Entrance Examination: Ph.D. Mathematics/Applied Mathematics, 2018

Hall Ticket Number ☐☐☐☐☐☐☐☐

Time: 2 hours
Max. Marks: 80

Part A: 40 Marks
Part B: 40 Marks

Instructions

1. Write your Hall Ticket Number in the OMR Answer Sheet given to you. Also write the Hall Ticket Number in the space provided above.
2. Answers are to be marked on the OMR sheet.
3. Please read the instructions carefully before marking your answers on the OMR answer sheet.
4. Hand over the OMR answer sheet at the end of the examination.
5. No additional sheets will be provided. Rough work can be done in the question paper itself/space provided at the end of the booklet.
6. Calculators are not allowed.
7. There are a total of 40 questions in **PART A** and **PART B** together.
8. Each correct answer carries 2 marks.
9. The appropriate answer should be coloured with either a blue or a black ball point or a sketch pen. DO NOT USE A PENCIL.
10. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ the set of complex numbers, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers, and $\mathbb{N}$ the set of all natural numbers.
11. This book contains **9** pages including this page and excluding pages for the rough work. Please check that your paper has all the pages.

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Part-A

1. Let X be a finite set with $|X| = 6$. Then the number of equivalence relations on X such that each equivalence class has at least three elements in it is:

   (A) 10.
   (B) 11.
   (C) 20.
   (D) 21.

2. The number of ways to select four distinct integers from $\{11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$ such that no two are consecutive integers is:

   (A) $\frac{9!}{4! \times 5!}$.
   (B) $\frac{8!}{4! \times 4!}$.
   (C) $\frac{7!}{4! \times 3!}$.
   (D) $\frac{6!}{4! \times 2!}$.

3. Let X be a nonempty finite set and $A, B \subset X$. Let $n = |(A \setminus B) \cup (B \setminus A)|$ and $k = |X \setminus (A \cap B)|$. If $n \neq 0$ and $k = 5n$, then $|X \setminus (A \cup B)|$ is equal to:

   (A) $4n$.
   (B) $3n$.
   (C) $2n$.
   (D) n.

4. Suppose $\mathbb{N} = \bigcup_{n=1}^{\infty} A_n$, where A_n's are nonempty and pairwise disjoint. Then,

   (A) A_n is a finite set for each $n \in \mathbb{N}$.
   (B) A_n is a finite set for all but finitely many $n \in \mathbb{N}$.
   (C) A_n is a finite set for infinitely many $n \in \mathbb{N}$.
   (D) None of the above.

5. Let (n_k) be a strictly increasing sequence of natural numbers and let $A = \{n_m - n_k : k < m\}$. Then, which of the following is TRUE?

   (A) $\mathbb{N} \setminus A$ is always finite.
   (B) If $\mathbb{N} \setminus A$ is finite, then $n_{k+1} = n_k + 1$ for all large $k \in \mathbb{N}$.
   (C) If $\mathbb{N} \setminus A$ is finite, then $\sup\{n_{k+1} - n_k : k \in \mathbb{N}\} < \infty$.
   (D) None of the above.

6. Let A, B, C, D be nonempty finite sets with $|B| < |C| = |D| < |A|$. Let $f_1 : A \to B$, $f_2 : B \to C$, $f_3 : C \to D$, and $f_4 : D \to A$ be functions. Then which of the following is FALSE?

   (A) $f_2 \circ f_1$ is never surjective.

   (B) $f_3 \circ f_2$ is never surjective.

   (C) $f_4 \circ f_3$ is never injective.

   (D) $f_1 \circ f_4$ is never injective.

7. If $k, m, n, p \in \mathbb{N}$ are such that $3k + 2p < 4k + m < 3n + 2p < 2n + 3p$, then the largest among k, m, n, p is:

   (A) k.

   (B) m.

   (C) n.

   (D) p.

8. The average age of the three granddaughters of a king is equal to the age of the second granddaughter, and is also equal to one-seventh of the age of the king. Then the average age of the king and his second granddaughter is equal to:

   (A) twice the average age of the first and third granddaughters.

   (B) thrice the average age of the first and third granddaughters.

   (C) four times the average age of the first and third granddaughters.

   (D) six times the average age of the first and third granddaughters.

9. In a code language, DEER, LION, TIGER, and ELEPHANT are coded respectively as OIIT, ARPE, HRMIT, and IAINGKEH. Then the code word for LEOPARD is:

   (A) AIPNKTO.

   (B) AIPNKOT.

   (C) AIPKNTO.

   (D) AIPKNOT.

