) = G(t) + P(t)G(\tau(t))$, we have

$$\begin{aligned} F(t) - F(\tau(t)) &= G(t) + P(t)G(\tau(t)) - G(\tau(t)) - P(\tau(t))G(\tau(\tau(t))) \\ &= G(t) + [P(t) - 1]G(\tau(t)) - P(\tau(t))G(\tau(\tau(t))). \end{aligned} \tag{3.4}$$

Let $\tau(t_n)$ be a sequence of points such that

$$\lim_{n\to\infty} \tau(t_n) = \infty, \qquad \lim_{n\to\infty} G(\tau(t_n)) = 0. \tag{3.5}$$

We shall prove the result for range (1)(the proofs for the ranges (2), (3) are similar and will be omitted). Now by replacing t by t_n and taking limit as $n \to \infty$ in (3.4), we obtain

$$\lim_{n\to\infty}[G(t_n) + [P(t_n) - 1]G(\tau(t_n)) - P(\tau(t_n))G(\tau(\tau(t_n)))] = 0.$$

Using (3.5) and considering the fact that $P(t)$ is bounded, we get

$$\lim_{n\to\infty}[G(t_n) - P(\tau(t_n))G(\tau(\tau(t_n)))] = 0.$$

Since $G(t_n) > 0$ and $P(\tau(t_n))G(\tau(\tau(t_n))) \leqslant 0$, then it implies that

$$\lim_{n\to\infty} P(\tau(t_n))G(\tau(\tau(t_n))) = 0.$$

Therefore,

$$\begin{aligned} L &= \lim_{n\to\infty} F(\tau(t_n)) \\ &= \lim_{n\to\infty}[G(\tau(t_n)) + P(\tau(t_n))G(\tau(\tau(t_n)))] \\ &= 0. \end{aligned}$$

□

Oscillation results for the range $\mathbf{0 \leqslant p(t) \leqslant p_1 < 1}$.

Theorem 3.2.5. *Let $0 \leqslant p(t) \leqslant p_1 < 1$. Suppose (H_1) and (H_2) hold. If for some* $\rho \in C^1([t_0, \infty), (0, \infty))$,

(H_6) $\limsup_{t\to\infty} \int_{t_4}^{t} \Big[k_1 q(s)\rho(s)G(1-p_1)G\Big(\int_{t_3}^{\alpha(s)} \frac{\int_{t_2}^{u}\frac{1}{a(\theta)}d\theta}{b(u)}du\Big)G\Big(\frac{1}{\int_{t_2}^{\alpha(s)}\frac{du}{a(u)}}\Big)$

$\times \frac{\int_{t_2}^{\alpha(s)}\frac{du}{a(u)}}{\int_{t_2}^{s}\frac{du}{a(u)}} - \frac{(\rho'(s))^2 a(s)}{4\rho(s)}\Big]ds = \infty;$

(H_7) $\int_{t_0}^{\infty} q(t)dt = \infty;$

(H_8) $G(u)G(v) = G(uv), H(-u) = -H(u), u, v \in \mathbb{R}, \frac{G(u)}{u} \geqslant k_1$ *for* $u \neq 0$ *and some* $k_1 > 0$

also hold, then every solution of (3.1) *either oscillates or converges to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (3.1) on $[t_0, \infty)$, $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. Then there exists $t_1 > t_0$ such that $y(\alpha(t)) > 0$, $y(\beta(t)) > 0$, $y(\sigma(t)) > 0$ for $t \geqslant t_1$. Set

$$z(t) = y(t) + p(t)y(\sigma(t)), \tag{3.6}$$

and

$$k(t) = \int_{t}^{\infty} \frac{1}{b(s)} \int_{s}^{\infty} \frac{1}{a(\theta)} \int_{\theta}^{\infty} h(u)H(y(\beta(u)))dud\theta ds. \tag{3.7}$$

Note that condition (H_1) and the fact that H is bounded function implies that $k(t)$ exists for all t. Now let

$$v(t) = z(t) + k(t), \tag{3.8}$$

then

$$(a(t)(b(t)v'(t))')' = -q(t)G(y(\alpha(t))) \leqslant 0 (\not\equiv 0) \tag{3.9}$$

for $t \geqslant t_1$. Clearly, $v(t)$, $b(t)v'(t)$, $a(t)(b(t)v'(t))'$ are monotonic functions and of constant sign on $[t_2, \infty)$, $t_2 \geqslant t_1$. If $v(t) > 0$ for $t \geqslant t_2$, in view of Lemma 3.2.1, any one of the cases (a) or (b) holds. Suppose the case (a) holds.
Define

$$w(t) = \rho(t)\frac{a(t)(b(t)v'(t))'}{b(t)v'(t)}. \tag{3.10}$$

Then $w(t) > 0$ for $t \geqslant t_2$. Since $v'(t) > 0$ in case (a) and $k'(t) < 0$, therefore $z'(t) > 0$. Now, $(1-p_1)z(t) < (1-p(t))z(t) < z(t) - p(t)z(\sigma(t)) < y(t) - p(t)p(\sigma(t))y(\sigma(\sigma(t)))$, which implies

$$(1-p_1)z(t) < y(t). \tag{3.11}$$

Since $(a(t)(b(t)v'(t))')' \leqslant 0$ and in case (a), $v'(t) > 0, (b(t)v'(t))' > 0$, then we obtain

$$b(t)v'(t) > \int_{t_2}^{t} \frac{a(s)(b(s)v'(s))'}{a(s)} ds \geqslant a(t)(b(t)v'(t))' \int_{t_2}^{t} \frac{ds}{a(s)}, \tag{3.12}$$

implies that

$$\left(\frac{b(t)v'(t)}{\int_{t_2}^{t} \frac{ds}{a(s)}} \right)' < 0. \tag{3.13}$$

Since $z'(t) > v'(t)$, then integrating $z'(t) > 0$ from $t_3(> t_2)$ to t and by using (3.13), we get

$$\begin{aligned} z(t) &> \int_{t_3}^{t} z'(s) ds \\ &\geqslant \int_{t_3}^{t} \frac{b(s)v'(s) \int_{t_2}^{s} \frac{du}{a(u)}}{\int_{t_2}^{s} \frac{du}{a(u)} b(s)} ds \\ &\geqslant \frac{b(t)v'(t)}{\int_{t_2}^{t} \frac{du}{a(u)}} \int_{t_3}^{t} \frac{\int_{t_2}^{s} \frac{du}{a(u)}}{b(s)} ds. \end{aligned}$$

Since G is nondecreasing, then

$$G(z(t)) \geqslant G(b(t)v'(t)) G\left(\frac{1}{\int_{t_2}^{t} \frac{du}{a(u)}} \right) G\left(\int_{t_3}^{t} \frac{\int_{t_2}^{s} \frac{du}{a(u)}}{b(s)} ds \right). \tag{3.14}$$

Differentiating (3.10), we obtain

$$w'(t) = \rho'(t) \frac{a(t)(b(t)v'(t))'}{b(t)v'(t)} + \rho(t) \frac{(a(t)(b(t)v'(t))')'}{b(t)v'(t)} - \rho(t) \frac{a(t)((b(t)v'(t))')^2}{(b(t)v'(t))^2}.$$

It follows from the equations (3.9), (3.10), (3.11) and (3.14) that

$$\begin{aligned} w'(t) &\leqslant \frac{\rho'(t)w(t)}{\rho(t)} - \frac{\rho(t)q(t)G(1-p_1)G(z(\alpha(t)))}{b(t)v'(t)} - \frac{w^2(t)}{\rho(t)a(t)} \\ &\leqslant \frac{\rho'(t)w(t)}{\rho(t)} - \frac{\rho(t)q(t)G(1-p_1)G(b(\alpha(t))v'(\alpha(t)))L_1(\alpha(t))}{b(t)v'(t)} - \frac{w^2(t)}{\rho(t)a(t)}, \end{aligned}$$

for $t \geqslant t_4 > t_3$, where

$$L_1(\alpha(t)) = G\left(\frac{1}{\int_{t_2}^{\alpha(t)} \frac{du}{a(u)}}\right) G\left(\int_{t_3}^{\alpha(t)} \frac{\int_{t_2}^{s} \frac{du}{a(u)}}{b(s)} ds\right).$$

Using (H_8) and (3.13), we obtain

$$w'(t) \leqslant \frac{\rho'(t)w(t)}{\rho(t)} - k_1\rho(t)q(t)G(1-p_1)L_1(\alpha(t))L_2(t) - \frac{w^2(t)}{\rho(t)a(t)},$$

where

$$L_2(t) = \frac{\int_{t_2}^{\alpha(t)} \frac{du}{a(u)}}{\int_{t_2}^{t} \frac{du}{a(u)}}.$$

Thus,

$$w'(t) \leqslant -\rho(t)q(t)G(1-p_1)k_1L_1(\alpha(t))L_2(t) + \frac{a(t)(\rho'(t))^2}{4\rho(t)}.$$

Integrating from t_4 to t, we obtain

$$\int_{t_4}^{t} \left[k_1\rho(s)q(s)G(1-p_1)L_1(\alpha(s))L_2(s) - \frac{a(s)(\rho'(s))^2}{4\rho(s)}\right] ds \leqslant w(t_4) - w(t),$$

taking lim sup as $t \to \infty$, we get a contradiction due to condition (H_6).

Suppose case (b) holds. We may note that $\lim_{t\to\infty} v(t)$ exists and equal to l (say). We claim $l = 0$. If not, then for every $\epsilon > 0$, there exists $t_3 > t_2$ such that $l < v(t) < l+\epsilon$ for $t \geqslant t_3$. Choose $0 < \epsilon < \frac{l(1-p_1)}{1+p_1}$. Since $\lim_{t\to\infty} k(t) = 0$, then for the same chosen ϵ, $k(t) < \epsilon$ for $t \geqslant t_4 > t_3$. Thus,

$$\begin{aligned} y(t) &= v(t) - p(t)y(\sigma(t)) - k(t) \\ &> v(t) - p(t)v(\sigma(t)) - k(t) \\ &> l - p_1(l+\epsilon) - \epsilon \end{aligned}$$

for $t \geqslant t_5 > t_4$. That is,

$$y(t) > (l-\epsilon) - p_1(l+\epsilon) > k_2(l+\epsilon) > k_2v(t). \tag{3.15}$$

By the choice of ϵ we can show that $k_2 > 0$. Using (3.15) in (3.9), we obtain

$$(a(t)(b(t)v'(t))')' \leqslant -q(t)G(k_2)G(v(\alpha(t))) \tag{3.16}$$

for $t \geqslant t_6 > t_5$. Since $\lim_{t\to\infty} a(t)(b(t)v'(t))'$ exists, then integrating (3.16) from t_6 to t, we get

$$\int_{t_6}^{\infty} q(t)dt < \infty,$$

a contradiction to (H_7). Hence $l = 0$, which implies $\lim_{t\to\infty} v(t) = 0$ and hence $\lim_{t\to\infty} z(t) = 0$. Since $y(t) \leqslant z(t)$, then, $\lim_{t\to\infty} y(t) = 0$.

Finally, we assume that $y(t) < 0$ for $t \geqslant t_0$. From (H_8), we note that $G(-u) = -G(u)$ and $H(-u) = -H(u), u \in \mathbb{R}$. Indeed, $G(1)G(1) = G(1)$ and $G(-1)G(-1) = G(1)$ implies that $G(-1) = -1$ and $G(1) = 1$. Hence putting $x(t) = -y(t)$ for $t \geqslant t_0$, we obtain $x(t) > 0$ and

$$(a(t)(b(t)(x(t) + p(t)x(\sigma(t)))')')' + q(t)G(x(\alpha(t))) - h(t)H(x(\beta(t))) = 0.$$

Proceeding as above, we can show that every solution of (3.1) oscillates or converges to zero as $t \to \infty$. This completes the proof of the theorem.

$\square$

Example 3.2.6. Consider

$$\left(\frac{1}{t}\left(t^{\frac{1}{2}}\left(y(t) + \frac{1}{2}y(t-\pi)\right)'\right)'\right)' + \left(\frac{t^{-\frac{1}{2}}}{2} + \frac{3t^{-\frac{5}{2}}}{8} + t^{-\frac{9}{2}}\right)y\left(t - \frac{7\pi}{2}\right)$$

$$-t^{\frac{-9}{2}}\left(1 + \sin^2\left(t - \frac{3\pi}{2}\right)\right)\frac{y\left(t - \frac{3\pi}{2}\right)}{1 + y^2\left(t - \frac{3\pi}{2}\right)} = 0, \tag{3.17}$$

for $t \geqslant 12$. It is easy to verify that the hypothesis of Theorem 3.2.5 are satisfied. Thus, every solution of (3.17) either oscillates or tends to zero as $t \to \infty$. Indeed, $y(t) = \sin t$ is such an oscillatary solution of (3.17).

Theorem 3.2.7. *Let* $0 \leqslant p(t) \leqslant p_1 < 1$. *Suppose that* (H_1), (H_3), $(H_6) - (H_8)$ *and*

(H_9) $\limsup_{t\to\infty} \int_{t_4}^{t} \left[k_1 q(s)\delta(s)G(1-p_1)G\left(\int_{t_3}^{\alpha(s)} \frac{du}{b(u)}\right) - \frac{1}{4\delta(s)a(s)}\right] ds = \infty$;
where $\delta(t) = \int_t^\infty \frac{ds}{a(s)}$

hold, then every solution of (3.1) *either oscillates or converges to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (3.1) on $[t_0, \infty)$, $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. (The proof in the case $y(t) < 0$ eventually is similar and will be omitted.) Then there exists $t_1 > t_0$ such that $y(\alpha(t)) > 0, y(\beta(t)) > 0, y(\sigma(t)) > 0$ for $t \geqslant t_1$. Setting (3.6), (3.7), (3.8) we get (3.9). Consequently, $v(t), b(t)v'(t), a(t)(b(t)v'(t))'$ are monotonic functions and of constant sign on $[t_2, \infty)$, $t_2 \geqslant t_1$. Let $v(t) > 0$ for $t \geqslant t_2$. In view of Lemma 3.2.2 any one of the cases $(a) - (c)$ hold. Suppose case (a) or (b) holds. Then we can get the conclusion of Theorem 3.2.7 by applying the proof of Theorem 3.2.5. Suppose case (c) holds. By (3.9) we can say that $a(t)(b(t)v'(t))'$ is monotonic nonincreasing, thus we get $a(t)(b(t)v'(t))' \leqslant a(t_2)(b(t_2)v'(t_2))'$ for $t \geqslant t_2 > t_1$. Dividing by $a(t)$ and further integrating from t_2 to t, we get

$$0 < b(t)v'(t) \leqslant b(t_2)v'(t_2) + a(t_2)(b(t_2)v'(t_2))' \int_{t_2}^{t} \frac{ds}{a(s)}.$$

Taking $t \to \infty$, we get

$$0 \leqslant b(t_2)v'(t_2) + a(t_2)(b(t_2)v'(t_2))' \int_{t_2}^{\infty} \frac{ds}{a(s)}.$$

Replacing t_2 by t, we obtain

$$1 \geqslant -\frac{a(t)(b(t)v'(t))'}{b(t)v'(t)} \int_{t}^{\infty} \frac{ds}{a(s)}. \tag{3.18}$$

Define

$$\phi(t) = \frac{a(t)(b(t)v'(t))'}{b(t)v'(t)}. \tag{3.19}$$

Then, $\phi(t) < 0$ for $t \geqslant t_2$. Hence by using (3.18) and (3.19), we obtain

$$-\delta(t)\phi(t) \leqslant 1, \tag{3.20}$$

where $\delta(t) = \int_t^\infty \frac{ds}{a(s)}$. Differentiating (3.19) and using (3.9) and (3.11), we obtain

$$\begin{aligned} \phi'(t) &= \frac{(a(t)(b(t)v'(t))')'}{b(t)v'(t)} - \frac{a(t)((b(t)v'(t))')^2}{(b(t)v'(t))^2} \\ &\leqslant -\frac{q(t)G(y(\alpha(t)))}{b(t)v'(t)} - \frac{\phi^2(t)}{a(t)} \\ &\leqslant -\frac{q(t)G(1-p_1)G(z(\alpha(t)))}{b(t)v'(t)} - \frac{\phi^2(t)}{a(t)}. \end{aligned} \tag{3.21}$$

Since $z'(s) > v'(s)$ and $v'(s) > \frac{b(t)v'(t)}{b(s)}$ for $s < t$, then

$$z'(s) > \frac{b(t)v'(t)}{b(s)}. \tag{3.22}$$

Integrating the last inequality from $t_3(> t_2)$ to t, we get

$$z(t) > z(t) - z(t_3) \geqslant b(t)v'(t) \int_{t_3}^t \frac{ds}{b(s)}. \tag{3.23}$$

Using (3.23) in (3.21), we get

$$\phi'(t) \leqslant -\frac{q(t)G(1-p_1)G\Big(b(\alpha(t))v'(\alpha(t)) \int_{t_3}^{\alpha(t)} \frac{ds}{b(s)}\Big)}{b(t)v'(t)} - \frac{\phi^2(t)}{a(t)}, \tag{3.24}$$

for $t \geqslant t_4 > t_3$. Since $(b(t)v'(t))' < 0$ and then using (H_8) in the last inequality, we get

$$\phi'(t) \leqslant -q(t)G(1-p_1)k_1 G\Big(\int_{t_3}^{\alpha(t)} \frac{ds}{b(s)}\Big) - \frac{\phi^2(t)}{a(t)}.$$

Multiplying both sides by $\delta(t)$ and integrating from t_4 to t, we obtain

$$\begin{aligned} \delta(t)\phi(t) - \delta(t_4)\phi(t_4) + \int_{t_4}^t \frac{\phi(s)}{a(s)} ds + \int_{t_4}^t \delta(s)q(s)G(1-p_1)k_1 G\Big(\int_{t_3}^{\alpha(s)} \frac{du}{b(u)}\Big) ds \\ + \int_{t_4}^t \frac{\phi^2(s)\delta(s)ds}{a(s)} \leqslant 0, \end{aligned}$$

it follows that

$$\int_{t_4}^t \Big[k_1 q(s)\delta(s)G(1-p_1)G\Big(\int_{t_3}^{\alpha(s)} \frac{du}{b(u)}\Big) - \frac{1}{4\delta(s)a(s)}\Big] ds \leqslant 1 + \phi(t_4)\delta(t_4), \tag{3.25}$$

a contradiction to (H_9). Hence proof of the Theorem is complete. □

Example 3.2.8. Consider

$$\Big(t^{\frac{4}{3}}\Big(y(t) + \frac{1}{2}y\Big(\frac{t}{2}\Big)\Big)''\Big)' + \frac{1}{k_1}\Big(\frac{t^{-\frac{5}{3}}}{18} + \frac{2t^{-1}}{3}\Big) y\Big(t - \frac{\pi}{2}\Big)$$

$$-t^{-5}(1+t^2)\frac{y(t-\pi)}{1+y^2(t-\pi)}=0, \tag{3.26}$$

for $t \geqslant 4$ and some $k_1 > 0$. Conditions $(H_1), (H_3), (H_6)-(H_8)$ and (H_9) are satisfied, so equation (3.26) satisfies the hypothesis of Theorem 3.2.7. Hence, every solution of (3.26) either oscillates or tends to zero as $t \to \infty$.

Theorem 3.2.9. *Let* $0 \leqslant p(t) \leqslant p_1 < 1$. *Suppose that* (H_1), (H_4), $(H_6)-(H_9)$ *and*

(H_{10}) $\quad \int_{t_3}^{\infty} \frac{1}{b(t)} \int_{t_3}^{t} \frac{1}{a(s)} \int_{t_3}^{s} q(u)G(\xi(\alpha(u)))dudsdt = \infty$

hold, then every solution of (3.1) *either oscillates or tends to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (3.1) on $[t_0, \infty)$, $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. (The proof in the case $y(t) < 0$ eventually is similar and will be omitted.) Then there exists $t_1 > t_0$ such that $y(\alpha(t)) > 0, y(\beta(t)) > 0, y(\sigma(t)) > 0$ for $t \geqslant t_1$. Setting (3.6), (3.7), (3.8) we get (3.9). Consequently, $v(t), b(t)v'(t), a(t)(b(t)v'(t))'$ are monotonic functions and of constant sign on $[t_2, \infty)$, where, $t_2 \geqslant t_1$. Let (H_4) holds. If $v(t) > 0$ for $t \geqslant t_2$, then any one of $(a)-(d)$ holds. The proof of the cases (a) and (b) follows from Theorem 3.2.5 and case (c) follows from Theorem 3.2.7.

Suppose case (d) holds. Since $(b(t)v'(t))' < 0$ in case (d), then $b(s)v'(s) \leqslant b(t)v'(t)$ for $s \geqslant t > t_2$. Integrating from t to s, we obtain

$$-v(t) < v(s) - v(t) \leqslant b(t)v'(t)\int_t^s \frac{du}{b(u)}.$$

Since $\lim_{s\to\infty} v(s)$ exists and assume it is positive, then for $s \to \infty$, we get

$$v(t) \geqslant -\xi(t)b(t)v'(t) > L\xi(t),$$

where $\xi(t) = \int_t^{\infty} \frac{du}{b(u)}$ and $L = -(b(t_2)v'(t_2))$. From (3.15) we can show that $y(t) > k_2 v(t)$, therefore $y(t) > k_2 L\xi(t)$. From (3.9) it is evident that

$$0 \geqslant (a(t)(b(t)v'(t))')' + q(t)G(k_2)G(L)G(\xi(\alpha(t)))$$

for $t \geqslant t_3 > t_2$. Integrating from t_3 to t, we get

$$0 > a(t)(b(t)v'(t))' + \int_{t_3}^{t} q(s)G(k_2)G(L)G(\xi(\alpha(s)))ds.$$

Further integrating the above inequality from t_3 to t, we get

$$0 > b(t)v'(t) + \int_{t_3}^{t} \frac{1}{a(s)} \int_{t_3}^{s} q(u)G(k_2)G(L)G(\xi(\alpha(u)))duds.$$

Again integrating from t_3 to t, we get

$$0 \geqslant v(t) - v(t_3) + G(k_2)G(L) \int_{t_3}^{t} \frac{1}{b(s)} \int_{t_3}^{s} \frac{1}{a(u)} \int_{t_3}^{u} q(v)G(\xi(\alpha(v)))dvduds.$$

Since $\lim_{t\to\infty} v(t)$ exists, then we obtain

$$\int_{t_3}^{\infty} \frac{1}{b(t)} \int_{t_3}^{t} \frac{1}{a(s)} \int_{t_3}^{s} q(u)G(\xi(\alpha(u)))dudsdt < \infty,$$

which is a contradiction to (H_{10}).

If $\lim_{t\to\infty} v(t) = 0$, which implies $\lim_{t\to\infty} z(t) = 0$. Since $y(t) \leqslant z(t)$, so also $\lim_{t\to\infty} y(t) = 0$. This complete the proof of the theorem. □

Example 3.2.10. Consider

$$\left(e^{\frac{t}{4}}\left(e^{\frac{t}{2}}\left(y(t) + \frac{1}{2}y(t-\pi)\right)'\right)'\right)' + \left(e^{t} + \frac{e^{\frac{t3}{2}}e^{\frac{t}{4}}}{64k_1}\right)y\left(\frac{t}{2}\right)$$

$$-e^{-t}\frac{y(t-\frac{\pi}{2})}{1+y^2(t-\frac{\pi}{2})} = 0, \tag{3.27}$$

for $t \geqslant 4$ and for some $k_1 > 0$. It is easy to verify that the conditions $(H_1), (H_4), (H_6)-(H_9)$ and (H_{10}) are satisfied, so equation (3.27) satisfies the hypothesis of Theorem 3.2.9. Thus, every solution of (3.27) either oscillates or tends to zero as $t \to \infty$.

Theorem 3.2.11. *Let $0 \leqslant p(t) \leqslant p_1 < 1$. Suppose that (H_1), (H_5), $(H_6) - (H_8)$ and (H_{10}) hold. Then every solution of* (3.1) *either oscillates or converges to zero as $t \to \infty$.*

Proof. Assume that (3.1) has a nonoscillatory solution on $[t_0, \infty)$, $t_0 \geqslant 0$ and let it be $y(t)$. Hence, $y(t) > 0$ or < 0 for $t \geqslant t_0$. Suppose that $y(t) > 0$ for $t \geqslant t_0$. Let there exists $t_1 > t_0$ such that $y(\alpha(t)) > 0, y(\beta(t)) > 0, y(\sigma(t)) > 0$ for $t \geqslant t_1$. Setting (3.6), (3.7), (3.8) we get (3.9). Consequently, $v(t), b(t)v'(t), a(t)(b(t)v'(t))'$ are monotonic functions on $[t_2, \infty), t_2 \geqslant t_1$. If $v(t) > 0$ for $t \geqslant t_2$, by Lemma 3.2.3 any one of the cases $(a), (b), (d)$ holds. Proofs for cases (a) and (b) follows from Theorem 3.2.5 and for case (d) follows from Theorem 3.2.9. Thus proof of the theorem is complete. □

Example 3.2.12. Consider

$$\left(e^{-t}\left(e^{\frac{t}{4}}\left(y(t)+\frac{1}{2}y(t-\pi)\right)'\right)'\right)'+e^{\frac{7t}{8}}y\left(\frac{t}{2}\right)$$

$$-e^{-2t}\frac{y\left(t-\frac{\pi}{2}\right)}{1+y^2\left(t-\frac{\pi}{2}\right)}=0, \tag{3.28}$$

for $t \geqslant 4$. It is easy to verify that the conditions $(H_1), (H_5), (H_6)-(H_8)$ and (H_{10}) are satisfied, so equation (3.28) satisfies the hypothesis of Theorem 3.2.11. Thus, every solution of (3.28) either oscillates or tends to zero as $t \to \infty$.

Oscillation results for the range $\mathbf{-1 < p_2 \leqslant p(t) \leqslant 0}$.

Theorem 3.2.13. *Let $-1 < p_2 \leqslant p(t) \leqslant 0$. Suppose (H_1), (H_2), (H_7) and (H_8) hold. If for some $\rho \in C^1([t_0,\infty),(0,\infty))$,*

$$(H_{11})\ \limsup_{t\to\infty}\int_{t_4}^{t}\left[k_1 q(s)\rho(s)G\left(\int_{t_3}^{\alpha(s)}\frac{\int_{t_2}^{u}\frac{1}{a(\theta)}d\theta}{b(u)}du\right)G\left(\frac{1}{\int_{t_2}^{\alpha(s)}\frac{du}{a(u)}}\right)\frac{\int_{t_2}^{\alpha(s)}\frac{du}{a(u)}}{\int_{t_2}^{s}\frac{du}{a(u)}}\right.$$
$$\left.-\frac{(\rho'(s))^2a(s)}{4\rho(s)}\right]ds=\infty$$

also hold, then every solution of (3.1) *either oscillates or converges to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a nonoscillatory solution of (3.1) on $[t_0,\infty)$, $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. (The proof in the case $y(t) < 0$ eventually is similar and will be omitted.) Suppose that $y(t) > 0$ for $t \geqslant t_0$. Proceeding as in Theorem 3.2.5 we obtain (3.9). Consequently, $v(t), b(t)v'(t), a(t)(b(t)v'(t))'$ are monotonic functions on $[t_2,\infty)$, $t_2 \geqslant t_1$. If $v(t) > 0$ for $t \geqslant t_2$, then by Lemma 3.2.1 any one of the cases (a) or (b) holds. Suppose case (a) holds. Defining $w(t)$ as in (3.10) we obtain $w(t) > 0$ for $t \geqslant t_2$. Since $v'(t) > 0$, it implies that $z'(t) > 0$ due to $k'(t) < 0$. Therefore, either $z(t) >$ *or* < 0 for $t \geqslant t_3 > t_2$. Suppose $z(t) > 0$ for $t \geqslant t_3$. Thus,

$$y(t) \geqslant z(t). \tag{3.29}$$

Proceeding as in Theorem 3.2.5, from (3.12), (3.13) we obtain (3.14). Differentiating $w(t)$, we obtain

$$w'(t) = \rho'(t)\frac{a(t)(b(t)v'(t))'}{b(t)v'(t)} + \rho(t)\frac{(a(t)(b(t)v'(t))')'}{b(t)v'(t)} - \rho(t)\frac{a(t)((b(t)v'(t))')^2}{(b(t)v'(t))^2}.$$

It follows from (3.9), (3.10), (3.14) and (3.29) that

$$\begin{aligned} w'(t) &\leqslant \frac{\rho'(t)w(t)}{\rho(t)} - \frac{\rho(t)q(t)G(z(\alpha(t)))}{b(t)v'(t)} - \frac{w^2(t)}{\rho(t)a(t)} \\ &\leqslant \frac{\rho'(t)w(t)}{\rho(t)} - \frac{\rho(t)q(t)G(b(\alpha(t))v'(\alpha(t)))L_1(\alpha(t))}{b(t)v'(t)} - \frac{w^2(t)}{\rho(t)a(t)}, \end{aligned}$$

where,

$$L_1(\alpha(t)) = G\left(\frac{1}{\int_{t_2}^{\alpha(t)} \frac{du}{a(u)}}\right) G\left(\int_{t_3}^{\alpha(t)} \frac{\int_{t_2}^{s} \frac{du}{a(u)}}{b(s)} ds\right).$$

Using (H_8) and (3.13), we obtain

$$w'(t) \leqslant \frac{\rho'(t)w(t)}{\rho(t)} - \rho(t)q(t)k_1L_1(\alpha(t))L_2(t) - \frac{w^2(t)}{\rho(t)a(t)},$$

where,

$$L_2(t) = \frac{\int_{t_2}^{\alpha(t)} \frac{du}{a(u)}}{\int_{t_2}^{t} \frac{du}{a(u)}}.$$

Hence,

$$w'(t) \leqslant -\rho(t)q(t)k_1L_1(\alpha(t))L_2(t) + \frac{a(t)(\rho'(t))^2}{4\rho(t)}.$$

Integrating from $t_4(> t_3)$ to t, we obtain

$$\int_{t_4}^{t} \left[\rho(s)q(s)k_1L_1(\alpha(s))L_2(s) - \frac{a(s)(\rho'(s))^2}{4\rho(s)}\right] ds \leqslant w(t_4) - w(t),$$

taking $\limsup$ as $t \to \infty$, we get a contradiction due to condition (H_{11}).

Next, if $z(t) < 0$ for $t \geqslant t_3$, then $\lim_{t\to\infty} z(t)$ exists as $z'(t) > 0$. Let it be l_1. Hence $-\infty < l_1 \leqslant 0$. Note that $z(t) < 0$ implies

$$y(t) \ < \ -p(t)y(\sigma(t)) < y(\sigma(t)).$$

Hence, $y(t)$ is bounded function. Therefore,

$$\begin{aligned}0 \geqslant \limsup_{t\to\infty} z(t) &\geqslant \limsup_{t\to\infty}(y(t) + p_2 y(\sigma(t)))\\ &\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_2 y(\sigma(t)))\\ &= \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} y(\sigma(t))\\ &= (1+p_2)\limsup_{t\to\infty} y(t).\end{aligned}$$

Since $(1 + p_2) > 0$ then $\limsup_{t\to\infty} y(t) = 0$. So also $\lim_{t\to\infty} y(t) = 0$.

Suppose case (b) holds. Now $\lim_{t\to\infty} v(t)$ exists implies $\lim_{t\to\infty} z(t)$ exists. Hence $z(t)$ is bounded. Claim that $y(t)$ is bounded. If this is not the case, then there is an increasing sequence $\{\eta_n\}_{n=1}^{\infty}$ such that $\eta_n \to \infty$ and $y(\eta_n) \to \infty$ as $n \to \infty$ and $y(\eta_n) = max\{y(t) : t_2 \leq t \leq \eta_n\}$. We may choose n large enough such that $\sigma(\eta_n) > t_2$. Hence,

$$z(\eta_n) \;\;\geq\;\; y(\eta_n) + p_2 y(\sigma(\eta_n)) \geq (1 + p_2)y(\eta_n),$$

a contradiction to the boundedness of $z(t)$. We will claim, $\liminf_{t\to\infty} y(t) = 0$. Let $\liminf_{t\to\infty} y(t) = l_2 > 0$. Then for some $\epsilon > 0$, there exists $t_3 > t_2$ such that $y(t) > l_2 - \epsilon > 0$ for $t \geqslant t_3$. Hence from (3.9), we get

$$(a(t)(b(t)v'(t))')' \leqslant -q(t)G(l_2 - \epsilon)$$

for $t \geqslant t_4 > t_3$. Since $\lim_{t\to\infty} a(t)(b(t)v'(t))'$ exists, then

$$\int_{t_4}^{\infty} q(t)dt < \infty,$$

a contradiction to (H_7). Hence, by Lemma 3.2.4, $\lim_{t\to\infty} z(t) = 0$. Thus,

$$\begin{aligned}0 = \limsup_{t\to\infty} z(t) &\geqslant \limsup_{t\to\infty}(y(t) + p_2 y(\sigma(t)))\\ &\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_2 y(\sigma(t)))\\ &= \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} y(\sigma(t))\\ &= (1+p_2)\limsup_{t\to\infty} y(t).\end{aligned}$$

Since $(1 + p_2) > 0$, then it implies $\limsup_{t\to\infty} y(t) = 0$. So also $\lim_{t\to\infty} y(t) = 0$.

Next, suppose $v(t) < 0$ for $t \geqslant t_2$. In view of Lemma 3.2.1 any one of the case (b) or (d) holds. In both the cases (b) and (d) $v'(t) < 0$. Thus $-\infty \leqslant \lim_{t\to\infty} v(t) < 0$.

Claim $y(t)$ is bounded. If this is not the case, then there is an increasing sequence $\{\eta_n\}_{n=1}^{\infty}$ such that $\eta_n \to \infty$ and $y(\eta_n) \to \infty$ as $n \to \infty$ and $y(\eta_n) = max\{y(t) : t_2 \leq t \leq \eta_n\}$. We may choose n large enough such that $\sigma(\eta_n) > t_2$. Hence

$$v(\eta_n) \;\geq\; y(\eta_n) + p_2 y(\sigma(\eta_n)) + k(\eta_n) \geq (1 + p_2)y(\eta_n) + k(\eta_n).$$

Since $(1 + p_2) > 0$, then $v(\eta_n) > 0$ for large n which is a contradiction. Hence $v(t)$ is bounded. If $-\infty < \lim_{t\to\infty} v(t) = (l_3) < 0$, then $-\infty < \lim_{t\to\infty} z(t) < 0$. Thus,

$$\begin{aligned} 0 > \limsup_{t\to\infty} z(t) &\geqslant \limsup_{t\to\infty}(y(t) + p_2 y(\sigma(t))) \\ &\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_2 y(\sigma(t))) \\ &= \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} y(\sigma(t)) \\ &= (1 + p_2) \limsup_{t\to\infty} y(t). \end{aligned}$$

Since $(1 + p_2) > 0$, then it implies $\limsup_{t\to\infty} y(t) < 0$, a contradiction to the fact that $y(t) > 0$. Hence the theorem is proved. □

Example 3.2.14. Consider

$$\left(\frac{1}{t}\left(t^{\frac{1}{2}}\left(y(t) - \frac{1}{2}y(t - 2\pi)\right)'\right)'\right)' + \left(\frac{t^{-\frac{1}{2}}}{2} + \frac{3t^{-\frac{5}{2}}}{8} + t^{-\frac{9}{2}}\right) y\left(t - \frac{7\pi}{2}\right)$$

$$-t^{-\frac{9}{2}}\left(1 + \sin^2\left(t - \frac{3\pi}{2}\right)\right)\frac{y\left(t - \frac{3\pi}{2}\right)}{1 + y^2\left(t - \frac{3\pi}{2}\right)} = 0, \tag{3.30}$$

for $t \geqslant 12$. It is easy to verify that the conditions $(H_1), (H_2), (H_7), (H_8)$ and (H_{11}) are satisfied, so equation (3.30) satisfies the hypothesis of Theorem 3.2.13. Thus, every solution of (3.30) either oscillates or tends to zero as $t \to \infty$. Indeed, $y(t) = \sin t$ is such a solution of (3.30).

Theorem 3.2.15. *Let* $-1 < p_2 \leqslant p(t) \leqslant 0$. *Suppose that* (H_1), (H_3), (H_7), (H_8), (H_{11}) *and*

(H_{12}) $$\limsup_{t\to\infty}\int_{t_4}^{t}\left[k_1q(s)\delta(s)G\Big(\int_{t_3}^{\alpha(s)}\frac{du}{b(u)}\Big)-\tfrac{1}{4\delta(s)a(s)}\right]ds=\infty,$$
where $\delta(t)=\int_t^\infty\frac{ds}{a(s)}$

hold, then every solution of (3.1) *either oscillates or converges to zero as* $t\to\infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (3.1) on $[t_0,\infty)$, $t_0\geqslant 0$, say $y(t)$ is an eventually positive solution. (The proof in the case $y(t)<0$ eventually is similar and will be omitted.) Suppose that $y(t)>0$ for $t\geqslant t_0$. Proceeding as Theorem 3.2.5 we obtain (3.9). Consequently, $v(t), b(t)v'(t), a(t)(b(t)v'(t))'$ are monotonic functions on $[t_2,\infty)$, $t_2\geqslant t_1$. If $v(t)>0$ for $t\geqslant t_2$, by Lemma 3.2.2 any one of the cases (a)-(c) holds. Suppose case (a) or (b) holds. Then proceeding same as in Theorem 3.2.13 we obtained our desired results. Let case (c) holds. As in case (c), $v'(t)>0$, it implies that $z'(t)>0$ for $t\geqslant t_2$ due to $k'(t)<0$. Suppose $z(t)>0$ for $t\geqslant t_3>t_2$. Proceeding as in the proof of Theorem 3.2.7, we obtain (3.19) and (3.20). Differentiating (3.19), we obtain

$$\phi'(t) = \frac{(a(t)(b(t)v'(t))')'}{b(t)v'(t)}-\frac{a(t)((b(t)v'(t))')^2}{(b(t)v'(t))^2}. \tag{3.31}$$

From (3.29) we obtain $y(t)\geqslant z(t)$. Using (3.9), (3.19) and (3.29) in (3.31), we obtain

$$\phi'(t) \leqslant -\frac{q(t)G(z(\alpha(t)))}{b(t)v'(t)}-\frac{\phi^2(t)}{a(t)}. \tag{3.32}$$

Using (3.22) and (3.23) in (3.32), we obtain

$$\phi'(t)\leqslant -\frac{q(t)G\Big(b(\alpha(t))v'(\alpha(t))\int_{t_3}^{\alpha(t)}\frac{ds}{b(s)}\Big)}{b(t)v'(t)}-\frac{\phi^2(t)}{a(t)}. \tag{3.33}$$

Since $(b(t)v'(t))'<0$, then using (H_8) in the last inequality, we get

$$\phi'(t)\leqslant -q(t)k_1G\Big(\int_{t_3}^{\alpha(t)}\frac{ds}{b(s)}\Big)-\frac{\phi^2(t)}{a(t)}$$

for $t\geqslant t_4>t_3$. Multiplying both sides by $\delta(t)$ and integrating from t_4 to t, we obtain

$$\delta(t)\phi(t)-\delta(t_4)\phi(t_4)+\int_{t_4}^{t}\frac{\phi(s)}{a(s)}ds+\int_{t_4}^{t}\delta(s)q(s)k_1G\Big(\int_{t_3}^{\alpha(s)}\frac{du}{b(u)}\Big)ds$$
$$+\int_{t_4}^{t}\frac{\phi^2(s)\delta(s)ds}{a(s)}\leqslant 0.$$

It follows from (3.20) that

$$\int_{t_4}^{t}\left[k_1 q(s)\delta(s)G\bigg(\int_{t_3}^{\alpha(s)}\frac{du}{b(u)}\bigg)-\frac{1}{4\delta(s)a(s)}\right]ds \leqslant 1+\phi(t_4)\delta(t_4), \tag{3.34}$$

a contradiction to (H_{12}).

If $z(t) < 0$ for $t \geqslant t_3$, then $y(t)$ is bounded. Since $z'(t) > 0$, then $\lim_{t\to\infty} z(t)$ exists. Now $-\infty < \lim_{t\to\infty} z(t) = l_4 \leqslant 0$. Thus,

$$\begin{aligned} 0 \geqslant \limsup_{t\to\infty} z(t) &= \limsup_{t\to\infty}(y(t)+p(t)y(\sigma(t))) \\ &\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_2 y(\sigma(t))) \\ &= \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} y(\sigma(t)) \\ &= (1+p_2)\limsup_{t\to\infty} y(t). \end{aligned}$$

Since $(1+p_2) > 0$, we obtain $\limsup_{t\to\infty} y(t) = 0$. Hence, $\lim_{t\to\infty} y(t) = 0$.

If $v(t) < 0$ for $t \geqslant t_2$, by Lemma 3.2.2 any one of the cases $(b), (c), (d)$ holds. For the cases (b) and (d) we may proceed as Theorem 3.2.13 to obtain the desired results. Suppose case (c) is true. Now $v(t) < 0$ and $v'(t) > 0$ for $t \geqslant t_2$ implies $\lim_{t\to\infty} v(t)$ exists. let it be l_5. Now $-\infty < l_5 \leqslant 0$ implies $\lim_{t\to\infty} z(t) \leqslant 0$, as $\lim_{t\to\infty} z(t) = \lim_{t\to\infty} v(t)$. Note that $v(t) < 0$ implies $y(t)$ is bounded. Hence,

$$\begin{aligned} 0 \geqslant \limsup_{t\to\infty} z(t) &\geqslant \limsup_{t\to\infty}(y(t)+p_2 y(\sigma(t))) \\ &\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_2 y(\sigma(t))) \\ &= \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} y(\sigma(t)) \\ &= (1+p_2)\limsup_{t\to\infty} y(t) \end{aligned}$$

as $(1+p_2) > 0$, we obtain $\lim_{t\to\infty} y(t) = 0$. Hence the theorem is proved. □

Example 3.2.16. Consider

$$\left(t^{\frac{4}{3}}\left(y(t)-\frac{1}{2}y\left(\frac{t}{2}\right)\right)''\right)' + \frac{1}{k_1}\left(\frac{t^{-\frac{5}{3}}}{36}+\frac{t^{-1}}{3}\right)y\left(t-\frac{\pi}{2}\right)$$

$$-t^{-5}(1+t^2)\frac{y(t-\pi)}{1+y^2(t-\pi)} = 0, \tag{3.35}$$

for $t \geqslant 4$ and for some $k_1 > 0$. It is easy to verify that the conditions $(H_1), (H_3), (H_7), (H_8)$, (H_{11}) and (H_{12}) are satisfied, so equation (3.35) satisfies the hypothesis of Theorem 3.2.15. Thus, every solution of (3.35) either oscillates or tends to zero as $t \to \infty$.

Theorem 3.2.17. *Let* $-1 < p_2 \leqslant p(t) \leqslant 0$. *Suppose* $(H_1), (H_4), (H_7), (H_8), (H_{11}), (H_{12})$ *and*

(H_{13}) $\quad \int_{t_4}^{\infty} \frac{1}{b(t)} \int_{t_4}^{t} \frac{1}{a(s)} \int_{t_4}^{s} q(u)dudsdt = \infty$

hold, then every solution of (3.1) *either oscillates or converges to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (3.1) on $[t_0, \infty)$, $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. (The proof in the case $y(t) < 0$ eventually is similar and will be omitted.) Suppose that $y(t) > 0$ for $t \geqslant t_0$. Proceeding as in Theorem 3.2.5 we obtain (3.9). Consequently, $v(t), b(t)v'(t), a(t)(b(t)v'(t))'$ are monotonic functions on $[t_2, \infty)$, $t_2 \geqslant t_1$. If $v(t) > 0$ for $t \geqslant t_2$, then any one of the cases (a)-(d) holds. For the cases (a) and (b) the proofs follows from Theorem 3.2.13 and for case (c) it follows from Theorem 3.2.15. If case (d) holds, now as $\lim_{t\to\infty} v(t)$ exists in case (d), it implies that $\lim_{t\to\infty} z(t)$ exists. Claim $\liminf_{t\to\infty} y(t) = 0$. If not, let it be l_6, where $l_6 > 0 (\neq 0)$. For some $\epsilon > 0$, there exists $t_3 > t_2$ such that $y(t) > l_6 - \epsilon > 0$ for $t \geqslant t_3$. From (3.9) it follows that

$$(a(t)(b(t)v'(t))')' \leqslant -q(t)G(l_6 - \epsilon)$$

for $t \geqslant t_4 > t_3$. As in case (d), $(b(t)v'(t))' < 0$ and $v'(t) < 0$, integrating from t_4 to t, we obtain

$$0 > a(t_4)(b(t_4)v'(t_4))' \geqslant a(t)(b(t)v'(t))' + G(l_6 - \epsilon) \int_{t_4}^{t} q(s)ds.$$

Further integrating from t_4 to t, we obtain

$$0 > b(t_4)v'(t_4) \geqslant b(t)v'(t) + G(l_6 - \epsilon) \int_{t_4}^{t} \frac{1}{a(s)} \int_{t_4}^{s} q(u)duds.$$

Again integrating from t_4 to t, we obtain

$$0 \geqslant v(t) - v(t_4) + G(l_6 - \epsilon) \int_{t_4}^{t} \frac{1}{b(s)} \int_{t_4}^{s} \frac{1}{a(u)} \int_{t_4}^{u} q(v)dvduds.$$

Since $\lim_{t\to\infty} v(t)$ exists, then it implies that

$$\int_{t_4}^{\infty} \frac{1}{b(t)} \int_{t_4}^{t} \frac{1}{a(s)} \int_{t_4}^{s} q(u)dudsdt < \infty,$$

a contradiction to (H_{13}). Hence, $l_6 = 0$. By Lemma 3.2.4 it follows that $\lim_{t\to\infty} z(t) = 0$. Thus, $z(t)$ is bounded. We can prove that $y(t)$ is also bounded. Thus,

$$\begin{aligned} 0 = \limsup_{t\to\infty} z(t) &\geqslant \limsup_{t\to\infty}(y(t) + p_2 y(\sigma(t))) \\ &\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_2 y(\sigma(t))) \\ &= \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} y(\sigma(t)) \\ &= (1 + p_2) \limsup_{t\to\infty} y(t). \end{aligned}$$

Since $(1 + p_2) > 0$, then it follows that $\lim_{t\to\infty} y(t) = 0$.

