Entrance Examination: M.Sc. Mathematics, 2017

Hall Ticket Number ☐☐☐☐☐☐☐☐

Time: 2 hours
Max. Marks: 100

Part A:25 Marks
Part B:75 Marks

Instructions

1. Write your Hall Ticket Number in the OMR Answer Sheet given to you. Also write the Hall Ticket Number in the space provided above.
2. Answers are to be marked on the OMR sheet.
3. Please read the instructions carefully before marking your answers on the OMR answer sheet.
4. Hand over the OMR answer sheet at the end of the examination.
5. The question paper can be taken by the candidate at the end of the examination.
6. No additional sheets will be provided. Rough work can be done in the question paper itself/space provided at the end of the booklet.
7. Calculators are not allowed.
8. There are a total of 50 questions in **PART A** and **PART B** together.
9. **There is a negative marking in PART A. Each correct answer carries 1 mark and each wrong answer carries −0.33 mark. Each question in PART A has only one correct option.**
10. **There is no negative marking in PART B. Each correct answer carries 3 marks. In PART B some questions have more than one correct option. All the correct options have to be marked in OMR sheet, otherwise zero marks will be credited.**
11. The appropriate answer(s) should be coloured with either a blue or a black ball point or a sketch pen. DO NOT USE A PENCIL.
12. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ the set of complex numbers, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers and $\mathbb{N}$ the set of all natural numbers.
13. This book contains 10 pages including this page and excluding pages for the rough work. Please check that your paper has all the pages.

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Part-A

1. Let $S = \{0, 1, 2, \ldots, 12\}$. Define a binary operation $*$ on S by $a * b=$ remainder when ab is divided by 13 (remainder when x is divided by 13 is the unique number y in S such that $x - y$ is divisible by 13). Then the smallest n such that $2^n = 1$ in S is

   (A) 3. (B) 4. (C) 6. (D) 12.

2. Let x, y, z be elements in a group such that $z = y^3x^2y^{-3}$. The order of x is 12. Then the order of z is

   (A) 2. (B) 3. (C) 6. (D) 12.

3. Let $\mathbb{Q}$ be the set of rationals and define a binary operation on $\mathbb{Q}$ by $a * b = a + b - ab$. Then

   (A) $(\mathbb{Q}, *)$ is a group.

   (B) $(S, *)$ is a group for some subset $S \subseteq \mathbb{Q}$.

   (C) $(\mathbb{Q}, *)$ is not associative.

   (D) $(\mathbb{Q}, *)$ has no identity.

4. An element a in a group has order 100. Then the order of a^{65} is

   (A) 5. (B) 10. (C) 20. (D) 25.

5. Consider the following two statements.

   S_1 : There exists a 3×3 matrix with integer entries with eigenvalues 1, w, w^2 (cube roots of 1).

   S_2 : A 2×2 matrix with eigenvalues w and w^2 cannot have all real entries.

   Then

   (A) both S_1 and S_2 are true.

   (B) S_1 is true but S_2 is false.

   (C) S_1 is false but S_2 is true.

   (D) both S_1 and S_2 are false.

6. Let $\begin{pmatrix} 3 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 12 \\ 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \\ 2 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 9 \\ 0 \\ 1 \\ 2 \end{pmatrix}$ be five vectors in $\mathbb{R}^4$. The maximum number of linearly independent vectors among them is

   (A) 2. (B) 3. (C) 4. (D) 5.

7. The determinant of A where

$$A = \begin{pmatrix} 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & c & d \\ 0 & 0 & 0 & e & f \\ p & q & r & s & t \\ v & w & x & y & z \end{pmatrix}$$

   is

   (A) $aderw$. (B) $ad(pw - vq)$. (C) $ad - bc$. (D) 0.

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8. Let $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$ be three non-zero vectors in $\mathbb{R}^3$ such that $\vec{x}_i \cdot \vec{x}_j = 0$ when $i \neq j$. Then

(A) such vectors cannot exist.

(B) such vectors form an orthonormal basis of $\mathbb{R}^3$.

(C) the vectors must have integer coordinates.

(D) the vectors must be linearly independent.

9. Let $f : [a, b] \to \mathbb{R}$ be continuous. Consider the following statements

S_1 : If $f \geq 0$ and $\displaystyle\int_a^b f(x)dx = 0$ then $f \equiv 0$.

S_2 : If $\displaystyle\int_c^d f(x)dx = 0$, $\forall\, [c, d] \subseteq [a, b]$ then $f \equiv 0$.