10. In a coding, the vowels A,E,I,O,U are permuted among themselves, and the remaining letters of the English alphabet are permuted among themselves. Then a possible code word for CROCODILE is:

    (A) PYAPAMELY.

    (B) PYAPIMELO.

    (C) PYAYAMELO.

    (D) PYAPAMELO.

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11. In a coding, each letter is replaced by a word consisting of three or more letters. If the coding for the words END, NET, and TEN are respectively ENEENEEEEN, ENEEENEEENE, and EENEENEENEE. Then the code word for DEN is:

(A) EENENEENEE.

(B) ENEEENENEE.

(C) EENEENEENEE.

(D) EENENEEENE.

12. In a particular type of coding, the words HILL, RIVER, and TREE are coded respectively as AAEM, NSLEN, and SSNY. Then the code word for THRILL is:

(A) AAEMNY.

(B) AAENMY.

(C) AAESMY.

(D) AAELMY.

13. In a particular type of coding, SKY and CLOUDS are written respectively as TMB and DNRYIY. Then the code word for RAIN is:

(A) SBKQ.

(B) SBLQ.

(C) SCKR.

(D) SCLR.

14. Fill appropriately the seventh entry in the sequence: $1, 6, 15, 28, 45, 6\cdot$ __, ...

(A) 87.

(B) 91.

(C) 95.

(D) 99.

15. Fill appropriately the eighth entry in the sequence: $1, 3, 6, 11, 18, 29, 42,$ __, ...

(A) 57.

(B) 58.

(C) 59.

(D) 60.

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16. Let $(Y_n)_{n=1}^{\infty}$ be a sequence of subsets of a set X. The negation of the statement "*there is an infinite set $M \subset \mathbb{N}$ such that Y_m's are pairwise disjoint for $m \in M$*" is:

    (A) There is an infinite set $M \subset \mathbb{N}$ such that $Y_k \cap Y_m \neq \emptyset$ for every $k, m \in M$.

    (B) For every infinite set $M \subset \mathbb{N}$, there are $k, m \in M$ such that $Y_k \cap Y_m \neq \emptyset$.

    (C) There is a nonempty finite set $M \subset \mathbb{N}$ such that Y_m's are pairwise disjoint for $m \in M$.

    (D) $\bigcap_{m \in M} Y_m \neq \emptyset$ for every infinite set $M \subset \mathbb{N}$.

17. Let $f_n : \mathbb{R} \to \mathbb{R}$ be functions for $n \in \mathbb{N}$. The negation of the statement "*at most finitely many f_n's are infinitely often differentiable*" is:

    (A) Infinitely many f_n's are infinitely often differentiable.

    (B) Infinitely many f_n's are differentiable only finitely many times.

    (C) All but finitely many f_n's are infinitely often differentiable.

    (D) All but finitely many f_n's are differentiable only finitely many times.

18. A fair die with numbers 1 to 6 written on its six faces is thrown twice. Which of the following events has the highest probability?

    (A) At least one throw produces the number 3.

    (B) Both throws produce the same number.

    (C) Both throws produce odd numbers.

    (D) Both throws produce numbers ≥ 4.

19. An example of a surjective map f from $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$ is:

    (A) $f(k, m, n) = k + m + n$.

    (B) $f(k, m, n) = \max\{k, m, n\} - \min\{k, m, n\}$.

    (C) $f(k, m, n) = \max\{k, n\} - \min\{k + m, n\}$.

    (D) $f(k, m, n) = \max\{k, m + n\} - \min\{k, m\}$.

20. Let $f, g : \mathbb{R} \to \mathbb{R}$ be bounded functions, and $h : \mathbb{R} \to \mathbb{R}$ be a function which is bounded below but not bounded above, i.e., $-\infty < \inf\{h(x) : x \in \mathbb{R}\} \leq \sup\{h(x) : x \in \mathbb{R}\} = \infty$. Then an example of a bounded function $F : \mathbb{R} \to \mathbb{R}$ is:

    (A) $F(x) = \max\{f(x), g(x) - h(x)\} - \min\{g(x), f(x) - h(x)\}$.

    (B) $F(x) = \max\{f(x), g(x) + h(x)\} - \min\{g(x), h(x) - f(x)\}$.

    (C) $F(x) = \max\{f(x), g(x) - h(x)\} - \min\{g(x), h(x) - f(x)\}$.

    (D) $F(x) = \max\{f(x), g(x) + h(x)\} - \min\{g(x), h(x) + f(x)\}$.

Part-B

21. Let A be a 5×5 complex matrix such that $A^2 = A$. Which of the following is TRUE?

(A) $(I - A)^2 = I - 2A$.