Next suppose that $v(t) < 0$ for $t \geqslant t_2$, then $y(t)$ is also bounded. Note that $\lim_{t\to\infty} v(t) = \lim_{t\to\infty} z(t)$. Let $\lim_{t\to\infty} z(t) = l_7$. In case (a) and (c) $v'(t) > 0$. Hence, $-\infty < \lim_{t\to\infty} v(t) \leqslant 0$. Thus,

$$\begin{aligned} 0 \geqslant \limsup_{t\to\infty} z(t) &\geqslant \limsup_{t\to\infty}(y(t) + p_2 y(\sigma(t))) \\ &\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_2 y(\sigma(t))) \\ &= \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} y(\sigma(t)) \\ &= (1 + p_2) \limsup_{t\to\infty} y(t). \end{aligned}$$

Since $(1 + p_2) > 0$, then it implies $\limsup_{t\to\infty} y(t) = 0$. So also $\lim_{t\to\infty} y(t) = 0$.

In case (b) and (d) $v'(t) < 0$, thus $-\infty \leqslant \lim_{t\to\infty} v(t) < 0$. Note that $l_7 = -\infty$ case is not possible due to boundedness of $y(t)$.
If $-\infty < l_7 < 0$, then

$$\begin{aligned} 0 > \limsup_{t\to\infty} z(t) &\geqslant \limsup_{t\to\infty}(y(t) + p_2 y(\sigma(t))) \\ &\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_2 y(\sigma(t))) \\ &= \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} y(\sigma(t)) \\ &= (1 + p_2) \limsup_{t\to\infty} y(t). \end{aligned}$$

Since $(1+p_2) > 0$, then it implies $\limsup_{t\to\infty} y(t) < 0$, a contradiction to the fact that $y(t) > 0$. This completes the proof of the theorem. □

Example 3.2.18. Consider

$$\left(e^{\frac{t}{4}}\left(e^{\frac{t}{2}}\left(y(t) - \frac{1}{2}y\left(t-\pi\right)\right)'\right)'\right)' + \left(e^{t} + \frac{e^{\frac{t_3}{2}}e^{\frac{t}{4}}}{128k_1}\right)y\left(\frac{t}{2}\right)$$

$$-e^{-t}\frac{y(t-\frac{\pi}{2})}{1+y^2(t-\frac{\pi}{2})} = 0, \tag{3.36}$$

for $t \geqslant 4$ and for some $k_1 > 0$. It is easy to verify that the conditions $(H_1), (H_4), (H_7), (H_8)$ and $(H_{11}) - (H_{13})$ are satisfied, so equation (3.36) satisfies the hypothesis of Theorem 3.2.17. Thus, every solution of (3.36) oscillates or tends to zero as $t \to \infty$.

Theorem 3.2.19. *Let $-1 < p_2 \leqslant p(t) \leqslant 0$. If $(H_1), (H_5), (H_7), (H_8), (H_{11})$ and (H_{13}) hold, then every solution of* (3.1) *either oscillates or converges to zero as $t \to \infty$.*

Proof. Assume that (3.1) has a nonoscillatory solution on $[t_0, \infty)$, $t_0 \geqslant 0$ and let it be $y(t)$. Hence, $y(t) > 0$ or < 0 for $t \geqslant t_0$. Suppose that $y(t) > 0$ for $t \geqslant t_0$. Proceeding as Theorem 3.2.5 we obtain (3.9). Consequently, $v(t), b(t)v'(t), a(t)(b(t)v'(t))'$ are monotonic functions on $[t_2, \infty)$, $t_2 \geqslant t_1$. Suppose $v(t) > 0$ for $t \geqslant t_2$. Then from Lemma 3.2.3 any one of the cases $(a), (b), (d)$ holds. Suppose case (a) or (b) holds. The proof for these cases follows from the proof of Theorem 3.2.13. If case (d) holds, then the proof for this case follows from Theorem 3.2.17.
If $v(t) < 0$ for $t \geqslant t_2$, then also from Lemma 3.2.3 any one of (a), (b) or (d) holds. The proofs for these cases follows from Theorem 3.2.17. □

Example 3.2.20. Consider

$$\left(e^{-t}\left(e^{\frac{t}{4}}\left(y(t) - \frac{1}{2}y(t-\pi)\right)'\right)'\right)' + e^{\frac{7t}{8}}y\left(\frac{t}{2}\right)$$

$$-e^{-2t}\frac{y\left(t-\frac{\pi}{2}\right)}{1+y^2\left(t-\frac{\pi}{2}\right)} = 0, \tag{3.37}$$

for $t \geqslant 4$. It is easy to verify that the conditions $(H_1), (H_5), (H_7), (H_8)$, (H_{11}) and (H_{13}) are satisfied, so equation (3.37) satisfies the hypothesis of Theorem 3.2.19. Thus, every solution of (3.37) either oscillates or tends to zero as $t \to \infty$.

3.3 Odd Higher Order NDDE; $n \geqslant 3$

In this section, we are concerned with the oscillatory and asymptotic behavior of all solutions of the higher order nonlinear neutral delay differential equations of the form

$$(a(t)(b(t)(y(t)+p(t)y(\sigma(t)))')')^{(n-2)}+q(t)G(y(\alpha(t)))$$

$$-h(t)H(y(\beta(t)))=0, \tag{3.38}$$

where $a, b, q \in C([t_0,\infty),(0,\infty))$, $h \in C([t_0,\infty),[0,\infty))$, p, σ, α, $\beta \in C([t_0,\infty),\mathbb{R})$, $\sigma(t) \leqslant t$, $\alpha(t) \leqslant t$, $\beta(t) \leqslant t$, $\lim_{t\to\infty}\sigma(t)=\infty$, $\lim_{t\to\infty}\alpha(t)=\infty$, $\lim_{t\to\infty}\beta(t)=\infty$, G and $H \in C(\mathbb{R},\mathbb{R})$ with $uG(u)>0$, $vH(v)>0$ for u, $v \neq 0$, $n(\geqslant 3)$ is odd number, H is bounded, G is non-decreasing under the assumptions $(H_2),(H_3),(H_4),(H_5)$ and

$$\int_{t_0}^{\infty}\frac{1}{b(t)}\int_{t}^{\infty}\frac{1}{a(s)}\int_{s}^{\infty}u^{n-3}h(u)\,duds dt<\infty, \tag{H_{14}}$$

for $0 \leqslant p(t) \leqslant p_1 < 1$ and $-1 < p_2 \leqslant p(t) \leqslant 0$.
Clearly, equations (3.1) and (3.2) are particular cases of equations (3.38).

By a solution of (3.38) we understand a function $y(t) \in C([T_y,\infty)), T_y \geqslant t_0 \geqslant 0$ such that $y(t)+p(t)y(\sigma(t)) \in C^1([T_y,\infty))$, $b(t)(y(t)+p(t)y(\sigma(t)))' \in C^1([T_y,\infty))$, $a(t)(b(t)(y(t)+p(t)y(\sigma(t)))')' \in C^{(n-2)}([T_y,\infty))$ and satisfies (3.38) on $[T_y,\infty)$. We consider only those solutions $y(t)$ of (3.38) which satisfies $\sup\{|y(t)|; t \geqslant T\} > 0$ for every $T \geqslant T_y$. We assume that (3.38) has such a solution. A solution of (3.38) is said to be oscillatory if it has arbitrarily large zeros on $[T_y,\infty)$; otherwise, it is called nonoscillatory.

3.4 Oscillation Criteria for Odd Higher Order Homogeneous Equation

In this section, sufficient conditions are obtained for oscillatory and asymptotic behavior of all solutions of (3.38). We need the following conditions and lemma for our use in the sequel.

(H_{15}) $G(-u) = -G(u), H(-u) = -H(u)$ for $u \in \mathbb{R}$;

(H_{16}) $\int_{t^*}^{\infty} \frac{1}{a(t)} \int_{t^*}^{t} (t-s)^{n-3} q(s) ds dt = \infty$;

(H_{17}) $\int_{t^*}^{\infty} \frac{1}{b(t)} \int_{t^*}^{t} \frac{1}{a(v)} \int_{t^*}^{v} (v-s)^{n-3} q(s) ds dv dt = \infty$; $t^* \geq t_0$.

Lemma 3.4.1. [51], ([56], p. 193) Let $y \in C^{(n)}([0,\infty),\mathbb{R})$ be of constant sign. Let $y^{(n)}(t)$ be of constant sign and $\not\equiv 0$ in any interval $[T,\infty)$, $T \geq 0$, and $y^{(n)}(t)y(t) \leqslant 0$. Then there exists a number $t_0 \geqslant 0$ such that the functions $y^{(j)}(t)$, $j = 1,2,...,n-1$ are of constant sign on $[t_0,\infty)$ and there exists a number $k \in \{1,3,...,n-1\}$ when n is even or $k \in \{0,2,..,n-1\}$ when n is odd such that

$$y(t)y^{(j)}(t) > 0 \quad for \;\; j = 0,1,2,...,k, \quad t \geq t_0$$
$$(-1)^{n+j-1} y(t)y^{(j)}(t) > 0 \;\; for \;\; j = k+1, k+2, ..., n-1, \quad t \geq t_0.$$

Oscillation results for the range $\mathbf{0 \leqslant p(t) \leqslant p_1 < 1}$.

Theorem 3.4.2. *Let $0 \leqslant p(t) \leqslant p_1 < 1$. Suppose (H_2), (H_7), (H_{14}) and (H_{15}) hold, then every solution of* (3.38) *either oscillates or converges to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a non-oscillatory solution of (3.38) on $[t_0,\infty)$, $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. Then there exists $t_1 > t_0$ such that $y(\alpha(t)) > 0$, $y(\beta(t)) > 0$, $y(\sigma(t)) > 0$ for $t \geqslant t_1$. Set $z(t)$ as in (3.6) and

$$k(t) = \frac{1}{(n-3)\,!} \int_t^{\infty} \frac{1}{b(s)} \int_s^{\infty} \frac{1}{a(\theta)} \int_{\theta}^{\infty} (u-\theta)^{n-3} h(u) H(y(\beta(u))) du d\theta ds. \quad (3.39)$$

Note that condition (H_{14}) and the fact that H is bounded function implies that $k(t)$ exists for all t. Let us define

$$v(t) = z(t) + k(t), \quad (3.40)$$

where $k(t)$ is defined as in (3.39), then

$$w^{(n-2)}(t) = -q(t)G(y(\alpha(t))) \leqslant 0 (\not\equiv 0), \quad (3.41)$$

where

$$w(t) = a(t)(b(t)v'(t))' \quad (3.42)$$

for $t \geqslant t_1$, here $w^{(n-2)}(t)$ represents the $(n-2)^{th}$ derivative of $'w'$ w.r.t $'t'$. Clearly, $w^{(n-3)}(t), w^{(n-4)}(t), ..., w'(t), w(t)$ are monotonic functions and of constant sign for $t \in [t_2, \infty)$, $t_2 \geqslant t_1$.

If $w(t) > 0$ for $t \geqslant t_2$, then in view of Lemma 3.4.1, $w^{(n-3)}(t) > 0$ for $t \geqslant t_2$.

Now $w(t) > 0$ implies $(b(t)v'(t))' > 0$ for $t \geqslant t_2$, which in turn implies $b(t)v'(t)$ is eventually monotonic function. Since $b(t) > 0$, then either $v'(t) > 0$ or < 0 for $t \geqslant t_3 > t_2$.

Case I. If $v'(t) > 0$ for $t \geqslant t_3$, then $z'(t) > 0$ eventually. Therefore, $(1-p_1)z(t) < (1-p(t))z(t) < z(t) - p(t)z(\sigma(t)) < y(t) - p(t)p(\sigma(t))y(\sigma(\sigma(t)))$, which implies

$$(1-p_1)z(t) < y(t)$$

for $t \geqslant t_4 > t_3$. From (3.41), $z'(t) > 0$ and by using the last inequality, we obtain

$$w^{(n-2)}(t) \leqslant -q(t)G((1-p_1)z(t_4))$$

for $t \geqslant t_5 > t_4$. Then integrating the preceeding inequality from $'t_5'$ to $'t'$ we obtain

$$\infty > w^{(n-3)}(t_5) > -w^{(n-3)}(t) + w^{(n-3)}(t_5) \geqslant G((1-p_1)z(t_4)) \int_{t_5}^{t} q(s)ds.$$

Since $\lim_{t\to\infty} w^{(n-3)}(t) < \infty$, then taking $t \to \infty$ in the last inequality we have $\int_{t_5}^{\infty} q(t)dt < \infty$, a contradiction to (H_7).

Case II. If $v'(t) < 0$ for $t \geqslant t_3$, we may note that $\lim_{t\to\infty} v(t)$ exists and equal to l (say). We will claim $l = 0$. If it is not true, then for every $\epsilon > 0$, there exists $t_4 > t_3$ such that $l < v(t) < l + \epsilon$ for $t \geqslant t_4$. Choose $0 < \epsilon < \frac{l(1-p_1)}{1+p_1}$. Since $\lim_{t\to\infty} k(t) = 0$, then for the same chosen ϵ, $k(t) < \epsilon$ for $t \geqslant t_5 > t_4$. Thus,

$$\begin{aligned} y(t) &= v(t) - p(t)y(\sigma(t)) - k(t) \\ &> v(t) - p(t)v(\sigma(t)) - k(t) \\ &> l - p_1(l+\epsilon) - \epsilon \end{aligned}$$

for $t \geqslant t_6 > t_5$. Now,

$$y(t) > (l-\epsilon) - p_1(l+\epsilon) > k_2(l+\epsilon) > k_2 v(t)\Big(> k_2 l\Big). \tag{3.43}$$

By the choice of ϵ, we can show that $k_2 > 0$. Using (3.43) in (3.41), we obtain

$$w^{(n-2)}(t) \leqslant -q(t)G(k_2 l) \tag{3.44}$$

for $t \geqslant t_7 > t_6$. Integrating (3.44) from $'t_7'$ to $'t'$, we obtain

$$\infty > w^{(n-3)}(t_7) > -w^{(n-3)}(t) + w^{(n-3)}(t_7) \geqslant G(k_2 l)\int_{t_7}^{t} q(s)ds.$$

Since $\lim_{t\to\infty} w^{(n-3)}(t) < \infty$, then taking $t \to \infty$ in the last inequality we obtain $\int_{t_7}^{\infty} q(t)dt < \infty$, a contradiction to (H_7). Therefore, $\lim_{t\to\infty} v(t) = 0$ and hence $\lim_{t\to\infty} z(t) = 0$. Since $y(t) \leqslant z(t)$, then $\lim_{t\to\infty} y(t) = 0$.

If $w(t) < 0$ for $t \geqslant t_2$, then $(b(t)v'(t))' < 0$ for $t \geqslant t_2$. Thus, $v'(t) > 0$ or $v'(t) < 0$ for $t \geqslant t_3 > t_2$.

Case III. Suppose $v'(t) > 0$ for $t \geqslant t_3$. Since $w^{(n-2)}(t) \leqslant 0$ eventually, then $w^{(n-3)}(t) > 0$ or < 0 eventually.

Subcase (i): Suppose $w^{(n-3)}(t) > 0$ eventually. Now $v'(t) > 0$ and $k'(t) < 0$ implies $z'(t) > 0$. Therefore, $(1-p_1)z(t) < (1-p(t))z(t) < z(t) - p(t)z(\sigma(t)) < y(t) - p(t)p(\sigma(t))y(\sigma(\sigma(t)))$, which implies

$$(1-p_1)z(t) < y(t) \tag{3.45}$$

for $t \geqslant t_3$. From (3.41) and $z'(t) > 0$, we obtain

$$w^{(n-2)}(t) \leqslant -q(t)G((1-p_1)z(t_3))$$

for $t \geqslant t_4 > t_3$. Then integrating the last inequality from $'t_4'$ to $'t'$, we obtain

$$\infty > w^{(n-3)}(t_4) > -w^{(n-3)}(t) + w^{(n-3)}(t_4) \geqslant G((1-p_1)z(t_3))\int_{t_4}^{t} q(s)ds.$$

Since $\lim_{t\to\infty} w^{(n-3)}(t) < \infty$, then taking $t \to \infty$ in the last inequality we have $\int_{t_4}^{\infty} q(t)dt < \infty$, a contradiction to (H_7).

Subcase (ii): If $w^{(n-3)}(t) < 0$ eventually, then from (3.41) we can conclude that $w^{(n-4)}(t) < 0, ..., w'(t) < 0$ for large t. Since $w'(t) < 0$ for $t > t_4(> t_3)$, then $w(t) < w(t_4)$, that is,

$$a(t)(b(t)v'(t))' < a(t_4)(b(t_4)v'(t_4))'.$$

Integrating the preceeding inequality from $'t_4'$ to $'t'$, we obtain

$$b(t)v'(t) < b(t_4)v'(t_4) + a(t_4)(b(t_4)v'(t_4))' \int_{t_4}^{t} \frac{ds}{a(s)}.$$

Using (H_2) in the preceeding inequality, we obtain $b(t)v'(t) \to -\infty$ as $t \to \infty$, a contradiction to the fact that $v'(t) > 0$.

Case IV. Suppose $v'(t) < 0$ for $t \geqslant t_3$. Then, integrating $(b(t)v'(t))' < 0$ twice from $'t_3'$ to $'t'$, we obtain

$$v(t) \leqslant v(t_3) + b(t_3)v'(t_3) \int_{t_3}^{t} \frac{ds}{b(s)}.$$

Using (H_2) in the preceeding inequality, we obtain $v(t) \to -\infty$ as $t \to \infty$, a contradiction to the fact that $v(t) > 0$.

Finally, we suppose that $y(t) < 0$ for $t \geqslant t_0$. From (H_{15}), we note that $G(-u) = -G(u)$ and $H(-u) = -H(u), u \in \mathbb{R}$. Hence putting $x(t) = -y(t)$ for $t \geqslant t_0$, we obtain $x(t) > 0$ and

$$(a(t)(b(t)(x(t) + p(t)x(\sigma(t)))')')^{(n-2)} + q(t)G(x(\alpha(t))) - h(t)H(x(\beta(t))) = 0.$$

Proceeding as above, we can show that every solution of (3.38) either oscillates or converges to zero as $t \to \infty$. This completes the proof of the theorem.

□

Example 3.4.3. Consider the fifth order differential equation

$$\left(y(t) + \frac{1}{2}y(t-\pi)\right)^{(v)} + \left(\frac{1}{2} + e^{-t}\right)y\left(t - \frac{\pi}{2}\right)$$

$$-e^{-t}\left(1+\sin^2\left(t-\frac{\pi}{2}\right)\right)\frac{y\left(t-\frac{\pi}{2}\right)}{1+y^2\left(t-\frac{\pi}{2}\right)}=0, \quad t\geq 4. \tag{3.46}$$

It is easy to verify that the hypothesis of Theorem 3.4.2 are satisfied. Thus, every solution of (3.46) either oscillates or tends to zero as $t\to\infty$. Indeed, $y(t)=\sin t$ is such an oscillatary solution of (3.46).

Theorem 3.4.4. *Let* $0\leqslant p(t)\leqslant p_1<1$. *Suppose that* $(H_3),(H_7),(H_{14})$, (H_{15}) *and* (H_{16}) *hold, then every solution of* (3.38) *either oscillates or converges to zero as* $t\to\infty$.

Proof. Let $y(t)$ be a non-oscillatory solution of (3.38) on $[t_0,\infty)$, $t_0\geqslant 0$, say $y(t)$ is an eventually positive solution. (The proof in case $y(t)<0$ eventually is similar and will be omitted.) Then there exists $t_1>t_0$ such that $y(\alpha(t))>0, y(\beta(t))>0, y(\sigma(t))>0$ for $t\geqslant t_1$. Setting $z(t),k(t)$, $v(t)$ as in (3.6), (3.39) and (3.40) respectively, we get (3.41) and (3.42) for $t\geqslant t_1$. Clearly, $w^{(n-3)}(t),w^{(n-4)}(t),..,w'(t),w(t)$ are monotonic functions and of constant sign for $t\in[t_2,\infty)$, $t_2\geqslant t_1$.

If $w(t)>0$ for $t\geqslant t_2$, in view of Lemma 3.4.1, $w^{(n-3)}(t)>0$ for $t\geqslant t_2$. Now $w(t)>0$ implies $(b(t)v'(t))'>0$ for $t\geqslant t_2$, which in turn implies $b(t)v'(t)$ is monotonic function. Thus, $v'(t)>0$ or <0 for $t\geqslant t_3>t_2$.

Case I. If $v'(t)>0$ for $t\geqslant t_3$, proceeding same as in Case I of Theorem 3.4.2, we obtain a contradiction due to (H_7).

Case II. If $v'(t)<0$ for $t\geqslant t_3$, proceeding same as in Case II of Theorem 3.4.2, we obtain $\lim_{t\to\infty}y(t)=0$.

If $w(t)<0$ for $t\geqslant t_2$, then $(b(t)v'(t))'<0$ for $t\geqslant t_2$. Thus $v'(t)>0$ or $v'(t)<0$ for $t\geqslant t_3>t_2$.

Case III. Suppose $v'(t)>0$ for $t\geqslant t_3$. Since $w^{(n-2)}(t)\leqslant 0$ eventually, then either

$w^{(n-3)}(t) > 0$ or < 0 eventually.

Subcase (i): If $w^{(n-3)}(t) > 0$ eventually, proceeding same as in *Subcase (i)* of Case III of Theorem 3.4.2, we obtain a contradiction due to (H_7).

Subcase (ii): If $w^{(n-3)}(t) < 0$ eventually, then from (3.41) it implies that $w^{(n-4)}(t) < 0, ..., w'(t) < 0$ for large t. Now $v'(t) > 0$ implies $z'(t) > 0$ eventually. Therefore from (3.41) and (3.45), we obtain

$$0 \geqslant w^{(n-2)}(t) + q(t)G((1-p_1)z(t_3))$$

for $t \geqslant t_4 > t_3$. Integrating the last inequality consecutively $(n-2)$ times from $'t_4'$ to $'t'$, we obtain

$$0 > w(t_4) \geqslant w(t) + \frac{1}{(n-3)\,!}\int_{t_4}^{t}(t-s)^{n-3}q(s)G((1-p_1)z(t_3))ds.$$

Hence,

$$0 > (b(t)v'(t))' + \frac{1}{(n-3)\,!}\frac{1}{a(t)}\int_{t_4}^{t}(t-s)^{n-3}q(s)G((1-p_1)z(t_3))ds.$$

Further integrating the preceeding inequality from $'t_4'$ to $'t'$, we obtain

$$b(t_4)v'(t_4) \geqslant b(t)v'(t) + \frac{1}{(n-3)\,!}\int_{t_4}^{t}\frac{1}{a(v)}\int_{t_4}^{v}(v-s)^{n-3}q(s)G((1-p_1)z(t_3))dsdv.$$

Since $\lim_{t\to\infty} b(t)v'(t) < \infty$, then from the last inequality for large t, we obtain

$$\frac{1}{(n-3)\,!}\int_{t_4}^{\infty}\frac{1}{a(t)}\int_{t_4}^{t}(t-s)^{n-3}q(s)G((1-p_1)z(t_3))dsdt < \infty,$$

a contradiction to (H_{16}).

Case IV. Since $v'(t) < 0$ and $(b(t)v'(t))' < 0$ for $t \geqslant t_3$, then using (H_3) and proceeding same as in Case IV of Theorem 3.4.2, we obtain a contradiction to the fact that $v(t) > 0$. Hence the proof of the theorem is complete. □

Theorem 3.4.5. *Let* $0 \leqslant p(t) \leqslant p_1 < 1$. *Suppose that* (H_4), (H_7) *and* $(H_{14}) - (H_{17})$ *hold, then every solution of* (3.38) *either oscillates or tends to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a non-oscillatory solution of (3.38) on $[t_0, \infty)$, $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. (The proof in case $y(t) < 0$ eventually is similar and will be omitted.) Then there exists $t_1 > t_0$ such that $y(\alpha(t)) > 0, y(\beta(t)) > 0, y(\sigma(t)) > 0$ for $t \geqslant t_1$. Setting $z(t), k(t)$, $v(t)$ as in (3.6), (3.39) and (3.40) respectively, we get (3.41) and (3.42) for $t \geqslant t_1$. Clearly, $w^{(n-3)}(t), w^{(n-4)}(t), .., w'(t), w(t)$ are monotonic functions and of constant sign for $t \in [t_2, \infty)$, $t_2 \geqslant t_1$.

If $w(t) > 0$ for $t \geqslant t_2$, in view of Lemma 3.4.1, $w^{(n-3)}(t) > 0$ for $t \geqslant t_2$. Now $w(t) > 0$ implies $(b(t)v'(t))' > 0$ for $t \geqslant t_2$, which in turn implies $b(t)v'(t)$ is monotonic function. Thus, $v'(t) > 0$ or < 0 for $t \geqslant t_3 > t_2$.

Case I. If $v'(t) > 0$ for $t \geqslant t_3$ then $z'(t) > 0$ eventually. Therefore, $(1 - p_1)z(t) < (1 - p(t))z(t) < z(t) - p(t)z(\sigma(t)) < y(t) - p(t)p(\sigma(t))y(\sigma(\sigma(t)))$, which implies

$$(1 - p_1)z(t) < y(t)$$

for $t \geqslant t_4 > t_3$. From (3.41), $z'(t) > 0$ and by using the last inequality, we obtain

$$w^{(n-2)}(t) \leqslant -q(t)G((1 - p_1)z(t_4))$$

for $t \geqslant t_5 > t_4$. Then integrating the preceeding inequality from $'t_5'$ to $'t'$, we obtain

$$\infty > w^{(n-3)}(t_5) > -w^{(n-3)}(t) + w^{(n-3)}(t_5) \geqslant G((1 - p_1)z(t_4)) \int_{t_5}^{t} q(s)ds.$$

Since $\lim_{t\to\infty} w^{(n-3)}(t) < \infty$, then taking $t \to \infty$ in the last inequality we have $\int_{t_5}^{\infty} q(t)dt < \infty$, a contradiction to (H_7).

Case II. If $v'(t) < 0$ for $t \geqslant t_3$, then proceeding same as in Case II of Theorem 3.4.2, we obtain $\lim_{t\to\infty} y(t) = 0$.

If $w(t) < 0$, then $(b(t)v'(t))' < 0$ for $t \geqslant t_2$. Thus, $v'(t) > 0$ or $v'(t) < 0$ for $t \geqslant t_3 > t_2$.

Case III. Suppose $v'(t) > 0$ for $t \geqslant t_3$. Since $w^{(n-2)}(t) \leqslant 0$ eventually, then either

$w^{(n-3)}(t) > 0$ or < 0 eventually.

Subcase (i): If $w^{(n-3)}(t) > 0$ eventually, then proceeding same as in *Subcase (i)* of Case III of Theorem 3.4.2, we obtain a contradiction due to (H_7).

Subcase (ii): If $w^{(n-3)}(t) < 0$ eventually, then proceeding same as in *Subcase (ii)* of Case III of Theorem 3.4.4, we obtain a contradiction (H_{16}).

Case IV. Suppose $v'(t) < 0$ for $t \geqslant t_3$. Since, $w^{(n-2)}(t) \leqslant 0$ eventually, then either $w^{(n-3)}(t) > 0$ or < 0 eventually.

Subcase (iii): If $w^{(n-3)}(t) < 0$ eventually, then, $w^{(n-4)}(t) < 0, ..., w'(t) < 0, w(t) < 0$ eventually. Since $\lim_{t\to\infty} v(t)$ exists, let $0 < \lim_{t\to\infty} v(t) < \infty$. Hence, from (3.41) and (3.43), we obtain

$$0 \geqslant w^{(n-2)}(t) + q(t)G(k_2 l) \tag{3.47}$$

for $t \geqslant t_4 > t_3$. Integrating (3.47) consecutively $(n-2)$ times from $'t_4'$ to $'t'$, we obtain

$$0 > w(t_4) \geqslant w(t) + \frac{1}{(n-3)\,!}\int_{t_4}^{t}(t-s)^{n-3}q(s)G(k_2 l)ds.$$

Hence,

$$0 > (b(t)v'(t))' + \frac{1}{a(t)}\frac{1}{(n-3)\,!}\int_{t_4}^{t}(t-s)^{n-3}q(s)G(k_2 l)ds.$$

Further integrating the preceeding inequality from $'t_4'$ to $'t'$ and considering the fact that $v'(t) < 0$, we obtain

$$0 > b(t_4)v'(t_4) \;\;\geqslant\;\; b(t)v'(t) + \frac{1}{(n-3)\,!}\int_{t_4}^{t}\frac{1}{a(\theta)}\int_{t_4}^{\theta}(\theta-s)^{n-3}q(s)G(k_2 l)dsd\theta.$$

Again integrating the last inequality from $'t_4'$ to $'t'$, we get

$$v(t_4) \;\;\geqslant\;\; v(t) + \frac{1}{(n-3)\,!}\int_{t_4}^{t}\frac{1}{b(u)}\int_{t_4}^{u}\frac{1}{a(\theta)}\int_{t_4}^{\theta}(\theta-s)^{n-3}q(s)G(k_2 l)dsd\theta du.$$

Since $\lim_{t\to\infty} v(t) < \infty$, then it implies for large t

$$\frac{G(k_2 l)}{(n-3)\,!}\int_{t_4}^{\infty}\frac{1}{b(u)}\int_{t_4}^{u}\frac{1}{a(\theta)}\int_{t_4}^{\theta}(\theta-s)^{n-3}q(s)dsd\theta du < \infty,$$

a contradiction to (H_{17}).
If $\lim_{t\to\infty} v(t) = 0$, then $\lim_{t\to\infty} z(t) = 0$. Thus, $\lim_{t\to\infty} y(t) = 0$ as $y(t) \leqslant z(t)$.

Subcase (iv): Suppose $w^{(n-3)}(t) > 0$ for $t \geqslant t_4 > t_3$. If $0 < \lim_{t\to\infty} v(t) < \infty$, then from (3.41) and (3.43), we have

$$\int_{t_4}^{\infty} q(t)dt < \infty,$$

a contradiction to (H_7). Hence $\lim_{t\to\infty} v(t) = 0$. Thus, $\lim_{t\to\infty} y(t) = 0$. Hence proof of the theorem is complete. □

Theorem 3.4.6. *Let $0 \leqslant p(t) \leqslant p_1 < 1$. Suppose that (H_5), (H_7), $(H_{14}), (H_{15})$ and (H_{17}) hold. Then every solution of* (3.38) *either oscillates or converges to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a non-oscillatory solution of (3.38) on $[t_0, \infty)$, $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. (The proof in case $y(t) < 0$ eventually is similar and will be omitted.) Then there exists $t_1 > t_0$ such that $y(\alpha(t)) > 0, y(\beta(t)) > 0, y(\sigma(t)) > 0$ for $t \geqslant t_1$. Setting $z(t), k(t)$, $v(t)$ as in (3.6), (3.39) and (3.40) respectively, we get (3.41) and (3.42) for $t \geqslant t_1$. Clearly, $w^{(n-3)}(t), w^{(n-4)}(t), .., w'(t), w(t)$ are monotonic functions and of constant sign for $t \in [t_2, \infty)$, $t_2 \geqslant t_1$.

If $w(t) > 0$ for $t \geqslant t_2$, in view of Lemma 3.4.1, $w^{(n-3)}(t) > 0$ for $t \geqslant t_2$. Now $w(t) > 0$ implies $(b(t)v'(t))' > 0$ for $t \geqslant t_2$, which in turn implies $b(t)v'(t)$ is monotonic function. Thus, $v'(t) > 0$ or < 0 for $t \geqslant t_3 > t_2$.

Case I. If $v'(t) > 0$ for $t \geqslant t_3$, then proceeding same as in Case I of Theorem 3.4.5, we obtain a contradiction due to (H_7).

Case II. If $v'(t) < 0$ for $t \geqslant t_3$, then proceeding same as in Case II of Theorem 3.4.2, we obtain $\lim_{t\to\infty} y(t) = 0$.

If $w(t) < 0$, for $t \geqslant t_2$, then $(b(t)v'(t))' < 0$ for $t \geqslant t_2$. Thus, $v'(t) > 0$ or $v'(t) < 0$ for $t \geqslant t_3 > t_2$.

Case III. Suppose $v'(t) > 0$ for $t \geqslant t_3$. Since $w^{(n-2)}(t) \leqslant 0$ eventually, then either $w^{(n-3)}(t) > 0$ or < 0 eventually.

Subcase (i): If $w^{(n-3)}(t) > 0$ eventually, then proceeding same as in *Subcase (i)* of Case III of Theorem 3.4.2, we obtain a contradiction due to (H_7).

Subcase (ii): If $w^{(n-3)}(t) < 0$ eventually, then using (H_5) and proceeding same as in *Subcase (ii)* of Case III of Theorem 3.4.2, we obtain a contradiction to the fact that $v'(t) > 0$.

Case IV. Suppose $v'(t) < 0$ for $t \geqslant t_3$. Since, $w^{(n-2)}(t) \leqslant 0$ eventually, then either $w^{(n-3)}(t) > 0$ or < 0 eventually.

Subcase (iii): Suppose $w^{(n-3)}(t) < 0$ eventually. If $0 < \lim_{t\to\infty} v(t) < \infty$, then proceeding same as in *Subcase (iii)* of Case IV of Theorem 3.4.5, we obtain a contradiction due to (H_{17}).
If $\lim_{t\to\infty} v(t) = 0$, then we obtain $\lim_{t\to\infty} y(t) = 0$.

Subcase (iv): If $w^{(n-3)}(t) > 0$ eventually, then proceeding same as in *Subcase (iv)* of Case IV of Theorem 3.4.5, we obtain $\lim_{t\to\infty} y(t) = 0$. Hence proof of the theorem is complete. □

Example 3.4.7. Consider the equation

$$\left(e^{-\frac{t}{8}}\left(e^{\frac{t}{4}}\left(y(t)+\frac{1}{2e^{\pi}}y(t-\pi)\right)'\right)'\right)' + \left(\frac{63}{64}e^{\frac{7t}{8}} + e^{-\frac{5t}{4}}\right)y^7\left(\frac{t}{4}\right)$$

$$-\frac{e^{-2t}(1+e^{-2t+\pi})}{e^{\frac{\pi}{2}}}\frac{y\left(t-\frac{\pi}{2}\right)}{1+y^2\left(t-\frac{\pi}{2}\right)} = 0, \quad t \geq 4. \tag{3.48}$$

It is easy to verify that the conditions $(H_5), (H_7), (H_{14}), (H_{15})$ and (H_{17}) are satisfied, so equation (3.48) satisfies the hypothesis of Theorem 3.4.6. Thus, every solution of (3.48) either oscillates or tends to zero as $t \to \infty$. Indeed, $y(t) = e^{-t}$ is such a solution of (3.48).

Oscillation results for the range $-1 < p_2 \leqslant p(t) \leqslant 0$.

Theorem 3.4.8. *Let $-1 < p_2 \leqslant p(t) \leqslant 0$. Suppose that (H_2), (H_7), (H_{14}) and (H_{15}) hold. Then every solution of* (3.38) *either oscillates or converges to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a non-oscillatory solution of (3.38) on $[t_0, \infty)$, $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. (The proof in case $y(t) < 0$ eventually is similar and will be omitted.) Then there exists $t_1 > t_0$ such that $y(\alpha(t)) > 0, y(\beta(t)) > 0, y(\sigma(t)) > 0$ for $t \geqslant t_1$. Setting $z(t), k(t)$, $v(t)$ as in (3.6), (3.39) and (3.40) respectively, we get (3.41) and (3.42) for $t \geqslant t_1$. Clearly, $w^{(n-3)}(t), w^{(n-4)}(t), .., w'(t), w(t)$ are monotonic functions and of constant sign for $t \in [t_2, \infty)$, $t_2 \geqslant t_1$.

If $w(t) > 0$ for $t \geqslant t_2$, in view of Lemma 3.4.1, $w^{(n-3)}(t) > 0$ for $t \geqslant t_2$. Now $w(t) > 0$ implies $(b(t)v'(t))' > 0$ for $t \geqslant t_2$, which in turn implies $b(t)v'(t)$ is monotonic function. Thus, $v'(t) > 0$ or < 0 for $t \geqslant t_3 > t_2$.

Case I. Suppose $v'(t) > 0$ for $t \geqslant t_3$. Now $v'(t) > 0$ and $k'(t) < 0$ implies that $z'(t) > 0$ eventually. Hence, $z(t) > 0$ or < 0 for $t \geq t_4 > t_3$.

Subcase (i): If $z(t) > 0$ for $t \geq t_4$, then

$$y(t) \geqslant z(t)$$

for $t \geqslant t_4 > t_3$. Using the last inequality in (3.41), we obtain

$$w^{(n-2)}(t) \leqslant -q(t)G(z(t_4)).$$

Thus integrating this from $'t_5'(> t_4)$ to $'t'$, we obtain

$$\infty > w^{(n-3)}(t_5) > -w^{(n-3)}(t) + w^{(n-3)}(t_5) \geqslant G(z(t_4)) \int_{t_5}^{t} q(s)ds.$$

Since $\lim_{t\to\infty} w^{n-3}(t) < \infty$, then taking limit as $t \to \infty$ in the last inequality we obtain

$$\int_{t_5}^{\infty} q(t)dt < \infty,$$

a contradiction to (H_7).

Subcase (ii): If $z(t) < 0$ for $t \geqslant t_4 > t_3$, then $\lim_{t\to\infty} z(t)$ exists. Note that $y(t)$ is bounded. Hence,

$$\begin{aligned} 0 \geqslant \limsup_{t\to\infty} z(t) &\geqslant \limsup_{t\to\infty}(y(t) + p_2 y(\sigma(t))) \\ &\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_2 y(\sigma(t))) \\ &= \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} y(\sigma(t)) \\ &= (1 + p_2)\limsup_{t\to\infty} y(t). \end{aligned}$$

Since $(1 + p_2) > 0$, then it implies $\limsup_{t\to\infty} y(t) = 0$. So also $\lim_{t\to\infty} y(t) = 0$.

Case II. If $v'(t) < 0$ for $t \geqslant t_3$, then two cases are possible: $v(t) > 0$ or < 0 for $t \geqslant t_4 > t_3$.

Subcase (iii): If $v(t) > 0$ for $t \geqslant t_4$, then $\lim_{t\to\infty} v(t)$ exists and equal to l_1 (say). We will claim $l_1 = 0$. If it is not true, then for every $\epsilon > 0$, there exists $t_5 > t_4$ such that $l_1 < v(t) < l_1 + \epsilon$ for $t \geqslant t_5$. Choose $0 < \epsilon < l_1$. Since $\lim_{t\to\infty} k(t) = 0$, then for the same chosen ϵ, $k(t) < \epsilon$ for $t \geqslant t_6 > t_5$. Thus,

$$v(t) - y(t) - k(t) = p(t)y(\sigma(t)) \leqslant 0$$

for $t \geqslant t_7 > t_6$. Hence,

$$l_1 - \epsilon < v(t) - k(t) \leqslant y(t).$$

From (3.41), we obtain

$$w^{(n-2)}(t) \leqslant -q(t)G(l_1 - \epsilon)$$

for $t \geqslant t_8 > t_7$. Thus integrating the last inequality from $'t_8'$ to $'t'$, we obtain

$$\infty > w^{(n-3)}(t_8) > -w^{(n-3)}(t) + w^{(n-3)}(t_8) \geqslant G(l_1 - \epsilon)\int_{t_8}^{t} q(s)ds.$$

Since $\lim_{t\to\infty} w^{n-3}(t) < \infty$, then taking limit as $t \to \infty$ in the preceeding inequality we get a contradiction to (H_7). Hence, $\lim_{t\to\infty} v(t) = 0$ and $\lim_{t\to\infty} z(t) = 0$. Hence $z(t)$ is bounded function. We can show that $y(t)$ is also bounded function. Thus,

$$\begin{aligned}
0 = \limsup_{t\to\infty} z(t) &\geqslant \limsup_{t\to\infty}(y(t) + p_2 y(\sigma(t))) \\
&\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_2 y(\sigma(t))) \\
&= \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} y(\sigma(t)) \\
&= (1 + p_2) \limsup_{t\to\infty} y(t).
\end{aligned}$$

Since $(1 + p_2) > 0$, then it implies $\limsup_{t\to\infty} y(t) = 0$. Hence, $\lim_{t\to\infty} y(t) = 0$.

Subcase (iv): Suppose $v(t) < 0$ for $t \geqslant t_4$, as $v'(t) < 0$ so $-\infty \leqslant \lim_{t\to\infty} v(t) < 0$. Thus, $-\infty \leqslant \lim_{t\to\infty} z(t)(= l_2) < 0$. If $l_2 = -\infty$, then we get a contradiction due to the boundedness of $y(t)$.

If $-\infty < l_2 < 0$, then

$$\begin{aligned}
0 > \limsup_{t\to\infty} z(t) &\geqslant \limsup_{t\to\infty}(y(t) + p_2 y(\sigma(t))) \\
&\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_2 y(\sigma(t))) \\
&= \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} y(\sigma(t)) \\
&= (1 + p_2) \limsup_{t\to\infty} y(t).
\end{aligned}$$

Since $(1 + p_2) > 0$, then it implies $\limsup_{t\to\infty} y(t) < 0$, a contradiction to the fact that $y(t) > 0$.

If $w(t) < 0$ for $t \geqslant t_2$, then $(b(t)v'(t))' < 0$ for $t \geqslant t_2$. Thus, $v'(t) > 0$ or $v'(t) < 0$ for $t \geqslant t_3 > t_2$.

Case III. Suppose $v'(t) > 0$ for $t \geqslant t_3$. Since $w^{(n-2)}(t) \leqslant 0$ eventually, then $w^{(n-3)}(t) > 0$ or < 0 eventually.

Subcase (v): Suppose $w^{(n-3)}(t) > 0$ eventually. Now $v'(t) > 0$ and $k'(t) < 0$

implies that $z'(t) > 0$ eventually. Hence, $z(t) > 0$ or < 0 eventually.
If $z(t) > 0$ eventually, then

$$y(t) \geqslant z(t) \tag{3.49}$$

for $t \geqslant t_4 > t_3$. Using (3.49) in (3.41), we obtain

$$w^{(n-2)}(t) \leqslant -q(t)G(z(t_4)).$$

Thus integrating this from $'t_5'(> t_4)$ to $'t'$, we obtain

$$\infty > w^{(n-3)}(t_5) > -w^{(n-3)}(t) + w^{(n-3)}(t_5) \geqslant G(z(t_4)) \int_{t_5}^{t} q(s)ds.$$

Since $\lim_{t\to\infty} w^{n-3}(t) < \infty$, then taking limit as $t \to \infty$ in the last inequality we obtain

$$\int_{t_5}^{\infty} q(t)dt < \infty,$$

a contradiction to (H_7).
If $z(t) < 0$ for $t \geqslant t_4 > t_3$, then $\lim_{t\to\infty} z(t)$ exists. Let it be l_3. So $-\infty < l_3 \leqslant 0$. We may note that $y(t)$ is bounded. Hence,

$$\begin{aligned}
0 \geqslant \limsup_{t\to\infty} z(t) &\geqslant \limsup_{t\to\infty}(y(t) + p_2 y(\sigma(t))) \\
&\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_2 y(\sigma(t))) \\
&= \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} y(\sigma(t)) \\
&= (1 + p_2) \limsup_{t\to\infty} y(t).
\end{aligned}$$

Since $(1 + p_2) > 0$, then it implies $\limsup_{t\to\infty} y(t) = 0$. So also $\lim_{t\to\infty} y(t) = 0$.

Subcase (vi): If $w^{(n-3)}(t) < 0$ eventually, then proceeding same as in Subcase (ii) of Case III of Theorem 3.4.2, we obtain a contradiction due to $v'(t) > 0$.

Case IV. Suppose $v'(t) < 0$ and $(b(t)v'(t))' < 0$ for $t \geqslant t_3$, then integrating $(b(t)v'(t))' < 0$ twice consecutively from t_3 to t, we obtain

$$v(t) \leqslant v(t_3) + b(t_3)v'(t_3) \int_{t_3}^{t} \frac{ds}{b(s)}.$$

Using (H_2) in the last inequality, we obtain $v(t) \to -\infty$ as $t \to \infty$. Hence $v(t) < 0$ for large t. It is easy to show that $y(t)$ is bounded, hence $v(t)$ is bounded, a contradiction. Hence proof of the theorem is complete. □

Example 3.4.9. Consider the fifth order equation

$$\left(y(t) - \frac{1}{2}y(t-2\pi)\right)^{(v)} + \left(\frac{1}{2} + e^{-t}\right)y\left(t - \frac{\pi}{2}\right)$$

$$-e^{-t}\left(1 + \sin^2\left(t - \frac{\pi}{2}\right)\right)\frac{y\left(t - \frac{\pi}{2}\right)}{1 + y^2\left(t - \frac{\pi}{2}\right)} = 0, \tag{3.50}$$

for $t \geqslant 4$. It is easy to verify that the conditions $(H_2), (H_7), (H_{14})$ and (H_{15}) are satisfied. So equation (3.50) satisfies the hypothesis of Theorem 3.4.8. Thus, every solution of (3.50) either oscillates or tends to zero as $t \to \infty$. Indeed, $y(t) = \sin t$ is an oscillatory solution of (3.50).