Then

(A) both S_1 and S_2 are true.

(B) S_1 is true but S_2 is false.

(C) S_1 is false but S_2 is true.

(D) both S_1 and S_2 are false.

10. Consider the following statements

S_1 : A bounded sequence of real numbers must have a convergent subsequence.

S_2 : A Cauchy sequence of real numbers must be bounded.

Then

(A) both S_1 and S_2 are true.

(B) S_1 is true but S_2 is false.

(C) S_1 is false but S_2 is true.

(D) both S_1 and S_2 are false.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and non-negative function. Consider the following statements:

S_1: If $\exists\, c \in (0, 1)$ such that $f(c) = 100$ then $\int_0^1 f(x)dx \geq \frac{1}{2}$.

S_2: If $\int_0^1 f(x)dx > \frac{1}{2}$ then $f(c) > \frac{1}{2}$ for some $c \in (0, 1)$.

S_3: If $\int_0^1 f(x)dx = \frac{1}{2}$ then $\exists\, c \in (0, 1)$ such that $f(c) = \frac{1}{2}$.

Then

(A) only S_1 is false.

(B) only S_2 is false.

(C) only S_3 is false.

(D) S_1, S_2 and S_3 are false.

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12. Let $X = \left(0, \frac{1}{3}\right) \cap \mathbb{Q}$ and $Y = \left(\frac{1}{9}, \frac{1}{3}\right) \cap \mathbb{Q}$. Then

(A) every function $f : X \to Y$ fails to be injective.

(B) every function $f : Y \to X$ fails to be surjective.

(C) every function $f : X \to Y$ satisfies $f(x) \geq x$, $\forall\, x \in X$.

(D) none of the above.

13. Which of the following sets is countable?

(A) The set of all rational numbers $\mathbb{Q}$.

(B) The set of all subsets of $\mathbb{N}$.

(C) The set of all real numbers.

(D) The set of irrational numbers.

14. If $f : \mathbb{R} \to \mathbb{R}$ is a cubic polynomial then which of the following is an empty set?

(A) $\{x \in \mathbb{R} : f(x^3) = \sqrt{2}\}$.

(B) $\{x \in \mathbb{R} : f(x+m) = f(x+n), \forall\, m, n \in \mathbb{N}\}$.

(C) $\{x \in \mathbb{R} : f'(x) = f'(y)$ for some $y \in \mathbb{R} \setminus \{x\}\}$, where f' is the derivative of f.

(D) $\{x \in \mathbb{R} : f(x^3 + x) = 0\}$.

15. Consider the statement: If $f : \mathbb{R} \to \mathbb{R}$ is a polynomial of even degree then it is not surjective. Which of these would follow from this?

(A) If $f : \mathbb{R} \to \mathbb{R}$ is a surjective polynomial function, then its degree should be odd.

(B) If $f : \mathbb{R} \to \mathbb{R}$ is a non-surjective polynomial function, then its degree can be any integer.

(C) All non-surjective functions have to be polynomials of even degree.

(D) If $f : \mathbb{C} \to \mathbb{C}$ is a polynomial of even degree, then it is not surjective.

16. If the line $\frac{x-7}{2} = -(y+3) = (z-4)$ intersects the planes $6x + 4y - 5z = 4$ and $x - 5y + \alpha z = 12$ in the same point then α is

(A) -2. (B) 0. (C) 2. (D) 4.

17. The equation of the plane through the points $(1, 0, -1)$, $(3, 2, 2)$ and parallel to the line $\frac{x-1}{1} = \frac{y-1}{-2} = \frac{z-2}{3}$ is

(A) $4x - y - 2z - 6 = 0$.

(B) $x - z = 0$.

(C) $x - 2y + z - 1 = 0$.

(D) none of the above.

18. Let $f : (0, 1) \to \mathbb{R}$ be a differentiable function such that $f'(x) \neq 0$, $\forall\, x \in (0, 1)$. Then which of the following statements is **False**?

(A) f is monotone. (B) f is one-one.

(C) f is uniformly continuous. (D) None of these.

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19. Let $f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q}. \end{cases}$ The function f at the point $x = 0$

(A) is differentiable.

(B) has the left derivative but not the right derivative.

(C) has the right derivative but not the left derivative.

(D) has neither the left derivative nor the right derivative.