(B) $Rank(A) \neq Rank(I - A)$.

(C) Either $A = 0$ or $A = I$.

(D) A has at least two non-real eigenvalues.

22. Let A and B be two 5×5 complex matrices. Which of the following is FALSE?

(A) $Trace(AB) = Trace(BA)$.

(B) $Det(AB) = Det(BA)$.

(C) If $AB = 0$, then $BA = 0$.

(D) If $Rank(AB) = 5$, then $Rank(BA) = 5$.

23. Let $\alpha = (1\,2)(3\,4\,5)$ and $\beta = (1\,2\,3\,4\,5\,6)$ be two permutations from the group S_6 of all permutations of the set $\{1, \ldots, 6\}$. Which of the following is FALSE?

(A) $\alpha\beta \neq \beta\alpha$.

(B) α is not conjugate to β in S_6.

(C) The subgroups $\langle\alpha\rangle$ and $\langle\beta\rangle$ are not isomorphic to each other.

(D) $\langle\alpha\rangle \cap \langle\beta\rangle$ is the trivial group.

24. In the ring $\mathbb{Z}[\sqrt{-3}]$, the element $1 + \sqrt{-3}$ is:

(A) an irreducible element.

(B) a prime element.

(C) a unit.

(D) an idempotent.

25. Which of the following is TRUE?

(A) There is a finite field F with $|F| = 6$.

(B) There is a finite field F with $|F| = 9$.

(C) There is a finite field F with $|F| = 12$.

(D) There is a finite field F with $|F| = 15$.

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26. For a function $f : \mathbb{R} \to \mathbb{R}$, which of the following is FALSE?

(A) If f is continuous and bounded, then f is uniformly continuous.

(B) If f is uniformly continuous, then $f \circ f$ is uniformly continuous.

(C) If f is continuous and $\lim_{x\to\infty} f(x) = 999 = \lim_{x\to-\infty} f(x)$, then f is uniformly continuous.

(D) If f is Lipschitz continuous, then f is uniformly continuous.

27. If $f : [a, b] \to \mathbb{R}$ is a continuous map, which of the following is TRUE?

(A) If $|f(b) - f(a)| \leq b - a$, then f has a fixed point in $[a, b]$.

(B) If f is increasing and $f(b) \leq b$, then f has a fixed point in $[a, b]$.

(C) If f is a polynomial of odd degree, then f has a fixed point in $[a, b]$.

(D) If $[a, b] \subset f([a, b])$, then f has a fixed point in $[a, b]$.

28. Let (x_n) be a sequence of real numbers. Then no subsequence of (x_n) is a Cauchy sequence if and only if:

(A) $\liminf_{n\to\infty} |x_n| > 0$.

(B) $\limsup_{n\to\infty} |x_n| = \infty$.

(C) $\lim_{n\to\infty} |x_n| = \infty$.

(D) $\liminf_{n\to\infty} |x_{n+1} - x_n| > 0$.

29. Let μ be the Lebesgue measure on $\mathbb{R}$, and $K \subset \mathbb{R}$ be a nowhere dense compact set. Then which of the following is TRUE?

(A) It is possible to have $\mu(K) = \infty$.

(B) $\mu(K)$ can be arbitrarily large, but $\mu(K) < \infty$ always.

(C) $\mu(K)$ can be positive, but $\mu(K) \leq 1$ always.

(D) $\mu(K) = 0$ always.

30. Let $(X, \mathcal{A}, \mu)$ be a measure space and $f_n : X \to [0, \infty]$ be measurable functions for $n \in \mathbb{N}$. Then which of the following is TRUE?

(A) $\int_X (\lim_{n\to\infty} f_n) d\mu \leq \lim_{n\to\infty} \int_X f_n d\mu$.

(B) $\int_X (\liminf_{n\to\infty} f_n) d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu$.

(C) $\liminf_{n\to\infty} \int_X f_n d\mu \leq \int_X (\liminf_{n\to\infty} f_n) d\mu$.

(D) $\int_X (\lim_{n\to\infty} f_n) d\mu \leq \limsup_{n\to\infty} \int_X f_n d\mu$.

31. If X is a metric space and $A \subset X$, then which of the following is FALSE?

   (A) If A is compact, then A is closed in X.

   (B) If X is separable, then A is separable.

   (C) If A is connected and dense in X, then X is connected.

   (D) If A is closed and bounded in X, then A is compact.

32. Let X_r be topological spaces for $r \in \mathbb{R}$, and consider $X := \prod_{r \in \mathbb{R}} X_r$ with product topology. Which of the following is FALSE?

   (A) If each X_r is compact, then X is compact.