Theorem 3.4.10. *Let $-1 < p_2 \leqslant p(t) \leqslant 0$. Suppose that (H_3), (H_7) and $(H_{14})-(H_{16})$ hold. Then every solution of* (3.38) *either oscillates or converges to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a non-oscillatory solution of (3.38) on $[t_0, \infty)$, $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. (The proof in case $y(t) < 0$ eventually is similar and will be omitted.) Then there exists $t_1 > t_0$ such that $y(\alpha(t)) > 0, y(\beta(t)) > 0, y(\sigma(t)) > 0$ for $t \geqslant t_1$. Setting $z(t), k(t)$, $v(t)$ as in (3.6), (3.39) and (3.40) respectively, we get (3.41) and (3.42) for $t \geqslant t_1$. Clearly, $w^{(n-3)}(t), w^{(n-4)}(t), .., w'(t), w(t)$ are monotonic functions and of constant sign for $t \in [t_2, \infty)$, $t_2 \geqslant t_1$.

If $w(t) > 0$ for $t \geqslant t_2$, in view of Lemma 3.4.1, $w^{(n-3)}(t) > 0$ for $t \geqslant t_2$. Now $w(t) > 0$ implies $(b(t)v'(t))' > 0$ for $t \geqslant t_2$, which in turn implies $b(t)v'(t)$ is monotonic function. Thus, $v'(t) > 0$ or < 0 for $t \geqslant t_3 > t_2$.

Case I. If $v'(t) > 0$ for $t \geqslant t_3$, then $z'(t) > 0$ and hence two cases are possible: $z(t) > 0$ or < 0 for $t \geqslant t_4 > t_3$.

Subcase (i): If $z(t) > 0$ for $t \geqslant t_4$, then proceeding same as in *Subcase (i)* of Case I of Theorem 3.4.8, we obtain a contradiction due to (H_7).

Subcase (ii): If $z(t) < 0$ for $t \geqslant t_4$, then proceeding same as in *Subcase (ii)* of Case I of Theorem 3.4.8, we obtain $\lim_{t\to\infty} y(t) = 0$.

Case II. If $v'(t) < 0$ for $t \geqslant t_3$, we have two cases; $v(t) > 0$ or $v(t) < 0$ for $t \geqslant t_4 > t_3$.

Subcase (iii): If $v(t) > 0$ for $t \geqslant t_4$, then $\lim_{t\to\infty} v(t) < \infty$. Proceeding same as in *Subcase (iii)* of Case II of Theorem 3.4.8, we get $\lim_{t\to\infty} y(t) = 0$.

Subcase (iv): If $v(t) < 0$ for $t \geqslant t_4$, then proceeding same as in *Subcase* (iv) of Case II of Theorem 3.4.8, we get a contradiction due to $y(t) > 0$.

If $w(t) < 0$ for $t \geqslant t_2$, then $(b(t)v'(t))' < 0$ for $t \geqslant t_2$. Thus, $v'(t) > 0$ or $v'(t) < 0$ for $t \geqslant t_3 > t_2$.

Case III. Suppose $v'(t) > 0$ for $t \geqslant t_3$. Since $w^{(n-2)}(t) \leqslant 0$ eventually, then either $w^{(n-3)}(t) > 0$ or < 0 eventually.

Subcase (v): Suppose $w^{(n-3)}(t) > 0$ eventually. Now $v'(t) > 0$ and $k'(t) < 0$ implies that $z'(t) > 0$. Hence $z(t) > 0$ or < 0 eventually.

If $z(t) > 0$ eventually, then proceeding same as in *Subcase (v)* of Case III of Theorem 3.4.8, we obtain a contradiction due to (H_7).

If $z(t) < 0$ eventually, then proceeding same as in *Subcase (v)* of Case III of Theorem 3.4.8, we get $\lim_{t\to\infty} y(t) = 0$.

Subcase (vi): Suppose $w^{(n-3)}(t) < 0$ eventually. Since $z'(t) > 0$ eventually, then $z(t) >$ or < 0 eventually.

If $z(t) > 0$ eventually, then proceeding same as in *Subcase (ii)* of Case III of Theorem 3.4.4, we get a contradiction due to (H_{16}).

If $z(t) < 0$ eventually, then $\lim_{t\to\infty} z(t)$ exists. Note that $y(t)$ is bounded. Hence,

$$\begin{aligned} 0 \geqslant \limsup_{t\to\infty} z(t) &\geqslant \limsup_{t\to\infty}(y(t) + p_2 y(\sigma(t))) \\ &\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_2 y(\sigma(t))) \\ &= \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} y(\sigma(t)) \\ &= (1 + p_2) \limsup_{t\to\infty} y(t), \end{aligned}$$

which implies $\lim_{t\to\infty} y(t) = 0$.

Case IV. If $v'(t) < 0$ for $t \geqslant t_3$, then using (H_3) and proceeding same as in Case IV of Theorem 3.4.8, we get a contradiction to the fact that $y(t)$ is bounded. Hence proof of the theorem is complete. □

Theorem 3.4.11. *Let* $-1 < p_2 \leqslant p(t) \leqslant 0$. *Suppose that* (H_4), (H_7) *and* $(H_{14}) - (H_{17})$ *hold. Then every solution of* (3.38) *either oscillates or converges to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (3.38) on $[t_0, \infty)$, $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. (The proof in case $y(t) < 0$ eventually is similar and will be omitted.) Then there exists $t_1 > t_0$ such that $y(\alpha(t)) > 0, y(\beta(t)) > 0, y(\sigma(t)) > 0$ for $t \geqslant t_1$. Setting $z(t), k(t)$, $v(t)$ as in (3.6), (3.39) and (3.40) respectively, we get (3.41) and (3.42) for $t \geqslant t_1$. Clearly, $w^{(n-3)}(t), w^{(n-4)}(t), .., w'(t), w(t)$ are monotonic functions and of constant sign for $t \in [t_2, \infty)$, $t_2 \geqslant t_1$.

If $w(t) > 0$ for $t \geqslant t_2$, in view of Lemma 3.4.1, $w^{(n-3)}(t) > 0$ for $t \geqslant t_2$. Now $w(t) > 0$ implies $(b(t)v'(t))' > 0$ for $t \geqslant t_2$, which in turn implies $b(t)v'(t)$ is monotonic function. Thus, $v'(t) > 0$ or < 0 for $t \geqslant t_3 > t_2$.

Case I. If $v'(t) > 0$ for $t \geqslant t_3$, then $z'(t) > 0$ eventually, thus we have two cases; $z(t) > 0$ or $z(t) < 0$ for $t \geqslant t_4 > t_3$.

Subcase (i): If $z(t) > 0$ for $t \geqslant t_4$, then proceeding same as in *Subcase (i)* of Case I of Theorem 3.4.8, we obtain a contradiction due to (H_7).

Subcase (ii): If $z(t) < 0$ for $t \geqslant t_4$, then $\lim_{t\to\infty} z(t)$ exists. Let it be l_4. Now $-\infty < l_4 \leqslant 0$. Note that $y(t)$ is bounded. Hence,

$$\begin{aligned} 0 \geqslant \limsup_{t\to\infty} z(t) &\geqslant \limsup_{t\to\infty}(y(t) + p_2 y(\sigma(t))) \\ &\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_2 y(\sigma(t))) \\ &= \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} y(\sigma(t)) \\ &= (1+p_2)\limsup_{t\to\infty} y(t). \end{aligned}$$

Since $(1+p_2) > 0$, then it implies $\limsup_{t\to\infty} y(t) = 0$. So also $\lim_{t\to\infty} y(t) = 0$.

Case II. If $v'(t) < 0$ for $t \geqslant t_3$, then we have two cases: $v(t) > 0$ or $v(t) < 0$ for $t \geqslant t_4 > t_3$.

Subcase (iii): If $v(t) > 0$ for $t \geqslant t_4$, then proceeding same as in *Subcase (iii)* of Case II of Theorem 3.4.8, we get $\lim_{t\to\infty} y(t) = 0$.

Subcase (iv): If $v(t) < 0$ for $t \geqslant t_4$, then proceeding same as in *Subcase (iv)* of Case II of Theorem 3.4.8, we get a contradiction due to $y(t) > 0$.

If $w(t) < 0$ for $t \geqslant t_2$, then $(b(t)v'(t))' < 0$ for $t \geqslant t_2$. Thus, $v'(t) > 0$ or $v'(t) < 0$ for $t \geqslant t_3 > t_2$.

Case III. Suppose $v'(t) > 0$ for $t \geqslant t_3$. Since $w^{(n-2)}(t) \leqslant 0$ eventually, then either $w^{(n-3)}(t) > 0$ or < 0 eventually.

Subcase (v): Suppose $w^{(n-3)}(t) > 0$ eventually. Since $v'(t) > 0$ for $t \geqslant t_3$, then $z'(t) > 0$ eventually. Thus we have two cases: $z(t) > 0$ or $z(t) < 0$ for $t \geqslant t_4 > t_3$.

If $z(t) > 0$ for $t \geqslant t_4$, then proceeding same as in *Subcase (v)* of Case III of Theorem 3.4.8, we get a contradiction to (H_7).

If $z(t) < 0$ for $t \geqslant t_4$, then proceeding same as in *Subcase (v)* of Case III of Theorem 3.4.8, we get $\lim_{t\to\infty} y(t) = 0$.

Subcase (vi): Suppose $w^{(n-3)}(t) < 0$ eventually. Since $v'(t) > 0$ for $t \geqslant t_3$, then $z'(t) > 0$ eventually. Thus we have two cases; $z(t) > 0$ or $z(t) < 0$ for $t \geqslant t_4 > t_3$.

If $z(t) > 0$ for $t \geqslant t_4$, then proceeding same as in *Subcase (ii)* of Case III of Theorem 3.4.4, we get a contradiction due to (H_{16}).

If $z(t) < 0$ for $t \geqslant t_4$, then $\lim_{t\to\infty} z(t)$ exists. Note that $y(t)$ is bounded. Hence,

$$\begin{aligned} 0 \geqslant \limsup_{t\to\infty} z(t) &\geqslant \limsup_{t\to\infty}(y(t) + p_2 y(\sigma(t))) \\ &\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_2 y(\sigma(t))) \\ &= \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} y(\sigma(t)) \\ &= (1 + p_2) \limsup_{t\to\infty} y(t), \end{aligned}$$

which implies $\lim_{t\to\infty} y(t) = 0$.

Case IV. Suppose $v'(t) < 0$ for $t \geqslant t_3$. Since $w^{(n-2)}(t) \leqslant 0$ eventually, then we have two cases $w^{(n-3)}(t) > 0$ or $w^{(n-3)}(t) < 0$ eventually.

Subcase (vii): Suppose $w^{(n-3)}(t) > 0$ eventually. Since $v'(t) < 0$, thus $v(t) > 0$ or $v(t) < 0$ for $t \geqslant t_4 > t_3$.

If $v(t) > 0$ for $t \geqslant t_4$, then $\lim_{t\to\infty} v(t)$ exists and equal to l_5 (say). We will claim $l_5 = 0$. If it is not true, then for every $\epsilon > 0$, there exists $t_5 > t_4$ such that $l_5 < v(t) < l_5 + \epsilon$ for $t \geqslant t_5$. Choose $0 < \epsilon < l_5$. Since $\lim_{t\to\infty} k(t) = 0$, then for the same chosen ϵ, $k(t) < \epsilon$ for $t \geqslant t_6 > t_5$. Thus,

$$v(t) - y(t) - k(t) = p(t)y(\sigma(t)) \leqslant 0$$

for $t \geqslant t_7 > t_6$. Hence,

$$l_5 - \epsilon < v(t) - k(t) \leqslant y(t).$$

From (3.41), we obtain

$$w^{(n-2)}(t) \leqslant -q(t)G(l_5 - \epsilon)$$

for $t \geqslant t_8 > t_7$. Thus integrating the last inequality from $'t_8'$ to $'t'$, we obtain

$$\infty > w^{(n-3)}(t_8) > -w^{(n-3)}(t) + w^{(n-3)}(t_8) \geqslant G(l_5 - \epsilon) \int_{t_8}^{t} q(s)ds.$$

Since $\lim_{t\to\infty} w^{n-3}(t) < \infty$, then taking limit as $t \to \infty$ in the preceeding inequality we get a contradiction to (H_7). Hence, $\lim_{t\to\infty} v(t) = 0$ and $\lim_{t\to\infty} z(t) = 0$. Hence $z(t)$ is also bounded. We can show that $y(t)$ is bounded. Thus,

$$\begin{aligned} 0 = \limsup_{t\to\infty} z(t) &\geqslant \limsup_{t\to\infty}(y(t) + p_2 y(\sigma(t))) \\ &\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_2 y(\sigma(t))) \\ &= \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} y(\sigma(t)) \\ &= (1 + p_2)\limsup_{t\to\infty} y(t). \end{aligned}$$

Since $(1 + p_2) > 0$, then $\limsup_{t\to\infty} y(t) = 0$. Hence $\lim_{t\to\infty} y(t) = 0$.

Suppose $v(t) < 0$ for $t \geqslant t_4$, as $v'(t) < 0$, thus, $-\infty \leqslant \lim_{t\to\infty} v(t) < 0$. Hence, $-\infty \leqslant \lim_{t\to\infty} z(t)(= l_6) < 0$. If $l_6 = -\infty$, then we get a contradiction due to boundedness of $y(t)$.

If $-\infty < l_6 < 0$, then

$$\begin{aligned} 0 > \limsup_{t\to\infty} z(t) &\geqslant \limsup_{t\to\infty}(y(t) + p_2 y(\sigma(t))) \\ &\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_2 y(\sigma(t))) \\ &= \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} y(\sigma(t)) \\ &= (1+p_2)\limsup_{t\to\infty} y(t). \end{aligned}$$

Since $(1+p_2) > 0$, then $\limsup_{t\to\infty} y(t) < 0$, a contradiction to the fact that $y(t) > 0$.

Subcase (viii): Suppose $w^{(n-3)}(t) < 0$ eventually, then from (3.41) we can conclude that $w^{(n-4)}(t) < 0, ..., w'(t) < 0$ for large t.
If $v(t) > 0$ eventually, then $\lim_{t\to\infty} v(t) < \infty$ and equal to l_7 (say). We will claim $l_7 = 0$. If it is not true, then for every $\epsilon > 0$, there exists $t_4 > t_3$ such that $l_7 < v(t) < l_7 + \epsilon$ for $t \geqslant t_4$. Choose $0 < \epsilon < l_7$. Since $\lim_{t\to\infty} k(t) = 0$, then for the same chosen ϵ, $k(t) < \epsilon$ for $t \geqslant t_5 > t_4$. Thus,

$$v(t) - y(t) - k(t) = p(t)y(\sigma(t)) \leqslant 0$$

for $t \geqslant t_6 > t_5$. Hence,

$$l_7 - \epsilon < v(t) - k(t) \leqslant y(t).$$

Therefore using the last inequality in (3.41), we obtain

$$0 \geqslant w^{(n-2)}(t) + q(t)G(l_7 - \epsilon)$$

for $t \geqslant t_7 > t_6$. Integrating the last inequality consecutively $(n-2)$ times from $'t_7'$ to $'t'$, we obtain

$$0 > w(t_7) \geqslant w(t) + \frac{1}{(n-3)\,!}\int_{t_7}^{t}(t-s)^{n-3}q(s)G(l_7-\epsilon)ds.$$

Hence,

$$0 > (b(t)v'(t))' + \frac{1}{a(t)}\frac{1}{(n-3)\,!}\int_{t_7}^{t}(t-s)^{n-3}q(s)G(l_7-\epsilon)ds.$$

Further integrating the preceeding inequality from $'t_7'$ to $'t'$ and considering the fact that $v'(t) < 0$, we obtain

$$0 > b(t_7)v'(t_7) \geqslant b(t)v'(t) + \frac{1}{(n-3)\,!}\int_{t_7}^{t}\frac{1}{a(\theta)}\int_{t_7}^{\theta}(\theta - s)^{n-3}q(s)G(l_7 - \epsilon)dsd\theta.$$

Again integrating the last inequality from $'t_7'$ to $'t'$, we get

$$v(t_7) \geqslant v(t) + \frac{1}{(n-3)\,!}\int_{t_7}^{t}\frac{1}{b(u)}\int_{t_7}^{u}\frac{1}{a(\theta)}\int_{t_7}^{\theta}(\theta - s)^{n-3}q(s)G(l_7 - \epsilon)dsd\theta du.$$

Since $\lim_{t\to\infty} v(t) < \infty$, then it implies that

$$\frac{1}{(n-3)\,!}G(l_7 - \epsilon)\int_{t_7}^{\infty}\frac{1}{b(u)}\int_{t_7}^{u}\frac{1}{a(\theta)}\int_{t_7}^{\theta}(\theta - s)^{n-3}q(s)dsd\theta du < \infty,$$

a contradiction to (H_{17}). Hence $\lim_{t\to\infty} v(t) = 0$ and $\lim_{t\to\infty} z(t) = 0$. Hence,

$$\begin{aligned}
0 = \limsup_{t\to\infty} z(t) &\geqslant \limsup_{t\to\infty}(y(t) + p_2 y(\sigma(t)))\\
&\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_2 y(\sigma(t)))\\
&= \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} y(\sigma(t))\\
&= (1 + p_2)\limsup_{t\to\infty} y(t).
\end{aligned}$$

Since $(1 + p_2) > 0$, then $\limsup_{t\to\infty} y(t) = 0$ and hence $\lim_{t\to\infty} y(t) = 0$.

If $v(t) < 0$ eventually, then $-\infty \leqslant \lim_{t\to\infty} v(t) < 0$. If $-\infty < \lim_{t\to\infty} v(t) < 0$, then $-\infty < \lim_{t\to\infty} z(t) < 0$. Hence,

$$\begin{aligned}
0 > \limsup_{t\to\infty} z(t) &\geqslant \limsup_{t\to\infty}(y(t) + p_2 y(\sigma(t)))\\
&\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_2 y(\sigma(t)))\\
&= \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} y(\sigma(t))\\
&= (1 + p_2)\limsup_{t\to\infty} y(t),
\end{aligned}$$

which implies $\limsup_{t\to\infty} y(t) < 0$, a contradiction to the fact that $y(t) > 0$.
$\lim_{t\to\infty} v(t) = -\infty$, then we obtain a contradiction to the boundedness of $y(t)$. Hence the proof of the theorem is complete. □

Theorem 3.4.12. *Let* $-1 < p_2 \leqslant p(t) \leqslant 0$. *Suppose that* (H_5), (H_7), (H_{14}), (H_{15}) *and* (H_{17}) *hold. Then every solution of* (3.38) *either oscillates or converges to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a non-oscillatory solution of (3.38) on $[t_0, \infty)$, $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. (The proof in case $y(t) < 0$ eventually is similar and will be omitted.) Then there exists $t_1 > t_0$ such that $y(\alpha(t)) > 0, y(\beta(t)) > 0, y(\sigma(t)) > 0$ for $t \geqslant t_1$. Setting $z(t), k(t)$, $v(t)$ as in (3.6), (3.39) and (3.40) respectively, we get (3.41) and (3.42) for $t \geqslant t_1$. Clearly, $w^{(n-3)}(t), w^{(n-4)}(t), .., w'(t), w(t)$ are monotonic functions and of constant sign for $t \in [t_2, \infty)$, $t_2 \geqslant t_1$.

If $w(t) > 0$ for $t \geqslant t_2$, in view of Lemma 3.4.1, $w^{(n-3)}(t) > 0$ for $t \geqslant t_2$. Now $w(t) > 0$ implies $(b(t)v'(t))' > 0$ for $t \geqslant t_2$, which in turn implies $b(t)v'(t)$ is monotonic function. Thus, $v'(t) > 0$ or < 0 for $t \geqslant t_3 > t_2$.

Case I. If $v'(t) > 0$ for $t \geqslant t_3$, then $z'(t) > 0$ eventually, thus we have two cases: $z(t) > 0$ or $z(t) < 0$ for $t \geqslant t_4 > t_3$.

Subcase (i): If $z(t) > 0$ for $t \geqslant t_4$, then proceeding same as in *Subcase (i)* of Case I of Theorem 3.4.8, we get a contradiction due to (H_7).

Subcase (ii): If $z(t) < 0$ for $t \geqslant t_4$, then proceeding same as in *Subcase (ii)* of Case I of Theorem 3.4.8, we get $\lim_{t\to\infty} y(t) = 0$.

Case II. If $v'(t) < 0$ for $t \geqslant t_3$, then we have two cases; $v(t) > 0$ or $v(t) < 0$ for $t \geqslant t_4 > t_3$.

Subcase (iii): If $v(t) > 0$ for $t \geqslant t_4$, then proceeding same as in *Subcase (iii)* of Case II of Theorem 3.4.8, $\lim_{t\to\infty} y(t) = 0$.

Subcase (iv): If $v(t) < 0$ for $t \geqslant t_4$, then proceeding same as in *Subcase (iv)* of

Case II of Theorem 3.4.8, we get a contradiction due to $y(t) > 0$.

If $w(t) < 0$ for $t \geqslant t_2$, then $(b(t)v'(t))' < 0$ for $t \geqslant t_2$. Thus, $v'(t) > 0$ or $v'(t) < 0$ for $t \geqslant t_3 > t_2$.

Case III. Suppose $v'(t) > 0$ for $t \geqslant t_3$. Since, $w^{(n-2)}(t) \leqslant 0$ eventually, then either $w^{(n-3)}(t) > 0$ or < 0 eventually.

Subcase (v): Suppose $w^{(n-3)}(t) > 0$ eventually. Now $v'(t) > 0$ and $k'(t) < 0$ implies that $z'(t) > 0$ eventually. Hence, $z(t) > 0$ or < 0 eventually.

If $z(t) > 0$ eventually, then proceeding same as in *Subcase (v)* of Case III of Theorem 3.4.8, we get a contradiction to (H_7).

If $z(t) < 0$ for $t \geqslant t_4 > t_3$, then proceeding same as in *Subcase (v)* of Case III of Theorem 3.4.8, we get $\lim_{t\to\infty} y(t) = 0$.

Subcase (vi): If $w^{(n-3)}(t) < 0$ eventually, then using (H_5) and proceeding same as in *Subcase (ii)* of Case III of Theorem 3.4.2, we get a contradiction due to $v'(t) > 0$.

Case IV. Suppose $v'(t) < 0$ eventually. Since $w^{(n-2)}(t) \leqslant 0$ eventually, we have two cases; $w^{(n-3)}(t) > 0$ or $w^{(n-3)}(t) < 0$ eventually.

Subcase (vii): If $w^{(n-3)}(t) > 0$ eventually, then proceeding same as in *Subcase (vii)* of Case IV of Theorem 3.4.11 for $v(t) > 0$ part, we get $\lim_{t\to\infty} y(t) = 0$ and for $v(t) < 0$ part we get a contradiction due to $y(t) > 0$.

Subcase (viii): If $w^{(n-3)}(t) < 0$ eventually, then first consider If $v(t) > 0$ eventually, then $\lim_{t\to\infty} v(t) < \infty$.

If $0 < \lim_{t\to\infty} v(t) < \infty$, then proceeding same as in *Subcase (viii)* of Case IV of Theorem 3.4.11, we get a contradiction due to (H_{17}).

If $\lim_{t\to\infty} v(t) = 0$, then $\lim_{t\to\infty} y(t) = 0$.

If $v(t) < 0$ eventually, then proceeding same as in *Subcase (viii)* of Case IV of Theorem 3.4.11, we get a contradiction due to $y(t) > 0$. Hence proof of the theorem is complete. □

Example 3.4.13. Consider the differential equation

$$\left(e^{-\frac{t}{8}}\left(e^{\frac{t}{4}}\left(y(t) - \frac{1}{2e^{\pi}}y(t-\pi)\right)'\right)'\right)' + \left(\frac{21}{64}e^{\frac{7t}{8}} + e^{-\frac{5t}{4}}\right)y^7\left(\frac{t}{4}\right)$$

$$-\frac{e^{-2t}(1+e^{-2t+\pi})}{e^{\frac{\pi}{2}}}\frac{y\left(t-\frac{\pi}{2}\right)}{1+y^2\left(t-\frac{\pi}{2}\right)} = 0, \quad t \geq 4. \tag{3.51}$$

It is easy to verify that the conditions $(H_5), (H_7), (H_{14}), (H_{15})$ and (H_{17}) are satisfied, so equation (3.51) satisfies the hypothesis of Theorem 3.4.12. Thus, every solution of (3.51) either oscillates or tends to zero as $t \to \infty$. Indeed, $y(t) = e^{-t}$ is such a solution of (3.51).

3.5 Conclusion

Note that by putting $G(u) = u$ and $k_1 = 1$ condition (H_6) of Theorem 3.2.5, condition (H_9) of Theorem 3.2.7 reduces to the condition (2.1) of Theorem 2.1 and the condition (2.14) of Theorem 2.2 of Li, Zhang and Xing [60] respectively.

The prototype of G satisfying $\frac{G(u)}{u} \geq k_1 > 0$ for $u \neq 0$ and some $k_1 > 0$ may take $G(u) = u(1+u^2)$ and $G(u) = u(k_1 + e^u)$ for $k_1 > 0$.

In Theorem 3.2.17, we may note that (H_{13}) together with (H_4) imply (H_7). But the converse need not be true.

In Theorem 3.4.5 and Theorem 3.4.11, we may note that (H_{17}) together with (H_4)

imply (H_{16}). But the converse need not be true.

In this chapter we have studied the oscillatory and asymptotic nature of the solutions of equations (3.1) and (3.38) respectively for the ranges $0 \le p(t) \le p_1 < 1$ and $-1 < p_2 \le p(t) \le 0$ with Riccati Transformation technique. One can investigate the behaviour of solutions of equations (3.1) and (3.38) for the ranges $1 \le p(t) < \infty$ and $-\infty < p(t) \le -1$.

Chapter 4

Oscillatory and Asymptotic Behavior of Solutions of Fourth Order Nonlinear Neutral Delay Differential Equations

4.1 Fourth Order NDDE with $\int_0^\infty \frac{t}{r(t)}dt = \infty$.

In this chapter, we consider fourth order nonlinear neutral delay differential equations with positive and negative coefficients of the form

$$(r(t)(y(t) + p(t)y(t-\tau))'')'' + q(t)G(y(t-\alpha)) - h(t)H(y(t-\beta)) = 0 \tag{4.1}$$

and its associated forced equations

$$(r(t)(y(t) + p(t)y(t-\tau))'')'' + q(t)G(y(t-\alpha)) - h(t)H(y(t-\beta)) = f(t), \tag{4.2}$$

where $r, q \in C([0,\infty),(0,\infty))$, $p \in C([0,\infty),\mathbb{R})$, $h \in C([0,\infty),[0,\infty))$, $f \in C([0,\infty),\mathbb{R})$, G and $H \in C(\mathbb{R},\mathbb{R})$ with $uG(u) > 0$, $vH(v) > 0$ for u, $v \neq 0$, H is bounded, G is non-decreasing, $\tau > 0$, $\alpha > 0$ and $\beta > 0$.

The objective of this work is to study oscillatory and asymptotic behavior of the functional differential equations (4.1) and (4.2) under the assumption

$$(H_0) \quad \int_0^\infty \frac{t}{r(t)}dt = \infty.$$

Because (4.1) and (4.2) are highly nonlinear, it is interesting to study both the equations under (H_0). If $h(t) \equiv 0$, then (4.1) and (4.2) reduce to

$$(r(t)(y(t) + p(t)y(t-\tau))'')'' + q(t)G(y(t-\alpha)) = 0 \tag{4.3}$$

and

$$(r(t)(y(t) + p(t)y(t-\tau))'')'' + q(t)G(y(t-\alpha)) = f(t) \tag{4.4}$$

respectively.

In [69], Parhi and Tripathy have studied the oscillatory and asymptotic behaviour of solutions of (4.3) and (4.4) under the assumption (H_0). Their work showed that if $q(t) < 0$, then it would be possible to obtain analogous results for oscillatory and asymptotic behaviour of solutions of (4.3) and (4.4). The problem remains open as to what happens if $q(t)$ changes sign. In particular, if $q(t) = q^+(t) - q^-(t)$, where $q^+(t) = \max\{0, q(t)\}$ and $q^-(t) = \max\{0, -q(t)\}$, then (4.3) and (4.4) can be viewed as

$$(r(t)(y(t) + p(t)y(t-\tau))'')'' + q^+(t)G(y(t-\alpha)) - q^-(t)G(y(t-\alpha)) = 0 \tag{4.5}$$

and

$$(r(t)(y(t) + p(t)y(t-\tau))'')'' + q^+(t)G(y(t-\alpha)) - q^-(t)G(y(t-\alpha)) = f(t) \tag{4.6}$$

respectively. Clearly, (4.5) and (4.6) are particular case of (4.1) and (4.2) respectively. Hence to enclose our prediction, the present work is devoted to study the more general equations of the type (4.1) and (4.2) rather than (4.5) and (4.6). On the other hand, (4.3) and (4.4) are special cases of (4.1) and (4.2) respectively and hence study of (4.1) and (4.2) are more illustrative in view of (H_0). The results in this section are new and generalize the earlier work of [69].

Keeping in view of the above fact, the motivation of the present work has come from the work of Parhi and Tripathy [69]. Since last decade, the study of the behaviour of the solutions of functional differential and difference equations with positive and negative coefficients of first, second and higher order is a major concerned of area of research. Most of the work dealt with the existence of positive solutions of the functional equations. However, much attention has not given to oscillation results during this period. To the best of our knowledge there are no results till date on forth order nonlinear neutral differential equations with positive and negative coefficients.

By a *solution* of (4.1) (or (4.2)) we understand a function $y \in C([-\rho, \infty), \mathbb{R})$ such that $(y(t) + p(t)y(t-\tau))$ is twice continuously differentiable, $(r(t)(y(t) + p(t)y(t-\tau)))$ is twice continuously differentiable and (4.1)(or (4.2)) is satisfied for $t \geq 0$, where $\rho = max\{\tau, \alpha, \beta\}$ and $\sup\{|y(t)|; t \geq t_0\} > 0$ for every $t \geq t_0$. A solution of (4.1) (or (4.2)) is said to be *oscillatory* if it has arbitrarily large zeros; otherwise, it is called *nonoscillatory.*

4.2 Oscillation Criteria for Homogeneous Equation with $\int_0^\infty \frac{t}{r(t)} dt = \infty$.

In this section, sufficient conditions are obtained for oscillatory and asymptotic behaviour of all solutions of (4.1) under the assumption (H_0). We need the following lemmas for our use in the sequel.

Lemma 4.2.1. ([69, Lemma 2.1]) Let (H_0) hold. Let u be a twice continuously differentiable function on $[0, \infty)$ such that $r(t)u''(t)$ is twice continuously differentiable and $(r(t)u''(t))'' \leq 0 (\not\equiv 0)$ for large t. If $u(t) > 0$ ultimately, then one of the cases (a) or (b) holds for large t, and if $u(t) < 0$ ultimately, then one of the cases (b), (c), (d) or

(e) holds for large t, where

$$(a)\ u'(t) > 0,\ u''(t) > 0\ and\ (r(t)u''(t))' > 0,$$

$$(b)\ u'(t) > 0,\ u''(t) < 0\ and\ (r(t)u''(t))' > 0,$$

$$(c)\ u'(t) < 0,\ u''(t) < 0\ and\ (r(t)u''(t))' > 0,$$

$$(d)\ u'(t) < 0,\ u''(t) < 0\ and\ (r(t)u''(t))' < 0,$$

$$(e)\ u'(t) < 0,\ u''(t) > 0\ and\ (r(t)u''(t))' > 0.$$

Lemma 4.2.2. ([69, Lemma 2.2]) Let the conditions of Lemma 4.2.1 hold. If $u(t) > 0$ ultimately, then $u(t) > R_T(t)(r(t)u''(t))'$ for $t \geq T \geq 0$, where $R_T(t) = \int_T^t \frac{(t-s)(s-T)}{r(s)} ds$.

Remark 4.2.3. Notice that $R_T(t)$ is increasing function.

Lemma 4.2.4. [42] Let F, G, P : $[t_0, \infty) \to \mathbb{R}$ and $c \in \mathbb{R}$ be such that $F(t) = G(t) + P(t)G(t-c)$, for $t \geq t_0 + \max\{0, c\}$. Assume that there exists numbers P_1, P_2, P_3 and $P_4 \in \mathbb{R}$ such that $P(t)$ is in one of the following ranges:

$$(1)\quad -\infty < P_1 \leq P(t) \leq 0,\quad (2)\ \ 0 \leq P(t) \leq P_2 < 1,\quad (3)\ \ 1 < P_3 \leq P(t) \leq P_4 < \infty.$$

Suppose that $G(t) > 0$ for $t \geq t_0$, $\liminf_{t\to\infty} G(t) = 0$ and that $\lim_{t\to\infty} F(t) = L \in \mathbb{R}$ exists. Then $L = 0$.

The results in this section will make use of the following conditions on the functions in equations (4.1) and (4.2):
(H_1) $\int_0^\infty \frac{s}{r(s)} \int_s^\infty th(t)dtds < \infty$;
(H_2) there exists $\lambda > 0$ such that $G(u) + G(v) \geq \lambda G(u+v), u > 0, v > 0$;
(H_3) $G(u)G(v) = G(uv)$ for $u, v \in \mathbb{R}$ and $H(-u) = -H(u)$ for $u \in \mathbb{R}$;
(H_4) G is sublinear and $\int_0^c \frac{du}{G(u)} < \infty$ for all $c > 0$;
(H_5) $\int_\tau^\infty Q(t)dt = \infty$, $Q(t) = min\{q(t), q(t-\tau)\}$ for $t \geq \tau$.

Theorem 4.2.5. *Assume that conditions* $(H_0) - (H_5)$ *hold,* $\tau \leq \alpha$*,* p_1, p_2 *and* p_3 *are positive real numbers. If (i)* $0 \leq p(t) \leq p_1 < 1$ *or* $1 < p_2 \leq p(t) \leq p_3 < \infty$ *holds, then every solution of* (4.1) *either oscillatory or converges to zero as* $t \to \infty$.

Proof. Assume that (4.1) has a nonoscillatory solution on $[t_0, \infty)$, $t_0 \geq 0$ and let it be $y(t)$. Hence, $y(t) > 0$ or < 0 for $t \geq t_0$. Suppose that $y(t) > 0$ for $t \geq t_0$. Define the functions

$$z(t) = y(t) + p(t)y(t-\tau), \tag{4.7}$$

$$K(t) = \int_t^\infty \frac{s-t}{r(s)} \int_s^\infty (\theta - s)h(\theta)H(y(\theta-\beta))d\theta ds. \tag{4.8}$$

Note that condition (H_1) and the fact that H is bounded function implies that $K(t)$ exists for all t. Now if

$$w(t) = z(t) - K(t) = y(t) + p(t)y(t-\tau) - K(t), \tag{4.9}$$

then from (4.1), we obtain

$$(r(t)w''(t))'' = -q(t)G(y(t-\alpha)) \leq 0 (\not\equiv 0), \tag{4.10}$$

for $t \geq t_0 + \rho$. Clearly, $w(t), w'(t), r(t)w''(t), (r(t)w''(t))'$ are monotonic functions on $[t_1, \infty), t_1 \geq t_0 + \rho$. In view of Lemma 4.2.1, we have to consider two cases, viz., $w(t) > 0$ or $w(t) < 0$ for $t \geq t_1$. Suppose the former holds. By the Lemma 4.2.1, any one of the cases (a) or (b) holds. Using (H_2) and (H_3), we obtain

$$\begin{aligned} 0 &= (r(t)w''(t))'' + q(t)G(y(t-\alpha)) + G(p_1)(r(t-\tau)w''(t-\tau))'' \\ &+ G(p_1)q(t-\tau)G(y(t-\tau-\alpha)) \\ &\geq (r(t)w''(t))'' + G(p_1)(r(t-\tau)w''(t-\tau))'' + \lambda Q(t)G(y(t-\alpha) + p_1 y(t-\alpha-\tau)) \\ &\geq (r(t)w''(t))'' + G(p_1)(r(t-\tau)w''(t-\tau))'' + \lambda Q(t)G(z(t-\alpha)) \end{aligned} \tag{4.11}$$

for $t \geq t_2 > t_1$. From (4.8), it follows that $K(t) > 0$ and $K'(t) < 0$, and so $w(t) > 0$ implies $w(t) < z(t)$ for $t \geq t_2$. Therefore, (4.11) yields

$$(r(t)w''(t))'' + G(p_1)(r(t-\tau)w''(t-\tau))'' + \lambda Q(t)G(w(t-\alpha)) \leq 0, \tag{4.12}$$

for $t \geq t_2$, that is

$$\begin{aligned} 0 &\geq (r(t)w''(t))'' + G(p_1)(r(t-\tau)w''(t-\tau))'' \\ &+ \lambda Q(t)G(R_T(t-\alpha)(r(t-\alpha)w''(t-\alpha))') \end{aligned}$$

due to Lemma 4.2.2, for $t \geq T + \rho > t_2$. Hence

$$\begin{aligned} 0 &\geq (r(t)w''(t))'' + G(p_1)(r(t-\tau)w''(t-\tau))'' \\ &+ \lambda Q(t)G(R_T(t-\alpha))G((r(t-\alpha)w''(t-\alpha))'), \end{aligned}$$

that is,

$$\begin{aligned} \lambda Q(t)G(R_T(t-\alpha)) &\leq -[G((r(t-\alpha)w''(t-\alpha))')]^{-1}(r(t)w''(t))'' \\ &- G(p_1)[G((r(t-\alpha)w''(t-\alpha))')]^{-1}(r(t-\tau)w''(t-\tau))'' \\ &\leq -[G((r(t)w''(t))')]^{-1}(r(t)w''(t))'' \\ &- G(p_1)[G((r(t-\tau)w''(t-\tau))')]^{-1}(r(t-\tau)w''(t-\tau))''. \end{aligned}$$

Since $\lim_{t\to\infty}(r(t)w''(t))' < \infty$, then using (H_4) the above inequality becomes

$$\int_{T+\rho}^{\infty} Q(t)G(R_T(t-\alpha))dt < \infty,$$

which contradicts (H_5) since $R_T(t)$ is monotonic increasing function.

Next, we assume that $w(t) < 0$ for $t \geq t_1$. Then $z(t) - K(t) < 0$ implies $y(t) \leq z(t) = y(t) + p(t)y(t-\tau) < K(t)$. Thus, $y(t)$ is bounded since $K(t)$ is bounded and monotonic. By the Lemma 4.2.1, any one of the cases (b), (c), (d) or (e) holds.

Consider the case (b). Since $\lim_{t\to\infty} K(t)$ exists, $\lim_{t\to\infty} w(t)$ exists, and so $\lim_{t\to\infty} z(t)$ exists. Furthermore, $\lim_{t\to\infty}(r(t)w''(t))'$ exists, and an integration of (4.10) implies

$$\int_{t_1}^{\infty} Q(t)G(y(t-\alpha))dt < \infty.$$

Hence, it is easy to verify that $\liminf_{t\to\infty} y(t) = 0$ due to (H_5). It then follows from Lemma 4.2.4 that $\lim_{t\to\infty} z(t) = 0$. Thus, $\lim_{t\to\infty} y(t) = 0$ since $z(t) \geq y(t)$.

To see that cases (c) and (d) are not possible, first note that $w(t) < 0$, $y(t)$ is bounded, $\lim_{t\to\infty} K(t)$ exists and hence $\lim_{t\to\infty} w(t)$ exists. On the otherhand, integrating successively, $w''(t) < 0$ from t_1 to $t(\geq t_1)$, yields $\lim_{t\to\infty} w(t) = -\infty$, which is a contradiction.

Consider the case (e). In this case $r(t)w''(t)$ is nondecreasing on $[t_1, \infty)$. Hence for $t \geq t_1$, $r(t)w''(t) \geq r(t_1)w''(t_1)$, that is,

$$tw''(t) \geq \frac{t}{r(t)}r(t_1)w''(t_1). \tag{4.13}$$

Integrating (4.13) from t_1 to t, we obtain

$$tw'(t) \geq w(t) - w(t_1) + t_1 w'(t_1) + r(t_1)w''(t_1) \int_{t_1}^{t} \frac{s}{r(s)} ds,$$

that is, $tw'(t) > 0$ for large t due to (H_0), a contradiction.

Finally, we assume that $y(t) < 0$ for $t \geq t_0$. From (H_3), we note that $G(-u) = -G(u)$ and $H(-u) = -H(u), u \in \mathbb{R}$. Indeed, $G(1)G(1) = G(1)$ and $G(-1)G(-1) = G(1)$ implies that $G(-1) = -1$ and $G(1) = 1$. Hence putting $x(t) = -y(t)$ for $t \geq t_0$, we obtain $x(t) > 0$ and

$$(r(t)(x(t) + p(t)x(t-\tau))'')'' + q(t)G(x(t-\alpha)) - h(t)H(x(t-\beta)) = 0.$$

Proceeding as above, we can show that every solution of (4.1) either oscillates or converges to zero as $t \to \infty$. This completes the proof of the theorem. □

Example 4.2.6. Consider

$$(y(t) + e^{-3}y(t-3))^{(iv)} + e^{-2}y^{\frac{1}{3}}(t-6)$$

$$-e^{-\frac{1}{5}}(2e^{-\frac{4t}{5}} + e^{-\frac{2t}{15}} + 2e^{2}e^{-\frac{14t}{5}} + e^{2}e^{-\frac{32t}{15}})\frac{y^{\frac{1}{5}}(t-1)}{1+y^{2}(t-1)} = 0, \tag{4.14}$$

for $t \geqslant 7$. It is easy to verify that the hypothesis of Theorem 4.2.5 are satisfied. Thus, every solution of (4.14) either oscillates or tends to zero as $t \to \infty$. Indeed, $y(t) = e^{-t}$ is such a solution of (4.14).

The following corollary is immediate.

Corollary 4.2.7. *Under the conditions of Theorem 4.2.5, every unbounded solution of (4.1) oscillates.*

Theorem 4.2.8. *Assume that conditions $(H_0) - (H_3), (H_5)$, $\tau \leq \alpha$ and*
(H_6) $\quad \frac{G(x_1)}{x_1^{\sigma}} \geq \frac{G(x_2)}{x_2^{\sigma}}$ *for* $\quad x_1 \geq x_2 > 0$ *and* $\sigma \geq 1$ *hold.*
If (i) $0 \leq p(t) \leq p_1 < 1$ or (ii) $1 < p_2 \leq p(t) \leq p_3 < \infty$ holds, then every solution of (4.1) is either oscillatory or tends to zero as $t \to \infty$.

Proof. Proceeding as in the proof of Theorem 4.2.5, we obtain

$$(r(t)w''(t))'' + G(p_1)(r(t-\tau)w''(t-\tau))'' + \lambda Q(t)G(z(t-\alpha)) \leq 0 \tag{4.15}$$

for $t \geq t_2$. In view of (4.10) and Lemma 4.2.1, $w(t)$ is nondecreasing, there exists $k > 0$ and $t_3 > t_2$ such that $w(t) > k$ for $t \geq t_3$. Hence use of (H_6) along with Lemma 4.2.2, we obtain

$$\begin{aligned} G(w(t-\alpha)) &= (G(w(t-\alpha))/w^\sigma(t-\alpha))w^\sigma(t-\alpha) \\ &\geq (G(k)/k^\sigma)(w^\sigma(t-\alpha)) \\ &> (G(k)/k^\sigma)R_T^\sigma(t-\alpha)((r(t-\alpha)w''(t-\alpha))')^\sigma \end{aligned} \tag{4.16}$$

for $t \geq T+\alpha > t_3+\alpha$. Using (4.16) in (4.15), we obtain

$$\begin{aligned} \lambda(G(k)/k^\sigma)R_T^\sigma(t-\alpha)Q(t)((r(t-\alpha)w''(t-\alpha))')^\sigma &< \lambda Q(t)G(w(t-\alpha)) \\ &\leq \lambda Q(t)G(z(t-\alpha)) \\ &\leq -(r(t)w''(t))'' \\ &\quad -G(p_1)(r(t-\tau)w''(t-\tau))'', \end{aligned}$$

that is,

$$\begin{aligned} \lambda(G(k)/k^\sigma)R_T^\sigma(t-\alpha)Q(t) &< -[(r(t-\alpha)w''(t-\alpha))']^{-\sigma}[(r(t)w''(t))'' \\ &+ G(p_1)(r(t-\tau)w''(t-\tau))''] \\ &< -((r(t)w''(t))')^{-\sigma}(r(t)w''(t))'' \\ &- G(p_1)((r(t-\tau)w''(t-\tau))')^{-\sigma}(r(t-\tau)w''(t-\tau))'' \end{aligned}$$

for $t \geq T+\alpha$. Since $\lim_{t\to\infty}(r(t)w''(t))'$ exists and $R_T(t)$ is nondecreasing, then proceeding as in the proof of Theorem 4.2.5, we obtain

$$\int_{T+\alpha}^{\infty} R_T^\sigma(t-\alpha)Q(t)dt < \infty,$$

which contradict (H_5). The proof in case $w(t) < 0$ is same as in Theorem 4.2.5. Thus the theorem is proved. □

Corollary 4.2.9. *Under the conditions of Theorem 4.2.8, every unbounded solution of* (4.1) *oscillates.*

In our next theorem we are able to replace conditions (H_3) and (H_4) in Theorem 4.2.5 with conditions (H_7) and (H_8) below.

Theorem 4.2.10. *Assume that conditions* (H_0)-(H_2), (H_5), $\tau \leq \alpha$ *and*
(H_7) $G(u)G(v) \geq G(uv)$ *for* $u > 0, v > 0$;
(H_8) $G(-u) = -G(u), H(-u) = -H(u), u \in \mathbb{R}$ *hold.*
If (i) $0 \leq p(t) < p_1 < 1$ *or (ii)* $1 < p_2 \leq p(t) \leq p_3 < \infty$ *holds, then every solution of* (4.1) *either oscillates or converges to zero as* $t \to \infty$.