20. The value of integral $\iint_S \left(\nabla \times \vec{F}\right) \cdot \hat{n} dS$ where $\vec{F} = (x^2 + y - 4)\hat{\imath} + 3xy\hat{\jmath} + (2xz + z^2)\hat{k}$ ans S is the surface of the hemisphere $x^2 + y^2 + z^2 = 16$ above the xy-plane is

(A) 0. (B) -16π. (C) 16π. (D) 32π.

21. Consider $f : [-1, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin(x), & x \in [-1,1] \cap \mathbb{Q}, \\ 0, & x \notin [-1,1] \cap \mathbb{Q}. \end{cases}$$

Then at $x = 0$, f is

(A) continuous and Riemann integrable in every interval (a, b) containing 0.

(B) continuous and Riemann integrable in exactly one interval (a, b) containing 0.

(C) continuous but not Riemann integrable in any interval containing x.

(D) discontinuous.

22. A ring A is called *local* if it has a unique maximal ideal. Which of the following is a local ring?

(A) $\mathbb{Z}$. (B) $\mathbb{C}[X]$. (C) $\mathbb{C}[X]/(X^2)$. (D) $\mathbb{Z}/6\mathbb{Z}$.

23. Let $M_2(\mathbb{R})$ be the set all 2×2 matrices with real coefficients. Then which of the following maps is a ring homomorphism?

(A) $\phi : M_2(\mathbb{R}) \to \mathbb{R}$, given by $\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a$.

(B) $\phi : \mathbb{R} \to M_2(\mathbb{R})$, given by $\phi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

(C) $\phi : M_2(\mathbb{R}) \to \mathbb{R}$, given by $\phi(A) = \mathrm{Tr}(A)$.

(D) $\phi : M_2(\mathbb{R}) \to \mathbb{R}$ given by $\phi(A) = \det(A)$.

24. Which of the following limits exists in $\mathbb{R}$?

(A) $\lim_{x \to 0^+} \frac{1}{x}$.

(B) $\lim_{x \to 0^+} \left|\sin \frac{1}{x}\right|$.

(C) $\lim_{x \to 0^+} x \sin \frac{1}{x}$.

(D) $\lim_{x \to 0^+} \frac{1}{x} \sin \frac{1}{x}$.

25. Let $\{a_n\} \to a$ and $\{b_n\} \to b$ in $\mathbb{R}$. Consider the following statements.

S_1 : If $a_n \leq b$ and $b_n \leq a$ for infinitely many $n \in \mathbb{N}$ then $a = b$.

S_2 : If $a_n \leq b_n$ for infinitely many $n \in \mathbb{N}$ then $a \leq b$.

Then

(A) both S_1 and S_2 are true.
(B) S_1 is true but S_2 is false.
(C) S_1 is false but S_2 is true.
(D) both S_1 and S_2 are false.

Part-B

26. Let A be the closure in $\mathbb{R}$ of the set $\{n + \frac{1}{m} + \frac{1}{k} : n, m, k \in \mathbb{N}\}$. For $S \subset \mathbb{R}$, let S' denote the set of limit points of S in $\mathbb{R}$. Then

(A) $\mathbb{N} \subseteq A'$.
(B) $\mathbb{N} \cup \{n + \frac{1}{m} : m \in \mathbb{N}\} \subseteq A'$.
(C) $\mathbb{N} \cup \{n + \frac{1}{m} : m \in \mathbb{N}\} \cup \{n + \frac{1}{m} + \frac{1}{k} : n, m, k \in \mathbb{N}\} \subseteq A'$.
(D) $\mathbb{N} \cup \{n + \frac{1}{m} + \frac{1}{k} + \frac{1}{\ell} : n, m, k, \ell \in \mathbb{N}\} \subseteq A'$.

27. The equation of the sphere through the circle $x^2 + y^2 + z^2 = 1$, $2x + 4y + 5z = 6$ and touching the plane $z = 0$ is

(A) $x^2 + y^2 + z^2 - 2x - 4y - 5z + 5 = 0$.
(B) $5(x^2 + y^2 + z^2) - 2x - 4y - 5z + 1 = 0$.
(C) $x^2 + y^2 + z^2 + 2x + 4y + 5z - 5 = 0$.
(D) $5(x^2 + y^2 + z^2) + 2x + 4y + 5z - 1 = 0$.

28. The volume of revolution of the region between $y = x^2$ and $y = 5x$ when revolved about the x-axis is

(A) $5^4\pi/3$. (B) $(2/3)5^4\pi$. (C) $5^4/3$. (D) $(2/3)5^4$.