   (B) If each X_r is connected, then X is connected.

   (C) If each X_r is a Hausdorff space, then X is a Hausdorff space.

   (D) If each X_r is metrizable, then X is metrizable.

33. Let $f : \mathbb{C} \to \mathbb{C}$ be $f(z) = e^z$ and $U = \{x + iy \in \mathbb{C} : -1 < x < 1 \text{ and } y \in \mathbb{R}\}$. Then, the set $f(U)$ is:

   (A) an open ball in $\mathbb{C}$.

   (B) an open annulus in $\mathbb{C}$.

   (C) an open half-space in $\mathbb{C}$.

   (D) the complement of a closed ball in $\mathbb{C}$.

34. Let $\sum_{n=0}^{\infty} a_n z^n$ be a complex power series with radius of convergence equal to 25. If we define $\quad = 5n^2 a_n$ for $n = 0, 1, 2, \ldots$, then the radius of convergence of the power series $\sum_{n=0} b_n z^n$ is equal to:

   (A) 25.

   (B) 5.

   (C) 1.

   (D) 0.

35. Consider the Banach space $(C[0,1], \|\cdot\|_\infty)$, where $C[0,1] := \{f : [0,1] \to \mathbb{R} : f \text{ is continuous}\}$ and $\|\cdot\|_\infty$ is the supremum norm on $C[0,1]$ defined as $\|f\|_\infty = \sup\{|f(t)| : t \in [0,1]\}$. Let $\phi : C[0,1] \longrightarrow \mathbb{R}$ be $\phi(f) = f(1/4) - 2f(3/4)$. Then,

   (A) ϕ is neither linear nor continuous.

   (B) ϕ is linear but not continuous.

   (C) ϕ is continuous but not linear.

   (D) ϕ is linear and continuous.

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36. Let X, Y be Banach spaces, $T : X \to Y$ be a map, and let $G(T) = \{(x, Tx) : x \in X\}$ be the graph of T in $X \times Y$. Then Closed graph theorem s.'ys the following:

(A) If T is linear and $G(T)$ is closed in $X \times Y$, then T is an open map.

(B) If $G(T)$ is closed in $X \times Y$, then T is linear and continuous.

(C) If T is linear and $G(T)$ is closed in $X \times Y$, then T is continuous.

(D) If T is continuous and $G(T)$ is closed in $X \times Y$, then T is linear.

37. The initial value problem $y' = 3y^{2/3}$, $y(0) = 0$, has

(A) a unique solution.

(B) exactly two solutions.

(C) more than two, but only finitely many, solutions.

(D) infinitely many solutions.

38. The sets of all eigenvalues and eigenfunctions of the Sturm-Liouville system $y'' + \lambda y = 0$, $0 \le x \le \pi$, $y(0) = 0 = y'(\pi)$ are given by:

(A) $\{(2n-1)^2/4 : n \in \mathbb{N}\}$ and $\{\sin(n - 1/2)x : n \in \mathbb{N}\}$.

(B) $\{(2n-1)^2/2 : n \in \mathbb{N}\}$ and $\{\sin(n - 1/2)x : n \in \mathbb{N}\}$.

(C) $\{(2n+1)^2/4 : n \in \mathbb{N}\}$ and $\{\sin(n + 1/2)x : n \in \mathbb{N}\}$.

(D) $\{(2n+1)^2/2 : n \in \mathbb{N}\}$ and $\{\sin(n + 1/2)x : n \in \mathbb{N}\}$.

39. If u i a solution of

$u_{tt}(x,t) = u_{xx}(x,t)$, $x \in (0,1)$, $t > 0$,

$u(x,0) = x^2(1-x)^2$, $x \in (0,1)$,

$u(0,t) = u(1,t) = 0$ for every $t > 0$,

then $u(1/2, 3/2)$ is equal to:

(A) 0.

(B) 1/2.

(C) -1/2.

(D) 1.

40. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map, and u be the solution of

$u_t(x,t) = u_{xx}(x,t)$, $x \in \mathbb{R}$, $t > 0$,

$u(x,0) = f(x)$, $x \in \mathbb{R}$.

Then which of the following is FALSE?

(A) $u \in C^\infty(\mathbb{R} \times (0,\infty))$.

(B) If f is bounded, then u is bounded.

(C) If $\int_0^\infty f(x)dx = 2$, then $\int_0^\infty u(x,t)dx = 1$ for every $t > 0$.

(D) If $f(x) > 0$ for every $x \in \mathbb{R}$, then $u(x,t) > 0$ for every $(x,t) \in \mathbb{R} \times (0,\infty)$.