Proof. Proceeding as in the proof of the Theorem 4.2.8, in case $w(t) > 0$ we have (4.15) for $t \geq t_2$. Since $w(t)$ is nondecreasing, then there exist $k > 0$ and $t_3 > t_2$ such that $w(t) > k$ for $t \geq t_3$, that is, $z(t) \geq w(t) > k$ for $t \geq t_3$. Consequently, inequality (4.15) yields

$$\lambda G(k) \int_{t_3}^{\infty} Q(t)dt < \infty,$$

a contradiction to (H_5). The rest of the proof is similar to the Theorem 4.2.5. This completes the proof of the theorem. □

Example 4.2.11. Consider

$$(y(t) + e^{-t-2\pi}y(t-2\pi))^{(iv)} + (4+7e^{-t})e^{-4\pi}y(t-4\pi)$$

$$-24e^{-t-\frac{\pi}{2}}(1+e^{-2t+\pi}\cos^2 t)\frac{y\left(t-\frac{\pi}{2}\right)}{1+y^2\left(t-\frac{\pi}{2}\right)} = 0, \tag{4.17}$$

for $t \geq 13$. Clearly, $(H_0)-(H_2), (H_5), (H_7)$ and (H_8) are satisfied. Hence by Theorem 4.2.10 every solution of (4.17) either oscillates or converges to zero as $t \to \infty$. In particular, $y(t) = e^{-t}\sin t$ is such a solution of (4.17).

Corollary 4.2.12. *Under the conditions of Theorem 4.2.10, every unbounded solutions of* (4.1) *oscillates.*

Remark 4.2.13. In Theorems 4.2.5 and Corollary 4.2.7, G is sublinear only, whereas in Theorem 4.2.8 and Corollary 4.2.9, G is superlinear. But in Theorem 4.2.10, G could be linear, sublinear or superlinear.

Next, we consider the case where $p(t)$ is negative. Here p_4, p_5 and p_6 are negative and real numbers.

Theorem 4.2.14. *Let* $-1 < p_4 \le p(t) \le 0$, (H_0), (H_1), (H_3), (H_4) *and*
(H_9) $\int_0^\infty q(t)dt = \infty$
hold, then every solution of (4.1) *either oscillates or tends to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (4.1). Because of (H_3), without loss of generality we may suppose that $y(t) > 0$ for $t \ge t_0 > 0$. Setting as in (4.7), (4.8) and (4.9) we obtain (4.10) for $t \ge t_0 + \rho$. By Lemma 4.2.1, $w(t)$ is monotonic on $[t_1, \infty)$, $t_1 \ge t_0 + \rho$. If $w(t) > 0$ for $t \ge t_1$, then any one of the cases (a) or (b) of Lemma 4.2.1 holds. Consequently, $w(t) > R_T(t)(r(t)w''(t))'$ for $t \ge t_2 > t_1$ by Lemma 4.2.2. Moreover, $w(t) \le y(t)$ since $p(t) \le 0$ implies that $y(t) > R_T(t)(r(t)w''(t))'$ for $t \ge t_2$ and hence (4.10) becomes

$$\int_{t_2+\alpha}^{\infty} q(t)G(R_T(t-\alpha))dt < \infty,$$

which contradict (H_9) since G, and R_T are increasing functions. Hence, $w(t) < 0$ for $t \ge t_1$, and so any one of the cases (b), (c), (d) or (e) of Lemma 4.2.1 holds.

We claim that $y(t)$ is bounded. If this is not the case, then there is an increasing sequence $\{\eta_n\}_{n=1}^{\infty}$ such that $\eta_n \to \infty$ and $y(\eta_n) \to \infty$ as $n \to \infty$ and $y(\eta_n) = max\{y(t) : t_1 \le t \le \eta_n\}$. We may choose n large enough such that $\eta_n - \tau > t_1$. Hence

$$0 \ \ge \ w(\eta_n) \ge y(\eta_n) + p_4 y(\eta_n - \tau) - K(\eta_n) \ge (1 + p_4)y(\eta_n) - K(\eta_n).$$

Since $K(\eta_n)$ is bounded and $(1 + p_4) > 0$, then $w(\eta_n) > 0$ for large n which is a contradiction.

The proof of the cases (c), (d) and (e) cannot hold are similar to the corresponding cases in the proof of Theorem 4.2.5. If (b) holds, then as in proof of Theorem 4.2.5

we obtain $\liminf_{t\to\infty} y(t) = 0$. Hence, $\lim_{t\to\infty} z(t) = 0$ by Lemma 4.2.4. Consequently,

$$\begin{aligned} 0 = \limsup_{t\to\infty} z(t) &\geq \limsup_{t\to\infty}(y(t) + p_4 y(t-\tau)) \\ &\geq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_4 y(t-\tau)) \\ &= \limsup_{t\to\infty} y(t) + p_4 \limsup_{t\to\infty} y(t-\tau) \\ &= (1+p_4)\limsup_{t\to\infty} y(t). \end{aligned}$$

Since $(1 + p_4) > 0$, $\limsup_{t\to\infty} y(t) = 0$. Hence, $\lim_{t\to\infty} y(t) = 0$. This completes the proof of the theorem. □

Corollary 4.2.15. *Under the conditions of Theorem 4.2.14, every unbounded solution of* (4.1) *oscillates.*

Theorem 4.2.16. *Assume that conditions* $(H_0), (H_1), (H_3), (H_4)$ *and* (H_9) *hold. If* $-\infty < p_5 \leq p(t) \leq p_6 < -1$, *then every bounded solution of* (4.1) *either oscillates or tends to zero as* $t \to \infty$.

Proof. Let $y(t)$ be bounded nonoscillatory solution of (4.1), on $[t_0, \infty), t_0 \geq 0$. With (4.7), (4.8) and (4.9) as above, we obtain (4.10) for $t \geq t_0 + \rho$. Hence from Lemma 4.2.1, $w(t)$ is monotonic on $[t_1, \infty), t_1 \geq t_0 + \rho$. If $w(t) > 0$ for $t \geq t_1$, then one of the cases (a) or (b) of Lemma 4.2.1 holds. Consequently, $w(t) > R_T(r(t)w''(t))'$ for $t \geq T > t_1$ by Lemma 4.2.2. Moreover, $w(t) \leq y(t)$. Choose $t_2 \in [T, \infty)$ such that $t - \alpha \geq T$ for all $t \in [t_2, \infty)$. Then, $y(t-\alpha) > R_T(t-\alpha)(r(t-\alpha)w''(t-\alpha))'$ for $t \geq t_2$ and (4.10) becomes

$$\int_{t_2}^{\infty} q(t)G(R_T(t-\alpha))dt < \infty,$$

which contradicts (H_9) since G, R_T are inceasing. Hence, $w(t) < 0$ for $t \geq t_1$, so one of the cases (b), (c), (d) or (e) of Lemma 4.2.1 holds.

In case (b), since $w(t) < 0, w'(t) > 0$, and $\lim_{t\to\infty} K(t)$ exists, we have $\lim_{t\to\infty} z(t)$ exists. Furthermore, $\lim_{t\to\infty}(r(t)w''(t))'$ exists. Integrating (4.10) from t_2 to t, we obtain

$$\int_{t_2}^{\infty} q(t)G(y(t-\alpha))dt < \infty,$$

which implies that $\liminf_{t\to\infty} y(t) = 0 = \liminf_{t\to\infty} y(t-\alpha)$ due to (H_9). Hence, $\lim_{t\to\infty} z(t) = 0$ by Lemma 4.2.4. Therefore,

$$\begin{aligned} 0 = \liminf_{t\to\infty} z(t) &= \liminf_{t\to\infty}(y(t) + p(t)y(t-\alpha)) \\ &\le \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_6 y(t-\alpha)) \\ &= \limsup_{t\to\infty} y(t) + p_6 \limsup_{t\to\infty} y(t-\alpha) \\ &= (1+p_6)\limsup_{t\to\infty} y(t). \end{aligned}$$

Since $(1+p_6) < 0$, we have $\limsup_{t\to\infty} y(t) \le 0$, so $\lim_{t\to\infty} y(t) = 0$.

Cases (c) and (d) are not possible since $w(t) < 0$ for $t \ge t_1$, $y(t)$ is bounded and $\lim_{t\to\infty} K(t)$ exists.

If Case(e) holds, we have $r(t)w''(t)$ is nondecreasing on $[t_1, \infty)$. Hence $t > t_2 \ge t_1$, $r(t)w''(t) \ge r(t_2)w''((t_2) > 0$, so

$$tw''(t) \ge r(t_2)w''(t)\frac{t}{r(t)}.$$

Integrating the above inequality from t_2 to t, we obtain

$$tw'(t) \ge w(t) - w(t_2) + t_2 w'(t_2) + r(t_2)w''(t_2)\int_{t_2}^{t} \frac{s}{r(s)}ds,$$

that is, $tw'(t) > 0$ for large t due to (H_0), is a contradiction. This completes the proof of theorem. □

4.3 Oscillation Criteria for Non-Homogeneous Equation with $\int_0^\infty \frac{t}{r(t)}dt = \infty$.

This section is devoted to study the oscillatory and asymptotic behavior of solutions of forced equations (4.2) with suitable forcing functions. Our attention is restricted to the forcing functions which are eventually change sign. we have the following hypotheses regarding f(t).

(H_{10}) There exists F $\in C^2([0,\infty),\mathbb{R})$ such that $-\infty < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \infty$, $rF'' \in C^2([0,\infty),\mathbb{R})$ and $(rF'')'' = f$.

(H_{11}) There exists F $\in C^2([0,\infty),\mathbb{R})$ such that $\liminf_{t\to\infty} F(t) = -\infty, \limsup_{t\to\infty} F(t) = \infty, rF'' \in C^2([0,\infty),\mathbb{R})$ and $(rF'')'' = f$.

Theorem 4.3.1. *Let $0 \le p(t) \le p_1 < \infty$. Assume that (H_0)-(H_2), (H_7), (H_8) and (H_{11}) hold. If*

$$(H_{12}) \qquad \limsup_{t\to\infty} \int_\alpha^t Q(s)G(F(s-\alpha))ds = +\infty \quad \text{and} \quad \liminf_{t\to\infty} \int_\alpha^t Q(s)G(F(s-\alpha))ds = -\infty$$

hold, then every solution of equation (4.2) is oscillatory.

Proof. Let $y(t)$ be a non oscillatory solution of (4.2) such that $y(t) > 0$ for $t \ge t_0 > 0$. Defining $z(t), K(t), w(t)$ as in (4.7), (4.8) and (4.9), respectively, equation (4.2) becomes

$$(r(t)w''(t))'' + q(t)G(y(t-\alpha)) = f(t). \tag{4.18}$$

Let

$$v(t) = w(t) - F(t) = z(t) - K(t) - F(t). \tag{4.19}$$

Then, for $t \ge t_0 + \rho$, equation (4.2) becomes

$$(r(t)v''(t))'' = -q(t)G(y(t-\alpha)) \le 0 (\not\equiv 0). \tag{4.20}$$

Thus, $v(t)$ is monotonic on $[t_1, \infty)$, $t_1 > t_0 + \rho$. Suppose $v(t) > 0$ for $t \ge t_1$ so that Case (a) or (b) of Lemma 4.2.1 holds. Then $z(t) > K(t) + F(t) > F(t)$ for $t \ge t_1$. Applying (H_2) and (H_7) yields

$$\begin{aligned} 0 &= (r(t)v''(t))'' + q(t)G(y(t-\alpha)) + G(p_1)(r(t-\tau)v''(t-\tau))'' \\ &+ G(p_1)q(t-\tau)G(y(t-\alpha-\tau)) \\ &\ge (r(t)v''(t))'' + G(p_1)(r(t-\tau)v''(t-\tau))'' + \lambda Q(t)G(y(t-\alpha) + p_1 y(t-\alpha-\tau)) \\ &\ge (r(t)v''(t))'' + G(p_1)(r(t-\tau)v''(t-\tau))'' + \lambda Q(t)G(z(t-\alpha)) \\ &\ge (r(t)v''(t))'' + G(p_1)(r(t-\tau)v''(t-\tau))'' + \lambda Q(t)G(F(t-\alpha)) \end{aligned} \tag{4.21}$$

for $t \geq t_2 > t_1$. Integrating the inequality (4.21) from $t_2 + \alpha$ to t and taking $\limsup$ as $t \to \infty$, we get

$$\limsup_{t\to\infty} \int_{t_2+\alpha}^{t} Q(s)G(F(s-\alpha))ds < \infty,$$

which is a contradiction to (H_{12}).

Therefore, $v(t) < 0$ for $t \geq t_1$. Thus any one of the cases (b), (c), (d) or (e) of Lemma 4.2.1 holds. Since $v(t) < 0$, then for each these cases $z(t) - K(t) < F(t)$. Thus, (H_{11}) implies $\liminf_{t\to\infty} z(t) = -\infty$, a contradiction to the fact that $z(t) > 0$. This completes the proof of the theorem. □

Example 4.3.2. Consider

$$(y(t) + (e^{-4t} + 1)y(t - 2\pi))^{(iv)} + 4e^{4\pi}y(t - 4\pi)$$

$$-100e^{-4t+\theta_1-2\pi}(1 + e^{2t-2\theta_1}\sin^2(t - \theta_1))\frac{y(t-\theta_1)}{1 + y^2(t-\theta_1)} = -4e^{t-2\pi}\sin t, \qquad (4.22)$$

for $t \geq 74$, where $\tan\theta_1 = \frac{24}{7}$. Indeed, if we choose $F(t) = e^{t-2\pi}\sin t$, then $(r(t)F''(t))'' = f(t)$.

Clearly, $(H_0) - (H_2), (H_7)$, (H_8), (H_{11}) and (H_{12}) are satisfied. Hence Theorem 4.3.1 can be applied to (4.22), that is, every solution of (4.22) oscillates. Indeed, $y(t) = e^t \sin t$ is such a solution of (4.22).

Theorem 4.3.3. *Let $-1 < p(t) \leq 0$. Suppose that (H_0), (H_1), (H_8) and (H_{11}) hold. If*

$$(H_{13}) \qquad \limsup_{t\to\infty} \int_{\alpha}^{t} q(s)G(F(s-\alpha))ds = +\infty \quad \textit{and}$$
$$\liminf_{t\to\infty} \int_{\alpha}^{t} q(s)G(F(s-\alpha))ds = -\infty$$

also hold, then every bounded solution of (4.2) *oscillates.*

Proof. Proceeding as in the proof of the Theorem 4.3.1, we obtain (4.20) for $t \geq t_1 \geq t_0 + \rho$. Thus, $v(t)$ is monotonic, so $v(t) > 0$ or $v(t) < 0$ for large t. If $v(t) > 0$ for $t \geq t_1$, then either case (a) or case (b) of Lemma 4.2.1 holds for $t \geq t_1$. Since $v(t)$ is monotonic, $z(t) > z(t) - K(t) > F(t)$ implies that $z(t) > F(t)$, so $y(t) > z(t) > F(t)$ for $t \geq t_1$. Choose $t_2 \in [t_1, \infty)$ such that $t - \alpha \geq t_2$ for all $t \in [t_2, \infty)$. Hence, for $t \geq t_2$, $y(t-\alpha) > z(t-\alpha) > F(t-\alpha)$. From (4.20), we have

$$q(t)G(F(t-\alpha)) \leq q(t)G(y(t-\alpha)) = -(r(t)v''(t))''$$

for $t \geq t_2$. An integration yields a contradiction to (H_{13}).

Now assume $v(t) < 0$ for $t \geq t_1$. Thus, $z(t) - K(t) < F(t)$ and condition (H_{11}) implies that $\liminf_{t\to\infty} z(t) = -\infty$. This contradicts the fact that $y(t)$ is bounded. This completes the proof of the theorem. □

Theorem 4.3.4. *Assume that* (H_0)-(H_2), (H_7), (H_8), (H_{10}) *and* (H_{12}) *hold. If* $0 \leq p(t) \leq p_1 < \infty$ *holds, then every unbounded solution of* (4.2) *oscillates.*

Proof. Let $y(t)$ be an unbounded nonoscillatory solution of (4.2) such that $y(t) > 0$ for $t \geq t_0$. Using (4.7)-(4.9) and (4.18), we obtain, inequality (4.20) for $t \geq t_0 + \rho$. Thus, $v(t)$ is monotonic on $[t_1, \infty)$, $t_1 > t_0 + \rho$. First assume $v(t) > 0$ for all $t \geq t_1$. Proceeding as in the proof of the Theorem 4.3.1, we obtain contradiction. Hence, $v(t) < 0$ for $t \geq t_1$. From Lemma 4.2.1, it follows that any one of the cases (b), (c), (d) or (e) holds. In case (b), $\lim_{t\to\infty} v(t)$ exists and hence $z(t) = v(t) + K(t) + F(t)$, implies that

$$y(t) \leq v(t) + K(t) + F(t). \tag{4.23}$$

That is, $y(t)$ is bounded, which is a contradiction. For each of the cases (c), (d) or (e), $v(t)$ is a nonincreasing function on $[t_1, \infty)$, so let $\lim_{t\to\infty} v(t) = l$, $l \in [-\infty, 0)$. If $l = -\infty$, then (4.23) yields

$$\begin{aligned}
\liminf_{t\to\infty} y(t) &\leq \limsup_{t\to\infty} v(t) + \liminf_{t\to\infty}(K(t) + F(t)) \\
&\leq \limsup_{t\to\infty} v(t) + \limsup_{t\to\infty} K(t) + \liminf_{t\to\infty} F(t) \\
&= \lim_{t\to\infty} v(t) + \lim_{t\to\infty} K(t) + \liminf_{t\to\infty} F(t)
\end{aligned}$$

that is, $\liminf_{t\to\infty} y(t) = -\infty$, which is a contradiction.

If $-\infty < l < 0$, then in cases (c) and (d), $v'(t)$ is decreasing. Successive integrations of $v''(t)$ again show that $\lim_{t\to\infty} v(t) = -\infty$, a contradiction. If Case (e) holds, $y(t) \le v(t) + K(t) + F(t) \le K(t) + F(t)$, which contradicts the unboundedness of $y(t)$. This completes the proof of the theorem. □

Example 4.3.5. Consider

$$(y(t) + (e^{-t} + 1)y(t - 2\pi))^{(iv)} + e^t y(t - 2\pi)$$

$$-e^{-t}\frac{y(t - 4\pi)}{1 + y^2(t - 4\pi)} = \sin t \tag{4.24}$$

for $t \geqslant 13$. It is easy to verify that the hypothesis of Theorem 4.3.4 are satisfied. Thus, every solution of (4.24) either oscillates or tends to zero as $t \to \infty$.

Our final theorem in this section gives sufficient conditions for equation (4.2) to have a bounded positive solution.

Theorem 4.3.6. *Let $0 \le p(t) \le p_1 < 1$, (H_1) and (H_{10}) hold with*

$$-\frac{1}{8}(1 - p_1) < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \frac{1}{4}(1 - p_1).$$

Furthermore, assume that G and H are Lipschitzian on the intervals of the form $[b, c]$, $0 < b < c < \infty$. If

$$\int_0^\infty \frac{s}{r(s)} \int_s^\infty tq(t)dtds < \infty$$

holds, then (4.2) admits a positive bounded solution.

Proof. It is possible to choose $t_0 > 0$ large enough such that for $t \ge t_0 > 0$,

$$\int_{t_0}^\infty \frac{t}{r(t)} \int_t^\infty sh(s)dsdt < \frac{1 - p_1}{4L}$$

and

$$\int_{t_0}^\infty \frac{t}{r(t)} \int_t^\infty sq(s)dsdt < \frac{1 - p_1}{4L},$$

where $L = max\{L_1, L_2, G(1), H(1)\}$ and L_1, L_2 are Lipschitz constants of G and H on $[\frac{1}{8}(1 - p_1), 1]$ respectively. Let $X = BC([t_0, \infty), \mathbb{R})$. Then X is a Banach Space with respect to supremum norm defined by

$$||x|| = \sup_{t\ge t_0}\{|x(t)|\}.$$

Let

$$S = \{x \in X : \frac{1}{8}(1-p_1) \le x(t) \le 1, t \ge t_0\}.$$

Hence S is a complete metric space. For $y \in S$, we define

$$Ty(t) = \begin{cases} Ty(t_0+\rho), t \in [t_0, t_0+\rho] \\ -p(t)y(t-\tau) + \frac{1}{2}(1+p_1) + F(t) + K(t) \\ -\int_t^\infty (\frac{s-t}{r(s)} \int_s^\infty (u-s)q(u)G(y(u-\alpha))du)ds, t \ge t_0+\rho \end{cases}$$

Indeed,

$$\begin{aligned} K(t) &= \int_t^\infty \frac{s-t}{r(s)} \int_s^\infty (u-s)h(u)H(y(u-\beta))duds \\ &\le H(1)\int_t^\infty \frac{s}{r(s)} \int_s^\infty uh(u)duds < \frac{1}{4}(1-p_1) \end{aligned}$$

implies that

$$Ty(t) < \frac{1+p_1}{2} + \frac{1-p_1}{4} + \frac{1-p_1}{4} = 1.$$

On the other hand,

$$\int_t^\infty \frac{s-t}{r(s)} \int_s^\infty (u-s)q(u)G(y(u-\alpha))duds < \frac{1-p_1}{4}$$

implies that

$$Ty(t) > -p_1 + \frac{1}{2}(1+p_1) - \frac{1}{8}(1-p_1) - \frac{1}{4}(1-p_1) = \frac{1}{8}(1-p_1).$$

Hence $Ty \in S$, that is, $T : S \to S$.

Next, we show that T is continuous. Let $y_k(t) \in S$ such that $\lim_{k\to\infty} \|y_k(t) - y(t)\| = 0$ for all $t \ge t_0$. Because S is closed, $y(t) \in S$. Indeed,

$$\begin{aligned} |(Ty_k) - (Ty)| &\le p(t)|y_k(t-\tau) - y(t-\tau)| \\ &+ \left| \int_t^\infty \frac{s-t}{r(s)} \int_s^\infty (u-s)q(u)[-G(y_k(u-\alpha)) + G(y(u-\alpha))]duds \right| \\ &+ \left| \int_t^\infty \frac{s-t}{r(s)} \int_s^\infty (u-s)h(u)[H(y_k(u-\beta)) - H(y(u-\beta))]duds \right| \end{aligned}$$

$$
\begin{aligned}
&\le p_1|y_k(t-\tau)-y(t-\tau)| \\
&+\int_t^\infty \frac{s-t}{r(s)}\int_s^\infty (u-s)q(u)|G(y_k(u-\alpha))-G(y(u-\alpha))|duds \\
&+\int_t^\infty \frac{s-t}{r(s)}\int_s^\infty (u-s)h(u)|H(y_k(u-\beta))-H(y(u-\beta))|duds \\
&\le p_1||y_k-y||+L_1||y_k-y||\int_t^\infty \frac{s}{r(s)}\int_s^\infty uq(u)duds \\
&+L_2||y_k-y||\int_t^\infty \frac{s}{r(s)}\int_s^\infty uh(u)duds,
\end{aligned}
$$

implies that

$$
\|(Ty_k)-(Ty)\|\le \|y_k-y\|\left[p_1+\frac{1-p_1}{4}+\frac{1-p_1}{4}\right]\to 0
$$

as $k\to\infty$. Hence T is continuous.

In order to apply Schauder's fixed point theorem [42] we need to show that Ty is precompact. Let $y\in S$. For $t_2\ge t_1$,

$$
\begin{aligned}
|(Ty)(t_2)-(Ty)(t_1)| &\le |p(t_2)y(t_2-\tau)-p(t_1)y(t_1-\tau)|+|F(t_2)-F(t_1)| \\
&+\Big|\int_{t_2}^\infty \frac{s-t_2}{r(s)}\int_s^\infty (u-s)h(u)H(y(u-\beta))duds \\
&-\int_{t_1}^\infty \frac{s-t_1}{r(s)}\int_s^\infty (u-s)h(u)H(y(u-\beta))duds\Big| \\
&+\Big|\int_{t_2}^\infty \frac{s-t_2}{r(s)}\int_s^\infty (u-s)q(u)G(y(u-\alpha))duds \\
&-\int_{t_1}^\infty \frac{s-t_1}{r(s)}\int_s^\infty (u-s)q(u)G(y(u-\alpha))duds\Big| \\
&\le |p(t_2)y(t_2-\tau)-p(t_1)y(t_1-\tau)|+|F(t_2)-F(t_1)| \\
&+\Big|\int_{t_2}^\infty \frac{s-t_1}{r(s)}\int_s^\infty (u-s)h(u)H(y(u-\beta))duds \\
&-\int_{t_1}^\infty \frac{s-t_1}{r(s)}\int_s^\infty (u-s)h(u)H(y(u-\beta))duds\Big| \\
&+\Big|\int_{t_2}^\infty \frac{s-t_1}{r(s)}\int_s^\infty (u-s)q(u)G(y(u-\alpha))duds \\
&-\int_{t_1}^\infty \frac{s-t_1}{r(s)}\int_s^\infty (u-s)q(u)G(y(u-\alpha))duds\Big|
\end{aligned}
$$

$$
\begin{aligned}
&= |p(t_2)y(t_2-\tau) - p(t_1)y(t_1-\tau)| + |F(t_2) - F(t_1)| \\
&+ \left| \int_{t_1}^{t_2} \frac{s-t_1}{r(s)} \int_s^\infty (u-s)h(u)H(y(u-\beta))duds \right| \\
&+ \left| \int_{t_1}^{t_2} \frac{s-t_1}{r(s)} \int_s^\infty (u-s)q(u)G(y(u-\alpha))duds \right| \\
&\to 0 \qquad\qquad as \;\; t_2 \to t_1.
\end{aligned}
$$

Thus, Ty is precompact. By Schauder's fixed point theorem T has a fixed point, that is, $Ty = y$. Consequently, $y(t)$ is a solution of (4.2) on $[\frac{1}{8}(1-p_1), 1]$. This completes the proof of the theorem. □

Remark 4.3.7. Theorems similar to Theorem 4.3.6 can be proved in other ranges of $p(t)$. Moreover, using other fixed point theorems such as Banach Fixed Point Theorem or Krasnosel'skii's Fixed Point Theorem etc. one can establish the existence of bounded positive solution of (4.2).

4.4 Fourth Order NDDE with $\int_0^\infty \frac{t}{r(t)}dt < \infty$.

In this section, we would like to study the oscillatory and asymptotic behaviour of the solutions of (4.1)/(4.2), under the assumption

$$(H_{14}) \qquad\qquad \int_0^\infty \frac{t}{r(t)}dt < \infty.$$

If $h(t) \equiv 0$, then (4.1) and (4.2) reduce to (4.3) and (4.4) respectively. In [68], Parhi and Tripathy have studied (4.3) and (4.4), under the assumption (H_{14}). If $h(t) \not\equiv 0$, then nothing is known about the behaviour of solutions of (4.1)/(4.2). Therefore, an attempt has been made to study the qualitative behaviour of the solutions of (4.1) and (4.2) under the assumption (H_1) and (H_{14}). We need the following lemms for our use in the sequel.

Lemma 4.4.1.[68] Let (H_{14}) hold. If $u(t)$ is an eventually positive twice continuously differentiable function such that $r(t)u''(t)$ is twice continuously differentiable and

$(r(t)u''(t))'' \leq 0$ $(\not\equiv 0)$ for large t, then one of the following cases holds for large t:

$$(a) \quad u'(t) > 0, u''(t) > 0 \quad and \quad (r(t)u''(t))' > 0,$$

$$(b) \quad u'(t) > 0, u''(t) < 0 \quad and \quad (r(t)u''(t))' > 0,$$

$$(c) \quad u'(t) > 0, u''(t) < 0 \quad and \quad (r(t)u''(t))' < 0,$$

$$(d) \quad u'(t) < 0, u''(t) > 0 \quad and \quad (r(t)u''(t))' > 0.$$

Lemma 4.4.2.[68] Suppose that the conditions of Lemma 4.4.1 hold. Then (i) the following inequalities hold for large t in the case (c) of Lemma 4.4.1:

$$u'(t) \geq -(r(t)u''(t))'R(t), u'(t) \geq -r(t)u''(t)\int_t^\infty \frac{ds}{r(s)},$$
$$u(t) \geq ktu'(t) \quad and \quad u(t) \geq -k(r(t)u''(t))'tR(t),$$

where $k > 0$ is a constant and $R(t) = \int_t^\infty \frac{s-t}{r(s)}ds$
and

$$(ii) \quad u(t) \geq r(t)u''(t)R(t)$$

for large t in case (d) of Lemma 4.4.1.

Lemma 4.4.3.[68] If the conditions of Lemma 4.4.1 hold, then there exist constants $k_1 > 0$ and $k_2 > 0$ such that $k_1R(t) \leq u(t) \leq k_2t$ for large t.

Lemma 4.4.4.[68] Let (H_{14}) hold. Suppose that $z(t)$ be a real valued twice continuously differentiable function on $[0, \infty)$, such that $r(t)z''(t)$ is twice continuously differentiable with $(r(t)z''(t))'' \leq 0$ $(\not\equiv 0)$ for large t. If $z(t) > 0$ eventually, then one of the following cases holds for large t:

$$(a) \quad z'(t) > 0, z''(t) > 0 \quad and \quad (r(t)z''(t))' > 0,$$

$$(b) \quad z'(t) > 0, z''(t) < 0 \quad and \quad (r(t)z''(t))' > 0,$$

$$(c) \quad z'(t) > 0, z''(t) < 0 \quad and \quad (r(t)z''(t))' < 0,$$

$$(d) \quad z'(t) < 0, z''(t) > 0 \quad and \quad (r(t)z''(t))' > 0.$$

If $z(t) < 0$ for large t, then either one of the cases (b) - (d) holds or one of the following cases holds for large t:

$$(e) \quad z'(t) < 0, z''(t) < 0 \quad and \quad (r(t)z''(t))' > 0,$$

$$(f) \quad z'(t) < 0, z''(t) < 0 \quad and \quad (r(t)z''(t))' < 0.$$

4.5 Oscillation Criteria for Homogeneous Equation with $\int_0^\infty \frac{t}{r(t)}dt < \infty$.

In this section, sufficient conditions are established for oscillation and asymptotic behaviour of solutions of (4.1) under the assumption (H_{14}). For our purpose, we need the following assumptions:

(H_{15}) $\int_{t_0}^\infty b(t)Q(t)G(R(t-\alpha))dt = \infty$, where $b(t) = \min\{R^\gamma(t), R^\gamma(t-\tau)\}$, $\gamma > 1$, $t_0 \geq \rho > 0$,

(H_{16}) $\int_{t_0}^\infty R^\gamma(t)G(R(t-\alpha))q(t)dt = \infty$, $\gamma > 1$, $t_0 \geq \rho > 0$.

Remark 4.5.1. Since $R(t) < \int_t^\infty \frac{s}{r(s)}ds$, then $R(t) \to 0$ as $t \to \infty$ in view of (H_{14}).

Remark 4.5.2. The prototype of G satisfying (H_2), (H_7) and (H_8) is

$G(u) = (a + b|u|^\gamma)|u|^\mu sgnu$, where $a \geqslant 1, b \geqslant 1, \gamma \geq 0$ and $\mu \geq 0$. However the prototype of G satisfying (H_2) and (H_3) is $G(u) = |u|^\lambda sgnu$, $\lambda > 0$. This G also satisfies the assumption (H_2), (H_7) and (H_8).

Theorem 4.5.3. Let $0 \leq p(t) \leq p_1 < 1$ or $1 < p_2 \leq p(t) \leq p_3 < \infty$. Suppose that (H_1) - (H_3), (H_5), (H_{14}) and (H_{15}) hold. Then every solution of (4.1) either oscillates or tends to zero as $t \to \infty$.

Proof. Due to Remark 4.5.1, $b(t) \to 0$ as $t \to \infty$. Hence (H_{15}) implies that

$$\int_{t_0}^\infty Q(t)G(R(t-\alpha))dt = \infty. \tag{4.25}$$

Assume that $y(t)$ is a nonoscillatory solution of (4.1). Then $y(t) > 0$ or $y(t) < 0$ for $t \geq t_0 > \rho$. Let $y(t) > 0$ for $t \geq t_0$. Setting $z(t), K(t), w(t)$ as in (4.7), (4.8), (4.9) respectively, we obtain (4.10) for $t \geq t_0 + \rho$. Consequently, $w(t)$, $w'(t)$, $(r(t)w''(t))$, $(r(t)w''(t))'$ are monotonic on $[t_1, \infty), t_1 \geq t_0 + \rho$. It follows that, either $w(t) > 0$ or < 0 for $t \geq t_1$. Let $w(t) > 0$ for $t \geqslant t_1$. By the Lemma 4.4.1, any one of the cases (a), (b), (c) or (d) holds. Suppose that any one of the cases (a), (b) or (d) holds. Using

(H_2), (H_3) and (H_5), equation (4.1) becomes

$$\begin{aligned} 0 &= (r(t)w''(t))'' + q(t)G(y(t-\alpha)) + G(p_1)(r(t-\tau)w''(t-\tau))'' \\ &+G(p_1)q(t-\tau)G(y(t-\tau-\alpha)) \\ &\geq (r(t)w''(t))'' + G(p_1)(r(t-\tau)w''(t-\tau))'' + \lambda Q(t)G(y(t-\alpha) \\ &+p_1y(t-\alpha-\tau)) \\ &\geq (r(t)w''(t))'' + G(p_1)(r(t-\tau)w''(t-\tau))'' + \lambda Q(t)G(z(t-\alpha)) \end{aligned}$$

for $t \geq t_2 > t_1$, where we have used the fact that $z(t) \leq y(t)+p_1y(t-\tau)$. From (4.8), it follows that $K(t) > 0, K'(t) < 0$, and hence $\lim_{t\to\infty} K(t)$ exists due to (H_1). Further, $w(t) > 0$ for $t \geq t_1$ implies that $w(t) < z(t)$ for $t \geq t_2$ and thus the last inequality yields

$$(r(t)w''(t))'' + G(p_1)(r(t-\tau)w''(t-\tau))'' + \lambda Q(t)G(w(t-\alpha)) \leq 0,$$

for $t \geq t_2$, that is,

$$(r(t)w''(t))'' + G(p_1)(r(t-\tau)w''(t-\tau))'' + \lambda G(k_1)Q(t)G(R(t-\alpha)) \leq 0$$

due to (H_3) and Lemma 4.4.3, for $t \geq t_3 > t_2$. Integrating the above inequality from t_3 to ∞, we get

$$\lambda G(k_1)\int_{t_3}^{\infty} Q(t)G(R(t-\alpha))dt < \infty,$$

a contradiction to (4.25). Next, we assume that the case (c) holds. Upon using Lemmas 4.4.2 and 4.4.3, we have

$$k(-r(t)w''(t))'tR(t) \leq w(t) \leq k_2t$$

for $t \geq t_4 > t_3$. Hence,

$$\begin{aligned} -[((-r(t)w''(t))')^{1-\gamma}]' &= (\gamma-1)((-r(t)w''(t))')^{-\gamma}(-r(t)w''(t))'' \\ &\geq (\gamma-1)L^{\gamma}R^{\gamma}(t)q(t)G(y(t-\alpha)), \end{aligned} \tag{4.26}$$

where $L = \frac{k}{k_2} > 0$. Therefore,

$$\begin{aligned}
-[((-r(t)w''(t))')^{1-\gamma}]' &- G(p_1)[((-r(t-\tau)w''(t-\tau))')^{1-\gamma}]' \\
&\geq (\gamma-1)L^{\gamma}[R^{\gamma}(t)q(t)G(y(t-\alpha)) \\
&+ G(p_1)R^{\gamma}(t-\tau)q(t-\tau)G(y(t-\tau-\alpha))] \\
&\geq \lambda(\gamma-1)L^{\gamma}b(t)Q(t)G(z(t-\alpha)) \\
&\geq \lambda(\gamma-1)L^{\gamma}b(t)Q(t)G(w(t-\alpha)) \\
&\geq \lambda(\gamma-1)L^{\gamma}G(k_1)b(t)Q(t)G(R(t-\alpha))
\end{aligned}$$

implies that

$$\lambda(\gamma-1)G(k_1)L^{\gamma}\int_{t_4}^{\infty} b(t)Q(t)G(R(t-\alpha))dt < \infty,$$

which contradicts (H_{15}). Let $w(t) < 0$ for $t \geqslant t_1$. Since $z(t) < K(t)$ and $K(t)$ is bounded, then $y(t)$ is bounded. From Lemma 4.4.4, it follows that any one of the cases (b) - (f) holds for $t \geq t_2 > t_1$. In the cases (e) and (f) of Lemma 4.4.4, $\lim_{t\to\infty} w(t) = -\infty$ which contradicts the fact that $y(t)$ is bounded and $w(t)$ is bounded. Consider the case (b) or (c), where $-\infty < \lim_{t\to\infty} w(t) \leq 0$. Consequently,

$$\begin{aligned}
0 \geq \lim_{t\to\infty} w(t) &= \limsup_{t\to\infty}[z(t) - K(t)] \\
&\geq \limsup_{t\to\infty}[y(t) - K(t)] \\
&\geq \limsup_{t\to\infty} y(t) - \lim_{t\to\infty} K(t) \\
&= \limsup_{t\to\infty} y(t)
\end{aligned}$$

implies that $\lim_{t\to\infty} y(t) = 0$. We may note that $\lim_{t\to\infty} K(t) = 0$. Finally, let the case (d) of Lemma 4.4.4 hold. Then $\lim_{t\to\infty}(r(t)w''(t))'$ exists. Hence integrating (4.10) from t_2 to ∞, we obtain

$$\int_{t_2}^{\infty} q(t)G(y(t-\alpha))dt < \infty,$$

that is,

$$\int_{t_2}^{\infty} Q(t)G(y(t-\alpha))dt < \infty. \tag{4.27}$$

If $\liminf_{t\to\infty} y(t) > 0$, then (4.27) yields,

$$\int_{t_2}^{\infty} Q(t)dt < \infty$$

which contradicts (H_5) due to Remark 4.5.1. Hence $\liminf_{t\to\infty} y(t) = 0$. Since $\lim_{t\to\infty} w(t)$ exists, using Lemma 4.2.4, we obtain $\lim_{t\to\infty} w(t) = 0 = \lim_{t\to\infty} z(t)$. Moreover, $z(t) \geq y(t)$ implies that $\lim_{t\to\infty} y(t) = 0$.

If $y(t) < 0$ for $t \geq t_0$, then we set $x(t) = -y(t)$ for $t \geq t_0$ and

$$(r(t)(x(t) + p(t)x(t-\tau))'')'' + q(t)G(x(t-\alpha)) - h(t)H(x(t-\beta)) = 0.$$

Proceeding as above, we obtain similar conclusion. This completes the proof of the theorem.

Example 4.5.4. Consider

$$\begin{aligned}(e^t(y(t) + \frac{1}{2}y(t-\pi))'')'' + e^{\frac{5t}{2}} y\left(t - \frac{\pi}{2}\right)& \\ -e^{-t}\frac{y(t-\frac{\pi}{2})}{1 + y^2(t-\frac{\pi}{2})} &= 0 \end{aligned} \tag{4.28}$$

for $t \geq 4$. Clearly (H_1) - (H_3), (H_5), (H_{14}) hold. Moreover,

$$(H_{15}) \qquad \int_{t_0}^{\infty} b(t)Q(t)G(R(t-\alpha))dt = \int_{t_0}^{\infty} e^{-\frac{3t}{2}} e^{\frac{5t-5\pi}{2}} e^{-(t-\frac{\pi}{2})}dt = \int_{t_0}^{\infty} dt = \infty$$

is satisfied, hence by Theorem 4.5.3, every solution of (4.28) either oscillates or tends to zero as $t \to \infty$.

Remark 4.5.5. From Theorem 4.5.3, it revels that $y(t)$ is bounded in the case $w(t) < 0$ for $t \geq t_1$, which further converges to zero as $t \to \infty$. However, this fact has not been observed in the other case. Hence we have proved the following theorem.

Theorem 4.5.6. Let $0 \leq p(t) \leq p_1 < \infty$. Suppose that (H_1) - (H_3), (H_5), (H_{14}) and (H_{15}) hold, then every unbounded solution of (4.1) oscillates.

Remark 4.5.7. Since $R(t) \to 0$ as $t \to \infty$, then (H_{16}) implies that

$$\int_{t_0}^{\infty} G(R(t-\alpha))q(t)dt = \infty. \tag{4.29}$$

Hence,

$$\int_{t_0}^{\infty} q(t)dt = \infty. \tag{4.30}$$

Theorem 4.5.8. Let $-1 < p_4 \leq p(t) \leq 0$. If (H_1), (H_8), (H_{14}) and (H_{16}) hold, then every solution of (4.1) either oscillates or tends to zero as $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (4.1) such that $y(t) > 0$ for $t \geq t_0 > 0$. Setting $z(t)$, $K(t)$ and $w(t)$ as in (4.7), (4.8) and (4.9) we obtain (4.10) for $t \geq t_0 + \rho$. Hence $w(t)$ is monotonic on $[t_1, \infty), t_1 \geq t_0 + \rho$. Let $w(t) > 0$ for $t \geq t_1$. Suppose that one of the cases (a), (b) or (d) of Lemma 4.4.1 holds for $t \geq t_1$. From Lemma 4.4.3, we have $y(t) \geq w(t) \geq k_1 R(t)$ for $t \geq t_2 > t_1$ and hence (4.10) yields

$$\int_{t_3}^{\infty} q(t)G(R(t-\alpha))dt < \infty, \qquad t_3 > t_2 + \alpha,$$

a contradiction to (4.29). Next we consider the case (c). Proceeding as in the proof of Theorem 4.5.3, we obtain (4.26). Further, $y(t) \geq w(t) \geq k_1 R(t)$ for $t \geq t_2$ by Lemma 4.4.3. Consequently, for $t \geq t_3 > t_2 + \alpha$,

$$-[((-r(t)w''(t))')^{1-\gamma}]' \geq (\gamma - 1)L^{\gamma}G(k_1)R^{\gamma}(t)q(t)G(R(t-\alpha)). \tag{4.31}$$

Integrating (4.31) from t_3 to ∞, we get

$$\int_{t_3}^{\infty} q(t)R^{\gamma}(t)G(R(t-\alpha))dt < \infty,$$

a contradiction to (H_{16}).

If $w(t) < 0$ for $t \geq t_1$, then $y(t)$ is bounded ultimately. Hence $z(t)$ is bounded and so also $w(t)$. In what follows, none of the cases (e) and (f) of Lemma 4.4.4 arises. In the case (b) or (c), $-\infty < \lim_{t\to\infty} w(t) \leq 0$. Using the fact that $\lim_{t\to\infty} K(t) = 0$, we have $\lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t)$. Hence

$$\begin{aligned} 0 \geq \lim_{t\to\infty} w(t) &= \lim_{t\to\infty} z(t) \\ &= \limsup_{t\to\infty} [y(t) + p(t)y(t-\tau)] \\ &\geq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty} (p_4 y(t-\tau)) \\ &= \limsup_{t\to\infty} y(t) + p_4 \limsup_{t\to\infty} y(t-\tau) \\ &= (1+p_4) \limsup_{t\to\infty} y(t) \end{aligned}$$

implies that $\limsup_{t\to\infty} y(t) = 0$, that is, $\lim_{t\to\infty} y(t) = 0$. Let the case (d) hold. Since $\lim_{t\to\infty}(r(t)w''(t))'$ exists, then (4.10) yields that

$$\int_{t_2}^{\infty} q(t)G(y(t-\alpha))dt < \infty. \tag{4.32}$$

If $\liminf_{t\to\infty} y(t) > 0$, then it follows from (4.32) that

$$\int_{t_2}^{\infty} q(t)dt < \infty$$

which contradicts (4.30). Hence $\liminf_{t\to\infty} y(t) = 0$. Using Lemma 4.2.4, we assert that $\lim_{t\to\infty} w(t) = 0 = \lim_{t\to\infty} z(t)$. Proceeding as above, we may show that $\limsup_{t\to\infty} y(t) = 0$ and hence $\lim_{t\to\infty} y(t) = 0$.

If $y(t) < 0$ for $t \geq t_0$, then one may proceed as above to obtain $\liminf_{t\to\infty} y(t) = 0$, that is $\lim_{t\to\infty} y(t) = 0$. This completes the proof of the theorem.

Theorem 4.5.9. Let $-\infty < p(t) \leq 0$. If $(H_1), (H_8), (H_{14})$ and (H_{16}) hold, then every unbounded solution of (4.1) is oscillatory.
The proof of the theorem follows from the proof of Theorem 4.5.8. Hence the details are omitted.

4.6 Oscillation Criteria for Non-Homogeneous Equation with $\int_0^{\infty} \frac{t}{r(t)} dt < \infty$.

This section is concerned for the oscillation and asymptotic behavior of solutions of (4.2) with suitable forcing functions. We restrict our forcing functions which are allowed to change the sign eventually.

Theorem 4.6.1. Let $0 \leq p(t) \leq p_1 < \infty$ holds. Assume that $(H_1), (H_2), (H_3), (H_{11}), (H_{14})$,

$$(H_{17}) \qquad \limsup_{t\to\infty} \int_{\alpha}^{t} b(s)Q(s)G(F(s-\alpha))ds = +\infty$$
$$\textit{and} \qquad \liminf_{t\to\infty} \int_{\alpha}^{t} b(s)Q(s)G(F(s-\alpha))ds = -\infty$$

hold, then every solution of (4.2) oscillates.