29. The equation of the plane which is perpendicular to the plane $5x + 3y + 6z + 8 = 0$ and contains a line of intersection of the planes $x + 2y + 3z - 4 = 0$, $2x + y - z + 5 = 0$ is

(A) $6x + 18y - 14z = 1$.
(B) $75x - 123y - z = 5$.
(C) $51x + 15y - 50z + 173 = 0$.
(D) $23x - 5y - 50z = 1$.

30. Let $f : (0,1) \to \mathbb{R}$ be a twice differentiable function. Which of the following statements are **False**?

(A) If f is bounded then f' is bounded.

(B) If f and f' are bounded then f'' is bounded.

(C) If $f(x) > 0$ and $f'(x) > 0$, $\forall\, x \in (0,1)$, then $f''(x) > 0$, $\forall\, x \in (0,1)$.

(D) If f'' is a polynomial then f is a polynomial.

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31. Let A be a non-empty subset of $\mathbb{R}$. Which of the following statements are **True**?

(A) If A is closed and dense in $\mathbb{R}$ then $A = \mathbb{R}$.

(B) If A is closed then A is infinite.

(C) If A is open then A is infinite.

(D) If A is infinite then A is either open or closed or dense in $\mathbb{R}$.

32. Let $A \subset \mathbb{R}$. Let A° denote the set of interior points of A and A' denote the set of all limit points of A in $\mathbb{R}$. Then which of the following statements are **True**?

(A) If A is finite then A° is empty.

(B) If A is finite then A' is empty.

(C) If A is countably infinite then A^0 is countably infinite.

(D) If A is countably infinite then A' is countably infinite.

33. Let $f : [-1, 1] \to \mathbb{R}$ be continuous on $[-1, 1]$ and twice differentiable on $(-1, 1)$. Let $f(0) = 0$, $f(-1) = -1$ and $f'(1/2) = 1$. Then which of the following are always **True**?

(A) there exists $c \in (-1, 0)$ such that $f'(c) = 1$.

(B) there exists $c \in (-1, 1)$ such that $f''(c) = 0$.

(C) $f''(c) = 0$ for all $c \in (-1, 1)$.

(D) $f(1) > 0$.

34. Let $x_n \in (0, \infty)$ be such that $\sum_{n=1}^{\infty} x_n^2 < \infty$. Which of the following series are convergent?

(A) $\sum_{n=1}^{\infty} \frac{1}{x_n^2 + n}$.

(B) $\sum_{n=1}^{\infty} \frac{x_n}{x_n^2 + 1}$.

(C) $\sum_{n=1}^{\infty} \frac{1}{x_n^2 + n^2}$.

(D) $\sum_{n=1}^{\infty} \frac{x_n}{n^{1/3}}$.

35. Let $a_n \leq b_n$ then which of the following statements are **True**?

(A) If $\sum b_n$ converges then $\sum a_n$ always converges.

(B) If $\sum b_n$ diverges then $\sum a_n$ always diverges.

(C) If $a_n \geq 0$ and $\sum b_n$ converges then $\sum a_n$ always converges.

(D) If $a_n \geq 0$ and $\sum b_n$ diverges then $\sum a_n$ always diverges.

36. Which of the following functions $f : \mathbb{R} \to \mathbb{R}$ are differentiable at $x = 1$?

(A) $f(x) = \left| |x| - \frac{1}{2} \right|$.

(B) $f(x) = \max\{|x-1|, |x+1|\}$.

(C) $f(x) = |x-1| + e^x$.

(D) $f(x) = |e^x - 1|$.

37. Which of the following permutations in S_5 are even?

(A) $(1\,2\,3\,4\,5)$. (B) $(1\,2\,3\,4)(1\,3)$. (C) $(1\,2)(3\,4)(4\,5)$. (D) $(1\,2\,3\,4)$.

38. The center of $\mathrm{GL}_2(\mathbb{R})$ is

(A) $\left\{ \begin{pmatrix} a & c \\ 0 & a \end{pmatrix} : a \neq 0,\ c \in \mathbb{R} \right\}$. (B) $\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \neq 0 \right\}$.

(C) $\mathrm{GL}_2(\mathbb{R})$. (D) $\{A \in \mathrm{GL}_2(\mathbb{R}) : \det(A) = 1\}$.

39. Which of the following statements are **True**?

(A) Every group whose order is the power of a prime has a non-trivial centre.

(B) Every group of order p^2 is cyclic, where p is a prime number.