Proof. Note that condition (H_{17}) implies (H_{12}). Suppose on the contrary that $y(t)$ is a non-oscillatory solution of (4.2) such that $y(t) > 0$ for $t \geq t_0 > \rho$. Setting as in (4.7), (4.8), (4.9) and (4.19), equation (4.2) becomes (4.20) for $t \geq t_0 + \rho$. Consequently, $v(t)$ is monotonic on $[t_1, \infty), t_1 > t_0 + \rho$. Let $v(t) > 0$ for $t \geq t_1$. Then

$$z(t) > K(t) + F(t) > F(t). \tag{4.33}$$

In view of equation (4.2), it is easy to verify that

$$\begin{aligned}
0 &= (r(t)v''(t))'' + q(t)G(y(t-\alpha)) + G(p_1)(r(t-\tau)v''(t-\tau))'' \\
&+ G(p_1)q(t-\tau)G(y(t-\alpha-\tau)) \\
&\geq (r(t)v''(t))'' + G(p_1)(r(t-\tau)v''(t-\tau))'' + \lambda Q(t)G(z(t-\alpha))
\end{aligned}$$

due to (H_2) and (H_3). Using (4.33), the last inequality yields,

$$0 \geq (r(t)v''(t))'' + G(p_1)(r(t-\tau)v''(t-\tau))'' + \lambda Q(t)G(F(t-\alpha)), \tag{4.34}$$

for $t \geq t_2 > t_1$. Assume that one of the cases (a), (b) and (d) of Lemma 4.4.1 holds. Then integrating (4.34), we obtain

$$\limsup_{t\to\infty} \int_{t_2}^{t} Q(s)G(F(s-\alpha))ds < \infty,$$

a contradiction to (H_{12}). Consider the case (c) of Lemma 4.4.1. From Lemma 4.4.3 it follows that

$$k(-r(t)v''(t))'tR(t) \leq v(t) \leq k_2 t, \qquad t \geq t_3 > t_2.$$

Hence in view of (4.26), we have

$$\begin{aligned}
-[((-r(t)v''(t))')^{1-\gamma}]' &- G(p_1)[((-r(t-\tau)v''(t-\tau))')^{1-\gamma}]' \\
&\geq \lambda(\gamma-1)L^{\gamma}b(t)Q(t)G(z(t-\alpha)) \\
&\geq \lambda(\gamma-1)L^{\gamma}b(t)Q(t)G(F(t-\alpha)),
\end{aligned}$$

for $t \geq t_3$. Integrating the last inequality, we obtain

$$\limsup_{t\to\infty} \int_{t_3}^{t} b(s)Q(s)G(F(s-\alpha))ds < \infty,$$

which contradicts (H_{17}). Consequently, $v(t) < 0$ for $t \geq t_1$. Thus any one of the cases (b) - (f) of Lemma 4.4.4 holds. If $v(t) < 0$, then $z(t) - K(t) < F(t)$. Hence, $\liminf_{t\to\infty} z(t) < 0$, a contradiction to the fact that $z(t) > 0$.

If $y(t) < 0$ for $t \geq t_0$, we set $x(t) = -y(t)$ to obtain $x(t) > 0$ for $t \geq t_0$ and

$$(r(t)(x(t) + p(t)x(t-\tau))'')'' + q(t)G(x(t-\alpha)) - h(t)H(x(t-\beta)) = \widetilde{f}(t)$$

due to (H_3), where $\widetilde{f}(t) = -f(t)$. If we set $\widetilde{F}(t) = -F(t)$, then $\widetilde{F}(t)$ changes sign and $(r(t)\widetilde{F}''(t))'' = \widetilde{f}(t)$. Proceeding as above we obtain a contradiction. Hence the theorem is proved.

Example 4.6.2. Consider

$$\begin{aligned}(e^t(y(t) + e^{-4t+\pi}y(t-2\pi))'')'' + 8e^{t+2\pi}y(t-2\pi)\\ -50e^{-3t+\frac{3\pi}{2}}(1+e^{2t-5\pi}\cos^2 t)\frac{y(t-\frac{5\pi}{2})}{1+y^2(t-\frac{5\pi}{2})} = 6e^{2t}\cos t\end{aligned} \tag{4.35}$$

for $t \geq 9$. Indeed, if we choose $F(t) = (\frac{e^t}{25})(9\sin t - 12\cos t)$, then it is easy to verify that $(r(t)F'')'' = f(t) = 6e^{2t}\cos t$. Also

$$\limsup_{t\to\infty} \int_{2\pi}^{t} b(s)Q(s)F(s-2\pi)ds = \infty,$$

where $F(t) = \frac{3}{5}e^t \sin(t-\theta_1)$ and $\tan\theta_1 = \frac{4}{3}$.

Clearly, $(H_1) - (H_3)$, $(H_{11}), (H_{14})$ and (H_{17}) are satisfied. Hence by Theorem 4.6.1, every solution of (4.35) is oscillatory. In particular, $y(t) = e^t \sin t$ is such an oscillatory solution of (4.35).

Theorem 4.6.3. Let $-1 < p(t) \leq 0$ holds. Suppose that $(H_1), (H_8), (H_{11})$ and (H_{14}) hold, then every bounded solution of (4.2) is oscillatory.

Proof. For the sake of contradiction, let $y(t)$ be a nonoscillatory bounded solution of (4.2) such that $y(t) > 0$ for $t \geq t_0 > \rho$. The case $y(t) < 0$ can be similarly dealt with. Let us set $v(t)$ as in (4.19), so that we get (4.20). Consequently, $v(t)$ is monotonic on $[t_1, \infty)$. Let $v(t) > 0$ for $t \geq t_1$. Then one of the cases (a) - (d) of Lemma 4.4.1 holds. Indeed, $v(t) > 0$ implies $0 < v(t) + K(t) = z(t) - F(t)$. Hence,

$$z(t) > F(t).$$

Thus, using (H_{11}) we get a contradiction due to the boundedness of $z(t)$.

Next, we assume that $v(t) < 0$ for $t \geq t_1$. Then one of the cases (b) - (f) of Lemma 4.4.4 holds. Indeed, $z(t) - K(t) < F(t)$. Therefore, $z(t) < K(t) + F(t)$ implies that

$$\begin{aligned}
\liminf_{t\to\infty} z(t) &\leq \liminf_{t\to\infty}(K(t)+F(t)) \\
&\leq \limsup_{t\to\infty} K(t) + \liminf_{t\to\infty} F(t) \\
&= \lim_{t\to\infty} K(t) + \liminf_{t\to\infty} F(t) \\
&\to -\infty,
\end{aligned}$$

a contradiction to the fact that $z(t)$ is bounded. This completes the proof of the theorem.

Theorem 4.6.4. Let $-1 < p_4 \leq p(t) \leq 0$. Suppose that (H_1), (H_3), (H_{10}), (H_{14}) and

$$(H_{18}) \qquad \begin{aligned}
\limsup_{t\to\infty} \int_\alpha^t R^\gamma(s)q(s)G(F(s-\alpha))ds &= +\infty \\
\liminf_{t\to\infty} \int_\alpha^t R^\gamma(s)q(s)G(F(s-\alpha))ds &= -\infty
\end{aligned}$$

hold, then every unbounded solution of (4.2) is oscillatory.

Proof. Note that condition (H_{18}) implies (H_{13}). For the sake of contradiction, let $y(t)$ be a nonoscillatory unbounded solution of (4.2) such that $y(t) > 0$ for $t \geq t_0 > \rho$. The case $y(t) < 0$ can be similarly dealt with. Let's set $v(t)$ as in (4.19), so that we get (4.20). Consequently, $v(t)$ is monotonic on $[t_1, \infty)$. Let $v(t) > 0$ for $t \geq t_1$. Then one of the cases (a) - (d) of Lemma 4.4.1 holds. Suppose case (a), (b), (d) hold. Indeed, $v(t) > 0$ implies $y(t) > z(t) > K(t) + F(t) > F(t)$. Hence, from (4.20) we get

$$\limsup_{t\to\infty} \int_\alpha^t q(s)G(F(s-\alpha))ds < \infty.$$

Thus, we get a contradiction due to (H_{13}).
If case (c) holds, then from (4.26), we obtain

$$-[((-r(t)v''(t))')^{1-\gamma}]' \geq (\gamma-1)L^\gamma R^\gamma(t)q(t)G(y(t-\alpha))$$

for $t \geq t_4 > t_3$. Hence integrating we get a contradiction due to (H_{18}).

Next, we assume that $v(t) < 0$ for $t \geq t_1$. Then one of the cases (b) - (f) of Lemma 4.4.4 holds. Since $y(t)$ is unbounded, then there is an increasing sequence $\{\eta_n\}_{n=1}^\infty$ such that $\eta_n \to \infty$ and $y(\eta_n) \to \infty$ as $n \to \infty$ and $y(\eta_n) = max\{y(t) : t_1 \le t \le \eta_n\}$. We may choose n large enough such that $\eta_n - \tau > t_1$. Hence

$$z(\eta_n) \;\ge\; y(\eta_n) + p_4 y(\eta_n - \tau) \ge (1 + p_4)y(\eta_n).$$

Indeed, $v(t) < 0$ implies $z(t) - K(t) < F(t)$. Therefore, $z(t) < K(t) + F(t)$ implies that

$$\begin{aligned}\infty = \limsup_{t\to\infty} z(\eta_n) &\le \limsup_{t\to\infty}(K(\eta_n) + F(\eta_n))\\ &\le \limsup_{t\to\infty} K(\eta_n) + \limsup_{t\to\infty} F(\eta_n)\\ &= \lim_{t\to\infty} K(\eta_n) + \limsup_{t\to\infty} F(\eta_n)\\ &< \infty,\end{aligned}$$

a contradiction. This completes the proof of the theorem.

Theorem 4.6.5. Let $1 < b_1 \le p(t) \le b_2 < \frac{1}{2}{b_1}^2$ and (H_1) hold. Suppose that (H_{10}) holds with

$$-\frac{(b_1 - 1)}{16b_2} < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \frac{(b_1 - 1)}{8b_2}.$$

Furthermore, assume that G and H are Lipschitzian on the intervals of the form $[b, c]$, $0 < b < c < \infty$. If

$$\int_0^\infty \frac{s}{r(s)} \int_s^\infty tq(t)dtds < \infty$$

holds, then (4.2) admits a positive bounded solution.

Proof. It is possible to choose $T_0 > 0$ large enough such that

$$\int_{T_0}^\infty \frac{s}{r(s)} \int_s^\infty tq(t)dtds < \frac{b_1 - 1}{16b_2G(1)}$$

and

$$\int_{T_0}^\infty \frac{s}{r(s)} \int_s^\infty th(t)dtds < \frac{b_1 - 1}{4b_1H(1)}.$$

Let $X = BC([T_0, \infty), \mathbb{R})$. Then X is a Banach Space with respect to supremum norm defined by

$$||x|| = \sup_{t\ge T_0}\{|x(t)|\}.$$

Let

$$S = \{x \in X : \frac{b_1 - 1}{8b_1b_2} \le x(t) \le 1, t \ge T_0\}.$$

Hence S is a closed bounded convex subset of X. Define two maps Ω_1 and Ω_2 on S as follows;

$$\Omega_1 y(t) = \begin{cases} \Omega_1 y(T_1), T_0 \le t \le T_1 \\ \\ -\frac{y(t+\tau)}{p(t+\tau)} + \frac{2{b_1}^2+b_1-1}{4b_1p(t+\tau)}, t \ge T_1 \end{cases}$$

and

$$\Omega_2 y(t) = \begin{cases} \Omega_2 y(T_1), T_0 \le t \le T_1 \\ \\ \frac{F(t+\tau)}{p(t+\tau)} + \frac{K(t+\tau)}{p(t+\tau)} \\ -\frac{1}{p(t+\tau)} \int_{t+\tau}^{\infty} (\frac{s-(t+\tau)}{r(s)} \int_s^{\infty} (u-s)q(u)G(y(u-\alpha))du)ds, t \ge T_1, \end{cases}$$

where $K(t)$ is defined in (4.8). Indeed,

$$\begin{aligned} K(t) &= \int_t^{\infty} \frac{s-t}{r(s)} \int_s^{\infty} (u-s)h(u)H(y(u-\beta))duds \\ &\le H(1) \int_t^{\infty} \frac{s}{r(s)} \int_s^{\infty} uh(u)duds < \frac{b_1-1}{4b_1} \end{aligned}$$

implies that

$$\begin{aligned} \Omega_1 y(t) + \Omega_2 y(t) &\le \frac{2{b_1}^2+b_1-1}{4{b_1}^2} + \frac{b_1-1}{8b_1b_2} + \frac{b_1-1}{4{b_1}^2} \\ &= \frac{{b_1}^2+b_1-1}{2{b_1}^2} + \frac{b_1-1}{8b_1b_2} \\ &\le \frac{{b_1}^2+b_1-1}{2{b_1}^2} + \frac{b_1-1}{8{b_1}^2} = \frac{4{b_1}^2+5b_1-5}{8{b_1}^2} < 1 \end{aligned}$$

and

$$\begin{aligned} \Omega_1 y(t) + \Omega_2 y(t) &\ge -\frac{1}{b_1} + \frac{2{b_1}^2+b_1-1}{4b_1b_2} - \frac{b_1-1}{16b_1b_2} - \frac{b_1-1}{16b_1b_2} \\ &= -\frac{1}{b_1} + \frac{2{b_1}^2+b_1-1}{4b_1b_2} - \frac{b_1-1}{8b_1b_2} \\ &= -\frac{1}{b_1} + \frac{4{b_1}^2+b_1-1}{8b_1b_2} \ge \frac{(b_1-1)}{8b_1b_2}, \end{aligned}$$

that is, $\Omega_1 y + \Omega_2 y \in S$. It is easy to verify that Ω_1 is a contraction mapping.

To show Ω_2 is completely continuous on S, we need to show that Ω_2 is continuous and maps bounded set into relatively compact sets. Next, we show that Ω_2 is continuous. Let $\{y_j(t)\}$ be the sequence of continuous functions defined on S such that $||y_j - y|| \to 0$ for all $j \to \infty$. Because S is closed and bounded, $(y_j - y) \in S$ and

$$\begin{aligned}|\Omega_2 y_j(t) - \Omega_2 y(t)| \;\leq\;& \frac{1}{p(t+\tau)} \int_{t+\tau}^{\infty} \frac{s-t-\tau}{r(s)} \int_s^{\infty} (u-s)h(u)|H(y_j(u-\beta)) - \\ & H(y(u-\beta))|duds + \frac{1}{p(t+\tau)} \int_{t+\tau}^{\infty} \frac{s-t-\tau}{r(s)} \int_s^{\infty} (u-s) \\ & q(u)|G(y_j(u-\alpha)) - G(y(u-\alpha))|duds.\end{aligned}$$

Because G and H are Lipschitz functions, then it follows that $||\Omega_2 y_j - \Omega_2 y|| \to 0$ as $j \to \infty$. To show $\Omega_2 S$ is relatively compact, it is sufficient to show that the family of functions $\{\Omega_2 x : x \in S\}$ is uniformly bounded and equicontinuous on $[T_1, \infty)$. It is clear that Ω_2 is uniformly bounded. Suppose $y \in S$ and let there exist t_1, $t_2 > 0$ such that for $t_1 \geq t_2 \geq T_1$,

$$\begin{aligned}|(\Omega_2 y)(t_1) - (\Omega_2 y)(t_2)| \;\leq\;& \left|\frac{F(t_1+\tau)}{p(t_1+\tau)} - \frac{F(t_2+\tau)}{p(t_2+\tau)}\right| \\ & + \left|\frac{K(t_1+\tau)}{p(t_1+\tau)} - \frac{K(t_2+\tau)}{p(t_2+\tau)}\right| \\ & + \left|\frac{1}{p(t_2+\tau)} \int_{t_2+\tau}^{\infty} \frac{s-t_2-\tau}{r(s)} \int_s^{\infty} (u-s)q(u)G(y(u-\alpha))duds \right. \\ & \left. - \frac{1}{p(t_1+\tau)} \int_{t_1+\tau}^{\infty} \frac{s-t_1-\tau}{r(s)} \int_s^{\infty} (u-s)q(u)G(y(u-\beta))duds\right| \\ & \to 0\end{aligned}$$

as $t_2 \to t_1$. It implies that $\Omega_2 S$ is relatively compact. Hence verifying all the required conditions of Krasnoselskii's fixed point theorem [49, Lemma 3] it yields that $\Omega_1 + \Omega_2$ has a fixed point in S, that is ,

$$y(t) = \begin{cases} -\frac{y(t+\tau)}{p(t+\tau)} + \frac{2{b_1}^2+b_1-1}{4b_1 p(t+\tau)} + \frac{K(t+\tau)}{p(t+\tau)} \\ -\frac{1}{p(t+\tau)} \int_{t+\tau}^{\infty} (\frac{s-(t+\tau)}{r(s)} \int_s^{\infty} (u-s)q(u)G(y(u-\alpha))du)ds + \frac{F(t+\tau)}{p(t+\tau)}. \end{cases}$$

Clearly, $y(t)$ is a solution of (4.2) on $[\frac{b_1-1}{8b_1 b_2}, 1]$. This completes the proof of the theorem.

Remark 4.6.6. Theorems similar to Theorem 4.6.5 can be proved in the other ranges of $p(t)$.

4.7 Conclusion

In Section 4.2, the oscillatory and asymptotic behaviour of solutions of equations (4.1) has been studied for the ranges $0 \leq p(t) \leq p_1 < 1$ or $1 < p_2 \leq p(t) \leq p_3 < \infty$, $-1 < p_4 \leq p(t) \leq 0$ with the assumption $\int_0^\infty \frac{t}{r(t)}dt = \infty$. However, in Section 4.5, we can observe that if $\int_0^\infty \frac{1}{r(t)}dt < \infty$, then some different conditions has been used in each theorem to obtain similar kind of results as obtained in the Section 4.2. We may observe that in Theorem 4.2.16 for the range $-\infty < p_5 \leq p(t) \leq p_6 < -1$, we have obtained similar kind of results for bounded solutions.

In Theorem 4.3.1 when $\int_0^\infty \frac{t}{r(t)}dt = \infty$, we used condition (H_{12}) but in Theorem 4.6.1 when $\int_0^\infty \frac{t}{r(t)}dt < \infty$, we used the condition (H_{17}). Note that condition (H_{17}) implies (H_{12}) but converse need not be true. For example, if $b(t) = e^{-3t}, Q(t) = e^t, G(F(t-\alpha)) = e^t \sin t$, then (H_{12}) holds but (H_{17}) is not true. Similarly condition (H_{18}) implies (H_{13}).

Note that in this chapter linear, sub-linear and super-linear aspects of the function G has been used to study the behaviour of solutions of the homogeneous NDDE (4.1). We fail to answer the behaviour of solutions of equation (4.1) under the assumption $\int_0^\infty \frac{t}{r(t)}dt = \infty$ ($\int_0^\infty \frac{t}{r(t)}dt < \infty$) for $p(t) = \pm 1$. This needs further investigation by using some new technique.

Chapter 5

Oscillation Results for Fourth Order Nonlinear Neutral Delay Differential Equations with Quasi-derivatives

5.1 Quasi-derivative with $\int_0^\infty \frac{1}{r_n(t)}dt = \infty$.

In this chapter, we consider the fourth order nonlinear neutral equations with quasi-derivatives of the form

$$L_4(y(t) + p(t)y(t-\tau)) + q(t)G(y(t-\alpha)) - h(t)H(y(t-\beta)) = 0 \tag{5.1}$$

and

$$L_4(y(t) + p(t)y(t-\tau)) + q(t)G(y(t-\alpha)) - h(t)H(y(t-\beta)) = f(t) \tag{5.2}$$

for various ranges of $p(t)$, where $p \in C([0,\infty),\mathbb{R})$, $q \in C([0,\infty),(0,\infty))$, $h \in C([0,\infty)$ $,[0,\infty))$, $f \in C([0,\infty),\mathbb{R})$, $G \in C(\mathbb{R},\mathbb{R})$ and $H \in C(\mathbb{R},\mathbb{R})$, G is nondecreasing and H is bounded with $uG(u) > 0, vH(v) > 0$ for $u, v \neq 0$, $\tau > 0$ and $\alpha, \beta > 0$. In this chapter, an attempt is made to study (5.1) and (5.2) under the assumption

$$(H_0) \qquad \int_0^\infty \frac{1}{r_n(t)}dt = \infty, \quad n = 1, 2, 3,$$

where $r_n \in C([0,\infty),(0,\infty))$ for various ranges of $p(t)$. For equations (5.1) and (5.2) we define quasi-derivative as follows:
Let $z(t) = y(t)+p(t)y(t-\tau)$, $L_0z(t) = z(t)$, $L_1z(t) = r_1(t)\frac{d}{dt}L_0z(t)$, $L_2z(t) = r_2(t)\frac{d}{dt}L_1z(t)$, $L_3z(t) = r_3(t)\frac{d}{dt}L_2z(t)$, $L_4z(t) = \frac{d}{dt}L_3z(t)$.

By a solution of (5.1)/(5.2) we understand a function $y \in C([-\rho,\infty),\mathbb{R})$ such that $(y(t)+p(t)y(t-\tau))$ is continuously differentiable, L_1, L_2, L_3 are differentiable operator and (5.1)/(5.2) is satisfied for $t \geq 0$, where $\rho = max\{\tau,\alpha,\beta\}$ and $\sup\{|y(t)| : t \geq t_0\} > 0$ for every $t \geq t_0$. A solution of (5.1)/(5.2) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory.

5.2 Oscillation Criteria for Homogeneous Equation with $\int_0^\infty \frac{1}{r_n(t)}dt = \infty$.

In this section, sufficient conditions are obtained for oscillatory and asymptotic behaviour of all solutions of (5.1) under the assumption (H_0). We need the following lemmas for our use in the sequel.

Lemma 5.2.1. [87] Let (H_0) hold. Let u be a continuously differentiable function on $[0,\infty)$ such that L_1u, L_2u, L_3u are continuously differentiable functions and $L_4u \leq 0(\not\equiv 0)$ for large t. If $u(t) > 0$ ultimately, then one of the cases (a) or (b) holds for large t, and if $u(t) < 0$ ultimately, then one of the cases (b), (c), (d) or (e) holds for large t, where

(a) $L_1u(t) > 0$, $L_2u(t) > 0$ and $L_3u(t) > 0$,
(b) $L_1u(t) > 0$, $L_2u(t) < 0$ and $L_3u(t) > 0$,
(c) $L_1u(t) < 0$, $L_2u(t) < 0$ and $L_3u(t) > 0$,
(d) $L_1u(t) < 0$, $L_2u(t) < 0$ and $L_3u(t) < 0$,
(e) $L_1u(t) < 0$, $L_2u(t) > 0$ and $L_3u(t) > 0$,
(f) $L_1u(t) < 0$, $L_2u(t) > 0$ and $L_3u(t) < 0$,

(g) $L_1u(t) > 0$, $L_2u(t) > 0$ and $L_3u(t) < 0$,

(h) $L_1u(t) > 0$, $L_2u(t) < 0$ and $L_3u(t) < 0$.

Proof. The proof of lemma follows from [52] for $n = 4$. Hence the details are omitted. □

Lemma 5.2.2. [87] Let the conditions of Lemma 5.2.1 hold. Assume that $r_1'(t) \geq 0$ and $r_3'(t) \geq 0$. If $u(t) > 0$ ultimately, then for $t \geq T \geq 0$, $u(t) \geq R(t,T)L_3u(t)$, where,

$$R(t,T) = \int_T^t \frac{1}{(r_1(\theta)r_3(\theta))} \left(\int_T^\theta \frac{(s-T)}{r_2(s)} ds \right) d\theta.$$

Theorem 5.2.3. *Let $0 \leq p(t) \leq p_1 < 1$ or $1 < p_2 \leq p(t) \leq p_3 < \infty$, $\tau \leq \alpha$, $r_1'(t), r_3'(t) > 0$ and (H_0) hold. If*

(H_1) $\int_0^\infty \frac{1}{r_1(s_2)} \int_{s_2}^\infty \frac{1}{r_2(s_1)} \int_{s_1}^\infty \frac{1}{r_3(s)} \int_s^\infty h(\theta) d\theta ds ds_1 ds_2 < \infty$,

(H_2) there exists $\lambda > 0$ such that $G(u) + G(v) \geq \lambda G(u+v), u > 0, v > 0$,

(H_3) $G(u)G(v) = G(uv)$ and $H(-u) = -H(u)$ for $u, v \in \mathbb{R}$,

(H_4) G is sublinear and $\int_0^c \frac{du}{G(u)} < \infty$ for all $c > 0$,

(H_5) $\int_\tau^\infty Q(t)dt = \infty$, $Q(t) = min\{q(t), q(t-\tau)\}$ for $t \geq \tau$

also hold, then every solution of (5.1) *either oscillates or converges to zero as $t \to \infty$.*

Proof. Assume that (5.1) has a non-oscillatory solution on $[t_0, \infty)$, $t_0 \geq 0$ and let it be $y(t)$. Hence $y(t) > 0$ or < 0 for $t \geq t_0$. Suppose that $y(t) > 0$ for $t \geq t_0$. Setting

$$z(t) = y(t) + p(t)y(t-\tau), \tag{5.3}$$

$$K(t) = \int_t^\infty \frac{1}{r_1(s_2)} \int_{s_2}^\infty \frac{1}{r_2(s_1)} \int_{s_1}^\infty \frac{1}{r_3(s)} \int_s^\infty h(\theta)H(y(\theta-\beta)) d\theta ds ds_1 ds_2 \tag{5.4}$$

and

$$w(t) = z(t) - K(t) = y(t) + p(t)y(t-\tau) - K(t), \tag{5.5}$$

we obtain

$$L_4w(t) = -q(t)G(y(t-\alpha)) \leq 0 (\not\equiv 0), \tag{5.6}$$

for $t \geq t_0 + \rho$. Consequently, $w(t), L_1w(t), L_2w(t), L_3w(t)$ are monotonic functions on $[t_1, \infty), t_1 \geq t_0 + \rho$. In what follows, we have two cases, viz. $w(t) > 0$ or $w(t) < 0$ for

$t \geq t_1$. Suppose the former holds. By the Lemma 5.2.1, any one of the cases (a) or (b) holds. Using (H_2) and (H_3), equation (5.1) can viewed as

$$\begin{aligned} 0 &= L_4w(t) + q(t)G(y(t-\alpha)) + G(p_1)L_4w(t-\tau) + G(p_1)q(t-\tau)G(y(t-\tau-\alpha)) \\ &\geq L_4w(t) + G(p_1)L_4w(t-\tau) + \lambda Q(t)G(y(t-\alpha) + p_1y(t-\alpha-\tau)) \\ &\geq L_4w(t) + G(p_1)L_4w(t-\tau) + \lambda Q(t)G(z(t-\alpha)) \end{aligned}$$

for $t \geq t_2 > t_1$. From (5.4), it follows that $K(t) > 0$ and $K'(t) < 0$ and hence $w(t) > 0$ for $t \geq t_1$ implies that $w(t) < z(t)$ for $t \geq t_2$. Therefore, the last inequality yields that

$$L_4w(t) + G(p_1)L_4w(t-\tau) + \lambda Q(t)G(w(t-\alpha)) \leq 0,$$

for $t \geq t_2$, that is,

$$0 \geq L_4w(t) + G(p_1)L_4w(t-\tau) + \lambda Q(t)G(R(t-\alpha, T)L_3w(t-\alpha))$$

due to Lemma 5.2.2, for $t \geq T + \rho > t_2$. Hence,

$$0 \geq L_4w(t) + G(p_1)L_4w(t-\tau) + \lambda Q(t)G(R(t-\alpha, T))G(L_3w(t-\alpha)),$$

that is,

$$\begin{aligned} \lambda Q(t)G(R(t-\alpha, T)) &\leq -[G(L_3w(t-\alpha))]^{-1}L_4w(t) \\ &- G(p_1)[G(L_3w(t-\alpha))]^{-1}L_4w(t-\tau) \\ &\leq -[G(L_3w(t))]^{-1}L_4w(t) \\ &- G(p_1)[G(L_3w(t-\tau))]^{-1}L_4w(t-\tau). \end{aligned}$$

Because $\lim_{t\to\infty} L_3w(t) < \infty$, then using (H_4), the above inequality becomes

$$\int_{T+\rho}^{\infty} Q(t)G(R(t-\alpha, T))dt < \infty,$$

which contradicts (H_5), where we have used the fact that $R(t, T)$ is monotonic increasing function.

Next, we suppose that the later holds. Then

$$y(t) \leq z(t) = y(t) + p(t)y(t-\tau) < K(t),$$

that is, $y(t)$ is bounded since $K(t)$ is bounded and monotonic. By the Lemma 5.2.1, any one of the cases (b), (c), (d) or (e) hold.

Consider the case (b).

Since $\lim_{t\to\infty} K(t)$ exists and $\lim_{t\to\infty} w(t)$ exists, then $\lim_{t\to\infty} z(t)$ exists. Further, $\lim_{t\to\infty} L_3 w(t)$ exists implies that

$$\int_{t_1}^{\infty} Q(t)G(y(t-\alpha))dt < \infty$$

and hence it is easy to verify that $\liminf_{t\to\infty} y(t) = 0$ due to (H_5). Consequently, it follows from Lemma 4.2.4 that $\lim_{t\to\infty} z(t) = 0$. Since $z(t) \geq y(t)$, then $\lim_{t\to\infty} y(t) = 0$.

Consider the case (c) and (d).

These two cases are not possible due to the fact that $w(t) < 0$, $y(t)$ is bounded, $\lim_{t\to\infty} K(t)$ exists and hence $\lim_{t\to\infty} w(t)$ exists. On the otherhand, integrating successively $L_2 w(t) < 0$ from t_1 to t, we get $\lim_{t\to\infty} w(t) = -\infty$, a contradiction.

Consider the case (e).

$L_2 w(t)$ is nondecreasing on $[t_1, \infty)$. Hence for $t \geq t_1$, $L_2 w(t) \geq L_2 w(t_1)$, that is,

$$r_2(t)\frac{d}{dt}L_1 w(t) \geq L_2 w(t_1). \tag{5.7}$$

Integrating (5.7) from t_1 to t, we obtain

$$L_1 w(t) \geq L_1 w(t_1) + L_2 w(t_1) \int_{t_1}^{t} \frac{1}{r_2(s)} ds,$$

that is, $L_1 w(t) > 0$ for large t due to (H_0), a contradiction.

Finally, we suppose that $y(t) < 0$ for $t \geq t_0$. From (H_3), we note that $G(-u) = -G(u)$ and $H(-u) = -H(u)$, $u \in \mathbb{R}$. Hence putting, $x(t) = -y(t)$ for $t \geq t_0$, we obtain $x(t) > 0$ and

$$L_4(x(t) + p(t)x(t-\tau)) + q(t)G(x(t-\alpha)) - h(t)H(x(t-\beta)) = 0.$$

Proceeding as above, we can show that every solution of (5.1) oscillates or converges to zero as $t \to \infty$. This completes the proof of the theorem. □

Theorem 5.2.4. *If all the conditions of Theorem 5.2.3 are satisfied, then every unbounded solution of* (5.1) *oscillates.*

The proof of the theorem follows from the proof of the Theorem 5.2.3 and hence the details are omitted.

Theorem 5.2.5. *Let* $0 \leq p(t) \leq p_1 < 1$ *or* $1 < p_2 \leq p(t) \leq p_3 < \infty$, $\tau \leq \alpha$ *and* $(H_0) - (H_3)$, (H_5) *hold. Assume that*
(H_6) $\frac{G(x_1)}{x_1^\sigma} \geq \frac{G(x_2)}{x_2^\sigma}$ *for* $x_1 \geq x_2 > 0$ *and* $\sigma \geq 1$.
Then every solution of (5.1) *either oscillates or tends to zero as* $t \to \infty$.

Proof. Proceeding as in the proof of Theorem 5.2.3, we obtain

$$L_4 w(t) + G(p_1) L_4 w(t-\tau) + \lambda Q(t) G(z(t-\alpha)) \leq 0, \tag{5.8}$$

for $t \geq t_2$. Using the fact that $w(t)$ is nondecreasing, there exists $k > 0$ and $t_3 > t_2$ such that $w(t) > k$ for $t \geq t_3$. Hence use of (H_6) along with Lemma 5.2.2 we obtain for $t \geq T + \alpha > t_3 + \alpha$,

$$\begin{aligned} G(w(t-\alpha)) &= (G(w(t-\alpha))/w^\sigma(t-\alpha)) w^\sigma(t-\alpha) \\ &\geq (G(k)/k^\sigma) w^\sigma(t-\alpha) \\ &\geq (G(k)/k^\sigma) R^\sigma(t-\alpha, T)(L_3 w(t-\alpha))^\sigma. \end{aligned} \tag{5.9}$$

Thus using (5.9) in (5.8), we get

$$\begin{aligned} \lambda(G(k)/k^\sigma) R^\sigma(t-\alpha, T) Q(t)(L_3 w(t-\alpha))^\sigma &< \lambda Q(t) G(w(t-\alpha)) \\ &\leq \lambda Q(t) G(z(t-\alpha)) \\ &\leq -L_4 w(t) - G(p_1) L_4 w(t-\tau), \end{aligned}$$

that is,

$$\begin{aligned}\lambda(G(k)/k^{\sigma})R^{\sigma}(t-\alpha,T)Q(t) &< -(L_3w(t-\alpha))^{-\sigma}[L_4w(t)\\ &+ G(p_1)L_4w(t-\tau)]\\ &< -(L_3w(t))^{-\sigma}L_4w(t)\\ &- G(p_1)(L_3w(t-\tau))^{-\sigma}L_4w(t-\tau)\end{aligned}$$

for $t \geq T+\alpha$. Since $\lim_{t\to\infty} L_3w(t)$ exists and $R(t,T)$ is nondecreasing, then proceeding as in the proof of Theorem 5.2.3, we obtain

$$\int_{T+\alpha}^{\infty} R^{\sigma}(t-\alpha,T)Q(t)dt < \infty,$$

a contradiction due to (H_5). The rest of the proof follows from Theorem 5.2.3. Thus proof of the theorem is complete. □

Theorem 5.2.6. *Let $0 \leq p(t) \leq p_1 < \infty$ and $\tau \leq \alpha$ hold. If (H_0) - (H_3), (H_5) and (H_6) hold, then every unbounded solution of* (5.1) *oscillates.*

The proof of the theorem follows from the Theorem 5.2.5. Hence the details are omitted.

Theorem 5.2.7. *Let $0 \leq p(t) \leq p_1 < 1$ or $1 < p_2 \leq p(t) \leq p_3 < \infty$, $\tau \leq \alpha$. Suppose that (H_0), (H_1), (H_2), (H_5) and*
(H_7) $G(u)G(v) \geq G(uv)$ for $u > 0, v > 0$,
(H_8) $G(-u) = -G(u)$, $H(-u) = -H(u)$, $u \in \mathbb{R}$
hold. Then every solution of (5.1) *either oscillates or converges to zero as $t \to \infty$.*

Proof. Proceeding as in the proof of the Theorem 5.2.5 we have (5.8) for $t \geq t_2$. Since $w(t)$ is nondecreasing, then there exist $k > 0$ and $t_3 > t_2$ such that $w(t) > k$ for $t \geq t_3$, that is, $z(t) \geq w(t) > k$ for $t \geq t_3$. Consequently, (5.8) yields

$$\lambda G(k) \int_{t_3}^{\infty} Q(t)dt < \infty,$$

a contradiction to (H_5). The rest of the proof follows from Theorem 5.2.3. This completes the proof of the theorem. □

Remark 5.2.8. In Theorems 5.2.3 - 5.2.4, G is sublinear only, whereas in Theorems 5.2.5 and 5.2.6, G is superlinear. But in Theorem 5.2.7, G could be linear, sublinear or superlinear.

Theorem 5.2.9. *Let $0 \leq p(t) \leq p_1 < \infty$. If (H_0), (H_1), (H_2), (H_5), (H_7) and (H_8) hold, then every unbounded solution of* (5.1) *is oscillatory.*

Theorem 5.2.10. *Let $-1 < p_4 \leq p(t) \leq 0$. If (H_0), (H_1), (H_3), (H_4) and*
(H_9) $\int_0^\infty q(t)dt = \infty$
hold, then every solution of (5.1) *either oscillates or tends to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a nonoscillatory solution of (5.1). Because of (H_3), without loss of generality we may assume that $y(t) > 0$ for $t \geq t_0 > 0$. Setting as in (5.3), (5.4) and (5.5) we obtain (5.6) for $t \geq t_0 + \rho$. Hence $w(t)$ is monotonic on $[t_1, \infty)$, $t_1 \geq t_0 + \rho$. If $w(t) > 0$ for $t \geq t_1$, then any one of the cases (a) or (b) of Lemma 5.2.1 holds. Consequently, $w(t) \geq R(t,T)L_3w(t)$ for $t \geq t_2 > t_1$ due to Lemma 5.2.2. Moreover, $w(t) \leq y(t)$ implies that $y(t) \geq R(t,T)L_3w(t)$ for $t \geq t_2$ and hence (5.6) becomes

$$\int_{t_2+\alpha}^{\infty} q(t)G(R(t-\alpha,T))dt < \infty,$$

a contradiction to (H_9). Hence $w(t) < 0$ for $t \geq t_1$. Then any one of the cases (b), (c), (d) or (e) of Lemma 5.2.1 holds. We claim that $y(t)$ is bounded. If not, let there be an increasing sequence $\{\eta_n\}_{n=1}^{\infty}$ such that $\eta_n \to \infty$ and $y(\eta_n) \to \infty$ as $n \to \infty$ and $y(\eta_n) = max\{y(t) : t_1 \leq t \leq \eta_n\}$. We may choose n large enough such that $\eta_n - \tau > t_1$. Hence,

$$\begin{aligned} w(\eta_n) &\geq y(\eta_n) + p_4 y(\eta_n - \tau) - K(\eta_n) \\ &\geq (1 + p_4)y(\eta_n) - K(\eta_n). \end{aligned}$$

Since $K(\eta_n)$ is bounded and $(1 + p_4) > 0$, then $w(\eta_n) > 0$ for large n which is a contradiction. Thus our claim holds and the cases (c), (d) and (e) are easy to verify following to Theorem 5.2.3. Using same type of reasoning as in the case (b) of Theorem 5.2.3, we obtain $\liminf_{t\to\infty} y(t) = 0$ and hence $\lim_{t\to\infty} z(t) = 0$ due to Lemma 4.2.4.

Consequently,

$$\begin{aligned} 0 = \limsup_{t\to\infty} z(t) &\geq \limsup_{t\to\infty}(y(t) + p_4 y(t-\tau)) \\ &\geq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_4(y(t-\tau)) \\ &= \limsup_{t\to\infty} y(t) + p_4 \limsup_{t\to\infty} y(t-\tau) \\ &= (1+p_4)\limsup_{t\to\infty} y(t) \end{aligned}$$

implies that $\limsup_{t\to\infty} y(t) = 0$. Hence, $\lim_{t\to\infty} y(t) = 0$. This completes the proof of the theorem. □

Example 5.2.11. Consider the differential equation

$$\Bigg((1+t)\Bigg((1+t)\Bigg(y(t) - \frac{1}{2}y(t-1)\Bigg)'\Bigg)''\Bigg)' + 2y^{\frac{1}{3}}(t-2) - \frac{1}{(1+t)^3}\frac{y(t-4)}{1+y^2(t-4)} = 0 \tag{5.10}$$

for $t \geq 5$. Clearly $(H_0), (H_1), (H_3), (H_4)$ and (H_9) of Theorem 5.2.10 are satisfied. Hence every solution of Theorem (5.10) either oscillates or tends to zero as $t \to \infty$.

Theorem 5.2.12. *Let $-1 < p_4 \leq p(t) \leq 0$. If (H_0), (H_1), (H_3), (H_4) and (H_9) hold, then every unbounded solution of* (5.1) *is oscillatory.*

Theorem 5.2.13. *Let $-\infty < p(t) < -1$. Assume that (H_0), (H_1), (H_3), (H_4) and (H_9) hold. Then the following statements are hold:*

(i) if $-\infty < p_5 \leq p(t) \leq p_6 < -1$, then every bounded solution of (5.1) *either oscillates or tends to zero as $t \to \infty$.*

(ii) if $-\infty < p_5 \leq p(t) \leq -1$, then unbounded solutions of (5.1) *oscillate.*

The proof of the theorem follows from the proofs of the Theorems 5.2.10 and 5.2.12. Hence the details are omitted.

5.3 Oscillation Criteria for Non-Homogeneous Equation with $\int_0^\infty \frac{1}{r_n(t)}dt = \infty$.

This section is devoted to study the oscillatory and asymptotic behavior of solutions of forced equations (5.2) with suitable forcing function. Our attention is restricted to the forcing functions which are changing sign eventually. We have the following hypotheses regarding f(t):

(H_{10}) There exists F, a real-valued continuously differentiable function on $[0, \infty)$ such that $F(t)$ changes sign, $r_1 F'$, $r_2(r_1 F')'$, $r_3(r_2(r_1 F')')'$ are all real-valued continuously differentiable functions on $[0, \infty)$ consecutively and $(r_3(r_2(r_1 F')')')' = f$ and $-\infty < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \infty$.

(H_{11}) All conditions are same as (H_{10}), only difference is $\liminf_{t\to\infty} F(t) = -\infty$, $\limsup_{t\to\infty} F(t) = \infty$.

Theorem 5.3.1. *Let $0 \le p(t) \le p_1 < \infty$. Assume that (H_0), (H_1), (H_2), (H_7), (H_8) and (H_{11}) hold. If*

$$(H_{12}) \qquad \limsup_{t\to\infty} \int_\alpha^t Q(s)G(F(s-\alpha))ds = +\infty \quad \textit{and}$$
$$\liminf_{t\to\infty} \int_\alpha^t Q(s)G(F(s-\alpha))ds = -\infty$$

hold, then every solution of (5.2) is oscillatory.

Proof. Let $y(t)$ be a non oscillatory solution of (5.2) such that $y(t) > 0$ for $t \ge t_0 > 0$. Setting as in (5.3), (5.4) and (5.5), let

$$V(t) = w(t) - F(t) = z(t) - K(t) - F(t). \tag{5.11}$$

Hence for $t \ge t_0 + \rho$, (5.2) becomes

$$L_4 V(t) = -q(t)G(y(t-\alpha)) \le 0 (\not\equiv 0). \tag{5.12}$$

Thus, $V(t)$ is monotonic on $[t_1, \infty)$, $t_1 > t_0 + \rho$. Suppose $V(t) > 0$ for $t \ge t_1$ so that Case (a) or (b) of Lemma 5.2.1 holds. Then $z(t) > K(t) + F(t) > F(t)$ for $t \ge t_1$. Applying (H_2) and (H_7) yields

$$\begin{aligned} 0 &= L_4V(t) + q(t)G(y(t-\alpha)) + G(p_1)L_4V(t-\tau) \\ &+ G(p_1)q(t-\tau)G(y(t-\alpha-\tau)) \\ &\ge L_4V(t) + G(p_1)L_4V(t-\tau) + \lambda Q(t)G(y(t-\alpha) + p_1 y(t-\alpha-\tau)) \\ &\ge L_4V(t) + G(p_1)L_4V(t-\tau) + \lambda Q(t)G(z(t-\alpha)) \\ &\ge L_4V(t) + G(p_1)L_4V(t-\tau) + \lambda Q(t)G(F(t-\alpha)) \end{aligned} \tag{5.13}$$

for $t \geq t_2 > t_1$. Integrating the inequality (5.13) from $t_2 + \alpha$ to t and taking *limsup* as $t \to \infty$, we get

$$\limsup_{t\to\infty} \int_{t_2+\alpha}^{t} Q(s)G(F(s-\alpha))ds < \infty,$$

which is a contradiction to (H_{12}). Consequently, $V(t) < 0$ for $t \geq t_1$. Thus any one of the cases (b), (c), (d) or (e) of Lemma 5.2.1 holds. Hence, $z(t) < K(t) + F(t)$ and

$$\begin{aligned}
\liminf_{t\to\infty} z(t) &\leq \liminf_{t\to\infty}[K(t) + F(t)] \\
&\leq \limsup_{t\to\infty} K(t) + \liminf_{t\to\infty} F(t) \\
&= \lim_{t\to\infty} K(t) + \liminf_{t\to\infty} F(t) \to -\infty,
\end{aligned}$$

a contradiction to the fact that $z(t) > 0$.

If $y(t) < 0$ for $t \geq t_0$, we set $x(t) = -y(t)$ to obtain $x(t) > 0$ for $t \geq t_0$ and

$$L_4(x(t) + p(t)x(t-\tau)) + q(t)G(x(t-\alpha)) - h(t)H(x(t-\beta)) = \tilde{f}(t),$$

where $\tilde{f}(t) = -f(t)$. If $\tilde{F}(t) = -F(t)$, then $\tilde{F}(t)$ changes sign and $(r_3(r_2(r_1\tilde{F}')')')' = \tilde{f}(t)$. Proceeding as above we obtain a contradiction. This completes the proof of the theorem. □

Example 5.3.2. Consider the differential equation

$$\begin{aligned}
&L_4(y(t) + e^{-6t}y(t-\pi)) + 2e^{2\pi}y(t-2\pi) \\
&-676e^{\theta-6t}(1 + e^{2(t-\pi-\theta)}\sin^2(t-\theta-\pi))\frac{y(t-\theta-\pi)}{1+y^2(t-\theta-\pi)} = -2e^t \sin t \qquad (5.14)
\end{aligned}$$

for $t \geq 49$, where $\tan\theta = \frac{120}{119}$. Indeed, if we choose $F(t) = \frac{e^t \sin t}{2}$, then

$$Q(t)F(t-2\pi) = e^t \sin t.$$

Hence,

$$\begin{aligned}
\limsup_{t\to\infty} \int_{2\pi}^{t} Q(s)F(s-2\pi) &= \limsup_{t\to\infty} \left\{ \frac{e^t \sin(t-\frac{\pi}{4})}{\sqrt{2}} + \frac{e^{2\pi}}{2} \right\} \\
&\to \infty.
\end{aligned}$$

Thus (H_{12}) holds. Clearly $(H_0) - (H_2), (H_7), (H_8)$ and (H_{11}) of Theorem 5.3.1 are satisfied. Hence every solution of (5.14) either oscillates or tends to zero as $t \to \infty$. Thus, $y(t) = e^t \sin t$ is such a solution of (5.14).

Theorem 5.3.3. *Let $-1 < p(t) \leq 0$. If (H_0), (H_1), (H_8) and (H_{11}) hold, then every bounded solution of* (5.2) *oscillates.*

Proof. Proceeding as in the proof of the Theorem 5.3.1, we obtain $V(t) > 0$ or < 0 when $y(t) > 0$ for $t \geq t_1 > t_0 + \rho$. If $V(t) > 0$, then any one of the cases (a) or (b) of Lemma 5.2.1 holds for $t \geq t_1$. Indeed, $V(t) > 0$, that is, $0 < V(t) + K(t) = z(t) - F(t)$. Hence,

$$\begin{aligned} \limsup_{t\to\infty} z(t) &\geq \limsup_{t\to\infty} F(t) \\ &\to \infty, \end{aligned}$$

a contradiction to the fact that $z(t)$ is bounded. Ultimately, $V(t) < 0$ for $t \geq t_1$. Indeed, $z(t) - K(t) < F(t)$. Therefore, $z(t) < K(t) + F(t)$ implies that

$$\begin{aligned} \liminf_{t\to\infty} z(t) &\leq \liminf_{t\to\infty}(K(t) + F(t)) \\ &\leq \limsup_{t\to\infty} K(t) + \liminf_{t\to\infty} F(t) \\ &= \lim_{t\to\infty} K(t) + \liminf_{t\to\infty} F(t) \\ &\to -\infty, \end{aligned}$$

a contradiction to the fact that $z(t)$ is bounded. Hence the theorem is proved. □

Theorem 5.3.4. *Let $0 \leq p(t) \leq p_1 < \infty$. Assume that (H_0), (H_1), (H_2), (H_7), (H_8), (H_{10}) and (H_{12}) hold. Then every unbounded solution of* (5.2) *oscillates.*

Proof. Suppose on the contrary that $y(t)$ is an unbounded nonoscillatory solution of (5.2) such that $y(t) > 0$ for $t \geq t_0$. Setting as in (5.3)-(5.5) and (5.11) we obtain (5.12) for $t \geq t_0 + \rho$. Hence $V(t)$ is monotonic on $[t_1, \infty)$, $t_1 > t_0 + \rho$. Proceeding as in the proof of the Theorem 5.3.1, we obtain contradiction when $V(t) > 0$ for $t \geq t_1$. Hence $V(t) < 0$ for $t \geq t_1$. From Lemma 5.2.1, it follows that any one of the cases (b), (c), (d) or (e) holds. In case (b), $\lim_{t\to\infty} V(t)$ exists. Hence,

$$z(t) = V(t) + K(t) + F(t),$$

implies that

$$y(t) \leq V(t) + K(t) + F(t), \tag{5.15}$$

that is, $y(t)$ is bounded, which is absurd. For each of the cases (c), (d) and (e), $V(t)$ is a nonincreasing on $[t_1, \infty)$. Let $\lim_{t\to\infty} V(t) = C$, $C \in [-\infty, 0)$. If $C = -\infty$, then (5.15) yields

$$\begin{aligned} \liminf_{t\to\infty} y(t) &\leq \limsup_{t\to\infty} V(t) + \liminf_{t\to\infty}(K(t) + F(t)) \\ &\leq \limsup_{t\to\infty} V(t) + \limsup_{t\to\infty} K(t) + \liminf_{t\to\infty} F(t) \\ &= \lim_{t\to\infty} V(t) + \lim_{t\to\infty} K(t) + \liminf_{t\to\infty} F(t), \end{aligned}$$

that is, $\liminf_{t\to\infty} y(t) = -\infty$, which is absurd. The contradiction is obvious when $-\infty < C < 0$ as $0 < y(t) \leq z(t) = V(t)+K(t)+F(t)$ implies that $y(t)$ is bounded, since $V(t), K(t), F(t)$ all are bounded, a contradiction to the fact that $y(t)$ is unbounded.

The case $y(t) < 0$ for $t \geq t_0$ is similar. This completes the proof of the theorem. □

5.4 Quasi-derivative with $\int_0^\infty \frac{1}{r_n(t)} dt < \infty$.

In this section, we consider the nonlinear neutral equations with quasi-derivatives of the form (5.1) and (5.2) under the assumption

$$(H_{13}) \qquad \int_0^\infty \frac{1}{r_n(t)} dt < \infty, n = 1, 2, 3,$$

for various ranges of $p(t)$.

5.5 Oscillation Criteria for Homogeneous Equation with $\int_0^\infty \frac{1}{r_n(t)} dt < \infty$.

In this section, sufficient conditions are obtained for oscillatory and asymptotic behaviour of bounded solutions of (5.1) under the assumption (H_{13}).

Theorem 5.5.1. *Let $0 \leq p(t) \leq p_1 < 1$ or $1 < p_2 \leq p(t) \leq p_3 < \infty$ holds. If (H_1), (H_3), (H_9), (H_{13}) and*

(H_{14}) $\int_{t_3}^{\infty} \frac{1}{r_1(t)} \int_{t_3}^{t} \frac{1}{r_2(s)} \int_{t_3}^{s} \frac{1}{r_3(u)} \int_{t_3}^{u} q(\theta) d\theta du ds dt = \infty,$

(H_{15}) $\int_{t_3}^{\infty} \frac{1}{r_3(t)} \int_{t_3}^{t} q(s) ds dt = \infty,$

(H_{16}) $\int_{t_3}^{\infty} \frac{1}{r_2(t)} \int_{t_3}^{t} \frac{1}{r_3(s)} \int_{t_3}^{s} q(u) du ds dt = \infty$

also hold, then every bounded solution of (5.1) *either oscillates or converges to zero as* $t \to \infty$.

Proof. Suppose on the contrary that $y(t)$ is a non-oscillatory bounded solution of (5.1) such that $y(t) > 0$ for $t \geq t_0$. The case $y(t) < 0$ for $t \geq t_0$ is similar hence omitted. Using (5.3), (5.4) and (5.5), we obtain (5.6) for $t \geq t_0 + \rho$. Consequently, $w(t), L_1w(t), L_2w(t), L_3w(t)$ are monotonic functions on $[t_1, \infty), t_1 \geq t_0 + \rho$. Then any one of $(a) - (h)$ hold.

Suppose cases (a) or (b) holds. In both the cases $L_1w(t) > 0$. Hence $w'(t) > 0$ for $t \geq t_1$. Thus, either $w(t) > 0$ or < 0 for $t \geq t_2 > t_1$.

Case I. If $w(t) > 0$ for $t \geq t_2$, then $0 < \lim_{t\to\infty} w(t) \leq \infty$.

Subcase (i): If $0 < \lim_{t\to\infty} w(t) < \infty$, then $0 < \lim_{t\to\infty} z(t) < \infty$. Claim $\liminf_{t\to\infty} y(t) = 0$. If not, let it be $l_1 > 0$. For some $\epsilon > 0$, there exists $t_3 > t_2$ such that $y(t) > l_1 - \epsilon > 0$ for $t \geq t_3$. Hence (5.6) implies

$$L_4w(t) \leq -q(t)G(l_1 - \epsilon).$$

Since $\lim_{t\to\infty} L_3w(t) < \infty$, then we obtain $\int_{t_3}^{\infty} q(t)dt < \infty$, a contradiction to (H_9). Hence, Lemma 4.2.4 implies $\lim_{t\to\infty} z(t) = 0$. Hence $y(t) \leq z(t)$, so also, $\lim_{t\to\infty} y(t) = 0$.

Subcase (ii): If $\lim_{t\to\infty} w(t) = \infty$, then we obtain a contradiction to the fact that $y(t)$ and hence $w(t)$ is bounded.

Case II. If $w(t) < 0$ for $t \geq t_2$, then $0 \leq \lim_{t\to\infty} w(t) < \infty$.

Subcase (iii): If $0 < \lim_{t\to\infty} w(t) < \infty$, then $0 < \lim_{t\to\infty} z(t) < \infty$. We can show that

$\liminf_{t\to\infty} y(t) = 0$. Hence by Lemma 4.2.4, $\lim_{t\to\infty} z(t) = 0$. Thus, $\lim_{t\to\infty} y(t) = 0$.

Subcase (iv): If $\lim_{t\to\infty} w(t) = 0$, then $\lim_{t\to\infty} z(t) = 0$. Hence $\lim_{t\to\infty} y(t) = 0$.

If case (c) or (e) hold, then $L_1 w(t) < 0$, hence $w'(t) < 0$ for $t \geq t_1$. Thus, $z'(t) < K'(t) < 0$ for $t \geq t_1$, which implies $\lim_{t\to\infty} z(t) < \infty$. As before, we can show that $\liminf_{t\to\infty} y(t) = 0$. By Lemma 4.2.4 $\lim_{t\to\infty} z(t) = 0$. Thus, $\lim_{t\to\infty} y(t) = 0$.

Suppose case (d) holds. Since $L_1 w(t) < 0$, then $w'(t) < 0$ for $t \geq t_1$. Hence, $z'(t) < K'(t) < 0$ for $t \geq t_2 > t_1$. Hence, $\lim_{t\to\infty} z(t) < \infty$. Thus, $\lim_{t\to\infty} w(t) < \infty$. Claim $\liminf_{t\to\infty} y(t) = 0$. If it is not possible, let it be $l_2 > 0$. For some $\epsilon > 0$, there exists $t_2 > t_1$ such that $y(t) > l_2 - \epsilon > 0$ for $t \geq t_2$. Hence (5.6) implies

$$L_4 w(t) \leq -q(t) G(l_2 - \epsilon).$$

Integrating the last inequality from $t_3 (> t_2)$ to t, we obtain

$$0 > L_3 w(t_3) \geq L_3 w(t) + G(l_2 - \epsilon) \int_{t_3}^{t} q(s) ds.$$

Again integrating the last inequality from t_3 to t, we get

$$0 > L_2 w(t_3) \geq L_2 w(t) + G(l_2 - \epsilon) \int_{t_3}^{t} \frac{1}{r_3(s)} \int_{t_3}^{s} q(u) du ds.$$

Further integrating the preceeding inequality from t_3 to t, we obtain

$$0 > L_1 w(t_3) \geq L_1 w(t) + G(l_2 - \epsilon) \int_{t_3}^{t} \frac{1}{r_2(s)} \int_{t_3}^{s} \frac{1}{r_3(u)} \int_{t_3}^{u} q(v) dv du ds.$$

Since $\lim_{t\to\infty} w(t) < \infty$, integrating the last inequality from t_3 to t, we obtain

$$w(t_3) \geq w(t) + G(l_2 - \epsilon) \int_{t_3}^{t} \frac{1}{r_1(s)} \int_{t_3}^{s} \frac{1}{r_2(u)} \int_{t_3}^{u} \frac{1}{r_3(v)} \int_{t_3}^{v} q(\theta) d\theta dv du ds,$$

a contradiction to (H_{14}). Hence, by Lemma 4.2.4, $\lim_{t\to\infty} z(t) = 0$, thus $\lim_{t\to\infty} y(t) = 0$.

If case (f) or (g) hold, then $\lim_{t\to\infty} L_2 w(t) < \infty$. Since $y(t)$, $z(t)$ and $w(t)$ are all bounded. Since $w(t)$ is monotonic, then $\lim_{t\to\infty} w(t) < \infty$. Hence $\lim_{t\to\infty} z(t) < \infty$.

Claim $\liminf_{t\to\infty} y(t) = 0$. If not, let it be $l_3 > 0$. For some $\epsilon > 0$, there exists $t_2 > t_1$ such that $y(t) > l_3 - \epsilon > 0$ for $t \geq t_2$. Hence (5.6) implies

$$L_4 w(t) \leq -q(t)G(l_3 - \epsilon).$$

Integrating the last inequality from $t_3(> t_2)$ to t, we obtain

$$0 > L_3 w(t_3) \geq L_3 w(t) + G(l_3 - \epsilon)\int_{t_3}^{t} q(s)ds.$$

Again integrating the preceeding inequality from t_3 to t, we get

$$L_2 w(t_3) \geq L_2 w(t) + G(l_3 - \epsilon)\int_{t_3}^{t} \frac{1}{r_3(s)}\int_{t_3}^{s} q(u)duds.$$

Since $\lim_{t\to\infty} L_2 w(t) < \infty$, we get a contradiction due to (H_{15}). Thus by Lemma 4.2.4, $\lim_{t\to\infty} z(t) = 0$. So also $\lim_{t\to\infty} y(t) = 0$.

Suppose case (h) hold, then $\lim_{t\to\infty} L_1 w(t) < \infty$. Since $y(t)$, $z(t)$ and $w(t)$ are all bounded and $w(t)$ is monotonic, then $\lim_{t\to\infty} w(t) < \infty$. Hence $\lim_{t\to\infty} z(t) < \infty$. Claim $\liminf_{t\to\infty} y(t) = 0$. If not, let it be $l_4 > 0$. For some $\epsilon > 0$, there exists $t_2 > t_1$ such that $y(t) > l_4 - \epsilon > 0$ for $t \geq t_2$. Integrating the last inequality from $t_3(> t_2)$ to t, we obtain

$$0 > L_3 w(t_3) \geq L_3 w(t) + G(l_4 - \epsilon)\int_{t_3}^{t} q(s)ds.$$

Again integrating the last inequality from t_3 to t, we get

$$0 > L_2 w(t_3) \geq L_2 w(t) + G(l_4 - \epsilon)\int_{t_3}^{t} \frac{1}{r_3(s)}\int_{t_3}^{s} q(u)duds.$$

Further integrating the preceeding inequality from t_3 to t, we obtain

$$L_1 w(t_3) \geq L_1 w(t) + G(l_4 - \epsilon)\int_{t_3}^{t} \frac{1}{r_2(s)}\int_{t_3}^{s} \frac{1}{r_3(u)}\int_{t_3}^{u} q(v)dvduds.$$

Since $\lim_{t\to\infty} L_1 w(t) < \infty$, we get a contradiction due to (H_{16}). Thus by Lemma 4.2.4, $\lim_{t\to\infty} z(t) = 0$, so also $\lim_{t\to\infty} y(t) = 0$.

□

Example 5.5.2.

$$(e^{\frac{t}{3}}(e^{\frac{t}{3}}(e^{\frac{t}{3}}(y(t) + e^{-1}y(t-1))')')')' + \frac{e^{2t}}{e^{15}}y^5(t-3)$$

$$-e^{(-2-2t)}(1+e^{(4-2t)})\frac{y(t-2)}{1+y^2(t-2)}=0 \tag{5.16}$$

for $t \geq 4$. Clearly (H_1), $(H_3), (H_9), (H_{13}), (H_{14})$-$(H_{16})$ of Theorem 5.5.1 are satisfied. Hence every solution of (5.16) either oscillates or tends to zero as $t \to \infty$. Thus, $y(t) = e^{-t}$ is such a solution of (5.16).

Theorem 5.5.3. *Let $-\infty < p_4 \leq p(t) \leq p_5 < -1$ holds. If (H_1), (H_3), (H_9) and (H_{13})-(H_{16}) also hold, then every bounded solution of* (5.1) *either oscillates or converges to zero as $t \to \infty$.*

Proof. Suppose on the contrary that $y(t)$ is a bounded non-oscillatory solution of (5.1) such that $y(t) > 0$ for $t \geq t_0$. Using (5.3), (5.4) and (5.5) we obtain (5.6) for $t \geq t_0 + \rho$. Consequently, $w(t), L_1w(t), L_2w(t), L_3w(t)$ are monotonic functions on $[t_1, \infty), t_1 \geq t_0 + \rho$. Then any one of the cases $(a) - (h)$ hold.

Suppose cases (a) or (b) holds. In both the cases $L_1w(t) > 0$ and hence $w'(t) > 0$ for $t \geq t_1$. Thus, either $w(t) > 0$ or < 0 for $t \geq t_2 > t_1$.

Case I. If $w(t) > 0$ for $t \geq t_2$, then $0 < \lim_{t\to\infty} w(t) \leq \infty$.

Subcase (i): If $0 < \lim_{t\to\infty} w(t) < \infty$, then proceeding same as in *Subcase (i)* of *Case I* of Theorem 5.5.1, we obtain $\lim_{t\to\infty} z(t) = 0$. Hence,

$$\begin{aligned} 0 = \liminf_{t\to\infty} z(t) &\leq \liminf_{t\to\infty}(y(t) + p_5 y(t-\tau)) \\ &\leq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_5(y(t-\tau)) \\ &= \limsup_{t\to\infty} y(t) + p_5 \limsup_{t\to\infty} y(t-\tau) \\ &= (1+p_5)\limsup_{t\to\infty} y(t). \end{aligned} \tag{5.17}$$

Since $(1 + p_5) < 0$, then $\lim_{t\to\infty} y(t) = 0$.

Subcase (ii): If $\lim_{t\to\infty} w(t) = \infty$, then we obtain a contradiction to the fact that $y(t)$ and hence $w(t)$ is bounded.

Case II. If $w(t) < 0$ for $t \geq t_2$, then $0 \leq \lim_{t\to\infty} w(t) < \infty$.

Subcase (iii): If $0 < \lim_{t\to\infty} w(t) < \infty$, then $0 < \lim_{t\to\infty} z(t) < \infty$. We can show that $\liminf_{t\to\infty} y(t) = 0$. Hence by Lemma 4.2.4, $\lim_{t\to\infty} z(t) = 0$. Thus, by (5.17) $\lim_{t\to\infty} y(t) = 0$.

Subcase (iv): If $\lim_{t\to\infty} w(t) = 0$, then $\lim_{t\to\infty} z(t) = 0$. Hence by (5.17) $\lim_{t\to\infty} y(t) = 0$.

Suppose case (c) or (e) hold, then $L_1 w(t) < 0$, so also $w'(t) < 0$ for $t \geq t_1$. Thus $z'(t) < K'(t) < 0$ for $t \geq t_1$, which implies $z(t) > 0$ or < 0 for $t \geq t_2 > t_1$.

If $z(t) > 0$ for $t \geq t_2$, then $\lim_{t\to\infty} z(t) < \infty$. Proceeding same as in Case (c) and (e) of Theorem 5.5.1, we can show that $\liminf_{t\to\infty} y(t) = 0$. By Lemma 4.2.4 $\lim_{t\to\infty} z(t) = 0$. Thus by (5.17), $\lim_{t\to\infty} y(t) = 0$.

If $z(t) < 0$ for $t \geq t_2$, then $-\infty \leq \lim_{t\to\infty} z(t) < 0$. If $-\infty < \lim_{t\to\infty} z(t) < 0$, we can show that $\liminf_{t\to\infty} y(t) = 0$. By Lemma 4.2.4, $\lim_{t\to\infty} z(t) = 0$. Thus, $\lim_{t\to\infty} y(t) = 0$. If $\lim_{t\to\infty} z(t) = -\infty$, then we get a contradiction due to the boundedness of $y(t)$.

Suppose case (d) holds. Then $L_1 w(t) < 0$, so also $w'(t) < 0$ for $t \geq t_1$. Thus, $z'(t) < K'(t) < 0$ for $t \geq t_1$, which implies $z(t) > 0$ or < 0 for $t \geq t_2 > t_1$.

If $z(t) > 0$ for $t \geq t_2$, then $\lim_{t\to\infty} z(t) < \infty$. Proceeding same as in case (d) of Theorem 5.5.1 and using (5.17), we obtain $\lim_{t\to\infty} y(t) = 0$.

If $z(t) < 0$ for $t \geq t_2$, then $-\infty \leq \lim_{t\to\infty} z(t) < 0$. If $-\infty < \lim_{t\to\infty} z(t) < 0$, then using (H_{14}) and (5.17), we get $\lim_{t\to\infty} y(t) = 0$.

If $\lim_{t\to\infty} z(t) = -\infty$, then we get a contradiction due to the boundedness of $y(t)$.

Suppose case (f) or (g) holds. Now $y(t)$ bounded implies $z(t)$ and $w(t)$ are bounded. Since $w(t)$ is monotonic function, then $\lim_{t\to\infty} w(t) < \infty$. Hence, $\lim_{t\to\infty} z(t) < \infty$. Proceeding same as cases (f) and (g) of theorem 5.5.1 and using (H_{15}), (5.17), we get $\lim_{t\to\infty} y(t) = 0$.

Suppose case (h) holds. Now $y(t)$ is bounded implies $z(t)$ and $w(t)$ are bounded. Since $w(t)$ is monotonic function, then $\lim_{t\to\infty} w(t) < \infty$. Hence, $\lim_{t\to\infty} z(t) < \infty$. Proceeding same as in case (h) of Theorem 5.5.1 and using (H_{16}), (5.17), we get $\lim_{t\to\infty} y(t) = 0$. This completes the proof of the theorem. □

Theorem 5.5.4. *Let $-1 < p_6 \leq p(t) \leq 0$, (H_1), (H_3), (H_9) and (H_{13})-(H_{16}) hold, then every bounded solution of* (5.1) *either oscillates or converges to zero as $t \to \infty$.*

Proof. Suppose on the contrary that $y(t)$ is a bounded non-oscillatory solution of (5.1) such that $y(t) > 0$ for $t \geq t_0$. Using (5.3), (5.4) and (5.5) we obtain (5.6) for $t \geq t_0 + \rho$. Consequently, $w(t), L_1w(t), L_2w(t), L_3w(t)$ are monotonic functions on $[t_1, \infty), t_1 \geq t_0 + \rho$. Then any one of the cases $(a) - (h)$ hold.

Suppose case (a) or (b) holds. In both the cases $L_1w(t) > 0$, hence $w'(t) > 0$ for $t \geq t_1$. Thus, $w(t) > 0$ or < 0 for $t \geq t_2 > t_1$.

Case I. If $w(t) > 0$ for $t \geq t_2$, then $0 < \lim_{t\to\infty} w(t) \leq \infty$.

Subcase (i): If $0 < \lim_{t\to\infty} w(t) < \infty$, then proceeding same as in *Subcase (i)* of *Case I* of Theorem 5.5.1, we obtain $\lim_{t\to\infty} z(t) = 0$. Hence,

$$\begin{aligned} 0 = \limsup_{t\to\infty} z(t) &\geq \limsup_{t\to\infty}(y(t) + p_6 y(t-\tau)) \\ &\geq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_6(y(t-\tau)) \\ &= \limsup_{t\to\infty} y(t) + p_6 \limsup_{t\to\infty} y(t-\tau) \\ &= (1+p_6)\limsup_{t\to\infty} y(t). \end{aligned} \tag{5.18}$$

Since $(1 + p_6) > 0$, then $\lim_{t\to\infty} y(t) = 0$.

Subcase (ii): If $\lim_{t\to\infty} w(t) = \infty$, then we obtain a contradiction to the fact that $y(t)$ and hence $w(t)$ is bounded.

Case II. If $w(t) < 0$ for $t \geq t_2$, then $0 \leq \lim_{t\to\infty} w(t) < \infty$.

Subcase (iii): If $0 < \lim_{t\to\infty} w(t) < \infty$, then $0 < \lim_{t\to\infty} z(t) < \infty$. We can show that $\liminf_{t\to\infty} y(t) = 0$. Hence by Lemma 4.2.4 $\lim_{t\to\infty} z(t) = 0$. Thus, by (5.18) $\lim_{t\to\infty} y(t) = 0$.

Subcase (iv): If $\lim_{t\to\infty} w(t) = 0$, then $\lim_{t\to\infty} z(t) = 0$. Hence by (5.18) $\lim_{t\to\infty} y(t) = 0$.

Suppose case (c) or (e) hold, then $L_1 w(t) < 0$, so also $w'(t) < 0$ for $t \geq t_1$. Thus, $z'(t) < K'(t) < 0$ for $t \geq t_1$, which implies $z(t) > 0$ or < 0 for $t \geq t_2 > t_1$.

If $z(t) > 0$ for $t \geq t_2$, then $\lim_{t\to\infty} z(t) < \infty$. Proceeding same as in Case (c) and (e) of Theorem 5.5.1, we can show that $\liminf_{t\to\infty} y(t) = 0$. By Lemma 4.2.4, $\lim_{t\to\infty} z(t) = 0$, thus, $\lim_{t\to\infty} y(t) = 0$.

If $z(t) < 0$ for $t \geq t_2$, then $-\infty \leq \lim_{t\to\infty} z(t) < 0$. If $-\infty < \lim_{t\to\infty} z(t) < 0$, then we can show that $\liminf_{t\to\infty} y(t) = 0$. By Lemma 4.2.4, $\lim_{t\to\infty} z(t) = 0$. Thus, $\lim_{t\to\infty} y(t) = 0$. If $\lim_{t\to\infty} z(t) = -\infty$, then we get a contradiction due to the boundedness of $y(t)$.

Suppose case (d) holds. then $L_1 w(t) < 0$, so also $w'(t) < 0$ for $t \geq t_1$. Thus $z'(t) < K'(t) < 0$ for $t \geq t_1$, which implies $z(t) > 0$ or < 0 for $t \geq t_2 > t_1$.

If $z(t) > 0$ for $t \geq t_2$, then $\lim_{t\to\infty} z(t) < \infty$. Proceeding same as in case (d) in Theorem 5.5.1 and using (5.18), we obtain $\lim_{t\to\infty} y(t) = 0$.

If $z(t) < 0$ for $t \geq t_2$, then $-\infty \leq \lim_{t\to\infty} z(t) < 0$. If $-\infty < \lim_{t\to\infty} z(t) < 0$, then using (H_{14}) and (5.18), we get $\lim_{t\to\infty} y(t) = 0$.
If $\lim_{t\to\infty} z(t) = -\infty$, then we obtain a contradiction due to the boundedness of $y(t)$.

Suppose case (f) or (g) holds. Now $y(t)$ is bounded implies $z(t)$ and $w(t)$ are bounded. Since $w(t)$ is monotonic function, then $\lim_{t\to\infty} w(t) < \infty$. Hence $\lim_{t\to\infty} z(t) < \infty$. Proceeding same as case (f) and (g) of Theorem 5.5.1 and using (H_{15}), (5.18), we get $\lim_{t\to\infty} y(t) = 0$.

Suppose case (h) holds. Now $y(t)$ is bounded implies $z(t)$ and $w(t)$ are bounded. Since $w(t)$ is monotonic function, then $\lim_{t\to\infty} w(t) < \infty$. Hence $\lim_{t\to\infty} z(t) < \infty$. Proceeding same as case (h) of theorem 5.5.1 and using (H_{16}), (5.18), we get $\lim_{t\to\infty} y(t) = 0$. This completes the proof of the theorem. □

5.6 Oscillation Criteria for Non-Homogeneous Equation with $\int_0^\infty \frac{1}{r_n(t)} dt < \infty$.

This section is devoted to study the oscillatory and asymptotic behavior of solutions of forced equations (5.2) with suitable forcing function.

Theorem 5.6.1. *Let $0 \leq p(t) \leq p_1 < \infty$ holds. Assume that (H_1), (H_3), (H_{11}) and (H_{13}) hold, then bounded solution of (5.2) is oscillatory.*

Proof. Proceeding same as Theorem 5.3.1 we obtain (5.11) and (5.12). Thus $V(t)$ is monotonic on $[t_1, \infty)$, $t_1 > t_0 + \rho$. Suppose $V(t) > 0$ for $t \geq t_1$, then $0 < V(t) + K(t) = z(t) - F(t)$. Hence,

$$\limsup_{t\to\infty} z(t) \geq \limsup_{t\to\infty} F(t) \to \infty,$$

a contradiction to the fact that $y(t)$ and hence $z(t)$ is bounded.

If $V(t) < 0$ for $t \geq t_1$, then $z(t) < K(t) + F(t)$, so also $\liminf_{t\to\infty} z(t) \to -\infty$, a contradiction to the fact that $z(t) > 0$. This completes the proof of the theorem. □

Theorem 5.6.2. *Let $-\infty < p_4 \leq p(t) \leq p_5 \leq 0$ holds. Assume that (H_1), (H_3), (H_{11}) and (H_{13}) hold, then bounded solution of* (5.2) *is oscillatory.*

The proof of the Theorem 5.6.2 is similar to that of Theorem 5.6.1. Hence the details are omitted.

5.7 Conclusion

We may observe that if $r_2(t) = r(t)$ and $r_1(t) = r_3(t) = 1$, the equation (5.1) and (5.2) reduces to (4.1) and (4.2) respectively.

The results obtained for the equations (5.1) and (5.2) under the assumption $\int_0^\infty \frac{1}{r_n(t)}dt = \infty$; $n = 1, 2, 3$ are similar to the results obtained for the equations (4.1) and (4.2) under the assumption $\int_0^\infty \frac{t}{r(t)}dt = \infty$. Whereas the results obtained for the equations (5.1) and (5.2) under the assumption $\int_0^\infty \frac{1}{r_n(t)}dt < \infty$; $n = 1, 2, 3$ are not similar to the results obtained for the equations (4.1) and (4.2) under the assumption $\int_0^\infty \frac{t}{r(t)}dt < \infty$.

We may note that the study of the oscillatory and asymptotic behaviour of solutions (5.1) and (5.2) under the assumption $\int_0^\infty \frac{1}{r_n(t)}dt < \infty$ includes all the cases (a)-(h).

Moreover, (H_{14}) together with (H_{13}) implies (H_{16}). Again, (H_{16}) together with (H_{13}) implies (H_{15}). Further, (H_{15}) together with (H_{13}) implies (H_9). But converse need not be true.

Chapter 6

Oscillation Results for n^{th} Order Nonlinear Neutral Delay Differential Equations with $n \geqslant 4$, n is Even

6.1 Even Higher Order NDDE with $\int_0^\infty \frac{1}{r(t)}dt = \infty$.

In the Chapter 4, Section 4.1, we have studied fourth order neutral delay differential equations (4.1) and (4.2) under the assumption

$$\int_0^\infty \frac{t}{r(t)}dt = \infty$$

for various ranges of $p(t)$.

In this section, we are concerned with the oscillatory and asymptotic behavior of solutions of the higher order nonlinear neutral delay differential equations with positive and negative coefficients of the form

$$\begin{aligned}(r(t)(y(t) + p(t)y(t-\tau))'')^{(n-2)} + q(t)G(y(t-\alpha)) \\ -h(t)H(y(t-\beta)) = 0\end{aligned} \qquad (6.1)$$

and

$$(r(t)(y(t)+p(t)y(t-\tau))'')^{(n-2)}+q(t)G(y(t-\alpha)) \\ -h(t)H(y(t-\beta))=f(t) \tag{6.2}$$

under the assumption

$$\int_0^\infty \frac{1}{r(t)}dt=\infty \tag{H_0}$$

for various ranges of $p(t)$, where $r, q \in C([0,\infty),(0,\infty))$, $p \in C([0,\infty),\mathbb{R})$ and $h \in C([0,\infty),[0,\infty))$, $f \in C([0,\infty),\mathbb{R})$, G and $H \in C(\mathbb{R},\mathbb{R})$ with $uG(u)>0$, $vH(v)>0$ for u, $v \neq 0$, H is bounded, G is non-decreasing, $\tau>0$, $\alpha>0$, $\beta>0$, $n \geqslant 4$ and n is even. Clearly, equations (6.1) and (6.2) generalizes the results obtained by Tripathy, Panigrahi and Basu [88].

By a solution of (6.1)/(6.2) we understand a function $y \in C([-\rho,\infty),\mathbb{R})$ such that $(y(t)+p(t)y(t-\tau))$ is twice continuously differentiable, $(r(t)(y(t)+p(t)y(t-\tau))'')$ is $(n-2)$ times continuously differentiable, where $\rho = max\{\tau,\alpha,\beta\}$ and satisfies (6.1)/(6.2) on $[0,\infty)$. We consider only those solutions $y(t)$ of (6.1)/(6.2) which satisfies $\sup\{|y(t)|; t \geqslant t_0\}>0$ for every $t \geqslant t_0$. We assume that (6.1)/(6.2) has such a solution. A solution of (6.1)/(6.2) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory.

6.2 Oscillation Criteria for Homogeneous Equation with $\int_0^\infty \frac{1}{r(t)}dt=\infty$.

In this section, sufficient conditions are obtained for oscillatory and asymptotic behavior of solutions of (6.1).

Remark 6.2.1. Note that (H_0) implies

$$\int_0^\infty \frac{t}{r(t)}dt=\infty,$$

but converse need not be true. For example if $r(t) = 1 + t^2$, then (H_0) does not hold, whereas $\int_0^\infty \frac{t}{r(t)}dt = \infty$ holds true.

Theorem 6.2.2. *Let* $0 \leqslant p(t) \leqslant p_1 < 1$ *or* $1 < p_2 \leqslant p(t) \leqslant p_3 < \infty$*. Suppose*

(H_1) $\int_0^\infty \frac{t}{r(t)} \int_t^\infty u^{n-3}h(u)dudt < \infty$;

(H_2) *there exists* $\lambda > 0$ *such that* $G(u) + G(v) \geq \lambda G(u + v)$ *for* $u > 0, v > 0$;

(H_3) $G(u)G(v) = G(uv), H(-u) = -H(u),\ u, v \in \mathbb{R}$;

(H_4) $\int_\tau^\infty Q(t)dt = \infty$, *where* $Q(t) = \min\{q(t), q(t-\tau); t \geqslant \tau\}$

hold, then every bounded solution of (6.1) *either oscillates or converges to zero as* $t \to \infty$.

Proof. Note that (H_4) implies (H_5), where

(H_5) $\int_\tau^\infty q(t)dt = \infty$.

Let $y(t)$ be a non-oscillatory bounded solution of (6.1) on $[t_0, \infty)$, $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. Set

$$z(t) = y(t) + p(t)y(t - \tau), \tag{6.3}$$

and

$$k(t) = \frac{1}{(n-3)\,!} \int_t^\infty \frac{s-t}{r(s)} \int_s^\infty (u-s)^{n-3}h(u)H(y(u-\beta))duds. \tag{6.4}$$

Note that condition (H_1) and the fact that H is bounded function implies that $k(t)$ exists for all t. Now if we let

$$v(t) = z(t) - k(t), \tag{6.5}$$

then

$$w^{(n-2)}(t) = -q(t)G(y(t-\alpha)) \leqslant 0(\not\equiv 0), \tag{6.6}$$

where

$$w(t) = r(t)v''(t) \tag{6.7}$$

for $t \geqslant t_0 + \rho$, $w^{(n-2)}(t)$ represents the $(n-2)^{th}$ derivative of $'w'$ w.r.t $'t'$. Clearly, $w^{(n-3)}(t), w^{(n-4)}(t), .., w'(t), w(t)$ are monotonic functions and of constant sign for $t \in [t_1, \infty)$, $t_1 \geqslant t_0 + \rho$.

If $w(t) > 0$ for $t \geqslant t_1$, in view of Lemma 3.4.1, $w^{(n-3)}(t) > 0$ for $t \geqslant t_1$. Now $w(t) > 0$ implies $v''(t) > 0$ for $t \geqslant t_1$, which in turn implies $v'(t)$ is monotonic function. Thus, $v'(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$.

Case I. If $v'(t) > 0$ for $t \geqslant t_2$, then integrating $v''(t) > 0$ twice consecutively from t_2 to t, we obtain

$$v(t) \geqslant v(t_2) + v'(t_2)(t - t_2)$$

which implies $v(t) \to \infty$ as $t \to \infty$. Since $y(t)$ is bounded, so also $z(t)$ is bounded. Hence $v(t)$ is bounded, a contradiction.

Case II. If $v'(t) < 0$ for $t \geqslant t_2$, then $z'(t) < 0$ eventually. Hence, $\lim_{t\to\infty} z(t) < \infty$. Claim $\liminf_{t\to\infty} y(t) = 0$. If it is not true, let $\liminf_{t\to\infty} y(t) = l_1 > 0$, then for some $\epsilon > 0$, there exists $t_3 > t_2$ such that

$$y(t) > (l_1 - \epsilon) > 0 \tag{6.8}$$

for $t \geqslant t_3$. Using (6.8) in (6.6), we obtain

$$w^{(n-2)}(t) \leqslant -q(t)G(l_1 - \epsilon) \tag{6.9}$$

for $t \geqslant t_4 > t_3$. Hence, integrating (6.9) from t_4 to t, we obtain

$$\infty > w^{(n-3)}(t_4) > -w^{(n-3)}(t) + w^{(n-3)}(t_4) \geqslant G(l_1 - \epsilon) \int_{t_4}^{t} q(s)ds.$$

Since $\lim_{t\to\infty} w^{(n-3)}(t) < \infty$, then taking limit as $t \to \infty$ we obtain $\int_{t_4}^{\infty} q(t)dt < \infty$, a contradiction to (H_5). Hence by Lemma 4.2.4, we get $\lim_{t\to\infty} z(t) = 0$. Thus $\lim_{t\to\infty} y(t) = 0$, as $y(t) \leqslant z(t)$.

If $w(t) < 0$ for $t \geqslant t_1$, then $v''(t) < 0$ for $t \geqslant t_1$. Thus, $v'(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$.

Case III. Suppose $v'(t) > 0$ for $t \geqslant t_2$. Since $w^{(n-2)}(t) \leqslant 0$ eventually, then $w^{(n-3)}(t) > 0$ or < 0 eventually.

Subcase (i): Suppose $w^{(n-3)}(t) > 0$ eventually. Since $v'(t) > 0$, then either $v(t) > 0$ or < 0 for $t \geqslant t_3 > t_2$. If $v(t) > 0$ for $t \geqslant t_3$, then using (H_2) and (H_3) in (6.6), we obtain

$$\begin{aligned} 0 &= w^{(n-2)}(t) + q(t)G(y(t-\alpha)) + G(p_1)w^{(n-2)}(t-\tau) \\ &+ G(p_1)q(t-\tau)G(y(t-\tau-\alpha)) \\ &\geqslant w^{(n-2)}(t) + G(p_1)w^{(n-2)}(t-\tau) + \lambda Q(t)G(y(t-\alpha) + p_1 y(t-\alpha-\tau)) \\ &\geqslant w^{(n-2)}(t) + G(p_1)w^{(n-2)}(t-\tau) + \lambda Q(t)G(z(t-\alpha)) \\ &\geqslant w^{(n-2)}(t) + G(p_1)w^{(n-2)}(t-\tau) + \lambda Q(t)G(v(t-\alpha)) \end{aligned}$$

for $t \geqslant t_4 > t_3$. Now $v'(t) > 0$ and $v(t) > 0$ imply $v(t) > k_1$ for $t \geqslant t_5 > t_4$. Hence from last inequality, we obtain

$$0 \geqslant w^{(n-2)}(t) + G(p_1)w^{(n-2)}(t-\tau) + \lambda Q(t)G(k_1)$$

for $t \geqslant t_6 > t_5$. Integrating the preceeding inequality from t_6 to t, we obtain

$$\begin{aligned} \infty &> w^{(n-3)}(t_6) + G(p_1)w^{(n-3)}(t_6-\tau) > -w^{(n-3)}(t) + w^{(n-3)}(t_6) \\ &- G(p_1)w^{(n-3)}(t-\tau) + G(p_1)w^{(n-3)}(t_6-\tau) \geqslant \lambda G(k_1)\int_{t_6}^{t} Q(s)ds. \end{aligned}$$

Since $\lim_{t\to\infty} w^{(n-3)}(t) < \infty$, then taking $t \to \infty$ in the last inequality we obtain $\int_{t_6}^{\infty} Q(t)dt < \infty$, a contradiction to (H_4).

If $v(t) < 0$ for $t \geqslant t_3$, then $\lim_{t\to\infty} v(t)$ exists, which implies $\lim_{t\to\infty} z(t)$ exists. Claim $\liminf_{t\to\infty} y(t) = 0$. If not, $\liminf_{t\to\infty} y(t) = l_2 > 0$, then for some $\epsilon > 0$, there exists $t_4 > t_3$ such that

$$y(t) > (l_2 - \epsilon) > 0$$

for $t \geqslant t_4$. Using the last inequality in (6.6), we obtain

$$w^{(n-2)}(t) \leqslant -q(t)G(l_2 - \epsilon)$$

for $t \geqslant t_5 > t_4$. Hence integrating the last inequality from t_5 to t, we obtain

$$\infty > w^{(n-3)}(t_5) > -w^{(n-3)}(t) + w^{(n-3)}(t_5) \geqslant G(l_2 - \epsilon) \int_{t_5}^{t} q(s)ds.$$

Since $\lim_{t\to\infty} w^{(n-3)}(t) < \infty$, then taking limit as $t \to \infty$ we have $\int_{t_5}^{\infty} q(t)dt < \infty$, a contradiction to (H_5). Hence by Lemma 4.2.4, we obtain $\lim_{t\to\infty} z(t) = 0$. Thus $\lim_{t\to\infty} y(t) = 0$, as $y(t) \leqslant z(t)$.

Subcase (ii): If $w^{(n-3)}(t) < 0$ eventually, then from (6.6) we obtain $w^{(n-4)}(t) < 0, ..., w'(t) < 0$ for large t. Since $w'(t) < 0$ for $t > t_3 > t_2$, then $w(t) < w(t_3)$, that is,

$$r(t)v''(t) < r(t_3)v''(t_3).$$

Integrating the last inequality from $'t_3'$ to $'t'$, we obtain

$$v'(t) < v'(t_3) + r(t_3)v''(t_3) \int_{t_3}^{t} \frac{ds}{r(s)},$$

which implies $v'(t) \to -\infty$ as $t \to \infty$, a contradiction to the fact that $v'(t) > 0$ eventually.

Case IV. If $v'(t) < 0$ for $t \geqslant t_2$, then integrating $v''(t) < 0$ twice consecutively from t_2 to t, we obtain

$$v(t) \leqslant v(t_2) + v'(t_2)(t - t_2),$$

which implies $\lim_{t\to\infty} v(t) = -\infty$. Since $z(t)$ is bounded, then $v(t)$ is bounded, a contradiction.

Finally, we suppose that $y(t) < 0$ for $t \geqslant t_0$. From (H_3), we note that $G(-u) = -G(u)$ and $H(-u) = -H(u), u \in \mathbb{R}$. Hence putting $x(t) = -y(t)$ for $t \geqslant t_0$, we obtain $x(t) > 0$ and

$$(r(t)(x(t) + p(t)x(t - \tau))'')^{(n-2)} + q(t)G(x(t - \alpha)) - h(t)H(x(t - \beta)) = 0.$$

Proceeding as above, we can show that every bounded solution of (6.1) oscillates or converges to zero as $t \to \infty$. This completes the proof of the theorem.

□

Example 6.2.3. Consider the sixth order differential equation

$$\left(y(t)+\frac{2}{3}y(t-\pi)\right)^{(vi)}+\left(\frac{1}{3}+e^{-t}\right)y(t-2\pi)$$

$$-e^{-t}(1+\sin^2 t)\frac{y(t-2\pi)}{1+y^2(t-2\pi)}=0, \tag{6.10}$$

for $t \geqslant 7$. Clearly, $(H_1)-(H_4)$ are satisfied. Hence by Theorem 6.2.2 every bounded solution of (6.10) either oscillates or converges to zero as $t \to \infty$. In particular, $y(t) = \sin t$ is such a bounded oscillatory solution of (6.10).

Theorem 6.2.4. *Let $-\infty < p_4 \leqslant p(t) \leqslant p_5 < -1$. Suppose that (H_1), (H_3) and (H_5) hold, then every bounded solution of* (6.1) *either oscillates or converges to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a non-oscillatory bounded solution of (6.1) on $[t_0, \infty)$, $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. (The proof in case $y(t) < 0$ eventually is similar and will be omitted.) Setting $z(t), k(t)$, $v(t)$ as in (6.3), (6.4) and (6.5) respectively, we obtain (6.6) and (6.7) for $t \geqslant t_0+\rho$. Clearly, $w^{(n-3)}(t), w^{(n-4)}(t), .., w'(t), w(t)$ are monotonic functions and of constant sign for $t \in [t_1, \infty)$, $t_1 \geqslant t_0+\rho$.

If $w(t) > 0$ for $t \geqslant t_1$, in view of Lemma 3.4.1, $w^{(n-3)}(t) > 0$ for $t \geqslant t_1$. Now $w(t) > 0$ implies $v''(t) > 0$ for $t \geqslant t_1$, which in turn implies $v'(t)$ is monotonic function. Thus, $v'(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$.

Case I. If $v'(t) > 0$ for $t \geqslant t_2$, then proceeding same as in Case I of Theorem 6.2.2, we obtain $v(t) \to \infty$, a contradiction due to the fact that $y(t)$ is bounded.

Case II. If $v'(t) < 0$ for $t \geqslant t_2$, then $v'(t) < 0$ implies $z'(t) < 0$ eventually. Hence,

$z(t) > 0$ or < 0 eventually.

Subcase (i): If $z(t) > 0$ eventually, then $\lim_{t\to\infty} z(t) < \infty$. Proceeding same as in Case II of Theorem 6.2.2, we obtain $\lim_{t\to\infty} z(t) = 0$.
Thus,

$$\left.\begin{aligned} 0 &= \liminf_{t\to\infty} z(t) \\ &\leqslant \liminf_{t\to\infty}[y(t) + p_5 y(t-\tau)] \\ &\leqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}[p_5 y(t-\tau)] \\ &= (1+p_5)\limsup_{t\to\infty} y(t). \end{aligned}\right\} \tag{6.11}$$

Since $(1 + p_5) < 0$, which implies $\lim_{t\to\infty} y(t) = 0$.

Subcase (ii): If $z(t) < 0$ eventually, then $-\infty \leqslant \lim_{t\to\infty} z(t) < 0$ due to $z'(t) < 0$. Let $-\infty < \lim_{t\to\infty} z(t) < 0$. Claim $\liminf_{t\to\infty} y(t) = 0$. If not, let $\liminf_{t\to\infty} y(t) = l_3 > 0$, then for some $\epsilon > 0$, there exists $t_3 > t_2$ such that

$$y(t) > (l_3 - \epsilon) > 0$$

for $t \geqslant t_3$. Using the last inequality in (6.6), we obtain

$$w^{(n-2)}(t) \leqslant -q(t)G(l_3 - \epsilon)$$

for $t \geqslant t_4 > t_3$. Hence, integrating the preceeding inequality from t_4 to t, we obtain

$$\infty > w^{(n-3)}(t_4) > -w^{(n-3)}(t) + w^{(n-3)}(t_4) \geqslant G(l_3 - \epsilon) \int_{t_4}^{t} q(s)ds.$$

Since $\lim_{t\to\infty} w^{(n-3)}(t) < \infty$, then taking limit as $t \to \infty$ we obtain $\int_{t_4}^{\infty} q(t)dt < \infty$, a contradiction to (H_5). Hence by Lemma 4.2.4, we obtain $\lim_{t\to\infty} z(t) = 0$. Thus, from (6.11), we obtain $\lim_{t\to\infty} y(t) = 0$.
If $\lim_{t\to\infty} z(t) = -\infty$, then we get a contradiction due to the fact that $y(t)$ is bounded.