(C) Every non-abelian group of order 6 is isomorphic to S_3.

(D) Every finite group of order n is isomorphic to a subgroup of S_n.

40. The automorphism group of $\mathbb{Z}/10\mathbb{Z}$ is isomorphic to

(A) $\mathbb{Z}/4\mathbb{Z}$. (B) $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. (C) $\mathbb{Z}/10\mathbb{Z}$. (D) S_4.

41. Which of the following are integral domains?

(A) $\mathbb{Z}$.

(B) $\mathbb{Z}[X]/(X^2 - 3)$.

(C) The ring of $n \times n$ matrices with entries in $\mathbb{R}$.

(D) The ring of continuous functions on the unit interval $[0, 1] \in \mathbb{R}$.

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42. The integer 11213 is a prime number and $11213 = 82^2 + 67^2$. Which of the following statements are **True** in the Gaussian ring $\mathbb{Z}[i]$?

(A) The ideal generated by $(82 + 67i)$ is a maximal ideal.

(B) $\pm 82 \pm 67i$, $\pm 67 \pm 82i$ are irreducible elements of $\mathbb{Z}[i]$.

(C) The Gaussian ring $\mathbb{Z}[i]$ is not a PID.

(D) $\mathbb{Z}[i]/(82 + 67i)$ is isomorphic to $\mathbb{Z}/11213\mathbb{Z}$.

43. Let A be a $n \times n$ matrix with real coefficients. Then A is non-singular if and only if

(A) for all $n \times p$ matrices B, $\text{rank}(AB) = \text{rank}(B)$.

(B) A has a left inverse.

(C) $\text{Tr}(A) = 0$.

(D) the row rank of A is n.

44. If a matrix A in $M_{n\times n}(\mathbb{R})$ where $n \geq 2$ is nilpotent, then which of the following statements are **True**?

(A) $I + A$ is non-singular.

(B) If A and B commute then AB is nilpotent.

(C) If P be any invertible matrix then PA is nilpotent.

(D) All eigenvalues of A are zero.

45. The solution of $2xy\ dx + \tan(x^2)\ dy + y^2 \sec(x^2)\ dy = 0$ is

(A) $y^2 + y\sin(x^2)$ =constant.

(B) $y^3 + 3y\sin(x^2)$ =constant.

(C) $x^3 + 3x\tan(y^2)$ =constant.

(D) $x^2 + x\sec(y^2)$ =constant.

46. The differential equation $\left(\frac{1}{x^2} + \frac{1}{y^2}\right) dx + \left(\frac{Ax+1}{y^3}\right) dy = 0$ is exact and has the of the form

(A) $A = -2$ and $2x^2 - 2y^2 - x = cxy^2$ where c is a constant.

(B) $A = 2$ and $2x^2 - 2y^2 - x = cxy$ where c is a constant.

(C) $A = -2$ and $2x^2 + 2y^2 - x = cxy$ where c is a constant.

(D) $A = 2$ and $2x^2 + 2y^2 - x = cxy^2$ where c is a constant.

47. Let (x_n) be a sequence in $\mathbb{R}$. Consider the following statements:

S_1: If $\{x_n\}$ is a Cauchy sequence , $(x_{3n+1}) \to a$ and $(x_{6n}) \to b$, then $a = b$.

S_2: If every subsequence $\{x_{n_k}\}$ has a convergent subsequence $\left\{x_{n_{k_l}}\right\}$, then $\{x_n\}$ is convergent.

Then

(A) both S_1 and S_2 are true.

(B) S_1 is true but S_2 is false.

(C) S_1 is false and S_2 is true

(D) both S_1 and S_2 are false.

48. If v_1, v_2, v_3 are linearly dependent vectors in a vector space V over a field F. If v_3 is not a linear combination of v_1 and v_2 then

(A) v_1 is a linear combination of v_2 and v_3.

(B) $\dim(V) \leq 2$.

(C) v_3 must be a zero vector.

(D) v_1 and v_2 are linearly dependent.

49. The number of words consisting of 4 letters from the letters of word CHEESE so that no two E's are together is

(A) 24. (B) 42. (C) 88. (D) 94.

50. Suppose a bag contains 4 white balls and 3 black balls. If two draws of 2 balls are successively made then the probability of getting 2 white balls at first draw and 2 black balls at second draw when the balls drawn at first draw are replaced is

(A) 3/7. (B) 1/7. (C) 19/49. (D) 2/49.

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