If $w(t) < 0$ for $t \geqslant t_1$, then $v''(t) < 0$ for $t \geqslant t_1$. Thus, $v'(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$.

Case III. Suppose $v'(t) > 0$ for $t \geqslant t_2$. Since $w^{(n-2)}(t) \leqslant 0$ eventually, then $w^{(n-3)}(t) > 0$

or < 0 eventually.

Subcase (iii): Suppose $w^{(n-3)}(t) > 0$ eventually. Since $v'(t) > 0$, then $v(t) > 0$ or < 0 for $t \geqslant t_3 > t_2$. If $v(t) > 0$ for $t \geqslant t_3$, then $z(t) > 0$ eventually. Hence from (6.6), we obtain

$$\begin{aligned} 0 &= w^{(n-2)}(t) + q(t)G(y(t-\alpha)) \\ &\geqslant w^{(n-2)}(t) + q(t)G(z(t-\alpha)) \\ &\geqslant w^{(n-2)}(t) + q(t)G(v(t-\alpha)) \end{aligned}$$

for $t \geqslant t_4 > t_3$. Now $v'(t) > 0$ and $v(t) > 0$ imply $v(t) > k_1$ for $t \geqslant t_5 > t_4$. Hence from the last inequality, we obtain

$$0 \geqslant w^{(n-2)}(t) + q(t)G(k_1)$$

for $t \geqslant t_6 > t_5$. Integrating the preceeding inequality from t_6 to t, we obtain

$$\infty > w^{(n-3)}(t_6) > -w^{(n-3)}(t) + w^{(n-3)}(t_6) \geqslant G(k_1) \int_{t_6}^{t} q(s)ds.$$

Since $\lim_{t\to\infty} w^{(n-3)}(t) < \infty$, then taking $t \to \infty$ in the last inequality we obtain $\int_{t_6}^{\infty} q(t)dt < \infty$, a contradiction to (H_5).

If $v(t) < 0$ for $t \geqslant t_3$, then $\lim_{t\to\infty} v(t)$ exists. Hence $\lim_{t\to\infty} z(t)$ exists. Proceeding same as in *Subcase (i)* of Case III of Theorem 6.2.2 for the $v(t) < 0$ part, we obtain $\lim_{t\to\infty} z(t) = 0$. Thus, from (6.11) we obtain $\lim_{t\to\infty} y(t) = 0$.

Subcase (iv): If $w^{(n-3)}(t) < 0$ eventually, then proceeding same as in *Subcase (ii)* of Case III of Theorem 6.2.2 we obtain a contradiction to the fact that $v'(t) > 0$.

Case IV. If $v'(t) < 0$ for $t \geqslant t_2$, then integrating $v''(t) < 0$ twice consecutively we obtain $\lim_{t\to\infty} v(t) = -\infty$, a contradiction due to the fact that $v(t)$ is bounded. This completes the proof of the theorem. □

Example 6.2.5. Consider the differential equation

$$\Big(y(t)+(e^{-t}-2)y(t-1)\Big)^{(iv)}+\frac{2e-1}{e^{-2}}y(t-2)$$

$$-16e^{-(t+3)+1}(1+e^{-2(t-3)})\frac{y(t-3)}{1+y^2(t-3)}=0, \qquad (6.12)$$

for $t \geqslant 4$. Clearly, $(H_1),(H_3)$ and (H_5) are satisfied. Hence by Theorem 6.2.4 every bounded solution of (6.12) either oscillates or converges to zero as $t \to \infty$. In particular, $y(t) = e^{-t}$ is such a solution of (6.12).

Theorem 6.2.6. *Let $-1 < p_6 \leqslant p(t) \leqslant 0$. Suppose that (H_1), (H_3) and (H_5) hold, then every bounded solution of* (6.1) *either oscillates or converges to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a non-oscillatory bounded solution of (6.1) on $[t_0,\infty)$, $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. (The proof in case $y(t) < 0$ eventually is similar and will be omitted.) Setting $z(t), k(t)$, $v(t)$ as in (6.3), (6.4) and (6.5) respectively, we obtain (6.6) and (6.7) for $t \geqslant t_0+\rho$. Clearly, $w^{(n-3)}(t), w^{(n-4)}(t), .., w'(t), w(t)$ are monotonic functions and of constant sign for $t \in [t_1,\infty)$, $t_1 \geqslant t_0+\rho$.

If $w(t) > 0$ for $t \geqslant t_1$, in view of Lemma 3.4.1, $w^{(n-3)}(t) > 0$ for $t \geqslant t_1$. Now $w(t) > 0$ implies $v''(t) > 0$ for $t \geqslant t_1$, which in turn implies $v'(t)$ is monotonic function. Thus, $v'(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$.

Case I. If $v'(t) > 0$ for $t \geqslant t_2$, then proceeding same as in Case I of Theorem 6.2.2, we obtain $v(t) \to \infty$, a contradiction due to the fact that $y(t)$ is bounded.

Case II. If $v'(t) < 0$ for $t \geqslant t_2$, then $z'(t) < 0$ eventually. Thus $z(t) > 0$ or < 0 eventually.

Subcase (i): If $z(t) > 0$ eventually, then $\lim_{t\to\infty} z(t) < \infty$. Proceeding same as in Case II of Theorem 6.2.2, we obtain $\lim_{t\to\infty} z(t) = 0$. Thus,

$$\left.\begin{aligned} 0 = \limsup_{t\to\infty} z(t) &\geqslant \limsup_{t\to\infty}(y(t) + p_6 y(t-\tau)) \\ &\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_6 y(t-\tau)) \\ &= \limsup_{t\to\infty} y(t) + p_6 \limsup_{t\to\infty} y(t-\tau) \\ &= (1+p_6)\limsup_{t\to\infty} y(t). \end{aligned}\right\} \tag{6.13}$$

Since $(1 + p_6) > 0$, then $\lim_{t\to\infty} y(t) = 0$.

Subcase (ii): If $z(t) < 0$ eventually, then $-\infty \leqslant \lim_{t\to\infty} z(t) < 0$. If $-\infty < \lim_{t\to\infty} z(t) < 0$, then proceeding same as in *Subcase (ii)* of Case II of Theorem 6.2.4, we obtain $\lim_{t\to\infty} z(t) = 0$. Hence by (6.13), we obtain $\lim_{t\to\infty} y(t) = 0$.

If $\lim_{t\to\infty} z(t) = -\infty$, then we get a contradiction due to the fact that $y(t)$ and hence $z(t)$ is bounded.

If $w(t) < 0$ for $t \geqslant t_1$, then $v''(t) < 0$ for $t \geqslant t_1$. Thus, $v'(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$.

Case III. Suppose $v'(t) > 0$ for $t \geqslant t_2$. Since $w^{(n-2)}(t) \leqslant 0$ eventually, then either $w^{(n-3)}(t) > 0$ or < 0 eventually.

Subcase (iii): Suppose $w^{(n-3)}(t) > 0$ eventually. Since $v'(t) > 0$, then $v(t) > 0$ or < 0 for $t \geqslant t_3 > t_2$.

If $v(t) > 0$ for $t \geqslant t_3$, then $z(t) > 0$ eventually. Proceeding same as in *Subcase (iii)* of Case III of Theorem 6.2.4 we obtain a contradiction due to (H_5).

If $v(t) < 0$ for $t \geqslant t_3$, then $\lim_{t\to\infty} v(t)$ exists. Hence, $\lim_{t\to\infty} z(t)$ exists. Proceeding same as in *Subcase (i)* of Case III of Theorem 6.2.2 for the $v(t) < 0$ part we obtain $\lim_{t\to\infty} z(t) = 0$. Thus from (6.13) we obtain $\lim_{t\to\infty} y(t) = 0$.

Subcase (iv): If $w^{(n-3)}(t) < 0$ eventually, then proceeding same as in *Subcase (ii)* of Case III of Theorem 6.2.2 we obtain a contradiction to the fact that $v'(t) > 0$.

Case IV. If $v'(t) < 0$ for $t \geqslant t_2$, then integrating $v''(t) < 0$ twice consecutively we obtain $\lim_{t\to\infty} v(t) = -\infty$, a contradiction due to the fact that $v(t)$ is bounded. This completes the proof of the theorem. □

Theorem 6.2.7. *Let* $0 \leqslant p(t) \leqslant p_1 < 1$ *or* $1 < p_2 \leqslant p(t) \leqslant p_3 < \infty$. *Suppose* $(H_1) - (H_4)$ *hold, then every solution of* (6.1) *either oscillates or converges to zero as* $t \to \infty$.

Proof. Suppose $y(t)$ be a nonoscillatory solution of (6.1) on $[t_0, \infty)$ $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. (The proof in case $y(t) < 0$ eventually is similar and will be omitted.) Proceeding same as in Theorem 6.2.2, we obtain $w^{(n-3)}(t), w^{(n-4)}(t), .., w'(t)$, $w(t)$ are monotonic functions and of constant sign for $t \in [t_1, \infty)$, $t_1 \geqslant t_0 + \rho$.

If $w(t) > 0$ for $t \geqslant t_1$, in view of Lemma 3.4.1, $w^{(n-3)}(t) > 0$ for $t \geqslant t_1$. Now $w(t) > 0$ implies $v''(t) > 0$ for $t \geqslant t_1$, which in turn implies $v'(t)$ is monotonic function. Thus, $v'(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$.

Case I. If $v'(t) > 0$ for $t \geqslant t_2$, then integrating $v''(t) > 0$ twice consecutively from t_2 to t, we obtain

$$v(t) \geqslant v(t_2) + v'(t_2)(t - t_2)$$

which implies $v(t) \to \infty$ as $t \to \infty$. Thus, $v(t) > 0$ for $t \geqslant t_3 > t_2$, then using (H_2) and (H_3) in (6.6), we obtain

$$\begin{aligned} 0 &= w^{(n-2)}(t) + q(t)G(y(t-\alpha)) + G(p_1)w^{(n-2)}(t-\tau) \\ &+ G(p_1)q(t-\tau)G(y(t-\tau-\alpha)) \\ &\geqslant w^{(n-2)}(t) + G(p_1)w^{(n-2)}(t-\tau) + \lambda Q(t)G(y(t-\alpha) + p_1 y(t-\alpha-\tau)) \\ &\geqslant w^{(n-2)}(t) + G(p_1)w^{(n-2)}(t-\tau) + \lambda Q(t)G(z(t-\alpha)) \\ &\geqslant w^{(n-2)}(t) + G(p_1)w^{(n-2)}(t-\tau) + \lambda Q(t)G(v(t-\alpha)) \end{aligned}$$

for $t \geqslant t_4 > t_3$. Now $v'(t) > 0$ and $v(t) > 0$ imply $v(t) > k_1$ for $t \geqslant t_5 > t_4$. Hence from last inequality, we obtain

$$0 \geqslant w^{(n-2)}(t) + G(p_1)w^{(n-2)}(t-\tau) + \lambda Q(t)G(k_1)$$

for $t \geqslant t_6 > t_5$. Integrating the preceeding inequality from t_6 to t, we obtain

$$\begin{aligned}\infty &> w^{(n-3)}(t_6) + G(p_1)w^{(n-3)}(t_6-\tau) > -w^{(n-3)}(t) + w^{(n-3)}(t_6)\\ &- G(p_1)w^{(n-3)}(t-\tau) + G(p_1)w^{(n-3)}(t_6-\tau) \geqslant \lambda G(k_1)\int_{t_6}^{t} Q(s)ds.\end{aligned}$$

Since $\lim_{t\to\infty} w^{(n-3)}(t) < \infty$, then taking $t \to \infty$ in the last inequality we obtain $\int_{t_6}^{\infty} Q(t)dt < \infty$, a contradiction to (H_4).

Case II. If $v'(t) < 0$ for $t \geqslant t_2$, then proceeding same as in Case II of Theorem 6.2.2, we obtain $\lim_{t\to\infty} y(t) = 0$.

If $w(t) < 0$ for $t \geqslant t_1$, then $v''(t) < 0$ for $t \geqslant t_1$. Thus, $v'(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$.

Case III. Suppose $v'(t) > 0$ for $t \geqslant t_2$. Since $w^{(n-2)}(t) \leqslant 0$ eventually, then $w^{(n-3)}(t) > 0$ or < 0 eventually.

Subcase (i): Suppose $w^{(n-3)}(t) > 0$ eventually. Since $v'(t) > 0$, then either $v(t) > 0$ or < 0 for $t \geqslant t_3 > t_2$. If $v(t) > 0$ for $t \geqslant t_3$, then proceeding same as in *Subcase (i)* of Case III of Theorem 6.2.2 for $v(t) > 0$ part, we obtain a contradiction due to (H_4).

If $v(t) < 0$ for $t \geqslant t_3$, then proceeding same as in *Subcase (i)* of Case III of Theorem 6.2.2 for $v(t) < 0$ part, we obtain $\lim_{t\to\infty} y(t) = 0$.

Subcase (ii): If $w^{(n-3)}(t) < 0$ eventually, then proceeding same as in *Subcase (ii)* of Case III of Theorem 6.2.2, we obtain a contradiction to the fact that $v'(t) > 0$ eventually.

Case IV. If $v'(t) < 0$ for $t \geqslant t_2$, then integrating $v''(t) < 0$ twice consecutively from t_2 to t, we obtain

$$v(t) \leqslant v(t_2) + v'(t_2)(t - t_2),$$

which implies $\lim_{t\to\infty} v(t) = -\infty$. Hence, $v(t) < 0$ for $t \geq t_3 > t_2$. Thus, $0 < z(t) < k(t)$, which implies $z(t)$ is bounded, so also $v(t)$, a contradiction. This completes the proof of the theorem.

□

Theorem 6.2.8. *Let* $-1 < p_6 \leqslant p(t) \leqslant 0$. *Suppose that* (H_1), (H_3) *and* (H_5) *hold, then every solution of* (6.1) *either oscillates or converges to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a non-oscillatory solution of (6.1) on $[t_0, \infty)$, $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. (The proof in case $y(t) < 0$ eventually is similar and will be omitted.) Setting $z(t), k(t)$, $v(t)$ as in (6.3), (6.4) and (6.5) respectively, we obtain (6.6) and (6.7) for $t \geqslant t_0 + \rho$. Clearly, $w^{(n-3)}(t), w^{(n-4)}(t), .., w'(t), w(t)$ are monotonic functions and of constant sign for $t \in [t_1, \infty)$, $t_1 \geqslant t_0 + \rho$.

If $w(t) > 0$ for $t \geqslant t_1$, in view of Lemma 3.4.1, $w^{(n-3)}(t) > 0$ for $t \geqslant t_1$. Now $w(t) > 0$ implies $v''(t) > 0$ for $t \geqslant t_1$, which in turn implies $v'(t)$ is monotonic function. Thus, $v'(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$.

Case I. If $v'(t) > 0$ for $t \geqslant t_2$, then proceeding same as in Case I of Theorem 6.2.2, we obtain $v(t) \to \infty$. Thus, $v(t) > 0$ for $t \geq t_3 > t_2$. Note that if $v(t) > 0$ for $t \geqslant t_3$, then $z(t) > 0$ eventually. Hence from (6.6), we obtain

$$\begin{aligned} 0 &= w^{(n-2)}(t) + q(t)G(y(t-\alpha)) \\ &\geqslant w^{(n-2)}(t) + q(t)G(z(t-\alpha)) \\ &\geqslant w^{(n-2)}(t) + q(t)G(v(t-\alpha)) \end{aligned}$$

for $t \geqslant t_4 > t_3$. Now $v'(t) > 0$ and $v(t) > 0$ imply $v(t) > k_1$ for $t \geqslant t_5 > t_4$. Hence from the last inequality, we obtain

$$0 \geqslant w^{(n-2)}(t) + q(t)G(k_1)$$

for $t \geqslant t_6 > t_5$. Integrating the preceeding inequality from t_6 to t, we obtain

$$\infty > w^{(n-3)}(t_6) > -w^{(n-3)}(t) + w^{(n-3)}(t_6) \geqslant G(k_1) \int_{t_6}^{t} q(s)ds.$$

Since $\lim_{t\to\infty} w^{(n-3)}(t) < \infty$, then taking $t \to \infty$ in the last inequality we obtain $\int_{t_6}^{\infty} q(t)dt < \infty$, a contradiction to (H_5).

Case II. If $v'(t) < 0$ for $t \geqslant t_2$, then $z'(t) < 0$ eventually. Thus $z(t) > 0$ or < 0 eventually.

Subcase (i): If $z(t) > 0$ eventually, then $\lim_{t\to\infty} z(t) < \infty$. Proceeding same as in *Subcase (i)* of Case II of Theorem 6.2.6, we obtain $\lim_{t\to\infty} y(t) = 0$.

Subcase (ii): If $z(t) < 0$ eventually, then $-\infty \leqslant \lim_{t\to\infty} z(t) < 0$. If $-\infty < \lim_{t\to\infty} z(t) < 0$, then proceeding same as in *Subcase (ii)* of Case II of Theorem 6.2.4, we obtain $\lim_{t\to\infty} z(t) = 0$. Hence by (6.13), we obtain $\lim_{t\to\infty} y(t) = 0$.

If $\lim_{t\to\infty} z(t) = -\infty$, then $z(t) < 0$ for $t \geq t_3 > t_2$. Thus, $y(t) < y(t-\tau)$ for $t \geq t_3$. Hence $y(t)$ and $z(t)$ are bounded, a contradiction.

If $w(t) < 0$ for $t \geqslant t_1$, then $v''(t) < 0$ for $t \geqslant t_1$. Thus, $v'(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$.

Case III. Suppose $v'(t) > 0$ for $t \geqslant t_2$. Since $w^{(n-2)}(t) \leqslant 0$ eventually, $w^{(n-3)}(t) > 0$ or < 0 eventually.

Subcase (iii): Suppose $w^{(n-3)}(t) > 0$ eventually. Since $v'(t) > 0$, then $v(t) > 0$ or < 0 for $t \geqslant t_3 > t_2$.

If $v(t) > 0$ for $t \geqslant t_3$, then $z(t) > 0$ eventually. Proceeding same as in *Subcase (iii)* of Case III of Theorem 6.2.4 we obtain a contradiction due to (H_5).

If $v(t) < 0$ for $t \geqslant t_3$, then $\lim_{t\to\infty} v(t)$ exists. Hence, $\lim_{t\to\infty} z(t)$ exists. Proceeding same as in *Subcase (i)* of Case III of Theorem 6.2.2 for the $v(t) < 0$ part we obtain $\lim_{t\to\infty} z(t) = 0$. Thus from (6.13) we obtain $\lim_{t\to\infty} y(t) = 0$.

Subcase (iv): If $w^{(n-3)}(t) < 0$ eventually, then proceeding same as in *Subcase (ii)* of Case III of Theorem 6.2.2 we obtain a contradiction to the fact that $v'(t) > 0$.

Case IV. If $v'(t) < 0$ for $t \geqslant t_2$, then integrating $v''(t) < 0$ twice consecutively we obtain $\lim_{t\to\infty} v(t) = -\infty$. Thus, $v(t) < 0$ for $t \geq t_3 > t_2$. We claim that $y(t)$ is bounded. If this is not the case, then there is an increasing sequence $\{\eta_n\}_{n=1}^{\infty}$ such that $\eta_n \to \infty$ and $y(\eta_n) \to \infty$ as $n \to \infty$ and $y(\eta_n) = max\{y(t) : t_3 \leq t \leq \eta_n\}$. We may choose n large enough such that $\eta_n - \tau > t_3$. Hence,

$$v(\eta_n) \geq y(\eta_n) + p_6 y(\eta_n - \tau) - k(\eta_n) \geq (1 + p_6)y(\eta_n) - k(\eta_n).$$

Since $k(\eta_n)$ is bounded and $(1 + p_6) > 0$, then $v(\eta_n) > 0$ for large n which is a contradiction. Hence $y(t)$ is bounded, which implies $v(t)$ is bounded, a contradiction. This completes the proof of the theorem. □

6.3 Oscillation Criteria for Non-Homogeneous Equation with $\int_0^\infty \frac{1}{r(t)} dt = \infty$.

This section is devoted to study the oscillatory and asymptotic behavior of solutions of forced equations (6.2) with suitable forcing functions. Our attention is restricted to the forcing functions which are eventually change sign. We have the following hypotheses regarding f(t):

(H_6) There exists F $\in C^2([0,\infty),\mathbb{R})$ such that $-\infty < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty}$F(t) $< \infty$, $rF'' \in C^{(n-2)}([0,\infty),\mathbb{R})$ and $(rF'')^{(n-2)} = f$.

(H_7) There exists F $\in C^2([0,\infty),\mathbb{R})$ such that $\liminf_{t\to\infty} F(t) = -\infty, \limsup_{t\to\infty} F(t) = \infty, rF'' \in C^{(n-2)}([0,\infty),\mathbb{R})$ and $(rF'')^{(n-2)} = f$.

Theorem 6.3.1. *Let* $0 \leqslant p(t) \leqslant p_1 < \infty$. *Suppose that* $(H_1) - (H_3)$, (H_7) *and*

(H_8) $\limsup_{t\to\infty} \int_\alpha^t Q(s)G(F(s-\alpha))ds = +\infty$, $\liminf_{t\to\infty} \int_\alpha^t Q(s)G(F(s-\alpha))ds = -\infty$

hold. Then every solution of (6.2) *oscillates.*

Proof. Let $y(t)$ be a non-oscillatory solution of (6.2) on $[t_0,\infty)$, $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. (The proof in case $y(t) < 0$ eventually is similar and will be omitted.) Setting $z(t), k(t)$, $v(t)$ as in (6.3), (6.4) and (6.5) respectively, from (6.2) we obtain

$$w_1^{(n-2)}(t) = -q(t)G(y(t-\alpha)) \leqslant 0 (\not\equiv 0), \tag{6.14}$$

where

$$w_1(t) = r(t)u''(t) \tag{6.15}$$

and

$$u(t) = v(t) - F(t) \tag{6.16}$$

for $t \geqslant t_0 + \rho$, where $\rho = max\{\tau,\alpha,\beta\}$. Clearly, $w_1^{(n-3)}(t), w_1^{(n-4)}(t), .., w_1'(t), w_1(t)$ are monotonic functions and of constant sign for $t \in [t_1,\infty)$, $t_1 \geqslant t_0 + \rho$.

If $w_1(t) > 0$ for $t \geqslant t_1$, in view of Lemma 3.4.1, $w_1^{(n-3)}(t) > 0$ for $t \geqslant t_1$. Now $w_1(t) > 0$ implies $u''(t) > 0$ for $t \geqslant t_1$, which in turn implies $u'(t)$ is monotonic function. Thus, $u'(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$.

Case I. If $u'(t) > 0$ for $t \geqslant t_2$, then integrating $u''(t) > 0$ twice consecutively from t_2 to t, we obtain

$$u(t) \geqslant u(t_2) + u'(t_2)(t - t_2),$$

which implies $u(t) \to \infty$ as $t \to \infty$. Thus, $u(t) > 0$ for large t. Hence, $z(t) > k(t) + F(t) > F(t)$ for $t \geqslant t_3 > t_2$. Using (H_2) and (H_3), we obtain from (6.14)

$$\begin{aligned} 0 &= w_1^{(n-2)}(t) + q(t)G(y(t-\alpha)) + G(p_1)w_1^{(n-2)}(t-\tau) \\ &+ G(p_1)q(t-\tau)G(y(t-\tau-\alpha)) \\ &\geqslant w_1^{(n-2)}(t) + G(p_1)w_1^{(n-2)}(t-\tau) + \lambda Q(t)G(y(t-\alpha) + p_1 y(t-\alpha-\tau)) \\ &\geqslant w_1^{(n-2)}(t) + G(p_1)w_1^{(n-2)}(t-\tau) + \lambda Q(t)G(z(t-\alpha)) \\ &\geqslant w_1^{(n-2)}(t) + G(p_1)w_1^{(n-2)}(t-\tau) + \lambda Q(t)G(F(t-\alpha)) \end{aligned}$$

for $t \geqslant t_4 > t_3$. Integrating the preceeding inequality from t_4 to t, we obtain

$$\begin{aligned} \infty &> w_1^{(n-3)}(t_4) + G(p_1)w_1^{(n-3)}(t_4-\tau) > -w_1^{(n-3)}(t) + w_1^{(n-3)}(t_4) \\ &- G(p_1)w_1^{(n-3)}(t-\tau) + G(p_1)w_1^{(n-3)}(t_4-\tau) \geqslant \lambda \int_{t_4}^{t} Q(s)G(F(s-\alpha))ds. \end{aligned}$$

Taking limsup as $t \to \infty$ in the last inequality we obtain $\limsup_{t\to\infty} \int_{t_4}^{t} Q(s)G(F(s-\alpha))ds < \infty$, a contradiction to (H_8).

Case II. If $u'(t) < 0$ for $t \geqslant t_2$, then $u(t) > 0$ or $u(t) < 0$ for $t \geqslant t_3 > t_2$.

Subcase (i): If $u(t) > 0$ for $t \geqslant t_3$, then proceeding same as in Case I we obtain a contradiction due to (H_8).

Subcase (ii): If $u(t) < 0$ for $t \geqslant t_3$, then

$$z(t) - k(t) < F(t).$$

Thus $\liminf_{t\to\infty} z(t) = -\infty$, a contradiction to the fact that $z(t) > 0$.

If $w_1(t) < 0$ for $t \geqslant t_1$, then $u''(t) < 0$ for $t \geqslant t_1$. Thus, $u'(t) > 0$ or $u'(t) < 0$ for $t \geqslant t_2 > t_1$.

Case III. Suppose $u'(t) > 0$ for $t \geqslant t_2$. Since $w_1^{(n-2)}(t) \leqslant 0$ eventually, then $w_1^{(n-3)}(t) > 0$ or < 0 eventually.

Subcase (iii): Suppose $w_1^{(n-3)}(t) > 0$ eventually. Since $u'(t) > 0$, then $u(t) > 0$ or $u(t) < 0$ for $t \geqslant t_3 > t_2$.

Suppose $u(t) > 0$ for $t \geqslant t_3$. Hence, $z(t) > k(t) + F(t) > F(t)$ for $t \geqslant t_3 > t_2$. Using (H_2) and (H_3), from (6.14) we obtain

$$\begin{aligned} 0 &= w_1^{(n-2)}(t) + q(t)G(y(t-\alpha)) + G(p_1)w_1^{(n-2)}(t-\tau) \\ &+ G(p_1)q(t-\tau)G(y(t-\tau-\alpha)) \\ &\geqslant w_1^{(n-2)}(t) + G(p_1)w_1^{(n-2)}(t-\tau) + \lambda Q(t)G(y(t-\alpha) + p_1 y(t-\alpha-\tau)) \\ &\geqslant w_1^{(n-2)}(t) + G(p_1)w_1^{(n-2)}(t-\tau) + \lambda Q(t)G(z(t-\alpha)) \\ &\geqslant w_1^{(n-2)}(t) + G(p_1)w_1^{(n-2)}(t-\tau) + \lambda Q(t)G(F(t-\alpha)) \end{aligned}$$

for $t \geqslant t_4 > t_3$. Integrating the preceeding inequality from t_4 to t, we obtain

$$\begin{aligned} \infty &> w_1^{(n-3)}(t_4) + G(p_1)w_1^{(n-3)}(t_4-\tau) > -w_1^{(n-3)}(t) + w_1^{(n-3)}(t_4) \\ &- G(p_1)w_1^{(n-3)}(t-\tau) + G(p_1)w_1^{(n-3)}(t_4-\tau) \geqslant \lambda \int_{t_4}^{t} Q(s)G(F(s-\alpha))ds. \end{aligned}$$

Taking $\limsup$ as $t \to \infty$ in the last inequality, we obtain $\limsup_{t\to\infty} \int_{t_4}^{t} Q(s)G(F(s-\alpha))ds < \infty$, a contradiction to (H_8).

If $u(t) < 0$ for $t \geqslant t_3$, then we get $\liminf_{t\to\infty} z(t) \to -\infty$, a contradiction to the fact that $z(t) > 0$.

Subcase (iv): If $w_1^{(n-3)}(t) < 0$ eventually, then from (6.14) we can conclude that $w_1^{(n-4)}(t) < 0, ..., w_1'(t) < 0$ for large t. Since $w_1'(t) < 0$ for $t > t_3 > t_2$, then

$w_1(t) < w_1(t_3)$, that is,

$$r(t)u''(t) < r(t_3)u''(t_3).$$

Integrating the last inequality from $'t_3'$ to $'t'$, we obtain

$$u'(t) < u'(t_3) + r(t_3)u''(t_3)\int_{t_3}^{t}\frac{ds}{r(s)},$$

which implies $u'(t) \to -\infty$ as $t \to \infty$, a contradiction to the fact that $u'(t) > 0$ eventually.

Case IV. If $u'(t) < 0$ for $t \geqslant t_2$, then integrating $u''(t) < 0$ from t_2 to t, we obtain

$$u(t) \leqslant u(t_2) + u'(t_2)(t - t_2),$$

therefore, $\lim_{t\to\infty} u(t) = -\infty$. Thus, $u(t) < 0$ for $t \geqslant t_3 > t_2$. Hence, $\liminf_{t\to\infty} z(t) = -\infty$, a contradiction due to the fact that $z(t) > 0$. Hence proof of the theorem is complete. □

Example 6.3.2. Consider

$$(y(t) + e^{-5t}y(t-2\pi))^{(iv)} + 2e^{2\pi}y(t-2\pi)$$

$$-289e^{-5t+\theta-2\pi}(1 + e^{2t-2\theta}\sin^2(t-\theta))\frac{y(t-\theta)}{1+y^2(t-\theta)} = -2e^t\sin t, \tag{6.17}$$

for $t \geqslant 57$, where $\tan\theta = \frac{240}{161}$. Indeed, if we choose $F(t) = \frac{e^t}{2}\sin t$, then $(r(t)F''(t))'' = f(t)$. Clearly, $(H_1) - (H_3), (H_7)$ and (H_8) are satisfied. Hence, by Theorem 6.3.1 every solution of (6.17) oscillates. Indeed, $y(t) = e^t \sin t$ is such a solution of (6.17).

Theorem 6.3.3. *Let* $0 \leqslant p(t) \leqslant p_1 < \infty$. *Suppose that* $(H_1) - (H_3)$, (H_6) *and* (H_8) *hold. Then every unbounded solution of* (6.2) *oscillates.*

Proof. Let $y(t)$ be an unbounded non-oscillatory solution of (6.2) on $[t_0, \infty)$, $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. (The proof in case $y(t) < 0$ eventually is similar and will be omitted.) Setting $z(t), k(t)$, $v(t)$ as in (6.3), (6.4) and (6.5) respectively, we obtain (6.14), (6.15) and (6.16) for $t \geqslant t_0 + \rho$, where $\rho = max\{\tau, \alpha, \beta\}$.

Clearly, $w_1^{(n-3)}(t), w_1^{(n-4)}(t), .., w_1'(t), w_1(t)$ are monotonic functions and of constant sign for $t \in [t_1, \infty)$, $t_1 \geqslant t_0 + \rho$.

If $w_1(t) > 0$ for $t \geqslant t_1$, in view of Lemma 3.4.1, $w_1^{(n-3)}(t) > 0$ for $t \geqslant t_1$. Now $w_1(t) > 0$ implies $u''(t) > 0$ for $t \geqslant t_1$, which in turn implies $u'(t)$ is monotonic function. Thus, $u'(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$.

Case I. If $u'(t) > 0$ for $t \geqslant t_2$, then proceeding same as in Case I of Theorem 6.3.1, we obtain a contradiction due to (H_8).

Case II. If $u'(t) < 0$ for $t \geqslant t_2$, then $u(t) > 0$ or $u(t) < 0$ for $t \geqslant t_3 > t_2$.

Subcase (i): If $u(t) > 0$ for $t \geqslant t_3$, then proceeding same as in Case I of Theorem 6.3.1, we obtain a contradiction due to (H_8).

Subcase (ii): If $u(t) < 0$ for $t \geqslant t_3$, then $y(t) \leqslant z(t) < k(t) + F(t)$. Hence, $y(t)$ is bounded, a contradiction to our assumption.

Case III. Suppose $u'(t) > 0$ for $t \geqslant t_2$. Since $w_1^{(n-2)}(t) \leqslant 0$ eventually, then $w_1^{(n-3)}(t) > 0$ or < 0 eventually.

Subcase (iii): Suppose $w_1^{(n-3)}(t) > 0$ eventually. Since $u'(t) > 0$, then either $u(t) > 0$ or $u(t) < 0$ for $t \geqslant t_3 > t_2$.

If $u(t) > 0$ for $t \geqslant t_3$, then proceeding same as in *Subcase (iii)* of Case III of Theorem 6.3.1, we obtain a contradiction due to (H_8).

If $u(t) < 0$ for $t \geqslant t_3$, then $y(t)$ is bounded, a contradiction to our assumption.

Subcase (iv): If $w_1^{(n-3)}(t) < 0$ eventually, then proceeding same as in *Subcase (iv)* of Case III of Theorem 6.3.1, we obtain a contradiction due to $u'(t) > 0$.

Case IV. If $u'(t) < 0$ for $t \geqslant t_2$, then we get $u(t) \to -\infty$ as $t \to \infty$. Thus, $u(t) < 0$ for $t \geqslant t_4 > t_3$, a contradiction due to the unboundedness of $y(t)$. This completes the proof of the theorem. $\square$

Theorem 6.3.4. *Let* $-\infty < p_4 \leqslant p(t) \leqslant p_5 \leqslant 0$. *Suppose that* $(H_1), (H_3)$ *and* (H_7) *hold, then every bounded solution of* (6.2) *oscillates.*

Proof. Let $y(t)$ be a non-oscillatory bounded solution of (6.2) on $[t_0, \infty)$, $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. (The proof in case $y(t) < 0$ eventually is similar and will be omitted.) Setting $z(t), k(t)$, $v(t)$ as in (6.3), (6.4) and (6.5) respectively, we obtain (6.14), (6.15) and (6.16) for $t \geqslant t_0 + \rho$, where $\rho = max\{\tau, \alpha, \beta\}$. Clearly, $w_1^{(n-3)}(t), w_1^{(n-4)}(t), .., w_1'(t), w_1(t)$ are monotonic functions and of constant sign for $t \in [t_1, \infty)$, $t_1 \geqslant t_0 + \rho$.

If $w_1(t) > 0$ for $t \geqslant t_1$, in view of Lemma 3.4.1, $w_1^{(n-3)}(t) > 0$ for $t \geqslant t_1$. Now $w_1(t) > 0$ implies $u''(t) > 0$ for $t \geqslant t_1$, which in turn implies $u'(t)$ is eventually monotonic function. Thus, $u'(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$.

Case I. If $u'(t) > 0$ for $t \geqslant t_2$, then integrating $u''(t) > 0$ twice consecutively from t_2 to t, we obtain

$$u(t) \geqslant u(t_2) + u'(t_2)(t - t_2),$$

which implies $u(t) \to \infty$ as $t \to \infty$. Hence $u(t) > 0$ for $t \geqslant t_3 > t_2$. Thus, $z(t) > F(t)$. Using (H_7), we get a contradiction due to the boundedness of $z(t)$.

Case II. If $u'(t) < 0$ for $t \geqslant t_2$, then we have two cases: $u(t) > 0$ or $u(t) < 0$ for $t \geqslant t_3 > t_2$.

Subcase (i): If $u(t) > 0$ for $t \geqslant t_3$, then we obtain a contradiction to the boundedness of $z(t)$.

Subcase (ii): If $u(t) < 0$ for $t \geqslant t_3$, then we get $\liminf_{t\to\infty} z(t) = -\infty$, a contradiction to the boundedness of $z(t)$.

If $w_1(t) < 0$ for $t \geqslant t_1$, then $u''(t) < 0$ for $t \geqslant t_1$. Thus, $u'(t) > 0$ or $u'(t) < 0$ for $t \geqslant t_2 > t_1$.

Case III. If $u'(t) > 0$ for $t \geqslant t_2$, then we have two cases: $u(t) > 0$ or $u(t) < 0$ for $t \geqslant t_3 > t_2$.

Subcase (iii): If $u(t) > 0$ for $t \geqslant t_3$, then $z(t) > F(t)$. Thus by using (H_7) we obtain a contradiction due to the boundedness of $z(t)$.

Subcase (iv): If $u(t) < 0$ for $t \geqslant t_3$, then $z(t) < k(t)+F(t)$. Hence $\liminf_{t\to\infty} z(t) = -\infty$, a contradiction due to the boundedness of $z(t)$.

Case IV. If $u'(t) < 0$ for $t \geqslant t_2$, we may note that $u''(t) < 0$, then integrating from t_2 to t, we obtain

$$u(t) \leqslant u(t_2) + u'(t_2)(t - t_2),$$

therefore, $\lim_{t\to\infty} u(t) = -\infty$. Hence, $u(t) < 0$ for $t \geqslant t_3 > t_2$. Thus, $z(t) < k(t)+F(t)$. Hence, $\liminf_{t\to\infty} z(t) = -\infty$, a contradiction due to the boundedness of $z(t)$. This completes the proof of the theorem. □

Example 6.3.5. Consider

$$\left(y(t) - \frac{e^{-t}}{2} y(t-2\pi)\right)^{(vi)} + (1+e^t)y(t-4\pi)$$

$$-4e^{-t}\left(1+\sin^2\left(t-\frac{\pi}{2}\right)\right)\frac{y\left(t-\frac{\pi}{2}\right)}{1+y^2\left(t-\frac{\pi}{2}\right)} = e^t \sin t, \tag{6.18}$$

for $t \geqslant 13$. Indeed, if we choose $F(t) = \frac{e^t cost}{8}$, then $F^{(vi)}(t) = f(t)$. Clearly, $(H_1) - (H_3)$ and (H_7) are satisfied. Hence by Theorem 6.3.4 every bounded solution of (6.18) oscillates. Indeed, $y(t) = \sin t$ is such a solution of (6.18).

Theorem 6.3.6. *Let $0 \leq p(t) \leq p_1 < 1$, (H_1) and (H_6) hold with*

$$-\frac{1}{8}(1-p_1) < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \frac{1}{4}(1-p_1). \tag{6.19}$$

Furthermore, assume that G and H are Lipschitzian on the intervals of the form $[b, c]$, $0 < b < c < \infty$. If

$$\int_{t_0}^{\infty} \frac{s}{r(s)} \int_{s}^{\infty} t^{n-3} q(t) dt ds < \infty,$$

then (6.2) *admits a positive bounded solution.*

Proof. It is possible to choose $t_1 > t_0$ large enough such that for $t \geq t_1$,

$$\frac{1}{(n-3)\,!} \int_{t_1}^{\infty} \frac{t}{r(t)} \int_{t}^{\infty} s^{n-3} h(s) ds dt < \frac{1-p_1}{4L} \tag{6.20}$$

and

$$\frac{1}{(n-3)\,!} \int_{t_1}^{\infty} \frac{t}{r(t)} \int_{t}^{\infty} s^{n-3} q(s) ds dt < \frac{1-p_1}{4L}, \tag{6.21}$$

where $L = max\{L_1, L_2, G(1), H(1)\}$ and L_1, L_2 are Lipschitz constants of G and H on $[\frac{1}{8}(1-p_1), 1]$ respectively. Let $X = BC([t_0, \infty), \mathbb{R})$. Then X is a Banach Space with respect to supremum norm defined by

$$||x|| = \sup_{t \geq t_0}\{|x(t)|\}.$$

Let

$$S = \{x \in X : \frac{1}{8}(1-p_1) \leq x(t) \leq 1, t \geq t_0\}.$$

Hence S is a complete metric space. For $y \in S$, we define

$$Ty(t) = \begin{cases} Ty(t_0+\rho), t \in [t_0, t_0+\rho] \\ -p(t)y(t-\tau) + \frac{1}{2}(1+p_1) + F(t) + k(t) \\ -\frac{1}{(n-3)\,!}\int_t^{\infty}(\frac{s-t}{r(s)}\int_s^{\infty}(u-s)^{n-3}q(u)G(y(u-\alpha))du)ds, t \geq t_0+\rho. \end{cases}$$

Using (6.20), we obtain

$$\begin{aligned} k(t) &= \frac{1}{(n-3)\,!}\int_t^\infty \frac{s-t}{r(s)}\int_s^\infty (u-s)^{n-3}h(u)H(y(u-\beta))duds \\ &< \frac{H(1)}{(n-3)\,!}\int_t^\infty \frac{s}{r(s)}\int_s^\infty u^{n-3}h(u)duds \\ &< \frac{1}{4}(1-p_1). \end{aligned} \tag{6.22}$$

Note that $H(1)$ denotes the bound for H on $[\frac{1}{8}(1-p_1), 1]$. Hence by using (6.19), (6.22), we obtain

$$Ty(t) < \frac{1+p_1}{2} + \frac{1-p_1}{4} + \frac{1-p_1}{4} = 1.$$

On the other hand, from (6.21), we get

$$\frac{1}{(n-3)\,!}\int_t^\infty \frac{s-t}{r(s)}\int_s^\infty (u-s)^{n-3}q(u)G(y(u-\alpha))duds < \frac{1-p_1}{4}. \tag{6.23}$$

Further using (6.19), (6.23), it follows that

$$Ty(t) > -p_1 + \frac{1}{2}(1+p_1) - \frac{1}{8}(1-p_1) - \frac{1}{4}(1-p_1) = \frac{1}{8}(1-p_1).$$

Hence $Ty \in S$, that is, $T : S \to S$.

Further for $x, y \in S$, we have

$$\begin{aligned} | \quad & Ty(t) - Tx(t)| \\ \leqslant \quad & |-p(t)||y(t-\tau) - x(t-\tau)| \\ + \quad & \left|\frac{1}{(n-3)\,!}\int_t^\infty \frac{s-t}{r(s)}\int_s^\infty (u-s)^{n-3}q(u)[G(x(u-\alpha)) - G(y(u-\alpha))]duds\right| \\ + \quad & \left|\frac{1}{(n-3)\,!}\int_t^\infty \frac{s-t}{r(s)}\int_s^\infty (u-s)^{n-3}h(u)[H(y(u-\beta)) - H(x(u-\beta))]duds\right| \\ \leqslant \quad & p_1|y(t-\tau) - x(t-\tau)| \\ + \quad & \frac{1}{(n-3)}\int_t^\infty \frac{s-t}{r(s)}\int_s^\infty (u-s)^{n-3}q(u)|G(x(u-\alpha)) - G(y(u-\alpha))|duds \\ + \quad & \frac{1}{(n-3)\,!}\int_t^\infty \frac{s-t}{r(s)}\int_s^\infty (u-s)^{n-3}h(u)|H(y(u-\beta)) - H(x(u-\beta))|duds \\ \leqslant \quad & p_1||y-x|| + L_1||y-x||\frac{1}{(n-3)\,!}\int_t^\infty \frac{s}{r(s)}\int_s^\infty u^{n-3}q(u)duds \\ + \quad & L_2||y-x||\frac{1}{(n-3)\,!}\int_t^\infty \frac{s}{r(s)}\int_s^\infty u^{n-3}h(u)duds, \end{aligned}$$

implies that

$$||Ty - Tx|| \leqslant ||y - x|| \left[p_1 + \frac{1 - p_1}{4} + \frac{1 - p_1}{4} \right] = \frac{(1 + p_1)}{2} \|y - x\|$$

for every $x, y \in S$. Thus, T is a contraction. By Banach fixed point theorem [42] T has a fixed point, that is, $Ty = y$. Consequently, $y(t)$ is a solution of (6.2) on $[\frac{1}{8}(1 - p_1), 1]$. This completes the proof of the theorem. □

Remark 6.3.7. The results similar to that of Theorem 6.3.6 can be obtained for all ranges of $p(t)$. Moreover, one can establish the existence of bounded positive solution by using various fixed point theorems.

6.4 Higher Order NDDE with $\int_0^\infty \frac{1}{r(t)} dt < \infty$.

In the Chapter 4, Section 4.4, we have studied the oscillatory and asymptotic behaviour of solutions of the fourth order neutral delay differential equations (4.1) and (4.2) under the assumption

$$\int_0^\infty \frac{t}{r(t)} dt < \infty.$$

In this section, we are concerned with the oscillatory and asymptotic behavior of solutions of the higher order nonlinear neutral delay differential equations of the form (6.1) and (6.2) under the assumption

$$\int_0^\infty \frac{1}{r(t)} dt < \infty \tag{H_9}$$

for various ranges of $p(t)$.

6.5 Oscillation Criteria for Homogeneous Equation with $\int_0^\infty \frac{1}{r(t)} dt < \infty$.

In this section, sufficient conditions are obtained for oscillatory and asymptotic behavior of all solutions of (6.1).

Theorem 6.5.1. *Let* $0 \leqslant p(t) \leqslant p_1 < 1$ *or* $1 < p_2 \leqslant p(t) \leqslant p_3 < \infty$. *Suppose* (H_1)-(H_4) *and*

(H_{10}) $\int_{t_5}^{\infty} \frac{1}{r(t)} \int_{t_5}^{t} (t-s)^{n-3} q(s) ds dt = \infty$

hold, then every bounded solution of (6.1) *either oscillates or tends to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a non-oscillatory bounded solution of (6.1) on $[t_0, \infty)$, $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. (The proof in case $y(t) < 0$ eventually is similar and will be omitted.) Set $z(t), k(t), v(t)$ as in (6.3), (6.4), (6.5), we obtain (6.6) and (6.7) for $t \geqslant t_0 + \rho$, where $\rho = max\{\tau, \alpha, \beta\}$, $w^{(n-2)}(t)$ represents the $(n-2)^{th}$ derivative of $'w'$ w.r.t $'t'$. Clearly, $w^{(n-3)}(t), w^{(n-4)}(t), .., w'(t), w(t)$ are monotonic functions and of constant sign for $t \in [t_1, \infty)$, $t_1 \geqslant t_0 + \rho$.

If $w(t) > 0$ for $t \geqslant t_1$, in view of Lemma 3.4.1, $w^{(n-3)}(t) > 0$ for $t \geqslant t_1$. Now $w(t) > 0$ implies $v''(t) > 0$ for $t \geqslant t_1$, which in turn implies $v'(t)$ is monotonic function. Thus, $v'(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$.

Case I. If $v'(t) > 0$ for $t \geqslant t_2$, then proceeding same as in Case I of Theorem 6.2.2, we obtain a contradiction due to the boundedness of $y(t)$.

Case II. If $v'(t) < 0$ for $t \geqslant t_2$, we may note that $v'(t) < 0$ implies $z'(t) < 0$ eventually. Hence, $\lim_{t\to\infty} z(t) < \infty$. Proceeding same as in Case II of Theorem 6.2.2, we obtain a contradiction due to $\lim_{t\to\infty} y(t) = 0$.

If $w(t) < 0$ for $t \geqslant t_1$, then $v''(t) < 0$ for $t \geqslant t_1$. Thus, $v'(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$.

Case III. Suppose $v'(t) > 0$ for $t \geqslant t_2$. Since $w^{(n-2)}(t) \leqslant 0$ eventually, then $w^{(n-3)}(t) > 0$ or < 0 eventually.

Subcase (i): Suppose $w^{(n-3)}(t) > 0$ eventually. Since $v'(t) > 0$, then $v(t) > 0$ or < 0 for $t \geqslant t_3 > t_2$.

If $v(t) > 0$ for $t \geqslant t_3$, then proceeding same as in Subcase (i) of Case III of Theorem 6.2.2 for $v(t) > 0$ part, we obtain a contradiction due to (H_4).

If $v(t) < 0$ for $t \geqslant t_3$, then proceeding same as in Subcase (i) of Case III of Theorem 6.2.2 for $v(t) < 0$ part, we obtain $\lim_{t\to\infty} y(t) = 0$.

Subcase (ii): If $w^{(n-3)}(t) < 0$ eventually, then from (6.6) we can conclude that $w^{(n-4)}(t) < 0, ..., w'(t) < 0$ for large t. Now, $v'(t) > 0$ implies $v(t) > 0$ or < 0 for $t \geqslant t_3 > t_2$.

If $v(t) > 0$ for $t \geqslant t_3$, then $0 < \lim_{t\to\infty} v(t) \leqslant \infty$ due to the fact that $v'(t) > 0$ eventually. If $0 < \lim_{t\to\infty} v(t) < \infty$, then $\lim_{t\to\infty} z(t) < \infty$. Claim $\liminf_{t\to\infty} y(t) = 0$. If not, let it be $l_4 > 0$. Then for some $\epsilon > 0$ there exists $t_4 > t_3$ such that

$$y(t) > (l_4 - \epsilon) > 0$$

for $t \geqslant t_4$. Therefore using the last inequality in (6.6), we obtain

$$0 \geqslant w^{(n-2)}(t) + q(t)G(l_4 - \epsilon)$$

for $t \geqslant t_5 > t_4$. Integrating the last inequality consecutively $(n-2)$ times from $'t_5'$ to $'t'$, we obtain

$$0 > w(t_5) \geqslant w(t) + \frac{1}{(n-3)\,!}\int_{t_5}^{t}(t-s)^{n-3}q(s)G(l_4 - \epsilon)ds.$$

Hence,

$$0 > v''(t) + \frac{1}{(n-3)\,!}\frac{1}{r(t)}\int_{t_5}^{t}(t-s)^{n-3}q(s)G(l_4 - \epsilon)ds.$$

Further integrating the preceeding inequality from $'t_5'$ to $'t'$, we obtain

$$v'(t_5) \geqslant v'(t) + \frac{1}{(n-3)\,!}\int_{t_5}^{t}\frac{1}{r(v)}\int_{t_5}^{v}(v-s)^{n-3}q(s)G(l_4 - \epsilon)dsdv.$$

Since $\lim_{t\to\infty} v'(t) < \infty$, then integrating the last inequality from $'t_5'$ to $'t'$ and taking $t \to \infty$, we obtain

$$\frac{1}{(n-3)\,!}\int_{t_5}^{\infty}\frac{1}{r(t)}\int_{t_5}^{t}(t-s)^{n-3}q(s)G(l_4 - \epsilon)dsdt < \infty,$$

a contradiction to (H_{10}). Hence by Lemma 4.2.4 $\lim_{t\to\infty} z(t) = 0$. Thus $\lim_{t\to\infty} y(t) = 0$, as $y(t) \leqslant z(t)$.

If $\lim_{t\to\infty} v(t) = \infty$, then we obtain a contradiction due to the fact that $y(t)$ is bounded.

If $v(t) < 0$ for $t \geqslant t_3$, then $-\infty < \lim_{t\to\infty} v(t) \leqslant 0$ due to the fact that $v'(t) > 0$ eventually. If $-\infty < \lim_{t\to\infty} v(t) < 0$, then $-\infty < \lim_{t\to\infty} z(t) < 0$, a contradiction to the fact that $z(t) > 0$.
If $\lim_{t\to\infty} v(t) = 0$, then $\lim_{t\to\infty} z(t) = 0$. Hence, $\lim_{t\to\infty} y(t) = 0$, since $y(t) \leqslant z(t)$.

Case IV. If $v'(t) < 0$, then integrating $v''(t) < 0$ twice consecutively from t_2 to t, we obtain $\lim_{t\to\infty} v(t) = -\infty$, a contradiction due to the fact that $v(t)$ is bounded. Hence proof of the theorem is complete. □

Example 6.5.2. Consider the differential equation

$$\left(e^t(y(t) + e^{-2t}y(t-2\pi))''\right)'' + \left(3e^{-t} + e^t\right) y\left(t - \frac{3\pi}{2}\right)$$

$$-8e^{-t}(1 + sin^2(t-\pi))\frac{y(t-\pi)}{1 + y^2(t-\pi)} = 0 \tag{6.24}$$

for $t \geqslant 7$. Clearly, $(H_1) - (H_4)$ and (H_{10}) are satisfied. Hence by Theorem 6.5.1 every bounded solution of (6.24) either oscillates or converges to zero as $t \to \infty$. In particular, $y(t) = \sin t$ is such a bounded oscillatory solution of (6.24).

Theorem 6.5.3. *Let $-\infty < p_4 \leqslant p(t) \leqslant p_5 < -1$. Suppose that (H_1), (H_3), (H_5) and (H_{10}) hold, then every bounded solution of* (6.1) *either oscillates or converges to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a non-oscillatory bounded solution of (6.1) on $[t_0, \infty)$, $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. (The proof in case $y(t) < 0$ eventually is similar and will be omitted.) Setting $z(t), k(t)$ and $v(t)$ as in (6.3), (6.4), (6.5) we obtain (6.6) and (6.7) respectively for $t \geqslant t_0 + \rho$. Clearly, $w^{(n-3)}(t), w^{(n-4)}(t), .., w'(t), w(t)$ are monotonic functions and of constant sign for $t \in [t_1, \infty)$, $t_1 \geqslant t_0 + \rho$.

If $w(t) > 0$ for $t \geqslant t_1$, in view of Lemma 3.4.1, $w^{(n-3)}(t) > 0$ for $t \geqslant t_1$. Now $w(t) > 0$ implies $v''(t) > 0$ for $t \geqslant t_1$, which in turn implies $v'(t)$ is monotonic function. Thus, $v'(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$.

Case I. If $v'(t) > 0$ for $t \geqslant t_2$, then proceeding same as in Case I of Theorem 6.2.2, we obtain a contradiction due to the fact that $y(t)$ is bounded.

Case II. If $v'(t) < 0$ for $t \geqslant t_2$, we may note that $v'(t) < 0$ implies $z'(t) < 0$ eventually. Hence, $z(t) > 0$ or < 0 for $t \geqslant t_3 > t_2$.

Subcase (i): If $z(t) > 0$ for $t \geqslant t_3$. Hence, $\lim_{t\to\infty} z(t) < \infty$. Proceeding same as in Subcase (i) of Case II of Theorem 6.2.4, we obtain $\lim_{t\to\infty} y(t) = 0$.

Subcase (ii): If $z(t) < 0$ for $t \geqslant t_3$, then $-\infty \leqslant \lim_{t\to\infty} z(t) < 0$ due to $z'(t) < 0$. If $-\infty < \lim_{t\to\infty} z(t) < 0$, then proceeding same as in Subcase (ii) of Case II of Theorem 6.2.4, we obtain $\lim_{t\to\infty} y(t) = 0$.
If $\lim_{t\to\infty} z(t) = -\infty$, we get a contradiction due to the fact that $y(t)$ and $z(t)$ are bounded.

If $w(t) < 0$ for $t \geqslant t_1$, then $v''(t) < 0$ for $t \geqslant t_1$. Thus, $v'(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$.

Case III. Suppose $v'(t) > 0$ for $t \geqslant t_2$. Since $w^{(n-2)}(t) \leqslant 0$ eventually, then either $w^{(n-3)}(t) > 0$ or < 0 eventually.

Subcase (iii): Suppose $w^{(n-3)}(t) > 0$ eventually. Since $v'(t) > 0$, then we have two cases: $v(t) > 0$ or < 0 for $t \geqslant t_3 > t_2$.

If $v(t) > 0$ for $t \geqslant t_3$, then proceeding same as in Subcase (iii) of Case III of Theorem 6.2.4 for $v(t) > 0$ part, we obtain a contradiction due to (H_5).

If $v(t) < 0$ for $t \geqslant t_3$, then proceeding same as in Subcase (iii) of Case III of Theorem 6.2.4 for $v(t) < 0$ part, we obtain $\lim_{t\to\infty} y(t) = 0$.

Subcase (iv): If $w^{(n-3)}(t) < 0$ eventually, then from (6.6) we can conclude that $w^{(n-4)}(t) < 0, ..., w'(t) < 0$ for large t. Since $v'(t) > 0$, then $v(t) > 0$ or < 0 for $t \geqslant t_3 > t_2$.

If $v(t) > 0$ for $t \geqslant t_3$, then $z(t) > 0$ eventually. Now $0 < \lim_{t\to\infty} v(t) \leqslant \infty$ due to the fact that $v'(t) > 0$ eventually. If $0 < \lim_{t\to\infty} v(t) < \infty$, then $0 < \lim_{t\to\infty} z(t) < \infty$. Suppose $\lim_{t\to\infty} z(t) = l_5 > 0$, then for some $\epsilon > 0$ there exists $t_4 > t_3$ such that

$$z(t) > (l_5 - \epsilon) > 0$$

for $t \geqslant t_4$. Since $y(t) > z(t)$, using this fact in the last inequality, from (6.6) we obtain

$$0 \geqslant w^{(n-2)}(t) + q(t)G(l_5 - \epsilon)$$

for $t \geqslant t_5 > t_4$. Integrating the last inequality consecutively $(n-2)$ times from $'t_5'$ to $'t'$, we obtain

$$0 > w(t_5) \geqslant w(t) + \frac{1}{(n-3)\,!} \int_{t_5}^{t} (t-s)^{n-3} q(s)G(l_5 - \epsilon)ds.$$

Hence,

$$0 > v''(t) + \frac{1}{(n-3)\,!} \frac{1}{r(t)} \int_{t_5}^{t} (t-s)^{n-3} q(s)G(l_5 - \epsilon)ds.$$

Further integrating the preceeding inequality from $'t_5'$ to $'t'$, we obtain

$$v'(t_5) \geqslant v'(t) + \frac{1}{(n-3)\,!} \int_{t_5}^{t} \frac{1}{r(v)} \int_{t_5}^{v} (v-s)^{n-3} q(s)G(l_5 - \epsilon)dsdv.$$

Since $\lim_{t\to\infty} v'(t) < \infty$, then integrating the last inequality from $'t_5'$ to $'t'$ and taking $t \to \infty$, we obtain

$$\frac{1}{(n-3)\,!} \int_{t_5}^{\infty} \frac{1}{r(t)} \int_{t_5}^{t} (t-s)^{n-3} q(s)G(l_5 - \epsilon)dsdt < \infty,$$

a contradiction to (H_{10}).
If $\lim_{t\to\infty} v(t) = \infty$, then we get a contradiction due to the fact that $y(t)$ and $v(t)$ are bounded.

If $v(t) < 0$ for $t \geqslant t_3$, then $-\infty < \lim_{t\to\infty} v(t) \leqslant 0$ due to the fact that $v'(t) > 0$ eventually. If $-\infty < \lim_{t\to\infty} v(t) < 0$, then $-\infty < \lim_{t\to\infty} z(t) < 0$. Hence,

$$z(t) < l_6 < 0 \tag{6.25}$$

for $t \geqslant t_4 > t_3$. Further $z(t) > p(t)y(t-\tau) > p_4 y(t-\tau)$ for $t \geqslant t_4$. From (6.25), we obtain $y(t-\alpha) > (p_4)^{-1} l_6$ for $t \geqslant t_5 > t_4$. From (6.6) we obtain

$$0 \geqslant w^{(n-2)}(t) + q(t)G((p_4)^{-1} l_6)$$

for $t \geqslant t_5 > t_4$. Integrating the last inequality consecutively $(n-2)$ times from $'t_5'$ to $'t'$, we obtain

$$0 > w(t_5) \geqslant w(t) + \frac{1}{(n-3)\,!} \int_{t_5}^{t} (t-s)^{n-3} q(s) G((p_4)^{-1} l_6) ds.$$

Hence,

$$0 > v''(t) + \frac{1}{(n-3)\,!} \frac{1}{r(t)} \int_{t_5}^{t} (t-s)^{n-3} q(s) G((p_4)^{-1} l_6) ds.$$

Further integrating the preceeding inequality from $'t_5'$ to $'t'$, we obtain

$$v'(t_5) \geqslant v'(t) + \frac{1}{(n-3)\,!} \int_{t_5}^{t} \frac{1}{r(v)} \int_{t_5}^{v} (v-s)^{n-3} q(s) G((p_4)^{-1} l_6) ds dv.$$

Since $\lim_{t\to\infty} v'(t) < \infty$, then integrating the last inequality from $'t_5'$ to $'t'$ and taking $t \to \infty$, we obtain

$$\frac{1}{(n-3)\,!} \int_{t_5}^{\infty} \frac{1}{r(t)} \int_{t_5}^{t} (t-s)^{n-3} q(s) G((p_4)^{-1} l_6) ds dt < \infty,$$

a contradiction to (H_{10}).

Case IV. If $v'(t) < 0$, then integrating $v''(t) < 0$ twice consecutively we obtain $\lim_{t\to\infty} v(t) = -\infty$, a contradiction due to the fact that $y(t)$ is bounded. Hence proof of the theorem is complete. □

Theorem 6.5.4. *Let* $-1 < p_6 \leqslant p(t) \leqslant 0$. *Suppose that* (H_1), (H_3), (H_5) *and* (H_{10}) *hold, then every bounded solution of* (6.1) *either oscillates or converges to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a non-oscillatory bounded solution of (6.1) on $[t_0, \infty)$, $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. (The proof in case $y(t) < 0$ eventually is similar and will be omitted.) Setting $z(t), k(t)$ and $v(t)$ as in (6.3), (6.4), (6.5) we obtain (6.6) and (6.7) respectively for $t \geqslant t_0 + \rho$. Clearly, $w^{(n-3)}(t), w^{(n-4)}(t), .., w'(t), w(t)$ are monotonic functions and of constant sign for $t \in [t_1, \infty)$, $t_1 \geqslant t_0 + \rho$.

If $w(t) > 0$ for $t \geqslant t_1$, in view of Lemma 3.4.1, $w^{(n-3)}(t) > 0$ for $t \geqslant t_1$. Now $w(t) > 0$ implies $v''(t) > 0$ for $t \geqslant t_1$, which in turn implies $v'(t)$ is monotonic function. Thus, $v'(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$.

Case I. If $v'(t) > 0$ for $t \geqslant t_2$, then proceeding same as in Case I of Theorem 6.2.2, we obtain a contradiction due to the fact that $y(t)$ is bounded.

Case II. Suppose $v'(t) < 0$ for $t \geqslant t_2$. we may note that $v'(t) < 0$ implies $z'(t) < 0$ eventually. Hence, $z(t) > 0$ or < 0 eventually.

Subcase (i): If $z(t) > 0$ eventually, then proceeding same as in Subcase (i) of Case II of Theorem 6.2.6, we obtain $\lim_{t\to\infty} y(t) = 0$.

Subcase (ii): If $z(t) < 0$ eventually, then proceeding same as in Subcase (ii) of Case II of Theorem 6.2.6, we obtain $\lim_{t\to\infty} y(t) = 0$.

If $\lim_{t\to\infty} z(t) = -\infty$, we get a contradiction due to the fact that $y(t)$ and $z(t)$ are bounded.

If $w(t) < 0$ for $t \geqslant t_1$, then $v''(t) < 0$ for $t \geqslant t_1$. Thus, $v'(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$.

Case III. Suppose $v'(t) > 0$ for $t \geqslant t_2$. Since $w^{(n-2)}(t) \leqslant 0$ eventually, then $w^{(n-3)}(t) > 0$ or < 0 eventually.

Subcase (iii): Suppose $w^{(n-3)}(t) > 0$ eventually. Since $v'(t) > 0$, then we have two cases: $v(t) > 0$ or < 0 for $t \geqslant t_3 > t_2$.

If $v(t) > 0$ for $t \geqslant t_3$, then $z(t) > 0$ eventually. Proceeding same as in *Subcase (iii)* of Case III of Theorem 6.2.4 for $v(t) > 0$ part, we obtain a contradiction due to (H_5).

If $v(t) < 0$ for $t \geqslant t_3$, then $\lim_{t\to\infty} v(t)$ exists. Hence, $\lim_{t\to\infty} z(t)$ exists. Proceeding same as in *Subcase (iii)* of Case III of Theorem 6.2.4 for the $v(t) < 0$ part and using (6.13), we obtain $\lim_{t\to\infty} y(t) = 0$.

Subcase (iv): If $w^{(n-3)}(t) < 0$ eventually, then from (6.6) we can conclude that $w^{(n-4)}(t) < 0, ..., w'(t) < 0$ for large t. Now, $v'(t) > 0$ implies $v(t) > 0$ or < 0 for $t \geqslant t_3 > t_2$.

If $v(t) > 0$ for $t \geqslant t_3$, $0 < \lim_{t\to\infty} v(t) \leqslant \infty$ due to the fact that $v'(t) > 0$ eventually. If $0 < \lim_{t\to\infty} v(t) < \infty$, then proceeding same as in *Subcase (iv)* of Case III of Theorem 6.5.3 we obtain a contradiction due to (H_{10}).

If $\lim_{t\to\infty} v(t) = \infty$, then we obtain a contradiction due to the fact that $y(t)$ and $v(t)$ are bounded.

If $v(t) < 0$ for $t \geqslant t_3$, then $-\infty < \lim_{t\to\infty} v(t) \leqslant 0$ due to the fact that $v'(t) > 0$ eventually. If $-\infty < \lim_{t\to\infty} v(t) < 0$, then $-\infty < \lim_{t\to\infty} z(t) < 0$. Hence

$$z(t) < l_7 < 0 \tag{6.26}$$

for $t \geqslant t_4 > t_3$. Further $z(t) > p(t)y(t-\tau) > p_6 y(t-\tau)$ for $t \geqslant t_4$. From (6.26), we obtain $y(t-\alpha) > (p_6)^{-1} l_7$ for $t \geqslant t_5 > t_4$. From (6.6) we get

$$0 \geqslant w^{(n-2)}(t) + q(t)G((p_6)^{-1} l_7)$$

for $t \geqslant t_5 > t_4$. Integrating the last inequality consecutively $(n-2)$ times from $'t_5'$ to

$'t'$, we obtain

$$0 > w(t_5) \geqslant w(t) + \frac{1}{(n-3)\,!}\int_{t_5}^{t}(t-s)^{n-3}q(s)G((p_6)^{-1}l_7)ds.$$

Hence

$$0 > v''(t) + \frac{1}{(n-3)\,!}\frac{1}{r(t)}\int_{t_5}^{t}(t-s)^{n-3}q(s)G((p_6)^{-1}l_7)ds.$$

Further integrating the preceeding inequality from $'t_5'$ to $'t'$, we obtain

$$v'(t_5) \geqslant v'(t) + \frac{1}{(n-3)\,!}\int_{t_5}^{t}\frac{1}{r(v)}\int_{t_5}^{v}(v-s)^{n-3}q(s)G((p_6)^{-1}l_7)dsdv.$$

Since $\lim_{t\to\infty} v'(t) < \infty$, then integrating the last inequality from $'t_5'$ to $'t'$ and taking $t \to \infty$, we obtain

$$\frac{1}{(n-3)\,!}\int_{t_5}^{\infty}\frac{1}{r(t)}\int_{t_5}^{t}(t-s)^{n-3}q(s)G((p_6)^{-1}l_7)dsdt < \infty,$$

a contradiction to (H_{10}).

Case IV. If $v'(t) < 0$, then integrating $v''(t) < 0$ twice consecutively we obtain $\lim_{t\to\infty} v(t) = -\infty$, a contradiction due to the fact that $y(t)$ is bounded. □

Example 6.5.5. Consider the differential equation

$$(e^t(y(t) - 3e^{-\frac{\pi}{2}}y(t-\pi))'')'' + \left(e^{-\frac{t}{2}} + e^t\right)\frac{e^{-\frac{\pi}{4}}}{8}y\left(t - \frac{\pi}{2}\right)$$

$$-\frac{e^{-\frac{t}{2}}(1+e^{-t}e^2)}{8e}\frac{y(t-2)}{1+y^2(t-2)} = 0 \tag{6.27}$$

for $t \geqslant 4$. Clearly, (H_1), (H_3), (H_5) and (H_{10}) are satisfied. Hence by Theorem 6.5.4 every bounded solution of (6.27) either oscillates or converges to zero as $t \to \infty$. In particular, $y(t) = e^{-\frac{t}{2}}$ is such a solution of (6.27).

6.6 Oscillation Criteria for Non-Homogeneous Equation with $\int_0^\infty \frac{1}{r(t)}dt < \infty$.

This section is devoted to study the oscillatory and asymptotic behavior of solutions of forced equations (6.2) with suitable forcing functions.

Theorem 6.6.1. *Let* $0 \leqslant p(t) \leqslant p_1 < \infty$. *Suppose that* (H_1), (H_3) *and* (H_7) *hold. Then every bounded solution of* (6.2) *oscillates.*

Proof. Let $y(t)$ be a non-oscillatory bounded solution of (6.2) on $[t_0, \infty)$, $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. (The proof in case $y(t) < 0$ eventually is similar and will be omitted.) Setting $z(t), k(t)$ and $v(t)$ as in (6.3), (6.4), (6.5) respectively, we obtain (6.14), (6.15) and (6.16) for $t \geqslant t_0 + \rho$, where $\rho = max\{\tau, \alpha, \beta\}$. Clearly, $w_1^{(n-3)}(t), w_1^{(n-4)}(t), .., w_1'(t), w_1(t)$ are monotonic functions and of constant sign for $t \in [t_1, \infty)$, $t_1 \geqslant t_0 + \rho$.

Suppose $w_1(t) > 0$ for $t \geqslant t_1$. Now $w_1(t) > 0$ implies $u''(t) > 0$ for $t \geqslant t_1$, which in turn implies $u'(t)$ is eventually monotonic function. Thus, $u'(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$.

Case I. If $u'(t) > 0$ for $t \geqslant t_2$, then proceeding same as in Case I of Theorem 6.3.1, we obtain $u(t) > 0$. Hence $z(t) > F(t)$, using (H_7) we obtain a contradiction due to the fact that $y(t)$ is bounded.

Case II. If $u'(t) < 0$ for $t \geqslant t_2$, then we have two cases: $u(t) > 0$ or $u(t) < 0$ for $t \geqslant t_3 > t_2$.

Subcase (i): If $u(t) > 0$ for $t \geqslant t_3$, then we obtain a contradiction due to the boundedness of $z(t)$.

Subcase (ii): If $u(t) < 0$ for $t \geqslant t_3$, then we get $\liminf_{t\to\infty} z(t) = -\infty$, a contradiction due to the boundedness of $z(t)$.

If $w_1(t) < 0$ for $t \geqslant t_1$, then $u''(t) < 0$ for $t \geqslant t_1$. Thus, $u'(t) > 0$ or $u'(t) < 0$ for $t \geqslant t_2 > t_1$.

Case III. If $u'(t) > 0$ for $t \geqslant t_2$, then we have two cases: $u(t) > 0$ or $u(t) < 0$ for $t \geqslant t_3 > t_2$.

Subcase (iii): If $u(t) > 0$ for $t \geqslant t_3$, then $z(t) > F(t)$. Hence by using (H_7) we obtain a contradiction due to the boundedness of $z(t)$.

Subcase (iv): If $u(t) < 0$ for $t \geqslant t_3$, then $z(t) < k(t) + F(t)$, then we obtain $\liminf_{t\to\infty} z(t) = -\infty$, a contradiction due to the boundedness of $z(t)$.

Case IV. If $u'(t) < 0$ for $t \geqslant t_2$, we may note that $u''(t) < 0$, then integrating from t_2 to t, we obtain

$$u(t) \leqslant u(t_2) + u'(t_2)(t - t_2),$$

therefore, $\lim_{t\to\infty} u(t) = -\infty$. Hence, $u(t) < 0$ for $t \geqslant t_3 > t_2$, so also $z(t) < k(t)+F(t)$. Thus we obtain $\liminf_{t\to\infty} z(t) = -\infty$, a contradiction due to the boundedness of $z(t)$. This completes the proof of the theorem. □

Example 6.6.2. Consider

$$(e^t(y(t) + e^{-5t}y(t - 2\pi))'')'' + (e^{2t} + 2e^t)y\left(t - \frac{3\pi}{2}\right)$$

$$-442e^{-4t}(1 + \sin^2(t - \theta))\frac{y(t - \theta)}{1 + y^2(t - \theta)} = e^{2t}\cos t \tag{6.28}$$

for $t \geqslant 51$, where $\tan\theta = \frac{342}{280}$. Indeed, if we choose $F(t) = \frac{e^t}{150}(9\sin t - 12\cos t)$, then $(r(t)F''(t))'' = f(t)$. Clearly, (H_1), (H_3) and (H_7) are satisfied. Hence by Theorem

6.6.1 every bounded solution of (6.28) oscillates. Indeed, $y(t) = \sin t$ is such a solution of (6.28).

Theorem 6.6.3. *Let* $-\infty < p_4 \leqslant p(t) \leqslant p_5 \leqslant 0$*. Suppose that* $(H_1), (H_3)$ *and* (H_7) *hold, then every bounded solution of* (6.2) *oscillates.*

Proof. Let $y(t)$ be a non-oscillatory bounded solution of (6.2) on $[t_0, \infty)$, $t_0 \geqslant 0$, say $y(t)$ is an eventually positive solution. (The proof in case $y(t) < 0$ eventually is similar and will be omitted.) Setting $z(t), k(t)$ and $v(t)$ as in (6.3), (6.4), (6.5) respectively, we obtain (6.14), (6.15) and (6.16) for $t \geqslant t_0 + \rho$, where $\rho = max\{\tau, \alpha, \beta\}$. Clearly, $w_1^{(n-3)}(t), w_1^{(n-4)}(t), .., w_1'(t), w_1(t)$ are monotonic functions and of constant sign for $t \in [t_1, \infty)$, $t_1 \geqslant t_0 + \rho$.

If $w_1(t) > 0$ for $t \geqslant t_1$, in view of Lemma 3.4.1, $w_1^{(n-3)}(t) > 0$ for $t \geqslant t_1$. Now $w_1(t) > 0$ implies $u''(t) > 0$ for $t \geqslant t_1$, which in turn implies $u'(t)$ is eventually monotonic function. Thus, $u'(t) > 0$ or < 0 for $t \geqslant t_2 > t_1$.

Case I. If $u'(t) > 0$ for $t \geqslant t_2$, then proceeding same as in Case I in Theorem 6.3.1, we obtain $u(t) > 0$, so also $z(t) > F(t)$. Using (H_7) we obtain a contradiction due to the fact that $y(t)$ is bounded.

Case II. If $u'(t) < 0$ for $t \geqslant t_2$, then we have two cases: $u(t) > 0$ or $u(t) < 0$ for $t \geqslant t_3 > t_2$.

Subcase (i): If $u(t) > 0$ for $t \geqslant t_3$, then we obtain a contradiction due to the boundedness of $z(t)$.

Subcase (ii): If $u(t) < 0$ for $t \geqslant t_3$, then we get $\liminf_{t\to\infty} z(t) = -\infty$, a contradiction due to the boundedness of $z(t)$.

If $w_1(t) < 0$ for $t \geqslant t_1$, then $u''(t) < 0$ for $t \geqslant t_1$. Thus, $u'(t) > 0$ or $u'(t) < 0$

for $t \geqslant t_2 > t_1$.

Case III. If $u'(t) > 0$ for $t \geqslant t_2$, then we have two cases: $u(t) > 0$ or $u(t) < 0$ for $t \geqslant t_3 > t_2$.

Subcase (iii): If $u(t) > 0$ for $t \geqslant t_3$, then $z(t) > F(t)$. Hence by using (H_7) we obtain a contradiction due to the boundedness of $z(t)$.

Subcase (iv): If $u(t) < 0$ for $t \geqslant t_3$, then $z(t) < k(t)+F(t)$. Hence, $\liminf_{t\to\infty} z(t) = -\infty$, a contradiction due to the boundedness of $z(t)$.

Case IV. If $u'(t) < 0$ for $t \geqslant t_2$, we may note that $u''(t) < 0$, then integrating from t_2 to t, we obtain

$$u(t) \leqslant u(t_2) + u'(t_2)(t - t_2).$$

Thus, $\lim_{t\to\infty} u(t) = -\infty$. Hence, $u(t) < 0$ for $t \geqslant t_3 > t_2$, so also $z(t) < k(t) + F(t)$. Thus we obtain $\liminf_{t\to\infty} z(t) = -\infty$, a contradiction due to the boundedness of $z(t)$. This completes the proof of the theorem. □

6.7 Conclusion

In Section 6.2, the oscillatory and asymptotic behaviour of the bounded solutions of equations (6.1) has been studied for the ranges $0 \leq p(t) \leq p_1 < 1$ or $1 < p_2 \leq p(t) \leq p_3 < \infty$, $-\infty < p_4 \leq p(t) \leq p_5 < -1$ and $-1 < p_6 \leq p(t) \leq 0$ under the assumption $\int_0^\infty \frac{1}{r(t)} dt = \infty$. In Section 6.5, we can observe that if $\int_0^\infty \frac{1}{r(t)} dt < \infty$, an extra condition (H_{10}) is required to obtain similar kind of results as obtained in the Section 6.2.

In Theorem 6.2.7 and 6.2.8 we have proved that every solution of (6.1) either oscillates or converges to zero as $t \to \infty$.

Moreover, in Theorem 6.3.1, every solution of (6.2) oscillates under the assumption $\liminf_{t\to\infty} F(t) = -\infty$, $\limsup_{t\to\infty} F(t) = \infty$, $\int_0^\infty \frac{1}{r(t)} dt = \infty$ whereas in Theorem

6.6.1 when $\liminf_{t\to\infty} F(t) = -\infty$, $\limsup_{t\to\infty} F(t) = \infty$, $\int_0^\infty \frac{1}{r(t)}dt < \infty$, only bounded solutions of (6.2) are oscillatory.

Bibliography

[1] R. P. Agarwal, "Difference Equations and Inequalities: Theory Methods and Application", Dekker, New york, 1992.

[2] R. P. Agarwal, P. J. Y. Wong, "Advanced Topics in Difference Equations", Kluwer Academic, Dordrecht, 1997.

[3] R. P. Agarwal, M. Bohner, *Basic calculus on time scales and some of its applications*, Results Math., **35 (1-2)** (1999), 3–22.

[4] R. P. Agarwal, M. F. Aktas, A. Tiryaki, *On oscillation criteria for third order nonlinear delay differential equations*, Archiv. Math., **45** (2009), 1–18.

[5] P. M. Atici, D. C. Biles, Alex Lebedinskky, *An application of time scales to economics*, Math. Comput. Model., **43** (2006), 718–726.

[6] F. M. Atici, F. Uysal, *A production inventory model of HMMS on time scales,* Appl. Math. Lett., **21(3)** (2008), 236–243.

[7] F.V. Atkinson, *On second-order nonlinear*, Pacific J. Math., **5** (1955), 643–647.

[8] B. Baculíková, J. Džurina, *On the asymptotic behavior of a class of third-order nonlinear neutral differential equations*, Cent. Euro. J. Math., **8** (2010), 1091–1103.

[9] B. Baculíková, J. Džurina, *Oscillation of third-order functional differential equations*, Elect. J. Qual. Theor. Diff. Eqns., **43** (2010), 1–10.

[10] B. Baculíková, J. Džurina, *Oscillation of third-order neutral differential equations*, Math. Comput. Model., **52** (2010), 215–226.

[11] B. Bacul*í*ková, J. D*ž*urina, Oscillation of third-order nonlinear differential equations, Appl. Math. Lett., **24** (2011), 466–470.

[12] Y. Bai, L. Liu, *New oscillation criteria for second-order neutral delay differential equations with positive and negative coefficients*, Abst. Appl. Anal., **564068**, (2010), 1–11.

[13] Z. Bartosiewicz, E. Pawluszewicz, *Realization of nonlinear control systems on time scales,* IEEE Trans, Automat. Control., **53** (2008), 571–575.

[14] A. Bellen, N. Guglielmi and A. E. Ruchli, *Methods for linear systems of circuit delay differential equations of neutral type*, IEEE Trans. Circ. Syst-I, **46** (1999), 212–216.

[15] R. Bellman, K. L. Cooke, *"Differential-Difference Equations"*, Academic Press, New York, 1963.

[16] S. Belohorec, *Oscillations solutions of certern nonlinear differential equations of second-order*, Mat. Fyz. Casopis Sloven Akad. Vied., **11** (1961), 250–255.

[17] E. Boe, H. C. Chang, *Dynamics of delayed systems under feedback control*, Chem. Eng. Sci., **44** (1989), 1281–1294.

[18] M. Bohner, A. Peterson, "Dynamic Equations on Time Scales: An Introduction with Applications", Birkhäuser Boston Inc., 2001.

[19] M. Bohner, A. Peterson, "Advances in Dynamic Equations on Time Scales", Birkhäuser Boston Inc., 2003.

[20] R. K. Brayton and R. A. Willoughby, *On the numerical integration of a symmetric system of difference-differential equations of neutral type*, J. Math. Anal. Appl., **18** (1967), 182–189.

[21] R. K. Brayton, *Nonlinear oscillations in a distributed network*, Quart. Appl. Math., **24** (1967), 289–301.

[22] W. E. Brumley, *On the asymptotic behaviour of solutions of differential-difference equations of neutral type*, J. Diff. Eqs., **7** (1970), 175–188.

[23] F. Burkowski, *Oscillation theorems for second order nonlinear functional differential equation*, J. Math. Anal. Appl., **33** (1971), 258–262.

[24] T. Candan, S. R. Dahiya, *Oscillation of third order functional differential equations with delay*, Elect. J. Diff. Eqns., **10** (2003), 79–88.

[25] T. Candan, R. S. Dahiya, *Functional differential equations of third-order*, Elect. J. Diff. Eqns., **12** (2005), 47–56.

[26] M. Cecchi, Z. Dosla, M. Marini, *Asymptotic behaviour of solutions of third order differential equations*, Archiv. Math., **33** (1997), 99–108.

[27] Q . Chuanxi, G. Ladas, *Oscillation in differential equations with positive and negative coefficients*, Canad. Math. Bull., **33** (1990), 442–450.

[28] R. D. Driver, *Existence and continuous dependence of solutions of a neutral functional-differential equation*, Arch. Rational Mech. Anal., **19** (1965), 149–166.

[29] R. D. Driver, *Ordinary and Delay Differential Equations*, Springer-Verlag, New York, 1976.

[30] R. D. Driver, *A mixed neutral systems*, Nonl. Anal, Theo. Meth. Appl., **8** (1984), 155–158.

[31] E. R. Duke, *"Solving higher order dynamic equations on time scales as first order systems M.Sc. thesis"*, Marshall University, 2006.

[32] J. Dźurina, Koŝice, *Asymptotic properties of third order delay differential equations*, Czech. Math. J, **45** (1995), 443–448.

[33] J. Džurina, R. Kotorov́a, *Properties of third-order trinomial differential equations with delay arguments*, Non. Anal. Theor. Meth. Appl., **71** (2009), 1995–2002.

[34] R. E. Edwards, *Functional Analysis*, Holt, Rinehart and Winston, Inc. New York, 1965.

[35] L. Erbe, *Existence of oscillatory solutions and asymptotic behavior for a class of third-order linear differential equations*, Pac. J. Math., **64** (1976), 369–385.

[36] L. H. Erbe, Q. Kong, B. G. Zhang, *Oscillation Theory of Functional Differential Equations*, Dekker, New York, 1995.

[37] K. Gopalasamy, "Stability and Oscillation in Population Dynamics", Kluwer Academic Publishers Boston, 1992.

[38] S. R. Grace, R. P. Agarwal, R. Pavani, E. Thandapani, *On the oscillation of certain third-order nonlinear functional differential equations*, Appl. Math. Comput., **202** (2008), 102–112.

[39] J. R. Graef, R. Savithri, E. Thandapani, *Oscillatory properties of third-order neutral delay differential equations*, Disc. Cont. Dyn. Syst. A, (2003), 342–350.

[40] M. Gregušš, "Third Order Linear Differential Equations", D. Reidel Publishing Company, Boston, 1987.

[41] C. P. Gupta, *Existence and uniqueness results for the bending of an elastic beam equation at resonance*, J. Math. Anal. Appl., **35** (1988), 208–225.

[42] I. Györi, G. Ladas, *"Oscillation Theory of Delay Differential Equation with Application"*, Claredon Press, Oxford, 1991.

[43] J. K. Hale, *"Theory of Functional Differential Equations"*, Springer-Verlag, New York, 1977.

[44] M. Hanan, *Oscillation criteria for a third-order linear differential equations*, Paci. J. Math., **11** (1961), 919–944.

[45] P. Hartman, A. Wintner, *Linear differential and difference equations with monotone solutions*, Amer. J. Math., **75** (1953), 731–743.

[46] Gary D. Jones, *Oscillation criteria for third order differential equations*, SIAM J. Math. Anal., **7** (1976), 13–15.

[47] M. A. Jones, B. Song, D. M. Thomas, *Controlling wound healing through debridement*, Math. Comput. Model., **40** (2004), 1057–1064.

[48] V. Kac, P. Cheung, *"Quantum Calculus" Universitent*, Springer - Verlag, New York (2002).

[49] B. Karpuz and Ö. Öcalan, *Necessary and sufficient conditions on asymptotic behaviour of solutions of forced neutral delay dynamic equations*, Nonlinear Anal., **71** (2009), 3063–3071.

[50] B. Karpuz, J. V. Manjolvić, Ö. Öcalan. Y. Shoukaku, *Oscillation criteria for a class of second-order neutral delay differential equations*, Appl. Math. Comput., **210** (2009), 303–312.

[51] I. T. Kiguradze, *On the oscillation of solutions of the equation* $\frac{d^m(u)}{dt^m} + a(t)u^m signu = 0$, Mat. Sb., **65** (1964), 172–187.

[52] I. T. Kiguradze, T. A. Chanturia, *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Kluwer Academic, Dordrecht, 1993.

[53] Y. Kuang, *"Delay Differential Equations: With Applications in Population Dynamic"*, Mathematics in Science and Engneering, **191**, 1993.

[54] T. Kusano, M. Naito, *Nonlinear oscillation of fourth order differential equations*, Canad. J. Math., **4** (1976), 840–852.

[55] T. Kusano, M. Naito, *On fourth order nonlinear oscillations*, J. London Math. Soc., **14 (2)** (1976), 91–105.

[56] G. S. Ladde, V. Lakhsmikantham, B. G. Zhang, *"Oscillation Theory of Differential Equations with Deviating Arguments"*, Marcel Dekker, Inc. New York, 1987.

[57] A. C. Lazer, *The behaviour of solutions of the differential equation* $y''' + q(t)y' + p(t)y = 0$, Pacific J. Math., **17** (1966), 435–466.

[58] W. T. Li, H. S. Quan, *Oscillation of higher order neutral differential equations with positive and negative coefficients*, Ann. Diff. Eqn., **2** (1995), 70–76.

[59] W. T. Li, J. Yan, *Oscillation of first order neutral differential equations with positive and negative coefficients*, Collect. Math., **50** (1999), 199–209.

[60] T. Li, C. Zhang, G. Xing, *Oscillation of third-order neutral delay differential equations*, Abst. Appl. Anal., **569201** (2012).

[61] X. Lin, *Oscillation of second order nonlinear neutral differential equations*, J. Math. Anal. Appl., **309** (2005), 442–452.

[62] A. B. Malinowska, D. F. M. Torres, *The Hahn quantum variational calculus*, J. optim. Theory Appl., **147** (2010), 419–442.

[63] J. Manojlović, Y. Shoukaku, T. Tanigawa, N. Yoshida, *Oscillatory behavior of second order neutral differential equations with positive and negative coefficients*, Appl. Appld. Math., **3** (2008), 1–17.

[64] H. P. Mckean, *N*agumo's equation, Advances in Math, **4** (1970), 209–223.

[65] O. Öcalan, *Oscillation of forced neutral differential equations with positive and negative coefficients*, Compu. Math. Appl., **54** (2007), 1411–1421.

[66] O. Öcalan, *Oscillation of neutral differential equation with positive and negative coefficients*, J. Math. Anal. Appl., **331** (2007), 644–654.

[67] S. Panigrahi, R. Basu, *Oscillation results for third order nonlinear mixed neutral differential equations*, (Accepted in Math. Slovaca).

[68] N. Parhi, A. K. Tripathy, *On oscillatory fourth order nonlinear neutral differential equations - I,* Math. Slovaca, **54** (2004), 389–410.

[69] N. Parhi, A. K. Tripathy, *On oscillatory fourth order nonlinear neutral differential equations - II,* Math. Slovaca, **55** (2005), 183–202.

[70] N. Parhi, S. Padhi, *Asymptotic behaviour of a class of third order differential equations*, Math. Slovaca, **50** (2000), 315–333.

[71] N. Parhi, P. C. Das, *On the zeros of solutions of nonhomogeneous third order differential equations*, Czech. Math. J, **41** (1991), 641–652.

[72] N. Parhi, R. N. Rath, *On oscillation of solutions of forced nonlinear neutral differential equations of higher order*, Czech. Math. J., **53** (2003), 805–829.

[73] N. Parhi, R. N. Rath, *On oscillation criteria for forced nonlinear higher order neutral differential equations*, Czech. Math. J., **54** (2004), 369–388.

[74] E. P. Popov, *"Automatic Regulation and Control"*, Nauka Mascow (in Russian), 1966.

[75] Y. Sahiner, *On oscillation of second-order neutral type delay differential equations*, Appl. Math. Comput., **150** (2004), 697–706.

[76] S. H. Saker, *Oscillation criteria of third-order nonlinear delay differential equations*, Math. Slovaca, **56** (2006), 433–450.

[77] S. H. Saker, J. Džurina, *On the oscillation of certain class of third order nonlinear delay differential equations*, Math. Bohem., **135** (2010), 225–237.

[78] W. Shi, P. Wang, *Oscillation criteria of a class of second-order neutral functional differential equations*, Appl. Math. Comput., **146** (2003), 211–226.

[79] M. Slemrod, E. F. Infante, *Asymptotic stability criteria for linear systems of difference-differential equations on neutral type and their discrete analogues*, J. Math. Anal. Appl., **38** (1972), 399–415.

[80] W. Snow, *Existence, uniqueness and stability for non-linear differential-difference equations in the neutral case*, N. Y. U. Courant Ins. Math. Sci. Rep. IMM-NYU **328**, (February, 1965).

[81] C. Soria-Hoyo, F. Pontiga, A. Castellanos, *A PIC based procedure for the integration of multiple time scale problems in gas discharge physics*, J. Comput. Phys., **228** (2009), 1017–1029.

[82] S. Tanaka, *A Oscillation theorem for a class of even order neutral delay differential equations*, J. Math. Anal. Appl., **273** (2002), 172–189.

[83] R. D. Terry, P. K. Wong, *Oscillation properties of a fourth order delay differential equation*, Funk. Ekvac., **15** (1972), 209–220.

[84] E. Thandapani, T. Li, *On the oscillation of third-order quasi-linear neutral functional differential equations*, Arch. Math., **47** (2011), 181–199.

[85] A. Tiryaki, M. F. Aktas, *Oscillation criteria of a certain class of third order nonlinear delay differential equations with damping*, J. Math. Anal. Appl., **325** (2007), 54–68.

[86] A. K. Tripathy, *O*scillation properties of a class of neutral differential equations with positive and negative coefficients, Fasc. Math., **45** (2010), 133–155.

[87] A. K. Tripathy, *Oscillation theorems for fourth order nonlinear neutral equations with quasi-derivatives-I*, Int. J. Math. Sci. Appl., **1** (2011), 377–393.

[88] A. K. Tripathy, S. Panigrahi, R. Basu, *Oscillatory and asymptotic behaviour of fourth order nonlinear neutral delay differential equations*, Fasc. Math., **52** (2014), 155–174.

[89] R. A. Usami, *A uniqueness theorem for a boundary value problem*, Proc. Amer. Math. Soc., **77** (1979), 329–335.

[90] S. D. Vreeke, G. M. Sandquist, *P*hase plane analysis of reactor kinetics, Neuclar Sci. Engner., **42** (1970), 259–305.

[91] J. S. Wong, *Necessary and sufficient conditions for oscillation of second order neutral functional differential equations*, J. Math. Anal. Appl., **252** (2000), 342–352.

[92] J. Yan, *Oscillations of second order neutral functional differential equations*, J. Math. Anal. Appl., **83** (1997), 27–41.

[93] E. M. E. Zayed, M. A. El-Moneam, *Some oscillation criteria for second order nonlinear functional ordinary differential equations*, Acta Math. Sci., **27B(3)** (2007), 602–610.